CWI Tracts

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Topological dynamix

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PREFACE

This book is an almost exact copy of the authors' dissertation, written under supervision of Prof.Dr. P.C. Baayen and Dr. J. de Vries, presented in Amsterdam at the Vrije Universiteit in 1982.

The subject matter consists of several topics from abstract topological dynamics that are closely related to the structure theory for minimal sets. By no means is it intended to be complete (for instance the reader will look in vain for topics like skew products and cohomology). The central themes are:

a) quasifactors of minimal ttgs

b) (weak) disjointness of homomorphisms of ttgs

c) the equicontinuous structure relation.

The notion of a minimal transformation group has existed as such for more than 50 years, but the structure theory is quite a young branch of mathematical research. Mainly under the influence of J. AUSLANDER, R. ELLIS and H. FURSTENBERG that theory arose in the sixties and, supplemented by the works of, among others, S. GLASNER, D.C. MCMAHON, W.A. VEECH and T.S. WU, it was developed further in the seventies. In the framework of this book it is unfeasible to draw a complete picture of the history of the subject. However, arguments concerning readability and notation and also the need for a consistent reference called for an extensive introduction in the form of chapter I. This chapter also contains some easy thoughts about semi-openness of homomorphisms that are helpful in the chapters IV and VII.

In chapter II the action on the hyperspace is introduced as are quasifactors and the circle operation.

The third chapter, as well, is chiefly introductory. The main theme here is to determine the equicontinuous structure relation in situations with enough almost periodicity to use the $\mathfrak{F}$-topologies as introduced by H. FURSTENBERG in [F63]. The purpose of this chapter is not only the introduction of the necessary notions but also the unification of the current approaches. The fourth and fifth chapters are devoted to a special form of proximality: high proximality. In chapter IV the highly proximal extensions themselves are being studied. In particular, the lifting of homomorphisms to open homomorphisms through highly proximal extensions is being considered as in the question of what kind of properties are invariant under this process. Moreover, some attention is paid to the Maximal Highly Proximal extension of a minimal ttg. In chapter V this will be studied more deeply by considering the structure of MHP generators. These MHP generators are certain closed subsets of the universal minimal ttg generating the MHP extensions as quasifactors. The MHP generator that generates the universal HPI ttg is constructed.

Disjointness and disjointness relations are the main subject of chapter VI. Two minimal ttgs are called disjoint if the cartesian product again is minimal. A typical result for this chapter is $\mathbf{PI}\cap\mathbf{P}^{\perp}\subseteq\mathbf{D}^{\perp\perp}$ in words: a minimal PI ttg which is disjoint from every minimal proximal ttg also is disjoint from every minimal ttg that is disjoint from every minimal distal ttg. The results are put together in two pictures. The results are also applied to the question whether or not two minimal ttgs are disjoint if they do not have a common nontrivial factor.

In chapter VII weak disjointness is being considered (two minimal ttgs are called weakly disjoint if the cartesian product is ergodic). An important role is played by homomorphisms with an additional measure structure: RIM extensions. Among others it is shown that for open RIM extensions of minimal ttgs the regionally proximal relation is an equivalence relation. Another question that is dealt with is to what extent weak disjointness of homomorphisms is implied by the disjointness of their maximal almost periodic factors.

The final chapter is mainly devoted to a study of a sharp form of regional proximality. In particular, the question is studied whether or not the equality of the regionally proximal relation and the sharply regionally proximal relation implies that the regionally proximal relation is an equivalence relation. The answer turns out to be the affirmative if the extension is open and also if the spaces are metric.

The chapters IV and V contain the results of research done in collaboration with J. Auslander [AW 81], and the results in chapter VIII and in VII.3. have been obtained together with J. Auslander, D.C. McMahon and T.S. Wu [AMWW ?]. Reading through the text one will encounter the reference [VW ?]. This concerns a not yet existent book, to be written by J. De Vries (originally planned to be written by J. De Vries and the present author). In that monograph the preliminaries for the structure theory will be dealt with in detail. It will also contain the results on the structure of minimal ttgs known up to the present day. After its completion, it will be a good introduction to the present book.

ACKNOWLEDGEMENTS

Lots of thoughts put together, sorted out or thrown away. To what purpose? What does it help? To me this is unimportant compared to the pure existence of this book. One might learn some facts about topological dynamics by reading it, but one hardly gets to know the depths, the failures and the despair leading to these 300 pages of satisfaction. It is quite unlikely I would have overcome the barriers without the encouragement of several cheerleaders:

Cor Baayen, my promotor, who believed in air-bubbles. I only saw the air, he somehow caught sight of a bubble, cherished it and gave it time and room to develop. Andries Brouwer, indispensable by embodying fun in mathematics and kicking me through the final crawlway. Joe Auslander, the initiator, active and passive, of so many topics in this book and the one that triggered my mathematical self-confidence during my stay at College Park. Ta Sun Wu and the gigantic amount of his stimulating letters; I took over several of his suggestions but his drafts still carry the potential for another book. But above all Jan de Vries. His careful reading and, as a result of that, his suggestions and the many corrections he made, were invaluable. The time he spent sifting out the manuscript and the interest he showed have been an enormous stimulus. Several other mathematicians contributed either thoughts or a fruitful environment for math-thinking, like Doug McMahon, Bob Ellis, Bill Veech, Eli Glasner, the members of the working group on topological dynamics in Amsterdam (serving as guinea-pigs) and of course T. S. McWoulander.

But in order to be important for the realization of this book one's contribution needn't be of a mathematical nature. The Mathematical Centre enabled me to work on the subject and to do the type-setting myself. Han, Gert-Jan, Jaap, Teus and most of all Bert assisted me in learning UNIX and TROFF, in using them and in getting the prints as they are.

The final stage in the birth of this book has been taken care of by Tobias (bloody diagrams) the printers Jan, Jos, Jaap en Frank, and, as coordinator, Dick.

I owe a last word of gratitude to all my friends, who showed interest and compassion. And most of all to my homemates Henny, Yob, Renee and the pets; they suffered from my absence and, worse, from my absence in presence.

Jaap

A variety of circumstances have resulted in a considerably delayed appearance of this monograph. For this we apologize to author and readers.

Managing editors

CONTENTS

I

BASICS, PRELIMINARIES AND GENERALITIES

The branch of mathematics called topological dynamics mainly emerged from the qualitative theory of differential equations. It studies classical dynamics from a topological point of view. This development was initiated by H. POINCARE and carried on by G.D. BIRKHOFF in the first decades of this century [Bi 27]. The latter explicitly generalized notions from the qualitative theory of autonomous differential equations to those for one parameter groups of transformations on abstract spaces. To him we owe notions like minimality and recurrence.

At about the same time the study of geodesics lead to the concept of symbolic dynamics ([H 98], [Mo 21,66]). Other related branches of mathematics at that time were the theory of measure preserving transformations and that of almost periodic functions.

At the end of the forties W.H. GOTTSCHALK and G.A. HEDLUND generalized the classical dynamical systems to arbitrary topological transformation groups (i.e., actions of arbitrary topological groups on arbitrary topological spaces) thus unifying many aspects of the mathematics mentioned above [GH 55].

From 1960 on the activity in the field of topological dynamics grew rapidly under the impact of the work of R. ELLIS and H. FURSTENBERG.

As our main interest is the structure theory of minimal transformation groups and their classification, this presentation of the basics of topological dynamics

and its concepts is chosen from that point of view. We do not pretend any completeness, in fact we try to omit everything not strictly needed for our purposes.
In the first section of this chapter we present the basic definitions of transformation groups and of several dynamical notions, with some of their most important properties. The second section deals with the algebraic approach to the asymptotic behavior of the action of a certain topological group T as developed mainly by R. ELLIS ; i.e., we discuss or rather picture the semigroup action of the universal ambit $\mathfrak{S}_T$ for T . In section 3. we shall prepare us for the comparison of transformation groups with each other (or, rather, that of homomorphisms of topological transformation groups with the same codomain), e.g. see IV.4. and VII.3..

If references are given, we let references to monographs prevail above others. The reader is assumed to be familiar with standard notions in general topology such as can be found in [Wi 70], [Du 66] and [Kl 55].

I.1. TRANSFORMATION GROUPS

In this section we shall define some basic notions in topological dynamics, as far as they are of interest for our purposes, which is mainly the structure theory of minimal transformation groups. No efforts to completeness and selfcontainedness are made; on the contrary, as the material is completely standard only the most urgently needed concepts and properties are discussed. The reader interested in details or eager for the motivation of this kind of mathematics is referred to such well organized texts as [B 75/79], [E 69] and [VW ?].

A *topological transformation group* (*ttg* for short) is a triple $<T,X,\pi>$, where T is a topological group, the *phase group*; X is a nonempty topological space, the *phase space*; and $\pi: T\times X \to X$, the *action*, is a (jointly) continuous map, such that

a) $\pi(e,x)=x$ for every $x\in X$, where $e\in T$ is the unit element;
b) $\pi(s,\pi(t,x))=\pi(st,x)$ for every $x\in X$ and $s,t\in T$.

If T is a topological group then T_d denotes the topological group with the same underlying group as T , but provided with the discrete topology.

Clearly, if $<T,X,\pi>$ is a ttg, then $<T_d,X,\pi>$ is a ttg too.

Let $<T,X,\pi>$ be a ttg. Then the map $\pi^t : X \to X$ defined by $\pi^t(x) := \pi(t,x)$ $(x \in X)$ is a homeomorphism and $(\pi^t)^{\leftarrow} = \pi^{t^{-1}}$ for every $t \in T$. So we can consider T as a topological homeomorphism group for X. The map $\pi_x : T \to X$ defined by $\pi_x(t) = \pi(t,x)$ $(t \in T)$ is a continuous map for every $x \in X$. We call $\pi_x[T]$ the *orbit* of x, and $\overline{\pi_x[T]}$ the *orbit closure* of x.

Unless stated otherwise, we assume T to be an arbitrary, but fixed, Hausdorff topological group; the phase space X of a ttg $<T,X,\pi>$ will always be a compact Hausdorff (CT_2) space with unique uniformity $\mathcal{U}_X$.
Whenever misunderstanding is unlikely, which is almost always the case, we shall suppress the action symbol and write the action as a "multiplication". So $tx := \pi(t,x)$ for every $x \in X$, $t \in T$; then the axioms for a ttg (apart from continuity) can be expressed as follows:

a) $ex = x$ for every $x \in X$, where $e \in T$ is the unit element in T;
b) $s(tx) = (st)x$ for every $x \in X$, $s,t \in T$.

As a consequence, the orbit and orbit closure of x are denoted by Tx and $\overline{Tx}$ respectively.

The phase group and the action being understood, we shall denote a ttg by its phase space only, but in a different font (script capitals). Thus $\mathcal{X}$ will always denote the ttg with X as a phase space and (the fixed) phase group T (if misunderstanding is unlikely).

A subset A of X is called (T-) *invariant* if

$$TA = \{ta \mid t \in T, a \in A\} \subseteq A \ ;$$

A is called *minimal* if A is nonempty, closed and T-invariant and A is minimal under that condition; i.e., if $B \subseteq X$ is nonempty, closed and T-invariant, and if $B \subseteq A$, then $B = A$.
Clearly, if A is T-invariant then $A = TA$, and the sets A°, $\overline{A}$ and $X \setminus A$ are easily seen to be T-invariant. If A is a nonempty closed invariant subset of X, then we may restrict the action of T on X to an action of T on A; i.e., $\mathcal{A} := <T,A,\pi|_{T\times A}>$ is a ttg. Such a ttg $\mathcal{A}$ is called a *subttg* of $\mathcal{X}$. A ttg $\mathcal{X}$ is called *minimal,* if X is a minimal subset of X, and so $\mathcal{X}$ is minimal iff $\mathcal{X}$ does not have nontrivial subttgs. Note that by a straightforward application of Zorn's lemma it follows that every ttg has a minimal subttg.

1.1. **THEOREM.** *Let* $\mathfrak{X}$ *be a ttg. The following statements are equivalent:*

a) $\mathfrak{X}$ *is a minimal ttg;*

b) *every* $x \in X$ *has a dense orbit; i.e.,* $X = \overline{Tx}$ *for every* $x \in X$ *;*

c) $X = TU$ *for every nonempty open set* $U \subseteq X$ *;*

d) *for every nonempty open* $U \subseteq X$ *, there is a finite subset* $F \subseteq T$ *with* $X = FU$ *.* □

A nonempty closed invariant subset A of X is called *point transitive* if there is an $a \in A$ such that $A = \overline{Ta}$; and such a point a is called a *transitive point* for A . In addition, $\mathfrak{X}$ is called a *point transitive ttg* if X is a point transitive subset of X . Obviously a minimal ttg is point transitive and every point in its phase space is a transitive point.

A nonempty closed invariant subset A of X is called *ergodic* if A does not have a proper invariant closed subset with nonempty interior (in A); and a ttg $\mathfrak{X}$ is *ergodic* if X is an ergodic subset of X . We could paraphrase this by saying that $\mathfrak{X}$ is ergodic if $\mathfrak{X}$ does not have a proper "substantial" subttg. Clearly every point transitive ttg is ergodic; hence every minimal ttg is ergodic. Under several conditions the converse is true (see 1.2.b resp. 1.17.) but not always (see 4.9. resp. II.1.10,11.).

1.2. **THEOREM.** *Let* $\mathfrak{X}$ *be a ttg.*

a) $\mathfrak{X}$ *is ergodic iff* $X = \overline{TU}$ *for every nonempty open* $U \subseteq X$ *iff for all nonempty open* U *and* V *in* X *there exists a* $t \in T$ *with* $U \cap tV \neq \emptyset$ *.*

b) *If* X *has a countable pseudobase, the following statements are equivalent:*

(i) $\mathfrak{X}$ *is ergodic;*

(ii) $\mathfrak{X}$ *is point transitive;*

(iii) *there is a dense* G_δ*-set of transitive points in* X *.*

[Note that a collection $\mathfrak{B}$ of open sets in X is called a *pseudobase* if for every open set $U \subseteq X$ there is a $B \in \mathfrak{B}$ with $B \subseteq U$ [Wi 70].] □

Let Λ be an index set and let for every $\lambda \in \Lambda$ a ttg $\mathfrak{X}_\lambda$ be given. Then we define the *product ttg* $\mathfrak{X} = \Pi\{\mathfrak{X}_\lambda \mid \lambda \in \Lambda\}$ as follows:

The phase space X of $\mathfrak{X}$ is given by $X = \Pi\{X_\lambda \mid \lambda \in \Lambda\}$ and the action of T on X by $tx = t\,(x_\lambda)_{\lambda \in \Lambda} = (tx_\lambda)_{\lambda \in \Lambda}$ for every $t \in T, x \in X$; i.e., the action of T on X is defined coordinatewise.

Clearly, $\mathfrak{X}$ is a ttg.

One could ask several questions about products, for instance (cf. [F 67]):

(i) when is the product of two minimal ttgs again minimal?
(ii) when is the product of an ergodic ttg and a minimal ttg ergodic?

In chapter VI we discuss problems related to (i) and in chapter VII we deal with variations on question (ii) (see also the discussion about (weak) disjointness in section I.3.).
Note that if $\mathfrak{X}$ is a minimal ttg, $\mathfrak{X} \times \mathfrak{X}$ is not minimal unless $\mathfrak{X} = \{\star\}$ (where $\{\star\}$ denotes the *trivial one point ttg*), for $\Delta_X \subseteq X \times X$ is a nonempty closed invariant subset of $X \times X$. However, if $\mathfrak{X}$ is ergodic it can occur that $\mathfrak{X} \times \mathfrak{X}$ is again ergodic; such a ttg is called *weakly mixing* (e.g. 4.8.).

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be ttgs (for T) and let $\phi: X \to Y$ be a mapping. Then ϕ is called *equivariant* if $\phi(tx) = t\,\phi(x)$ for every $x \in X$, $t \in T$; i.e., ϕ commutes with the actions (of T) on X and Y. A continuous equivariant map $\phi: X \to Y$ is called a *homomorphism of ttgs*; as such it will be denoted by $\phi: \mathfrak{X} \to \mathfrak{Y}$. If ϕ is surjective we use at random other terminologies like " ϕ is an *extension*", " $\mathfrak{X}$ is an *extension* of $\mathfrak{Y}$ " or " $\mathfrak{Y}$ is a *factor* of $\mathfrak{X}$ ". If $\phi: X \to Y$ is an equivariant homeomorphism, then $\phi: \mathfrak{X} \to \mathfrak{Y}$ is called an *isomorphism of ttgs*. For $\phi: \mathfrak{X} \to \mathfrak{Y}$ and $\psi: \mathfrak{Y} \to \mathfrak{Z}$, both homomorphisms of ttgs, the map $\theta := \psi \circ \phi$ is a homomorphism of ttgs and if ϕ is surjective, we call ψ a factor of θ (by ϕ).
Note that a ttg $\mathfrak{X}$ can be considered as a homomorphism from $\mathfrak{X}$ to $\{\star\}$. We call a property *absolute* or *relative* whenever we consider the property for ttgs or the corresponding property for homomorphisms of ttgs, respectively.

Let $\mathfrak{X}$ be a ttg and let R be an equivalence relation on X such that R as a subset of $X \times X$ is closed and invariant. It is not difficult to show that the map $\pi: T \times X/R \to X/R$, defined by $\pi(t, R[x]) = R[tx]$ for every $t \in T$, $x \in X$, is a continuous action of T on X/R. Hence $\mathfrak{Y} := \mathfrak{X}/R$ is a ttg and the quotient map $\kappa: \mathfrak{X} \to \mathfrak{Y}$ is a surjective homomorphism of ttgs with $R = \{(x_1, x_2) \in X \times X \mid \kappa(x_1) = \kappa(x_2)\}$.
Conversely, for a surjective homomorphism $\phi: \mathfrak{X} \to \mathfrak{Y}$ of ttgs we define

$$R_\phi := \{(x_1, x_2) \in X \times X \mid \phi(x_1) = \phi(x_2)\} = \bigcup \{\phi^{\leftarrow}(y) \times \phi^{\leftarrow}(y) \mid y \in Y\}.$$

Then R_ϕ is a nonempty invariant closed equivalence relation on X, $\mathfrak{R}_\phi$ is a subttg of $\mathfrak{X} \times \mathfrak{X}$, and $Y \cong X/R_\phi$ $(\mathfrak{Y} \cong \mathfrak{X}/\mathfrak{R}_\phi)$.
So there is a one to one correspondence between the surjective homomorphisms with domain $\mathfrak{X}$ and the invariant closed equivalence relations on X.

Recall that a map $f:X \to Y$ of topological spaces is called *semi-open* if $\mathrm{int}_Y\, \phi[U] \neq \varnothing$ whenever $\mathrm{int}_X\, U \neq \varnothing$.

1.3. **REMARK.** *Let* $\phi\colon \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of ttgs. Then:*

a) *if* $A \subseteq X$ *is closed and invariant then* $\phi[A]$ *is closed and invariant; in particular, the image of an orbit closure is an orbit closure;*

b) $\phi[X]$ *is a nonempty closed invariant subset of* Y , *so* $\phi[\mathfrak{X}]$ *is a subttg of* $\mathfrak{Y}$ *;*

c) *if* $\mathfrak{Y}$ *is minimal then* ϕ *is a surjective homomorphism of ttgs;*

d) *if* $\mathfrak{Y}$ *is ergodic and* ϕ *is semi-open then* ϕ *is a surjective homomorphism of ttgs;*

e) *if* $\mathfrak{X}$ *is minimal, point transitive, ergodic or weakly mixing then* $\phi[\mathfrak{X}]$ *has the corresponding property.* □

Openness of homomorphisms plays an important role in our considerations; e.g. see sections IV.3. and VII.2. and the result in VIII.3.4.. Although openness is not always guaranteed, homomorphisms of minimal ttgs are open to a certain extent (besides the following result see also III.2.8.).

1.4. **THEOREM.** *Let* $\phi\colon \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of ttgs with* $\mathfrak{Y}$ *minimal.*

a) *If* $\mathfrak{X}$ *is minimal,* ϕ *is semi-open.*

b) *If* X *has a dense set of points with a minimal orbit closure then* ϕ *is semi-open.*

PROOF.

a) Let $U \subseteq X$ be nonempty and open and let $V \subseteq X$ be nonempty and open such that $V \subseteq \overline{V} \subseteq U$. Let $F \subseteq T$ be finite such that $FV = X$ (1.1.d). Then

$$Y = \phi[X] = \phi[F\overline{V}] = F.\phi[\overline{V}];$$

and so, for some $t \in F$, $t\,\phi[\overline{V}]$ has a nonempty interior. As left multiplication with t^{-1} is a homeomorphism, $\phi[\overline{V}] = t^{-1}t\,\phi[\overline{V}]$ has a nonempty interior, so $\phi[U]$ has a nonempty interior.

b) Let $U \subseteq X$ be nonempty and open and let $Z \subseteq X$ be minimal subset of X with $U \cap Z \neq \varnothing$. As (by a) $\phi|_Z$ is semi-open, it follows that $\phi|_Z\,[U \cap Z]$ has a nonempty interior in $\phi[Z] = Y$. Hence, after observing that $\phi|_Z\,[U \cap Z] \subseteq \phi[U]$, the proof is completed. □

1.5. **EXAMPLE.**

Let $\mathcal{X} = <T, X, \pi>$ be a ttg. Consider X^X equipped with the product topology. Under the composition of maps, X^X is a right semitopological semigroup, and X^X is a CT_2 space.
Define $\bar{\pi}: T \to X^X$ by $\bar{\pi}(t) = \pi^t$; i.e., represent the elements of T as homeomorphisms of X . Then the corestriction of $\bar{\pi}$ to $\bar{\pi}[T]$ is a continuous homomorphism of groups. Define

$$E(X) := E(<T, X, \pi>) := \mathrm{cl}_{X^X}\, \bar{\pi}[T] ,$$

then clearly $E(X)$ is a CT_2 space. One can show that $E(X)$ is a subsemigroup of the right semitopological semigroup X^X into which T is densely mapped by $\bar{\pi}$.
On $E(X)$ we can define an action $\tilde{\pi}$ of T by $\tilde{\pi}(t, f) := \pi^t \circ f$ for every $t \in T$, $f \in E(X)$. Clearly, $E(\mathcal{X}) := <T, E(X), \tilde{\pi}>$ is a subttg of the product ttg $\mathcal{X}^X$.
The set $E(X)$ as well as the ttg $E(\mathcal{X})$ are called the *enveloping semigroup* of $\mathcal{X}$. The following facts are standard (cf. [E 69], chapter 3):

a) *$E(\mathcal{X})$ is a point transitive ttg (every " $t \in T$ " is a transitive point) and $E(\mathcal{X})$ is minimal iff $E(X)$ is a group.*

b) *For every $x_0 \in X$ the map $\delta_{x_0}: E(\mathcal{X}) \to \mathcal{X}$, defined by $\delta_{x_0}(f) := f(x_0)$ for every $f \in E(X)$, is a homomorphism of ttgs; and $\delta_{x_0}[E(X)] = \overline{Tx_0}$.*

c) *If $\phi: \mathcal{X} \to \mathcal{Y}$ is a surjective homomorphism of ttgs, then there is a unique surjective homomorphism $\tilde{\phi}: E(\mathcal{X}) \to E(\mathcal{Y})$ such that for every $x_0 \in X$ we have $\phi \circ \delta_{x_0} = \delta_{\phi(x_0)} \circ \tilde{\phi}$, and $\tilde{\phi}$ is a semigroup homomorphism.*

One could paraphraze b by saying that $E(X)$ acts on every orbit closure in X in such a way that it extends the action of T ; $E(X)$ embodies the limit behavior of T .
The investigations with respect to the algebraic properties of this action of $E(X)$ on X , that were initiated by R. ELLIS ([E 60]) turned out to be rather important for topological dynamics. We shall deal with this in section I.2..

Another way of constructing ttgs is given by the notion of inverse limit:
Let ν be an ordinal and let $\mathcal{X}_\lambda$ be a ttg for every $\lambda < \nu$. A *tower of height* ν , or an *inverse system of height* ν will be a collection $\{\phi_\alpha^\beta \mid \alpha \leqslant \beta < \nu\}$ of

surjective homomorphisms $\phi_\alpha^\beta : \mathfrak{X}_\beta \to \mathfrak{X}_\alpha$ of ttgs such that for every $\alpha \leqslant \beta \leqslant \gamma < \nu$ we have $\phi_\alpha^\beta \circ \phi_\beta^\gamma = \phi_\alpha^\gamma$.
Let $X = \operatorname{inv}\lim\{X_\lambda \mid \lambda < \nu\}$ in the category of CT_2 spaces; we can represent X as the subset of $\Pi\{X_\lambda \mid \lambda < \nu\}$ consisting of all ν-tuples $(x_\lambda)_{\lambda<\nu}$ such that $\phi_\alpha^\beta(x_\beta) = x_\alpha$ for every $\alpha \leqslant \beta < \nu$. Denote the projections by $\phi_\lambda : X \to X_\lambda$, then $\phi_\alpha^\beta \circ \phi_\beta = \phi_\alpha$ for every $\alpha \leqslant \beta < \nu$. A base for the topology on X is formed by the collection

$$\{\phi_\lambda^{\leftarrow}[U] \mid U \text{ open in } X_\lambda, \lambda < \nu\}.$$

As all spaces are compact, X is a nonempty closed subset of $\Pi\{X_\lambda \mid \lambda < \nu\}$ and clearly X is T-invariant, so $\mathfrak{X}$ is a ttg and the projections $\phi_\lambda : \mathfrak{X} \to \mathfrak{X}_\lambda$ are homomorphisms of ttgs.
The homomorphism $\phi_0 : \mathfrak{X} \to \mathfrak{X}_0$ is called the *inverse limit* of $\{\phi_\alpha^\beta \mid \alpha \leqslant \beta < \nu\}$.
Note that if $\mathfrak{Z}$ is a ttg and

$$\mathfrak{X} = \operatorname{inv}\lim\{\phi_\alpha^\beta : \mathfrak{X}_\beta \to \mathfrak{X}_\alpha \mid \alpha \leqslant \beta < \nu\}$$

then

$$\mathfrak{Z} \times \mathfrak{X} = \operatorname{inv}\lim\{id_Z \times \phi_\alpha^\beta : \mathfrak{Z} \times \mathfrak{X}_\beta \to \mathfrak{Z} \times \mathfrak{X}_\alpha \mid \alpha \leqslant \beta < \nu\}.$$

It follows that

$$\mathfrak{X} \times \mathfrak{X} = \operatorname{inv}\lim\{\phi_\alpha^\beta \times \phi_\alpha^\beta : \mathfrak{X}_\beta \times \mathfrak{X}_\beta \to \mathfrak{X}_\alpha \times \mathfrak{X}_\alpha \mid \alpha \leqslant \beta < \nu\}.$$

1.6. **REMARK.** *let* $\{\phi_\alpha^\beta : \mathfrak{X}_\beta \to \mathfrak{X}_\alpha \mid \alpha \leqslant \beta < \nu\}$ *be an inverse system, and let* $\mathfrak{X} = \operatorname{inv}\lim \mathfrak{X}_\lambda$. *Then* $\mathfrak{X}$ *is minimal, ergodic or weakly mixing iff* $\mathfrak{X}$ *has that property for every* $\lambda < \nu$. □

Let $\mathfrak{X}$ be a ttg, then $\mathfrak{X}$ is called *strictly-quasi-separable* if $\mathfrak{X}$ is the inverse limit of ttgs with metric phase spaces and $\mathfrak{X}$ is called *quasi-separable* if $\mathfrak{X}$ is a factor of a strictly-quasi-separable ttg. Note that the definitions here are slightly different from the usual ones (e.g. [E 69], [K 71] and [K 72]).

1.7. **THEOREM.** ([K 72]) *If* T *is a locally compact* σ*-compact topological group, then every point transitive ttg (for* T *) is strictly-quasi-separable.* □

We shall now turn to some basical dynamical notions (after [GH 55]).
Fix a collection $\mathfrak{A}$ of subsets of T , the *admissible sets*, and let $\mathfrak{X}$ be a ttg. A point $x \in X$ is called (*locally*) *recursive* if for every $U \in \mathcal{V}_x$ there is an

$A \in \mathcal{Q}$ (and a $V \in \mathcal{V}_x$) such that $Ax \subseteq U$ $(AV \subseteq U)$; here $\mathcal{V}_x$ denotes the neighbourhood filter of x . The ttg $\mathfrak{X}$ is called *pointwise* (*locally*) *recursive* if every $x \in X$ is a (locally) recursive point. $\mathfrak{X}$ is called *uniformly recursive* if for every index $\alpha \in \mathcal{U}_X$ there is an $A \in \mathcal{Q}$ such that $Ax \subseteq \alpha(x)$ for every $x \in X$. The type of recursiveness we are interested in in this booklet is almost periodicity. In order to define almost periodicity we have to define a special collection of admissible sets. A subset B of T is called (*right*) *syndetic* if there is a compact subset K of T such that $KB = T$. If we let $\mathcal{Q}$ be the collection of syndetic subsets of T , recursiveness with respect to $\mathcal{Q}$ is called *almost periodicity*. As being syndetic depends on the topology of T , almost periodicity seems to depend on the topology of T ; however, it turns out it actually doesn't (see 1.9., 1.11.b and 1.12.). If T is endowed with the discrete topology, $B \subseteq T$ is syndetic if $T = FB$ for a finite subset F of T . Almost periodicity with respect to the discrete topology on T (T_d) is called *discrete almost periodicity*.

Note that if $\mathfrak{X} = <T, X, \pi>$ is a ttg for T then any statement about discrete almost periodicity concerning $\mathfrak{X}$ is in fact a statement about almost periodicity concerning $<T_d, X, \pi>$. However, a statement about almost periodicity concerning $<T_d, X, \pi>$ is only a statement about discrete almost periodicity concerning $<T, X, \pi>$ provided that $<T, X, \pi>$ is a ttg!

1.8. **REMARK.** *Let* $\mathfrak{X}$ *be a ttg and let* $x \in X$.

a) *If* $\mathfrak{X}$ *is uniformly almost periodic, then* $\mathfrak{X}$ *is pointwise locally almost periodic.*

b) *If* $x \in X$ *is a locally almost periodic point, then* x *is an almost periodic point.* □

In the sequel a pointwise locally almost periodic ttg will be called a *locally almost periodic* ttg.

The next theorem shows the dynamics interest of minimal ttgs.

1.9. **THEOREM.** *Let* $\mathfrak{X}$ *be a ttg and* $x \in X$. *Then the following statements are equivalent:*

a) $\overline{Tx}$ *is a minimal subset of* X ;

b) x *is a discrete almost periodic point in* X ;

c) x *is an almost periodic point in* X . □

1.10. **REMARK.** *Let* $\phi : \mathcal{X} \to \mathcal{Y}$ *be a homomorphism of ttgs.*

a) *If* $x \in X$ *is an almost periodic point in* X *then* $\phi(x)$ *is almost periodic in* Y .

b) *If* $y \in \phi[X]$ *is an almost periodic point in* Y *then there is an almost periodic point* $x \in X$ *with* $\phi(x) = y$.

c) *If* $\mathcal{X}$ *is pointwise almost periodic, then* $\phi[\mathcal{X}]$ *is.*

d) *If* $\mathcal{Z}$ *is the inverse limit of a tower consisting entirely of pointwise almost periodic ttgs, then* $\mathcal{Z}$ *is pointwise almost periodic.* □

For local almost periodicity we can formulate similar statements, but the proofs are substantially harder (e.g. [E 69], [MW 72] and VI.5.6.):

1.11. **THEOREM.** *Let* $\phi : \mathcal{X} \to \mathcal{Y}$ *be a surjective homomorphism of ttgs, and let* $x \in X$ *be a transitive point.*

a) *The point* $x \in X$ *is locally almost periodic iff* $x' \in X$ *is locally almost periodic for every* $x' \in X$ *; so* $\mathcal{X}$ *is locally almost periodic* ([GH 55] 4.31.).

b) *The point* $x \in X$ *is locally almost periodic iff* x *is discrete locally almost periodic* ([B 75/79] 2.8.43.).

c) *If* $x' \in X$ *is locally almost periodic and if* ϕ *is open, then* $\phi(x')$ *is locally almost periodic.*

d) *If* $\mathcal{X}$ *is locally almost periodic, then so is* $\mathcal{Y}$ (cf. III.5.6.).

e) *If* $\mathcal{Z}$ *is the inverse limit of a tower consisting entirely of minimal locally almost periodic ttgs, then* $\mathcal{Z}$ *is minimal and locally almost periodic* (cf. III.5.6.). □

The following theorem characterizes uniform almost periodicity in terms of equicontinuity or compact group actions.

1.12. **THEOREM.** *Let* $\mathcal{X} = <T, X, \pi>$ *be a ttg. Then the following statements are equivalent* (cf. [B 75/79] 2.8.3. and [E 69] 4.5.)*:*

a) $\mathcal{X}$ *is uniformly almost periodic;*

b) $\mathcal{X}$ *is discrete uniformly almost periodic;*

c) $\overline{\pi}[T]$ *is an equicontinuous family of homeomorphisms;*

d) $\overline{\pi}[T]$ *is a uniformly equicontinuous family of homeomorphisms;*

e) $E(X)$ *is a* CT_2 *topological group consisting of homeomorphisms of* X . □

1.13. **REMARK.**

a) *A factor of a uniformly almost periodic ttg is uniformly almost periodic.*

b) *A subttg of a uniformly almost periodic ttg is uniformly almost periodic.*

c) *Let* ν *be an ordinal and let* $\mathfrak{X}_\lambda$ *be a ttg for every* $\lambda<\nu$ *. Then* $\Pi\{\mathfrak{X}_\lambda \mid \lambda<\nu\}$ *is uniformly almost periodic iff* $\mathfrak{X}_\lambda$ *is uniformly almost periodic for every* $\lambda<\nu$ *.*

d) *The inverse limit of a tower consisting entirely of uniformly almost periodic ttgs is uniformly almost periodic.* □

The uniformly almost periodic ttgs are the "beautiful ones". To illustrate this: if the phase space of a uniformly almost periodic ttg is metrizable, there is a compatible metric such that the T-translations $\{\pi^t \mid t\in T\}$ are isometries. In order to indicate how special the uniformly almost periodic minimal ttgs are, consider bT , the Bohr compactification of T ; bT is the reflection of the topological group T in the category of CT_2 topological groups. Then $\mathfrak{S}=<T,bT,\mu>$ is a minimal ttg, with $\mu\colon T\times bT\to bT$ defined by $\mu(t,x)=\iota(t)x$, where $\iota\colon T\to bT$ is the reflection.

1.14. **THEOREM.** *Let* $\mathfrak{X}$ *be a minimal ttg. Then* $\mathfrak{X}$ *is uniformly almost periodic iff* $\mathfrak{X}\cong\mathfrak{S}/H$ *for some closed subgroup* H *of* bT *. In fact,* $\mathfrak{S}$ *is the universal uniformly almost periodic minimal ttg for* T *. It follows that the phase space* X *of a uniformly almost periodic minimal ttg* $\mathfrak{X}$ *is homogeneous (in the sense that for every* x *and* x' *in* X *there is a homeomorphism* $f\colon X\to X$ *with* $f(x)=x'$ *).* □

No wonder that uniformly almost periodic minimal ttgs play the role of a touchstone in the structure theory for minimal ttgs; i.e., one of the approaches is to investigate to what extent a certain ttg differs from being uniformly almost periodic. One of the first dynamical concepts that was attacked in this approach was that of distality.

Let $\mathfrak{X}$ be a ttg and let x_1 and x_2 be elements of X . Then x_1 and x_2 are called *proximal*, or (x_1,x_2) is called a *proximal pair* if $\overline{T(x_1,x_2)}\cap\Delta_X\neq\varnothing$; in other words, x_1 and x_2 are proximal if there is a net $\{t_i\}_i$ in T with $\lim t_ix_1=\lim t_ix_2$. If $x_1=x_2$ or if (x_1,x_2) is not a proximal pair then (x_1,x_2) is called a *distal pair*, and x_1 and x_2 are called *distal.* If (x_1,x_2) is distal for every $x_2\in X$ then x_1 is called a *distal point for* $\mathfrak{X}$. The ttg $\mathfrak{X}$ is called *distal* (*proximal*) if every pair in

$X \times X$ is distal (proximal), $\mathfrak{X}$ is called *point distal* if there is a transitive distal point for X.
Before we indicate the connection between distality and almost periodicity we shall state some generalities on distal and proximal ttgs; the proofs of 1.15.b and 1.18. depend on the algebraic theory in I.2..

1.15. **THEOREM.** *Let $\mathfrak{X}$ be a ttg. Then the following statements are equivalent:*

a) *$\mathfrak{X}$ is distal;*

b) *$E(X)$ is a group (hence $E(X)$ is distal and minimal;* cf. 1.4. and 1.16.*);* [E 69] 5.3., 5.9.*;*

c) *$\mathfrak{X}^n$ is pointwise almost periodic for every $n \in \mathbb{N}$.* ☐

1.16. **REMARK.** ([E 69] chapter 5)

a) *A factor of a distal (proximal) ttg is distal (proximal).*

b) *A subttg of a distal (proximal) ttg is distal (proximal).*

c) *A product of distal (proximal) ttgs is distal (proximal).*

d) *An inverse limit of distal (proximal) ttgs is distal (proximal).* ☐

An interesting (and nontrivial) result is the following:

1.17. **THEOREM.** *An ergodic and distal ttg is minimal* ([E 78] 1.9.). ☐

Part of the relation between uniform almost periodicity and distality is given by:

1.18. **THEOREM.** *A ttg $\mathfrak{X}$ is uniformly almost periodic iff $\mathfrak{X}$ is distal and locally almost periodic* ([E 69] 5.28.). ☐

That distality alone is not sufficient for uniform almost periodicity may be seen from 4.5.(iii).
In the case of minimal ttgs the relation between uniform almost periodicity and distality is given by the FURSTENBERG STRUCTURE THEOREM ((1.24.), abbreviated: FST), which is the germ of a considerable part of topological dynamics.

Before we can state FST in full generality, we shall discuss the relative versions of notions such as almost periodicity. So let $\phi: \mathfrak{X} \to \mathfrak{Y}$ be a surjective homomorphism of ttgs. The extension ϕ is called a *group extension* if there is a CT_2 topological group K and an action of K on X that commutes

with the action of T on X (i.e., $tkx = ktx$ for every $x \in X$, $t \in T$ and $k \in K$) such that, in addition, $\phi^{\leftarrow}\phi(x) = Kx$ for every $x \in X$.
The map ϕ is called an *almost periodic extension* if ϕ is a factor of a group extension.

1.19. **NOTE.** *A minimal ttg* $\mathfrak{X}$ *is uniformly almost periodic iff* $\psi : \mathfrak{X} \to \{\star\}$ *is an almost periodic extension* (1.14.).

In studying uniform almost periodicity, the (relativized) regionally proximal relation plays an important role. Define the *regionally proximal relation for* ϕ by

$$Q_\phi := \bigcap \{\overline{T\alpha \cap R_\phi} \mid \alpha \in \mathfrak{U}_X\}.$$

and let $Q_{\mathfrak{X}}$ be defined as $Q_{\mathfrak{X}} := Q_\psi$ where $\psi : \mathfrak{X} \to \{\star\}$. Then Q_ϕ is always a closed, T-invariant, reflexive and symmetric relation, but in general Q_ϕ is not an equivalence relation (see VIII.1.5.).
Note that $(x_1, x_2) \in Q_\phi$ iff there is a net $\{(x_1^i, x_2^i)\}_i$ in R_ϕ and a net $\{t_i\}_i$ in T such that

$$(x_1^i, x_2^i) \to (x_1, x_2) \quad \text{and} \quad t_i\,(x_1^i, x_2^i) \to (z, z) \quad \text{for some } z \in X.$$

Define the *equicontinuous structure relation* E_ϕ for ϕ as the smallest invariant closed equivalence relation that contains Q_ϕ; $E_{\mathfrak{X}} := E_\psi$ where $\psi : \mathfrak{X} \to \{\star\}$.
One of the main themes in the structure theory for minimal ttgs is the question: under what conditions is E_ϕ equal to Q_ϕ. The importance of this question may be illustrated by the following theorem.

1.20. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a surjective homomorphism of ttgs.*

a) *The following statements are equivalent:*
   (i) ϕ *is an almost periodic extension;*
   (ii) $Q_\phi = \Delta_X$ *;*
   (iii) *for every* $\alpha \in \mathfrak{U}_X$ *there is a* $\beta \in \mathfrak{U}_X$ *with* $T\alpha \cap R_\phi \subseteq \beta$.

b) *Let* $\kappa : \mathfrak{X} \to \mathfrak{X}/E_\phi$ *be the quotient homomorphism and let* $\psi : \mathfrak{X}/E_\phi \to \mathfrak{Y}$ *be such that* $\psi \circ \kappa = \phi$. *Then* ψ *is the maximal almost periodic factor of* ϕ. *I.e., if* $\theta : \mathfrak{Z} \to \mathfrak{Y}$ *is an almost periodic extension such that* ϕ *factorizes over* θ, *then* ψ *factorizes over* θ.

c) *If* X *is a metrizable space, then* ϕ *is almost periodic iff there exists a continuous map* $d : R_\phi \to \mathbb{R}$ *which is T-invariant (i.e.,* $d(tx, ty) = d(x, y)$ *for every t, x, y), such that* d *is a metric on each fiber (such a* ϕ *is called* isometric*!).*

PROOF. Cf. [V 77] 2.4.3., [E 69], and [MW 76] 1.1.. □

The homomorphism ϕ is called *distal* (*proximal*) if every pair $(x_1, x_2) \in R_\phi$ is a distal (proximal) pair, and ϕ is called *point distal* if there is a transitive point $x \in X$ such that (x, x') is distal for every $x' \in \phi^{\leftarrow}\phi(x)$ (then x is called a ϕ- *distal point*).
Define the (*relative*) *proximal relations* P_ϕ and $P_{\mathfrak{X}}$ for ϕ and $\mathfrak{X}$ respectively by

$$P_\phi := \bigcap \{T\alpha \cap R_\phi \mid \alpha \in \mathfrak{U}_X\} \quad \text{and} \quad P_{\mathfrak{X}} := \bigcap \{T\alpha \mid \alpha \in \mathfrak{U}_X\}.$$

Then clearly, $P_\phi = P_{\mathfrak{X}} \cap R_\phi$, P_ϕ is the collection of proximal pairs in R_ϕ ; and ϕ is distal (proximal) iff $P_\phi = \Delta_X$ ($P_\phi = R_\phi$). In general P_ϕ is not closed and not an equivalence relation (4.7.(iii)). If P_ϕ is closed it is an equivalence relation ([A 60]), but not the other way round ([S 70]).
We shall now state some properties of distal, proximal and almost periodic extensions. (In the proof of 1.23.a,b the algebraic theory of I.2. plays a role.)

1.21. **THEOREM.**

a) *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$, $\theta: \mathfrak{Z} \to \mathfrak{X}$ *and* $\psi: \mathfrak{Z} \to \mathfrak{Y}$ *be surjective homomorphisms of ttgs such that* $\phi = \psi \circ \theta$.
Then ϕ *is proximal iff* θ *and* ψ *are proximal.*
If ϕ *is distal (almost periodic) then* θ *is distal (almost periodic).*
If $\mathfrak{Y}$ *is pointwise almost periodic then* ϕ *is distal iff* θ *and* ψ *are distal.*
If $\mathfrak{X}$ *is minimal and* ϕ *is almost periodic then* θ *and* ψ *are almost periodic.*

b) *Let* Λ *be an index set and let for every* $\lambda \in \Lambda$ *a surjective homomorphism of ttgs* $\phi_\lambda: \mathfrak{X}_\lambda \to \mathfrak{Y}_\lambda$ *be given and let* $\phi: \Pi_\Lambda \mathfrak{X}_\lambda \to \Pi_\Lambda \mathfrak{Y}_\lambda$ *be defined coordinatewise. Then* ϕ *is distal, proximal or almost periodic iff* ϕ_λ *is such for every* $\lambda \in \Lambda$.

c) *Let* ϕ *be the inverse limit of a tower* $\{\phi_\alpha^\beta \mid \alpha \leqslant \beta < \nu\}$. *Then* ϕ *is distal (proximal) iff* $\phi_\alpha^{\alpha+1}$ *is distal (proximal) for every* $\alpha + 1 < \nu$.

PROOF. Cf. [B 75/79] 3.12.28.,29. and [VW ?]. □

In general, the composition of two almost periodic extensions fails to be almost periodic, as can be seen from 4.5.(iii) and FST (1.24.). Sometimes, however, an almost periodic extension of a uniformly almost periodic ttg can be shown to be uniformly almost periodic:

1.22. **REMARK.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a surjective homomorphism of ttgs. If* R_ϕ *is open and closed in* $X \times X$ *, then* $Q_\phi = Q_{\mathfrak{X}}$ *.*
In particular, if $\operatorname{card}(Y) < \aleph_0$ *and* ϕ *is almost periodic, then* $\mathfrak{X}$ *is uniformly almost periodic* (compare [MW 76] 2.1.).

PROOF. For some $\alpha_0 \in \mathfrak{U}_X$, $\overline{T\alpha} \subseteq R_\phi$ so $\overline{T\alpha} = \overline{T\alpha \cap R_\phi}$ for every $\alpha \subseteq \alpha_0$. Hence

$$Q_{\mathfrak{X}} = \bigcap \{\overline{T\alpha} \mid \alpha \in \mathfrak{U}_X\} = \bigcap \{\overline{T\alpha \cap R_\phi} \mid \alpha \in \mathfrak{U}_X\} = Q_\phi .$$

□

1.23. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of ttgs with* $\mathfrak{Y}$ *minimal.*

a) *If* ϕ *is distal then* $\mathfrak{X}$ *is pointwise almost periodic.*
b) *The extension* ϕ *is distal iff* R_ϕ *is pointwise almost periodic.*
c) *If* ϕ *is proximal then* $\mathfrak{X}$ *has a unique minimal subttg.*
d) *The extension* ϕ *is proximal iff* R_ϕ *has a unique minimal subttg.*

PROOF. Cf. [G 76] II.1.1.,2. and [VW ?]. □

We shall now formulate the Furstenberg Structure Theorem (FST).
Although H. FURSTENBERG did not prove FST in its fullest generality, we still call 1.24. "the Furstenberg Structure Theorem" to honour the father of the basic idea in revealing the structure of distality. (The same we do with the Veech Structure Theorem IV.1.13..)
At first FST was proven by H. FURSTENBERG in the absolute case and for metric ttgs [F 63]. R. ELLIS proved it in the relativized form for quasi-separable ttgs [E 69]. In [E 78] R. ELLIS also could get rid of the countability assumptions for the absolute case. The definitive version was proven by D.C. MCMAHON and T.S. WU [MW 81].

1.24. **THEOREM.** FST : *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs. Then* ϕ *is distal iff* ϕ *is the inverse limit of a tower consisting of almost periodic extensions.* □

1.25. **COROLLARY.** *A minimal ttg* $\mathfrak{X}$ *has a nontrivial distal factor iff it has a nontrivial uniformly almost periodic factor.* □

In some special cases, for instance for ttgs with manifolds as phase space and a decent topological group as phase group, one can calculate the height of the tower (in 1.24.), e.g. [IM ?], [R ?] and [B 75/79] section 3.17..

The structure of point distal homomorphisms of minimal ttgs is determined similar to FST , see the discussion in IV.1..

I.2. THE UNIVERSAL AMBIT

> For several properties of ttgs there exists a universal ttg with that property which is unique up to isomorphism. In particular, the universal point transitive ttg $\mathcal{S}_T$ and the universal minimal ttg $\mathcal{M}$ for a given topological group T are of considerable importance in topological dynamics.
> In this section we shall deal with $\mathcal{S}_T$, $\mathcal{M}$ and their technical impact on topological dynamics. But we shall also briefly discuss other universal ttgs.
> As the theory presented here is completely standard, and as it is only incorporated in this monograph for the sake of notation and reference, we shall omit proofs. For more details see [E 69] chapters 3 and 5, [B 75/79] section 1.4., [VW ?] and [G 76] chapter I.

In the sequel a ttg $\mathcal{X}$ together with a distinguished transitive point $x \in X$ will be called an *ambit*; notation: $(\mathcal{X},x)$. An *ambit morphism* $\phi:(\mathcal{X},x)\rightarrow(\mathcal{Y},y)$ will be a surjective homomorphism $\phi:\mathcal{X}\rightarrow\mathcal{Y}$ of point transitive ttgs, such that $\phi(x)=y$.
Note that every ambit morphism is unique.
As the phase space of a point transitive ttg is the image of βT_d , the Cech-Stone compactification of T_d , there can only be a set of essentially different ambits for T . So let $\mathbf{A}$ be a set of ambits for T , such that for every ambit $(\mathcal{X},x)$ there is an $(\mathcal{A},a)\in \mathbf{A}$ which is isomorphic to $(\mathcal{X},x)$. Let

$$\mathcal{Z}' := \Pi\{\mathcal{A} \mid (\mathcal{A},a)\in \mathbf{A}\} \qquad \text{and} \qquad z = (a)_{(\mathcal{A},a)\in \mathbf{A}} ,$$

and define $\mathcal{Z} := \overline{Tz}$. Then $(\mathcal{Z},z)$ is an ambit, which projects onto each ambit for T . Hence $(\mathcal{Z},z)$ is the (unique up to isomorphism) *universal ambit* for T ; say, $(\mathcal{Z},z)=(<T,Z,\zeta>,z)$.
We shall mention two other ways to describe the universal ambit.

Let $<T,X,\pi>$ be a topological transformation group with X a Hausdorff space which need not be compact. Then there exists a ttg $<T,\beta_T X,\tilde{\pi}>$ and a homomorphism $\iota_X : <T,X,\pi> \rightarrow <T,\beta_T X,\tilde{\pi}>$

with $\beta_T X$ a CT_2 space and $\iota_X[X]$ dense in $\beta_T X$, such that every dense equivariant map $\phi : < T, X, \pi > \to < T, Y, \sigma >$ with Y a CT_2 space, factorizes over $< T, \beta_T X, \tilde{\pi} >$, [dV 75].

If ι_X is an embedding, $< T, \beta_T X, \tilde{\pi} >$ is called the *T-compactification of* $< T, X, \pi >$. Under some mild conditions such a T-compactification exists, ([dV 77], [LV 80]). For example, if T is discrete and X is a Tychonoff space, then the action of T can be extended to βX so $< T, \beta X, \tilde{\pi} >$ is the T-compactification of $< T, X, \pi >$. If T is not discrete, then the extended action of T on βX may fail to be jointly continuous, however.

An other simple example is the T-compactification of $< T, T, \lambda >$ where λ denotes the multiplication on T. One can show that the map $\iota_T : < T, T, \lambda > \to < T, \beta_T T, \tilde{\lambda} >$ is an embedding, ([dV 75], [LV 80]). Clearly, $(< T, \beta_T T, \tilde{\lambda} >, \iota_T(e))$ is an ambit. As $\zeta_z : T \to Z$ is an equivariant map that takes e to z, it factorizes over $< T, \beta_T T, \tilde{\lambda} >$ say $\tilde{\zeta}_z : < T, \beta_T T, \tilde{\lambda} > \to < T, Z, \zeta >$, taking $\iota_T(x)$ to z. Hence $(< T, \beta_T T, \tilde{\lambda} >, \iota_T(e))$ is isomorphic to $(\mathfrak{Z}, z)$.

Note that this shows that T acts *effectively* on Z; i.e., for every $t \in T$ with $t \neq e$ there is a $z' \in Z$ with $tz' \neq z'$.

For a third description of the universal ambit consider $S(T)$, the Gel'fand dual of the Banach algebra $\mathrm{RUC}^*(T)$ of bounded right uniformly continuous functions on T. Then T can be densely embedded in $S(T)$ by assigning to each $t \in T$ the evaluation map $\delta_t : \mathrm{RUC}^*(T) \to \mathbb{C}$. One can show that the multiplication λ on T can be extended to a jointly continuous action μ of T on $S(T)$. Then $(< T, S(T), \mu >, \delta_e)$ is an ambit; moreover it turns out that $(< T, S(T), \mu >, \delta_e)$ is isomorphic to $(\mathfrak{Z}, z)$.

Using this characterization of the universal ambit, it can be shown that the action of T on Z is *strongly effective* in case T is locally compact ([V 77]); i.e., $tz' \neq z'$ for every $t \in T$ with $t \neq e$ and for every $z \in Z$.

In our studies the exact construction of the universal ambit will never play a role. The pure existence of a universal ambit for T in which $(< T, T, \lambda >, e)$ is densely embedded and which is unique up to isomorphism is sufficient.

We shall denote the point transitive ttg in the universal ambit by $\mathfrak{S}_T$, with phase space S_T, and we shall consider T as a subspace of S_T; the unit element e in T will always be the transitive point of the universal ambit.

2.1. **REMARK.** *The* CT_2 *space* S_T *has a semigroup structure which extends the group structure of* T *, such that the right translation* $\rho_p : \xi \mapsto \xi p : S_T \to S_T$ *is continuous for every* $p \in S_T$ *, and the left translation* $\lambda_t : \xi \mapsto t\xi : S_T \to S_T$ *is an homeomorphism for every* $t \in T$ *. Moreover, the right translations* ρ_p *are just the extensions to* S_T *of the right translations* $\rho_p |_T$ *induced by the action of* T *on* S_T *, and the left translations are just the ones induced by that action;* (see [VW ?] and [V 77] section 2.2.). □

As for every ttg $\mathcal{X}$ the pair $(E(\mathcal{X}), e)$ is an ambit (here e is id_X), there is an ambit morphism $\epsilon_X : (S_T, e) \to (E(\mathcal{X}), e)$ and $\epsilon_X : S_T \to E(X)$ is a semigroup homomorphism.

In a certain sense S_T acts on the phase space X of a ttg $\mathcal{X}$ (via $E(X)$): assign to $p \in S_T$ and $x \in X$ the element $\epsilon_X(p)(x)$ in X . This is a kind of right semitopological semigroup "semiaction", for S_T is a right semitopological semigroup which acts on X as a semigroup (and extends the action of T), but in general it lacks continuity.

As $\delta_x : E(\mathcal{X}) \to \mathcal{X}$ is a homomorphism of ttgs for every $x \in X$, the map $\rho_x := \delta_x \circ \epsilon_X : (S_T, e) \to (\mathcal{X}, x)$ is an ambit morphism; in particular, "evaluation" in x is a continuous map from S_T onto $\overline{Tx}$ for every $x \in X$. So for $p \in S_T$ and for a net $\{t_i\}_i$ in T converging to p in S_T , the net $\{t_i x\}_i$ converges to $\rho_x(p)$ in X for every $x \in X$. This observation is valid for every ttg $\mathcal{X}$, so we may interpret S_T as a universal enveloping semigroup; and so S_T embodies the universal limit behavior of T .

Define $px := \epsilon_X(p)(x) = \rho_x(p)$ for every $p \in S_T$, $x \in X$. Note that for every $p, q \in S_T$, $x \in X$ we have

a) $p(qx) = (pq)x$;
b) $\rho_x : r \mapsto rx : S_T \to X$ is continuous, but in general $\lambda_p : y \mapsto py : X \to X$ is not continuous.

If $\phi : \mathcal{X} \to \mathcal{Y}$ is a homomorphism of ttgs, then ϕ commutes with the "action" of S_T ; i.e., $\phi(px) = p\phi(x)$ for every $p \in S_T$, $x \in X$.

We can now apply the theory of compact right semitopological semigroups to reveal some of the structure of S_T . Although the statements to follow are valid in a more general setting, we shall state them just for S_T , except in the case of 2.6.. As enveloping semigroups are homomorphic images of S_T , this theory is easily transferable to the enveloping semigroups in general.

A subset E of S_T is called a *left ideal* if $S_T.E \subseteq E$; so a closed subset E of S_T is a left ideal iff E is T-invariant, and this in turn implies that the closure of a left ideal is a left ideal again. A typical example of a left ideal is a subset of S_T of the form $S_T.p$ $(= \overline{Tp})$ for a $p \in S_T$. This observation shows that every minimal left ideal is closed and that every left ideal contains a minimal left ideal (Zorn). Moreover, a subset E of S_T is a minimal left ideal iff E is a minimal subset of S_T . Minimal left ideals of S_T (which are subsemigroups of S_T) have a nice structure:

2.2. **THEOREM.** *Let I be a minimal left ideal in S_T and let $J = J(I)$ be the set of all idempotents in I . Then the following statements hold:*

a) $J \neq \varnothing$; (I is a closed subsemigroup of S_T *is sufficient!)*

b) $pv = p$ *for every* $p \in I$, $v \in J$;

c) *for every* $v \in J$ *the set* vI $(= \{vp \mid p \in I\})$ *is a subgroup of* I *with unit element* v , *and* $vI = \{p \in I \mid vp = p\}$;

d) *for every* $v, w \in J$ *the map* $\lambda_w : p \mapsto wp : vI \to wI$ *is an isomorphism of groups with inverse* λ_v ;

e) $\{vI \mid v \in J\}$ *is a partitioning of* I ;

f) *if* $u \in J$, *then every* $p \in I$ *has a unique representation as* $p = wa$, *where* $w \in J$, $a \in uI$. □

For convenience we establish some notation.
Let I be a minimal left ideal in S_T (or in $E(X)$ for some ttg $\mathfrak{X}$). Then we denote the set of all idempotents in I by $J(I)$.
Let $p \in S_T$; then λ_p will denote the left multiplication with p ($q \mapsto pq$, $q \in S_T$) and ρ_p will be the right multiplication with p (which is continuous). If $\mathfrak{X}$ is a ttg and $x \in X$ then ρ_x denotes the evaluation at x ("right multiplication" with x); i.e., $\rho_x : S_T \to X$ is defined by $\rho_x(q) = qx$ $(q \in S_T)$.

2.3. **THEOREM.** *Let I and K be minimal left ideals in S_T .*

a) *For every* $v \in J(I)$ *there is a unique* $v' \in J(K)$ *such that* $vv' = v'$ *and* $v'v = v$; *notation:* $v \sim v'$.

b) *For every* $v \in J(I)$ *the map* $\rho_v : K \to I$ *is a homeomorphism with inverse* $\rho_{v'} : I \to K$, *where* $v' \in J(K)$ *with* $v' \sim v$; *moreover,* ρ_v *is an isomorphism of semigroups and* ρ_v *is equivariant.*

c) *Fix* $u \in J(I)$ *and let* $p \in I$, *say* $p = va$ *for* $v \in J(I)$, $a \in uI$. *Then* $\rho_p : K \to I$ *is an equivariant homeomorphism with inverse* $\rho_q : I \to K$, *with* $q = va^{-1}v'$, *where* $v' \in J(K)$ *is such that* $v' \sim v$. □

2.4. **THEOREM.** *Let* I *be a minimal left ideal in* S_T *and let* $u \in J(I)$. *Every equivariant endomorphism* $\phi : I \to I$ *has the form* $\phi = \rho_a$ *for some* $a \in uI$. *In particular, it follows that every equivariant endomorphism of* I *is an isomorphism.* □

The minimal left ideals of S_T and their idempotents are closely related to the notion of almost periodicity; this is expressed in the next theorem.

2.5. **THEOREM.** *Let* $\mathfrak{X}$ *be a ttg and let* $x \in X$. *The following statements are equivalent:*

a) x *is an almost periodic point in* $\mathfrak{X}$;
b) $\overline{Tx}$ *is a minimal subset of* X;
c) *there exists a minimal left ideal* I *of* S_T *such that* $x \in Ix$;
d) *for every minimal left ideal* I *of* S_T *there is a* $v \in J(I)$ *with* $vx = x$. □

Note that if x is an almost periodic point in $\mathfrak{X}$, then $\overline{Tx} = Ix$ for every minimal left ideal I in S_T. Moreover, let $\mathfrak{X}$ be minimal and $x \in X$, then each minimal left ideal I of S_T is mapped homomorphically onto X by the map $\rho_x : S_T \to X$. This shows that every minimal left ideal I of S_T considered as a subttg $\mathfrak{I}$ of $\mathfrak{S}_T$ is a *universal minimal ttg.*
Let $\mathfrak{M}$ be the universal minimal ttg. As $\mathfrak{I}$ is a minimal ttg there is a homomorphism $\phi : \mathfrak{M} \to \mathfrak{I}$ of minimal ttgs. But $\mathfrak{I}$ is a universal minimal ttg; so there is a homomorphism $\psi : \mathfrak{I} \to \mathfrak{M}$ of minimal ttgs. Hence $\phi \circ \psi : I \to I$ is an endomorphism of I; which, by 2.4., implies that $\phi \circ \psi$ is an isomorphism. Consequently, $\mathfrak{M}$ and $\mathfrak{I}$ are isomorphic ttgs. Therefore, we may conclude that there exists a universal minimal ttg for T, which is unique up to isomorphism. This universal minimal ttg will be denoted by $\mathfrak{M}$ and its phase space by M. We shall always consider $\mathfrak{M}$ as a subttg of $\mathfrak{S}_T$, i.e., we consider M as a minimal left ideal in S_T. As such, M acts on every minimal ttg as a semigroup. Sometimes it is necessary to specify a particular minimal left ideal in S_T, which is used as the universal minimal ttg (for instance, if we want to apply 2.7. below).
In general the existence of $\mathfrak{M}$ and its structure suffice. So if no minimal left ideal is specified its choice is irrelevant and we just assume M to be some (fixed) minimal left ideal in S_T.
Note that 2.2. pictures the structure of M as a disjoint union of subgroups "centered around the idempotents" in M. We shall denote the set of those idempotents in M by J. Usually, for a fixed $u \in J$ we shall denote the subgroup uM by G; then for $v \in J$, $vM = vG$.

We shall end our considerations about compact right semitopological semigroups by mentioning the following result ([E 69]).

2.6. **THEOREM.** *Let* E *be a compact* T_1 *topological space provided with a group structure such that the maps* $\rho_x : y \mapsto yx : E \to E$ *are continuous* $(x \in E)$*, and let* M *be a nonempty closed subsemigroup of* E *. Then* M *is a subgroup of* E *.* ☐

We shall now relate the structure of S_T and M to the notions of proximality and distality.

2.7. **THEOREM.** *Let* $\mathfrak{X}$ *be a ttg. The following statements are equivalent for a pair* $(x,y) \in X \times X$ *:*

a) $(x,y) \in P_{\mathfrak{X}}$*;*

b) *there is a* $p \in S_T$ *with* $px = py$ *;*

c) *there is a minimal left ideal* I *in* S_T *such that* $px = py$ *for every* $p \in I$ *.*

Moreover, $\{vx \mid v \in S_T, vv = v\} \subseteq P_{\mathfrak{X}}[x]$*; if* $\mathfrak{X}$ *is minimal, then*

$$P_{\mathfrak{X}}[x] = \{vx \mid v \in J(I) \text{ for some m.l.i. } I \text{ in } S_T\} .$$ ☐

2.8. **REMARK.** *Let* $\mathfrak{X}$ *be a ttg. For every minimal left ideal* I *in* S_T *and for every* $v \in J(I)$ *we have that each pair in* vX *is a distal pair; here* $vX = \{vx \mid x \in X\}$ *.*

Paraphrased: if (x,y) *is a almost periodic point in* $X \times X$ *, then the pair* (x,y) *is a distal pair* (compare 1.15.c for $n = 2$ *).* ☐

2.9. **COROLLARY.**

a) *Let* $\mathfrak{X}$ *be proximal minimal ttg. Then the only equivariant endomorphism of* $\mathfrak{X}$ *is the identity* id_X *on* $\mathfrak{X}$ ([G 76] II.4.1.).

b) *Let* T *be an abelian group. then there are no nontrivial proximal minimal ttgs for* T (for a more general result see [G 76] II.3.4.). ☐

Let $\mathfrak{X}$ be a minimal ttg and let I be a minimal left ideal in S_T . Define $(S_T)_x := \{p \in S_T \mid px = x\}$, $I_x := I \cap (S_T)_x$ and $J_x(I) := J(I) \cap (S_T)_x$.

2.10. **REMARK.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs. A point* $x \in X$ *is a* ϕ*-distal point iff* $J_x = J_{\phi(x)}$ *. Hence* $x \in X$ *is a distal point iff* $J_x = J$ *and* $\mathfrak{X}$ *is distal iff* $uX = X$ *for every* $u \in J$ *.* ☐

Fix $u \in J$. Let $\mathfrak{X}$ be a minimal ttg and let $x \in X$ be such that $ux = x$. Then the *Ellis group* $\mathfrak{G}(\mathfrak{X},x)$ of $\mathfrak{X}$ with respect to x in G ($= uM$) is defined as

$$\mathfrak{G}(\mathfrak{X},x) := M_x \cap G = \{a \in G \mid ax = x\}.$$

Clearly, $\mathfrak{G}(\mathfrak{X},x)$ is a subgroup of G .

2.11. **NOTE** *that if* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *is a homomorphism of minimal ttgs and* $x = ux \in X$ *, then* $\mathfrak{G}(\mathfrak{X},x) \subseteq \mathfrak{G}(\mathfrak{Y},\phi(x))$.

2.12. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs and* $x = ux \in X$. *Then* ϕ *is distal iff* $\phi^{\leftarrow}\phi(px) = p\,\mathfrak{G}(\mathfrak{Y},\phi(x))x$ *for every* $p \in M$. *In particular,* $\mathfrak{X}$ *is distal iff* $X = pX$ *for every* $p \in M$. □

2.13. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs and let* $x = ux \in X$. *Then the following statements are equivalent:*

a) ϕ *is proximal;*

b) $\mathfrak{G}(\mathfrak{X},x) = \mathfrak{G}(\mathfrak{Y},\phi(x))$ *;*

c) *for every* $(x_1,x_2) \in R_\phi$ *there is a* $v \in J$ *with* $x_2 = vx_1$.

In particular, $\mathfrak{X}$ *is proximal iff* $\mathfrak{G}(\mathfrak{X},x) = G$ *iff* $X = Jx$ *iff* $uX = \{x\}$. □

From these observations (2.12. and 2.13.) it follows easily that if ϕ, ψ and θ are homomorphisms of minimal ttgs such that $\phi = \theta \circ \psi$, then ϕ is distal (proximal) iff θ and ψ are distal (proximal).

We shall proceed with some observations on other universal ttgs.
Let $\mathfrak{Y}$ be a minimal ttg. Then there is a set $\Lambda_{\mathfrak{Y}}$ of homomorphisms $\phi : \mathfrak{X} \to \mathfrak{Y}$ of minimal ttgs such that every minimal extension of $\mathfrak{Y}$ is isomorphic to a unique member of $\Lambda_{\mathfrak{Y}}$ (i.e., for every homomorphism $\theta : \mathfrak{W} \to \mathfrak{Y}$ of minimal ttgs there is a $\psi \in \Lambda_{\mathfrak{Y}}$ and an isomorphism ξ such that $\theta \circ \xi = \psi$). Let C be a property of homomorphisms of minimal ttgs and let $\Lambda_C := \{\phi \in \Lambda_{\mathfrak{Y}} \mid \phi$ has property $C\}$. Then every extension of $\mathfrak{Y}$ with property C is isomorphic to exactly one member of Λ_C (so Λ_C is the set of "essentially different" extensions of $\mathfrak{Y}$ with property C). Define

$$\mathfrak{Z}_C := \Pi\{\mathfrak{X}_\lambda \mid \lambda \in \Lambda_C, \lambda : \mathfrak{X}_\lambda \to \mathfrak{Y}\},$$

and let $\phi_C : \mathfrak{Z}_C \to \mathfrak{Y}^{|\Lambda_C|}$ be defined coordinatewise. Let $y_0 \in Y$ and

$u \in J_{y_0}$; and let $x_\lambda \in X_\lambda$ be such that $x_\lambda = ux_\lambda$ and $\lambda(x_\lambda) = y_0$. Then $z := (x_\lambda)_{\lambda \in \Lambda_C}$ is an almost periodic point in Z_C . So $W := \overline{Tz}$ is a minimal subset of Z_C which is mapped onto $\mathcal{Y}$ by ϕ_C (more precisely, Z_C is mapped onto the diagonal in $\mathcal{Y}^{|\Lambda_C|}$).
Let $\psi: \mathcal{W} \to \mathcal{Y}$ be the restriction of ϕ_C to W . Then, clearly, ψ factorizes over each $\lambda \in \Lambda_C$ by projection (i.e., each $\lambda \in \Lambda_C$ is a factor of ψ). This shows that $\psi: \mathcal{W} \to \mathcal{Y}$ is the universal minimal C-extension of $\mathcal{Y}$, provided that ψ has property C , and provided that uniqueness can be shown. For several properties C this can be guaranteed. For instance, if C stands for distality, proximality or almost periodicity, then ψ has property C by 1.21.b. The property of point distality behaves less well. But, if $\mathcal{Y}$ is distal then, for suitably chosen x_λ , the map $\psi: \mathcal{W} \to \mathcal{Y}$ is point distal. In all these cases it can be shown that ψ is unique up to isomorphism.
Thus we obtain the following theorem:

2.14. **THEOREM.** *Let $\mathcal{Y}$ be a minimal ttg. There exists a universal almost periodic (distal, proximal) extension of $\mathcal{Y}$, which is unique up to isomorphism.*
If $\mathcal{Y}$ is distal then there exists a universal minimal point distal extension of $\mathcal{Y}$ which is unique up to isomorphism.
In particular, there is a universal minimal almost periodic (distal, point distal, proximal) minimal ttg for T , which is unique up to isomorphism; notation: $\mathcal{E}_{(T)}$ ($\mathcal{D}_{(T)}$, $p\mathcal{D}_{(T)}$, $\mathcal{P}_{(T)}$) . □

Another construction of the universal almost periodic, distal or proximal minimal extensions of $\mathcal{Y}$ can be given as follows:
Let $\gamma: \mathcal{M} \to \mathcal{Y}$ be a homomorphism of minimal ttgs.
Then $\psi: \mathcal{M}/E_\gamma \to \mathcal{Y}$ is the universal almost periodic minimal extension of $\mathcal{Y}$ (cf. 1.20.b).
Define S_γ to be the smallest invariant closed equivalence relation in R_γ that contains P_γ . Then $\psi: \mathcal{M}/S_\gamma \to \mathcal{Y}$ is the universal distal minimal extension of $\mathcal{Y}$ (the P_γ-analogue of 1.20.b).
Observe that

$$JR_\gamma = \{v\,(x_1,x_2) \mid v \in J\,,\ (x_1,x_2) \in R_\gamma\}$$

is just the set of all almost periodic points in R_γ , and that JR_γ is invariant. Define N_γ to be the smallest invariant closed equivalence relation in R_γ that contains JR_γ ([B 75/79] 3.14.17.). Then $\psi: \mathcal{M}/N_\gamma \to \mathcal{Y}$ is the universal proximal minimal extension of $\mathcal{Y}$ (see also III.1.13.).

We shall end this section with a brief discussion of regularity (see [A 66], [Sh 74]). Often universal minimal extensions have a neat automorphism structure called regularity. A homomorphism $\phi: \mathcal{X} \to \mathcal{Y}$ of ttgs is called *regular* if for every almost periodic point $(x_1, x_2) \in R_\phi$ there is an equivariant endomorphism $\xi: \mathcal{X} \to \mathcal{X}$ such that $\xi(x_1) = x_2$. It follows that ϕ is regular iff for every $(x_1, x_2) \in R_\phi$ there exists an (equivariant) endomorphism $\xi: \mathcal{X} \to \mathcal{X}$ such that $(\xi(x_1), x_2) \in P_\phi$.

Clearly, if ϕ is a regular homomorphism of minimal ttgs then the endomorphisms ξ above are automorphisms. It is not difficult to show that a group extension of minimal ttgs is regular (see 2.17.) and, evidently, every proximal extension of minimal ttgs is regular.

2.15. **REMARK.** *Let* $\phi: \mathcal{X} \to \mathcal{Y}$ *be a regular homomorphism of minimal ttgs,* $u \in J$ *and* $x = ux \in X$. *Then* $\mathfrak{G}(\mathcal{X}, x)$ *is a normal subgroup of* $\mathfrak{G}(\mathcal{Y}, \phi(x))$. □

Let $\phi: \mathcal{X} \to \mathcal{Y}$ be a homomorphism of minimal ttgs. The *regularizer* Reg(ϕ) of ϕ is defined as follows: Let $u \in J$ and $y = uy \in Y$; and note that $u\phi^{\leftarrow}(y) = \{x \in X \mid ux = x, \phi(x) = y\} \neq \varnothing$. Define a point

$$z \in \Pi\{\mathcal{X}_\lambda \mid \mathcal{X}_\lambda = \mathcal{X}, \lambda \in u\phi^{\leftarrow}(y)\} = \mathcal{X}^{u\phi^{\leftarrow}(y)} \quad \text{by} \quad z = (x)_{x \in u\phi^{\leftarrow}(y)}.$$

Then, clearly, $z = uz$, so $X' := \overline{Tz}$ is a minimal subset of $X^{u\phi^{\leftarrow}(y)}$. Let $\theta: \mathcal{X}' \to \mathcal{X}$ be the projection and define

$$\mathrm{Reg}(\phi): \mathcal{X}' \to \mathcal{Y} \quad \text{by} \quad \mathrm{Reg}(\phi) = \phi \circ \theta.$$

It is not difficult to show that Reg(ϕ) is a regular homomorphism of minimal ttgs, and that ϕ is regular iff ϕ and Reg(ϕ) are equal up to isomorphism (i.e., θ is an isomorphism).

2.16. **REMARK.** *Let* T *be an abelian group. Then every minimal uniformly almost periodic ttg is regular.*

PROOF. Let $\mathcal{X}$ be a minimal uniformly almost periodic ttg. As T is abelian, every element of $E(X)$ commutes with every element of T. By 1.12., every element of $E(X)$ is a homeomorphism of X, and so $E(X)$ consists of equivariant endomorphisms. As $E(X)x = X$ for every $x \in X$, regularity follows. □

2.17. **REMARK.** *let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a group extension of minimal ttgs. Then* ϕ *is regular.*
In particular, the universal minimal almost periodic extension of $\mathfrak{Y}$ *is regular.*

PROOF. Let K be a CT_2 topological group such that K acts on X continuously and such that K and T commute and $\phi^{\leftarrow}\phi(x)=Kx$ for every $x\in X$. Then the elements of K are the equivariant endomorphisms that guarantee regularity. Let $\alpha_{\mathfrak{Y}}:\mathfrak{A}(\mathfrak{Y})\to\mathfrak{Y}$ be the universal minimal almost periodic extension of $\mathfrak{Y}$. Then $\alpha_{\mathfrak{Y}}$ is a factor of a group extension. As $\alpha_{\mathfrak{Y}}$ is universal it is a group extension itself. □

I.3. FIBERED PRODUCTS

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be ttgs. Then the dynamical properties of the cartesian product $\mathfrak{X}\times\mathfrak{Y}$ seem to reflect a certain correlation between the dynamical properties of $\mathfrak{X}$ and $\mathfrak{Y}$. For instance, if $\mathfrak{X}$ and $\mathfrak{Y}$ are minimal, then minimality of $\mathfrak{X}\times\mathfrak{Y}$ shows a kind of independency for $\mathfrak{X}$ and $\mathfrak{Y}$; in that case $\mathfrak{X}$ and $\mathfrak{Y}$ are called disjoint. This section is meant to provide definitions and techniques necessary for the study of disjointness and weak disjointness ($\mathfrak{X}\times\mathfrak{Y}$ ergodic) in the chapters VI and VII. In many cases only a sketch of proof is given.
The general setting is as follows:
Let $\phi:\mathfrak{X}\to\mathfrak{Z}$ and $\psi:\mathfrak{Y}\to\mathfrak{Z}$ be surjective homomorphisms of ttgs. Define $R_{\phi\psi}:=\{(x,y)\in X\times Y\mid\phi(x)=\psi(y)\}$, the fibered product. Clearly, $R_{\phi\psi}$ is closed and invariant and $R_{\phi\phi}=R_{\phi}$. This fibered product may be interpreted as the relative version of the cartesian product.
We shall comment on $R_{\phi\psi}$ throughout this section.

Let $\phi:\mathfrak{X}\to\mathfrak{Z}$ and $\psi:\mathfrak{Y}\to\mathfrak{Z}$ be homomorphisms of minimal ttgs. Then ϕ and ψ are called *disjoint* if $R_{\phi\psi}$ is a minimal subset of $X\times Y$; notation: $\phi\perp\psi$. If $\mathfrak{Z}$ is the trivial one point ttg ($\{\star\}$), then instead of $\phi\perp\psi$ we write $\mathfrak{X}\perp\mathfrak{Y}$; we say that the minimal ttgs $\mathfrak{X}$ and $\mathfrak{Y}$ are *disjoint*. If ϕ and ψ are not disjoint we write $\phi\not\perp\psi$.
Clearly, $\phi\perp id_Z$ and $\phi\not\perp\psi$ for every nontrivial factor ψ of ϕ (compare VI.1.1.). From 1.23.a,c it is easily deducible that a distal minimal ttg is disjoint from every proximal minimal ttg.

3.1. **REMARK.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Z}$ *be a homomorphism of minimal ttgs.*

a) *Let* $\psi:\mathfrak{Y}\to\mathfrak{Z}$ *be a homomorphism of minimal ttgs such that* $\phi\perp\psi$. *Then* $\phi\perp\theta$ *for every factor* θ *of* ψ.

b) *Let* $\{\psi_\alpha^\beta:\mathfrak{Y}_\beta\to\mathfrak{Y}_\alpha \mid \alpha<\beta<\nu\}$ *be an inverse system of homomorphisms of minimal ttgs, with* $\mathfrak{Y}_0=\mathfrak{Z}$ *and such that* $\psi_0^\beta\perp\phi$ *for every* $\beta<\nu$. *Let* $\psi=\operatorname{inv}\lim\psi_\alpha^\beta$*; then* $\psi\perp\phi$.

c) *Let* $\psi:\mathfrak{Y}\to\mathfrak{Z}$ *be a homomorphism of minimal ttgs with* $\phi\perp\psi$. *Then there is a homomorphism of minimal ttgs* $\theta:\mathfrak{W}\to\mathfrak{Z}$ *that factorizes over* ψ *and which is maximally disjoint from* ϕ. *That is,* $\phi\perp\theta$ *and* $\phi\not\perp\xi$ *for every proper minimal extension* ξ *of* θ.

PROOF.

a) Obvious.

b) This follows from 1.6. and from the easy observation that $R_{\phi\psi}=\operatorname{inv}\lim\{R_{\phi\psi_0^\beta}\mid\beta<\nu\}$; here the maps

$$\theta_\alpha^\beta:R_{\phi\psi_0^\beta}\to R_{\phi\psi_0^\alpha}\ \text{ are defined as }\ \theta_\alpha^\beta:=id_X\times\psi_\alpha^\beta|_{R_{\phi\psi_0^\beta}}.$$

c) Consider the collection Λ of homomorphisms $\xi:\mathfrak{Z}'\to\mathfrak{Z}$ of minimal ttgs with $\xi\perp\phi$, such that ξ factorizes over ψ ; i.e., $\xi=\psi\circ\lambda$ for some homomorphism λ. Define an ordering on Λ by: $\xi<\eta$ iff $\eta=\xi\circ\mu$ for some homomorphism μ ($\xi,\eta\in\Lambda$). By b, every chain in Λ has an upper bound in Λ. Hence, by Zorn's lemma, the assertion follows. ☐

Clearly, $R_{\phi\psi}$ is minimal iff $R_{\phi\psi}$ has a unique minimal subset and $R_{\phi\psi}$ has a dense subset of almost periodic points.
In order to know whether $R_{\phi\psi}$ contains a unique minimal subset we have:

3.2. **THEOREM.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Z}$ *and* $\psi:\mathfrak{Y}\to\mathfrak{Z}$ *be homomorphisms of minimal ttgs. Let* $u\in J$, $z_0\in uZ$, $x_0\in u\phi^{\leftarrow}(z_0)$ *and* $y_0\in u\psi^{\leftarrow}(z_0)$. *Let* $H=\mathfrak{G}(\mathfrak{X},x_0)$, $F=\mathfrak{G}(\mathfrak{Y},y_0)$ *and* $K=\mathfrak{G}(\mathfrak{Z},z_0)$ *be the Ellis groups of* $\mathfrak{X}$, $\mathfrak{Y}$ *and* $\mathfrak{Z}$ *with respect to* x_0, y_0 *and* z_0 *in* G. *Then* $R_{\phi\psi}$ *has a unique minimal subset iff* $HF=K$.

PROOF. Suppose that $R_{\phi\psi}$ has a unique minimal subset. Let $k\in K$; and remark that $(x_0,ky_0)=u\,(x_0,ky_0)$ is an almost periodic point in $R_{\phi\psi}$. As (x_0,y_0) is an almost periodic point too, there is an $a\in G$ such that $(x_0,ky_0)=a\,(x_0,y_0)$. Clearly, $a\in H$ and $a^{-1}k\in F$. So we have $k=aa^{-1}k\in HF$, which implies that $K\subseteq HF$. As $H\cup F\subseteq K$, it follows that $K=HF$.

Conversely, let $W \subseteq R_{\phi\psi}$ be a minimal subset of $R_{\phi\psi}$ and assume that $K = HF$. Clearly, for some $a \in G$ the point $(x_0, ay_0) \in W$. Hence $a \in K$; say $a = hf$ for certain $h \in H$ and $f \in F$. Then $(x_0, ay_0) = h\,(h^{-1}x_0, fy_0)$, and as $h^{-1} \in H$, $f \in F$, we have $(x_0, ay_0) = h\,(x_0, y_0)$. This shows that $W \cap \overline{T(x_0,y_0)} \neq \varnothing$. As $\overline{T(x_0,y_0)}$ is minimal (2.5.), it follows that $W = \overline{T(x_0,y_0)}$. □

3.3. **COROLLARY.**

a) *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be minimal ttgs. Then the following statements are equivalent:*
   (i) $X \times Y$ *has a unique minimal subset;*
   (ii) $\mathfrak{G}(\mathfrak{X},x).\mathfrak{G}(\mathfrak{Y},y) = G$ *for some* $x \in uX$ *and* $y \in uY$*;*
   (iii) $\mathfrak{G}(\mathfrak{X},x).\mathfrak{G}(\mathfrak{Y},y) = G$ *for every* $x \in uX$ *and* $y \in uY$.

b) *Let* H *and* F *be subgroups of* G *that can occur as Ellis groups of certain minimal ttgs. Then* $HF = G$ *iff* $HgF = G$ *for some* $g \in G$ *iff* $HgF = G$ *for every* $g \in G$. □

3.4. **REMARK.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Z}$ *be a homomorphism of minimal ttgs and let* $\psi : \mathfrak{Y} \to \mathfrak{Z}$ *be a proximal extension of (not necessarily minimal) ttgs. Then* $R_{\phi\psi}$ *has a unique minimal subset.*

PROOF. Define $\theta : \mathfrak{R}_{\phi\psi} \to \mathfrak{X}$ as the projection. Then θ is proximal. As $\mathfrak{X}$ is minimal, the remark follows from 1.23.c. □

Let $\phi : \mathfrak{X} \to \mathfrak{Z}$ and $\psi : \mathfrak{Y} \to \mathfrak{Z}$ be surjective homomorphisms of ttgs (not necessarily minimal). Then ϕ and ψ are said to satisfy the *generalized Bronstein condition* (gBc) if $\overline{JR_{\phi\psi}} = R_{\phi\psi}$; i.e., if the almost periodic points are dense in $R_{\phi\psi}$. If $\overline{JR_\phi} = R_\phi$ then ϕ is said to satisfy the *Bronstein condition* (Bc); we shall also say that ϕ is a Bc *map* or a Bc *extension*. Bc extensions turn out to behave nicely with respect to the regionally proximal relation and the interpolation of almost periodic factors, as will be made clear in 4.4. and III.3..

Note that if the pair (ϕ,ψ) satisfies gBc, then X and Y, being factors of $R_{\phi\psi}$, both have a dense subset of almost periodic points.

3.5. **REMARK.**

a) *Let* $\phi\colon \mathfrak{X}\to\mathfrak{Z}$ *be a homomorphism of minimal ttgs and let* $\psi\colon\mathfrak{Y}\to\mathfrak{Z}$ *be a proximal extension. Then* $\phi\perp\psi$ *iff* (ϕ,ψ) *satisfies* gBc (cf. 3.4.).

b) *In particular, a proximal homomorphism of minimal ttgs is a* Bc *extension iff it is an isomorphism.*

c) *Let* $\phi\colon\mathfrak{X}\to\mathfrak{Z}$ *be a homomorphism of minimal ttgs. If* $\overline{JR_\phi}$ *is an equivalence relation, then* $\phi=\xi\circ\eta$ *where* η *is a* Bc *extension and* ξ *is a proximal extension* (cf. the discussion just below 2.14.). □

In case $\mathfrak{Z}$ is a minimal ttg, the fact that ϕ and ψ satisfy the generalized Bronstein condition implies semi-openness for the canonical map $\theta\colon\mathfrak{R}_{\phi\psi}\to\mathfrak{Z}$, defined by $\theta(x,y)=\phi(x)=\psi(y)$ for all $(x,y)\in R_{\phi\psi}$ (1.4.b).

Semi-openness has the following technical advantage:

3.6. **LEMMA.** *Let* $\phi\colon\mathfrak{X}\to\mathfrak{Z}$ *and* $\psi\colon\mathfrak{Y}\to\mathfrak{Z}$ *be surjective homomorphisms of ttgs, such that the canonical map* $\theta\colon\mathfrak{R}_{\phi\psi}\to\mathfrak{Z}$ *is semi-open. Then for every nonempty open* $W\subseteq R_{\phi\psi}$ *there are nonempty open subsets* U *and* V *in* X *and* Y *such that*

$$\phi[U]=\psi[V] \text{ and } \varnothing\neq U\times V\cap R_{\phi\psi}\subseteq W\,.$$

PROOF. Let U' and V' be open subsets of X and Y such that $\varnothing\neq U'\times V'\cap R_{\phi\psi}\subseteq W$ and let

$$O:=\operatorname{int}(\theta[U'\times V'\cap R_{\phi\psi}])=\operatorname{int}(\phi[U']\cap\psi[V']).$$

Then $U:=U'\cap\phi^{\leftarrow}[O]$ and $V:=V'\cap\psi^{\leftarrow}[O]$ suffice. □

3.7. **COROLLARY.** *Let* $\phi\colon\mathfrak{X}\to\mathfrak{Z}$ *and* $\psi\colon\mathfrak{Y}\to\mathfrak{Z}$ *be surjective homomorphisms of ttgs, and let* W *be an arbitrary open set in* $R_{\phi\psi}$. *In each of the following cases we can find open sets* U *and* V *in* X *and* Y *such that* $\phi[U]=\psi[V]$ *and* $\varnothing\neq U\times V\cap R_{\phi\psi}\subseteq W$.

(i) $\mathfrak{Z}$ *is minimal and* ϕ *and* ψ *satisfy* gBc;

(ii) ϕ *is open and* ψ *is semi-open;*

(iii) $\mathfrak{X}$ *is minimal and* ψ *is open;*

(iv) $\mathfrak{Z}$ *is minimal,* ϕ *is open and* $Y=\overline{JY}$.

PROOF.

(i) Follows from 3.6. and 1.4.b.

(ii) It is easy to show that θ is semi-open; hence 3.6. applies.

(iii) Follows from (ii) and 1.4.b (interchange ϕ and ψ).

(iv) Follows from (ii) and 1.4.b. □

If a pair in $R_{\phi\psi}$ can be approximated by almost periodic points in $R_{\phi\psi}$, then it can be approximated by almost periodic points with a first coordinate in Tx (for some fixed $x \in X$), provided that ϕ is a homomorphism of minimal ttgs. This is shown in the next lemma.

3.8. **LEMMA.** *Let $\phi:\mathcal{X}\to\mathcal{Z}$ and $\psi:\mathcal{Y}\to\mathcal{Z}$ be surjective homomorphisms of ttgs with $\mathcal{X}$ minimal, and let $x \in X$ and $u \in J_x$. Then*

$$\overline{JR_{\phi\psi}} = \overline{T\{x\}\times u\,\psi^{\leftarrow}\phi(x)} \quad (= \overline{\{(tx,ty) \mid t\in T, y\in u\,\psi^{\leftarrow}\phi(x)\}}).$$

PROOF. As $\{x\}\times u\,\psi^{\leftarrow}\phi(x) \subseteq JR_{\phi\psi}$ the inclusion $\subseteq$ holds.
Conversely, let $(x_1,y_1)\in\overline{JR_{\phi\psi}}$ and let $U\times V\cap R_{\phi\psi}$ be a basic open neighbourhood of $(x_1,y_1)\in R_{\phi\psi}$; i.e., U and V are open neighbourhoods of x_1 and y_1 in X and Y. As $U\times V\cap\overline{JR_{\phi\psi}} \neq \varnothing$, there is a point $(x_2,y_2)\in U\times V\cap JR_{\phi\psi}$; say $(x_2,y_2)=v\,(x_2,y_2)$. By minimality of $\mathcal{X}$, there is an $a\in G$ with $x_2=vax$. So $(x_2,y_2)=va(x,a^{-1}y_2)$, and clearly, $(x,a^{-1}y_2)\in\{x\}\times u\,\psi^{\leftarrow}\phi(x)$. Hence

$$(x_2,y_2)\in U\times V\cap\overline{T(\{x\}\times u\,\psi^{\leftarrow}\phi(x))},$$

and as U and V are arbitrary, $(x_1,y_1)\in\overline{T(\{x\}\times u\,\psi^{\leftarrow}\phi(x))}$. □

In the same spirit we have the following result, the easy proof of which is omitted.

3.9. **LEMMA.** *Let $\phi:\mathcal{X}\to\mathcal{Z}$ and $\psi:\mathcal{Y}\to\mathcal{Z}$ be surjective homomorphisms of ttgs, with ψ an open map. If $x_0\in X$ is a transitive point and ϕ is semi-open, then $R_{\phi\psi}=\overline{T(\{x_0\}\times\psi^{\leftarrow}\phi(x_0))}$. In particular, if $\mathcal{X}$ is minimal, then $R_{\phi\psi}=T(\{x\}\times\psi^{\leftarrow}\phi(x))$ for every $x\in X$.* □

The results in 3.6. through 3.9. show that openness of maps as well as density of almost periodic points in $R_{\phi\psi}$ provide a (technically) convenient description of $R_{\phi\psi}$. Both aspects are almost "embodied" by the so called RIC extensions, which we shall define hereafter (see III.1. for properties of those RIC extensions).
A homomorphism $\phi:\mathcal{X}\to\mathcal{Z}$ of minimal ttgs is called a RIC- *extension* (abbreviation for Relatively InContractible) if $\phi\perp\psi$ for every proximal homomorphism $\psi:\mathcal{Y}\to\mathcal{Z}$ of minimal ttgs. If $\mathcal{Z}$ is the trivial one point ttg, $\mathcal{X}$ is called *incontractible*.

Note that ϕ is RIC iff $\phi \perp \kappa$, where $\kappa : \mathfrak{A}(\mathcal{Z}) \to \mathcal{Z}$ is the universal minimal proximal extension of $\mathcal{Z}$. In particular, it follows from 2.9. that every minimal ttg for an abelian group T is incontractible.
If, for a certain topological group T, the universal minimal ttg $\mathcal{M}_T$ is trivial, then every minimal ttg for T is incontractible; for, obviously, $\mathcal{X} \perp \{\star\}$. Such a topological group is called *strongly amenable* (the name will be clear from VII.1.11.).

It turns out that RIC extensions are open (III.1.4.) and that RIC extensions satisfy the Bronstein condition in a strong way (III.1.9. and III.1.5.).
It is still unsolved whether or not an open Bc extension is a RIC extension. We shall provide two partial results with respect to that question in III.1.9. and V.3.7..

Another concept in relating homomorphisms of ttgs (not necessarily minimal) is that of weak disjointness. Two surjective homomorphisms $\phi : \mathcal{X} \to \mathcal{Z}$ and $\psi : \mathcal{Y} \to \mathcal{Z}$ of ttgs are called *weakly disjoint* if $R_{\phi\psi}$ is an ergodic subset of $X \times Y$; notation: $\phi \mathrel{\dot{\_}} \psi$. If $\mathcal{Z}$ is the trivial one point ttg and $\phi \mathrel{\dot{\_}} \psi$, then we say $\mathcal{X}$ and $\mathcal{Y}$ are *weakly disjoint*; notation: $\mathcal{X} \mathrel{\dot{\_}} \mathcal{Y}$.
In contrast to the situation for disjointness, it is possible that a homomorphism of ttgs is weakly disjoint from itself. Such a homomorphism $\phi : \mathcal{X} \to \mathcal{Z}$ with $\phi \mathrel{\dot{\_}} \phi$ is called *weakly mixing.* If $\mathcal{Z}$ is trivial then $\mathcal{X}$ is called a *weakly mixing* ttg.
The following example of weakly disjoint ttgs and weakly mixing ttgs originates from S. GLASNER [G 75.1]. We shall defer the proof until VII.2.14., where a relativized version is given.

3.10. **EXAMPLE.** *Let $\mathcal{X}$ be a proximal minimal ttg. Then $\mathcal{X}$ is weakly disjoint from every minimal ttg. In particular, a proximal minimal ttg is weakly mixing* ([G 76] II.2.2.). □

3.11. **REMARK.** *A weakly mixing homomorphism of ttgs does not admit nontrivial almost periodic factors.*

PROOF. Let $\phi : \mathcal{X} \to \mathcal{Y}$ be a surjective weakly mixing homomorphism of ttgs. Then for every $\alpha \in \mathcal{U}_X$ we have $\overline{T\alpha \cap R_\phi} = R_\phi$. Hence $Q_\phi = R_\phi$, so $E_\phi = R_\phi$ and ϕ does not admit nontrivial almost periodic factors. □

I.4. MISCELLANEA

This section does not have a main theme. We intend to give some examples and we shall comment on the relations P_ϕ, Q_ϕ and E_ϕ for a homomorphism ϕ of minimal ttgs.

We shall need the following lemma ([AG 77] lemma II.2.; also compare VI.3.1. in here).

4.1. **LEMMA.** *Consider the next commutative diagram consisting of homomorphisms of minimal ttgs.*

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\mu"] \arrow[d, "\phi"'] & \mathfrak{Z} \arrow[d, "\psi"] \\
\mathfrak{Y} \arrow[r, "\nu"'] \arrow[ur, dashed, "\theta"] & \mathfrak{W}
\end{tikzcd}$$

Let ϕ be a proximal extension and let ψ be distal. Then there is a homomorphism $\theta: \mathfrak{Y} \rightarrow \mathfrak{Z}$ such that $\mu = \theta \circ \phi$ and $\nu = \psi \circ \theta$. □

Let $\phi: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a homomorphism of minimal ttgs. Since $P_\phi \subseteq Q_\phi \subseteq E_\phi$, $P_\phi \circ Q_\phi \cup Q_\phi \circ P_\phi \subseteq E_\phi$; sometimes, however, we have $E_\phi = Q_\phi \circ P_\phi$ (e.g. III.3.8. and VII.1.19., 1.20.). The following holds with respect to $Q_\phi \circ P_\phi$:

4.2. **REMARK.** *Let $\phi: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a homomorphism of minimal ttgs. Then*

$$Q_\phi \circ P_\phi = P_\phi \circ Q_\phi =$$

$\{(x_1, x_2) \in R_\phi \mid v(x_1, x_2) \in Q_\phi$ for some m.l.i. I in S_T and some $v \in J(I)\}$.

PROOF. Follows from 2.7. and the fact that Q_ϕ is closed and invariant. □

Consider the next commutative diagram of homomorphisms of minimal ttgs.

$$\begin{tikzcd}
\mathfrak{X} \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & \mathfrak{Y} \\
& \mathfrak{Z} \arrow[ur, "\theta"'] &
\end{tikzcd}$$

The following describes how P_ϕ and P_θ, Q_ϕ and Q_θ and E_ϕ and E_θ are related.

4.3. **THEOREM.** *In the situation above, the following statements hold:*

a) $\psi \times \psi[P_\phi] = P_\theta$*;*
b) $\psi \times \psi[Q_\phi] = Q_\theta$*;*
c) $\psi \times \psi[P_\phi \circ Q_\phi] = P_\theta \circ Q_\theta$*;*
d) $\psi \times \psi[E_\phi] = E_\theta$*;*
e) *for every* $x \in X$, $\psi[E_\phi[x]] = E_\theta[\psi(x)]$.

PROOF.

a) This is a straightforward relativization of [E 69] 5.22.3..
b) This is a straightforward relativization of [MW 80.2] 3.2..
c) Follows easily from b and 4.2..
e) [MW 83] 2.3..
d) Follows from e (but a direct proof is possible). □

In the previous section we already mentioned the use of dense sets of almost periodic points. In chapter III we shall discuss a technique that is perfectly fit to attack the regionally proximal relation in the situation of a Bc extension. In fact, it attacks the regionally proximal relation as far as the set $\overline{JR_\phi}$ is concerned.

To that end define for a homomorphism $\phi: \mathfrak{X} \to \mathfrak{Y}$ of minimal ttgs:

$$Q_\phi^* := \bigcap \{ \overline{T\alpha \cap \overline{JR_\phi}} \mid \alpha \in \mathfrak{U}_X \}$$

and note that $Q_\phi^* = \bigcap \{ \overline{T\alpha \cap JR_\phi} \mid \alpha \in \mathfrak{U}_X \}$. In other words, $(x_1, x_2) \in Q_\phi^*$ iff there is a net $\{(x_1^i, x_2^i)\}_i$ in JR_ϕ and there are $t_i \in T$ such that

$$(x_1^i, x_2^i) \to (x_1, x_2) \quad \text{and} \quad t_i(x_1^i, x_2^i) \to (x_1, x_1).$$

Clearly, Q_ϕ^* is a closed, invariant, reflexive and symmetric relation in $\overline{JR_\phi}$, and $Q_\phi^* \subseteq Q_\phi$; if ϕ satisfies the Bronstein condition, then $Q_\phi^* = Q_\phi$.

4.4. **LEMMA.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs, and let* (x_1, x_2) *be an almost periodic point in* Q_ϕ^* *; say* $(x_1, x_2) = u\,(x_1, x_2)$ *for some* $u \in J$. *Then there are nets* $\{x_2^i\}_i$ *in* $u\phi^{\leftarrow}\phi(x_1)$ *and* s_i *and* t_i *in* T *such that*

$$s_i(x_1, x_2^i) \to (x_1, x_2) \quad \text{and} \quad t_i(x_1, x_2^i) \to (x_1, x_1) \quad \textit{in} \quad \overline{JR_\phi}$$

while $s_i u \to u$ *and* $t_i u \to u$ *in* M.

PROOF. (See also [MW 74] 2.2.) By 3.8., $\overline{JR_\phi} = \overline{T(\{x_1\} \times u\phi^{\leftarrow}\phi(x_1))}$ and so it follows easily that

$$Q_\phi^* = \bigcap \{\overline{T\alpha \cap T(\{x_1\} \times u\phi^{\leftarrow}\phi(x_1))} \mid \alpha \in \mathfrak{U}_X\} .$$

This shows that we can find a net $\{z_\lambda\}_\lambda$ in $u\phi^{\leftarrow}\phi(x_1)$ and s_λ^1 and t_λ^1 in T such that

$$s_\lambda^1\,(x_1, z_\lambda) \to (x_1, x_2) \quad \text{and} \quad t_\lambda^1\,(x_1, z_\lambda) \to (x_1, x_1) .$$

Let $g_\lambda \in G$ be such that $z_\lambda = g_\lambda x_1$ and note that $g_\lambda \phi(x_1) = \phi(x_1)$. After passing to a suitable subnet we can find p_1 , p_2 , p_3 and p_4 in M such that

$$s_\lambda^1\,(u, g_\lambda) \to (p_1, p_2) \quad \text{and} \quad t_\lambda^1\,(u, g_\lambda) \to (p_3, p_4) ;$$

note that $p_1 x_1 = p_3 x_1 = p_4 x_1 = x_1$ and $p_2 x_1 = x_2$.
Choose nets $\{r_\mu\}_\mu$ and $\{r_\mu^1\}_\mu$ in T with $r_\mu \to u$ and $r_\mu^1 \to up_1 up_3^{-1}$. Then there are nets $\{s_\nu^2\}_\nu$ and $\{t_\nu^2\}_\nu$ in T (subnets of the product nets (s^1, r) , (t^1, r^1)) such that

$$s_\nu^2\,(u, g_\nu) \to (up_1, up_2) \quad \text{and} \quad t_\nu^2\,(u, g_\nu) \to (up_1, up_1 up_3^{-1} p_4) ,$$

for suitable $g_\nu \in \{g_\lambda \mid \lambda\}$. By continuity of right multiplication with up_1^{-1} we have $s_\nu^2 up_1^{-1} \to up_1 up_1^{-1} = u$ and $t_\nu^2 up_1^{-1} \to u$; hence

$$s_\nu^2 up_1^{-1}\,(u, up_1 g_\nu) \to (u, up_2) \quad \text{and} \quad t_\nu^2 up_1^{-1}\,(u, up_1 g_\nu) \to (u, up_1 up_3^{-1} p_4) .$$

But then we can find (sub)nets $\{s_i\}_i$ and $\{t_i\}_i$ in T and $g_i \in \{g_\nu \mid \nu\}$ such that

$$s_i\,(u, up_1 g_i) \to (u, up_2) \quad \text{and} \quad t_i\,(u, up_1 g_i) \to (u, up_1 up_3^{-1} p_4) .$$

Hence $s_i u \to u$ and $t_i u \to u$ in M ; and $up_1 g_i x_1 \in u\phi^{\leftarrow}\phi(x_1)$, while

$$s_i\,(x_1, up_1 g_i x_1) \to (x_1, up_2 x_1) = (x_1, x_2) \text{ and}$$

$$t_i\,(x_1, up_1 g_i x_1) \to (x_1, up_1 up_3^{-1} p_4 x_1) = (x_1, x_1) .$$

□

We shall turn to some examples. Although they are completely standard they can serve as a link to reality.
It is left as an exercise for the reader to check the properties of the ttgs mentioned here.

4.5. (i) Let X be the circle, considered as the unit interval I with end points identified. Define $\phi: X \to X$ by $\phi(x) = x + \alpha \pmod 1$ for some irrational $\alpha \in I$, and let the action of $\mathbb{Z}$ on X be defined by $(n,x) \mapsto \phi^n(x) = x + n\alpha \pmod 1$. Then $\mathcal{X}_\alpha$ is a ttg for $\mathbb{Z}$. $\mathcal{X}_\alpha$ is minimal; and as ϕ is an isometry, $\mathcal{X}_\alpha$ is uniformly almost periodic.

(ii) Let $Y = X \times X$ be the torus and define a homeomorphism $\psi: Y \to Y$ by $\psi(x_1, x_2) = (x_1 + \alpha, x_2 + \beta)$ for irrational $\alpha, \beta \in I$ such that α / β is irrational too. Again, let the action of $\mathbb{Z}$ on Y be defined by the iterates of ψ. Then $\mathcal{Y} = \mathcal{X}_\alpha \times \mathcal{X}_\beta$ is the product of two uniformly almost periodic minimal ttgs, so $\mathcal{Y}$ is uniformly almost periodic. As α/β is irrational, the point $(0,0)$ has a dense orbit in Y, and it follows that $\mathcal{Y}$ is minimal.

Note that this means that the uniformly almost periodic minimal ttgs $\mathcal{X}_\alpha$ and $\mathcal{X}_\beta$ are disjoint iff α and β are independent over $\mathbb{Q}$.

(iii) Let $Z = X \times X$ be the torus and define a homeomorphism $\theta: Z \to Z$ by $\theta(x_1, x_2) = (x_1 + \alpha, x_1 + x_2)$ for a transcendental $\alpha \in I$. Let the action of $\mathbb{Z}$ on Z be defined by the iterates of θ.

Clearly, the projection $\pi: \mathcal{Z} \to \mathcal{X}_\alpha$ is a homomorphism of ttgs; moreover, π is a group extension (every fiber is homeomorphic to the CT_2 group X). Hence $\mathcal{Z}$ is an almost periodic extension of a uniformly almost periodic minimal ttg and so $\mathcal{Z}$ is distal. As $\{(n\alpha, \frac{1}{2}\alpha(n^2 - n)) \mid n \in \mathbb{Z}\}$ is dense in Z it follows that $\mathcal{Z}$ is minimal. This ttg $\mathcal{Z}$ is not uniformly almost periodic, however, [F 63].

4.6. (i) Consider the uniformly almost periodic minimal ttg $\mathcal{X}_\alpha$ for $T = \mathbb{Z}$ as in 4.5.(i). Let $x_0 \in X$ and put $E = \mathbb{Z}x_0$, the orbit of x_0. Clearly, E is a proper dense subset of X. Split every $e \in E$ into two distinct points e^+, e^- and define

$$Y := (X \setminus E) \cup \{e^+ \mid e \in E\} \cup \{e^- \mid e \in E\}.$$

Let $\phi: Y \to X$ be the obvious identification map. Provide Y with a CT_2 topology by defining a base $\mathcal{B}$ as follows:

Every full original (under ϕ) of an open interval in X is an element of $\mathcal{B}$. For every $e \in E$ and every $\epsilon > 0$ the sets $(e - \epsilon, e) \cup \{e^+\}$ and $(e, e + \epsilon) \cup \{e^-\}$ are elements of $\mathcal{B}$. We can extend the action of $\mathbb{Z}$ on X to an action of $\mathbb{Z}$ on Y by defining

$$(n,x) \mapsto x + n\alpha \pmod 1 \quad \text{for every} \quad x \in X \setminus E \;;$$

$$(n,e^+) \mapsto (e+n\alpha)^+ \pmod 1, \quad (n,e^-) \mapsto (e+n\alpha)^- \pmod 1 \qquad (e \in E).$$

Then $\mathcal{Y}$ is a ttg for $\mathbb{Z}$ and as every orbit is dense, $\mathcal{Y}$ is minimal. The map $\phi: \mathcal{Y} \to \mathcal{X}$ is a homomorphism of minimal ttgs and ϕ is one-to-one in the points outside E (i.e., ϕ is almost one-to-one or almost automorphic). So ϕ is point distal, and every $x \in X \setminus E$ is not just a ϕ-distal point but even a distal point for $\mathcal{Y}$, i.e., $\mathcal{Y}$ is a point distal minimal ttg.
Also ϕ is proximal, for e^+ and e^- are proximal ($e \in E$). Hence $\mathcal{Y}$ is a proximal extension of a uniformly almost periodic ttg, a so called *proximal-equicontinuous* ttg. As ϕ is proximal in a special way, $\mathcal{Y}$ is even locally almost periodic (see VI.5.6.).

(ii) A point distal ttg does not have to be locally almost periodic, since every minimal distal ttg is point distal; e.g. $\mathcal{Z}$ in 4.5.(iii) is point distal. If $\mathcal{Z}$ were locally almost periodic, it would have been uniformly almost periodic by 1.18..

4.7. (i) Let $T := T(a,b)$ be the free group on two generators (a and b), and let X be the circle. Define $a: X \to X$ by $a(x) = x + \alpha \pmod 1$ for an irrational $\alpha \in I$ and define $b: X \to X$ by $b(x) = x^2$. Then a and b are homeomorphisms, and $\mathcal{X}$ is a ttg for T. By the action of a, $\mathcal{X}$ is minimal and by the action of b, $\mathcal{X}$ is proximal.

(ii) Let Y be the circle and define $c: Y \to Y$ by $c(y) = y + \tfrac{1}{2}\alpha$ (same α as in (i)) and define $d: Y \to Y$ by $d(y) = 2y^2$ for $0 \leqslant y \leqslant \tfrac{1}{2}$ and $d(y) = \tfrac{1}{2} + 2(y - \tfrac{1}{2})^2$ for $\tfrac{1}{2} \leqslant y < 1$..
By the rotation c, $\mathcal{Y}$ is a minimal ttg for T $(= T(c,d))$. Define $\phi: Y \to X$ by $\phi(y) = 2y \pmod 1$. Then ϕ is a homomorphism of minimal ttgs. Moreover ϕ is a group extension, the CT_2 group being the group consisting of two elements.
Note that $P_{\mathcal{Y}} = Y \times Y \setminus \{(y, y + \tfrac{1}{2}) \mid y \in Y\}$ and $Q_{\mathcal{Y}} = Y \times Y$, so $Q_{\mathcal{Y}} = E_{\mathcal{Y}} = P_{\mathcal{Y}} \circ P_{\mathcal{Y}}$.
This map ϕ as well as $\mathcal{Y}$ is called the *twofold covering* of the minimal proximal rotation. Obviously, we can define threefold and fourfold coverings similarly.

4.8. Let $\mathcal{X} = \langle X, \sigma \rangle$ be the shift transformation on two symbols, i.e., $X = \{0,1\}^{\mathbb{Z}}$ and $\sigma: X \to X$ is defined by $\sigma(x)_i = x_{i+1}$ for all $i \in \mathbb{Z}$.
Define blocks B_k for $k \in \mathbb{N}$ as follows:

$$B_0 = 00\ ;\quad B_1 = 0010\ ;\quad B_{n+1} = B_n B_n 1 B_n \quad \text{for every} \quad n \in \mathbb{N}\ ;$$

and let $Y \subseteq X$ be defined by

$Y = \{x \in X \mid$ every finite segment of x is a segment of B_k for some $k \in \mathbb{N}\}$.

Then Y is a closed shift invariant subset of X ; so $\mathcal{Y}$ is a ttg for $\mathbb{Z}$. It turns out that $\mathcal{Y}$ is a minimal weakly mixing ttg (cf. [J 82]). Moreover, $\mathcal{Y}$ is a *prime* ttg, i.e., $\mathcal{Y}$ does not have nontrivial factors.
For more details on this so called Chacon transformation $\mathcal{Y}$ see [J 82].

4.9. Let Z be a compact, nonseparable, nonmetric topological space and define $X = Z^{\mathbb{Z}}$. Let σ be the shift on X . Then $\mathcal{X}$ is a ttg for $\mathbb{Z}$. As X is not separable, X does not contain transitive points. But it is easy to see that $\mathcal{X}$ is ergodic.

I.5. REMARKS

In this section we shall briefly discuss some more or less isolated subjects, which are closely related to the material presented in the previous sections of this chapter.

5.1. In the literature one often encounters a function algebraic approach to topological dynamics, especially in the mathematical environment of R. ELLIS. It is just a matter of taste that we didn't adopt this approach.
In short it comes down to the following (see [E 69] chapters 9 and 10). Let $\mathcal{X}$ be a ttg and denote by $\mathcal{C}(X)$ the Banach algebra of all continuous complex valued functions on X provided with the supremum norm. As a point transitive ttg $\mathcal{X}$ is a factor of $\mathcal{S}_T$, we can consider $\mathcal{C}(X)$ as a subalgebra of $\mathcal{C} := \mathcal{C}(S_T)$. In this way there is a one-to-one correspondence between the point transitive ttgs and the so called T-subalgebras of $\mathcal{C}$. So the study of point transitive ttgs can be transformed into the study of certain subalgebras of $\mathcal{C}$. In this approach one rather studies point transitive ttgs with a fixed base point.

5.2. Let T be an arbitrary topological group. If $\mathcal{X}$ is a ttg for T_d , then $\mathcal{X}$ is a ttg for T except the (joint) continuity of the action. in general, the action will not be continuous; but under some conditions it is, as may be seen from the "theorem of Ellis" [E 57].

> *Let* T *be a locally compact* T_2 *topological group, and let* $< T_d, X, \pi >$ *be a ttg for* T_d. *If* $\pi: T \times X \to X$ *is separately continuous then* π *is jointly continuous, (hence* $\mathfrak{X}$ *is a ttg for* T *).*

This nontrivial result plays a role in the proof of 1.20.. The theorem is not stated here in its fullest generality. For a short and transparent proof see [T 79]. In [Cr 81] a "game theoretic" proof is given.

5.3. In section one we gave relative notions of distality, proximality and almost periodicity. We did not define relative local almost periodicity. This is studied in [MW 80.2]. There it turns out that a homomorphism $\phi: \mathfrak{X} \to \mathfrak{Y}$ of minimal ttgs is locally almost periodic if $\phi = \psi \circ \theta$ with θ highly proximal (see IV) and ψ almost periodic (for the absolute case this was shown in [MW 72], cf. VI.5.6.).

5.4. **NOTE.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of ttgs with* $\mathfrak{Y}$ *minimal. Suppose that* $x \in X$ *is a* ϕ*-distal point, i.e.,* $X = \overline{T(x)}$ *and* x *is distal from every* $x' \in X$ *with* $\phi(x) = \phi(x')$. *Then* $\mathfrak{X}$ *is minimal.*

Let $v \in J_y$ and note that $\phi(vx) = v\,\phi(x) = vy = y = \phi(x)$. By assumption x and vx are distal. As $vx = v.vx$ it follows from 2.7. that x and vx are proximal. Hence $x = vx$, and x is an almost periodic point (2.5.). So $X = \overline{Tx}$ is a minimal set.

Note that 4.6.(i) shows that point distal is not necessarily distal.
The corresponding notion of a point proximal homomorphism of minimal ttgs is not very useful, as is shown by the next observation:

> **NOTE.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs and suppose that* $x \in X$ *is a proximal point for* ϕ, *i.e.,* $(x, x') \in P_{\mathfrak{X}}$ *for every* $x' \in X$ *with* $(x, x') \in R_\phi$. *Then* ϕ *is proximal.*

Let $(x_1, x_2) \in R_\phi$, say $x_1 = px$ and $x_2 = qx$ for some p and q in M. Let $u \in J_x$; then

$$(x, up^{-1}qx) = up^{-1}(x_1, x_2) \in \overline{TR_\phi} = R_\phi .$$

By assumption, x and $up^{-1}qx$ are proximal. By 2.8., x and $up^{-1}qx$ are distal; hence $x = up^{-1}qx$. But then $upx = up\,(up^{-1}qx) = uqx$; and so, by 2.7., px and qx are proximal.

5.5. The notion of disjointness was introduced in [F 67] for not necessarily minimal ttgs. Two ttgs $\mathcal{X}$ and $\mathcal{Y}$ are called *disjoint* if for every ttg $\mathcal{Z}$ and all surjective homomorphisms $\phi: \mathcal{Z} \to \mathcal{X}$ and $\psi: \mathcal{Z} \to \mathcal{Y}$ the induced homomorphism $\theta: \mathcal{Z} \to \mathcal{X} \times \mathcal{Y}$ is surjective. Here θ is such that $\phi = \pi_1 \circ \theta$ and $\psi = \pi_2 \circ \theta$, where π_1 and π_2 are the projections.

$$\begin{tikzcd}
 & \mathcal{Z} \arrow[dl, "\phi"'] \arrow[d, "\theta"] \arrow[dr, "\psi"] & \\
\mathcal{X} & \mathcal{X} \times \mathcal{Y} \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \mathcal{Y}
\end{tikzcd}$$

Clearly disjointness is preserved under factors. If $\mathcal{X}$ and $\mathcal{Y}$ are disjoint then one of them has to be minimal. Moreover, if both $\mathcal{X}$ and $\mathcal{Y}$ are minimal, then $\mathcal{X}$ and $\mathcal{Y}$ are disjoint iff $\mathcal{X} \times \mathcal{Y}$ is minimal.
These facts are easy to verify, so their proofs are left for the reader.

5.6. The notion of weak disjointness first occurs in [P 72], with the same definition as we gave. A slightly different definition can be found in [M 78], where two ttgs $\mathcal{X}$ and $\mathcal{Y}$ are called weakly disjoint if $\mathcal{X} \times \mathcal{Y}$ is a point transitive ttg. Clearly the notions coincide if both X and Y have a countable pseudobase.
Of course this yields different definitions for the notion of weak mixing. Let $\mathcal{X}$ be a minimal ttg.

WM1 A ttg $\mathcal{X}$ is weakly mixing if $\mathcal{X} \times \mathcal{X}$ is ergodic ([P 72]).

WM2 A ttg $\mathcal{X}$ is weakly mixing if $\mathcal{X} \times \mathcal{X}$ is point transitive ([M 76.1]).

Other definitions occurring in the literature are:

WM3 A ttg $\mathcal{X}$ is weakly mixing if $Q_{\mathcal{X}} = X \times X$ ([B 75/79] 3.13.14.).

WM4 A ttg $\mathcal{X}$ is weakly mixing if $E_{\mathcal{X}} = X \times X$ ([E 81] 0.10.).

Clearly, WM2 $\Rightarrow$ WM1 $\Rightarrow$ WM3 $\Rightarrow$ WM4 and WM2 $\Rightarrow$ WM1 in case X has a countable pseudobase. If $\mathcal{X}$ is incontractible or if $\mathcal{X}$ admits an invariant measure, then WM1, WM3 and WM4 are equivalent (VII.3.11.).
Our definition of weak mixing will always be WM1.

5.7. In 4.5. through 4.9. we gave a few examples of ttgs and homomorphisms of ttgs. They just serve as an illustration. In the literature many other (and more sophisticated) examples can be found; we shall name a few and give some references.

(i) Many examples do exist based on shift systems e.g. 4.8.. In this area the intertwining of ergodic theory and topological dynamics is quite strong, [D 80], [Mt 71], [Mk 75].

(ii) Let Y and Z be CT_2 spaces and let $\sigma : Z \to Z$ be a homeomorphism. Suppose $h : Z \to \mathcal{H}(Y)$ is a continuous map from Z into the full homeomorphism group of Y (uniform topology). Define a homeomorphism ϕ on $X = Z \times Y$ by $\phi(z,y) = (\sigma(z), h(z)(y))$. Then X is called a *skew product* of Z and Y. In fact in 4.5.(iii) Z is a skew product of X and X, where $h : X \to \mathcal{H}(X)$ is defined by $h(x)(x') = x + x'$.

Many examples are made using skew product constructions e.g. [GW 79], [G 80], [GW 81].

(iii) Our example 4.6.(i) can be generalized considerably (see for instance IV.1.4.). In [M 76.1] and [M 78] many examples are constructed with the method which is discussed in IV.1.4..

(iv) In [B 75/79] one can find a lot of examples coming from the qualitative theory of differential equations.

(v) By way of anthology of other examples we shall just mention some papers in which interesting examples can be found. This list is not meant to be complete so a lot of other interesting examples may remain unmentioned: [E 65], [FKS 73], [G 74], [G 75.1], [M 76.2], [MW 72], [MW 76], [P 71], [S 70], [W 67].

II

HYPER TRANSFORMATION GROUPS

In the structure theory of minimal ttgs it turns out to be useful to study the behavior of subsets of the phase space under the given action. One of the first (rudimentary) occurrences of the hyperspace in that respect was in [V 70], in which the study of the phenomenon of the shrinking of a fiber to a point was started (cf. IV.1.).
In this chapter we shall briefly discuss the action of T on the hyperspace 2^X of the phase space X, which is induced by the action of T on X.
The first section is just an introduction with some emphasis on ergodicity. Recursiveness, in particular almost periodicity, is discussed in the second section. In the third one the induced action of S_T on the hyperspace ("the circle operation") is introduced, as are quasifactors. These notions will occur frequently in the sequel.

II.1. HYPERSPACES AND ERGODICITY

Many standard constructions do exist that build new ttgs out of old ones (cf. section I.1.). In this section we introduce the hyper ttg $2^{\mathcal{X}}$ induced by the ttg $\mathcal{X}$. We also define the so called "circle-action" (or "circle-operation") of S_T on $2^{\mathcal{X}}$. Both concepts play a major role in this booklet. We end this section with observations on ergodicity of $2^{\mathcal{X}}$.

Let X be a topological space. The *hyperspace* 2^X of X is defined to be the collection of all nonempty closed subsets of X .
On 2^X we can define the *Vietoris topology* as follows:
For an open set U in X define

$$<U>:=\{B\in 2^X\,|\,B\subseteq U\}\ \text{ and}$$

$$<U>^*:=\{B\in 2^X\,|\,B\cap U\neq\varnothing\}$$

and let

$$\mathbb{S}:=\{<U>\,|\,U \text{ open in } X\}\cup\{<U>^*\,|\,U\text{ open in }X\}\,.$$

The Vietoris topology on 2^X is the topology generated by the subbase $\mathbb{S}$.
Note that a base for the Vietoris topology is formed by the sets of the form

$$<U_1,\ldots,U_n>:=<\bigcup_{i=1}^{n}U_i>\cap\bigcap_{i=1}^{n}<U_i>^*$$

Note that $<U>^*=<X,U>$.

1.1. **THEOREM.** *Let X be a topological space.*

a) *If X is a T_1-space, then X can be homeomorphically embedded in 2^X by the map $x\mapsto\{x\}$.*
b) *X is metrizable iff 2^X is metrizable.*
c) *X is* CT_2 *iff 2^X is* CT_2 .

PROOF. Cf. [Mi 51]. □

1.2. Let X be a CT_2 space and let $\mathfrak{U}$ be the unique uniform structure for X . Then the Vietoris topology is just the uniform topology on 2^X induced by the unique uniform structure $\mathfrak{U}^*$, which is generated by the collection $\{\alpha^*\ |\ \alpha\in\mathfrak{U}\}$; here

$$\alpha^*:=\{(A,B)\in 2^X\times 2^X\ |\ A\subseteq\alpha(B)\ \text{ and }\ B\subseteq\alpha(A)\}\,.$$

For a proof of this we refer to [Mi 51].

Let $\phi:X\to Y$ be a closed continuous surjection. Then ϕ induces maps

$$2^{\phi}:2^X\to 2^Y\qquad\text{and}\qquad \phi_{\mathrm{ad}}:2^Y\to 2^X$$

defined by $2^{\phi}(A)=\phi[A]$ for all $A\in 2^X$ and $\phi_{\mathrm{ad}}(B)=\phi^{\leftarrow}[B]$ for all $B\in 2^Y$.

1.3. **THEOREM.** *Let* $\phi : X \to Y$ *be a continuous surjection of* CT_2 *spaces. Then*

a) 2^{ϕ} *is continuous;*

b) $\phi_{\mathrm{ad}}|_Y : Y \to 2^X$ *is an upper semi continuous* (u.s.c.) *map; i.e.,* $\{y \in Y \mid \phi_{\mathrm{ad}}(y) = \phi^{\leftarrow}(y) \subseteq U\}$ *is open in* Y *for open* U *in* X *;*

c) $\phi_{\mathrm{ad}}|_Y$ *is continuous in* $y \in Y$ *iff* ϕ *is open in every point of* $\phi^{\leftarrow}(y)$ *;*

d) ϕ_{ad} *is continuous iff* $\phi_{\mathrm{ad}}|_Y$ *is continuous iff* ϕ *is open;*

e) *if* X *is metrizable then there is a dense* G_δ*-set* Y' *in* Y *such that* ϕ_{ad} *is continuous in every point of* Y' *.*

PROOF. For a, b and d see [Mi 51]; c is straightforward. A proof for e can be found in [Fo 51]. □

1.4. **REMARK.** *Let* X *be a* CT_2 *space. The map* $\iota_n : X^n \to 2^X$ *defined by* $(x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\}$ *is continuous. Moreover, it is locally one-to-one in the points* $(x_1, \ldots, x_n)$ *with* $x_i \neq x_j$ *for all* $i \neq j$ *. Also note that* $\bigcup \{\iota_n[X^n] \mid n \in \mathbb{N}\}$ *is dense in* 2^X *.* □

The following remark on convergence in 2^X seems useful, the easy proof is omitted.

1.5. **REMARK.** *Let* $\{A_i\}_i$ *be a convergent net in* 2^X *. Then* $A = \lim A_i$ *in* 2^X *iff the following conditions are satisfied:*

(i) A *contains all convergence points of every net* $\{a_i\}_i$ *with* $a_i \in A_i$ *;*

(ii) *for every* $x \in A$ *there is a net* $\{a_j\}_j$ *with* $a_j \in A_j$ *(after passing to a suitable subnet* $\{A_j\}_j$ *of* $\{A_i\}_i$ *) such that* x *is a convergence point of* $\{a_j\}_j$ *.* □

Let $\mathfrak{X} = \langle T, X, \pi \rangle$ be a ttg (note that X is a CT_2 space unless stated otherwise). Then, clearly, $\langle T_d, 2^X, 2^{\pi} \rangle$ is a ttg, where the map $2^{\pi} : T_d \times 2^X \to 2^X$ is defined by

$$2^{\pi}(t, A) := \pi[\{t\} \times A]$$

for all $t \in T$ and $A \in 2^X$ (or, suppressing the action symbol, $(t, A) \mapsto tA$). Indeed, every homeomorphism π^t of X extends to a homeomorphism $2^{(\pi^t)} = (2^{\pi})^t$ of 2^X (by 1.3.a).

1.6. **THEOREM.** *Let* $\mathfrak{X} = < T, X, \pi >$ *be a ttg for an arbitrary topological group* T . *Then* $2^{\mathfrak{X}} := < T, 2^X, 2^\pi >$ *is a ttg and* $\mathfrak{X}$ *can be equivariantly embedded in* $2^{\mathfrak{X}}$.

PROOF. By the above, we only have to prove the continuity of $2^\pi : T \times 2^X \to 2^X$. Let $t \in T$ and $A \in 2^X$ and take a subbase neighbourhood $<U>$ of $2^\pi(t,A)$. Then $\pi[\{t\} \times A] \subseteq U$, so by continuity of π there are open neighbourhoods V and W of t and A in T and X , such that $\pi[V \times W] \subseteq U$. Hence $2^\pi[V \times <W>] \subseteq <U>$. Next consider a subbase neighbourhood $<U>^*$ of $2^\pi(t,A)$; i.e., there is an $a \in A$ with $\pi(t,a) \in \pi[\{t\} \times A] \cap U$. By continuity of π there are open neighbourhoods V and W of t and a in T and X such that $\pi[V \times W] \subseteq U$. Hence $2^\pi[V \times <W>^*] \subseteq <U>^*$. The second part of the statement is obvious. □

1.7. Note that $2^{\mathfrak{X}}$ contains $\mathfrak{X}$ as a closed invariant subset; thus $2^{\mathfrak{X}}$ is minimal iff $\mathfrak{X}$ is trivial. Further on, however, we shall see that $2^{\mathfrak{X}}$ can very well be ergodic and nontrivial (1.11.). We omit the easy proof of the following theorem.

1.8. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a surjective homomorphism of ttgs. Then*

a) $2^\phi : 2^{\mathfrak{X}} \to 2^{\mathfrak{Y}}$ *is a homomorphism of ttgs;*

b) $\phi_{\mathrm{ad}} : 2^{\mathfrak{Y}} \to 2^{\mathfrak{X}}$ *is an equivariant u.s.c. map; it is a homomorphism of ttgs iff* ϕ *is open;*

c) ϕ_{ad} *and* $\phi_{\mathrm{ad}}|_Y$ *are embeddings iff* ϕ *is open.* □

From now on we shall (again) forget about the action symbol, i.e.: if $\mathfrak{X}$ is a ttg then $2^{\mathfrak{X}}$ is a ttg and the action will be denoted by $(t,A) \mapsto tA$. However, this notation may cause some ambiguity with respect to the action of S_T on $\mathfrak{X}$ and $2^{\mathfrak{X}}$. To circumvent misunderstanding, we shall denote the action of S_T on $2^{\mathfrak{X}}$ by the "circle operation".
Let $A \in 2^X$ and $p \in S_T$, then

$$pA := \{pa \mid a \in A\} \text{ and}$$

$$p \circ A := \lim t_i A \text{ in } 2^X \text{ for some net } \{t_i\}_i \text{ in } T \text{ with } p = \lim t_i$$

$$= \lim t_i A \text{ in } 2^X \text{ for every net } \{t_i\}_i \text{ in } T \text{ with } p = \lim t_i .$$

1.9. **LEMMA.** *Let* $\mathfrak{X}$ *be a ttg,* $A \in 2^X$, $p, q \in S_T$ *and* $t \in T$.
a) *Let* $\{t_i\}_i$ *be a net in* T *with* $p = \lim t_i$. *Then*

$$p \circ A = \{x \in X \mid x = \lim t_j a_j \text{ for a subnet } \{t_j\}_j \text{ of } \{t_i\}_i \text{ and for } a_j \in A\}.$$

b) $pA \subseteq p \circ A$ *and* $tA = t \circ A$,
c) $p \circ (q \circ A) = pq \circ A$.

PROOF.
a) Clear from the definition above and 1.5..
b) Follows immediately from a.
c) Follows from the fact that S_T acts as a semigroup on 2^X . □

1.10. For nonempty subsets A of X which are not (necessarily) closed, we define $p \circ A := p \circ \overline{A}$. Clearly, (also if A is not closed)

$$p \circ A = \{x \in X \mid x = \lim t_j a_j \text{ for } a_j \in A \text{ and } t_j \to p\} .$$

Note that if A is finite we have $pA = p \circ A$ for all $p \in S_T$. As was mentioned earlier, $2^{\mathfrak{X}}$ can never be minimal (in a nontrivial way). We shall see now that $2^{\mathfrak{X}}$ can be ergodic (cf. 1.15.).

1.11. **THEOREM.** *For all* $n \in \mathbb{N}$ *let* $\mathfrak{X}^n$ *be an ergodic ttg. Then* $(2^{\mathfrak{X}})^n$ *is ergodic for all* $n \in \mathbb{N}$. [*Hence* $(2^{2^{\mathfrak{X}}})^n$ *is ergodic for all* $n \in \mathbb{N}$ *and so on.*]

PROOF. Let W^1 and W^2 be nonempty open sets in $(2^X)^n$. We have to find a $t \in T$ with $tW^1 \cap W^2 \neq \varnothing$. Let $m \in \mathbb{N}$ and open sets U^i_j and V^i_j in X for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ be such that

$$\varnothing \neq < U^1_1, \ldots, U^1_m > \times \cdots \times < U^n_1, \ldots, U^n_m > \subseteq W^1 \quad \text{and}$$

$$\varnothing \neq < V^1_1, \ldots, V^1_m > \times \cdots \times < V^n_1, \ldots, V^n_m > \subseteq W^2 .$$

As $\mathfrak{X}^{mn}$ is ergodic there is a $t \in T$ such that

$$t(U^1_1 \times \cdots \times U^n_m) \cap (V^1_1 \times \cdots \times V^n_m) = L \neq \varnothing ,$$

say $(x^1_1, \ldots, x^1_m, x^2_1, \ldots, x^n_m) \in L$. Then clearly

$$\{x^1_1, \ldots, x^1_m\} \times \cdots \times \{x^n_1, \ldots, x^n_m\} \in tW^1 \cap W^2 ;$$

so $tW^1 \cap W^2 \neq \varnothing$, which proves the theorem. □

1.12. **REMARK.** *Let* $\mathfrak{X}$ *be a ttg and* $n \in \mathbb{N}$. *If* $(2^{\mathfrak{X}})^n$ *is ergodic then* $\mathfrak{X}^n$ *is ergodic. In particular, if* $2^{\mathfrak{X}}$ *is ergodic (weakly mixing) then* $\mathfrak{X}$ *is ergodic (weakly mixing).*

PROOF. Let $U_1 \times \cdots \times U_n$ and $V_1 \times \cdots \times V_n$ be basic open (and nonempty) in X^n . Then

$$U := <U_1> \times \cdots \times <U_n> \quad \text{and} \quad V := <V_1> \times \cdots \times <V_n>$$

are open in $(2^X)^n$. So there is a $t \in T$ such that $tU \cap V \neq \emptyset$; hence there are $x_i \in U_i$ with $t\{x_i\} = \{tx_i\} \in <V_i>$. But then

$$t(x_1, \ldots, x_n) \in t(U_1 \times \cdots \times U_n) \cap V_1 \times \cdots \times V_n \ ,$$

and $\mathfrak{X}^n$ is ergodic. □

1.13. **LEMMA.** *Let* $\mathfrak{X}$ *be a ttg. If* $2^{\mathfrak{X}}$ *is ergodic then* $Q_{\mathfrak{X}} = X \times X$.

PROOF. Choose $\alpha \in \mathfrak{U}$ and U open in X such that $U \times U \subseteq \alpha$. Let $(x_1, x_2) \in X \times X$ and let V_1 and V_2 be open neighbourhoods of x_1 and x_2 in X . As the ttg $2^{\mathfrak{X}}$ is ergodic we can find a $t \in T$ such that $t<U> \cap <V_1, V_2> \neq \emptyset$. In particular, there are points y_1 and y_2 in U with $t(y_1, y_2) \in V_1 \times V_2$. Hence

$$\emptyset \neq V_1 \times V_2 \cap t(U \times U) \subseteq V_1 \times V_2 \cap T\alpha \ ,$$

So $X \times X \subseteq \overline{T\alpha}$ for all $\alpha \in \mathfrak{U}_X$ and, consequently, $X \times X = Q_X$. □

For 1.14. and 1.15. we need some results from chapter VII. which do not depend on the results in this section.

1.14. **COROLLARY.** *Let* $\mathfrak{X}$ *be minimal such that* $X \times X$ *has a dense subset of almost periodic points. If* $2^{\mathfrak{X}}$ *is ergodic then* $\mathfrak{X}$ *is weakly mixing.*

PROOF. By 1.13., $Q_X = X \times X$; hence by VII.3.17. (absolute case), $\mathfrak{X}$ is weakly mixing. □

1.15. **THEOREM.** *Let* $\mathfrak{X}$ *be a minimal ttg. If* $\mathfrak{X}$ *has an invariant measure, or if* $\mathfrak{X}$ *is incontractible, then the following statements are equivalent:*

a) $\mathfrak{X}$ *is weakly mixing;*

b) $\mathfrak{X}^n$ *is ergodic for all* $n \in \mathbb{N}$;

c) $2^{\mathfrak{X}}$ *is ergodic;*

d) $(2^{\mathfrak{X}})^n$ *is ergodic for all* $n \in \mathbb{N}$.

PROOF. By VII.3.11., the statements a and b are equivalent to " $E_{\mathfrak{X}} = X \times X$ ". By 1.11., d follows from b; and, of course, c is implied by d. Assume c; then, by 1.13., it follows that $Q_{\mathfrak{X}} = X \times X$. Hence $X \times X = Q_{\mathfrak{X}} \subseteq E_{\mathfrak{X}}$, so $E_{\mathfrak{X}} = X \times X$. □

1.16. In particular this means that the equivalence of a through d of 1.15. holds for every minimal ttg $\mathfrak{X}$ in the case of an amenable (e.g. abelian) phase group T (every minimal ttg for an amenable group has an invariant measure (cf. VII.1.11.)).

1.17. **LEMMA.** *Let $\mathfrak{X}$ be a ttg and let $n \in \mathbb{N}$. If $(2^{\mathfrak{X}})^n$ is ergodic then for any n open sets $V_1, \dots, V_n$ in X there is a minimal left ideal I in S_T with $p \circ V_i = X$ for all $p \in I$ and all $i \in \{1, \dots, n\}$.*

PROOF. As the collection

$$\{p \in S_T \mid p \circ V_i = X \text{ for } i \in \{1, \dots, n\}\ \}$$

is closed and T-invariant in S_T , we only have to prove it is nonempty. For every $\gamma \in \mathcal{U}_X$ choose a finite γ-dense set $\{x_1^\gamma, \dots, x_{n_\gamma}^\gamma\}$ in X , i.e.,

$$\bigcup \{\gamma(x_i^\gamma) \mid i \in \{1, \dots, n_\gamma\}\ \} = X .$$

Then $<V_1> \times \cdots \times <V_n>$ and $(<\gamma(x_1^\gamma), \dots, \gamma(x_{n_\gamma}^\gamma)>)^n$ are open sets in $(2^X)^n$. So there is a $t_\gamma \in T$ such that

$$t_\gamma <V_i> \cap <\gamma(x_1^\gamma), \dots, \gamma(x_{n_\gamma}^\gamma)> \neq \emptyset \quad \text{for } i \in \{1, \dots, n\} .$$

Note that this means that $t_\gamma V_i \cap \gamma(x_j^\gamma) \neq \emptyset$ for all $i \in \{1, \dots, n\}$ and all $j \in \{1, \dots, n_\gamma\}$. Let $p = \lim t_\gamma \in S_T$ (for a suitable subnet). Let $x \in X$ and let U be a neighbourhood of x in X ; choose $\alpha \in \mathcal{U}_X$ with $\overline{\alpha(x)} \subseteq U$ and $\beta \in \mathcal{U}_X$ with $\beta = \beta^{-1}$ and $\beta^3 \subseteq \alpha$. Then for all $\gamma \in \mathcal{U}_X$ with $\gamma \subseteq \beta$ there is an $x_\gamma \in \{x_1^\gamma, \dots, x_{n_\gamma}^\gamma\}$ with $\gamma(x_\gamma) \subseteq \alpha(x)$. Hence

$$\emptyset \neq t_\gamma V_i \cap \gamma(x_\gamma) \subseteq t_\gamma V_i \cap \alpha(x)$$

for all $\gamma \subseteq \beta$ and all $i \in \{1, \dots, n\}$. But then $p \circ V_i \cap \overline{\alpha(x)} \neq \emptyset$ for all $i \in \{1, \dots, n\}$ and so $p \circ V_i \cap U \neq \emptyset$. As U was arbitrary, $x \in \overline{p \circ V_i} = p \circ V_i$ for all $i \in \{1, \dots, n\}$. As $x \in X$ was arbitrary, $p \circ V_i = X$ for all $i \in \{1, \dots, n\}$. □

1.18. **THEOREM.** *Let $\mathfrak{X}$ be a ttg. Consider the following statements:*

a) *$\mathfrak{X}^n$ is ergodic for all $n \in \mathbb{N}$.*

b) *$(2^{\mathfrak{X}})^n$ is ergodic for all $n \in \mathbb{N}$.*

c) *For every finite collection $\{V_1, \ldots, V_n\}$ of open subsets of X there is a minimal left ideal I in S_T such that $p \circ V_i = X$ for all $p \in I$ and every $i \in \{1, \ldots, n\}$.*

d) *For every countable collection $\mathcal{V}$ of open subsets of X there is a minimal left ideal I in S_T such that $p \circ V = X$ for all $p \in I$ and every $V \in \mathcal{V}$.*

e) *There is a minimal left ideal I in S_T such that $p \circ V = X$ for all $p \in I$ and every open set V in X.*

The statements a , b , c and d are equivalent and they are implied by e . If X has a countable pseudobase (e.g., X is metric) then all five statements are equivalent.

PROOF.

e $\Rightarrow$ d $\Rightarrow$ c Trivial.

c $\Rightarrow$ a Let $U_1 \times \cdots \times U_n$ be a basic open set. By c there exists a $p \in S_T$ with $p \circ U_i = X$ for $i \in \{1, \ldots, n\}$. But then

$$X^n = p \circ U_1 \times \cdots \times p \circ U_n = p \circ (U_1 \times \cdots \times U_n) \subseteq \overline{T(U_1 \times \cdots \times U_n)}.$$

As $n \in \mathbb{N}$ and $U_1 \times \cdots \times U_n$ is arbitrary, $\mathfrak{X}^n$ is ergodic for all $n \in \mathbb{N}$.

a $\Rightarrow$ b Cf. 1.11..

b $\Rightarrow$ c Follows from 1.17..

c $\Rightarrow$ d Let $\mathcal{V} = \{V_i \mid i \in \mathbb{N}\}$, then for all $n \in \mathbb{N}$ we can find $p_n \in S_T$ such that $p_n \circ V_i = X$ for $1 \leqslant i \leqslant n$. Let $p = \lim p_n$, for a suitable subnet. Then clearly $p \circ V = X$ for all $V \in \mathcal{V}$. As $\{p \in S_T \mid p \circ V = X$ for all $V \in \mathcal{V}\}$ is a nonempty closed T-invariant subset of S_T, it contains a minimal left ideal.

d $\Rightarrow$ e Let $\mathfrak{B}$ be the countable pseudobase for X and let I be a minimal left ideal of S_T such that $p \circ B = X$ for all $p \in I$ and all $B \in \mathfrak{B}$. Let V be an open set in X, then there is a $B \in \mathfrak{B}$ with $B \subseteq V$. Hence $X = p \circ B \subseteq p \circ V$ so $p \circ V = X$. □

II.2. RECURSIVENESS

> In order to illustrate to what extent properties of ttgs relate to properties of the induced hyper ttgs, we shall in this section remark on recursiveness in hyper ttgs (also see [Ko 75]).

Fix a collection $\mathfrak{A}$ of subsets of T, to be called the admissible sets, and recall the definitions of (uniform) (pointwise) (local) recursiveness (just after I.1.7.).

2.1. **THEOREM.** *Let* $\mathfrak{X}$ *be a ttg. Then*

a) $2^{\mathfrak{X}}$ *is uniformly recursive iff* $\mathfrak{X}$ *is uniformly recursive;*

b) *if* $2^{\mathfrak{X}}$ *is pointwise recursive then* $\mathfrak{X}$ *is pointwise locally recursive.*

PROOF.

a) Suppose $\mathfrak{X}$ is uniformly recursive. Let $\alpha \in \mathcal{U}_X$ with $\alpha = \alpha^{-1}$ and remember that $\{\beta^* \mid \beta = \beta^{-1} \in \mathcal{U}_X\}$ forms a base for $\mathcal{U}^*$ (1.2.). Let $H \in \mathfrak{A}$ be such that $Hx \subseteq \alpha(x)$ for all $x \in X$ and let $A \in 2^X$. Then $hA \subseteq \alpha(A)$ and by symmetry, $A \subseteq \alpha(hA)$ for all $h \in H$, so $hA \in \alpha^*(A)$ for all $h \in H$; hence $2^{\mathfrak{X}}$ is uniformly recursive. Obviously, if $2^{\mathfrak{X}}$ is uniformly recursive then $\mathfrak{X}$ as a subttg is uniformly recursive too.

b) Let $x \in X$ and let $U \in \mathcal{V}_x$. If $V \in \mathcal{V}_x$ with $\overline{V} \subseteq U$, then $\overline{V} \in 2^X$ is a recursive point and $<U> \in \mathcal{V}_{\overline{V}}$. So there is an $H \in \mathfrak{A}$ with $H\overline{V} \subseteq <U>$, hence $H.V \subseteq U$, and x is a locally recursive point. □

2.2. **THEOREM.** *Let* T *be an abelian group. Then* $x \in X$ *is (locally) recursive in* $\mathfrak{X}$ *iff every finite subset of* Tx *is (locally) recursive in* $2^{\mathfrak{X}}$.

PROOF. We shall prove the theorem for local recursiveness; modification for recursiveness is obvious.

Suppose that $x \in X$ is a locally recursive point in $\mathfrak{X}$ and let $A = \{t_1 x, \ldots, t_n x\} \in 2^X$. Let O be a neighbourhood of A in 2^X and note that, without loss of generality, we may assume that $O = <U_1, \ldots, U_n>$ such that $t_i x \in U_i$ for all $i \in \{1, \ldots, n\}$ and $U_i = U_j$ iff $U_i \cap U_j \neq \emptyset$ (i.e., repetition in the U_i's is allowed!). Choose $V_i \in \mathcal{V}_x$ with $t_i V_i \subseteq U_i$ and let $V := \bigcap \{V_i \mid i \in \{1, \ldots, n\}\}$. As x is a locally recursive point, there is an $H \in \mathfrak{A}$ and a $W \in \mathcal{V}_x$ with $HW \subseteq V$. Hence

$$Ht_i W = t_i HW \subseteq t_i V \subseteq t_i V_i \subseteq U_i$$

and so

$$H. < t_i W, \ldots, t_n W > \subseteq < U_1, \ldots, U_n > .$$

Clearly, $< t_1 W, \ldots, t_n W >$ is a neighbourhood of A . □

2.3. **COROLLARY.** *Let* T *be an abelian group and let* $\mathcal{X}$ *be the orbit closure of a (locally) recursive point. Then* $2^{\mathcal{X}}$ *has a dense set of (locally) recursive points.*

PROOF. By 2.2., it is sufficient to prove that

$$\{A \in 2^X \mid A \subseteq Tx \text{ with } |A| < \aleph_0\}$$

is dense in 2^X . But this follows immediately from the fact that Tx is dense in X , because for every basic open set $< U_1, \ldots, U_n >$ in 2^X we have $U_i \cap Tx \neq \varnothing$ for $i \in \{1, \ldots, n\}$. □

2.4. **REMARK.**

a) *If* $(x_1, \ldots, x_n) \in X^n$ *is a (locally) recursive point in* $\mathcal{X}^n$ *then* $\{x_1, \ldots, x_n\}$ *is a (locally) recursive point in* $2^{\mathcal{X}}$.

b) *If* $\mathcal{X}^n$ *has a dense set of (locally) recursive points for all* $n \in \mathbb{N}$, *then* $(2^{\mathcal{X}})^n$ *has a dense set of (locally) recursive points for all* $n \in \mathbb{N}$.

PROOF. Follows immediately from 1.4.. □

Let $\mathcal{Q}$ be the collection of (left) syndetic subsets of T . Then the corresponding notion of recursiveness is called *almost periodicity.*

2.5. **REMARK.** *Let* $\mathcal{X}$ *be a ttg. Then* $(x_1, \ldots, x_n) \in X^n$ *is an almost periodic point in* $\mathcal{X}^n$ *iff* $\{x_1, \ldots, x_n\}$ *is an almost periodic point in* $2^{\mathcal{X}}$.

PROOF. Suppose $A = \{x_1, \ldots, x_n\}$ is an almost periodic point in $2^{\mathcal{X}}$. Then there is a minimal left ideal K in S_T and an idempotent $u \in J(K)$ such that $u \circ A = A$. As A is finite, $A = u \circ A = uA$; so $x_i = ux_i$ for all $i \in \{1, \ldots, n\}$. Hence, $(x_1, \ldots, x_n) = u(x_1, \ldots, x_n)$ and the point $(x_1, \ldots, x_n)$ is almost periodic in $\mathcal{X}^n$. The other way around is contained in 2.4.a. □

2.6. **THEOREM.** *$\mathfrak{X}$ is a distal ttg iff every finite subset of X is an almost periodic point in $2^{\mathfrak{X}}$.*

PROOF.

"If": Let x and y in X. Then $\{x,y\}$ is an almost periodic point in $2^{\mathfrak{X}}$. Suppose $x \neq y$ and let U and V be open neighbourhoods of x and y in X such that $\overline{U}\times\overline{V}\cap\Delta_X=\varnothing$. As $<U,V>$ is a neighbourhood of $\{x,y\}$ in $2^{\mathfrak{X}}$, we can find an $H\in\mathfrak{A}$ with $H.\{x,y\}\subseteq<U,V>$. But then $H.(x,y)\subseteq U\times V\cup V\times U$ and so $\mathrm{cl}_{X\times X}(H(x,y))\subseteq\overline{U}\times\overline{V}\cup\overline{V}\times\overline{U}$; hence, $\mathrm{cl}_{X\times X}(H(x,y))\cap\Delta_X=\varnothing$. Let K be a compact subset of T with $KH=T$. Then

$$K.\,\mathrm{cl}_{X\times X}(H(x,y))=\mathrm{cl}_{X\times X}(KH(x,y))=\overline{T(x,y)}$$

and clearly $K.\,\mathrm{cl}_{X\times X}(H(x,y))\cap\Delta_X=\varnothing$, so x and y are distal.

"Only if": Suppose $\mathfrak{X}$ is distal. Let $A=\{x_1,\ldots,x_n\}\subseteq X$, then $(x_1,\ldots,x_n)\in X^n$. As $\mathfrak{X}^n$ is distal, it is pointwise almost periodic (I.1.23.a). Hence $(x_1,\ldots,x_n)$ is almost periodic in $\mathfrak{X}^n$ and so by 2.4.a, A is an almost periodic point in $2^{\mathfrak{X}}$. □

2.7. **THEOREM.** *Let $\mathfrak{X}$ be a ttg. The following statements are equivalent:*

a) *$\mathfrak{X}$ is uniformly almost periodic;*
b) *$2^{\mathfrak{X}}$ is uniformly almost periodic;*
c) *$2^{\mathfrak{X}}$ is pointwise almost periodic.*

PROOF. (See also [Ko 75]) By 2.1.a, a and b are equivalent. As c follows from b, we only have to prove that c implies a:
By 2.1.b, $\mathfrak{X}$ is pointwise locally almost periodic, and by 2.6., $\mathfrak{X}$ is distal. So from I.1.18. it follows that $\mathfrak{X}$ is uniformly almost periodic. □

2.8. **REMARK.** *Let $\mathfrak{X}$ be a distal minimal ttg which is not almost periodic. Then for every $x\in X$ there is a neighbourhood U of x such that no closed neighbourhood V of x with $V\subseteq U$ is an almost periodic point in $2^{\mathfrak{X}}$.*

PROOF. (WU) Suppose that there is a $x\in X$ such that for every neighbourhood U of x there is a closed neighbourhood V of x with $V\subseteq U$ which is an almost periodic point in $2^{\mathfrak{X}}$. Then that x is a locally almost periodic point in $\mathfrak{X}$. For let U be an open neighbourhood of x and let V be a closed neighbourhood of x with $V\subseteq U$ which is almost periodic in $2^{\mathfrak{X}}$. Then $V\in<U>$ so there is a syndetic subset H of T such that

$hV \in <U>$ for all $h \in H$. Hence $HV^\circ \subseteq H\overline{V} = HV \subseteq U$. As $\mathfrak{X}$ is minimal and X contains a locally almost periodic point, $\mathfrak{X}$ is pointwise locally almost periodic (I.1.11.a). But then, as $\mathfrak{X}$ is distal, $\mathfrak{X}$ must be uniformly almost periodic (I.1.18.), which contradicts the assumption. □

II.3. QUASIFACTORS

Minimal subttgs of the hyper ttgs (quasifactors) are studied in this section. We state some easy facts and we introduce a kind of relativization of hyper ttgs ($2_\phi^{\mathfrak{X}}$). Especially the relation between an almost periodic homomorphism ϕ and the minimal subttgs of $2_\phi^{\mathfrak{X}}$ will be considered. We end this section with some technicalities on the circle operation and an observation on the points of openness of a homomorphism of minimal ttgs.

Let $\mathfrak{X}$ be a ttg. A *quasifactor* of $\mathfrak{X}$ is a minimal subttg of $2^{\mathfrak{X}}$. There are several obvious quasifactors. For instance the trivial ttg is a quasifactor of every ttg, it is the quasifactor generated by the phase space of the ttg. Also the minimal subttgs of $\mathfrak{X}^n$ $(n \in \mathbb{N})$ are quasifactors of $\mathfrak{X}$ (cf. 2.5.).
Let $\mathfrak{Z}$ be a quasifactor of $\mathfrak{X}$. Then Z is the orbit closure of some almost periodic point $A \in 2^X$; i.e., $\mathfrak{Z} = \mathfrak{QF}(A, \mathfrak{X}) := < T, QF(A, \mathfrak{X}), 2^\pi >$, where

$$QF(A, \mathfrak{X}) := \{p \circ A \mid p \in M\}$$

and we say that $\mathfrak{Z}$ is generated by A. Note that we can choose $A \in Z$ arbitrarily.
Remark that $\mathfrak{QF}(A, \mathfrak{X})$ is well defined only if $A \in 2^X$ is almost periodic; otherwise $QF(A, \mathfrak{X})$ depends on the choice of M in S_T.

3.1. **EXAMPLE.** *Consider example* I.4.7.((i) and (ii)), *the twofold covering of the proximal circle.*
   a) *The quasifactors of (this specific)* $\mathfrak{X}$ *are just* $\mathfrak{QF}(X, \mathfrak{X})$ $(\cong \{\star\})$ *and* $\mathfrak{QF}(\{x\}, \mathfrak{X})$ $(\cong \mathfrak{X})$.
   b) *The quasifactors of* $\mathfrak{Y}$ *are* $\{\star\}$, $\mathfrak{Y}$, $\mathfrak{QF}(\{0, ½\}, \mathfrak{Y})$ $(\cong \mathfrak{X})$ *and* $\mathfrak{QF}([0, ½], \mathfrak{Y})$ $(\cong \mathfrak{X})$.

PROOF.

a) Let $\mathfrak{Z}$ be a nontrivial quasifactor of $\mathfrak{X}$ (i.e., $\mathfrak{Z} \neq \{\star\}$ and $\mathfrak{Z} \neq \mathfrak{X}$). Then there is an $A \in Z$ and an ϵ with $0 < \epsilon < 1$ such that $A \subseteq [0, \epsilon]$. Applying b ($x \mapsto x^2$) infinitely many times shows that $\{0\} \in \overline{TA}$. Hence $\mathfrak{Z} = \mathfrak{QF}(\{x\}, \mathfrak{X})$.

b) Clearly, the subttgs of $2^{\mathfrak{Y}}$ mentioned are quasifactors of $\mathfrak{Y}$. To show that these are the only ones, the same argument as in a is used. □

3.2. There can be many quasifactors of a ttg $\mathfrak{X}$. For instance, if $\mathfrak{X}$ is uniformly almost periodic then every closed subset of X generates a quasifactor (2.7.). If $\mathfrak{X}^n$ has a dense subset of almost periodic points for every $n \in \mathbb{N}$, then there is a dense set of points in 2^X that generate quasifactors (1.4. and 2.5.). Note that this occurs if $\mathfrak{X}$ is minimal and incontractible (III.1.9.).

3.3. REMARK. *Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a homomorphism of ttgs.*

a) *If $\mathfrak{Z} = \mathfrak{QF}(A, \mathfrak{X})$ is a quasifactor of $\mathfrak{X}$, then $2^{\phi}[\mathfrak{Z}] = \mathfrak{QF}(\phi[A], \mathfrak{Y})$ is a quasifactor of $\mathfrak{Y}$. $2^{\phi}[\mathfrak{Z}]$ is trivial iff $\phi[A] = Y$ for some (hence all) $A \in Z$.*

b) *If $\mathfrak{W} = \mathfrak{QF}(B, \mathfrak{Y})$ is a quasifactor of $\mathfrak{Y}$, with $B \subseteq \phi[X]$ then there exists a quasifactor $\mathfrak{W}'$ of $\mathfrak{X}$ such that $2^{\phi}[\mathfrak{W}'] = \mathfrak{W}$.*

c) *If ϕ is open and surjective then every quasifactor of $\mathfrak{Y}$ is homeomorphic to a quasifactor of $\mathfrak{X}$.*

PROOF.

a) Follows from the continuity of 2^{ϕ}.

b) Define $\mathfrak{W}' := \mathfrak{QF}(u \circ \phi^{\leftarrow}[B], \mathfrak{X})$, then $2^{\phi}(u \circ \phi^{\leftarrow}[B]) = u \circ B$ hence $2^{\phi}[\mathfrak{W}'] = \mathfrak{W}$.

c) If ϕ is open then $\phi_{\mathrm{ad}} : 2^{\mathfrak{Y}} \to 2^{\mathfrak{X}}$ is a topological embedding. □

Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a homomorphism of ttgs. Then define

$$2^X_{\phi} := 2^{\phi^{\leftarrow}}[Y], \text{ i.e., } 2^X_{\phi} = \{A \in 2^X \mid \phi[A] = y \text{ for some } y \in Y\}.$$

It is easy to check that 2^X_{ϕ} is closed and invariant (so $2^{\mathfrak{X}}_{\phi}$ is a ttg) and that $\mathfrak{X}$ is embedded in $2^{\mathfrak{X}}_{\phi}$.

The relative version of 1.4. would be: R^n_{ϕ} is embedded in 2^X_{ϕ} for every $n \in \mathbb{N}$; where

$$R^n_{\phi} := \{(x_1, \ldots, x_n) \in X^n \mid \phi(x_1) = \cdots = \phi(x_n)\}.$$

It is readily shown that $\bigcup\{R^n_\phi \mid n \in \mathbb{N}\}$ is densely embedded in 2^X_ϕ .

The following theorem is a straightforward generalization (relativization) of 1.11. and 2.5.. We leave the proof (which is an obvious modification of that in the absolute case) for the reader.

3.4. **THEOREM.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a homomorphism of ttgs, and let* $\psi:=2^\phi|_{2^X_\phi}:2^{\mathfrak{X}}_\phi\to\mathfrak{Y}$. *If for all* $n\in\mathbb{N}$, R^n_ϕ *has a dense subset of almost periodic points (is ergodic), then* R^n_ψ *has a dense subset of almost periodic points (is ergodic) for all* $n\in\mathbb{N}$.
Consequently, if for all $n\in\mathbb{N}$, R^n_ϕ *has a dense subset of almost periodic points (is ergodic), then* 2^X_ϕ *, as a factor of* R^n_ψ *, has a dense subset of almost periodic points (is ergodic).* □

3.5. **THEOREM.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a surjective homomorphism of ttgs. Then the following statements are equivalent:*

a) $\phi:\mathfrak{X}\to\mathfrak{Y}$ *is almost periodic;*
b) $2^\phi:2^{\mathfrak{X}}_\phi\to\mathfrak{Y}$ *is almost periodic.*

If $\mathfrak{X}$ *is minimal then* a *and* b *are equivalent to*

c) $2^\phi:2^{\mathfrak{X}}_\phi\to\mathfrak{Y}$ *is distal;*
d) $2^{\mathfrak{X}}_\phi$ *is pointwise almost periodic.*

PROOF. Equivalence of a and b is a straightforward generalization of 2.7.. Suppose $\mathfrak{X}$ is minimal, then the implications b ⇒ c and c ⇒ d are obvious.

d ⇒ a ([Sh 76] 1.4.) If $2^{\mathfrak{X}}_\phi$ is pointwise almost periodic then, clearly, ϕ is distal. By I.1.20.a, it is sufficient to prove that $Q_\phi=\Delta_X$. So let $(x_1,x_2)\in Q_\phi\subseteq R_\phi$ and let $u\in J_{x_1}(=J_{x_2})$; then, by I.1.23.b, we have $(x_1,x_2)=u(x_1,x_2)\in uQ^*_\phi$. By I.4.4., we can find nets $\{t_i\}_i$, and $\{s_i\}_i$ in T and elements $x^i_2\in\phi^{\leftarrow}\phi(x_1)=u\phi^{\leftarrow}\phi(x_1)$ in such a way that $t_iu\to u$, $s_iu\to u$, $t_i(x_1,x^i_2)\to(x_1,x_2)$ and $s_i(x_1,x^i_2)\to(x_1,x_1)$. Let $x_3=\lim x^i_2\in\phi^{\leftarrow}\phi(x_1)$. Then, for each i_0 , $A_{i_0}:=\{x^i_2\mid i\geqslant i_0\}\cup\{x_3\}$ is closed and $A_{i_0}\in 2^X_\phi$. As 2^X_ϕ is pointwise almost periodic, there is a $v\in J$ with $v\circ A_{i_0}=A_{i_0}$. But $A_{i_0}\subseteq\phi^{\leftarrow}\phi(x_1)=u\phi^{\leftarrow}\phi(x_1)$ (I.2.12.), so we have

$$A_{i_0}=uA_{i_0}\subseteq u(u\circ A_{i_0})=uv(u\circ A_{i_0})\subseteq u(v\circ(u\circ A_{i_0}))=u(v\circ A_{i_0})=uA_{i_0}\,.$$

As $t_ix^i_2\to x_2$ we have $x_2\in u\circ A_{i_0}$ and similarly $x_1\in u\circ A_{i_0}$. By the choice of u , $x_1=ux_1$, $x_2=ux_2$ so $\{x_1,x_2\}\subseteq u(u\circ A_{i_0})=A_{i_0}$. Hence

$$\{x_1,x_2\}\subseteq\bigcap\{A_{i_0}\mid i_0\}=\{x_3\}\,,\ \text{i.e.,}\ x_1=x_2\,.$$ □

3.6. **THEOREM.** *Let* $\phi\colon \mathcal{X} \to \mathcal{Y}$ *be a homomorphism of ttgs with* $\mathcal{Y}$ *minimal. Then the following statements are equivalent:*

a) $\phi\colon \mathcal{X} \to \mathcal{Y}$ *is distal;*
b) *every finite* $A \in 2^X_\phi$ *is an almost periodic point in* $2^{\mathcal{X}}_\phi$ *;*
c) $\mathcal{X}$ *is pointwise almost periodic and* $2^\phi\colon \mathcal{Z} \to \mathcal{Y}$ *is distal for every quasifactor* $\mathcal{Z}$ *of* $\mathcal{X}$ *with* $Z \subseteq 2^X_\phi$ *.*

PROOF. The equivalence of a and b is an obvious modification of 2.6..

c $\Rightarrow$ a Let $(x_1, x_2) \in R_\phi$. Suppose that x_1 and x_2 are proximal; then $\overline{Tx_1} \cap \overline{Tx_2} \neq \emptyset$. As $\mathcal{X}$ is pointwise almost periodic, $\overline{Tx_1}$ and $\overline{Tx_2}$ are minimal. So $\overline{Tx_1} = \overline{Tx_2}$, and in particular $x_2 \in \overline{Tx_1}$. Now observe that the minimal subttg $< T, \overline{Tx_1} >$ of $\mathcal{X}$ can be considered as a quasifactor of $\mathcal{X}$, namely $< T, \overline{Tx_1} > \cong \mathfrak{WF}(\{x_1\}, \mathcal{X})$. By assumption, $2^\phi\colon \mathfrak{WF}(\{x_1\}, \mathcal{X}) \to \mathcal{Y}$ is distal, so $\phi|_{\overline{Tx_1}}\colon < T, \overline{Tx_1} > \to \mathcal{Y}$ is distal. Since $x_2 \in \overline{Tx_1}$ and $\phi(x_1) = \phi(x_2)$ it follows that x_1 and x_2 are distal.

a $\Rightarrow$ c ([AG 77] lemma II.1.) Note that from the assumption it follows that $\mathcal{X}$ is pointwise almost periodic (I.1.23.a) and that for all $y \in Y$, $u \in J_y$ we have $\phi^{\leftarrow}(y) = u\phi^{\leftarrow}(y)$ (I.2.12.). Let A and B be almost periodic points in $2^{\mathcal{X}}_\phi$ and suppose that they form a proximal pair while $2^\phi(A) = 2^\phi(B) = \{y\}$ so $\phi[A] = \phi[B] = y$. By I.2.7., there is a minimal left ideal I in S_T such that $p \circ A = p \circ B$ for all $p \in I$. In addition, let $u, v \in J(I)$ be such that $A = u \circ A$ and $B = v \circ B$; and note that $u, v \in J_y(I)$. By the distality of ϕ we have $A = vA$ so

$$A = vA \subseteq v \circ A = v \circ B = B \; .$$

Similarly $B \subseteq A$, hence $A = B$. □

3.7. **COROLLARY.** *Let* $\mathcal{X}$ *be a minimal ttg. Then* $\mathcal{X}$ *is uniformly almost periodic (distal) iff every quasifactor of* $\mathcal{X}$ *is uniformly almost periodic (distal).*

PROOF. Cf. 3.5. (3.6.). □

3.8. **REMARK.** *If* $\mathcal{Y}$ *is minimal and* $\phi\colon \mathcal{X} \to \mathcal{Y}$ *is distal, then every orbit closure in* $2^{\mathcal{X}}_\phi$ *contains a unique minimal subset. In particular, if* $\psi := 2^\phi|_{2^X_\phi}$ *then* P_ψ *is an equivalence relation.*

PROOF. Let $A \in 2^X_\phi$ and let I and K be minimal left ideals in S_T . Let $y = \phi[A]$ and let $u \in J_y(I)$ and $v \in J(K)$ with $u \sim v$, hence

$v \in J_y(K)$. As ϕ is a distal map, $v\phi^{\leftarrow}(y) = \phi^{\leftarrow}(y)$; so $vA = A$ and

$$u \circ A = u \circ vA \subseteq uv \circ A = v \circ A \quad (u \sim v) .$$

Similarly, $v \circ A \subseteq u \circ A$ and so $v \circ A = u \circ A$. Hence the minimal sets $\{p \circ A \mid p \in I\}$ and $\{p \circ A \mid p \in K\}$ are the same. Since every minimal subset of the orbit closure of A in 2_ϕ^X is of the form $\{p \circ A \mid p \in I'\}$ for some minimal left ideal I' in S_T , this proves the first statement.
Let $\psi := 2^\phi|_{2_\phi^X} : 2_\phi^{\mathcal{X}} \to \mathcal{Y}$ and suppose $(A,B) \in P_\psi$ and $(B,C) \in P_\psi$. Put $y = \psi(A) = \psi(B) = \psi(C)$, and let I and K be the minimal left ideals in S_T such that $p \circ A = p \circ B$ for all $p \in I$ and $p \circ B = p \circ C$ for all $p \in K$. Let $u \in J_y(I)$ and $v \in J_y(K)$ with $u \sim v$. Then, by the argument above, $u \circ A = v \circ A$, $u \circ B = v \circ B$ and $u \circ C = v \circ C$ so

$$u \circ A = u \circ B = v \circ B = v \circ C = u \circ C .$$

□

Let $\phi : \mathcal{X} \to \mathcal{Y}$ be a distal homomorphism of ttgs and let $\mathcal{Y}$ be minimal. Let $\mathrm{Reg}(\phi) : \mathcal{X}' \to \mathcal{Y}$ be the regularizer of ϕ (recall the definition just below I.2.15.); i.e., X' is the orbit closure of $z = (x)_{x \in \phi^{\leftarrow}(y)}$ in $X^{|\phi^{\leftarrow}(y)|}$ for some fixed $y \in Y$. Then z is an almost periodic point (note that, by distality of ϕ, $u\phi^{\leftarrow}(y) = \phi^{\leftarrow}(y)$ for all $u \in J_y$), so $\mathcal{X}'$ is minimal and $\mathrm{Reg}(\phi)$ is defined by $\mathrm{Reg}(\phi)(pz) = py$ for all $p \in M$. Note that if $\mathcal{Y} = \{\star\}$, then $\mathcal{X}' = E(\mathcal{X})$.

3.9. **REMARK.** *With notation as above (so* ϕ *is distal!):*

a) $\mathrm{Reg}(\phi)$ *is (well defined and) distal.*

b) *For* $a \in M$ *we have* $az = z$ *iff* $ax = x$ *for all* $x \in \phi^{\leftarrow}(y)$.

c) *Let* $A \subseteq \phi^{\leftarrow}(y)$, $u \in J_y$ *and* $a \in uM$. *Then* $az = z$ *implies* $u \circ A = a \circ A$.

PROOF. a and b are obvious.

c) Let $A \subseteq \phi^{\leftarrow}(y)$ and $az = z$ then $ax = x$ for all $x \in \phi^{\leftarrow}(y)$ so $aA = A$. Then $A = aA \subseteq a \circ A$, hence

$$u \circ A \subseteq u \circ (a \circ A) = ua \circ A = a \circ A .$$

Also $a^{-1}x = x$ for all $x \in \phi^{\leftarrow}(y)$, so similarly $u \circ A \subseteq a^{-1} \circ A$ and

$$a \circ A = a \circ (u \circ A) \subseteq a \circ (a^{-1} \circ A) = aa^{-1} \circ A = u \circ A .$$

□

3.10. **THEOREM.** *Let* $\phi:\mathcal{X}\to\mathcal{Y}$ *be a distal homomorphism of ttgs and let* $\mathcal{Y}$ *be minimal. Then for every quasifactor* $\mathcal{Z}$ *of* $\mathcal{X}$ *which is a subttg of* $2^{\mathcal{X}}_{\phi}$ *the map* $2^{\phi}:\mathcal{Z}\to\mathcal{Y}$ *is a factor of* $\mathrm{Reg}(\phi)$. *I.e., there is a homomorphism* $\theta:\mathcal{X}'\to\mathcal{Z}$ *with* $\mathrm{Reg}(\phi)=2^{\phi}\circ\theta$.

In case $\mathcal{Y}=\{\star\}$ *this means that every quasifactor of* $\mathcal{X}$ *is a factor of* $E(\mathcal{X})$.

PROOF. Let $y\in Y$ and let $z\in X'$ be as in the discussion just before 3.9.. Suppose that $\mathcal{Z}$ is a quasifactor of $\mathcal{X}$ with $Z\subseteq 2^{X}_{\phi}$. Let $A\in Z$ with $2^{\phi}(A)=y$ and define $\theta:\mathcal{X}'\to\mathcal{Z}$ by $\theta(pz)=p\circ A$ for all $p\in M$. It suffices to prove that θ is well defined. Let p and q in M be such that $pz=qz$. Then $upz=uqz$ and $py=qy$. By 3.9.c, it follows readily that $up\circ A=uq\circ A$; hence $p\circ A$ and $q\circ A$ are proximal. As

$$2^{\phi}(p\circ A)=py=qy=2^{\phi}(q\circ A)\,,$$

$p\circ A$ and $q\circ A$ are distal (3.6.c), hence $p\circ A=q\circ A$. □

The following facts concerning the "circle-arithmetics" are collected for the convenience of the reader and the author.

3.11. **REMARK.** *Let* $\mathcal{X}$ *be a minimal ttg. Then*

a) $u(u\circ A)=u(v\circ A)$ *for* $A\subseteq X$ *and for every* $u,v\in J$ *;*

b) $u\circ uA=u\circ vA$ *for* $A\subseteq X$ *and for every* $u,v\in J$ *;*

c) $p\circ A=w\circ pA$ *for* $A\subseteq vX$, $v\in J$ *and* $w\in J_p$.

If $\phi:\mathcal{X}\to\mathcal{Y}$ *is a homomorphism of minimal ttgs,* $y\in Y$, $p\in M$, $w\in J_p$ *and* $u,v\in J$ *then*

d) $p\circ v\phi^{\leftarrow}(y)=w\circ u\phi^{\leftarrow}(py)$ *;*

e) $p\circ\phi^{\leftarrow}(y)=w\circ\phi^{\leftarrow}(py)$.

PROOF.

a) As $u=uv$ and $v=vu$ (I.2.2.b),

$$u(u\circ A)=uv(u\circ A)\subseteq u(v\circ(u\circ A))=u(vu\circ A)=u(v\circ A)$$

and also $u(v\circ A)=uu(v\circ A)\subseteq u(u\circ(v\circ A))=u(uv\circ A)=u(u\circ A)$.

b) As $u=uv$ and $v=vu$ we have

$$u\circ uA=u\circ uvA\subseteq u\circ(u\circ vA)=u\circ vA$$

and $u\circ vA=u\circ vuA\subseteq u\circ v\circ uA=uv\circ uA=u\circ uA$.

c) Since $A\subseteq vX$, it follows that $A=vA$. So

$$p\circ A=p\circ vA=p\circ vp^{-1}pA\subseteq pvp^{-1}\circ pA$$

and, as $w \in J_p$ (which means that $w \in J$ with $wp = p$), $pvp^{-1} = wpvp^{-1} = wpwp^{-1} = w$; hence $p \circ A \subseteq w \circ pA$.
Conversely, $w \circ pA \subseteq w \circ (p \circ A) = wp \circ A = p \circ A$.

d) Let ϕ, y, p, u, v and w be as in the assumption. Then by c, $p \circ v\phi^{\leftarrow}(y) = w \circ p\phi^{\leftarrow}(y)$, and as $p\phi^{\leftarrow}(y) \subseteq w\phi^{\leftarrow}(py)$ it follows that $p \circ v\phi^{\leftarrow}(y) \subseteq w \circ w\phi^{\leftarrow}(py) = w \circ u\phi^{\leftarrow}(py)$ (b).
Conversely, $u = upvp^{-1}$, so

$$w \circ u\phi^{\leftarrow}(py) = w \circ upvp^{-1}\phi^{\leftarrow}(py) \subseteq wup \circ vp^{-1}\phi^{\leftarrow}(py) \subseteq$$

$$\subseteq wp \circ v\phi^{\leftarrow}(y) = p \circ v\phi^{\leftarrow}(y) .$$

e) Clearly, as $p \circ \phi^{\leftarrow}(y) \subseteq \phi^{\leftarrow}(py)$, we have

$$p \circ \phi^{\leftarrow}(y) = w \circ p \circ \phi^{\leftarrow}(y) \subseteq w \circ \phi^{\leftarrow}(py) .$$

Conversely, for $u' \in J_y$ we have $w \circ \phi^{\leftarrow}(py) = wpu'p^{-1} \circ \phi^{\leftarrow}(py)$ and $u'p^{-1} \circ \phi^{\leftarrow}(py) \subseteq \phi^{\leftarrow}(y)$. So $wpu'p^{-1} \circ \phi^{\leftarrow}(py) \subseteq wp \circ \phi^{\leftarrow}(y) = p \circ \phi^{\leftarrow}(y)$. □

We end this section with some observations on the points of openness for a homomorphism $\phi : \mathfrak{X} \to \mathfrak{Y}$ of minimal ttgs.

3.12. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs and let* $y \in Y$. *Then* $\{x \in \phi^{\leftarrow}(y) \mid \phi \text{ open in } x\} = \bigcap \{u \circ \phi^{\leftarrow}(y) \mid u \in J_y\}$.

PROOF. Note that ϕ is open in x iff for every net $\{y_i\}_i$ in Ty converging to y there is a net $\{x_i\}_i$ in X converging to x with $\phi(x_i) = y_i$ (Ty is dense in Y !). Suppose ϕ is open in $x \in \phi^{\leftarrow}(y)$ and let $u \in J_y$. Let $\{t_i\}_i$ be a net in T with $t_i \to u$. Then $t_i y \to y$. So by openness of ϕ in x , there are x_i in X such that $t_i x_i \to x$ and $\phi(x_i) = y$. This shows that $x = \lim t_i x_i \in u \circ \phi^{\leftarrow}(y)$ (1.9.a).
Conversely, let $x \in \bigcap \{u \circ \phi^{\leftarrow}(y) \mid u \in J_y\}$. Let $\{t_i y\}_i$ be a net in Ty converging to y and let $u \in J_y$. Then $\{t_i u\}_i$ converges to $p \in M$ (for a suitable subnet). Let $w \in J$ be such that $wp = p$; then $w \in J_y$, for

$$wpy = py = \lim t_i uy = \lim t_i y = y .$$

By assumption, $x \in w \circ \phi^{\leftarrow}(y)$ and, by 3.11.e,

$$w \circ \phi^{\leftarrow}(y) = w \circ \phi^{\leftarrow}(py) = wp \circ \phi^{\leftarrow}(y) ,$$

so $x \in wp \circ \phi^{\leftarrow}(y) = p \circ \phi^{\leftarrow}(y)$. As the net $\{t_i u\}_i$ converges to p , there are $x_i \in \phi^{\leftarrow}(y)$ such that $x = \lim t_i u x_i$. The arbitrary choice of the net $\{t_i y\}_i$ shows that ϕ is open in x . □

3.13. **COROLLARY.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs.*

a) *If* x *is a* ϕ*-distal point in* X *then* ϕ *is open in* x .

b) *If* ϕ *is distal then* ϕ *is open.*

PROOF.

a) If x is a ϕ-distal point then $J_x = J_{\phi(x)}$ (I.2.10.). So $x = ux$ for every $u \in J_{\phi(x)}$; hence $x \in u\phi^{\leftarrow}\phi(x) \subseteq u \circ \phi^{\leftarrow}\phi(x)$ for every $u \in J_{\phi(x)}$. But then, by 3.12., ϕ is open in x .

b) If ϕ is distal then every $x \in X$ is a ϕ-distal point. By a , ϕ is open in every $x \in X$; so ϕ is open. □

II.4. REMARKS

4.1. The notion of hyper ttg occurs naturally in topological dynamics. One could imagine that the action of T on closed subsets of X yields some extra information about $\mathfrak{X}$. In 1970 W.A. VEECH used a special kind of quasifactor ([V 70]) and R. ELLIS ([E 73]) and D.C. MCMAHON and T.S. WU ([MW 74]) mention the action of T on 2^X more or less explicitly. In [G 75.1], [G 74] and [G 76] S. GLASNER studies this action in more detail. However, all occurrences of hyper ttgs deal with hyper ttgs for discrete topological groups. S.C. KOO ([Ko 75]) was the first (and only one) to publish a proof of the fact that the topology of T didn't destroy the existence of hyper ttgs. His proof uses the uniform structure; we gave a proof (1.6.) using the Vietoris topology, which is "easier to handle".
The remainder of section II.1. is devoted to the question: what do we know if $2^{\mathfrak{X}}$ is ergodic. As far as we know no related results were published until now.

QUESTIONS

a) If $\mathfrak{X}$ is minimal and proximal then $\mathfrak{X}$ is weakly mixing (cf. [G 76] II.2.2. and, in here: VII.2.14.); what can be said about the ergodicity of $\mathfrak{X}^n$ for $n \geqslant 3$, and what about $2^{\mathfrak{X}}$? (Note that in general they are not ergodic!)

Note that "with respect to" this question the notions of totally proximal

[$2^{\mathcal{X}}$ has exactly two quasifactors] and extremally proximal [$2^{\mathcal{X}}$ has exactly two quasifactors of which $\{\star\}$ is isolated] were introduced in [G 74].

b) Is it possible to extend theorem 1.18. to a collection of statements in which not just a particular minimal left ideal can be chosen, but in which any minimal left ideal suffices?

4.2. In section II.2. we state some generalities on recursiveness in hyper ttgs. The main purpose was to give a hyperspace proof of 2.7. (see also 3.5.). Here we follow [Ko 75], but the proofs are shorter and easier (e.g., 2.5. and 2.6. compared to [Ko 75] theorem 4.2. and corollary 4.1.; and note that 2.6. is almost evident if we use the idempotents in S_T). Theorem 2.2. slightly generalizes [Ko 75] theorem 2.2.. The result in 2.8. is due to T.S. WU (private communication).

QUESTIONS

a) Can we weaken the condition on T in 2.2. and 2.3.?

b) By 1.4. and 2.5. we know that $2^{\mathcal{X}}$ has a dense set of almost periodic points if $\mathcal{X}^n$ has a dense subset of almost periodic points for all $n \in \mathbb{N}$. Under what extra conditions does the inverse implication hold?

The following example shows that extra conditions in the question b above are needed.

EXAMPLE: (S. GLASNER)

Let $X = \{0,1\}^{\mathbb{Z}}$ with the usual product topology. Let σ be the shift, i.e., $(\sigma(x))_n = x_{n+1}$ for all $n \in \mathbb{Z}$; and define $t_0 : X \to X$ by $t_0(x)[n] = x[n]$ for all $n \in \mathbb{Z} \setminus \{1\}$, $t_0(x)[1] = x[1]$ if $x[0] = 1$, $t_0(x)[1] = 1 - x[1]$ if $x[0] = 0$; and define $t_1 : X \to X$ by $t_1(x)[n] = x[n]$ for all $n \in \mathbb{Z} \setminus \{1\}$, $t_1(x)[1] = x[1]$ if $x[0] = 0$, $t_1(x)[1] = 1 - x[1]$ if $x[0] = 1$.
Let T be the group generated by σ, t_0 and t_1. Then $\mathcal{X} = <T, X>$ is minimal and proximal, so $\mathcal{X}^n$ does not have a dense subset of almost periodic points for all $n \in \mathbb{N}$ with $n \geqslant 2$. But $2^{\mathcal{X}}$ has a dense subset of almost periodic points! For:
Let $n \in \mathbb{N}$ and $\beta \in \{0,1\}^{2n+1}$. Define

$$A^n_\beta := \{x \in X \mid x[m.\,10^{10^n} - n\ ,\ m.\,10^{10^n} + n] = \beta \text{ for all } m \in \mathbb{N}\}\ .$$

Then one can show that A^n_β is an almost periodic point in $2^{\mathcal{X}}$. Moreover, choose $\beta_1, \ldots, \beta_l$ in $\{0,1\}^{2n+1}$ then $\bigcup \{A^n_{\beta_j} \mid j \in \{1, \ldots, l\}\}$ is an almost periodic point in $2^{\mathfrak{X}}$. But as $2^{\mathfrak{X}}$ has a dense set of points of this form, it follows that $2^{\mathfrak{X}}$ has a dense subset of almost periodic points.

4.3. In section II.3. we describe some facts about quasifactors and relativized hyperspaces. Remark 3.8. is based upon a note of T.S. WU (private communication) and it generalizes [G 79] 4.3.. Theorem 3.10. is a relativized version of [G 75.1] 2.5.; but the proof is different from the one there.

QUESTIONS

a) How do properties of ϕ reflect in properties of 2^{ϕ} ? In particular, what can be said about quasifactors of point distal or proximal ttgs? (cf. 3.7.).

b) With respect to 3.8.: is P_{ψ} closed?

c) If every quasifactor of $\mathfrak{X}$ is a factor of $E(\mathfrak{X})$, what does that imply for $\mathfrak{X}$?

III

$\mathfrak{F}$-TOPOLOGIES, A TOOL IN STRUCTURE THEORY

One of the most important issues in the structure theory of minimal ttgs is to determine the almost periodic factors of a given homomorphism ϕ, i.e., to understand E_ϕ. In general we do not know very much about E_ϕ, but there are conditions to be laid upon ϕ that enable us to describe E_ϕ precisely. One of them is the existence of a relatively invariant measure, which is treated in chapter VII, the other is ϕ being a RIC extension (Bronstein condition already suffices).

In 1973 I.U. BRONSTEIN proved that for an open Bc extension ϕ the regionally proximal relation is an equivalence relation ([B 73], in Russian, and not really recognized at that time). The method was in a certain sense elementary: he just uses properties of uniform structures and syndetic sets.
In 1977 W.A. VEECH published a proof of that fact (without openness) heavily depending on the construction of weak topologies on u-invariant parts of fibers (which was initiated by H. FURSTENBERG in [F 63]).
It turns out that these weak topologies ($\mathfrak{F}$-topologies) are perfectly fit to describe the regionally proximal relation in $\overline{JR}_\phi$.

We shall deal with RIC extensions in section 1., and among others we shall see that every map is a RIC extension up to proximality. In section 2. we describe the $\mathfrak{F}$-topologies and we use them in section 3. to understand E_ϕ for a Bc extension ϕ. Section 4. deals with PI extensions; there we apply the foregoing to the structure theory.

In this chapter no substantially new results can be found. It is more or less a recollection of what is known in this part of the theory, arranged in a way suitable for our purposes in other chapters, and some times slightly generalized (e.g. 3.10.a).

III.1. RIC EXTENSIONS

In the structure theory of minimal ttgs, RIC extensions play an important role. The reason will be clear in the sections III.3. and III.4. (also see VIII.1.4.). In short it comes down to the following observations:

Every map $\phi : \mathfrak{X} \rightarrow \mathfrak{Y}$ is RIC up to proximal extensions (1.11.), and RIC extensions behave nicely with respect to almost periodic factors (3.9.).

In this section we shall have a close look at RIC extensions.

Remember that an extension of minimal ttgs $\phi : \mathfrak{X} \rightarrow \mathfrak{Y}$ is called relatively incontractible (RIC) if $\phi \perp \psi$ for every proximal extension $\psi : \mathfrak{Z} \rightarrow \mathfrak{Y}$ of minimal ttgs (I. below 3.9.). For example a distal extension is RIC.

1.1. The following observation with respect to Ellis groups is useful. Let $\mathfrak{Y}$ be a minimal ttg, $u \in J$ and $y = uy \in Y$. Let $F = \mathfrak{G}(\mathfrak{Y}, y)$ be the Ellis group of $\mathfrak{Y}$ with respect to the point y (in $G = uM$). Then $u(u \circ F) = F$. The proof is as follows:
As $Fy = y$, and as $\rho_y : p \mapsto py : \mathfrak{M} \rightarrow \mathfrak{Y}$ is continuous, we have

$$(u \circ F)y = (u \circ \bar{F})y = u \circ \bar{F}y = u \circ \overline{Fy} = u \circ y = uy = y \ .$$

So $u(u \circ F)y = uy = y$, which shows that $u(u \circ F) \subseteq F$. As, clearly, $F = uF = uuF \subseteq u(u \circ F)$, it follows that $u(u \circ F) = F$.

1.2. **LEMMA.** *Let* $\mathfrak{Y}$ *be a minimal ttg,* $u \in J$ *and* $y = uy$. *Let* $F = \mathfrak{G}(\mathfrak{Y}, y)$ *be the Ellis group of* $\mathfrak{Y}$ *with respect to* y *(in* G*). Then* $\mathfrak{G}(\mathfrak{W}\mathfrak{F}(u \circ F, \mathfrak{M}), u \circ F) = F$ *and* $\kappa : \mathfrak{W}\mathfrak{F}(u \circ F, \mathfrak{M}) \rightarrow \mathfrak{Y}$ *is a proximal homomorphism of minimal ttgs, where* κ *is defined by* $\kappa(p \circ F) = py$ *for all* $p \in M$.

PROOF. Cf. [G 76] IX.3.3.. □

We shall give several descriptions of relative incontractibility in 1.3., 1.5. and 1.9.. In fact these are characterizations that can be used to define RIC extensions; indeed, 1.3.b and 1.3.c occur as such in the literature (cf. [G 76] and [V 77]). Our definition of RIC extensions, or better our choice of the equivalent statement to be definition is based on personal taste rather than theoretical considerations.

1.3. **THEOREM.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a homomorphism of minimal ttgs. Let* $x_0\in X$ *,* $y_0=\phi(x_0)$ *and* $u\in J_{x_0}$*; let* $F=\mathfrak{G}(\mathfrak{Y},y_0)$ *be the Ellis group of* $\mathfrak{Y}$ *with respect to* y_0 *in* G *. Then the following statements are equivalent:*

a) ϕ *is a* RIC *extension;*

b) $\phi^{\leftarrow}(py_0)=p\circ Fx_0$ *for every* $p\in M$ *;*

c) $\phi^{\leftarrow}(y)=v\circ v\phi^{\leftarrow}(y)$ *for every* $y\in Y$ *and* $v\in J_y$.

PROOF. The equivalence of a and b may be deduced from [G 76] X.1.3.. The equivalence of b and c is an exercise for the reader (use II.3.11.). ☐

1.4. **COROLLARY.** *A* RIC *extension* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *of minimal ttgs is open.*

PROOF. We shall show that $\phi_{\mathrm{ad}}:Y\to 2^X$ is a continuous map; hence, by II.1.3.d, ϕ is an open map. As follows:
By 1.3.b, for all $p\in M$ we have $\phi^{\leftarrow}(py_0)=p\circ Fx_0$. Hence the mapping $\xi:M\to 2^X$, defined by $p\mapsto\phi_{\mathrm{ad}}(py_0)$, is continuous. Since $\eta:p\mapsto py_0:M\to Y$ is a quotient map and $\xi=\phi_{\mathrm{ad}}\circ\eta$, it follows that ϕ_{ad} is continuous. ☐

In the literature the only proof of the next theorem is not quite correct so we provide the (easy) proof here.

1.5. **THEOREM.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a homomorphism of minimal ttgs. Then the following statements are equivalent:*

a) ϕ *is a* RIC *extension;*

b) *for every homomorphism* $\psi:\mathfrak{Z}\to\mathfrak{Y}$ *with* $Z=\overline{JZ}$ *we have that* (ϕ,ψ) *satisfies the generalized Bronstein condition;*

c) *for every homomorphism* $\psi:\mathfrak{Z}\to\mathfrak{Y}$ *of minimal ttgs, we have that* (ϕ,ψ) *satisfies the generalized Bronstein condition.*

PROOF.

a ⇒ b Let W be an open set in $R_{\phi\psi}$. As ϕ is open and $Z=\overline{JZ}$ it follows from I.3.7.(iv) that there are open sets U and V in X and Z

such that $\varnothing \neq U \times V \cap R_{\phi\psi} \subseteq W$ and $\phi[U] = \psi[V]$. Let $z \in V$ be an almost periodic point and let $v \in J_z$, then for $y = \psi(z)$ we have $v \in J_y$ and, by 1.3., $\phi^{\leftarrow}(y) = v \circ v\phi^{\leftarrow}(y)$. Let $x \in U$ be such that $\phi(x) = \psi(z) = y$, then $x \in v \circ v\phi^{\leftarrow}(y)$. Let $\{t_i\}_i$ be a net in T be such that $v = \lim t_i$ and let $x_i \in v\phi^{\leftarrow}(y)$ with $x = \lim t_i x_i$. Then $(x_i, z) = v\,(x_i, z)$ and $(x_i, z) \in R_{\phi\psi}$. As $(x, z) = \lim t_i(x_i, z)$ and as W is a neighbourhood of (x, z) in $R_{\phi\psi}$, it follows that $t_i(x_i, z) \in W$ eventually, and so that W contains an almost periodic point. So $R_{\phi\psi}$ has a dense subset of almost periodic points, hence (ϕ,ψ) satisfies gBc.

b $\Rightarrow$ c Trivial.

c $\Rightarrow$ a Let $\psi: \mathfrak{Z} \rightarrow \mathfrak{Y}$ be a proximal homomorphism of minimal ttgs. Then, by I.3.4., $R_{\phi\psi}$ has a unique minimal subttg. By assumption c, $R_{\phi\psi}$ has a dense subset of almost periodic points, hence $R_{\phi\psi}$ is minimal and so $\phi \perp \psi$. As ψ was arbitrary, ϕ is a RIC extension. □

1.6. **COROLLARY.** *Let $\phi: \mathfrak{X} \rightarrow \mathfrak{Z}$ be a* RIC *extension of minimal ttgs and let $\psi: \mathfrak{Y} \rightarrow \mathfrak{Z}$ be a homomorphism of minimal ttgs. Let $x_0 \in X$, $u \in J_{x_0}$, $y_0 = uy_0 \in \psi^{\leftarrow}\phi(x_0)$ and let H, F and K be the Ellis groups of $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ with respect to x_0, y_0 and $\phi(x_0)$ in G. Then $\phi \perp \psi$ iff $HF = K$.*

PROOF. By 1.5.c, we know that $R_{\phi\psi}$ has a dense subset of almost periodic points. Hence $R_{\phi\psi}$ is minimal iff it has a unique minimal subset. This is, by I.3.2., equivalent to $HF = K$. □

We say that a homomorphism $\phi: \mathfrak{X} \rightarrow \mathfrak{Y}$ satisfies the *n-fold Bronstein condition* for certain $n \in \mathbb{N}$ if

$$R_{\phi}^{n} := \{(x_1, \ldots, x_n) \in X^n \mid \phi(x_1) = \cdots = \phi(x_n)\}$$

has a dense subset of almost periodic points (notation: ϕ is n-Bc).

1.7. **COROLLARY.** *If $\phi: \mathfrak{X} \rightarrow \mathfrak{Y}$ is a* RIC *extension of minimal ttgs, then ϕ satisfies the n-fold Bronstein condition for every $n \in \mathbb{N}$.*

PROOF. For $n = 2$ the statement follows from 1.5.c.
Suppose that the statement is true for some $k \in \mathbb{N}$ with $k \geqslant 2$. So R_{ϕ}^{k} has a dense subset of almost periodic points. Define $\psi: \mathfrak{R}_{\phi}^{k} \rightarrow \mathfrak{Y}$ by $\psi(x_1, \ldots, x_k) = \phi(x_1)$. Then, by 1.5.b, $R_{\phi\psi}$ has a dense subset of almost periodic points. Clearly $R_{\phi\psi} \cong R_{\phi}^{k+1}$ so the statement is true for $k + 1$, which proves the corollary. □

1.8. In particular, it follows from 1.5. and 1.4. that every RIC extension is an open Bc extension. It is still an unsolved question whether or not an open Bc extension is a RIC extension. Some partial answers can be given:

(i) If ϕ is a regular homomorphism of minimal ttgs which is open and which satisfies the Bronstein condition, then ϕ is a RIC extension (V.3.7.).
(ii) Theorem 1.9. below.

1.9. **THEOREM.** *Let $\phi : \mathcal{X} \to \mathcal{Y}$ be a homomorphism of minimal ttgs. Then ϕ is a* RIC *extension iff ϕ is an open map that satisfies n-*Bc *for every $n \in \mathbb{N}$.*

PROOF. By 1.4. and 1.7. we only have to prove the "if"-part. So suppose ϕ is n-Bc for every $n \in \mathbb{N}$. First we show that for arbitrary $y \in Y$ and $u \in J$ we have

$$\bigcup \{t\,\{x_1, \ldots, x_n\} \mid t \in T,\ n \in \mathbb{N},\ x_i \in u\phi^{\leftarrow}(y)\} \text{ is dense in } 2^X_\phi \qquad ☜$$

(for 2^X_ϕ see the discussion just after II.3.3.).
Let U be a basic open set in 2^X_ϕ; i.e., let $m \in \mathbb{N}$ and let $U_1, \ldots, U_m$ be open sets in X such that $U := \langle U_1, \ldots, U_m \rangle \cap 2^X_\phi \neq \varnothing$ (see II.1.). Let $A \in U$. Then $A \cap U_i \neq \varnothing$ for $i \in \{1, \ldots, m\}$; say $x'_i \in A \cap U_i$. Hence

$$(x'_1, \ldots, x'_m) \in U_1 \times \cdots \times U_m \cap R^m_\phi\ ,$$

so $U_1 \times \cdots \times U_m \cap R^m_\phi$ is a nonempty open set in R^m_ϕ. As ϕ is m-Bc, there is an almost periodic point

$$v\,(x_1, \ldots, x_m) = (x_1, \ldots, x_m) \in U_1 \times \cdots \times U_m \cap R^m_\phi\ ,$$

(for some $v \in J$). Let $\phi(x_1) = y'$ and let $p \in vM$ with $y' = py$. Then $up^{-1}x_i \in u\phi^{\leftarrow}(y)$ for every $i \in \{1, \ldots, m\}$, and, clearly, we have

$$(x_1, \ldots, x_m) = vpup^{-1}(x_1, \ldots, x_m)\,.$$

Let $\{t_i\}_i$ be a net in T with $t_i \to vp$ then for some t_{i_0} we have

$$t_i up^{-1}(x_1, \ldots, x_m) \in U_1 \times \cdots \times U_m \cap R^m_\phi$$

for every $i \geqslant i_0$. For those i,

$$t_i\,\{up^{-1}x_1, \ldots, up^{-1}x_m\} \in \langle U_1, \ldots, U_m \rangle \cap 2^X_\phi\ ,$$

and ☜ holds.

If $y \in Y$, then $\phi^{\leftarrow}(y) \in 2^X_\phi$; so by ☞, there is a net $\{t_i\}_i$ in T and there are $\{x^i_1, \dots, x^i_{n_i}\} \in 2^X_\phi$ with $x^i_k \in u\phi^{\leftarrow}(y)$ for $k \in \{1, \dots, n_i\}$ such that $t_i\{x^i_1, \dots, x^i_{n_i}\} \to \phi^{\leftarrow}(y)$ in 2^X_ϕ. Let, for a suitable subnet, $p = \lim t_i u$, then

$$\phi^{\leftarrow}(y) = \lim t_i \{x^i_1, \dots, x^i_{n_i}\} \subseteq \lim t_i u\phi^{\leftarrow}(y) \subseteq \lim t_i u \circ u\phi^{\leftarrow}(y) =$$

$$= (\lim t_i u) \circ u\phi^{\leftarrow}(y) = p \circ u\phi^{\leftarrow}(y).$$

As $p \in M$, there is a $v \in J$ with $vp = p$. Then $v \in J_y$; for

$$y = \phi[\phi^{\leftarrow}(y)] \subseteq \phi[p \circ u\phi^{\leftarrow}(y)] = p \circ \phi[u\phi^{\leftarrow}(y)] = p \circ uy = py$$

so $y = py$ and $vy = vpy = py = y$. By II.3.11.c, we know

$$\phi^{\leftarrow}(y) \subseteq p \circ u\phi^{\leftarrow}(y) = v \circ vp\phi^{\leftarrow}(y).$$

As $vp\phi^{\leftarrow}(y) = v\phi^{\leftarrow}(y)$ we have $\phi^{\leftarrow}(y) \subseteq v \circ v\phi^{\leftarrow}(y)$. And so it follows that $\phi^{\leftarrow}(y) = v \circ v\phi^{\leftarrow}(y)$; for, obviously, $v \circ v\phi^{\leftarrow}(y) \subseteq \phi^{\leftarrow}(y)$.

We have shown that there exists a $v \in J_y$ with $\phi^{\leftarrow}(y) = v \circ v\phi^{\leftarrow}(y)$. In order to conclude that ϕ is a RIC extension we have to know that $\phi^{\leftarrow}(y) = w \circ w\phi^{\leftarrow}(y)$ for every $w \in J_y$. As ϕ is open, ϕ_{ad} is continuous so $\phi^{\leftarrow}(y) = w \circ \phi^{\leftarrow}(y)$ for every $w \in J_y$. Hence

$$\phi^{\leftarrow}(y) = w \circ \phi^{\leftarrow}(y) = w \circ (v \circ v\phi^{\leftarrow}(y)) = wv \circ v\phi^{\leftarrow}(y) = w \circ v\phi^{\leftarrow}(y),$$

and , by II.3.11.b, it follows that $\phi^{\leftarrow}(y) = w \circ v\phi^{\leftarrow}(y) = w \circ w\phi^{\leftarrow}(y)$ for every $w \in J_y$, which proves the theorem. □

1.10. **REMARK.**

a) *A factor of a* RIC *extension is a* RIC *extension.*

b) *The composition of two* RIC *extensions is a* RIC *extension.*

c) *The inverse limit of* RIC *extensions is a* RIC *extension.*

PROOF.

a) Immediate from the definition of RIC extensions and from I.3.1.a.

b) Let $\phi: \mathfrak{X} \to \mathfrak{Y}$ and $\psi: \mathfrak{Y} \to \mathfrak{Z}$ be RIC extensions. For $x_0 \in X$ and $u \in J_{x_0}$ let $y_0 = \phi(x_0)$, $z_0 = \psi(y_0)$ and let F and K be the Ellis groups of $\mathfrak{Y}$ and $\mathfrak{Z}$ with respect to y_0 and z_0 in G $(= uM)$. Then

$$(\psi \circ \phi)^{\leftarrow}(pz_0) = \phi^{\leftarrow}[\psi^{\leftarrow}(pz_0)] = \phi^{\leftarrow}[p \circ Ky_0],$$

and as ϕ is open, we have $\phi^{\leftarrow}[p \circ Ky_0] = p \circ \phi^{\leftarrow}[Ky_0]$, hence

$$(\psi\circ\phi)^{\leftarrow}(pz_0)=p\circ\phi^{\leftarrow}[Ky_0]=p\circ[\,\bigcup\{k\circ Fx_0\mid k\in K\}]\,.$$

By II.3.11.c $k\circ Fx_0=u\circ kFx_0$ so

$$\bigcup\{k\circ Fx_0\mid k\in K\}\subseteq u\circ KFx_0=u\circ Kx_0\,,$$

for by I.2.11., $F\subseteq K$. But then

$$(\psi\circ\phi)^{\leftarrow}(pz_0)\subseteq p\circ u\circ Kx_0=p\circ Kx_0\,;$$

clearly $p\circ Kx_0\subseteq(\psi\circ\phi)^{\leftarrow}(pz_0)$, so $(\psi\circ\phi)^{\leftarrow}(pz_0)=p\circ Kx_0$ and $\psi\circ\phi$ is a RIC extension by 1.3..

c) Follows immediately from b, I.3.1.b and the definition of RIC extensions. □

Note that the converse statement for b is not true. For, if T is abelian every minimal ttg for T is incontractible (note that T does not admit nontrivial proximal ttgs), but there do exist nontrivial proximal extensions between minimal ttgs. [E.g., by IV.2.8., $\mathfrak{E}^*$ is a nontrivial (highly) proximal extension of $\mathfrak{E}$ for every discrete topological group T with $|bT|\geqslant\aleph_0$.]

Now that we have some basical knowledge about RIC extensions, we shall discuss one of the phenomena that make them interesting, i.e., the fact that every homomorphism of minimal ttgs can be related to a RIC extension in a canonical way.

Let $\phi:\mathfrak{X}\to\mathfrak{Y}$ be a homomorphism of minimal ttgs and fix $u\in J$, $x_0=ux_0\in X$ and $y_0=\phi(x_0)$. Let $F=\mathfrak{G}(\mathfrak{Y},y_0)$ be the Ellis group of $\mathfrak{Y}$ with respect to y_0 in G . We define a (shadow) diagram EGS(ϕ) for ϕ as follows.

$$\begin{tikzcd}
\mathfrak{X}' \arrow[r, "\sigma"] \arrow[d, "\phi'"'] & \mathfrak{X} \arrow[d, "\phi"] \\
\mathfrak{Y}' \arrow[r, "\tau"'] & \mathfrak{Y}
\end{tikzcd}$$

Define a quasifactor $\mathfrak{Y}'$ of $\mathfrak{X}$ by $Y'=\{p\circ Fx_0\,|\,p\in M\}$ and let $X'=\{(x,A)\,|\,x\in A\in Y'\}$ be a subset of $X\times Y'$; $\sigma:X'\to X$ and $\phi':X'\to Y'$ are the projections and $\tau:Y'\to Y$ is defined by $\tau(p\circ Fx_0)=py_0$.

1.11. **REMARK.**

a) $Y' = \{v \circ v\phi^{\leftarrow}(y) \mid y \in Y.\ v \in J_y\}$ *and* $\tau: \mathcal{Y}' \to \mathcal{Y}$ *is a proximal homomorphism of minimal ttgs.*

b) $\mathcal{X}'$ *is a minimal ttg and* $\sigma: \mathcal{X}' \to \mathcal{X}$ *is a proximal extension.*

c) ϕ' *is a* RIC *extension.*

PROOF.

a) [EGS 75] 5.2. (use II.3.11. for the description of Y').

b) [EGS 75] 5.6..

c) [EGS 75] 5.9.1.. □

So our shadow diagram EGS(ϕ) is a commutative diagram consisting of homomorphisms of minimal ttgs. It shows that every homomorphism of minimal ttgs can be lifted to a RIC extension by means of proximal extensions.

The diagram EGS(ϕ) is minimal in the following sense

$$\begin{tikzcd}
\mathcal{X}'' \arrow[rr, "\lambda"] \arrow[dr, dashed, "\xi"'] \arrow[ddd, "\phi''"'] & & \mathcal{X} \arrow[ddd, "\phi"] \\
 & \mathcal{X}' \arrow[ur, "\sigma"'] \arrow[d, "\phi'"'] & \\
 & \mathcal{Y}' \arrow[dr, "\tau"] & \\
\mathcal{Y}'' \arrow[ur, dashed, "\eta"] \arrow[rr, "\mu"'] & & \mathcal{Y}
\end{tikzcd}$$

Consider the diagram above with $\phi'': \mathcal{X}'' \to \mathcal{Y}''$ a RIC extension of minimal ttgs and $\mu: \mathcal{Y}'' \to \mathcal{Y}$ proximal. Then there are maps $\eta: \mathcal{Y}'' \to \mathcal{Y}'$ and $\xi: \mathcal{X}'' \to \mathcal{X}'$ such that $\phi' \circ \xi = \eta \circ \phi'$.
The proof of this fact is left as an exercise for the reader.

Thus, indeed, EGS(ϕ) is in a certain sense the minimal lifting of ϕ to a RIC extension. Also we can construct a maximal lifting, but first we shall construct the universal proximal extension of a minimal ttg using an EGS shadow diagram (see also I.2.14. and the remark just below that item).

1.12. Let $\mathcal{Y}$ be a minimal ttg and let $\gamma: \mathcal{M} \to \mathcal{Y}$ be a homomorphism of minimal ttgs, say $\gamma(u) = y_0$; and let F be the Ellis group of $\mathcal{Y}$ with respect to y_0 in G . Construct EGS(γ).

$$\begin{tikzcd}
\mathcal{M} \cong \mathcal{M}' \arrow[r, "\sigma"] \arrow[d, "\gamma'"'] & \mathcal{M} \arrow[d, "\gamma"] \\
\mathfrak{A}(F) = \mathcal{Y}' \arrow[r, "\tau"'] & \mathcal{Y}
\end{tikzcd}$$

Then $Y' = \{p \circ Fu \mid p \in M\} = QF(u \circ F, \mathcal{M})$ (which will be denoted by $\mathfrak{A}(F)$); and $\mathcal{M}' \cong \mathcal{M}$, for $\mathcal{M}$ is the universal minimal ttg, so σ is an isomorphism. If we identify $\mathcal{M}'$ with $\mathcal{M}$ via σ, then it is clear that $\gamma' : \mathcal{M} \to \mathcal{Y}'$ is given by $\gamma'(p) = p \circ F$. Note that this implies that $\{p \circ F \mid p \in M\}$ is a partitioning of M.

1.13. **REMARK.**

a) *Every homomorphism* $\psi : \mathfrak{Z} \to \mathfrak{A}(F)$ *is a* RIC *extension.*

b) $\tau : \mathfrak{A}(F) \to \mathcal{Y}$ *is the universal minimal proximal extension of* $\mathcal{Y}$. *In particular,* $\mathfrak{A}(G) = \mathcal{P}_T$.

PROOF.

a) Let $\gamma' : \mathcal{M} \to \mathfrak{A}(F)$ be the map defined in 1.12., so $\gamma'(p) = p \circ F$ for $p \in M$. Let $\psi : \mathfrak{Z} \to \mathfrak{A}(F)$ be a homomorphism of minimal ttgs. Let $z_0 = uz_0 \in Z$ be such that $\psi(z_0) = u \circ F$ and define $\theta : \mathcal{M} \to \mathfrak{Z}$ by $\theta(p) = pz_0$ for every $p \in M$. Then $\gamma' = \psi \circ \theta$, so ψ is a factor of γ'. Hence, by 1.10.a, ψ is a RIC extension.

b) We know already that τ is a proximal extension. Let $\phi : \mathcal{X} \to \mathcal{Y}$ be a proximal homomorphism of minimal ttgs. Construct EGS(ϕ) and consider the next diagram.

$$\begin{tikzcd}
\mathcal{M} \arrow[rr] \arrow[dr, dashed] \arrow[ddd, "\xi"'] & & \mathcal{X} \arrow[ddd, "\phi"] \\
& \mathcal{X}' \arrow[ur] \arrow[d, "\phi'"'] & \\
& \mathcal{Y}' \arrow[dr] & \\
\mathfrak{A}(F) \arrow[ur, dashed, "\eta"] \arrow[rr, "\tau"'] & & \mathcal{Y}
\end{tikzcd}$$

Note that ϕ' is RIC and proximal; hence ϕ' is an isomorphism. By the discussion just above 1.12., it follows from the facts that ξ is RIC (1.13.a) and τ is proximal that there is a homomorphism $\eta : \mathfrak{A}(F) \to \mathcal{Y}'$. But then τ factorizes over ϕ, which shows that τ is universal. $\square$

1.14. Let $\phi:\mathcal{X}\to\mathcal{Y}$ be a homomorphism of minimal ttgs. Let $u\in J$, $x_0=ux_0\in X$, $y_0=\phi(x_0)$, $H=\mathfrak{G}(\mathcal{X},x_0)$ and $F=\mathfrak{G}(\mathcal{Y},y_0)$. Then the following shadow diagram $\mathfrak{A}(\phi)$ is the maximal lifting of ϕ to a RIC extension.

$$\begin{tikzcd}
\mathfrak{A}(H) \arrow[r, "\sigma"] \arrow[d, "\phi^{\mathfrak{A}}"'] & \mathcal{X} \arrow[d, "\phi"] \\
\mathfrak{A}(F) \arrow[r, "\tau"'] & \mathcal{Y}
\end{tikzcd}$$

Note that $\mathfrak{A}(H)=\mathfrak{W}\mathfrak{F}(u\circ H,\mathcal{M})$, $\sigma:\mathfrak{A}(H)\to\mathcal{X}$ is defined by $\sigma(p\circ H)=px_0$ and $\phi^{\mathfrak{A}}:\mathfrak{A}(H)\to\mathfrak{A}(F)$ is defined by $\phi^{\mathfrak{A}}(p\circ H)=p\circ F$.

That $\phi^{\mathfrak{A}}$ is well defined follows from 1.1. and:

1.15. **REMARK.** *Let H and F be subgroups of G. Then $H\subseteq u(u\circ F)$ iff the map $p\circ H\mapsto p\circ F:\mathfrak{A}(H)\to\mathfrak{A}(F)$ is a well defined homomorphism (which is* RIC*).*

PROOF. Let $p\circ H\mapsto p\circ F$ define a homomorphism. As, by II.3.11.c, $h\circ H=u\circ H$ for every $h\in H$, it follows that $h\circ F=u\circ F$ and so $h\in u\circ F$ for every $h\in H$; hence $H\subseteq u(u\circ F)$. Let $H\subseteq u(u\circ F)$; then $p\circ H\subseteq p\circ u(u\circ F)\subseteq p\circ F$ for every $p\in M$. Suppose that $p\circ H=q\circ H$, then $p\in p\circ H=q\circ H\subseteq q\circ F$. Choose a net $\{t_i\}_i$ in T with $t_i\to q$ and let $f_i\in F$ be such that $p=\lim t_if_i$. Then

$$p\circ F=(\lim t_if_i)\circ u\circ F=\lim t_i(u\circ F)=q\circ F$$

(II.3.11.c). □

III.2. $\mathfrak{F}$-TOPOLOGIES

The proof of the structure theorem for metric minimal distal ttgs as presented in [F 63] by H. FURSTENBERG had an enormous impact on the study of ttgs; may be it was even more important then the result itself. The big contribution to topological dynamics in that proof is the technique of the $\mathfrak{F}$-topology, a weaker topology on the phase space X of a minimal distal ttg $\mathfrak{X}$ to make the elements of $E(\mathfrak{X})$ homeomorphisms of X provided with the $\mathfrak{F}$-topology (compare I.1.12.e).
One can extend that technique to the construction of suitable (weak) topologies on the "maximal distal parts" of the phase space X of a minimal ttg $\mathfrak{X}$: Let $u \in J$, then one can construct an $\mathfrak{F}(\mathfrak{X}, u)$ topology on uX which is weaker then the relative topology, but still has nice properties.
In [E 67] R. ELLIS introduces a weakening of the topology on uX in a different way, the τ-topology, which is beautifully characterized in [EGS 75] using the circle operation. Also it is shown in [EGS 75] that the two topologies introduced by H. FURSTENBERG and R. ELLIS are in fact identical.
In this section we shall describe the $\mathfrak{F}$-topologies based on the τ-topologies. We do not intend to give a complete exposition of the subject, so most of the proofs will be omitted. For more details we refer to [V 77], [G 76], [EGS 75] and [VW ?].

We shall use almost the same notation as in [V 77].
Let T be an arbitrary topological group and fix a minimal left ideal I in S_T. Fix $u \in J(I)$ and let $V \subseteq T$ be a set such that $u \in \mathrm{int}_{S_T}\mathrm{cl}_{S_T} V$. define the open subset $V(u)$ of T by:

$$V(u) := \{t \in T \mid tu \in \mathrm{int}_I\,((\mathrm{cl}_{S_T} V) \cap I)\}.$$

2.1. **REMARK.** *With notation as above the following statements hold:*

a) *if* $u \in \mathrm{int}_{S_T}\mathrm{cl}_{S_T} V$ *then* $V(u)(u) = V(u)$*;*

b) *a base for the neighbourhoods of* u *in* I *is formed by the collection* $\{(\mathrm{cl}_{S_T} V) \cap I \mid V \subseteq T, u \in \mathrm{int}_{S_T}\mathrm{cl}_{S_T} V, V(u) = V\}$*;*

c) *let* $x = ux$ *be an almost periodic point in a ttg* $\mathfrak{X}$ *and let* $U \in \mathcal{V}_x$*, then there exists an open subset* V *of* T *such that* $u \in \mathrm{int}_{S_T}\mathrm{cl}_{S_T} V$*,* $V(u) = V$ *and* $Vx \subseteq U$*.*

PROOF. For a and b see [V 77] page 811 or [VW ?]; c follows immediately from b . □

Let $\mathcal{X}$ be a minimal ttg and let $x \in uX$. If $V \subseteq T$ is an open set such that $u \in \mathrm{int}_{S_T}\mathrm{cl}_{S_T} V$ and $V(u) = V$, and if U is a neighbourhood of x in X (provided with its original CT_2 topology) then define

$$[U,V] := V^{-1}U = \{t^{-1}U \mid t \in V\}.$$

Denote by $\mathcal{N}_x := \mathcal{N}_x^u$ the following collection of subsets of uX :

$$\{[U,V] \cap uX \mid U \in \mathcal{V}_x,\ V(u) = V \text{ open in } T \text{ with } u \in \mathrm{int}_{S_T}\mathrm{cl}_{S_T} V\}.$$

2.2. **REMARK.** *The collection* $\bigcup\{\mathcal{N}_x^u \mid x \in uX\}$ *of subsets of* uX *forms a base for a topology on* uX *, in which every* $\mathcal{N}_x^u$ *is a neighbourhood base for* x *. This topology will be called the* $\mathfrak{F}(\mathcal{X},u)$*-topology on* uX *.* □

The above description of the $\mathfrak{F}(\mathcal{X},u)$-topology is the one we shall use mostly (it is sometimes referred to as the *τ-topology*). Another description uses the circle operation.

Let $\mathcal{X}$ be a minimal ttg. Then define a closure operator on uX as follows: For $A \subseteq uX$ let

$$\mathrm{cl}_{\mathfrak{F}}(A) := u \circ A \cap uX = u(u \circ A)$$

(note that $u \circ A := u \circ \bar{A}$). Clearly, $\mathrm{cl}_{\mathfrak{F}}$ is a closure operator.

2.3. **REMARK.** *The topology on* uX *generated by the closure operator* $\mathrm{cl}_{\mathfrak{F}}$ *is just the* $\mathfrak{F}(\mathcal{X},u)$*-topology on* uX *.* □

The generalized Furstenberg method to introduce the $\mathfrak{F}$-topologies on uX is as follows:

Let $\mathcal{X}$ be a minimal ttg and let Σ be the set of continuous pseudometrics on X . For $\sigma \in \Sigma$ define a T-invariant upper semi continuous real valued map $F_\sigma : X \times X \to \mathbb{R}$ by

$$F_\sigma(x_1,x_2) = \inf\{\sigma(tx_1,tx_2) \mid t \in T\}$$

Then for every $x \in X$ and $\epsilon > 0$ the set

$$U(x,\sigma,\epsilon) := \{x' \in X \mid F_\sigma(x,x') < \epsilon\}$$

is an open set in X .

2.4. **REMARK.** *The collection* $\{U(x,\sigma,\epsilon) \cap uX \mid x \in uX, \sigma \in \Sigma, \epsilon > 0\}$ *of subsets of* uX *forms a base for the* $\mathfrak{F}(\mathcal{X},u)$*-topology on* uX *.* □

Almost everything studied in topological dynamics is essentially independent of the topology of the phase group T . Only the existence of ttgs, or better the joint continuity of actions does depend on it. So it will not be very surprising that the $\mathfrak{I}(\mathfrak{X},u)$-topology on uX does not depend on the topology of T ; as follows.

2.5. **REMARK.** *Let* $\mathfrak{X}$ *be a minimal ttg for* T *and let* I *and* K *be minimal left ideals in* S_T *and* S_{T_d} *respectively (* T_d *denotes the topological group* T *provided with the discrete topology). Then for every* $u \in J(I)$ *there is a* $v \in J(K)$ *such that*

$$(uX, \mathfrak{I}(\mathfrak{X},u)) = (vX, \mathfrak{I}(\mathfrak{X},v)) .$$

PROOF. First note that the sets $U(x,\sigma,\epsilon)$ do not depend on the topology of T , nor on I , K or $u \in J(I)$, $v \in J(K)$. So the remark is proven if for every $u \in J(I)$ we can find a $v \in J(K)$ with $vX = uX$.
Let $u \in J(I)$. As $< T_d , I >$ is a minimal ttg and as K is a minimal left ideal in S_{T_d} there is, by I.2.5.d, an idempotent $v \in J_u(K)$; i.e., $vu = u$. But then $uX = vuX \subseteq vX$. On the other hand, if $x' \in vX$ then $ux' \in uX \subseteq vX$; so, by I.2.8., x' and ux' are distal under T_d , hence under T (distality does not depend on the topology of T). By I.2.7., x' and ux' are proximal under T , so $ux' = x'$. This shows that every point in vX is u-invariant; i.e., $vX \subseteq uX$. □

In 2.2., 2.3. and 2.4. we gave three descriptions of the $\mathfrak{I}(\mathfrak{X},u)$-topology each of which has its own (dis)-advantages. The three together give a lot of nice properties. The easy proof of the following theorem is omitted.

2.6. **THEOREM.** *Let* $\mathfrak{X}$ *be a minimal ttg and let* $u \in J$. *Then*

a) $(uX, \mathfrak{I}(\mathfrak{X},u))$ *is a compact* T_1*-space;*

b) *the map* $\lambda_a : (uX, \mathfrak{I}(\mathfrak{X},u)) \to (uX, \mathfrak{I}(\mathfrak{X},u))$ *is a homeomorphism for every* $a \in G$ *(recall that* $\lambda_a(x) := ax$ *for every* $x \in uX$ *);*

c) *the map* $\lambda_v : (uX, \mathfrak{I}(\mathfrak{X},u)) \to (vX, \mathfrak{I}(\mathfrak{X},v))$ *is a homeomorphism for every* $v \in J$ *;*

d) *for every* $p \in M$ *and for* $w \in J$ *with* $wp = p$ *the map* $\lambda_p : (uX, \mathfrak{I}(\mathfrak{X},u)) \to (wX, \mathfrak{I}(\mathfrak{X},w))$ *is a homeomorphism.* □

Note that if $\mathfrak{X}$ is a minimal distal ttg then $uX = X$, so λ_a is a homeomorphism of $(X, \mathfrak{I}(\mathfrak{X},u))$. As $E(\mathfrak{X})$ is a group, $a \mapsto \lambda_a : uM \to E(\mathfrak{X})$ is a surjection, so $E(\mathfrak{X})$ can be seen as a group of homeomorphisms of $(X, \mathfrak{I}(\mathfrak{X},u))$.

2.7. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs and let* $u \in J$. *Then for the surjective map* $\phi_u = \phi|_{uX} : uX \to uY$ *we have*

a) ϕ_u *is continuous with respect to the* $\mathfrak{T}$*-topologies;*
b) ϕ_u *is closed with respect to the* $\mathfrak{T}$*-topologies;*
c) ϕ_u *is an* $\mathfrak{T}$*-homeomorphism iff* ϕ *is proximal.*

PROOF. a and b are easy exercises for the reader (use the τ-topology and the closure operator for a and b respectively). Statement c follows immediately from the observation that ϕ_u is one to one iff ϕ is proximal. □

2.8. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs and let* $u \in J$. *Then* $\phi_u : (uX, \mathfrak{T}(\mathfrak{X}, u)) \to (uY, \mathfrak{T}(\mathfrak{Y}, u))$ *is an open map.*

PROOF. Consider EGS(ϕ) restricted to the u-invariant parts.

$$\begin{tikzcd}
uX' \arrow[r, "\sigma_u"] \arrow[d, "\phi'_u"'] & uX \arrow[d, "\phi_u"] \\
uY' \arrow[r, "\tau_u"'] & uY
\end{tikzcd}$$

Then, by 2.7.c, it follows that σ_u and τ_u are $\mathfrak{T}$-homeomorphisms, so we may conclude that ϕ_u is an $\mathfrak{T}$-open map iff ϕ'_u is $\mathfrak{T}$-open. So it suffices to prove the theorem for the case that ϕ is a RIC extension.
Suppose that ϕ is a RIC extension and let $x = ux \in X$. Let $U \in \mathcal{V}_x$ and let V be an open subset of T with $u \in \mathrm{int}_{S_T}\mathrm{cl}_{S_T} V$ and $V = V(u)$, then $[U, V] \cap uX$ is a (basic) neighbourhood of x in $(uX, \mathfrak{T}(\mathfrak{X}, u))$. We shall prove that $\phi_u\,[[U, V] \cap uX] = [\phi[U], V] \cap uY$. As ϕ is an open map (1.4.) it follows that $[\phi[U], V] \cap uY$ is an $\mathfrak{T}(\mathfrak{Y}, u)$-neighbourhood of $\phi_u(x)$. Hence ϕ_u is an $\mathfrak{T}$-open map.
First note that

$$\phi_u\,[[U, V] \cap uX] \subseteq \phi[[U, V]] \cap \phi[uX] = [\phi[U], V] \cap uY .$$

Let $y = uy \in [\phi[U], V] \cap uY$, then $y = \phi(t^{-1}x')$ for some $t \in V$ and $x' \in U$. As ϕ is RIC we have $z := t^{-1}x' \in \phi^{\leftarrow}(y) = u \circ u\phi^{\leftarrow}(y)$. Let $\{t_i\}_i$ be a net in T with $t_i \to u$ and let $x_i \in u\phi^{\leftarrow}(y)$ be such that $z = \lim t_i x_i$. Since left multiplication with t is a homeomorphism we have $tt_i x_i \to tz = x'$ and $tt_i \to tu$, hence $tt_i u \to tu$. As $t \in V = V(u)$ we have $tu \in \mathrm{int}_M(\mathrm{cl}_{S_T}[V] \cap M)$, so $tt_i u \in \mathrm{int}_M(\mathrm{cl}_{S_T}[V] \cap M)$ eventually, hence

$tt_i \in V(u) = V$ eventually. Also $tt_i x_i \in U$ eventually, so we can find some i_0 such that $tt_{i_0}x_{i_0} \in U$ and $tt_{i_0} \in V$. This shows that

$$x_{i_0} = (tt_{i_0})^{-1}.tt_{i_0}x_{i_0} \in V^{-1}U \ ,$$

so $x_{i_0} \in [U, V] \cap u\phi^{\leftarrow}(y)$. Hence $x_{i_0} \in [U, V] \cap uX$, while $\phi(x_{i_0}) = y$ and so it follows that $y \in \phi_u\,[[U, V] \cap uX]$, which implies

$$\phi_u\,[[U, V] \cap uX] = [\phi[U], V] \cap uY$$

in case ϕ is a RIC extension. □

As every minimal ttg $\mathfrak{X}$ is a factor of $\mathfrak{M}$, it follows from 2.7. and 2.8. that $(uX, \mathfrak{F}(\mathfrak{X}, u))$ is an open, closed and continuous image of $(uM, \mathfrak{F}(\mathfrak{M}, u))$. So $(uM, \mathfrak{F}(\mathfrak{M}, u))$ plays a central role in the observations about $\mathfrak{F}$-topologies.
We shall collect a few theoretical aspects of $(uM, \mathfrak{F}(\mathfrak{M}, u))$.

2.9. **THEOREM.** *The group uM provided with the* $\mathfrak{F}(\mathfrak{M}, u)$*-topology is a* CT_1 *space with continuous right and left translations and with a continuous inversion (these are even homeomorphisms)* (cf. [V 77] 2.5.9.). □

The next theorem characterizes the Ellis groups as the $\mathfrak{F}(\mathfrak{M}, u)$-closed subgroups of uM .

2.10. **THEOREM.** *Let F be the Ellis group in uM of some minimal ttg* $\mathfrak{Y}$ *with respect to a certain point* $y = uy \in Y$ *. Then F is an* $\mathfrak{F}(\mathfrak{M}, u)$*-closed subgroup of uM and so all left and right translations as well as the inversion are* $\mathfrak{F}(\mathfrak{M}, u)$*-homeomorphisms.*
Moreover, every $\mathfrak{F}(\mathfrak{M}, u)$*-closed subgroup K of uM is the Ellis group of the minimal ttg* $\mathfrak{A}(K) := \mathfrak{W}(u \circ K, \mathfrak{M})$ *(which is maximally proximal in the sense that it admits no nontrivial minimal proximal extensions).*

PROOF. The first part of the theorem is immediate from 2.9., 2.3. and 1.1.. Let K be an $\mathfrak{F}(\mathfrak{M}, u)$-closed subgroup of uM . Then one shows easily, using II.3.11.c, that $K = \mathfrak{G}(\mathfrak{A}(K), u \circ K)$ which by 1.13.b proves the theorem. □

In the sequel we need the following technical lemma. For a proof see for instance [G 76] IX.1.10., 1.11.. Note that the techniques to be developed in section V.1. enable us to give an alternative (and easier) proof.

2.11. **LEMMA.** *Let* $u \in J$ *and consider* $(uM, \mathfrak{F}(\mathfrak{M}, u))$.

a) *If* A *and* B *are* $\mathfrak{F}(\mathfrak{M}, u)$*-closed subsets of* uM *then* AB *is an* $\mathfrak{F}(\mathfrak{M}, u)$*-closed subset of* uM .

b) *Let* $\{A_i \mid i \in \Lambda\}$ *be a collection of* $\mathfrak{F}(\mathfrak{M}, u)$*-closed subsets of* uM *, which is directed by inclusion and let* K *be an* $\mathfrak{F}(\mathfrak{M}, u)$*-closed subset of* uM *. Then for* $A := \bigcap \{A_i \mid i \in \Lambda\}$ *we have* $AK = \bigcap \{A_i K \mid i \in \Lambda\}$ *and* $KA = \bigcap \{KA_i \mid i \in \Lambda\}$. □

The reason why this "$\mathfrak{F}$-stuff" was invented is (somewhat hidden in) the theorem to follow, compare 2.12.b with I.1.12.e.

First we need a definition:

Let F be an $\mathfrak{F}(\mathfrak{M}, u)$-closed subgroup of uM , then define

$$\mathrm{H}(F) := \bigcap \{\mathrm{cl}_{\mathfrak{F}(\mathfrak{M}, u)}(F \cap U) \mid U \in \mathfrak{N}_u\} ,$$

where $\mathfrak{N}_u$ is the $\mathfrak{F}(\mathfrak{M}, u)$-neighbourhood filter of u in uM .

2.12. **THEOREM.** *With notation as above:*

a) $\mathrm{H}(F)$ *is an* $\mathfrak{F}(\mathfrak{M}, u)$*-closed normal subgroup of* F *;*

b) $F/\mathrm{H}(F)$ *provided with the quotient topology is a* CT_2 *topological group;*

c) $\mathrm{H}(F)$ *is the smallest* $\mathfrak{F}(\mathfrak{M}, u)$*-closed normal subgroup* K *of* F *, such that* F/K *is a* CT_2 *topological group.*

PROOF. Cf. [G 76] IX.1.9.. □

Let F be an $\mathfrak{F}(\mathfrak{M}, u)$-closed subgroup of $G = uM$, then define for every ordinal $\alpha \geqslant 1$ an $\mathfrak{F}(\mathfrak{M}, u)$-closed normal subgroup $\mathrm{H}_\alpha(F)$ of F as follows:

$\mathrm{H}_1(F) := \mathrm{H}(F)$;

let $\mathrm{H}_\alpha(F)$ be defined, then define

$\mathrm{H}_{\alpha+1}(F) := \mathrm{H}(\mathrm{H}_\alpha(F))$;

let α be a limit ordinal and let $\mathrm{H}_\beta(F)$ be defined for all $\beta < \alpha$, then define

$\mathrm{H}_\alpha(F) := \bigcap \{\mathrm{H}_\beta(F) \mid \beta < \alpha\}$.

As $\{\mathrm{H}_\alpha(F) \mid \alpha\}$ is a descending family of $\mathfrak{F}(\mathfrak{M}, u)$-closed subsets of uM , there is an ordinal ν , for which $\mathrm{H}_\nu(F) = \mathrm{H}_{\nu+1}(F)$. Then $\mathrm{H}_\gamma(F) = \mathrm{H}_\nu(F)$ for every $\gamma \geqslant \nu$; this $\mathrm{H}_\nu(F)$ will be denoted by F_∞. For normality of $\mathrm{H}_\alpha(F)$ in F see [EGS 75] prop. 3.13..

2.13. **LEMMA.** *Let* A *and* B *be* $\mathfrak{F}(\mathfrak{M},u)$*-closed subgroups of* $G = uM$.

a) *If* AB *is a group, then* $A.\mathrm{H}(AB) = A.\mathrm{H}(B)$ *(and, also,* $\mathrm{H}(AB).B = \mathrm{H}(A).B$ *).*

b) *If* AB *is a group, then* $AB_\infty = A.(AB)_\infty$ *; in particular, if* $AB = G$ *then* $AB_\infty = AG_\infty$.

c) *If* $AB\,\mathrm{H}(G) = G$ *, then* $ABG_\infty = G$ *(* AB *need not be a group!).*

PROOF.

a) [EGS 75] 3.12..

b) Straightforward corollary from a.

c) [EGS 76] 2.3.. □

2.14. **REMARK.** *Let* F *be an* $\mathfrak{F}(\mathfrak{M},u)$*-closed subgroup of* uM *, and let* $v \in J$ *. Then*

a) vF *is an* $\mathfrak{F}(\mathfrak{M},v)$*-closed subgroup of* vM *and* $\mathrm{H}(vF) = v\,\mathrm{H}(F)$ *, in particular* $(vF)_\infty = vF_\infty$ *;*

b) *for every* $p \in M$ *we have* $\mathrm{H}(pFp^{-1}) = p\,\mathrm{H}(F)p^{-1}$ *, where* $\mathrm{H}(pFp^{-1})$ *is calculated in* wM *for* $w \in J_p$.

PROOF. Follows easily from 2.6.c and 2.9.. □

After these observations about $(uM,\mathfrak{F}(\mathfrak{M},u))$ (or $(G,\mathfrak{F}(\mathfrak{M},u))$) we shall now return to the (more general) case of $(uX,\mathfrak{F}(\mathfrak{X},u))$ or rather to the u-invariant part of a fiber with the relative $\mathfrak{F}$-topology (in the spirit of [V 77]). Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a homomorphism of minimal ttgs. Let $y \in Y$ and $u \in J_y$, and let $F = \mathfrak{G}(\mathfrak{Y},y)$ be the Ellis group of $\mathfrak{Y}$ with respect to y in G . Then $u\phi^{\leftarrow}(y) = u\phi^{\leftarrow}\phi(x) = Fx$ for every $x \in \phi^{\leftarrow}(y)$. Define for every $x \in uX$ the set $E(x) := E(x,\phi,u) \subseteq u\phi^{\leftarrow}\phi(x)$ by

$$E(x) := \bigcap \{\mathrm{cl}_{\mathfrak{F}(\mathfrak{X},u)}(U \cap u\phi^{\leftarrow}\phi(x)) \mid U \in \mathfrak{N}_x\} .$$

Beware that $E(x)$ depends on the choices of M and $u \in J$.
In the remark to follow we link the approaches as can be found in [V 77] and in [G 76] and [EGS 75].

2.15. **REMARK.** *With notation as above:*

a) $E(x) = \mathrm{H}(F).x$ *for* $x \in u\phi^{\leftarrow}(y)$ *;*

b) $E(px) = pE(x)$ *for all* $p \in M$ *; where* $E(px) = E(px,\phi,v)$ *and* $v \in J$ *such that* $vp = p$ *;*

c) $\{E(x') \mid x' \in u\phi^{\leftarrow}\phi(x)\}$ *is a partitioning of* $u\phi^{\leftarrow}\phi(x)$.

PROOF.

a) Define the map $\gamma = \rho_x : \mathcal{M} \to \mathcal{X}$ by $\gamma(p) = px$. Then γ is a homomorphism of minimal ttgs. So by 2.7. and 2.8. the map $\gamma_u : (uM, \mathfrak{T}(\mathcal{M}, u)) \to (uX, \mathfrak{T}(\mathcal{X}, u))$ is an open, closed and continuous surjection. As $F = \gamma_u^{\leftarrow}[u\phi^{\leftarrow}\phi(x)]$, the restriction

$$\gamma_u \mid_F : (F, \mathfrak{T}(\mathcal{M}, u)) \to (u\phi^{\leftarrow}\phi(x), \mathfrak{T}(\mathcal{X}, u))$$

is an open, closed and continuous surjection too. But then

$$\{V \cap u\phi^{\leftarrow}\phi(x) \mid V \in \mathcal{N}_x\} = \{\gamma_u[U \cap F] \mid U \in \mathcal{N}_u\} ,$$

and as the collection $\{\mathrm{cl}_{\mathfrak{T}(\mathcal{M},u)}(U \cap F) \mid U \in \mathcal{N}_u\}$ is directed by inclusion and γ_u is closed and continuous, it follows easily that $E(x) = \mathrm{H}(F)x$.

b) Let $p \in M$ and $v \in J$ with $vp = p$, and define $y' = py$. Then $\mathfrak{G}(\mathcal{Y}, y') = pFp^{-1}$ is the Ellis group of $\mathcal{Y}$ with respect to y' in vM . Hence $E(px) = \mathrm{H}(pFp^{-1}).px$ and so by 2.14.,

$$E(px) = p\,\mathrm{H}(F)p^{-1}px = p\,\mathrm{H}(F)x$$

which by a proves that $E(px) = pE(x)$.

c) Let $z \in E(x')$, then $z \in \mathrm{H}(F)x'$, say $z = fx'$ for $f \in \mathrm{H}(F)$. But then

$$E(z) = \mathrm{H}(F)z = (\mathrm{H}(F)f^{-1})z = \mathrm{H}(F)f^{-1}z = \mathrm{H}(F)x' = E(x') .$$

$\square$

Similar to the definition of the normal subgroups $\mathrm{H}_\alpha(F)$ we can define subsets $E_\alpha(x) = E_\alpha(x, \phi, u)$ for every ordinal α , as follows:

$E_1(x) := E(x)$;

let $E_\alpha(x)$ defined, then define

$E_{\alpha+1}(x) := \bigcap \{\mathrm{cl}_{\mathfrak{T}(\mathcal{X},u)}(U \cap E_\alpha(x) \mid U \in \mathcal{N}_x\}$;

let α be a limit ordinal and let for every $\beta < \alpha$ the set $E_\beta(x)$ be defined, then define

$E_\alpha(x) := \bigcap \{E_\beta(x) \mid \beta < \alpha\}$.

As $\{E_\alpha(x) \mid \alpha\}$ is a descending family of $\mathfrak{T}(\mathcal{X}, u)$-closed subsets of $u\phi^{\leftarrow}\phi(x)$ there is an ordinal ν , for which $E_\nu(x) = E_{\nu+1}(x)$. For that ordinal ν we define $E_\infty(x) := E_\nu(x)$.

2.16. **REMARK.** *With notation as above.*

For every ordinal α *we have* $E_\alpha(x) = \mathrm{H}_\alpha(F)x$ *(* $F = \mathfrak{G}(\mathcal{Y}, \phi(x))$ *). In particular,* $E_\infty(x) = F_\infty x$.

PROOF. We prove the theorem by transfinite induction.
For $\alpha = 1$ the statement is true by 2.15.a.
Suppose α is a limit ordinal and let $E_\beta(x) = \mathrm{H}_\beta(F)x$ for every $\beta < \alpha$. Then

$$E_\alpha(x) = \bigcap\{E_\beta(x) \mid \beta < \alpha\} = \bigcap\{\mathrm{H}_\beta(F)x \mid \beta < \alpha\}.$$

As $\{\mathrm{H}_\beta(F) \mid \beta < \alpha\}$ is a family of $\mathfrak{F}(\mathfrak{M}, u)$-closed subsets of F, linearly ordered by inclusion, while $\gamma_u : F \to u\phi^{\leftarrow}\phi(x)$ is an $\mathfrak{F}$-closed and $\mathfrak{F}$-continuous map (γ as in the proof of 2.15.a) it follows that

$$\gamma_u\,[\mathrm{H}_\alpha(F)] = \gamma_u[\bigcap\{\mathrm{H}_\beta(F) \mid \beta < \alpha\}] = \bigcap\{\gamma_u[\mathrm{H}_\beta(F)] \mid \beta < \alpha\}.$$

Hence

$$\mathrm{H}_\alpha(F)x = \bigcap\{\mathrm{H}_\beta(F)x \mid \beta < \alpha\} = \bigcap\{E_\beta(x) \mid \beta < \alpha\} = E_\alpha(x).$$

Let $\alpha \geqslant 1$ be an ordinal and let $E_\alpha(x) = \mathrm{H}_\alpha(F)x$. Then it is easily checked that $\gamma_u^{\leftarrow}[\mathrm{H}_\alpha(F)x] = \mathrm{H}_\alpha(F).H$, where $H = \mathfrak{G}(\mathfrak{X}, x)$, the Ellis group of $\mathfrak{X}$ with respect to x in G. So $\gamma_u : \mathrm{H}_\alpha(F)H \to E_\alpha(x)$ is an $\mathfrak{F}$-open, $\mathfrak{F}$-closed and $\mathfrak{F}$-continuous surjection, which implies that

$$\gamma_u[\bigcap\{\mathrm{cl}_{\mathfrak{F}(\mathfrak{M},u)}(\mathcal{U} \cap \mathrm{H}_\alpha(F)H) \mid \mathcal{U} \in \mathfrak{N}_u\}] =$$

$$= \bigcap\{\mathrm{cl}_{\mathfrak{F}(\mathfrak{X},u)}(\mathcal{V} \cap E_\alpha(x)) \mid \mathcal{V} \in \mathfrak{N}_x\};$$

hence $\mathrm{H}(\mathrm{H}_\alpha(F)H)x = E_{\alpha+1}(x)$. Since $x = Hx$, it follows that $E_{\alpha+1}(x) = \mathrm{H}(\mathrm{H}_\alpha(F)H)Hx$ and so, by 2.13.a,

$$E_{\alpha+1}(x) = \mathrm{H}(\mathrm{H}_\alpha(F))Hx = \mathrm{H}_{\alpha+1}(F)x. \qquad \square$$

In order to shed some light on the foregoing $\mathfrak{F}$-manipulations we just mention the following result (e.g. see [G 76] IX.2.1.4.):

2.17. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a distal homomorphism of minimal ttgs. Then* ϕ *is almost periodic iff* $E(x) = \{x\}$ *for some (hence all)* $x \in X$. $\square$

We shall end this section with a rather technical theorem, which is the final blow in understanding the equicontinuous structure relation as will be shown in section III.3.. This result (2.20.) can be found in [V 77], hidden between other technicalities. The present form of 2.20. is due to T.S. WU.

Recall that $\phi : \mathfrak{X} \to \mathfrak{Y}$ is a homomorphism of minimal ttgs, $x \in X$, $u \in J_x$ and that $H = \mathfrak{G}(\mathfrak{X}, x)$ and $F = \mathfrak{G}(\mathfrak{Y}, \phi(x))$ are the Ellis groups of $\mathfrak{X}$ and $\mathfrak{Y}$ with respect to x and $\phi(x)$ in uM .
For $x' \in uX$ we denote the $\mathfrak{T}(\mathfrak{X}, u)$-neighbourhood system of x' in uX by $\mathcal{N}_{x'}$ and $\mathcal{N}^{\phi}_{x'}$ denotes the relative $\mathfrak{T}(\mathfrak{X}, u)$-neighbourhood system of x' in $u\phi^{\leftarrow}\phi(x)$. So if $x' \in u\phi^{\leftarrow}\phi(x) = Fx$, then

$$\mathcal{N}^{\phi}_{x'} = \{\mathcal{U} \cap Fx \mid \mathcal{U} \in \mathcal{N}_{x'}\} .$$

The $\mathfrak{T}(\mathfrak{M}, u)$-neighbourhood system of u in uM is denoted by $\mathcal{N}_u$.

2.18. **LEMMA.** *Let $\mathcal{V} \subseteq Fx$ be a nonempty $\mathfrak{T}(\mathfrak{X}, u)$-open subset of Fx (relative topology). Then* $\mathrm{cl}_{\mathfrak{T}(\mathfrak{X},u)}\mathrm{H}(F)\mathcal{V} = \mathrm{cl}_{\mathfrak{T}(\mathfrak{X},u)}\mathcal{V}$.

PROOF. Let $x' \in \mathcal{V}$; then $\mathcal{V} \in \mathcal{N}^{\phi}_{x'}$. By 2.15. and the fact that $Fx' = Fx$, we have

$$\mathrm{H}(F)x' = E(x') = \bigcap \{\mathrm{cl}_{\mathfrak{T}(\mathfrak{X},u)}(\mathcal{U} \cap Fx') \mid \mathcal{U} \in \mathcal{N}_{x'}\} =$$
$$= \bigcap \{\mathrm{cl}_{\mathfrak{T}(\mathfrak{X},u)}\mathcal{U} \mid \mathcal{U} \in \mathcal{N}^{\phi}_{x'}\} .$$

Hence $\mathrm{H}(F)x' = E(x') \subseteq \mathrm{cl}_{\mathfrak{T}(\mathfrak{X},u)}\mathcal{V}$. As $x' \in \mathcal{V}$ was arbitrary, we have $\mathrm{H}(F)\mathcal{V} \subseteq \mathrm{cl}_{\mathfrak{T}(\mathfrak{X},u)}\mathcal{V}$ and so

$$\mathrm{cl}_{\mathfrak{T}(\mathfrak{X},u)}\mathcal{V} \subseteq \mathrm{cl}_{\mathfrak{T}(\mathfrak{X},u)}\mathrm{H}(F)\mathcal{V} \subseteq \mathrm{cl}_{\mathfrak{T}(\mathfrak{X},u)}\mathcal{V} . \qquad \square$$

2.19. **LEMMA.** *There is an $\mathfrak{T}(\mathfrak{X}, u)$-neighbourhood base at x in Fx consisting of "symmetric" sets; i.e.: for every $\mathcal{V} \in \mathcal{N}^{\phi}_{x}$ there is a $\mathcal{V}_0 \in \mathcal{N}^{\phi}_{x}$ with $\mathcal{V}_0 \subseteq \mathcal{V}$ and $(\mathcal{V}_0)^{-1} := \{f^{-1}x \mid fx \in \mathcal{V}_0, f \in F\} = \mathcal{V}_0$. Note that $\mathrm{cl}_{\mathfrak{T}(\mathfrak{X},u)}\mathcal{V}$ is symmetric if $\mathcal{V} \in \mathcal{N}^{\phi}_{x}$ is symmetric (with respect to x).*

PROOF. A neighbourhood base at x in Fx is formed by the sets of the form $U(x, \sigma, \epsilon) \cap Fx$ with $\sigma \in \Sigma$ and $\epsilon > 0$. These sets $U(x, \sigma, \epsilon) \cap Fx$ are symmetric. For let $f \in F$ be such that $fx \in U(x, \sigma, \epsilon) \cap Fx$. Then $F_\sigma(fx, x) < \epsilon$ and so

$$F_\sigma(f^{-1}x, x) = F_\sigma(f^{-1}(x, fx)) = F_\sigma(x, fx) < \epsilon$$

hence $f^{-1}x \in U(x, \sigma, \epsilon) \cap Fx$. (The second equality follows from the definition of F_σ and from the almost periodicity of (x, fx) in $X \times X$.)

Let $V \in \mathcal{N}_x^{\phi}$ be a symmetric set with respect to x , then the set $W := \{f \in F \mid fx \in V\}$ is symmetric with respect to u . As the map $p \mapsto p^{-1} : F \to F$ is an $\mathfrak{F}(\mathcal{N}, u)$-homeomorphism it follows easily that $\mathrm{cl}_{\mathfrak{F}(\mathcal{N},u)} W$ is a symmetric set in F with respect to u . Since $\mathrm{cl}_{\mathfrak{F}(\mathcal{N},u)} V = (\mathrm{cl}_{\mathfrak{F}(\mathcal{N},u)} W).x$ we have that $\mathrm{cl}_{\mathfrak{F}(\mathcal{X},u)} V$ is symmetric. □

2.20. **THEOREM.** *With notation as above.*
Let $V \in \mathcal{N}_x^{\phi}$ *; then* $J\mathrm{H}(F)x \cap u \circ Fx \subseteq u \circ V$.

PROOF. By 2.19. we may assume V to be symmetric. Define $A := \mathrm{int}_{\mathfrak{F}(\mathcal{X},u)} \mathrm{cl}_{\mathfrak{F}(\mathcal{X},u)} V$ in the relative $\mathfrak{F}(\mathcal{X},u)$-topology on Fx . We claim that

$$\{A\} \cup \{gV \mid g \in F \text{ and } gx \notin \mathrm{cl}_{\mathfrak{F}(\mathcal{X},u)} V\}$$

is an $\mathfrak{F}(\mathcal{X},u)$-open covering of Fx . As follows:
Let $f \in F$ be such that $fx \notin A$; i.e.,

$$fx \in Fx \setminus A = \mathrm{cl}_{\mathfrak{F}(\mathcal{X},u)}(Fx \setminus \mathrm{cl}_{\mathfrak{F}(\mathcal{X},u)} V) .$$

So we can find a net $\{f_i x\}_i$ with $f_i x \in Fx \setminus \mathrm{cl}_{\mathfrak{F}(\mathcal{X},u)} V$ such that $f_i x \to fx$ in the $\mathfrak{F}(\mathcal{X},u)$-topology. Since

$$\lambda_{f^{-1}} : (Fx, \mathfrak{F}(\mathcal{X},u)) \to (Fx, \mathfrak{F}(\mathcal{X},u))$$

is a homeomorphism, $f^{-1} f_i x \to x$ in the $\mathfrak{F}(\mathcal{X},u)$-topology. As $V \in \mathcal{N}_x^{\phi}$, there is an i_0 with $f^{-1} f_{i_0} x \in V$ and by symmetry of V , $f_{i_0}^{-1} f x \in V$. Hence $fx \in f_{i_0} V$, where $f_{i_0} \in F$ is such that $f_{i_0} x \in Fx \setminus \mathrm{cl}_{\mathfrak{F}(\mathcal{X},u)} V$, which establishes our claim.
By compactness, there are finitely many $g_i \in F$ with $g_i x \notin \mathrm{cl}_{\mathfrak{F}(\mathcal{X},u)} V$, say $g_1, \ldots, g_n$, such that

$$Fx \subseteq A \cup \bigcup \{g_i V \mid i \in \{1, \ldots, n\}\}.$$

As $\{A\} \cup \{g_i V \mid i \in \{1, \ldots, n\}\}$ is a finite collection it follows that

$$u \circ Fx = u \circ (A \cup \bigcup \{g_i V \mid i \in \{1, \ldots, n\}\}) =$$

$$= u \circ A \cup \bigcup \{u \circ g_i V \mid i \in \{1, \ldots, n\}\} .$$

By II.3.11.c we know that $u \circ g_i V = g_i \circ V$, so

$$u \circ Fx = u \circ A \cup \bigcup \{g_i \circ V \mid i \in \{1, \ldots, n\}\} .$$

Now let $x' \in J\mathrm{H}(F)x \cap u \circ Fx$, say $x' = vpx$ for some $v \in J$ and $p \in \mathrm{H}(F)$. We shall prove that $x' = vpx \notin g_i \circ V$ for every $i \in \{1, \ldots, n\}$. It then follows that

$$x' \in u \circ A \subseteq u \circ \mathrm{cl}_{\mathfrak{F}(\mathfrak{X},u)}V = u \circ u(u \circ V) \subseteq u \circ V ,$$

which proves the theorem. Suppose $vpx \in g_i \circ V$, then

$$x = ux = up^{-1}vpx \in up^{-1}(g_i \circ V) \subseteq u(u \circ up^{-1}g_i V) = \mathrm{cl}_{\mathfrak{F}(\mathfrak{X},u)}up^{-1}g_i V .$$

As $\mathrm{H}(F)$ is a normal subgroup of F and $g_i \in F$ we can find $q \in \mathrm{H}(F)$ such that $up^{-1}g_i = g_i q$, so

$$x \in \mathrm{cl}_{\mathfrak{F}(\mathfrak{X},u)} g_i qV = g_i \mathrm{cl}_{\mathfrak{F}(\mathfrak{X},u)} qV \subseteq g_i \mathrm{cl}_{\mathfrak{F}(\mathfrak{X},u)} \mathrm{H}(F)V.$$

By 2.18. it follows that

$$x \in g_i \mathrm{cl}_{\mathfrak{F}(\mathfrak{X},u)} \mathrm{H}(F)V \subseteq g_i \mathrm{cl}_{\mathfrak{F}(\mathfrak{X},u)} V ,$$

hence $g_i^{-1}x \in \mathrm{cl}_{\mathfrak{F}(\mathfrak{X},u)}V$. Since by 2.19. $\mathrm{cl}_{\mathfrak{F}(\mathfrak{X},u)}V$ is symmetric we have $g_i x \in \mathrm{cl}_{\mathfrak{F}(\mathfrak{X},u)}V$, which contradicts the choice of g_i. □

III.3. THE EQUICONTINUOUS STRUCTURE RELATION

In this section we consider the equicontinuous structure relation for Bc extensions and we give a foretaste of chapter VII in proving that the equicontinuous structure relation E_ϕ is equal to the regionally proximal relation Q_ϕ in case of a Bc extension ϕ. This result is not new. In 1973 I.U. BRONSTEIN proved this for open Bc extensions [B 73], hence an EGS diagram and some easy observations as will be discussed in IV.4.3. finish the job. In 1977 another proof of this fact was given in [V 77], heavily depending on the techniques of $\mathfrak{F}$-topologies, whereas Bronstein's proof is "elementary". We give a slightly different proof, but, as in [V 77], the key is 2.20..

Let $\phi: \mathfrak{X} \to \mathfrak{Y}$ be a homomorphism of minimal ttgs, $x \in X$, $u \in J_x$ and let $F = \mathfrak{G}(\mathfrak{Y}, \phi(x))$ be the Ellis group of $\mathfrak{Y}$ with respect to $\phi(x)$ in uM.

We shall relate the sets

$$E(x)=E(x,\phi,u) \text{ and } Q_\phi[x]=\{x'\in\phi^{\leftarrow}\phi(x)\mid(x,x')\in Q_\phi\}.$$

with each other.

For $z\in X$ and $v\in J$ define the subset $L^v[z]$ of $\phi^{\leftarrow}\phi(uz)$ by

$$L^v[z]:=\bigcap\{v\circ\mathcal{U}\mid\mathcal{U}\in\mathcal{N}_{uz}^{\phi}\},$$

where $\mathcal{N}_{uz}^{\phi}$ is the $\mathfrak{T}(\mathcal{X},u)$-neighbourhood system of uz in $u\phi^{\leftarrow}\phi(z)$.

3.1. **REMARK.** *$E(x)=uL^u[x]=uL^v[x]$ for every $v\in J$.*

PROOF. Clearly, $E(x)\subseteq L^u[x]$; for $\mathrm{cl}_{\mathfrak{T}(\mathcal{X},u)}\mathcal{U}=u(u\circ\mathcal{U})\subseteq u\circ\mathcal{U}$ for every $\mathcal{U}\in\mathcal{N}_x^{\phi}$. So we have $E(x)=uE(x)\subseteq uL^u[x]$, and the equality $uL^u[x]=uL^v[x]$ follows from II.3.11.a.
Conversely, $uL^u[x]\subseteq u(u\circ\mathcal{U})$ for every $\mathcal{U}\in\mathcal{N}_x^{\phi}$, so $uL^u[x]\subseteq\mathrm{cl}_{\mathfrak{T}(\mathcal{X},u)}\mathcal{U}$ for every $\mathcal{U}\in\mathcal{N}_x^{\phi}$; hence $uL^u[x]\subseteq E(x)$. □

3.2. **LEMMA.** *Let $(x_1,x_2)\in R_\phi$ be an almost periodic point, and let U_1 and U_2 be open neighbourhoods of x_1 and x_2 in X. Then*

$$L^u[x_1]\times u\circ L^u[x_2]\subseteq\overline{T(U_1\times U_2\cap JR_\phi)}\subseteq\overline{T(U_1\times U_2\cap R_\phi)}.$$

PROOF. Let $v\in J$ be such that $(x_1,x_2)=(vx_1,vx_2)$. By 2.1.c, we can find an open set $V\subseteq T$ such that $v\in\mathrm{int}_{S_T}\mathrm{cl}_{S_T}V$, $V(v)=V$ and $Vx_2\subseteq U_2$. Define $\mathcal{U}_1\in\mathcal{N}_{vx_1}^{\phi}$ by

$$\mathcal{U}_1:=[U_1,V]\cap v\phi^{\leftarrow}\phi(x_1).$$

Choose $z\in\mathcal{U}_1$, then $z=t^{-1}z'$ for some $t\in V$ and $z'\in U_1$, while $\phi(z)=\phi(x_1)$. Hence $(z,x_2)\in JR_\phi$ and

$$(z,x_2)=t^{-1}(z',tx_2)\in t^{-1}(U_1\times Vx_2)\cap JR_\phi\subseteq T(U_1\times U_2)\cap JR_\phi,$$

so

$$\mathcal{U}_1\times\{x_2\}\subseteq T(U_1\times U_2)\cap JR_\phi=T(U_1\times U_2\cap JR_\phi).$$

If $x'\in\mathcal{U}_1$, then $x'=vx'$ and $(x',x_2)\in t_0(U_1\times U_2\cap JR_\phi)$ for some $t_0\in T$. By 2.1.c, there is an open set $V_1\subseteq T$ such that $v\in\mathrm{int}_{S_T}\mathrm{cl}_{S_T}V_1$, $V_1(v)=V_1$ and $V_1x'\subseteq t_0U_1$.

Define $\mathcal{U}_2 \in \mathfrak{N}^{\phi}_{vx_2}$ by

$$\mathcal{U}_2 := [t_0 U_2, V_1] \cap v\phi^{\leftarrow}\phi(x_2) .$$

As above, it follows that

$$\{x'\} \times \mathcal{U}_2 \subseteq Tt_0(U_1 \times U_2) \cap JR_{\phi} = T(U_1 \times U_2 \cap JR_{\phi}) .$$

But then $\{ux'\} \times u\mathcal{U}_2 \subseteq \overline{T(U_1 \times U_2 \cap JR_{\phi})}$ and so

$$\{ux'\} \times u \circ u\mathcal{U}_2 = u \circ (\{ux'\} \times u\mathcal{U}_2) \subseteq \overline{T(U_1 \times U_2 \cap JR_{\phi})} .$$

By 2.6.c, $u\mathcal{U}_2 \in \mathfrak{N}^{\phi}_{ux_2}$ so $L^u[x_2] \subseteq u \circ u\mathcal{U}_2$; hence

$$\{ux'\} \times L^u[x_2] \subseteq \overline{T(U_1 \times U_2 \cap JR_{\phi})} .$$

As $x' \in \mathcal{U}$ was arbitrary, we have

$$u\mathcal{U}_1 \times L^u[x_2] \subseteq \overline{T(U_1 \times U_2 \cap JR_{\phi})} ,$$

and so

$$u \circ u\mathcal{U}_1 \times u \circ L^u[x_2] = u \circ (u\mathcal{U}_1 \times L^u[x_2]) \subseteq \overline{T(U_1 \times U_2 \cap JR_{\phi})} .$$

Again by 2.6.c, $u\mathcal{U}_1 \in \mathfrak{N}^{\phi}_{ux_1}$; so $L^u[x_1] \subseteq u \circ u\mathcal{U}_1$; hence

$$L^u[x_1] \times u \circ L^u[x_2] \subseteq \overline{T(U_1 \times U_2 \cap JR_{\phi})} .$$

□

Remember the definition of $Q^*_{\phi} = \bigcap \{\overline{T(\alpha \cap JR_{\phi})} \mid \alpha \in \mathfrak{U}_X\}$, and note that $Q_{\phi} = Q^*_{\phi}$ if ϕ is a Bc extension (see the discussion just before I.4.4.).
The following notation will be used:

$$J_x \circ A := \bigcup \{v \circ A \mid v \in J_x\} ,$$

where A is a subset of a ttg. (For example: $J_x \circ u\phi^{\leftarrow}\phi(x)$, $J_x \circ Fx$, or $J_x \circ u\psi^{\leftarrow}(z)$.)
In chapter V. we present an extensive study of this "circle operation for sets".

3.3. **LEMMA.** *With notation as before, the following inclusions hold:*

a) $L^v[x] \times v \circ L^v[x] \subseteq Q^*_{\phi} \subseteq Q_{\phi}$ *for every* $v \in J$ *;*

b) $\bigcup \{L^w[x] \mid w \in J_x\} \subseteq \bigcap \{J_x \circ \mathcal{U} \mid \mathcal{U} \in \mathfrak{N}^{\phi}_x\} \subseteq Q^*_{\phi}[x] \subseteq Q_{\phi}[x]$*;*

c) $E(x) \subseteq uQ^*_{\phi}[x] \subseteq uQ_{\phi}[x]$*;*

d) $J_{\phi(x)}\mathrm{H}(F)x \subseteq Q^*_{\phi} \circ P_{\phi}[x] \subseteq E_{\phi}[x]$.

PROOF.

a) Note that the choice of $u \in J$ is not relevant in (the proof of) 3.2.. Let $v \in J$ and $\alpha \in \mathcal{U}_X$. As $(x,x) \in \alpha$ it follows from 3.2. that

$$L^v[x] \times v \circ L^v[x] \subseteq \overline{T(\alpha \cap JR_\phi)} .$$

Since $\alpha \in \mathcal{U}_X$ was arbitrary, we have

$$L^v[x] \times v \circ L^v[x] \subseteq \bigcap \{\overline{T(\alpha \cap JR_\phi)} \mid \alpha \in \mathcal{U}_X\} = Q_\phi^* \subseteq Q_\phi$$

for every $v \in J$.

b) As $L^w[x] \subseteq w \circ \boldsymbol{\mathcal{U}}$ for every $\boldsymbol{\mathcal{U}} \in \mathcal{N}_x^\phi$, we have

$$\bigcup \{L^w[x] \mid w \in J_x\} \subseteq J_x \circ \boldsymbol{\mathcal{U}}$$

for every $\boldsymbol{\mathcal{U}} \in \mathcal{N}_x^\phi$.

Let $\alpha \in \mathcal{U}_X$ and let $U \in \mathcal{V}_x$ be such that $U \times U \subseteq \alpha$. Let $V = V(u)$ be an open set in T with $u \in \mathrm{int}_{S_T}\mathrm{cl}_{S_T} V$ such that $Vx \subseteq U$. Define $\boldsymbol{\mathcal{V}} \in \mathcal{N}_x^\phi$ by $\boldsymbol{\mathcal{V}} := [U, V] \cap u\phi^{\leftarrow}\phi(x)$. Then

$$\{x\} \times \boldsymbol{\mathcal{V}} \subseteq T(U \times U \cap JR_\phi) ,$$

so

$$\{x\} \times (J_x \circ \boldsymbol{\mathcal{V}}) \subseteq \overline{T(U \times U \cap JR_\phi)} \subseteq \overline{T\alpha \cap JR_\phi} ;$$

hence

$$\{x\} \times \bigcap \{J_x \circ \boldsymbol{\mathcal{U}} \mid \boldsymbol{\mathcal{U}} \in \mathcal{N}_x^\phi\} \subseteq \{x\} \times (J_x \circ \boldsymbol{\mathcal{V}}) \subseteq \overline{T\alpha \cap JR_\phi} .$$

As α was arbitrary it follows that

$$\{x\} \times \bigcap \{J_x \circ \boldsymbol{\mathcal{U}} \mid \boldsymbol{\mathcal{U}} \in \mathcal{N}_x^\phi\} \subseteq \bigcap \{\overline{T\alpha \cap JR_\phi} \mid \alpha \in \mathcal{U}_X\} = Q_\phi^*$$

and so

$$\bigcup \{L^w[x] \mid w \in J_x\} \subseteq \bigcap \{J_x \circ \boldsymbol{\mathcal{U}} \mid \boldsymbol{\mathcal{U}} \in \mathcal{N}_x^\phi\} \subseteq Q_\phi^*[x] \subseteq Q_\phi[x] .$$

c) By 3.1., $E(x) = uL^u[x]$ so $E(x) \subseteq uQ_\phi^*[x] \subseteq uQ_\phi[x]$.

d) Let $x' \in J_{\phi(x)}\mathrm{H}(F)x$, say $x' = vpx$ for certain $v \in J_{\phi(x)}$ and $p \in \mathrm{H}(F)$. Then

$$px \in \mathrm{H}(F)x = E(x) \subseteq uQ_\phi^*[x] \subseteq Q_\phi^*[x] ,$$

so $(x,px) \in Q_\phi^*$ and $(vx,x') = (vx,vpx) \in Q_\phi^*$. As $(x,vx) \in P_\phi$ we have $(x,x') \in Q_\phi^* \circ P_\phi$; hence $J_{\phi(x)}\mathrm{H}(F)x \subseteq Q_\phi^* \circ P_\phi[x]$. □

3.4. **THEOREM.** *With notation as agreed upon earlier the following equations hold:* $E(x) = \mathrm{H}(F)x = uQ_\phi^*[x]$.
In particular, if ϕ *satisfies the Bronstein condition then* $E(x) = uQ_\phi[x]$.

PROOF. By 3.3.c and 2.15. we know already that $E(x) = \mathrm{H}(F)x \subseteq uQ_\phi^*[x]$. Let $x' \in uQ_\phi^*[x]$, i.e., $(x,x') = u(x,x') \in Q_\phi^*$. Applying I.4.4. there are nets $\{x'_i\}_i$ in $u\phi^{\leftarrow}\phi(x') = u\phi^{\leftarrow}\phi(x)$ and $\{t_i\}_i$ and $\{s_i\}_i$ in T such that

$$s_i(x,x'_i) \to (x,x'),\ t_i(x,x'_i) \to (x,x),\ s_i u \to u \text{ and } t_i u \to u .$$

Let $\mathcal{U}$ and $\mathcal{V}$ be $\mathfrak{T}(\mathfrak{X},u)$-neighbourhoods of x and x' in $u\phi^{\leftarrow}\phi(x)$, say

$$\mathcal{U} = [U,V] \cap u\phi^{\leftarrow}\phi(x) \in \mathfrak{N}_x^\phi \text{ and } \mathcal{V} = [U',V'] \cap u\phi^{\leftarrow}\phi(x) \in \mathfrak{N}_{x'}^\phi ,$$

where $U \in \mathcal{V}_x$, $U' \in \mathcal{V}_{x'}$ and $V = V(u)$, $V' = V'(u)$ are open sets in T with $u \in \mathrm{int}_{S_T}\mathrm{cl}_{S_T} V \cap \mathrm{int}_{S_T}\mathrm{cl}_{S_T} V'$. As $\mathrm{cl}_{S_T} V \cap M$ and $\mathrm{cl}_{S_T} V' \cap M$ are neighbourhoods of u in M (2.1.b), we can find an i_0 such that for every $i \geqslant i_0$ we have

$$s_i u \in \mathrm{int}_M(\mathrm{cl}_{S_T} V' \cap M) \text{ and } t_i u \in \mathrm{int}_M(\mathrm{cl}_{S_T} V \cap M) ,$$

so $s_i \in V'(u) = V'$ and $t_i \in V(u) = V$, while $s_i(x,x'_i) \in U \times U'$ and $t_i(x,x'_i) \in U \times U$. But then, for every $i \geqslant i_0$:

$$x'_i \in s_i^{-1}U' \subseteq (V')^{-1}.U' = [U',V'] \text{ and}$$

$$x'_i \in t_i^{-1}U \subseteq V^{-1}.U = [U,V]$$

so

$$x'_i \in [U,V] \cap u\phi^{\leftarrow}\phi(x) \cap [U',V'] \cap u\phi^{\leftarrow}\phi(x) = \mathcal{U} \cap \mathcal{V} .$$

Consequently, it follows that $x' \in \mathrm{cl}_{\mathfrak{T}(\mathfrak{X},u)}\mathcal{U}$ for every $\mathcal{U} \in \mathfrak{N}_x^\phi$, hence

$$x' \in \bigcap \{\mathrm{cl}_{\mathfrak{T}(\mathfrak{X},u)}\mathcal{U} \mid \mathcal{U} \in \mathfrak{N}_x^\phi\} = E(x) .$$

□

In order to characterize Q_ϕ^* we need the following observations with respect to the almost periodic points in R_ϕ (and $R_{\phi\psi}$).
Only for the following lemma (3.5.) and theorem (3.6.) we do not assume our choice (fixation) of ϕ and x.

3.5. **LEMMA.** *Let* $\phi:\mathcal{X}\to\mathcal{Z}$ *and* $\psi:\mathcal{Y}\to\mathcal{Z}$ *be homomorphisms of ttgs with* $\mathcal{X}$ *minimal and let* $u\in J$ *be arbitrary. Then* $(x,y)\in\overline{JR_{\phi\psi}}$ *iff* $y\in J_x\circ u\psi^{\leftarrow}\phi(x)$.

PROOF. Let $y\in J_x\circ u\psi^{\leftarrow}\phi(x)$, $v\in J_x$ with $y\in v\circ u\psi^{\leftarrow}\phi(x)$, and let $\{t_i\}_i$ be a net in T with $t_i\to v$. Then there are $y_i\in u\psi^{\leftarrow}\phi(x)$ such that $y=\lim t_iy_i$. As $(ux,y_i)\in JR_{\phi\psi}$ and $(x,y)=\lim t_i(ux,y_i)$, we have $(x,y)\in\overline{TJR_{\phi\psi}}=\overline{JR_{\phi\psi}}$.
Conversely, let $(x,y)\in\overline{JR_{\phi\psi}}$ and remember that by I.3.8. we have

$$\overline{JR_{\phi\psi}}=\overline{T(\{x\}\times v\psi^{\leftarrow}\phi(x))}\quad\text{for every } v\in J_x\ .$$

Let $\{t_i\}_i$ in T and $y_i\in v\psi^{\leftarrow}\phi(x)$ be such that $(x,y)=\lim t_i(x,y_i)$, and let $p\in M$ be the limit of $\{t_iv\}_i$ for a suitable subnet. Then

$$x=\lim t_ix=\lim t_ivx=(\lim t_iv)x=px$$

and

$$y=\lim t_iy_i=\lim t_ivy_i\in\lim t_ivv\psi^{\leftarrow}\phi(x)=p\circ v\psi^{\leftarrow}\phi(x)\,.$$

Let $w\in J$ be such that $p=wp$, then $w\in J_x$ and

$$p\circ v\psi^{\leftarrow}\phi(x)=wp\circ v\psi^{\leftarrow}\phi(x)=w\circ(up\circ v\psi^{\leftarrow}\phi(x))\,.$$

By II.3.11.b, we have

$$up\circ v\psi^{\leftarrow}\phi(x)=up\circ u\psi^{\leftarrow}\phi(x)=u\circ up\psi^{\leftarrow}\phi(x)\,.$$

As $px=x$, $up\psi^{\leftarrow}\phi(x)=u\psi^{\leftarrow}\phi(x)$; so $up\circ v\psi^{\leftarrow}\phi(x)=u\circ u\psi^{\leftarrow}\phi(x)$ and

$$\begin{aligned}y\in p\circ v\psi^{\leftarrow}\phi(x)&=w\circ(up\circ v\psi^{\leftarrow}\phi(x))=w\circ(u\circ u\psi^{\leftarrow}\phi(x))=\\&=w\circ u\psi^{\leftarrow}\phi(x)\subseteq J_x\circ u\psi^{\leftarrow}\phi(x)\,.\end{aligned}$$

□

3.6. **THEOREM.** *Let* $\phi:\mathcal{X}\to\mathcal{X}$ *and* $\psi:\mathcal{Y}\to\mathcal{Z}$ *be homomorphisms of ttgs, let* $\mathcal{X}$ *be minimal and* $u\in J$. *Then* ϕ *and* ψ *satisfy the generalized Bronstein condition iff* $\psi^{\leftarrow}(z)=J_x\circ u\psi^{\leftarrow}(z)$ *for every* $z\in Z$ *and every* $x\in\phi^{\leftarrow}(z)$. *In particular,* ϕ *satisfies the Bronstein condition iff* $\phi^{\leftarrow}\phi(x)=J_x\circ u\phi^{\leftarrow}\phi(x)$ *for every* $x\in X$.

PROOF. Follows immediately from 3.5.. □

The following theorem explicitely describes Q_ϕ^*, hence it describes Q_ϕ in case ϕ satisfies Bc.

3.7. **THEOREM.** *With notation as agreed upon earlier:*

$$Q_\phi^*[x] = J_{\phi(x)}\mathrm{H}(F)x \cap J_x \circ Fx = \bigcup \{L^w[x] \mid w \in J_x\} =$$

$$= \bigcap \{J_x \circ \mathcal{U} \mid \mathcal{U} \in \mathscr{N}_x^\phi\} .$$

PROOF. Clearly, $Q_\phi^* \subseteq \overline{JR_\phi}$; so by 3.5.,

$$Q_\phi^*[x] \subseteq J_x \circ u\phi^{\leftarrow}\phi(x) = J_x \circ Fx .$$

By 3.4., $Q_\phi^*[x] \subseteq J\mathrm{H}(F)x$, and so

$$Q_\phi^*[x] \subseteq J\mathrm{H}(F)x \cap \phi^{\leftarrow}\phi(x) = J_{\phi(x)}\mathrm{H}(F)x .$$

Consequently,

$$Q_\phi^*[x] \subseteq J_{\phi(x)}\mathrm{H}(F)x \cap J_x \circ Fx .$$

Next, observe that for $w \in J_x$, by II.3.11.b, $w \circ Fx = w \circ wFx$ and $\mathscr{N}_{wx}^\phi = \{w\mathcal{U} \mid \mathcal{U} \in \mathscr{N}_x^\phi\}$. So by 2.20., $J_{\phi(x)}\mathrm{H}(F)x \cap w \circ Fx \subseteq w \circ w\mathcal{U}$ for every $\mathcal{U} \in \mathscr{N}_x^\phi$. And as $w \circ w\mathcal{U} = w \circ \mathcal{U}$ (II.3.11.b), it follows that $J_{\phi(x)}\mathrm{H}(F)x \cap w \circ Fx \subseteq L^w[x]$; hence

$$J_{\phi(x)}\mathrm{H}(F)x \cap J_x \circ Fx \subseteq \bigcup \{L^w[x] \mid w \in J_x\} .$$

The proof is finished by applying 3.3.b. □

Define a subset S of R_ϕ by

$$S := \{(x_1, x_2) \in R_\phi \mid (ux_1, ux_2) \in Q_\phi^*\} .$$

Then clearly $Q_\phi^* \subseteq S \subseteq Q_\phi^* \circ P_\phi \subseteq E_\phi$.

3.8. **LEMMA.**

a) *S is an equivalence relation and $S[x] = J_{\phi(x)}\mathrm{H}(F)x$.*

b) *If $JQ_\phi \subseteq Q_\phi^*$ then $S = E_\phi = Q_\phi^* \circ P_\phi$. In particular, if ϕ is a* Bc *extension then $Q_\phi[x] = J_{\phi(x)}\mathrm{H}(F)x$.*

PROOF.

a) Clearly, $x' \in S[x]$ iff $ux' \in uQ_\phi^*[x] = E(x) = \mathrm{H}(F)x$, and so we have $S[x] = J_{\phi(x)}\mathrm{H}(F)x$.

Let (x_1, x_2) and $(x_2, x_3) \in S$ and let $a \in uM$ be such that $ax_2 = x$. Then $(ax_1, x) = a(x_1, x_2) \in Q_\phi^*$ and $(x, ax_3) \in Q_\phi^*$, so $ax_1 \in E(x)$ and $ax_3 \in E(x)$. By 2.15.c, $ax_3 \in E(ax_3) = E(ax_1)$; so, applying 3.4. to ax_1 in stead of x , it follows that $ax_3 \in uQ_\phi^*[ax_1]$. But then $u(x_1, x_3) \in Q_\phi^*$ and so $(x_1, x_3) \in S$.

b) If $JQ_\phi \subseteq Q_\phi^*$ then $Q_\phi \subseteq S$. By a, S is an equivalence relation, so

$$Q_\phi \circ Q_\phi \subseteq S \circ S = S \subseteq Q_\phi^* \circ P_\phi \subseteq Q_\phi \circ Q_\phi \,.$$

As Q_ϕ is closed and T-invariant, $S = Q_\phi \circ Q_\phi$ is closed and T-invariant. Since $Q_\phi \subseteq S \subseteq E_\phi$ it follows that $S = E_\phi = Q_\phi^* \circ P_\phi$. ☐

3.9. **THEOREM.** *If* ϕ *is a* Bc *extension, then* $E_\phi = Q_\phi$.

PROOF. If ϕ is a Bc extension, then $Q_\phi = Q_\phi^*$. By 3.8., we know $E_\phi[x] = S[x] = J_{\phi(x)}\mathrm{H}(F)x$, but also $E_\phi[x] \subseteq \phi^{\leftarrow}\phi(x) = J_x \circ Fx$ (3.6.). So $E_\phi[x] \subseteq J_{\phi(x)}\mathrm{H}(F)x \cap J_x \circ Fx$; hence by 3.7., $E_\phi[x] \subseteq Q_\phi^*[x] = Q_\phi[x]$. As the choice of x in the beginning of this section was arbitrary, it follows that $E_\phi = Q_\phi$. ☐

3.10. **REMARK.**

a) *If* ϕ *is a* Bc *extension, then* $E_\phi[x] \subseteq J_{x'} \circ \mathcal{U}$ *for every* $\mathcal{U} \in \mathfrak{N}_x^\phi$ *and every* $x' \in \phi^{\leftarrow}\phi(x)$.

b) *If* ϕ *is a* RIC *extension, then* $E_\phi[x] \subseteq u \circ \mathcal{U}$ *for every* $\mathcal{U} \in \mathfrak{N}_x^\phi$.

PROOF. By 3.8., $E_\phi[x] = J_{\phi(x)}\mathrm{H}(F)x$.

a) Since, by 3.6., $\phi^{\leftarrow}\phi(x) = J_{x'} \circ Fx$ it follows that

$$E_\phi[x] \subseteq J_{\phi(x)}\mathrm{H}(F)x \cap J_{x'} \circ Fx$$

and so by 2.20., $E_\phi[x] \subseteq J_{x'} \circ \mathcal{U}$ for every $\mathcal{U} \in \mathfrak{N}_x^\phi$ (compare the proof of 3.7.).

b) If ϕ is a RIC extension, then $\phi^{\leftarrow}\phi(x) = u \circ Fx$ (cf. 1.3.); hence

$$E_\phi[x] \subseteq J_{\phi(x)}\mathrm{H}(F)x \cap u \circ Fx$$

and so by 2.20., $E_\phi[x] \subseteq u \circ \mathcal{U}$ for every $\mathcal{U} \in \mathfrak{N}_x^\phi$. ☐

3.11. Now that we exactly know what the equicontinuous structure relation looks like for Bc extensions, it is not difficult to describe the maximal almost periodic factors of those extensions.

So let $\phi: \mathfrak{X} \to \mathfrak{Y}$ be a Bc extension and let $\kappa: \mathfrak{X} \to \mathfrak{X}/E_\phi$ be the quotient map, and $\theta: \mathfrak{X}/E_\phi \to \mathfrak{Y}$ the extension of $\mathfrak{Y}$ defined by E_ϕ . Let $H = \mathfrak{G}(\mathfrak{X}, x_0)$ and $F = \mathfrak{G}(\mathfrak{Y}, \phi(x_0))$ be the Ellis groups of $\mathfrak{X}$ and $\mathfrak{Y}$ with respect to $x_0 = ux_0$ and $\phi(x_0)$ in uM . Then

NOTE.

a) *The Ellis group* $\mathfrak{G}(\mathfrak{X}/E_\phi, \kappa(x_0))$ *of* $\mathfrak{X}/E_\phi$ *with respect to* $\kappa(x_0)$ *in* uM *is* $\mathrm{H}(F)H$ *;*

b) $M_{\kappa(x_0)} = J_{\phi(x_0)}\mathrm{H}(F)H$.

PROOF.

a) Let $a \in \mathrm{H}(F)H$, say $a = fh$ for some $f \in \mathrm{H}(F)$, $h \in H$. Then

$$a\kappa(x_0) = \kappa(ax_0) = \kappa(fhx_0) = \kappa(fx_0) .$$

By 3.3.d, we have $fx_0 \in E_\phi[x_0]$ so $\kappa(fx_0) = \kappa(x_0)$, which shows that $a \in \mathfrak{G}(\mathfrak{X}/E_\phi, \kappa(x_0))$.
Conversely, let $a \in \mathfrak{G}(\mathfrak{X}/E_\phi, \kappa(x_0))$, so $a\kappa(x_0) = \kappa(x_0)$. Then by 3.9. and 3.4., we have $ax_0 \in E_\phi[x_0] = Q_\phi^*[x_0]$, hence by 3.4., $ax_0 \in \mathrm{H}(F)x_0$, say $ax_0 = fx_0$ for $f \in \mathrm{H}(F)$. Hence $f^{-1}a \in H$ and so $a \in fH \subseteq \mathrm{H}(F)H$.

b) As $\theta: \mathfrak{X}/E_\phi \to \mathfrak{Y}$ is almost periodic, it is distal and so $\kappa(x_0)$ is a θ-distal point; hence by I.2.10., $J_{\kappa(x_0)} = J_{\theta(\kappa(x_0))} = J_{\phi(x_0)}$. Clearly,

$$M_{\kappa(x_0)} = J_{\kappa(x_0)} . \mathfrak{G}(\mathfrak{X}/E_\phi, \kappa(x_0)) = J_{\phi(x_0)}\mathrm{H}(F)H .$$

□

The easy proof of the following remark will be omitted (for "if" use I.2.13.).

3.12. **REMARK.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *be a* Bc *extension. Let* $x_0 \in X$, $u \in J_{x_0}$, $y_0 = \phi(x_0)$ *and let* $H = \mathfrak{G}(\mathfrak{X}, x_0)$ *and* $F = \mathfrak{G}(\mathfrak{Y}, y_0)$ *be the Ellis groups of* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *with respect to* x_0 *and* y_0 *in* uM . *Then* $E_\phi = R_\phi$ *iff* $\mathrm{H}(F)H = F$. □

More details on the equicontinuous structure relation for Bc extensions will be given in chapter VIII..

The final observation in this section concerns the Ellis group of the maximal almost periodic factor of a homomorphism $\phi: \mathfrak{X} \to \mathfrak{Y}$ of minimal ttgs that does not necessarily satisfy the Bronstein condition.

3.13. **THEOREM.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs,* $x_0 \in X$ *and let* H *and* F *be the Ellis groups of* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *with respect to* x_0 *and* $\phi(x_0)$. *Then* $K := \mathfrak{G}(\mathfrak{X}/E_\phi, E_\phi[x_0]) = HA_{\mathfrak{Y}}$, *where* $A_{\mathfrak{Y}}$ *is the Ellis group of the maximal almost periodic extension* $\alpha_{\mathfrak{Y}}: \mathfrak{A}(\mathfrak{Y}) \to \mathfrak{Y}$ *with respect to some* $z \in \alpha_{\mathfrak{Y}}^{\leftarrow}(\phi(x_0))$. *In particular, we have* $\mathfrak{G}(\mathfrak{X}/E_{\mathfrak{X}}, E_{\mathfrak{X}}[x_0]) = HE$, *where* E *is the Ellis group of the universal uniformly almost periodic minimal ttg* $\mathfrak{E}$.

PROOF. First observe that $\alpha_{\mathcal{Y}}: \mathfrak{E}(\mathcal{Y}) \to \mathcal{Y}$ is a regular extension (cf. I.2.17.). So $A_{\mathcal{Y}}$ is a normal subgroup of F and $HA_{\mathcal{Y}}$ is an $\mathfrak{F}(\mathcal{M}, u)$-closed subgroup of F. As the induced map $\theta: \mathcal{X}/E_{\phi} \to \mathcal{Y}$ is a factor of both $\alpha_{\mathcal{Y}}$ and ϕ, it follows easily that $HA_{\mathcal{Y}} \subseteq K$. If there exists an almost periodic extension $\mathcal{W}$ of $\mathcal{Y}$ between $\mathcal{X}$ and $\mathcal{Y}$ with Ellis group $HA_{\mathcal{Y}}$, then the theorem will be proven.
Consider the following diagram of homomorphisms of minimal ttgs:

$$\begin{tikzcd}
& \mathfrak{A}(A_{\mathcal{Y}}) \arrow[dl, "\pi_1"'] \arrow[rr, "\alpha"] \arrow[dd, "\xi"] & & \mathfrak{A}(HA_{\mathcal{Y}}) \arrow[dr, "\beta"] \arrow[ddll, "\nu"'] & \\
\mathfrak{E}(\mathcal{Y}) \arrow[dr, "\alpha_{\mathcal{Y}}"'] & & & & \mathfrak{A}(K) \arrow[dl, "\pi_2"] \\
& \mathcal{Y} & & \mathcal{X}/E_{\phi} \arrow[ll, "\theta"] &
\end{tikzcd}$$

Here π_1 and π_2 are the universal proximal extensions of $\mathfrak{E}(\mathcal{Y})$ and $\mathcal{X}/E_{\phi}$, and α and β are the obvious RIC extensions (1.15.). Define $\xi := \alpha_{\mathcal{Y}} \circ \pi_1$ and $\nu := \theta \circ \pi_2 \circ \beta$, and note that $\xi = \nu \circ \alpha$. Clearly, $E_{\xi} = R_{\pi_1} = P_{\xi}$, so from I.4.3. it follows that

$$E_{\nu} = \alpha \times \alpha[E_{\xi}] = \alpha \times \alpha[P_{\xi}] = P_{\nu}.$$

This shows that $\nu = \lambda \circ \mu$, where $\mu: \mathfrak{A}(HA_{\mathcal{Y}}) \to \mathfrak{A}(HA_{\mathcal{Y}})/E_{\nu}$ is proximal and $\lambda: \mathfrak{A}(HA_{\mathcal{Y}})/E_{\nu} \to \mathcal{Y}$ is almost periodic. From I.4.1. and the following diagram it follows that ϕ factorizes over $\mathfrak{A}(HA_{\mathcal{Y}})/E_{\nu}$, which proves the theorem (here γ is the obvious RIC extension (1.15.)).

$$\begin{tikzcd}
\mathfrak{A}(H) \arrow[r, "\gamma"] \arrow[d, "\pi_3"'] & \mathfrak{A}(HA_{\mathcal{Y}}) \arrow[r, "\mu"] & \mathfrak{A}(HA_{\mathcal{Y}})/E_{\nu} \arrow[d, "\lambda"] \\
\mathcal{X} \arrow[rr, "\phi"'] & & \mathcal{Y}
\end{tikzcd}$$

□

III.4. PI EXTENSIONS

One of the ways to tackle the problem of determining the structure of a minimal ttg is to build that ttg with elements we (pretend to) know. From this point of view H. FURSTENBERG and W.A. VEECH tried to understand distal and point distal ttgs respectively. Their method was generalized in [EGS 75] to the theory of PI extensions as will briefly be exposed in this section.

4.1. A homomorphism $\phi:\mathfrak{X}\to\mathfrak{Y}$ of minimal ttgs is a *strictly*-PI *extension* if there is an ordinal ν and a tower for ϕ of height ν, (i.e., an inverse system $\{\phi_\alpha^\beta \mid \alpha\leqslant\beta<\nu\}$ of homomorphisms $\phi_\alpha^\beta:\mathfrak{X}_\beta\to\mathfrak{X}_\alpha$ of minimal ttgs) such that:

a) $\mathfrak{X}_0=\mathfrak{Y}$, $\mathfrak{X}_\nu=\mathfrak{X}$ and $\phi=\operatorname{inv\,lim}\{\phi_\alpha^\beta \mid \alpha\leqslant\beta<\nu\}$;
b) for every $\alpha<\nu$ the map $\phi_\alpha^{\alpha+1}$ is either proximal or almost periodic.

The homomorphism ϕ is called a PI- *extension* if there is a strictly-PI extension $\psi:\mathfrak{Z}\to\mathfrak{Y}$ such that ϕ is a factor of ψ ; i.e., $\psi=\phi\circ\theta$ for some homomorphism $\theta:\mathfrak{Z}\to\mathfrak{X}$ of minimal ttgs.

4.2. **EXAMPLE.** *Let* A *and* F *be* $\mathfrak{T}(\mathfrak{M},u)$*-closed subgroups of* $G=uM$ *with* $A\subseteq F$ *. Then the homomorphism* $\phi:\mathfrak{A}(F_\infty A)\to\mathfrak{A}(F)$*, defined by* $\phi(p\circ F_\infty A)=p\circ F$ (cf. 1.15.) *is a strictly*-PI *extension. (Remember that* $\mathfrak{A}(K):=\mathfrak{W}\mathfrak{F}(u\circ K,\mathfrak{M})$ *for every subgroup* K *of* G *.)*

PROOF. We shall prove that $\phi_\alpha:\mathfrak{A}(\mathrm{H}_\alpha(F)A)\to\mathfrak{A}(F)$ is strictly-PI for every ordinal $\alpha\geqslant 0$, where $\mathrm{H}_0(F):=F$.
For $\alpha=0$ we have $\mathrm{H}_\alpha(F)A=FA=F$, and clearly $\phi_0:\mathfrak{A}(F)\to\mathfrak{A}(F)$ is a strictly-PI extension.
Suppose that $\phi_\beta:\mathfrak{A}(\mathrm{H}_\beta(F)A)\to\mathfrak{A}(F)$ is a strictly-PI extension. As $F_\infty A$ is a group and $F_\infty A\subseteq\mathrm{H}_\beta(F)A$ it follows from 1.15. and 1.13.a that the map $\psi:\mathfrak{A}(F_\infty A)\to\mathfrak{A}(\mathrm{H}_\beta(F)A)$ is a well defined RIC extension. Let $\kappa:\mathfrak{A}(F_\infty A)\to\mathfrak{A}(F_\infty A)/E_\psi$, then by 3.11.:

$$K:=\mathfrak{G}(\mathfrak{A}(F_\infty A)/E_\psi,\kappa(u\circ F_\infty A))=\mathrm{H}(\mathrm{H}_\beta(F)A)F_\infty A\ ,$$

and as $F_\infty A=AF_\infty$ it follows from 2.13.a that

$$K=\mathrm{H}(\mathrm{H}_\beta(F)A)AF_\infty=\mathrm{H}(\mathrm{H}_\beta(F))AF_\infty=\mathrm{H}_{\beta+1}(F)A\ .$$

By 1.13.b, $\mathfrak{A}(\mathrm{H}_{\beta+1}(F)A) \to \mathfrak{A}(F_\infty A)/E_\psi$ is a proximal extension, so the map $\theta : \mathfrak{A}(\mathrm{H}_{\beta+1}(F)A) \to \mathfrak{A}(\mathrm{H}_\beta(F)A)$ is strictly-PI. Hence $\phi_{\beta+1} = \phi_\beta \circ \theta$ is strictly-PI.
As an inverse limit of strictly-PI extensions is strictly-PI, the example is proven after the observation that $\phi = \operatorname{inv\,lim} \phi_\beta$. □

4.3. **THEOREM.** *Let* $\mathfrak{Y}$ *be a minimal ttg and let* $y_0 \in Y$, $u \in J_{y_0}$. *Then the map* $\phi : \mathfrak{A}(F_\infty) \to \mathfrak{Y}$ *defined by* $\phi(p \circ F_\infty) = py_0$ *is the universal* PI *extension of* $\mathfrak{Y}$ *; i.e., if* $\eta : \mathfrak{X} \to \mathfrak{Y}$ *is a* PI *extension and* $x_0 \in u\eta^{\leftarrow}(y_0)$ *then there is a homomorphism* $\nu : \mathfrak{A}(F_\infty) \to \mathfrak{X}$ *with* $\nu(u \circ F_\infty) = x_0$ *and* $\eta \circ \nu = \phi$. *Here* $F = \mathfrak{G}(\mathfrak{Y}, y_0)$ *is the Ellis group of* $\mathfrak{Y}$ *with respect to* y_0 *in* G .

PROOF. By 4.2. with $A = \{u\}$, it follows that $\mathfrak{A}(F_\infty) \to \mathfrak{A}(F)$ is strictly-PI and as $\mathfrak{A}(F) \to \mathfrak{Y}$ defined by $p \circ F \mapsto py_0$ is proximal by 1.13.b, it is clear that $\phi : \mathfrak{A}(F_\infty) \to \mathfrak{Y}$ defined by $p \circ F_\infty \mapsto p \circ F \mapsto py_0$ is strictly-PI.
We shall show that every strictly-PI extension of $\mathfrak{Y}$ is a factor of ϕ (no matter what base points are chosen). Note that it suffices to prove that for an arbitrary factor $\theta : \mathfrak{Z} \to \mathfrak{Y}$ of ϕ the map $\theta \circ \xi : \mathfrak{W} \to \mathfrak{Y}$ is a factor of ϕ for every proximal or almost periodic extension $\xi : \mathfrak{W} \to \mathfrak{Z}$ (proceed by induction).
Consider the following diagram of homomorphisms of minimal ttgs:

$$\begin{tikzcd}
\mathfrak{A}(F_\infty) \arrow[r, "\phi"] \arrow[d, dashed] \arrow[dr, "\kappa"] & \mathfrak{Y} \\
\mathfrak{W} \arrow[r, "\xi"'] & \mathfrak{Z} \arrow[u, "\theta"']
\end{tikzcd}$$

Let $z_0 \in u\theta^{\leftarrow}(y_0)$ and $K := \mathfrak{G}(\mathfrak{Z}, z_0)$. For some $a \in F$, $\kappa(a \circ F_\infty) = z_0$; and so, by I.2.11., $\mathfrak{G}(\mathfrak{A}(F_\infty), a \circ F_\infty) \subseteq K \subseteq F$. As F_∞ is a normal subgroup of F , we have that $F_\infty = \mathfrak{G}(\mathfrak{A}(F_\infty), a \circ F_\infty)$.
First suppose that ξ is proximal. Let $w_0 \in u\xi^{\leftarrow}(z_0)$; then by I.2.13. $K = \mathfrak{G}(\mathfrak{W}, w_0)$. But then by 1.15. and 1.13.b, there is a map

$$p \circ F_\infty (\mapsto p \circ K) \mapsto pw_0 : \mathfrak{A}(F_\infty) \to \mathfrak{W}$$

and so $\theta \circ \xi$ is a factor of ϕ .
Suppose that ξ is almost periodic and let $w_0 \in \xi^{\leftarrow}(z_0)$, then $w_0 = uw_0$. As ξ is RIC and $\mathfrak{W} = \mathfrak{W}/E_\xi$ it follows from 3.11.a that $\mathrm{H}(K) \subseteq \mathfrak{G}(\mathfrak{W}, w_0)$.

Since $F_\infty \subseteq K$ and $F_\infty = \mathrm{H}(F_\infty)$ we have

$$F_\infty = \mathrm{H}(F_\infty) \subseteq \mathrm{H}(K) \subseteq \mathfrak{G}(\mathfrak{W}, w_0)\,.$$

By 1.15., there is a homomorphism $\mathfrak{A}(F_\infty) \to \mathfrak{A}(\mathfrak{G}(\mathfrak{W}, w_0))$; hence $p \circ F_\infty \mapsto pw_0$ is well defined and $\theta \circ \xi$ is a factor of ϕ .
This shows that every strictly-PI extension of $\mathfrak{Y}$ is a factor of ϕ . But then every PI extension of $\mathfrak{Y}$ is a factor of ϕ . □

4.4. **THEOREM.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs. Let* $x_0 \in X$, $u \in J_{x_0}$ *and* $y_0 = \phi(x_0) \in Y$ *and let* $H = \mathfrak{G}(\mathfrak{X}, x_0)$ *and* $F = \mathfrak{G}(\mathfrak{Y}, y_0)$ *be the Ellis groups of* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *with respect to* x_0 *and* y_0 *in* G . *Then the following statements are equivalent:*

a) ϕ *is a factor of a strictly-*PI *extension under a proximal map; i.e., there is a strictly-*PI *extension* ψ *and a proximal extension* θ *with* $\psi = \phi \circ \theta$ *;*

b) ϕ *is a* PI *extension;*

c) $F_\infty \subseteq H$ *(equivalently:* $F_\infty = H_\infty$ *or* $E_\infty(x_0) = \{x_0\}$ *).*

PROOF.

a ⇒ b Trivial.

b ⇒ c Let ϕ be a PI extension. Then by 4.3., $\mathfrak{X}$ is a factor of $\mathfrak{A}(F_\infty)$, say $\xi: \mathfrak{A}(F_\infty) \to \mathfrak{X}$, and $\xi(u \circ F_\infty) = x_0$. By I.2.11., it follows that $F_\infty \subseteq H$.

The proof of the equivalence of $F_\infty \subseteq H$, $F_\infty = H_\infty$ and $E_\infty(x_0) = \{x_0\}$ is left as an exercise for the reader.

c ⇒ a If $F_\infty \subseteq H$, then $F_\infty H = H$. Hence by 4.2., the map

$$p \circ H \mapsto p \circ F : \mathfrak{A}(H) \to \mathfrak{A}(F)$$

is a strictly-PI extension. As the homomorphism $p \circ H \mapsto px_0 : \mathfrak{A}(H) \to \mathfrak{X}$ is proximal, the theorem is proven. □

4.5. **COROLLARY.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs. Then the property for* ϕ *of being a* PI *extension does not depend on the topology of* T *; i.e.,* $\phi: <T_d, X> \to <T_d, Y>$ *is a* PI *extension iff* $\phi: <T, X> \to <T, Y>$ *is a* PI *extension.*

PROOF. By 4.4. ϕ is a PI extension iff $E_\infty(x_0) = x_0$. As $E_\infty(x_0)$ is calculated in $(uX, \mathfrak{F}(\mathfrak{X}, u))$ and as the $\mathfrak{F}(\mathfrak{X}, u)$-topology does not depend on the topology of T (2.5.) the corollary follows. □

We shall now describe the construction of the "canonical PI tower" for a homomorphism $\phi: \mathcal{X} \to \mathcal{Y}$ of minimal ttgs. For full details and proofs see for example [G 76], [V 77] and [VW ?].

4.6. Let $\phi: \mathcal{X} \to \mathcal{Y}$ be a homomorphism of minimal ttgs, let $x_0 \in X$, $u \in J_{x_0}$ and $y_0 = \phi(x_0)$ and let $H = \mathfrak{G}(\mathcal{X}, x_0)$ and $F = \mathfrak{G}(\mathcal{Y}, y_0)$.
Define $\mathcal{X}_0 := \mathcal{X}$, $\mathcal{Y}_0 := \mathcal{Y}$ and $\phi_0 := \phi$, and note that we have $\mathfrak{G}(\mathcal{Y}_0, y_0) = \mathrm{H}_0(F)H (= F)$.
Let α be an ordinal and let $\phi_\alpha: \mathcal{X}_\alpha \to \mathcal{Y}_\alpha$, $x_\alpha \in uX_\alpha$, $y_\alpha = \phi_\alpha(x_\alpha)$ and the homomorphisms $\sigma'_\alpha: \mathcal{X}_\alpha \to \mathcal{X}$, $\tau'_\alpha: \mathcal{Y}_\alpha \to \mathcal{Y}$ be defined for α, such that σ'_α is proximal and $\sigma'_\alpha(x_\alpha) = x_0$, τ'_α is strictly-PI and $\tau'_\alpha(y_\alpha) = y_0$, while $\mathfrak{G}(\mathcal{Y}_\alpha, y_\alpha) = \mathrm{H}_\alpha(F)H$. Construct EGS($\phi_\alpha$), let $y'_\alpha := u \circ u\phi_\alpha^{\leftarrow}(y_\alpha)$ and $x'_\alpha := (x_\alpha, y'_\alpha)$.

$$\begin{tikzcd}
 & \mathcal{X}_{\alpha+1} = \mathcal{X}'_\alpha \arrow[r, "\sigma_\alpha"] \arrow[dl, "\phi_{\alpha+1}"'] \arrow[d, "\phi'_\alpha"] & \mathcal{X}_\alpha \arrow[d, "\phi_\alpha"] \\
\mathcal{X}'_\alpha/E_{\phi'_\alpha} = \mathcal{Y}_{\alpha+1} \arrow[r, "\xi_{\alpha+1}"'] & \mathcal{Y}'_\alpha \arrow[r, "\tau_\alpha"'] & \mathcal{Y}_\alpha
\end{tikzcd}$$

Let $\xi_{\alpha+1}: \mathcal{X}'_\alpha/E_{\phi'_\alpha} \to \mathcal{Y}'_\alpha$ be the maximal almost periodic factor of the RIC extension ϕ'_α. Then define $(\mathcal{X}_{\alpha+1}, x_{\alpha+1}) := (\mathcal{X}'_\alpha, x'_\alpha)$, $\mathcal{Y}_{\alpha+1} := \mathcal{X}'_\alpha/E_{\phi'_\alpha}$ and $\phi_{\alpha+1}: \mathcal{X}_{\alpha+1} \to \mathcal{Y}_{\alpha+1}$ as the quotient map. Furthermore let $y_{\alpha+1} := \phi_{\alpha+1}(x_{\alpha+1})$, $\sigma'_{\alpha+1} := \sigma'_\alpha \circ \sigma_\alpha$ and $\tau'_{\alpha+1} := \tau'_\alpha \circ \tau_\alpha \circ \xi_{\alpha+1}$. Then $\sigma'_{\alpha+1}$ is proximal, $\tau'_{\alpha+1}$ is strictly-PI and, by 3.11., we have $\mathfrak{G}(\mathcal{Y}_{\alpha+1}, y_{\alpha+1}) = \mathrm{H}(\mathrm{H}_\alpha(F)H)H$; hence by 2.13.a,

$$\mathfrak{G}(\mathcal{Y}_{\alpha+1}, y_{\alpha+1}) = \mathrm{H}(\mathrm{H}_\alpha(F))H = \mathrm{H}_{\alpha+1}(F)H .$$

If α is a limit ordinal such that $\phi_\beta: \mathcal{X}_\beta \to \mathcal{Y}_\beta$ is defined for every $\beta < \alpha$ as described above, then define $x_\alpha := (x_\beta)_{\beta<\alpha} \in \Pi\{X_\beta \mid \beta < \alpha\}$, $X_\alpha := \overline{T(x_\alpha)}$ and $y_\alpha := (y_\beta)_{\beta<\alpha} \in \Pi\{Y_\beta \mid \beta < \alpha\}$, $Y_\alpha := \overline{T(y_\alpha)}$. Then clearly $\mathcal{X}_\alpha$ and $\mathcal{Y}_\alpha$ are minimal ttgs, and $\mathfrak{G}(\mathcal{Y}_\alpha, y_\alpha) = \mathrm{H}_\alpha(F)H$. Define $\phi_\alpha: \mathcal{X}_\alpha \to \mathcal{Y}_\alpha$ as the induced ambit morphism, and let $\sigma'_\alpha := \mathrm{inv\,lim}\{\sigma'_\beta \mid \beta < \alpha\}$ and $\tau'_\alpha := \mathrm{inv\,lim}\{\tau'_\beta \mid \beta < \alpha\}$. Then σ'_α is proximal and τ'_α is strictly-PI.

4.7. In this construction there are two possibilities

A For some ordinal ν : $\mathrm{H}_\nu(F)H = H$.
Then ϕ_ν is proximal, the construction stops (the tower ends) at height ν

and the map $\psi := \tau'_\nu \circ \phi_\nu$ is a strictly-PI extension of which ϕ is a factor. This shows that ϕ is a PI extension.
Note that if ϕ is a PI extension, $F_\infty \subseteq H$, so there does exist an ordinal ν with $\mathrm{H}_\nu(F) \subseteq H$ (and so $\mathrm{H}_\nu(F)H = H$).

B $F_\infty H \neq H$.
Then the tower ends at height $\infty + 1$. For $\phi'_\infty : \mathcal{X}'_\infty \to \mathcal{Y}'_\infty$ does not admit a nontrivial almost periodic factor, which follows by 3.12. from the observation that $\mathfrak{S}(\mathcal{Y}'_\infty, y'_\infty) = F_\infty H$ and that $\mathrm{H}(F_\infty H)H = \mathrm{H}(F_\infty)H = F_\infty H$.
This leads to the situation depicted in the following diagram:

$$\begin{tikzcd}
\mathcal{X}'_\infty \arrow[r, "\sigma'_\infty"] \arrow[d, "\phi'_\infty"'] & \mathcal{X} \arrow[d, "\phi"] \\
\mathcal{Y}'_\infty \arrow[r, "\tau'_\infty"'] & \mathcal{Y}
\end{tikzcd}$$

where ϕ'_∞ is a RIC extension, but $E_{\phi'_\infty} = R_{\phi'_\infty}$, σ'_∞ is proximal and τ'_∞ is strictly-PI.
One could paraphrase this as follows: Every homomorphism ϕ is a PI extension modulo some junk in ϕ'_∞.

Much work is done in understanding the "junk" in ϕ'_∞ (e.g. [E 73], [EGS 75], [M 76.1] and [V 77]). For instance it turned out that ϕ'_∞ is a weakly mixing extension (see VII.3.23.) and that ϕ'_∞ is an isomorphism in case $P_\phi[x]$ is countable for some $x \in X$ ([G 76] for X is metric; [MN 80] in the general absolute case; open in the general relativized case).

4.8. NOTE. *If $\phi : \mathcal{X} \to \mathcal{Y}$ is a homomorphism of metric minimal ttgs then the height of the* PI-*tower for ϕ is countable.*

PROOF. By II.1.1.b, we know that every ttg in the PI-tower is metric. Consider $\mathcal{Y}_\infty$, then every map $\tau_\infty^\alpha : \mathcal{Y}_\infty \to \mathcal{Y}_\alpha$ defines a closed equivalence relation $R_{\tau_\infty^\alpha}$ on Y_∞. Clearly, the collection $\{R_{\tau_\infty^\alpha} \mid \alpha < \infty\}$ is a linearly ordered (by inclusion) collection of closed subsets. It is not difficult to see that there can be at most $c(Y_\infty \times Y_\infty)$ different subsets in that collection, where $c(Y_\infty \times Y_\infty)$ is the cellularity number of $Y_\infty \times Y_\infty$. As $c(Y_\infty \times Y_\infty)$ is smaller than $d(Y_\infty \times Y_\infty)$, the density number of $Y_\infty \times Y_\infty$, and as, by metrizability, $d(Y_\infty \times Y_\infty) \leqslant \aleph_0$, the remark follows. $\square$

We shall end this section with a remark (the proof of which is omitted cf. [VW ?]) that states that the canonical tower as presented here is just the tower presented in [V 77].

4.9. **REMARK.** *With notation as in 4.6.. For every* $\alpha \geqslant 0$ *we have*

$$Y_\alpha \cong QF(u \circ E_\alpha(x_0), \mathcal{X}) \text{ and } X_\alpha \cong \{(x,y') \mid x \in y' \in Y_\alpha\},$$

where $E_0(x_0) := u\phi^{\leftarrow}(y_o)$. *Then* $\sigma'_\alpha : \mathcal{X}_\alpha \to \mathcal{X}$ *and* $\phi_\alpha : \mathcal{X}_\alpha \to \mathcal{Y}_\alpha$ *are the projections and* $\tau'_\alpha : \mathcal{Y}_\alpha \to \mathcal{Y}$ *is defined as* $\tau'_\alpha := 2^\phi|_{Y_\alpha}$. □

III.5. REMARKS

The notion of RIC extension is introduced in [EGS 75] as an extension satisfying the property 1.3.b. In that paper the EGS(ϕ) diagram for ϕ is studied in a way leading towards the canonical PI tower for ϕ (4.6. and 4.7.). A similar approach can be found in [MW 74].
The relation Q^*_ϕ occurs in [B 75/79] and plays a major role in [B 75/79] section 3.13.; note that the notation differs: our Q^*_ϕ is denoted there by $Q(R\mathfrak{F})$.

With respect to the question whether or not the Bronstein condition implies relative incontractibility, the following observation can be made.

5.1. **REMARK.** *Let* $\phi : \mathcal{X} \to \mathcal{Y}$ *be a* Bc *extension of minimal ttgs.*
If $\mathcal{X} \cong \mathfrak{A}(F)$ *for some* $\mathfrak{F}(\mathcal{M}, u)$*-closed subgroup* F *of* G *then* ϕ *is a* RIC *extension.*

PROOF. Construct EGS(ϕ), then $\phi \circ \sigma = \tau \circ \phi'$ (notation as in the discussion just before 1.11.). As $\mathcal{X}$ does not admit nontrivial proximal extensions (1.13.b), σ is an isomorphism and so $\phi \circ \sigma$ is a Bc extension. But then τ, as a factor of $\phi \circ \sigma$, is a Bc extension; hence, by I.3.5.b, τ is an isomorphism. This shows that ϕ is a RIC extension. (Note, that ϕ is open and also that $\mathcal{Y} \cong \mathfrak{A}(F')$ for some subgroup F' of G with $F \subseteq F'$.) □

Some knowledge about Q^*_ϕ could be derived from the knowledge about RIC extensions; as is shown by the next theorem. But first we need a lemma.

5.2. **LEMMA.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a homomorphism of minimal ttgs and let* $\phi':\mathfrak{X}'\to\mathfrak{Y}'$ *be the "*RIC *lifting" of* ϕ *in its* EGS *diagram. Then* $\sigma\times\sigma[R_{\phi'}]=\overline{JR_\phi}$, *where* $\sigma:\mathfrak{X}'\to\mathfrak{X}$ *is the proximal map in* EGS(ϕ) (compare IV.4.5.).

PROOF. As ϕ' is a RIC extension, $R_{\phi'}=\overline{JR_{\phi'}}$ and so $\sigma\times\sigma[R_{\phi'}]\subseteq\overline{JR_\phi}$. Let $(x_1,x_2)\in JR_\phi$, say $(x_1,x_2)=v(x_1,x_2)$ for some $v\in J$. For $x'_1=vx'_1\in\sigma^{\leftarrow}(x_1)$, $x'_2=vx'_2\in\sigma^{\leftarrow}(x_2)$ we have $\phi'(x'_1)=v\phi'(x'_1)$ and $\phi'(x'_2)=v\phi'(x'_2)$, so $\phi'(x'_1)$ and $\phi'(x'_2)$ are distal. On the other hand

$$\tau\phi'(x'_1)=\phi\sigma(x'_1)=\phi(x_1)=\phi(x_2)=\phi\sigma(x'_2)=\tau\phi'(x'_2)\;;$$

so $\phi'(x'_1)$ and $\phi'(x'_2)$ are proximal. Hence $(x'_1,x'_2)\in R_{\phi'}$, which implies that $JR_\phi\subseteq\sigma\times\sigma[R_{\phi'}]$. ☐

5.3. **THEOREM.** *Let* ϕ,ϕ' *and* σ *be as in the lemma. Then* $\sigma\times\sigma[Q_{\phi'}]=Q^*_\phi$.

PROOF. From 5.2. it follows easily that $\sigma\times\sigma[Q_{\phi'}]\subseteq Q^*_\phi$. Let $(x_1,x_2)\in Q^*_\phi$ and let $\{(x^i_1,x^i_2)\}_i$ be a net in JR_ϕ and $\{t_i\}_i$ a net in T such that

$$(x^i_1,x^i_2)\to(x_1,x_2)\ \text{and}\ t_i(x^i_1,x^i_2)\to(x_1,x_1)\,.$$

Let $(\bar{x}^i_1,\bar{x}^i_2)\in R_{\phi'}$ be such that $\sigma\times\sigma(\bar{x}^i_1,\bar{x}^i_2)=(x^i_1,x^i_2)$. Then after passing to suitable subnets:

$$(\bar{x}^i_1,\bar{x}^i_2)\to(\bar{x}_1,\bar{x}_2)\ \text{and}\ t_i(\bar{x}^i_1,\bar{x}^i_2)\to(z_1,z_2)\,.$$

Clearly, $\sigma\times\sigma(\bar{x}_1,\bar{x}_2)=(x_1,x_2)$ and $\sigma\times\sigma(z_1,z_2)=(x_1,x_1)$, so z_1 and z_2 are proximal.
Let $\alpha\in\mathfrak{U}_{X'}$; then $(z_1,z_2)\in T\alpha\cap R_{\phi'}$ and so $t_i(\bar{x}^i_1,\bar{x}^i_2)\in T\alpha\cap R_{\phi'}$ eventually. Hence $(\bar{x}^i_1,\bar{x}^i_2)\in T\alpha\cap R_{\phi'}$ eventually; consequently, $(\bar{x}_1,\bar{x}_2)\in\overline{T\alpha\cap R_{\phi'}}$. This holds for every $\alpha\in\mathfrak{U}_{X'}$, so $(\bar{x}_1,\bar{x}_2)\in Q_{\phi'}$ and $(x_1,x_2)\in\sigma\times\sigma[Q_{\phi'}]$. ☐

In section 3. we have seen that one can understand a lot about the (relative) regionally proximal relation as far as enough almost periodicity is assumed. In particular, 3.7. shows that (with the usual notation):

$$Q^*_\phi[x]=J_{\phi(x)}\mathrm{H}(F)x\cap J_x\circ Fx\;.$$

For some points $x\in X$ we can be a little more specific as is shown in the next corollary (of 5.3.).

5.4. **COROLLARY.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a homomorphism of minimal ttgs, let* $x\in X$ *and* $u\in J_x$ *. If* $x\in\bigcap\{v\circ u\phi^{\leftarrow}\phi(x)\mid v\in J_{\phi(x)}\}$ *then* $Q^*_\phi[x]=J_{\phi(x)}\mathrm{H}(F)x$ *, where* $F=\mathfrak{G}(\mathfrak{Y},\phi(x))$ *is the Ellis group of* $\mathfrak{Y}$ *with respect to* $\phi(x)$ *in* G *.*

PROOF. Construct EGS(ϕ) and let $\phi':\mathfrak{X}'\to\mathfrak{Y}'$ and $\sigma:\mathfrak{X}'\to\mathfrak{X}$ be as usual (e.g. see 1.11. and the discussion preceding it). By 5.3., it follows that

$$Q^*_\phi[x]=\sigma[\bigcup\{Q_{\phi'}[x']\mid x'\in\sigma^{\leftarrow}(x)\}]=$$

$$=\sigma[\bigcup\{Q_{\phi'}[(x,v\circ u\phi^{\leftarrow}\phi(x))]\mid x\in v\circ u\phi^{\leftarrow}\phi(x),\ v\in J_{\phi(x)}\}]\,.$$

As ϕ' is a RIC extension (hence Bc), we know from 3.8. and 3.9. that

$$Q_{\phi'}[(x,v\circ u\phi^{\leftarrow}\phi(x))]=J_{v\circ u\phi^{\leftarrow}\phi(x)}\mathrm{H}(F).(x,v\circ u\phi^{\leftarrow}\phi(x))\,.$$

By assumption, $x\in v\circ u\phi^{\leftarrow}\phi(x)$ for every $v\in J_{\phi(x)}$, so

$$Q^*_\phi[x]=\sigma[\bigcup\{J_{v\circ u\phi^{\leftarrow}\phi(x)}\mathrm{H}(F).(x,v\circ u\phi^{\leftarrow}\phi(x))\mid v\in J_{\phi(x)}\}]=$$

$$=\bigcup\{J_{v\circ u\phi^{\leftarrow}\phi(x)}\mathrm{H}(F)x\mid v\in J_{\phi(x)}\}=J_{\phi(x)}\mathrm{H}(F)x\,.$$

□

As we do have some knowledge about Q^*_ϕ without restrictions on ϕ, one could ask whether that helps in determining E_ϕ without restrictions on ϕ. So we have the following (unsolved) question:

5.5. **QUESTION.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a homomorphism of minimal ttgs. Does* $Q_\phi=Q^*_\phi$ *imply that* Q_ϕ *is an equivalence relation?*

Related to 5.5. is the question whether Q^*_ϕ itself is an equivalence relation. some results concerning that question are gathered in 5.6.. The (almost obvious) proofs are omitted.

5.6. **REMARK.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a homomorphism of minimal ttgs.*

a) *Consider the following three statements:*
   (i) Q^*_ϕ *is an equivalence relation;*
   (ii) $Q^*_\phi\circ Q^*_\phi\subseteq\overline{JR_\phi}$*;*
   (iii) $\{x\}\times J_{\phi(x)}E(ux)\subseteq\overline{JR_\phi}$ *for every* $x\in X$ *(* $u\in J$ *fixed).*
   Then (i) and (ii) are equivalent and they are implied by (iii).

b) *If* $P_\phi\subseteq Q^*_\phi$ *, or equivalently* $P_\phi\subseteq\overline{JR_\phi}$*, then*
   (i) $Q^*_\phi\circ Q^*_\phi=Q^*_\phi\circ P_\phi$ *and* $Q^*_\phi\circ Q^*_\phi$ *is an equivalence relation;*
   (ii) *the three statements in* a *are equivalent.* □

In [B 77] and [MN 80] characterizations are given for PI ttgs. The philosophy there is to give descriptions that do not depend on the rather "abstract" ∞-construction. So they are presented as "internal" characterizations.

5.7. In order to describe the characterization of I.U. BRONSTEIN, define a *C-extension* to be a homomorphism $\phi: \mathcal{X} \to \mathcal{Y}$ of minimal ttgs such that every point transitive subttg of $\mathcal{R}_\phi$ which has a dense subset of almost periodic points is minimal.
In [B 77] the following theorem is proven:

> **THEOREM.** *Let* $\phi: \mathcal{X} \to \mathcal{Y}$ *be a homomorphism of minimal ttgs with* X *metric. Then* ϕ *is a* PI*-extension iff* ϕ *is a* C*-extension.* □

A slight generalization of this result will be given in the remarks on chapter VII. namely VII.4.6. through VII.4.8..

5.8. Let $\mathcal{X}$ be a minimal ttg and let $K \subseteq X$ be a subset of X containing at least two points (we shall call such a K *nontrivial*). A point $x \in K$ is said to be *strongly regionally proximal to* y *in* K if $y \in K$ and if there are nets $\{k_i\}_i$ in K and $\{t_i\}_i$ in T such that

$$(x, k_i) \to (x, y) \text{ and } t_i(x, k_i) \to (x, x)$$

(notation: $x \in \mathrm{SRP}(K, y)$).
In [MN 80] the following theorem is proven:

> **THEOREM.** *Let* $\mathcal{X}$ *be a minimal ttg. Then* $\mathcal{X}$ *is not a* PI *ttg iff for some* $w \in J$ *there is a closed nontrivial subset* K *of* X *such that* $K = \overline{wK}$ *and such that for some (each)* $x \in K$, $x \in \mathrm{SRP}(K, y)$ *for all* $y \in K$. □

This result together with the techniques developed in [E 78] enabled D.C. MCMAHON and L.J. NACHMAN to generalize the knowledge about metric PI ttgs to the nonmetric case. For instance they show that every minimal ttg that has a point with countable proximal cell is a PI ttg. In particular it follows that a point distal ttg is a PI ttg (Veech Structure Theorem).

IV

HIGH PROXIMALITY

This chapter is devoted to the study of a special kind of proximal extensions, namely, highly proximal extensions. These are extensions for which the points in any fiber are ″uniformly proximal″; i.e., the whole fiber shrinks to a point under the action of M on the hyperspace of the domain.

In the first section we picture the historical perspective of this chapter by way of a short (hence incomplete) description of almost automorphic extensions and the Veech Structure Theorem (the point distal equivalent of FST).
Then, in section 2., a purely topological characterization of high proximality: irreducibility, is discussed.
In the third section we relate highly proximal extensions to open extensions, via diagrams AG(ϕ) and *(ϕ), in a way similar to the relation between proximal extensions and RIC extensions, via EGS(ϕ) and AG(ϕ), as discussed in section III.1.. As a result of the comparison of AG(ϕ) and EGS(ϕ) it is shown that in the canonical PI tower for a point distal homomorphism of minimal ttgs the proximal extensions actually are highly proximal.
The forth section starts with some general considerations with respect to lifting properties in EGS and AG type diagrams. Using these general results we show for instance that the property of the relative regionally proximal relation being an equivalence relation is invariant under highly proximal lifting (by AG(ϕ) or *(ϕ)). The irreducibility result in IV.4.14. enables us to show

that disjointness and (to some extent) weak mixing are highly proximal lifting invariants. The intuitive outcome of section 4. is that in many cases we may study properties of homomorphisms of minimal ttgs just by studying those properties for open homomorphisms.
Section 5. deals with the highly proximal equivalent of PI extensions, namely HPI extensions. In section 6. we give information about what is (well) known and what is known by now.

Many of the results in this chapter can be found in [AG 77] and [AW 81].

The study of high proximality will be continued in chapter V. in a somewhat different way. There the maximally highly proximal extensions are related to certain closed subsemigroups in M .

IV.1. SOME HISTORY

> In the seventies one of the main issues in the structure theory of minimal ttgs was the Veech Structure Theorem. The objective was to find a structural concept for point distal homomorphisms of minimal ttgs in the same spirit as FST (I.1.24.).
> From this endeavour originated the study of almost automorphic extensions ([V 70]) and, in the generalization to nonmetric ttgs, the concept of high proximality ([E 73], [Sh 74,76], [AG 77], [AW 81]). Although the intention was different, this concept was in fact studied in [Ar 78] too.
> In this section we shall provide some background. Also two examples are given.

Let $\phi:\mathcal{X}\to\mathcal{Y}$ be a surjective homomorphism of ttgs. We call ϕ an *almost-automorphic* (a-a) *extension* if there is a transitive point $x\in X$ such that ϕ is one to one in x , i.e. $\phi^{\leftarrow}\phi(x)$ consists of a single point.

1.1. **REMARK.** *Let* $\phi:\mathcal{X}\to\mathcal{Y}$ *be a homomorphism of ttgs and let* $\mathcal{Y}$ *be minimal.*
   a) *If* ϕ *is an a-a extension, then* $\mathcal{X}$ *is minimal.*
   b) ϕ *is an a-a extension iff* ϕ *is proximal and point distal.*
   c) *If* ϕ *is open and a-a then* ϕ *is an isomorphism.*
   d) *If* X *is metric and* ϕ *is a-a, then there is a dense* G_δ*-set of points in which* ϕ *is one to one.*

PROOF.

a) Let $x \in X$ be a transitive point such that ϕ is one to one in x. As $\phi(x)$ is an almost periodic point, there is an almost periodic point $x' \in X$ with $\phi(x') = \phi(x)$. Since ϕ is one to one in x, we have $x = x'$ and so x is an almost periodic point with a dense orbit in X, so $\mathfrak{X}$ is minimal.

b) If ϕ is proximal and point distal, then clearly ϕ is one to one in the ϕ-distal points. If ϕ is a-a then ϕ is point distal, for every one-to-one-point for ϕ is a ϕ-distal point. As $\mathfrak{X}$ is minimal and as a one-to-one-point for ϕ is a ϕ-proximal point, it follows from the second part of I.5.4. that ϕ is proximal.

c) Let $x \in X$ be a one-to-one-point for ϕ and let $y \in Y$ and $p \in M$ be such that $py = \phi(x)$. By II.1.3.d, $\phi_{\mathrm{ad}} : \mathfrak{Y} \to 2^{\mathfrak{X}}$ is continuous, so

$$p \circ \phi^{\leftarrow}(y) = (p \circ \phi_{\mathrm{ad}}(y) = \phi_{\mathrm{ad}}(py)) = \phi^{\leftarrow}(py) = \phi^{\leftarrow}\phi(x) = \{x\}\,.$$

Then for $v \in J_y$:

$$\phi^{\leftarrow}(y) = v \circ \phi^{\leftarrow}(y) = vp^{-1} \circ (p \circ \phi^{\leftarrow}(y)) = \{vp^{-1}x\}$$

and ϕ is one to one in $\phi^{\leftarrow}(y)$.

d) Let (X,d) be a metric space and let

$$B(x,\epsilon) := \{x' \in X \mid d(x,x') < \epsilon\}\,.$$

Then for every $x \in X$ and every $n \in \mathbb{N}$ the set

$$A(x,n) := \{y \mid \phi^{\leftarrow}(y) \subseteq B(x,2^{-n})\}$$

is open by the upper semi continuity of ϕ_{ad}. Hence

$$A_n := \bigcup \{A(x,n) \mid x \in X\}$$

is open for all $n \in \mathbb{N}$. Clearly, $A := \phi^{\leftarrow} \bigcap \{A_n \mid n \in \mathbb{N}\}$ is the collection of points in which ϕ is one to one; since this set is invariant in X, it is dense. Moreover,

$$A = \phi^{\leftarrow} \bigcap \{A_n \mid n \in \mathbb{N}\} = \bigcap \{\phi^{\leftarrow}[A_n] \mid n \in \mathbb{N}\}\,.$$

Hence A is a dense G_δ-set. □

In [V 70] W.A. VEECH proved that every metric point distal ttg with a residual set of distal points can be obtained as a factor under an a-a extension of a strictly-AI ttg (i.e., a strictly-PI ttg in whose tower the proximal extensions are even almost automorphic). R. ELLIS proved in [E 73] the analogue of this for the relativized case without requiring the set of ϕ-distal points to be residual. He even generalized it by replacing the metrizability condition by a somewhat weaker countability assumption (strict-quasi separability). In doing so he implicitly gave the notion of high proximality.

Let $\phi : \mathscr{X} \to \mathscr{Y}$ be a homomorphism of minimal ttgs. Then ϕ is called *highly proximal* (*hp*) if for some $y \in Y$ there is a net $\{t_i\}_i$ in T such that the net $\{t_i \phi^{\leftarrow}(y)\}_i$ in 2^X converges to a singleton.

1.2. **REMARK.** *Let* $\phi : \mathscr{X} \to \mathscr{Y}$ *be a homomorphism of minimal ttgs and let* X *be metrizable. Then* ϕ *is a-a iff* ϕ *is hp.*

PROOF. Clearly an a-a extension is hp. For let $x \in X$ be a one-to-one-point for ϕ, then $t\phi^{\leftarrow}\phi(x) = \{tx\}$ so the constant net $\{t_i = t\}_i$ suffices.
Conversely, let ϕ be an hp extension and let X be metrizable. Let $y^* \in Y$ be such that ϕ_{ad} is continuous in y^* (II.1.3.e) and let $x^* \in \phi^{\leftarrow}(y^*)$. By assumption, there is a $y \in Y$ and a net $\{t_i\}_i$ in T such that $\{t_i \phi^{\leftarrow}(y)\}_i$ in 2^X converges to a singleton, say $\{x\}$. As $\mathscr{X}$ is minimal there is a net $\{s_j\}_j$ in T with $s_j x \to x^*$. So there is a (diagonal) net $\{t'_\lambda\}_\lambda$ in T with $\{t'_\lambda \phi^{\leftarrow}(y)\}_\lambda$ converges to $\{x^*\}$. Then $\lim t'_\lambda = \phi(x^*) = y^*$. Since ϕ_{ad} is continuous in y' we have

$$\{x^*\} = \lim t'_\lambda \phi^{\leftarrow}(y) = \phi^{\leftarrow}(\lim t'_\lambda y) = \phi^{\leftarrow}(y^*) ,$$

hence x^* is a one-to-one-point for ϕ. □

Note that there exists no absolute counterpart of high proximality; i.e., there is no such thing as a highly proximal ttg. For if $t_i X \to \{x\}$ in 2^X then we should have $\{x\} = \lim t_i X = \lim X = X$; i.e., X is trivial.

The ultimate form of the Veech Structure Theorem would be

1.3. **VST.** *Let* $\phi : \mathscr{X} \to \mathscr{Y}$ *be a point distal homomorphism of minimal ttgs. Then there are a minimal ttg* $\mathscr{X}'$ *and homomorphisms* $\sigma : \mathscr{X}' \to \mathscr{X}$ *and* $\tau : \mathscr{X}' \to \mathscr{Y}$ *such that* $\tau = \phi \circ \sigma$*,* σ *is hp and* τ *is strictly-*HPI *(i.e.,* τ *is strictly-*PI *and every proximal extension in the tower for* τ *is hp).*

$$\begin{tikzcd}
\mathscr{X}' \arrow[rr, "\sigma"] \arrow[dr, "\tau"'] & & \mathscr{X} \arrow[dl, "\phi"] \\
& \mathscr{Y} &
\end{tikzcd}$$

The theorem is known to be true for the absolute case ($\mathcal{Y} = \{\star\}$) [MN 80], [MW 81] and for the case that $\mathcal{X}$ is strictly-quasi separable, [E 73] (hence in case T is locally compact and σ-compact).

1.4. EXAMPLE.

Let T be a discrete topological group and let $\mathcal{X}$ be a minimal ttg for T. Let $x_0 \in X$. Then Tx_0 provided with the relative topology is a completely regular Hausdorff space. Let $Y = \beta(Tx_0)$, the Cech-Stone compactification of the orbit of x_0, and let $\phi: Y \to X$ be the canonical extension of the embedding $\iota: Tx_0 \to X$.

Since every continuous map $f: Tx_0 \to Z$, with Z a CT_2 space, extends to $\beta(Tx_0)$, T acts as a group of homeomorphisms on $\beta(Tx_0)$. So $\mathcal{Y}$ is a ttg and $\phi: \mathcal{Y} \to \mathcal{X}$ is a homomorphism of ttgs. As the remainder of Y, i.e., $\beta(Tx_0) \setminus Tx_0$, is mapped onto $X \setminus Tx_0$ (cf. [GJ 60] 6.11.) it follows that the map ϕ is an almost automorphic extension. Hence, by 1.1.a, $\mathcal{Y}$ is minimal.

Note that $\mathcal{Y}$ is the maximal almost automorphic extension of $\mathcal{X}$ which is one to one in the fiber of x_0.

1.5. EXAMPLE.

Let $T := \mathbb{Z}$, let Y be the circle (unit interval with end points identified) and let $\mathcal{Y} := \langle T, Y, \tilde{\alpha} \rangle$ be the rotation over an irrational angle ($\tilde{\alpha}(n,x) = x + n\alpha \pmod 1$, α irrational). Define $X := Y \times \{0,1\}$ and provide X with a 0-dimensional CT_2 topology as follows:

A neighbourhood base at $(x,0)$ is formed by the sets of the form

$$(x-\epsilon, x] \times \{0\} \cup (x-\epsilon, x) \times \{1\} \quad (\epsilon > 0),$$

and a neighbourhood base at $(x,1)$ by the sets of the form

$$(x, x+\epsilon) \times \{0\} \cup [x, x+\epsilon) \times \{1\} \quad (\epsilon > 0).$$

Define an action $\hat{\alpha}$ of $\mathbb{Z}$ on X by $\hat{\alpha}(n,(x,k)) = (x + n\alpha, k)$ for $k \in \{0,1\}$. Then $\mathcal{X} := \langle T, X, \hat{\alpha} \rangle$ is a minimal ttg (the Ellis minimal set [E 69] 5.29.).

Let $\phi: \mathcal{X} \to \mathcal{Y}$ be the projection; then ϕ is a two to one homomorphism of minimal ttgs, which is not open. Moreover, ϕ is proximal and, as every fiber is finite, ϕ is even highly proximal. But ϕ is not almost automorphic (from this it is clear that X is not metric!).

IV.2. IRREDUCIBILITY

For homomorphisms of minimal ttgs, the notion of high proximality turns out to be equivalent to the notion of irreducibility for maps known from general topology. So if equivariance is assumed, high proximality can be deduced from the topological properties of the map alone.
We shall construct the universal highly proximal extension of a ttg in a way similar (even equal) to the construction of projective covers (e.g., see [Wa 74]). This leads to the characterization of the Maximal Highly Proximal ttgs (MHP ttgs) as the Gleason spaces (in case $T = T_d$); and to the conclusion that a minimal distal ttg is never MHP (unless is it "trivial").

Let $f : X \to Y$ be a continuous surjection of CT_2 spaces. Then f is called *irreducible* if the only closed subset A of X with $f[A] = Y$ is X itself.

2.1. LEMMA. *Let $f : X \to Y$ be an irreducible map of CT_2 spaces. Then for every nonempty open U in X there exists a nonempty open U' in U such that $\overline{U} = \overline{U'}$ and $U' = f^{\leftarrow}f[U']$; in particular, $f[U']$ is open.*

PROOF. Let $U \subseteq X$ be open and nonempty and define $U' := f^{\leftarrow}[Y \setminus f[X \setminus U]]$. Then clearly, $U' = f^{\leftarrow}f[U'] \subseteq U$ and U' is open and nonempty by irreducibility. Let $x \in \overline{U}$ and $V \in \mathcal{V}_x$; then $U \cap V \neq \varnothing$ and open, so $V' := f^{-1}[Y \setminus f[X \setminus (U \cap V)]]$ is open and nonempty. Clearly, $V' \subseteq U' \cap U \cap V \subseteq U' \cap V$; hence $U' \cap V \neq \varnothing$. As V was arbitrary, $x \in \overline{U'}$. □

2.2. LEMMA. *Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a surjective homomorphism of ttgs and suppose that $\phi : X \to Y$ is irreducible.*

a) *If $\mathfrak{Y}$ is minimal then $\mathfrak{X}$ is minimal.*
b) *If $\mathfrak{Y}$ is point transitive then $\mathfrak{X}$ is point transitive.*
c) *If $\mathfrak{Y}$ is ergodic then $\mathfrak{X}$ is ergodic* (cf. VII.3.1.).
d) *If Y has a dense subset of almost periodic points then X has a dense subset of almost periodic points.*

PROOF.

a) Let $Z \subseteq X$ be a minimal subset. Then $\phi[Z] = Y$, so $Z = X$ by irreducibility.

b) Let $y \in Y$ have a dense orbit and let $x \in \phi^{\leftarrow}(y)$.
Then $\phi[\overline{Tx}] = \overline{Ty} = Y$; so by irreducibility, $\overline{Tx} = X$.

c) Let U be a nonempty open subset of X. By 2.1., there is a nonempty open $U' \subseteq U$ with $\phi[U']$ is open in Y. As $\mathcal{Y}$ is ergodic, $\overline{T\phi[U']} = Y$, so

$$Y = \overline{T\phi[U']} = \phi[\overline{TU'}] \subseteq \phi[\overline{TU}] .$$

By irreducibility of ϕ, it follows that $\overline{TU} = X$, so $\mathcal{X}$ is ergodic.

d) Let $X' \subseteq X$ and $Y' \subseteq Y$ be the collections of almost periodic points in X and Y respectively. By I.10.b, $\phi[X'] = Y'$, so $\phi[\overline{X'}] = \overline{Y'} = Y$. Irreducibility of ϕ implies $X = \overline{X'}$. □

The following theorem shows the dynamical properties of irreducible maps, it also explains why we are interested in irreducibility.

2.3. **THEOREM.** *Let $\phi: \mathcal{X} \to \mathcal{Y}$ be a homomorphism of minimal ttgs. The following statements are equivalent:*

a) *ϕ is highly proximal;*

b) *ϕ is irreducible;*

c) *$2^{\phi}: 2^{\mathcal{X}}_{\phi} \to \mathcal{Y}$ is proximal;*

d) *$2^{\mathcal{X}}_{\phi}$ has a unique minimal subset;*

e) *$p \circ \phi^{\leftarrow}(y) = \{px\}$ for all $y \in Y$, $x \in \phi^{\leftarrow}(y)$ and $p \in M$.*

PROOF.

a $\Rightarrow$ b Let $y \in Y$, $x \in X$ and the net $\{t_i\}_i$ in T be such that $\lim_{2^X} t_i \phi^{\leftarrow}(y) = \{x\}$. Let U be an arbitrary nonempty open set in X. Let $t \in T$ with $tx \in U$; then $t^{-1}U \in \mathcal{V}_x$. So for some i_0 we have $t_i \phi^{\leftarrow}(y) \subseteq t^{-1}U$ for all $i \geqslant i_0$. Hence U contains a fiber, so ϕ is irreducible.

b $\Rightarrow$ c Let A and B in 2^X_ϕ be such that $2^{\phi}(A) = 2^{\phi}(B) = y \in Y$ (i.e., A and B are closed subsets of $\phi^{\leftarrow}(y)$). Let $x \in X$, and for every $\alpha \in \mathcal{U}_X$ let U_α be an open set in X with $U_\alpha = \phi^{\leftarrow}\phi[U_\alpha] \subseteq \alpha(x)$ (2.1.). As $\phi[U_\alpha]$ is open and nonempty there is a $t_\alpha \in T$ with $t_\alpha y \in \phi[U_\alpha]$. So

$$t_\alpha \phi^{\leftarrow}(y) \subseteq \phi^{\leftarrow}\phi[U_\alpha] = U_\alpha \subseteq \alpha(x) .$$

Clearly, $t_\alpha \phi^{\leftarrow}(y) \to \{x\}$ in 2^X, and so

$$\lim_{2^X} t_\alpha A \subseteq \lim_{2^X} t_\alpha \phi^{\leftarrow}(y) = \{x\} .$$

Similarly $\lim_{2^X} t_\alpha B = \{x\}$, so A and B are proximal.

c $\Rightarrow$ d Follows from I.1.23.c.

d $\Rightarrow$ e As $X \subseteq 2^X_\phi$, $\mathcal{X}$ has to be the unique minimal subttg of $2^{\mathcal{X}}_\phi$. Since for all $y \in Y$ and $p \in M$ the set $p \circ \phi^{\leftarrow}(y)$ is an almost periodic point in 2^X_ϕ, we have $px \in p \circ \phi^{\leftarrow}(y) \in X$ for all $x \in \phi^{\leftarrow}(y)$; hence $p \circ \phi^{\leftarrow}(y) = \{px\}$.

e ⇒ a Trivial. □

2.4. **REMARK.**

a) *Let* $\phi: \mathcal{X} \to \mathcal{Y}$ *be a homomorphism of minimal ttgs. Then* ϕ *is open and highly proximal iff* ϕ *is an isomorphism.*

b) *Let* $\phi: \mathcal{X} \to \mathcal{Y}$ *and* $\psi: \mathcal{Y} \to \mathcal{Z}$ *be homomorphisms of minimal ttgs. Then* $\psi \circ \phi$ *is highly proximal iff* ϕ *and* ψ *are highly proximal.*

c) *Let* $\{\phi^\beta_\alpha: X_\beta \to X_\alpha \mid \alpha \leqslant \beta < \nu\}$ *be an inverse system (tower of height* ν *) consisting of homomorphisms of minimal ttgs. Then* $\phi = \operatorname{inv}\lim \phi^\beta_\alpha$ *is highly proximal iff every* ϕ^β_α *is highly proximal.*

PROOF.

a) If ϕ is an open map, then ϕ_{ad} is continuous. So for all $y \in Y$, $x \in \phi^{\leftarrow}(y)$ and $u \in J_y$ we have $\phi^{\leftarrow}(y) = u \circ \phi^{\leftarrow}(y) = \{ux\}$ (2.3.a), hence ϕ is one to one.

b and c Follow from the equivalence of a and b in 2.3.. □

2.5. **THEOREM.** *For every minimal ttg* $\mathcal{Y}$ *there is a universal highly proximal extension which is unique up to isomorphism (i.e., there is an highly proximal extension* $\chi: \mathcal{Y}^* \to \mathcal{Y}$ *such that for an arbitrary highly proximal extension* $\phi: \mathcal{X} \to \mathcal{Y}$ *there is a* $\psi: \mathcal{Y}^* \to \mathcal{X}$ *with* $\chi = \phi \circ \psi$ *).*

PROOF. As every minimal hp extension of $\mathcal{Y}$ is a factor of $\mathcal{M}$, there is only a set of essentially different hp extensions of $\mathcal{Y}$, say $\{\phi_\lambda: \mathcal{X}_\lambda \to \mathcal{Y} \mid \lambda \in \Lambda\}$. Define

$$\Phi: \Pi\{X_\lambda \mid \lambda \in \Lambda\} \to \mathcal{Y}^\Lambda \text{ by } \Phi((x_\lambda)_{\lambda \in \Lambda}) = (\phi_\lambda(x_\lambda))_{\lambda \in \Lambda} .$$

Let $X := \Phi^{\leftarrow}[\Delta^\Lambda_Y]$ and $\tilde{\phi} := \Phi|_X : X \to Y \cong \Delta^\Lambda_Y$. Then $\tilde{\phi}$ is a homomorphism of ttgs, which is proximal by I.1.21.b. Let $\mathcal{Z}$ be the unique minimal subttg of $\mathcal{X}$ and $\phi := \tilde{\phi}|_Z$. Clearly every hp extension of $\mathcal{Y}$ is a factor of $\mathcal{Z}$ under projection (up to isomorphism).

In 2.6. below it will be proven that ϕ *is an hp extension.*

So $\phi: \mathcal{Z} \to \mathcal{Y}$ is a universal minimal hp extension. We shall show that it is unique up to isomorphism.

Suppose $\phi': \mathcal{Z}' \to \mathcal{Y}$ is a universal minimal hp extension too. As ϕ' is an hp extension of $\mathcal{Y}$, there is a $\xi: \mathcal{Z} \to \mathcal{Z}'$ such that $\phi = \phi' \circ \xi$. As ϕ' is universal and ϕ is hp, there is a $\eta: \mathcal{Z}' \to \mathcal{Z}$ with $\phi' = \phi \circ \eta$ so $\phi = \phi \circ \eta \circ \xi$.

$$\begin{tikzcd}
\mathfrak{Z} \arrow[r, "\phi"] \arrow[d, "\xi"'] & \mathcal{Y} \\
\mathfrak{Z}' \arrow[r, "\eta"'] \arrow[ru, "\phi'"] & \mathfrak{Z} \arrow[u, "\phi"']
\end{tikzcd}$$

Let $z \in Z$, then $\phi(z) = \phi(\eta \circ \xi(z))$; by proximality of ϕ , the points z and $\eta \circ \xi(z)$ are proximal in $\mathfrak{Z}$. As $J_z \subseteq J_{\eta \circ \xi(z)}$ it follows that $(z, \eta \circ \xi(z))$ is an almost periodic point in $\mathfrak{Z} \times \mathfrak{Z}$; so $z = \eta \circ \xi(z)$. Hence $\eta \circ \xi = id_Z$ and so $\mathfrak{Z}$ and $\mathfrak{Z}'$ are isomorphic ttgs. □

2.6. LEMMA. *(With notation as above:)* $\phi : \mathfrak{Z} \to \mathcal{Y}$ *is highly proximal.*

PROOF. We shall show that every open set in Z contains a fiber of ϕ . Let $U \subseteq Z$ be basic open and nonempty; i.e., there are $\alpha_1, \ldots, \alpha_n \in \Lambda$ and open sets $U_i \subseteq X_{\alpha_i}$ such that $U = U' \cap Z \neq \emptyset$, where

$$U' := U_1 \times \cdots \times U_n \times \Pi\{X_\alpha \mid \alpha \in \Lambda \setminus \{\alpha_1, \ldots, \alpha_n\}\} .$$

Note that $\tilde{\phi}[U' \cap X] = \bigcap_{i=1}^{n} \phi_{\alpha_i}[U_i] \neq \emptyset$. As $U' \cap Z \neq \emptyset$, U' contains an almost periodic point, and so $W := \mathrm{int}_Y(\tilde{\phi}[U' \cap Z]) \neq \emptyset$ (I.1.4.a). By 2.1., there are open $V_i \subseteq X_i$ such that

$$\emptyset \neq V_1 = \phi_{\alpha_1}^{\leftarrow}\phi_{\alpha_1}[V_1] \subseteq U_1 \cap \phi_{\alpha_1}^{\leftarrow}[W]$$

and for $i \in \{1, \ldots, n\}$

$$\emptyset \neq V_i = \phi_{\alpha_i}^{\leftarrow}\phi_{\alpha_i}[V_i] \subseteq U_i \cap \phi_{\alpha_i}^{\leftarrow}\phi_{\alpha_{i-1}}[V_{i-1}] .$$

Define $V := V' \cap X$ with

$$V' := V_1 \times \cdots \times V_n \times \Pi\{X_\alpha \mid \alpha \in \Lambda \setminus \{\alpha_1, \ldots, \alpha_n\}\}$$

Then $V = \tilde{\phi}^{\leftarrow}\tilde{\phi}[V]$ is nonempty, hence $V \cap Z \neq \emptyset$ and $V \cap Z$ contains a fiber under ϕ . □

The universal (minimal) hp extension of a minimal ttg $\mathcal{Y}$ will be denoted by $\chi_{\mathcal{Y}} : \mathcal{Y}^* \to \mathcal{Y}$. If $\chi_{\mathcal{Y}}$ is an isomorphism $\mathcal{Y}$ will be called a *Maximally Highly Proximal ttg* (MHP *ttg*).

In section IV.3. we characterize the MHP ttgs in terms of quasifactors of $\mathcal{M}$.

In case T is provided with the discrete topology, we can give a topological characterization of MHP ttgs; as follows:

2.7. **THEOREM.** *Let T be a discrete topological group and let the ttg $\mathfrak{X} := < T, X >$ be minimal. Then $\mathfrak{X}$ is an* MHP *ttg iff X is extremally disconnected (the closure of every open set is open).*
In particular, the universal highly proximal extension of a minimal ttg is just its Gleason extension.

PROOF. It is well known ([Wi 70] 14.2.5.) that an irreducible extension of an extremally disconnected CT$_2$ space is a homeomorphism. Since the universal hp extension is irreducible, a minimal ttg with extremally disconnected phase space is an MHP ttg.
Conversely, let $\mathfrak{X}$ be a minimal ttg. Let $\chi_X : G(X) \to X$ be the Gleason extension of X (e.g. [Wi 70] 14.2.2.), then χ_X is irreducible. As every homeomorphism on X extends uniquely to a homeomorphism on $G(X)$ and as T is discrete it follows that $G(\mathfrak{X}) = < T, G(X) >$ is a ttg and that $\chi_X : G(\mathfrak{X}) \to \mathfrak{X}$ is an irreducible homomorphism of ttgs. By 2.2., $G(\mathfrak{X})$ is a minimal ttg, and $G(X)$ is extremally disconnected. If $\mathfrak{X}$ is an MHP ttg, the irreducible map $G(\mathfrak{X}) \to \mathfrak{X}$ is an isomorphism, so X is extremally disconnected. □

2.8. **COROLLARY.** *Let T be a discrete topological group, and let $\mathfrak{X} = < T, X >$ be a minimal ttg. If $\mathfrak{X}$ is a distal* MHP *ttg then X is finite.*

PROOF. Let $\mathfrak{X}$ be distal and let $\mathfrak{Y}$ be the maximal almost periodic factor of $\mathfrak{X}$. As $\kappa : X \to X/E_X \cong Y$ is open and X is extremally disconnected, Y is extremally disconnected too. However, $Y \cong bT/H$ for some closed subgroup H of the Bohr-compactification bT of T (I.1.14.); so Y is homogeneous. By [Ak 78] III section 3, it follows that an extremally disconnected homogeneous CT$_2$ space is finite. Hence Y is finite. Suppose $\kappa : \mathfrak{X} \to \mathfrak{Y}$ is nontrivial. Then κ is distal and, by FST (I.1.24.), it follows that $\theta : X/E_\kappa \to Y$ is a nontrivial almost periodic extension. As Y is finite it follows from I.1.22. that X/E_κ is almost periodic, which contradicts the assumption of $\mathfrak{Y}$ being the maximal almost periodic factor of $\mathfrak{X}$. □

IV.3. HIGHLY PROXIMAL LIFTING

Similar to the construction of the EGS diagram we construct an AG diagram ([AG 77]). The objective is to show that every homomorphism of minimal ttgs is open up to high proximality. Using an AG diagram we characterize the MHP ttgs as quasifactors of $\mathfrak{M}$ generated by the so-called MHP generators. Also we compare the EGS and AG diagrams and conclude that, in case ϕ is a point distal homomorphism of minimal ttgs, AG(ϕ) equals EGS(ϕ). Hence it follows that a point distal map is PI iff it is HPI.

3.1. **THEOREM.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a homomorphism of minimal ttgs and let* $\mathfrak{Z}$ *be a quasifactor of* $\mathfrak{Y}$. *Then there is a quasifactor* $\mathfrak{Z}'$ *of* $\mathfrak{X}$ *such that* $2^{\phi}:\mathfrak{Z}'\to\mathfrak{Z}$ *is highly proximal.*

PROOF. Let $A = u\circ A$ be an almost periodic point in 2^{Y} such that $\mathfrak{Z}=\mathfrak{W}\mathfrak{F}(A,\mathfrak{Y})$ and define $\mathfrak{Z}':=\mathfrak{W}\mathfrak{F}(u\circ\phi^{\leftarrow}[A],\mathfrak{X})$. Then $2^{\phi}:\mathfrak{Z}'\to\mathfrak{Z}$ is a homomorphism of minimal ttgs. Define

$$\mathcal{C}[A]:=\{B\in Z' \mid B\subseteq\phi^{\leftarrow}[A]\}\,.$$

Then $u\circ\phi^{\leftarrow}[A]\in\mathcal{C}[A]$, so $\mathcal{C}[A]$ is nonempty and clearly, $\mathcal{C}[A]$ is a closed subset of $\mathfrak{Z}'$. Let $\mathcal{K}$ be a chain in $\mathcal{C}[A]$ (ordered by inclusion). Then $L=\bigcap\{K \mid K\in\mathcal{K}\}$ is a lower bound for $\mathcal{K}$ in $\mathcal{C}[A]$, for $L\neq\varnothing$ and $L=\lim_{2^X}\mathcal{K}\in \mathrm{cl}_{2^X}\mathcal{C}[A]=\mathcal{C}[A]$. So, by Zorns lemma, there is a minimal element $C\in\mathcal{C}[A]$. Let $p\in M$ be such that $C=p\circ\phi^{\leftarrow}[A]$. Denote the circle operation of M on 2^{2^X} by $\square$.
We claim that $p\,\square\,(2^{\phi})^{\leftarrow}(A)=\{C\}$; note that

$$(2^{\phi})^{\leftarrow}(A)=\{q\circ\phi^{\leftarrow}[A] \mid q\circ A=A\}\,.$$

Let $B\in p\,\square\,(2^{\phi})^{\leftarrow}(A)$ and let $\{t_i\}_i$ be a net in T converging to p. Then, after passing to a suitable subnet, $B=\lim t_i q_i\circ\phi^{\leftarrow}[A]$ for certain $q_i\in M$ with $q_i\circ A=A$. As

$$\phi[q_i\circ\phi^{\leftarrow}[A]]=q_i\circ\phi[\phi^{\leftarrow}[A]]=q_i\circ A=A\ ,$$

we have $q_i\circ\phi^{\leftarrow}[A]\subseteq\phi^{\leftarrow}[A]$; hence

$$B=\lim t_i q_i\circ\phi^{\leftarrow}[A]\subseteq\lim t_i\phi^{\leftarrow}[A]=p\circ\phi^{\leftarrow}[A]=C\ .$$

But C was minimal in $\mathcal{C}[A]$ so $B=C$, which proves the claim.
This shows that $2^{\phi}:\mathfrak{Z}'\to\mathfrak{Z}$ is highly proximal. □

3.2. Note that in the above $\mathfrak{Z}=\{p\circ\phi^{\leftarrow}[B]\mid p\in M \text{ and } B\in\mathfrak{Z}\}$. For, let $p\in M$ and $B\in\mathfrak{Z}$ then $B=q\circ A$ for some $q\in M$ and also $A=vq^{-1}\circ B$ for $v\in J_q$. Hence

$$p\circ\phi^{\leftarrow}[B]=pqvq^{-1}\circ\phi^{\leftarrow}[B]\subseteq pq\circ\phi^{\leftarrow}[vq^{-1}\circ B]=pq\circ\phi^{\leftarrow}[A]\subseteq$$
$$\subseteq p\circ\phi^{\leftarrow}[q\circ A]=p\circ\phi^{\leftarrow}[B]\,.$$

3.3. Remark that $\mathcal{Y}$ can be represented as a quasifactor $\mathcal{Y}'$ of $\mathcal{X}$ up to a highly proximal extension τ by taking $\mathfrak{Z}=\mathcal{Y}$ (in 3.1.), as follows:
Define $Y'=\{p\circ\phi^{\leftarrow}(y)\mid p\in M\,, y\in Y\}$ and $\tau=2^{\phi}|_{Y'}:\mathcal{Y}'\to\mathcal{Y}$. Note that τ is one to one in $p\circ\phi^{\leftarrow}(y)\in Y'$ iff ϕ is open in all points of $\phi^{\leftarrow}(py)$ (use II.1.3.c resp. II.3.12., II.3.11.e to conclude that $\phi^{\leftarrow}(py)=q\circ\phi^{\leftarrow}(y)$ for every $q\in M$ with $py=qy$). In particular, τ is a homeomorphism iff ϕ is open.

3.4. Let $\phi:\mathcal{X}\to\mathcal{Y}$ be a homomorphism of minimal ttgs. We shall construct a shadow diagram AG(ϕ) for ϕ ,

$$\begin{tikzcd}
\mathcal{X}' \arrow[r, "\sigma"] \arrow[d, "\phi'"'] & \mathcal{X} \arrow[d, "\phi"] \\
\mathcal{Y}' \arrow[r, "\tau"'] & \mathcal{Y}
\end{tikzcd}$$

consisting of homomorphisms of minimal ttgs, with the following properties:
ag1 σ and τ are highly proximal;
ag2 ϕ' is open.
Moreover, the diagram is minimal under those properties.

Define $\mathcal{Y}'$ as the quasifactor representation of $\mathcal{Y}$ in $\mathcal{X}$, so

$$Y':=\{p\circ\phi^{\leftarrow}(y)\mid p\in M,\ y\in Y\}=\{p\circ\phi^{\leftarrow}(y_0)\mid p\in M\}$$

for some fixed $y_0\in Y$. Let $X':=\{(x,A)\in X\times Y'\mid x\in A\}$ and define $\sigma:X'\to X$ and $\phi':X'\to Y'$ as the projections, and let $\tau:=2^{\phi}|_{Y'}$.

3.5. **LEMMA.** *Let* $\mathfrak{Z}=\mathfrak{QF}(A\,,\mathcal{X})$ *be a quasifactor of* $\mathcal{X}$ *and let* $W\subseteq X\times Z$ *be defined by* $W:=\{(x,B)\mid x\in B\in Z\}$. *Then* W *is closed and* T*-invariant and the projection* $\pi:\mathcal{W}\to\mathfrak{Z}$ *is an open homomorphism of ttgs.*

PROOF. Let $(x,B) \notin W$, so $x \notin B$. Then there are open sets U and V in X such that $U \cap V = \varnothing$, $x \in U$ and $B \subseteq V$. Clearly, $U \times <V>$ is an open neighbourhood of the point (x,B) in $X \times Z$ and $U \times <V> \cap W = \varnothing$, so W is closed.
T-invariance is obvious.
Let $U' = U \times <V_1, \ldots, V_n> \cap W$ be a basic open set in W and note that, without loss of generality, we may assume that $U \subseteq \bigcup \{V_i \mid i \in \{1, \ldots, n\}\}$. It is easy to verify that

$$\pi[U'] = <V_1, \ldots, V_n, U> \cap Z \ ;$$

so π is open. □

From 3.5. it follows that, in 3.4., $\mathscr{X}' = <T, X'>$ is a ttg and that ϕ' is an open homomorphism of ttgs, which shows ag2. As τ is irreducible (3.1. and 3.3.) σ is irreducible too.
For, let $(U' = U \times <V_1, \ldots, V_n>) \cap X'$ be basic open, nonempty, and (without loss of generality) let $U \subseteq \bigcup \{V_i \mid i \in \{1, \ldots, n\}\}$. Then it is easily seen that $\phi'[U'] = <V_1, \ldots, V_n, U> \cap Y'$. So, by irreducibility of τ, there is a $y \in Y$ with $\tau^{\leftarrow}(y) \subseteq \phi'[U'] \subseteq <V_1, \ldots, V_n>$. Let $u \in J_y$; then $u \circ \phi^{\leftarrow}(y) \in \tau^{\leftarrow}(y)$, so $u \circ \phi^{\leftarrow}(y) \cap U \neq \varnothing$, say $x \in u \circ \phi^{\leftarrow}(y) \cap U$. But then $\{x\} \times \tau^{\leftarrow}(y) \subseteq U \times <V_1, \ldots, V_n>$. As $\sigma^{\leftarrow}(x) = (\{x\} \times \tau^{\leftarrow}(y)) \cap X'$, it follows that U' contains a full σ-fiber. Hence σ is irreducible and, by 2.2.a, it follows that $\mathscr{X}'$ is minimal; so by 2.3., σ is highly proximal, which shows ag1.

3.6. The diagram AG(ϕ) for ϕ is minimal under the conditions ag1 and ag2. Consider the following commutative diagram consisting of homomorphisms of minimal ttgs, with on the right hand side AG(ϕ).

$$\begin{tikzcd}
\mathscr{W} \arrow[rrrr, dashed, "\mu"] \arrow[drr, "\xi"'] \arrow[dd, "\psi"'] & & & & \mathscr{X}' \arrow[dll, "\sigma"'] \arrow[dd, "\phi'"] \\
& & \mathscr{X} \arrow[dd, "\phi"] & & \\
\mathscr{Z} \arrow[rrrr, dashed, "\nu"] \arrow[drr, "\eta"'] & & & & \mathscr{Y}' \arrow[dll, "\tau"] \\
& & \mathscr{Y} & &
\end{tikzcd}$$

Let ψ be open and let ξ and η highly proximal. Then there are homomorphisms μ and ν with $\sigma \circ \mu = \xi$ and $\tau \circ \nu = \eta$. As follows:

Let $y_0 \in Y$, $u \in J_{y_0}$, $z_0 = uz_0 \in \eta^{\leftarrow}(y_0)$, $w_0 = uw_0 \in \psi^{\leftarrow}(z_0)$ and let $x_0 = \xi(w_0)$. Define $\nu : \mathfrak{Z} \to \mathcal{Y}'$ by $\nu(pz_0) = p \circ \phi^{\leftarrow}(y_0)$, and $\mu : \mathcal{W} \to \mathcal{X}'$ by $\mu(pw_0) = (px_0, p \circ \phi^{\leftarrow}(y_0))$.
Note that in order to show minimality of AG(ϕ) it is sufficient to show that ν is well defined.
Suppose that $pz_0 = qz_0$. As η is highly proximal, $z_0 = u \circ \eta^{\leftarrow}(y_0)$. By continuity of ψ_{ad} (II.1.3.d), we have $\psi^{\leftarrow}(z_0) = u \circ \psi^{\leftarrow}\eta^{\leftarrow}(y_0) = u \circ \xi^{\leftarrow}\phi^{\leftarrow}(y_0)$; hence $\xi\psi^{\leftarrow}(z_0) = u \circ \phi^{\leftarrow}(y_0)$. Again, by continuity of ψ_{ad} , we have

$$p \circ \psi^{\leftarrow}(z_0) = \psi^{\leftarrow}(pz_0) = \psi^{\leftarrow}(qz_0) = q \circ \psi^{\leftarrow}(z_0) .$$

So

$$p \circ \phi^{\leftarrow}(y_0) = p \circ \xi\psi^{\leftarrow}(z_0) = \xi(p \circ \psi^{\leftarrow}(z_0)) = \xi(q \circ \psi^{\leftarrow}(z_0)) =$$
$$= q \circ \xi\psi^{\leftarrow}(z_0) = q \circ \phi^{\leftarrow}(y_0) ;$$

and ν is well defined.

With the help of the AG diagrams we can characterize the universal highly proximal extensions as quasifactors of $\mathcal{M}$.
Let $\mathcal{X}$ be a minimal ttg, $x_0 \in X$ and $u \in J_{x_0}$. Define $\gamma : \mathcal{M} \to \mathcal{X}$ by $\gamma(p) = px_0$. Consider AG(γ):

$$\begin{tikzcd}
\mathcal{M}' \arrow[r, "\sigma"] \arrow[d, "\gamma'"'] & \mathcal{M} \arrow[d, "\gamma"] \\
\mathcal{X}' \arrow[r, "\tau"'] & \mathcal{X}
\end{tikzcd}$$

so

$$X' = \{p \circ \gamma^{\leftarrow}(x_0) \mid p \in M\} = \{p \circ M_{x_0} \mid p \in M\} =$$
$$= \{p \circ \gamma^{\leftarrow}(x) \mid p \in M , x \in X\} = \{p \circ M_x \mid p \in M , x \in X\} ,$$

and

$$M' = \{(p, q \circ \gamma^{\leftarrow}(x_0)) \mid p, q \in M , p \in q \circ \gamma^{\leftarrow}(x_0)\} =$$
$$= \{(p, q \circ M_x) \mid p, q \in M , p \in q \circ M_x\} ,$$

while γ' is open, σ and τ are highly proximal.

3.7. **LEMMA.** *(Situation and notation as above.)*

a) $\{p \circ \gamma^{\leftarrow}(x) \mid p \in M, x \in X\}$ *is a partitioning of* M .

b) *The map* $\gamma^* : \mathfrak{M} \to \mathfrak{X}'$ *defined by* $p \mapsto p \circ \gamma^{\leftarrow}(x_0)$ *is an open homomorphism of minimal ttgs.*

c) $\mathfrak{X}'$ *is an MHP ttg.*

d) $\mathfrak{X}$ *is an MHP ttg iff* $\mathfrak{X} \cong \mathfrak{X}'$.

PROOF.

a) As $\mathfrak{M}$ is the universal minimal ttg, σ is an isomorphism. Now suppose that for some $p,q \in M$ and some $x,x' \in X$ we have $p \circ \gamma^{\leftarrow}(x) \cap q \circ \gamma^{\leftarrow}(x') \neq \varnothing$, say $r \in p \circ \gamma^{\leftarrow}(x) \cap q \circ \gamma^{\leftarrow}(x') \neq \varnothing$. Hence $(r, p \circ \gamma^{\leftarrow}(x))$ and $(r, q \circ \gamma^{\leftarrow}(x'))$ are elements of M' that are both mapped onto r by σ ; but σ is injective, so $p \circ \gamma^{\leftarrow}(x) = q \circ \gamma^{\leftarrow}(x')$.

b) Follows from the fact that σ is an isomorphism.

c) Suppose there is an hp extension $\psi : \mathfrak{Z} \to \mathfrak{X}'$. For $z = uz \in Z$ with $\psi(z) = u \circ \gamma^{\leftarrow}(x_0)$ let $\delta : \mathfrak{M} \to \mathfrak{Z}$ be defined by $\delta(p) = pz$. Then $\gamma^* = \psi \circ \delta$ and, since γ^* is open, ψ is open. By 2.4.a, it follows that ψ is an isomorphism.

d) If $\mathfrak{X}$ is an MHP ttg, $\tau : \mathfrak{X}' \to \mathfrak{X}$ is an isomorphism, so $\mathfrak{X} \cong \mathfrak{X}'$; the other way around is c. □

3.8. Let $C \subseteq M$ be an almost periodic point in 2^M . Then C is called a *Maximally Highly Proximal generator* (MHP *generator*) if $C \cap J \neq \varnothing$ and $\{p \circ C \mid p \in M\}$ is a partitioning of M .
We shall study MHP generators in chapter V.. The terminology is justified by the equivalence of a and d in the following theorem.

3.9. **THEOREM.** *Let* $\mathfrak{X}$ *be a minimal ttg. Then the following statements are equivalent:*

a) $\mathfrak{X}$ *is an* MHP *ttg (* $\mathfrak{X} = \mathfrak{X}^*$ *see the definition just before 2.7.);*

b) *every homomorphism* $\phi : \mathfrak{Y} \to \mathfrak{X}$ *of minimal ttgs is open;*

c) $\mathfrak{X}$ *is a factor of* $\mathfrak{M}$ *under an open homomorphism;*

d) $\mathfrak{X} \cong \mathfrak{W}\mathfrak{F}(C, \mathfrak{M})$ *for an* MHP *generator* C .

PROOF.

a $\Rightarrow$ b Consider AG(ϕ). Then $\mathfrak{X} \cong \mathfrak{X}'$, for $\mathfrak{X}$ is an MHP ttg; so $\mathfrak{Y} \cong \mathfrak{Y}'$, and $\phi = \phi'$ is open.

b $\Rightarrow$ c Trivial.

c $\Rightarrow$ d Let $\gamma : \mathfrak{M} \to \mathfrak{X}$ be open, say γ is defined by $\gamma(p) = px_0$ for some $x_0 \in uX$ and all $p \in M$. By II.1.3.d, $\gamma^{\leftarrow}(px_0) = p \circ \gamma^{\leftarrow}(x_0)$, so

$$\{p \circ \gamma^{\leftarrow}(x_0) \mid p \in M\} = \{\gamma^{\leftarrow}(px_0) \mid p \in M\}$$

which is a partitioning of M . As $u \in u \circ \gamma^{\leftarrow}(x_0)$, $u \circ \gamma^{\leftarrow}(x_0)$ is an MHP generator. By 3.3., $\gamma' := 2^{\gamma} : \mathfrak{W}\mathfrak{F}(u \circ \gamma^{\leftarrow}(x_0), \mathcal{M}) \to \mathcal{X}$ is hp. As γ' is a factor of γ , γ' is open and so γ' is an isomorphism.

d $\Rightarrow$ a Define $\gamma : \mathcal{M} \to \mathcal{X}$ by $\gamma(p) = p \circ C$. Then

$$\gamma^{\leftarrow}(p \circ C) = \{q \in M \mid q \circ C = p \circ C\} = p \circ C ,$$

as follows:
Let $p \circ C = q \circ C$. As $C \cap J \neq \varnothing$, $u \in u \circ C$ and so

$$q = qu \in q \circ u \circ C = qu \circ C = q \circ C = p \circ C .$$

Conversely, let $q \in p \circ C$. As $q \in q \circ C$, $q \in p \circ C \cap q \circ C$; but C is an MHP generator, so $p \circ C = q \circ C$.
This shows, with notation as in the discussion preceding 3.7., that $X = X'$ and so by 3.7. that $\mathcal{X}$ is an MHP ttg. □

3.10. Let $\phi : \mathcal{X} \to \mathcal{Y}$ be a homomorphism of minimal ttgs. We can construct a kind of maximal AG diagram for ϕ , which will be called $*(\phi)$, as follows:

$$\begin{tikzcd}
\mathcal{X}^* \arrow[r, "\chi_{\mathcal{X}}"] \arrow[d, "\phi^*"'] & \mathcal{X} \arrow[d, "\phi"] \\
\mathcal{Y}^* \arrow[r, "\chi_{\mathcal{Y}}"'] & \mathcal{Y}
\end{tikzcd}$$

Let $u \in J$ and choose $x_0 = ux_0 \in X$, $y_0 = \phi(x_0)$. Define $\gamma : \mathcal{M} \to \mathcal{X}$ by $\gamma(p) = px_0$ and $\delta : \mathcal{M} \to \mathcal{Y}$ by $\delta(p) = py_0$. Then $\delta = \phi \circ \gamma$. Analogues to III.1.13.b, but using 3.6., one shows that

$$\chi_{\mathcal{X}} : \mathcal{X}^* = \mathfrak{W}\mathfrak{F}(u \circ \gamma^{\leftarrow}(x_0), \mathcal{M}) \to \mathcal{X} \text{ and } \chi_{\mathcal{Y}} : \mathcal{Y}^* = \mathfrak{W}\mathfrak{F}(u \circ \delta^{\leftarrow}(y_0), \mathcal{M}) \to \mathcal{Y}$$

are the universal hp extensions of $\mathcal{X}$ and $\mathcal{Y}$. Define $\phi^* : \mathcal{X}^* \to \mathcal{Y}^*$ by $\phi^*(p \circ \gamma^{\leftarrow}(x_0)) = p \circ \delta^{\leftarrow}(x_0)$ for all $p \in M$. Then ϕ^* is well defined; for, let $p \circ \gamma^{\leftarrow}(x_0) = q \circ \gamma^{\leftarrow}(x_0)$, then clearly

$$p \circ \gamma^{\leftarrow}(x_0) = q \circ \gamma^{\leftarrow}(x_0) \subseteq p \circ \delta^{\leftarrow}(y_0) \cap q \circ \delta^{\leftarrow}(y_0) .$$

As $\{p \circ \delta^{\leftarrow}(y_0) \mid p \in M\}$ partitions M , we have $p \circ \delta^{\leftarrow}(y_0) = q \circ \delta^{\leftarrow}(y_0)$.
Obviously ϕ^* is open and the diagram commutes.
Note that ϕ^* is an isomorphism iff ϕ is hp (use 2.4.b).

In section IV.4. we shall search for properties on ϕ that will be preserved under hp lifting; i.e., properties of ϕ such that ϕ^* in $*(\phi)$ and ϕ' in AG(ϕ) have (almost) the same property.

3.11. If $\mathfrak{X}$ is metric, then AG(ϕ) consists entirely of homomorphisms of metric minimal ttgs (II.1.1.b); whereas, in general, $*(\phi)$ does not (cf. 2.7.).

The next theorem deals with the question whether or not AG(ϕ) and EGS(ϕ) coincide for a homomorphism $\phi: \mathfrak{X} \to \mathfrak{Y}$ of minimal ttgs.

3.12. **THEOREM.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs.*

a) AG(ϕ) *and* EGS(ϕ) *coincide iff for some* $y \in Y$ *and* $u \in J_y$ *we have* $u \circ \phi^{\leftarrow}(y) = u \circ u\phi^{\leftarrow}(y)$ *(e.g.* ϕ' *in* AG(ϕ) *is* RIC*).*

b) *If* ϕ *is open, then* AG(ϕ) *and* EGS(ϕ) *coincide iff* ϕ *is* RIC.

c) *If* $\mathfrak{X}$ *is metric, then* AG(ϕ) *and* EGS(ϕ) *coincide iff for some* $y \in Y : \bigcap \{u \circ u\phi^{\leftarrow}(y) \mid u \in J_y\} \neq \emptyset$.

PROOF.

a) If AG(ϕ) and EGS(ϕ) coincide, then $u \circ \phi^{\leftarrow}(y) = u \circ u\phi^{\leftarrow}(y)$, for all $y \in Y$ and $u \in J_y$. If $u \circ \phi^{\leftarrow}(y) = u \circ u\phi^{\leftarrow}(y)$ for some $y \in Y$ and $u \in J_y$, then

$$QF(u \circ \phi^{\leftarrow}(y), \mathfrak{M}) \cap QF(u \circ u\phi^{\leftarrow}(y), \mathfrak{M}) \neq \emptyset ,$$

so they are equal and AG(ϕ) equals EGS(ϕ).

b) Let ϕ be open; then ϕ' in AG(ϕ) is just ϕ . Clearly, AG(ϕ) and EGS(ϕ) coincide iff EGS(ϕ) reduces to ϕ , which is the case iff ϕ is RIC.

c) Let y be such that $\phi_{\mathrm{ad}}: \mathfrak{Y} \to 2^{\mathfrak{X}}$ is continuous in y (II.1.3.e). So for every $u \in J_y$ we have $u \circ \phi^{\leftarrow}(y) = \phi^{\leftarrow}(y)$. If AG(ϕ) and EGS(ϕ) coincide, then $u \circ \phi^{\leftarrow}(y) = u \circ u\phi^{\leftarrow}(y)$; hence $\phi^{\leftarrow}(y) = u \circ u\phi^{\leftarrow}(y)$ for every $u \in J_y$ and so

$$\phi^{\leftarrow}(y) = \bigcap \{u \circ u\phi^{\leftarrow}(y) \mid u \in J_y\} \neq \emptyset .$$

The other way around can be found in [V 77] 2.3.7.. □

3.13. **THEOREM.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *be a point distal homomorphism of minimal ttgs.*

a) *Let* $\psi: \mathfrak{Z} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs. If* ϕ *or* ψ *is open then* (ϕ,ψ) *satisfies the generalized Bronstein condition.*

b) *If* ϕ *is open then* ϕ *is* RIC.

PROOF.

a) Let $U \times V \cap R_{\phi\psi} \neq \varnothing$ be a basic open set in $R_{\phi\psi}$. By I.3.7.(iii), we may assume (without loss of generality) that $\phi[U]=\psi[V]$. Let $x \in U$ be a ϕ-distal point, then for $z \in V$ with $\phi(x)=\psi(z)$ we have $J_z \subseteq J_{\psi(z)} = J_{\phi(x)} = J_x$ (I.2.10.), so (x,z) is an almost periodic point in $R_{\phi\psi}$.

b) Let $\psi : \mathfrak{X} \to \mathcal{Y}$ be proximal. By a , $R_{\phi\psi}$ has a dense subset of almost periodic points. Define $\theta : \mathfrak{R}_{\phi\psi} \to \mathcal{X}$ as the projection. Then clearly θ is proximal, so $R_{\phi\psi}$ has a unique minimal subset. Hence $\mathfrak{R}_{\phi\psi}$ is minimal and $\phi \perp \psi$. So, by definition, ϕ is RIC. □

3.14. **COROLLARY.** *Let* $\phi : \mathcal{X} \to \mathcal{Y}$ *be a point distal homomorphism of minimal ttgs. Then*

a) AG(ϕ) *and* EGS(ϕ) *coincide;*

b) *the canonical* PI *tower for* ϕ *is an* HPI *tower.*

PROOF.

a) We show that the map ϕ' in AG(ϕ) is point distal. Then it follows from 3.13. that ϕ' is RIC and so that AG(ϕ) and EGS(ϕ) coincide (3.12.b). Let $x \in X$ be a ϕ-distal point, $y=\phi(x)$ and let $u \in J_x$. Then $(x, u \circ \phi^{\leftarrow}(y)) \in X'$ (in AG(ϕ)), and $(x, u \circ \phi^{\leftarrow}(y))$ is a ϕ'-distal point; as follows:

Let $(x', u \circ \phi^{\leftarrow}(y)) \in X'$; then by minimality of $\mathcal{X}'$, there is a $v \in J$ with $(x', u \circ \phi^{\leftarrow}(y)) = v(x', u \circ \phi^{\leftarrow}(y))$. So $v \in J_{x'} \subseteq J_{\phi(x')} = J_y$ and as $J_y = J_x$, $v \in J_x$. Hence $(x, u \circ \phi^{\leftarrow}(y)) = v(x, u \circ \phi^{\leftarrow}(y))$, so $(x, u \circ \phi^{\leftarrow}(y))$ and $(x', u \circ \phi^{\leftarrow}(y))$ are distal.

b) Follows immediately from a. □

3.15. **REMARK.** *Let* $\mathcal{X}$ *be a point distal* MHP *ttg, then* $\mathcal{X}$ *is a strictly* AI-*ttg, (i.e., every proximal extension in the strictly*-PI *tower for* $\mathcal{X}$ *is a-a).*

PROOF. By 1.3. (VST in the absolute case) and the fact that $\mathcal{X}$ is MHP, it follows that there is a strictly-HPI tower for $\mathcal{X}$. As $\mathcal{X}$ is point distal, every map in the tower has to be point distal, which is obvious for the almost periodic homomorphisms, but which can only occur for the highly proximal homomorphisms if they are almost automorphic (1.1.b). □

We shall conclude this section with a characterization of open maps which resembles the definition of RIC extensions (just after I.3.9.).

3.16. **THEOREM.** *Let* $\phi:\mathcal{X}\to\mathcal{Y}$ *be a homomorphism of minimal ttgs. Then* ϕ *is open iff* $\phi\perp\psi$ *for every hp extension* $\psi:\mathcal{Z}\to\mathcal{Y}$ *of minimal ttgs.*

PROOF.

"$\Rightarrow$" Let ϕ be open and let ψ be hp. Define $\theta:\mathfrak{R}_{\phi\psi}\to\mathcal{X}$ as the projection. We shall prove that θ is irreducible (and so, by 2.2.a, that $\phi\perp\psi$). Let $U\times V\cap R_{\phi\psi}$ be a nonempty basic open set in $R_{\phi\psi}$. By I.3.7.(iii), we may assume that $\phi[U]=\psi[V]$. By 2.1. and 2.3., there is a $y\in Y$ with $\psi^{\leftarrow}(y)\subseteq V$. For $x\in U$ with $\phi(x)=y$ we have

$$\theta^{\leftarrow}(x)=\{x\}\times\psi^{\leftarrow}(y)\subseteq U\times V\cap R_{\phi\psi}\,.$$

Hence θ is irreducible.

"$\Leftarrow$" Note that it is sufficient to prove that ϕ is open if $\phi\perp\chi_{\mathcal{Y}}$, where $\chi_{\mathcal{Y}}:\mathcal{Y}^*\to\mathcal{Y}$ is the universal hp extension of $\mathcal{Y}$. For $y\in Y$ and $u\in J_y$, we shall prove that $\phi^{\leftarrow}(y)\subseteq u\circ\phi^{\leftarrow}(y)$. Then, it follows that $\phi^{\leftarrow}(y)=u\circ\phi^{\leftarrow}(y)$ and, as y and $u\in J_y$ are arbitrary, ϕ is open (II.3.12.). Define $\gamma:\mathcal{M}\to\mathcal{Y}$ by $\gamma(p)=py$ for all $p\in M$. Then $\mathcal{Y}^*=\mathfrak{W}\mathfrak{F}(u\circ\gamma^{\leftarrow}(y),\mathcal{M})$ and $\chi_{\mathcal{Y}}(p\circ\gamma^{\leftarrow}(y))=py$ for all $p\in M$. Let $x\in\phi^{\leftarrow}(y)$, then $(x,u\circ\gamma^{\leftarrow}(y))\in R_{\phi\chi_{\mathcal{Y}}}$. So by minimality of $\mathfrak{R}_{\phi\chi_{\mathcal{Y}}}$, there is a $v\in J$ with $x=vx$ and $u\circ\gamma^{\leftarrow}(y)=v\circ\gamma^{\leftarrow}(y)$. As $v\in v\circ\gamma^{\leftarrow}(y)$ we have

$$x=vx\in(v\circ\gamma^{\leftarrow}(y))x=v\circ\gamma^{\leftarrow}(y).x=u\circ\gamma^{\leftarrow}(y).x\subseteq u\circ\phi^{\leftarrow}(y)\,,$$

for $\phi(\gamma^{\leftarrow}(y).x)=\gamma^{\leftarrow}(y).\phi(x)=\gamma^{\leftarrow}(y).y=y$. □

IV.4. LIFTING INVARIANTS

In this section we deal with the problem: what is left of ϕ after lifting it using AG (or EGS) type diagrams. We start with general considerations concerning this problem (4.2. and 4.3.). Those results that are interesting in their own right, lead to to the conclusions in 4.8., telling us about E_{ϕ^*}, Q_{ϕ^*} and P_{ϕ^*} in relation to E_ϕ, Q_ϕ and P_ϕ. After that we generalize the point of view to lifting a pair of homomorphisms, and we show that properties like " $\phi \perp \psi$ " are carried over to ϕ^* and ψ^* (4.16.).

4.1. Consider the following commutative diagram consisting of homomorphisms of minimal ttgs.

$$\begin{tikzcd}
\mathscr{X}' \arrow[r, "\sigma"] \arrow[d, "\phi'"'] & \mathscr{X} \arrow[d, "\phi"] \\
\mathscr{Y}' \arrow[r, "\tau"'] & \mathscr{Y}
\end{tikzcd}$$

Assume that σ is proximal and that $\sigma \times \sigma[R_{\phi'}] = R_\phi$.

4.2. **THEOREM.** *Under the circumstances of 4.1.:*

a) $\sigma \times \sigma[P_{\phi'}] = P_\phi$, *even* $P_{\phi'} = (\sigma \times \sigma)^{\leftarrow}[P_\phi] \cap R_{\phi'}$*;*

b) $\sigma \times \sigma[Q_{\phi'}] = Q_\phi$*;*

c) $Q_{\phi'} \circ P_{\phi'} = (\sigma \times \sigma)^{\leftarrow}[Q_\phi \circ P_\phi] \cap R_{\phi'}$, *so* $\sigma \times \sigma[Q_{\phi'} \circ P_{\phi'}] = Q_\phi \circ P_\phi$.

PROOF. Note that in all cases the inclusion $\subseteq$ is straightforward.

b) Let $(x_1, x_2) \in Q_\phi$ and let $\{(x_1^i, x_2^i)\}_i$ and $\{t_i\}_i$ be nets in R_ϕ and T, such that

$$(x_1^i, x_2^i) \to (x_1, x_2) \text{ and } t_i(x_1^i, x_2^i) \to (x_1, x_1).$$

Then there are $(\bar{x}_1^i, \bar{x}_2^i) \in R_{\phi'}$ with $\sigma \times \sigma(\bar{x}_1^i, \bar{x}_2^i) = (x_1^i, x_2^i)$. Let $(\bar{x}_1, \bar{x}_2) = \lim(\bar{x}_1^i, \bar{x}_2^i)$ and $(z_1, z_2) = \lim t_i(\bar{x}_1^i, \bar{x}_2^i)$, after passing to suitable subnets. Then $\sigma \times \sigma(\bar{x}_1, \bar{x}_2) = (x_1, x_2)$ and $\sigma \times \sigma(z_1, z_2) = (x_1, x_1)$, so $(z_1, z_2) \in P_{\mathscr{X}'} \cap R_{\phi'} = P_{\phi'}$. Let $\alpha \in \mathscr{U}_{X'}$ be open. Then $(z_1, z_2) \in P_{\phi'} \subseteq T\alpha \cap R_\phi$, so $t_i(\bar{x}_1^i, \bar{x}_2^i) \in T\alpha \cap R_\phi$ for all $i \geqslant i_\alpha$. Hence $(\bar{x}_1^i, \bar{x}_2^i) \in T\alpha \cap R_\phi$ for all $i \geqslant i_\alpha$, and so it follows that $(\bar{x}_1, \bar{x}_2) = \lim(\bar{x}_1^i, \bar{x}_2^i) \in \overline{T\alpha \cap R_\phi}$. As α was arbitrary, $(\bar{x}_1, \bar{x}_2) \in Q_{\phi'}$, and $(x_1, x_2) = \sigma \times \sigma(\bar{x}_1, \bar{x}_2) \subseteq \sigma \times \sigma[Q_{\phi'}]$.

c) Let $(x_1,x_2)\in Q_\phi \circ P_\phi$ and let $(x'_1,x'_2)\in R_{\phi'}$ be such that $\sigma\times\sigma(x'_1,x'_2)=(x_1,x_2)$. We shall prove that $(x'_1,x'_2)\in Q_{\phi'}\circ P_{\phi'}$. Let $z_3\in X$ be such that $(x_1,z_3)\in P_\phi$ and $(z_3,x_2)\in Q_\phi$. By I.2.7., there is a minimal left ideal $I\subseteq S_T$ with $px_1=pz_3$ for every $p\in I$. Let $v\in J_{x'_2}(I)$. Then, as $J_{x'_2}\subseteq J_{x_2}$,

$$(vz_3,x_2)=v(z_3,x_2)\in \overline{TQ_\phi}=Q_\phi\,.$$

Let $(z'_3,z'_2)\in Q_{\phi'}$ be such that $\sigma\times\sigma(z'_3,z'_2)=(vz_3,x_2)$ (by b !) and such that $(z'_3,z'_2)=v(z'_3,z'_2)$. Then

$$\sigma(vx'_1)=vx_1=vz_3=\sigma(z'_3) \text{ and } \sigma(x'_2)=x_2=\sigma(z'_2)\,,$$

so vx'_1 and z'_3 are proximal. As they are both v-invariant, they are distal too (I.2.8.), hence $vx'_1=z'_3$. Similarly, $x'_2=z'_2$. As $\phi'(vx'_1)=\phi'(x'_2)=\phi'(x'_1)$, we have $(x'_1,vx'_1)\in R_{\phi'}\cap P_{\mathfrak{X}'}=P_{\phi'}$. Since $(vx'_1,x'_2)=(z'_3,z'_2)\in Q_{\phi'}$, it follows that $(x'_1,x'_2)\in Q_{\phi'}\circ P_{\phi'}$.

a) The proof of this statement is a special case of the proof of c (replace z_3 by x_2). □

4.3. **THEOREM.** *Under the circumstances of 4.1.:*

a) *P_ϕ is a (closed) equivalence relation iff $P_{\phi'}$ is.*

b) *If $Q_\phi=\Delta_X$ then $Q_{\phi'}=P_{\phi'}(=E_{\phi'})$.*

c) *$E_\phi=Q_\phi\circ P_\phi$ iff $E_{\phi'}=Q_{\phi'}\circ P_{\phi'}$.*

d) *If $E_{\phi'}=Q_{\phi'}$ then $E_\phi=Q_\phi$.*

e) *If $E_\phi=Q_\phi\circ P_\phi$ then $\sigma\times\sigma[E_{\phi'}]=E_\phi$.*

PROOF.

a) From 4.2.a it follows that if P_ϕ is a (closed) equivalence relation then $P_{\phi'}$ is a (closed) equivalence relation too.

Suppose that $P_{\phi'}$ is an equivalence relation. First note that this is equivalent to:

$$(x'_1,x'_2)\in P_{\phi'} \text{ iff } (x'_1,x'_2)\in R_{\phi'} \text{ and } px'_1=px'_2 \text{ for every } p\in M\,. \qquad \text{☜}$$

Clearly, if ☜ holds then $P_{\phi'}$ is an equivalence relation.

Conversely, let $P_{\phi'}$ be an equivalence relation. Obviously, the "if"-part of ☜ is true. Let $(x'_1,x'_2)\in P_{\phi'}$ and let $u\in J_{x'_1}$. As $(x'_2,ux'_2)\in P_{\phi'}$ and as $P_{\phi'}$ is an equivalence relation, $(x'_1,ux'_2)\in P_{\phi'}$; so by I.2.8., $x'_1=ux'_2$. Hence for every $p\in M$, $px'_1=pux'_2=px'_2$. This proves ☜.

Let (x_1,x_2) and (x_2,x_3) be elements of P_ϕ, and let $(x_1^*,x_3^*)\in R_{\phi'}$ with $\sigma\times\sigma(x_1^*,x_3^*)=(x_1,x_3)$. By 4.2.a, we can find (x'_1,x'_2) and $(\overline{x}'_2,x'_3)$ in

$P_{\phi'}$ with $\sigma\times\sigma(x'_1,x'_2)=(x_1,x_2)$ and $\sigma\times\sigma(\bar{x}'_2,x'_3)=(x_2,x_3)$. Let $u\in J_{x_1^*}$ and $v\in J_{x_3^*}$; then by proximality of σ we have $x_1^*=ux'_1$ and $x_3^*=vx'_3$. As $P_{\phi'}$ is an equivalence relation it follows from ☞ that $x_1^*=ux'_1=ux'_2$ and $x_3^*=vx'_3=v\bar{x}'_2$. But then $(ux'_2,v\bar{x}'_2)\in R_{\phi'}$. Since $\sigma(ux'_2)=\sigma(u\bar{x}'_2)=ux_2$, we have by proximality of σ that $ux'_2=u\bar{x}'_2$; hence $(ux'_2,v\bar{x}'_2)\in P_{\mathfrak{X}'}$ and

$$(x_1^*,x_3^*)=(ux'_2,v\bar{x}'_2)\in R_{\phi'}\cap P_{\mathfrak{X}'}=P_{\phi'}\,.$$

Consequently

$$(x_1,x_3)=\sigma\times\sigma(x_1^*,x_3^*)\in\sigma\times\sigma[P_{\phi'}]=P_{\phi}\,.$$

If $P_{\phi'}$ is closed then, obviously, P_ϕ is closed.

b) As $Q_{\phi'}\subseteq(\sigma\times\sigma)^{\leftarrow}\sigma\times\sigma[Q_{\phi'}]$ and as, by 4.2.b, $\sigma\times\sigma[Q_{\phi'}]=Q_\phi$, we have by 4.2.a,

$$Q_{\phi'}\subseteq(\sigma\times\sigma)^{\leftarrow}[\Delta_X]\subseteq(\sigma\times\sigma)^{\leftarrow}[P_\phi]\cap R_{\phi'}=P_{\phi'}\,;$$

and so $Q_{\phi'}=P_{\phi'}(=E_{\phi'})$.

c) The "only if"-part follows from 4.2.c.

Conversely, suppose that for ϕ' we have $E_{\phi'}=Q_{\phi'}\circ P_{\phi'}$. We shall prove that $Q_\phi\circ P_\phi\circ Q_\phi\circ P_\phi\subseteq Q_\phi\circ P_\phi$. (Then, clearly, $Q_\phi\circ P_\phi$ is an equivalence relation and it is closed. Indeed,

$$Q_\phi\circ Q_\phi\subseteq Q_\phi\circ P_\phi\circ Q_\phi\circ P_\phi\subseteq Q_\phi\circ P_\phi\subseteq Q_\phi\circ Q_\phi\,,$$

so $Q_\phi\circ P_\phi=Q_\phi\circ Q_\phi$ and, obviously, $Q_\phi\circ Q_\phi$ is closed. Consequently, $E_\phi=Q_\phi\circ P_\phi$.) As follows:

Let (x_1,x_2) and (x_2,x_3) in $Q_\phi\circ P_\phi$ and let $(x_1^*,x_3^*)\in R_{\phi'}$ be such that $\sigma\times\sigma(x_1^*,x_3^*)=(x_1,x_3)$. By 4.2.c, there are (x'_1,x'_2) and $(\bar{x}'_2,x'_3)$ in $Q_{\phi'}\circ P_{\phi'}$ with $\sigma\times\sigma(x'_1,x'_2)=(x_1,x_2)$ and $\sigma\times\sigma(\bar{x}'_2,x'_3)=(x_2,x_3)$. Let $u\in J_{x_1^*}$ and $v\in J_{x_3^*}$; then, by I.2.8. and by proximality of σ, $x_1^*=ux'_1$ and $x_3^*=vx'_3$. So

$$(x_1^*,ux'_2)=u(x'_1,x'_2)\in\overline{T(Q_{\phi'}\circ P_{\phi'})}=\overline{TE_{\phi'}}=E_{\phi'}$$

and, similarly, $(v\bar{x}'_2,x_3^*)\in E_{\phi'}$. Clearly, $(ux'_2,v\bar{x}'_2)\in P_{\mathfrak{X}'}\cap R_{\phi'}=P_{\phi'}$; hence $(x_1^*,x_3^*)\in E_{\phi'}\circ P_{\phi'}\circ E_{\phi'}=E_{\phi'}$. Consequently,

$$(x_1,x_3)=\sigma\times\sigma(x_1^*,x_3^*)\in\sigma\times\sigma[E_{\phi'}]=\sigma\times\sigma[Q_{\phi'}\circ P_{\phi'}]=Q_\phi\circ P_\phi\,,$$

which proves the "if"-part.

d) Completely analoguous to the proof of c.

e) Follows from c and 4.2.c. □

4.4. We shall now look for situations in which the conditions of 4.1. are satisfied. To that end consider the following commutative diagram of homomorphisms of minimal ttgs.

$$\begin{tikzcd}
\mathscr{X}' \arrow[dr, "\phi'"] \arrow[dd, "\sigma"'] & & \mathscr{Z}' \arrow[dl, "\psi'"'] \arrow[dd, "\zeta"] \\
& \mathscr{Y}' \arrow[dd, "\tau"] & \\
\mathscr{X} \arrow[dr, "\phi"'] & & \mathscr{Z} \arrow[dl, "\psi"] \\
& \mathscr{Y} &
\end{tikzcd}$$

4.5. LEMMA. *Consider the diagram above. If* τ *is proximal and if* (ϕ,ψ) *satisfies the generalized Bronstein condition then* $\sigma\times\zeta[R_{\phi'\psi'}]=R_{\phi\psi}$. (compare III.5.2.)

PROOF. Clearly, $\sigma\times\zeta[R_{\phi'\psi'}]\subseteq R_{\phi\psi}$.
Conversely, let (x,z) be an almost periodic point in $R_{\phi\psi}$, say $(x,z)=u(x,z)$ for some $u\in J$, and let $(x',z')=u(x',z')\in X'\times Z'$ with $\sigma\times\zeta(x',z')=(x,z)$. Then $(\phi'(x'),\psi'(z'))=u(\phi'(x'),\psi'(z'))\in R_\tau$, for

$$\tau(\phi'(x'))=\phi\circ\sigma(x')=\phi(x)=\psi(z)=\psi\circ\zeta(z')=\tau(\psi'(z')).$$

As τ is a proximal map, $\phi'(x')=\psi'(z')$; so $(x',z')\in R_{\phi'\psi'}$. Therefore $JR_{\phi\psi}\subseteq\sigma\times\zeta[R_{\phi'\psi'}]\subseteq R_{\phi\psi}$. Since the almost periodic points are dense in $R_{\phi\psi}$ and $\sigma\times\zeta[R_{\phi'\psi'}]$ is closed, the lemma follows. □

4.6. LEMMA. *Let* $\phi:\mathscr{X}\to\mathscr{Y}$ *and* $\psi:\mathscr{Y}\to\mathscr{Z}$ *be homomorphisms of minimal ttgs with* ϕ *open and* ψ *hp. Let* $z\in Z$, $y\in\psi^{\leftarrow}(z)$ *and* $p\in M$. *Then* $\phi^{\leftarrow}(py)=p\circ\phi^{\leftarrow}\psi^{\leftarrow}(z)$.

PROOF. As ϕ_{ad} is continuous, $p\circ\phi^{\leftarrow}\psi^{\leftarrow}(z)=\phi^{\leftarrow}(p\circ\psi^{\leftarrow}(z))$. But ψ is hp so $p\circ\psi^{\leftarrow}(z)=\{py\}$. □

4.7. LEMMA. *Consider the diagram in 4.4.. Let* τ *be hp, and let* ϕ' *and* ψ' *be open. If* ϕ *or* ψ *is open then* $\sigma\times\zeta[R_{\phi'\psi'}]=R_{\phi\psi}$.

PROOF. Assume ϕ to be open, let $y\in Y$, $y'\in\tau^{\leftarrow}(y)$, and observe that

$$R_{\phi'\psi'}=\bigcup\{\phi'^{\leftarrow}(py')\times\psi'^{\leftarrow}(py')\mid p\in M\}.$$

By 4.6., it follows that

$$R_{\phi'\psi'} = \bigcup\{p\circ\phi'^{\leftarrow}\tau^{\leftarrow}(y)\times p\circ\psi'^{\leftarrow}\tau^{\leftarrow}(y) \mid p\in M\}\,.$$

As $\phi'^{\leftarrow}\tau^{\leftarrow}(y)=\sigma^{\leftarrow}\phi^{\leftarrow}(y)$ and $\psi'^{\leftarrow}\tau^{\leftarrow}(y)=\zeta^{\leftarrow}\psi^{\leftarrow}(y)$ we have

$$R_{\phi'\psi'} = \bigcup\{p\circ\sigma^{\leftarrow}\phi^{\leftarrow}(y)\times p\circ\zeta^{\leftarrow}\psi^{\leftarrow}(y) \mid p\in M\}\,.$$

So

$$\begin{aligned}\sigma\times\zeta[R_{\phi'\psi'}] &= \bigcup\{\sigma(p\circ\sigma^{\leftarrow}\phi^{\leftarrow}(y))\times\zeta(p\circ\zeta^{\leftarrow}\psi^{\leftarrow}(y)) \mid p\in M\} = \\ &= \bigcup\{p\circ\phi^{\leftarrow}(y)\times p\circ\psi^{\leftarrow}(y) \mid p\in M\} = \\ &= \bigcup\{\phi^{\leftarrow}(py)\times p\circ\psi^{\leftarrow}(y) \mid p\in M\}\end{aligned}$$

by openness of ϕ. Since $\psi^{\leftarrow}(py)=\bigcup\{q\circ\psi^{\leftarrow}(y) \mid q\in M \text{ with } qy=py\}$ it follows that

$$\phi^{\leftarrow}(py)\times\psi^{\leftarrow}(py)=\bigcup\{\phi^{\leftarrow}(qy)\times q\circ\psi^{\leftarrow}(y) \mid q\in M \text{ with } qy=py\}\,,$$

hence that $\sigma\times\zeta[R_{\phi'\psi'}]=\bigcup\{\phi^{\leftarrow}(py)\times\psi^{\leftarrow}(py) \mid p\in M\}=R_{\phi\psi}$. ☐

From 4.5. and 4.7. it follows that the conditions in 4.1. are satisfied in the following situations (notations as in 4.1.):

a) σ, τ proximal and ϕ satisfying the Bronstein condition. For instance: EGS(ϕ), $\mathfrak{A}(\phi)$, AG(ϕ) and *(ϕ) with ϕ a Bc extension.
b) σ proximal, τ hp, ϕ' and ϕ open. For instance: *(ϕ) with ϕ open.

4.8. **COROLLARY.** *Let $\phi:\mathfrak{X}\to\mathfrak{Y}$ be a homomorphism of minimal ttgs and let $\phi^*:\mathfrak{X}^*\to\mathfrak{Y}^*$ be the induced map between the universal highly proximal extensions of $\mathfrak{X}$ and $\mathfrak{Y}$ (as in *(ϕ), see* 3.10.*). Let ϕ be open or let ϕ satisfy the Bronstein condition then*
a) *P_{ϕ} is a (closed) equivalence relation iff P_{ϕ^*} is;*
b) *if $Q_{\phi}=\Delta_X$ then $Q_{\phi^*}=P_{\phi^*}(=E_{\phi^*})$;*
c) *$E_{\phi}=Q_{\phi}\circ P_{\phi}$ iff $E_{\phi^*}=Q_{\phi^*}\circ P_{\phi^*}$;*
d) *if $E_{\phi^*}=Q_{\phi^*}$ then $E_{\phi}=Q_{\phi}$;*
e) *if ϕ is almost periodic (distal) then $\phi^*=\theta\circ\kappa$, where κ is hp and θ is almost periodic (distal).*
(compare VIII.2.1.).

PROOF.

a, b, c and d are immediate from 4.3. and the discussion above.

e) Suppose ϕ is distal. Then P_ϕ is a closed equivalence relation, so P_{ϕ^*} is a closed equivalence relation by a, and $\phi^* = \theta \circ \kappa$ with κ the proximal quotient map defined by P_{ϕ^*} and θ distal. If ϕ is almost periodic, then $Q_\phi = \Delta_X$, so $Q_{\phi^*} = P_{\phi^*}$ and θ is even almost periodic. So we only have to prove that κ is hp. As follows:
Let $Z = X^*/P_{\phi^*}$ and define $\psi: \mathfrak{Z} \to \mathfrak{X}$ by $\psi(\kappa(x)) = \sigma(x)$.

$$\begin{tikzcd}
\mathfrak{X}^* \arrow[rr, "\sigma"] \arrow[dd, "\phi^*"'] \arrow[dr, "\kappa"] & & \mathfrak{X} \arrow[dd, "\phi"] \\
& \mathfrak{Z} \arrow[ur, dashed, "\psi"'] \arrow[dl, "\theta"] & \\
\mathfrak{Y}^* \arrow[rr, "\tau"'] & & \mathfrak{Y}
\end{tikzcd}$$

Observe that ψ and κ are hp if ψ is well defined (2.4.b).
Suppose that $\kappa(x) = \kappa(x')$ then $(x, x') \in R_{\phi^*}$, so $(\sigma(x), \sigma(x')) \in R_\phi$. As ϕ is distal (almost periodic), $\sigma(x)$ and $\sigma(x')$ are distal. As κ is proximal, $(x, x') \in P_{\phi^*}$, so $(\sigma(x), \sigma(x')) \in P_\phi$; hence $\sigma(x) = \sigma(x')$ and so $\psi(\kappa(x)) = \psi(\kappa(x'))$; i.e., ψ is well defined. □

4.9. Consider the next commutative diagram of homomorphisms of minimal ttgs, considered on the phase spaces.

$$\begin{tikzcd}
X' \arrow[rrr, "\sigma"] \arrow[dd, "\phi'"'] \arrow[dr, "\kappa'"] & & & X \arrow[dd, "\phi"] \arrow[dl, "\kappa"'] \\
& X'/E_{\phi'} \arrow[r, "\xi"'] \arrow[dl, "\theta'"] & X/E_\phi \arrow[dr, "\theta"'] & \\
Y' \arrow[rrr, "\tau"'] & & & Y
\end{tikzcd}$$

Let σ be proximal. Note that $\xi: X'/E_{\phi'} \to X/E_\phi$ always exists as a homomorphism of minimal ttgs, because $\sigma \times \sigma[E_{\phi'}] \subseteq E_\phi$.

4.10. **LEMMA.** *Consider the diagram of 4.9..*

a) *If* $\sigma\times\sigma[E_{\phi'}]=E_\phi$ *then* ξ *is proximal.*

b) *If* ξ *is proximal and* $\sigma\times\sigma[R_{\phi'}]=R_\phi$ *then* $\sigma\times\sigma[E_{\phi'}]=E_\phi$.

PROOF.

a) Suppose $\xi(\kappa'(x'_1))=\xi(\kappa'(x'_2))$. We shall show that $\kappa'(x'_1)$ and $\kappa'(x'_2)$ are proximal. As $\xi\circ\kappa'=\kappa\circ\sigma$, we have $(\sigma(x'_1),\sigma(x'_2))\in E_\phi$. By assumption, we can find $(z_1,z_2)\in E_{\phi'}$ with $\sigma\times\sigma(z_1,z_2)=\sigma\times\sigma(x'_1,x'_2)$. Then, by proximality of $\sigma\times\sigma$, it follows that (z_1,z_2) and (x'_1,x'_2) are proximal in $X'\times X'$; hence $(\kappa'(z_1),\kappa'(z_2))$ and $\kappa'(x'_1),\kappa'(x'_2))$ are proximal in $X'/E_{\phi'}\times X'/E_{\phi'}$. But as $(z_1,z_2)\in E_{\phi'}$, $\kappa'(z_1)=\kappa'(z_2)$, so $(\kappa'(x'_1),\kappa'(x'_2))$ is proximal to a point in the diagonal; i.e., $\kappa'(x'_1)$ and $\kappa'(x'_2)$ are proximal.

b) Let $(x_1,x_2)\in E_\phi$ and let $(x'_1,x'_2)\in R_{\phi'}$ be such that $\sigma\times\sigma(x'_1,x'_2)=(x_1,x_2)$. Then

$$\xi(\kappa'(x'_1))=\kappa\circ\sigma(x'_1)=\kappa(x_1)=\kappa(x_2)=\xi(\kappa'(x'_2)),$$

so $\kappa'(x'_1)$ and $\kappa'(x'_2)$ are proximal. As θ' is almost periodic and

$$(\kappa'(x'_1),\kappa'(x'_2))=\kappa'\times\kappa'(x'_1,x'_2)\in\kappa'\times\kappa'[R_{\phi'}]=R_{\theta'}$$

it follows that $\kappa'(x'_1)=\kappa'(x'_2)$ and so that $(x'_1,x'_2)\in E_{\phi'}$. ☐

4.11. In particular, 4.10 applies to AG(ϕ), EGS(ϕ) and $\mathfrak{A}(\phi)$ in case ϕ satisfies the Bronstein condition (compare 4.5. and III.5.2., 5.3.) and to *(ϕ) in case ϕ is open or ϕ is a Bc extension.

4.12. **THEOREM.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a homomorphism of minimal ttgs. Consider* *(ϕ) *and let the map* $\xi:X^*/E_{\phi^*}\to X/E_\phi$ *be as in 4.9.. If* $\sigma\times\sigma[E_{\phi^*}]=E_\phi$ *then* ξ *is highly proximal. In particular,* ξ *is highly proximal in each of the following cases:*

a) ϕ *is a* Bc *extension;*

b) ϕ *is open and* $E_\phi=Q_\phi\circ P_\phi$.

PROOF. First note that, by 4.5., 4.7., 4.3.e and III.3.9., both cases (a and b) imply that $\sigma\times\sigma[R_{\phi^*}]=R_\phi$ and $\sigma\times\sigma[E_{\phi^*}]=E_\phi$.

As $\theta:X/E_\phi\to Y$ is almost periodic (notation as in 4.9.), it follows from 4.8.e, that $\theta^*:(X/E_\phi)^*\to Y^*$ can be written as $\theta^*=\nu\circ\mu$, where $\mu:(X/E_\phi)^*\to Z$ is hp and $\nu:Z\to Y^*$ is almost periodic. Clearly, Z is a factor of X^*, and as $\theta': X^*/E_{\phi^*} \to Y^*$ is the maximal almost periodic extension of Y^* between X^* and Y^*, there is a map $\eta: X^*/E_{\phi^*} \to Z$ with $\theta' = \nu \circ \eta$. By I.1.21.a, η is almost periodic and so by 4.8.e, the map $\eta^*: (X^*/E_{\phi^*})^* \to Z^*$ can be written as $\eta^* = \alpha \circ \beta$, where β is hp and α is almost periodic. Note that by high proximality of μ, $Z^* = (X/E_\phi)^*$, so $\eta^* = \xi^*$. However, by the assumption, it follows from 4.10. that ξ is proximal; hence ξ^* is proximal. But then η^* is proximal and, by I.1.21.a, α is proximal, so α is an isomorphism and $\eta^* = \beta$ is highly proximal. As η^* is open, η^* is an isomorphism, so ξ is highly proximal (for $\xi^* = \eta^*$ is an isomorphism). □

4.13. **THEOREM.** *Consider the next commutative diagram consisting of homomorphisms of minimal ttgs. Let σ and ζ be highly proximal.*

$$\begin{tikzcd}
\mathscr{X}' \arrow[dr, "\phi'"] \arrow[dd, "\sigma"'] & & \mathscr{Z}' \arrow[dl, "\psi'"'] \arrow[dd, "\zeta"] \\
& \mathscr{Y}' \arrow[dd, "\tau"] & \\
\mathscr{X} \arrow[dr, "\phi"'] & & \mathscr{Z} \arrow[dl, "\psi"] \\
& \mathscr{Y} &
\end{tikzcd}$$

If ϕ' or ψ' is open, or if (ϕ',ψ') satisfies gBc, *then the map*

$$\sigma \times \zeta : R_{\phi'\psi'} \to \sigma \times \zeta[R_{\phi'\psi'}]$$

is irreducible.

PROOF. If ϕ' or ψ' is open, or if (ϕ',ψ') satisfies gBc then for every open $W \subseteq R_{\phi'\psi'}$ there are nonempty open U and V in X' and Z' such that $\phi'[U] = \psi'[V]$, while $U \times V \cap R_{\phi'\psi'} \subseteq W$ (by I.3.7.).

Let W be open in $R_{\phi'\psi'}$ and let U and V be as above. By 2.1., there is a nonempty open $U' = \sigma^{\leftarrow}\sigma[U'] \subseteq U$. As $\emptyset \neq (\phi'[U'])^\circ \subseteq \psi'[V]$ there is a nonempty open $V' = \zeta^{\leftarrow}\zeta[V']$ with $V' \subseteq V \cap \psi'^{\leftarrow}[(\phi'[U'])^\circ]$. Clearly, $U' \times V' = (\sigma \times \zeta)^{\leftarrow}(\sigma \times \zeta)[U' \times V']$, hence $U' \times V' \cap R_{\phi'\psi'}$ contains a full fiber under $\sigma \times \zeta : R_{\phi'\psi'} \to \sigma \times \zeta[R_{\phi'\psi'}]$. Since $U' \times V' \cap R_{\phi'\psi'} \subseteq W$ this shows that $\sigma \times \zeta : R_{\phi'\psi'} \to \sigma \times \zeta[R_{\phi'\psi'}]$ is irreducible. □

4.14. There are two "standard" diagrams of the type as exposed in 4.13..

A The one obtained by the *-construction.

Let $\phi:\mathfrak{X}\to\mathfrak{Y}$ and $\psi:\mathfrak{Z}\to\mathfrak{Y}$ be homomorphisms of minimal ttgs. Then we can construct $*(\phi,\psi)$ as follows (note that ϕ^* and ψ^* are open):

$$\begin{tikzcd}
\mathfrak{X}^* \arrow[r, "\phi^*"] \arrow[d, "\sigma = \chi_{\mathfrak{X}}"'] & \mathfrak{Y}^* \arrow[d, "\tau = \chi_{\mathfrak{Y}}"] & \mathfrak{Z}^* \arrow[l, "\psi^*"'] \arrow[d, "\zeta = \chi_{\mathfrak{Z}}"] \\
\mathfrak{X} \arrow[r, "\phi"'] & \mathfrak{Y} & \mathfrak{Z} \arrow[l, "\psi"]
\end{tikzcd}$$

B The one obtained by a double diagram construction.

Let $y_0 \in Y$ and $u \in J_{y_0}$. Define

$$Y' := \{(p\circ\phi^{\leftarrow}(y_0), p\circ\psi^{\leftarrow}(y_0)) \mid p\in M\} \subseteq 2^X \times 2^Z .$$

Then, clearly, $\mathfrak{Y}'$ is minimal and $\tau:\mathfrak{Y}'\to\mathfrak{Y}$ is hp, where τ is defined by $\tau(p\circ\phi^{\leftarrow}(y_0), p\circ\psi^{\leftarrow}(y_0)) = py_0$ for all $p\in M$. For let

$$\tau_X : \mathfrak{W}\mathfrak{F}(u\circ\phi^{\leftarrow}(y_0), \mathfrak{X}) \to \mathfrak{Y} \text{ and } \tau_Z : \mathfrak{W}\mathfrak{F}(u\circ\psi^{\leftarrow}(y_0), \mathfrak{Z}) \to \mathfrak{Y}$$

be the maps in AG(ϕ) and AG(ψ). Then $\tau^{\leftarrow}(y_0) \subseteq \tau_X^{\leftarrow}(y_0) \times \tau_Z^{\leftarrow}(y_0)$; hence

$$u\circ\tau^{\leftarrow}(y_0) \subseteq u\circ(\tau_X^{\leftarrow}(y_0)\times\tau_Z^{\leftarrow}(y_0)) =$$

$$= u\circ\tau_X^{\leftarrow}(y_0) \times u\circ\tau_Z^{\leftarrow}(y_0) = (u\circ\phi^{\leftarrow}(y_0), u\circ\psi^{\leftarrow}(y_0)) ,$$

and τ is highly proximal. Define X' and Z' by

$$X' := \{(x,(A,B)) \mid (A,B)\in Y' \text{ and } x\in A\}$$

$$Z' := \{(z,(A,B)) \mid (A,B)\in Y' \text{ and } z\in B\} .$$

Let $\phi':\mathfrak{X}'\to\mathfrak{Y}'$, $\sigma:\mathfrak{X}'\to\mathfrak{X}$, $\psi':\mathfrak{Z}'\to\mathfrak{Y}'$ and $\zeta:\mathfrak{Z}'\to\mathfrak{Z}$ be the projections. Using our knowledge about AG(ϕ) and AG(ψ) it is straightforward to show that ϕ' and ψ' are open and that σ and ζ are irreducible; hence that $\mathfrak{X}'$ and $\mathfrak{Z}'$ are minimal, and so that σ and ζ are hp.

This diagram will be called AG(ϕ,ψ). Note that AG(ϕ,ψ) reduces to (ϕ,ψ) if ϕ and ψ are open, and that $*(\phi,\phi)$ and AG(ϕ,ϕ) are just two times $*(\phi)$ and AG(ϕ) respectively.

4.15. Consider $*(\phi,\psi)$ and $AG(\phi,\psi)$, with notation as in 4.13..
If (ϕ,ψ) satisfies gBc or if ϕ or ψ is open then $\sigma\times\zeta[R_{\phi'\psi'}]=R_{\phi\psi}$, so $\sigma\times\zeta: R_{\phi'\psi'}\to R_{\phi\psi}$ is irreducible.
In particular, if ϕ is open or if ϕ is a Bc extension, then $\sigma\times\sigma: R_{\phi'}\to R_{\phi}$ is irreducible (in case ϕ is open this is only meaningful in the $*$ version).

4.16. **THEOREM.** *Let $\phi:\mathcal{X}\to\mathcal{Y}$ and $\psi:\mathcal{Z}\to\mathcal{Y}$ be homomorphisms of minimal ttgs. Let * refer to $*(\phi,\psi)$ and $'$ to $AG(\phi,\psi)$.*

a) *If (ϕ,ψ) satisfies gBc then (ϕ^*,ψ^*) and (ϕ',ψ') do.*
If $\phi\perp\psi$ then $\phi^*\perp\psi^*$ and $\phi'\perp\psi'$.

b) *Let ϕ or ψ be open. Then (ϕ,ψ) satisfies gBc iff (ϕ^*,ψ^*) satisfies gBc iff (ϕ',ψ') satisfies gBc.*

c) *Let ϕ or ψ be open or let (ϕ,ψ) satisfy gBc. Then*
$\phi\perp\psi$ iff $\phi^*\perp\psi^*$ iff $\phi'\perp\psi'$, and
$\phi\mathrel{\dot{\perp}}\psi$ iff $\phi^*\mathrel{\dot{\perp}}\psi^*$ iff $\phi'\mathrel{\dot{\perp}}\psi'$.

PROOF. Notation as in 4.14..
In all cases $\sigma\times\zeta[R_{\phi'\psi'}]=R_{\phi\psi}$ (4.5. and 4.7.), so $\sigma\times\zeta: R_{\phi'\psi'}\to R_{\phi\psi}$ is irreducible by 4.13.. The theorem now follows from 2.2.. ☐

4.17. **THEOREM.** *Let $\phi:\mathcal{X}\to\mathcal{Y}$ be a homomorphism of minimal ttgs. Let * refer to $*(\phi)$ and $'$ to $AG(\phi)$.*

a) *If ϕ is a* Bc *extension then ϕ^* and ϕ' are* Bc *extensions.*
If ϕ is open then ϕ is a Bc *extension iff ϕ^* is a* Bc *extension.*

b) *If ϕ is open or if ϕ is a* Bc *extension then ϕ is weakly mixing iff ϕ^* is weakly mixing iff ϕ' is weakly mixing.*

c) *If ϕ is open then ϕ is a* RIC *extension iff ϕ^* is a* RIC *extension.*

PROOF.

a and b Follow immediately from 4.16..

c) Let ϕ be open. Suppose that ϕ is a RIC extension and let $\kappa:\mathfrak{M}(\mathcal{Y}^*)\to\mathcal{Y}^*$ be the universal minimal proximal extension of $\mathcal{Y}^*$. Then $\tau\circ\kappa:\mathfrak{M}(\mathcal{Y}^*)\to\mathcal{Y}$ is proximal, so $\phi\perp\tau\circ\kappa$. Clearly, $(\tau\circ\kappa)^*=\kappa$, so by 4.16.a, it follows that $\phi^*\perp\kappa$; hence, by definition, ϕ is a RIC extension.
Conversely, suppose that ϕ^* is a RIC extension and let $\kappa':\mathfrak{M}(\mathcal{Y})\to\mathcal{Y}$ be the universal minimal proximal extension of $\mathcal{Y}$. Then there is a map $\eta:\mathfrak{M}(\mathcal{Y})\to\mathcal{Y}^*$ with $\tau\circ\eta=\kappa'$. As ϕ^* is a RIC extension, $\phi^*\perp\eta$, and by openness of ϕ, it follows from 4.16.c and the fact that $\eta=(\kappa')^*$ that $\phi\perp\kappa'$. Consequently, ϕ is a RIC extension. ☐

4.18. Note that, by 4.16.c with $\mathscr{Y} = \{\star\}$, it follows that $\mathscr{X} \perp \mathscr{Z}$ iff $\mathscr{X}' \perp \mathscr{Z}'$ whenever $\mathscr{X}$ and $\mathscr{X}'$ as well as $\mathscr{Z}$ and $\mathscr{Z}'$ are hp equivalent (two minimal ttgs are called *hp equivalent* if they have isomorphic MHP extensions). For, clearly, every map $\mathscr{X} \to \{\star\}$ is open.

IV.5. HPI EXTENSIONS

We shall briefly discuss HPI extensions. Among others we show for a homomorphism $\phi = \theta \circ \psi$ of minimal ttgs that ϕ is an HPI extension iff θ and ψ are HPI extensions.

5.1. In 1.3. we already mentioned the concept of an HPI extension. For completeness we shall define it again:
An extension $\phi: \mathscr{X} \to \mathscr{Y}$ of minimal ttgs is called a *strictly*-HPI *extension* if there is an ordinal ν and a tower $\{\phi_\alpha^\beta : \mathscr{X}_\beta \to \mathscr{X}_\alpha \mid \alpha \leqslant \beta \leqslant \nu\}$ consisting of homomorphisms of minimal ttgs with $\mathscr{X}_0 = \mathscr{Y}$ and $\mathscr{X}_\nu = \mathscr{X}$ such that for every ordinal $\alpha < \nu$ the extension $\phi_\alpha^{\alpha+1} : \mathscr{X}_{\alpha+1} \to \mathscr{X}_\alpha$ is either almost periodic or highly proximal.
An extension $\phi: \mathscr{X} \to \mathscr{Y}$ of minimal ttgs is called an HPI *extension* if there is a minimal ttg $\mathscr{X}'$ and homomorphisms $\theta: \mathscr{X}' \to \mathscr{X}$ and $\psi: \mathscr{X}' \to \mathscr{Y}$ such that $\psi = \phi \circ \theta$, θ is highly proximal and ψ is strictly-HPI (compare III.4.1.).

$$\begin{tikzcd}
\mathscr{X}' \arrow[rr, "\psi"] \arrow[dr, "\theta"'] & & \mathscr{Y} \\
& \mathscr{X} \arrow[ur, "\phi"'] &
\end{tikzcd}$$

5.2. **LEMMA.** *Let* $\phi: \mathscr{X} \to \mathscr{Y}$ *be an* HPI *extension of minimal ttgs. Then* $\phi^* : \mathscr{X}^* \to \mathscr{Y}^*$ *is a strictly-*HPI *extension.*

PROOF. Let $\mathscr{X}'$ be a minimal ttg such that there is an hp extension $\theta: \mathscr{X}' \to \mathscr{X}$ and a strictly-HPI extension $\psi: \mathscr{X}' \to \mathscr{Y}$. As $\mathscr{X}'^* = \mathscr{X}^*$ it is sufficient to prove that $\psi^* : \mathscr{X}'^* \to \mathscr{Y}^*$ is strictly-HPI (for, clearly, $\psi^* = \phi^*$).

Let $\{\psi_\alpha^\beta : \mathfrak{X}_\beta \to \mathfrak{X}_\alpha \mid \alpha \leqslant \beta \leqslant \nu\}$ be the HPI tower for ψ, so $\mathfrak{X}_\nu = \mathfrak{X}'$, $\mathfrak{X}_0 = \mathfrak{Y}$. Then $(\psi_\alpha^{\alpha+1})^*$ is either trivial (if $\psi_\alpha^{\alpha+1}$ is hp) or, by 4.8.e, $(\psi_\alpha^{\alpha+1})^* = \xi \circ \eta$ with η hp and ξ almost periodic (if $\psi_\alpha^{\alpha+1}$ is almost periodic). Hence $\{(\psi_\alpha^{\alpha+1})^* : \mathfrak{X}_{\alpha+1}^* \to \mathfrak{X}_\alpha^* \mid \alpha < \nu\}$ is an HPI tower for ψ^* and so ψ^* is strictly-HPI. □

5.3. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be an open* HPI *extension, then* ϕ *is a* RIC *extension.*

PROOF. As is shown in 5.2., ϕ^* has a tower consisting of extensions $(\psi_\alpha^{\alpha+1})^* = \xi \circ \eta : \mathfrak{X}_{\alpha+1}^* \to \mathfrak{X}_\alpha^*$ with ξ almost periodic and η hp, coming from almost periodic extensions $\psi_\alpha^{\alpha+1} : \mathfrak{X}_{\alpha+1} \to \mathfrak{X}_\alpha$. By 4.17.c, it follows that $(\psi_\alpha^{\alpha+1})^*$ is a RIC extension for every $\alpha < \nu$. As ϕ^* is the inverse limit of RIC extensions, ϕ^* itself is a RIC extension (III.1.10.c); hence, again by 4.17.c, ϕ is a RIC extension. □

For the following it would have been more elegant if we would have used pointed ttgs, especially to see that the diagrams involved are commutative. In spite of that, we don't, and leave the checking of the commutativity of the diagrams as exercises for the reader.

5.4. **THEOREM.** *Let* $\phi : \mathfrak{X}^* \to \mathfrak{Y}$ *be an* HPI *extension of minimal ttgs. Suppose that* $\phi = \theta \circ \psi$,

$$\begin{tikzcd}
\mathfrak{X}^* \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & \mathfrak{Y} \\
& \mathfrak{Z} \arrow[ur, "\theta"'] &
\end{tikzcd}$$

then ψ *is strictly* HPI.

PROOF. We shall prove that ψ^* is strictly-HPI. As $\psi = \chi_{\mathfrak{Z}} \circ \psi^*$, where $\chi_{\mathfrak{Z}} : \mathfrak{Z}^* \to \mathfrak{Z}$ is the canonical maximal hp extension of $\mathfrak{Z}$, it follows that ψ is strictly-HPI too.

First note that, by 5.2., ϕ^* is a strictly-HPI extension. So let

$$\{(\phi_\alpha^\beta)^* : \mathfrak{X}_\beta^* \to \mathfrak{X}_\alpha^* \mid \alpha \leqslant \beta \leqslant \nu\}, \text{ with } \mathfrak{X}_\nu^* = \mathfrak{X}^* \text{ and } \mathfrak{X}_0^* = \mathfrak{Y}^*$$

be a strictly-HPI tower for ϕ^* (as in the proof of 5.2.).

Let $\mathfrak{Z}_0 = \mathfrak{Z}_0^* := \mathfrak{Z}^*$ and define $\psi_0^\nu : \mathfrak{X}^* \to \mathfrak{Z}_0^*$ by $\psi_0^\nu := \psi^*$, and

$\theta_0 = \theta_0^* : \mathfrak{Z}_0^* \to \mathfrak{X}_0^* = \mathfrak{Y}^*$ by $\theta_0 := \theta^*$. Note that $(\phi_0^\nu)^* = \theta_0 \circ \psi_0^\nu$.
Suppose that $\mathfrak{Z}_\alpha^*$, $\psi_\alpha^\nu : \mathfrak{X}^* \to \mathfrak{Z}_\alpha^*$ and $\theta_\alpha : \mathfrak{Z}_\alpha^* \to \mathfrak{X}_\alpha^*$ are defined for all ordinals $\alpha < \beta$ in such a way that $(\phi_\alpha^\nu)^* = \theta_\alpha \circ \psi_\alpha^\nu$.
If β is a limit ordinal define $\mathfrak{Z}_\beta^*$, ψ_β^ν and θ_β by taking inverse limits.
Suppose that β is a nonlimit ordinal, then $\mathfrak{Z}_{\beta-1}^*$, $\psi_{\beta-1}^\nu$ and $\theta_{\beta-1}$ are defined such that $(\phi_{\beta-1}^\nu)^* = \theta_{\beta-1} \circ \psi_{\beta-1}^\nu$.

$$\begin{tikzcd}
\mathfrak{X}^* \arrow[rr, "(\phi_{\beta-1}^\nu)^*"] \arrow[dr, "\psi_{\beta-1}^\nu"'] & & \mathfrak{X}_{\beta-1}^* \\
& \mathfrak{Z}_{\beta-1} \arrow[ur, "\theta_{\beta-1}"'] &
\end{tikzcd}$$

Clearly $Q_{\psi_{\beta-1}^\nu} \subseteq Q_{(\phi_{\beta-1}^\nu)^*}$, hence $E_{\psi_{\beta-1}^\nu} \subseteq E_{(\phi_{\beta-1}^\nu)^*}$. Define $\mathfrak{Z}_\beta := \mathfrak{X}^*/E_{\psi_{\beta-1}^\nu}$. Then there is a map $\xi : \mathfrak{Z}_\beta \to \mathfrak{X}^*/E_{(\phi_{\beta-1}^\nu)^*}$; hence there is a map $\eta : \mathfrak{Z}_\beta \to \mathfrak{X}_\beta$ ($\mathfrak{X}_\beta \to \mathfrak{X}_{\beta-1}^*$ almost periodic in the tower for ϕ^*). Let $\theta_\beta := \eta^*$ and $\psi_{\beta-1}^\nu : \mathfrak{X}^* \to \mathfrak{Z}_\beta^*$ by $\psi_{\beta-1}^\nu = \kappa^*$, where $\kappa : \mathfrak{X}^* \to \mathfrak{X}^*/E_{\psi_{\beta-1}^\nu} = \mathfrak{Z}_\beta$ is the quotient map. It is readily seen that $(\phi_\beta^\nu)^* = \theta_\beta \circ \psi_\beta^\nu$. Observe that $\mathfrak{Z}_\beta \to \mathfrak{Z}_{\beta-1}^*$ is almost periodic (by definition of $\mathfrak{Z}_\beta$) and so that $\mathfrak{Z}_\beta^* \to \mathfrak{Z}_{\beta-1}^*$ is strictly-HPI.
By transfinite induction $\mathfrak{Z}_\nu^*$, ψ_ν^ν and θ_ν are defined such that $(\phi_\nu^\nu)^* = \theta_\nu \circ \psi_\nu^\nu$. As $(\phi_\nu^\nu)^* = id_{\mathfrak{X}^*}$, it follows that ψ_ν^ν is an isomorphism; hence $\mathfrak{X}^* \cong \mathfrak{Z}_\nu^*$ and $\mathfrak{Z}_\nu^*$ is a strictly-HPI extension of $\mathfrak{Z}^*$ (by construction), which proves the theorem after observing that $\mathfrak{Z}_\nu^* \to \mathfrak{Z}^*$ is just ψ^* . □

5.5. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs and suppose that* $\phi = \theta \circ \psi$ *. If* ϕ *is an* HPI *extension then so is* ψ .

PROOF. As $\phi^* = \theta^* \circ \psi^*$ and as, by 5.2., ϕ^* is (strictly) HPI, it follows from 5.4. that ψ^* is strictly-HPI. Let $\psi : \mathfrak{X} \to \mathfrak{Z}$, then ψ is a factor of $\chi_{\mathfrak{Z}} \circ \psi^*$ under an hp map (see the construction of the * diagram). As $\chi_{\mathfrak{Z}} \circ \psi^*$ is strictly-HPI, ψ is HPI. □

5.6. Note that for an HPI extension $\phi : \mathfrak{X} \to \mathfrak{Y}$ the diagrams AG(ϕ) and EGS(ϕ) coincide. For, ϕ^* is strictly-HPI and open, so ϕ^* is a RIC extension. Hence, ϕ' in AG(ϕ) is a RIC extension (4.17.). So, by 3.12.a, AG(ϕ) and EGS(ϕ) coincide.

5.7. **THEOREM.** *Let* $\phi : \mathcal{X}^* \to \mathcal{Y}$ *be a* PI *extension of minimal ttgs. Then* ϕ *is an* HPI *extension iff every open* ψ *such that* $\phi = \theta \circ \psi$ *(for some* θ*) is a* RIC *extension.*

PROOF. Suppose that ϕ is an HPI extension. Then, by 5.5., a map ψ as in the theorem is an HPI extension. Hence, as such a ψ is (assumed to be) open, it is a RIC extension by 5.3..
Now suppose ϕ is a PI extension such that every open ψ with $\phi = \theta \circ \psi$ for some θ is a RIC extension. In particular, $\phi^* : \mathcal{X}^* \to \mathcal{Y}^*$ is a RIC extension, for $\phi = \chi_{\mathcal{Y}} \circ \phi^*$ and ϕ^* is open. Let $\mathcal{Y}_1 = \mathcal{X}^*/E_{\phi^*}$. Then the map $\phi_1 : \mathcal{X}^* \to \mathcal{Y}_1$ has the property that its EGS diagram coincides with its AG diagram. For, clearly, the following diagram is the AG(ϕ_1) diagram.

$$\begin{tikzcd} & \mathcal{X}^* \arrow[dl, "\phi_1^*"'] \arrow[d, "\phi_1"] \\ \mathcal{Y}_1^* \arrow[r, "\chi_{\mathcal{Y}_1}"'] & \mathcal{Y}_1 \end{tikzcd}$$

As, by assumption, ϕ_1^* is a RIC extension it follows that this is also the EGS(ϕ_1) diagram. Iterating this procedure we construct the canonical PI tower for ϕ, and it consists entirely of highly proximal and almost periodic extensions. As ϕ is a PI extension, it follows that $\mathcal{X}^*_\infty = \mathcal{Y}_\infty$; but also, as all the proximal maps in the tower are hp, $\mathcal{X}^*_\infty = \mathcal{X}^*$. So $\mathcal{X}^* = \mathcal{Y}_\infty \to \mathcal{Y}$ is a strictly-PI extension, which consists of hp and almost periodic extensions, hence $\mathcal{X}^* \to \mathcal{Y}$ is strictly-HPI. □

5.8. Note that from 5.7. it follows that if $\phi : \mathcal{X} \to \mathcal{Y}$ is HPI then $\phi^* : \mathcal{X}^* \to \mathcal{Y}^*$ can be constructed by taking maximal almost periodic extensions under $\mathcal{X}^*$ and maximal highly proximal extensions successively.

5.9. **THEOREM.** *Let* $\phi : \mathcal{X} \to \mathcal{Y}$ *be an* HPI *extension of minimal ttgs. Let* θ *and* ψ *be homomorphisms such that* $\phi = \theta \circ \psi$. *Then* θ *is an* HPI *extension. (In other words: a factor of an* HPI *extension is an* HPI *extension.)*

PROOF. Let $\psi : \mathcal{X} \to \mathcal{Z}$ and $\theta : \mathcal{Z} \to \mathcal{Y}$. Define $\tilde{\phi} = \phi \circ \chi_{\mathcal{X}} : \mathcal{X}^* \to \mathcal{Y}$. We shall prove that $\theta \circ \chi_{\mathcal{Z}} : \mathcal{Z}^* \to \mathcal{Y}$ is an HPI extension. Hence, by 5.2., $\theta \circ \chi_{\mathcal{Z}}$ is strictly-HPI and, by definition, θ is an HPI extension.
As $\tilde{\phi}$ is a PI extension, $\theta \circ \chi_{\mathcal{Z}}$ (as a factor of $\tilde{\phi}$) is a PI extension. Let ξ

and η be homomorphisms such that $\theta\circ\chi_{\mathcal{Z}}=\eta\circ\xi$ and let ξ be open.

$$\begin{tikzcd}
\mathcal{X}^* \arrow[rr, "\chi_{\mathcal{X}}"] \arrow[rrrr, bend left, "\tilde{\phi}"] \arrow[dr, "\psi^*"'] \arrow[dd, "\xi\circ\psi^*"'] & & \mathcal{X} \arrow[rr, "\phi"] \arrow[dr, "\psi"] & & \mathcal{Y} \\
 & \mathcal{Z}^* \arrow[rr, "\chi_{\mathcal{Z}}"] \arrow[dl, "\xi"] & & \mathcal{Z} \arrow[ur, "\theta"'] & \\
\mathcal{W} \arrow[rrrruu, "\eta"'] & & & &
\end{tikzcd}$$

Then $\xi\circ\psi^*:\mathcal{X}^*\to\mathcal{W}$ is open; hence, as $\tilde{\phi}$ is HPI, it follows from 5.7. that $\xi\circ\psi^*$ is a RIC extension. Consequently, ξ is a RIC extension (III.1.10.a). By 5.7., it follows that $\theta\circ\chi_{\mathcal{Z}}$ is HPI. □

5.10. **COROLLARY.** *Let ϕ, ψ and θ be homomorphisms of minimal ttgs such that $\phi=\theta\circ\psi$. Then ϕ is an* HPI *extension iff θ and ψ are* HPI *extensions.*

PROOF. The "only if"-part follows from 5.5. and 5.9.; for the "if"-part use 5.2.. □

5.11. **THEOREM.** *Let $\phi:\mathcal{X}\to\mathcal{Y}$ be an* HPI *extension of minimal ttgs. Then the canonical* PI *tower for ϕ is an* HPI *tower.*

PROOF. Construct AG(ϕ):

$$\begin{tikzcd}
\mathcal{X}' \arrow[r, "\sigma"] \arrow[d, "\phi'"'] & \mathcal{X} \arrow[d, "\phi"] \\
\mathcal{Y}' \arrow[r, "\tau"'] & \mathcal{Y}
\end{tikzcd}$$

As σ is hp, $\phi\circ\sigma$ is an HPI extension. By 5.10., ϕ' is an HPI extension and as it is open, it is a RIC extension by 5.3.. So AG(ϕ) and EGS(ϕ) coincide. Define $\mathcal{X}_1:=\mathcal{X}'$, $\mathcal{Y}_1=\mathcal{X}'/E_{\phi'}$ and $\phi_1:\mathcal{X}_1\to\mathcal{Y}_1$ as the quotient map. Then, by 5.10., ϕ_1 is an HPI extension.

Iterating this procedure we construct the canonical PI tower for ϕ, which is build up by AG diagrams; i.e., the PI tower is an HPI tower. □

For the next theorem, which characterizes HPI extensions of metric minimal ttgs, we need the following lemma.

5.12. **LEMMA.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a surjective homomorphism of ttgs and let* X *have a dense subset of almost periodic points. If* $Y_0 \subseteq Y$ *is a residual set then* $\phi^{\leftarrow}[Y_0]$ *is residual in* X .

PROOF. Let $\{A_n \mid n \in \mathbb{N}\}$ be a collection of closed nowhere dense subsets of Y such that

$$Y \setminus Y_0 = \bigcup \{A_n \mid n \in \mathbb{N}\} .$$

Then clearly $X \setminus \phi^{\leftarrow}[Y_0] = \bigcup \phi^{\leftarrow}[A_n]$. So it is sufficient to prove that the full original of a nowhere dense closed subset in Y is nowhere dense in X . Let $A = \overline{A} \subseteq Y$ be nowhere dense. Suppose that $U \subseteq \phi^{\leftarrow}[A]$ for some nonempty open U in X , then $\phi[U] \subseteq \phi\phi^{\leftarrow}[A] = A$. As X has a dense subset of almost periodic points, ϕ is semi-open so $\phi[U]$ has a nonempty interior in Y (I.1.4.b), which contradicts the nowhere density of A . □

5.13. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of metric minimal ttgs. Then* ϕ *is an* HPI *extension iff* ϕ *is point distal.*

PROOF. If ϕ is point distal then, by 1.3. in the metric case, ϕ is an HPI extension.

Conversely, suppose that ϕ is an HPI extension. Then, by 5.11., the PI tower for ϕ is an HPI tower. As X is metric, the height of the tower is countable (III.4.8.), and all ttgs in it are metric. Hence there is a metric minimal ttg $\mathfrak{X}'$ and a map $\sigma : \mathfrak{X}' \to \mathfrak{X}$, which is highly proximal and for all $n \in \mathbb{N}$ there are metric minimal ttgs $\mathfrak{X}'_n$ and $\mathfrak{X}_n$ such that $\tau_n : \mathfrak{X}'_n \to \mathfrak{X}_n$ is hp and $\xi_n : \mathfrak{X}_n \to \mathfrak{X}'_{n-1}$ is almost periodic, with $\mathfrak{Y} = \mathfrak{X}_0$ and $\mathfrak{X}' = \operatorname{inv\,lim} \mathfrak{X}'_n$.

We shall prove that $\phi' := \phi \circ \sigma : \mathfrak{X}' \to \mathfrak{Y}$ is point distal; hence that ϕ is point distal. As all minimal ttgs are metric, the maps $\mathfrak{X}'_n \to \mathfrak{X}_n$ are almost automorphic. Let $W_n \subseteq X'_n$ be the collection of automorphic points. Then, by 1.1.d, W_n is residual in X'_n . Let $\phi_n : \mathfrak{X}' \to \mathfrak{X}'_n$; then, by 5.12., $\phi_n^{\leftarrow}[W_n]$ is residual. Hence

$$W := \bigcap \{\phi_n^{\leftarrow}[W_n] \mid n \in \mathbb{N}\}$$

is a residual subset of in X' . Let $x' \in W$ and define for every $n \in \mathbb{N}$ the points $x'_n := \phi_n(x')$ and $x_n := \tau_n(x'_n)$. Then, in particular,

$\phi'(x') = \tau_0 \circ \phi_0(x') = x_0$. As $x' \in W$, τ_0 is one to one in x'_0, hence $J_{x_0} = J_{x'_0}$. By distality of ξ_1, it follows that $J_{x_1} = J_{x'_0}$, so $J_{x_1} = J_{x_0}$. Countable induction shows that $J_{x'} = J_{x_0}$; hence x' is a ϕ'-distal point (I.2.10.). □

5.14. Let $\mathcal{Y}$ be a minimal ttg. Then there exists a universal HPI extension of $\mathcal{Y}$ as follows:
First take $\mathcal{Y}^*$ and let $\mathcal{X}_1$ be the maximal almost periodic extension of $\mathcal{Y}^*$ (under $\mathcal{M}$). Suppose $\mathcal{X}_\alpha$ is constructed, then construct $\mathcal{X}_{\alpha+1}$ as the maximal almost periodic extension of $\mathcal{X}_\alpha^*$. If α is a limit ordinal and if $\mathcal{X}_\beta$ is constructed for all $\beta < \alpha$, then define $\mathcal{X}_\alpha := \operatorname{inv}\lim\{\mathcal{X}_\beta \mid \beta < \alpha\}$. For some ordinal ν, $\mathcal{X}_{\nu+1} \cong \mathcal{X}_\nu^*$ (for there is just a set of essentially different minimal ttgs). Clearly, $\mathcal{X}_\nu^*$ is an HPI extension of $\mathcal{Y}$.
That this is a universal HPI extension of $\mathcal{Y}$ follows from the next observation: Let $\psi: \mathcal{W} \to \mathcal{Z}$ be a homomorphism of minimal ttgs. Let $\kappa_{\mathcal{W}}: \mathcal{W}' \to \mathcal{W}$ and $\kappa_{\mathcal{Z}}: \mathcal{Z}' \to \mathcal{Z}$ be the maximal almost periodic extensions of $\mathcal{W}$ and $\mathcal{Z}$ respectively. Then there is a $\theta: \mathcal{W}' \to \mathcal{Z}'$ such that $\kappa_{\mathcal{Z}} \circ \theta = \psi \circ \kappa_{\mathcal{W}}$. For let $\gamma_{\mathcal{W}}: \mathcal{M} \to \mathcal{W}$ and $\gamma_{\mathcal{Z}}: \mathcal{M} \to \mathcal{Z}$ be such that $\psi \circ \gamma_{\mathcal{W}} = \gamma_{\mathcal{Z}}$. Then $Q_{\gamma_{\mathcal{W}}} \subseteq Q_{\gamma_{\mathcal{Z}}}$, hence $E_{\gamma_{\mathcal{W}}} \subseteq E_{\gamma_{\mathcal{Z}}}$. As $\mathcal{W} = \mathcal{M}/E_{\gamma_{\mathcal{W}}}$ and $\mathcal{Z} = \mathcal{M}/E_{\gamma_{\mathcal{Z}}}$, this shows that there is a map $\theta: \mathcal{W}' \to \mathcal{Z}'$ with $\kappa_{\mathcal{Z}} \circ \theta = \psi \circ \kappa_{\mathcal{W}}$.
Obviously the universal HPI extension is unique up to isomorphism (note 5.5.). Let $\phi_\lambda: \mathcal{X}_\lambda^* \to \mathcal{Y}$ be an HPI extension ($\lambda \in \Lambda$). Using 2.6. and the corresponding property for almost periodic extensions it is routine to check that for every minimal

$$\mathcal{Z} \subseteq R_{\{\phi_\lambda \mid \lambda \in \Lambda\}} \subseteq \Pi\{\mathcal{X}_\lambda^* \mid \lambda \in \Lambda\}$$

the map $\mathcal{Z} \to \mathcal{Y}$ is HPI.

So we showed the following:

5.15. **COROLLARY.** *Let* $\mathcal{Y}$ *be a minimal ttg. Then there is a universal minimal* HPI *extension* $\phi: \mathcal{X} \to \mathcal{Y}$, *which is unique up to isomorphism, and* ϕ *is regular. In particular, there exists a universal minimal* HPI *ttg, which is unique up to isomorphism and which is regular* (see V.4. for an other construction). □

IV.6. REMARKS

6.1. Section 1. contains some generalities on almost automorphic extensions and highly proximal extensions, which can be found for instance in [V 70] and [Sh 76]. The main purpose was to give a glimpse at the historical context of the rest of this chapter.

The example in 1.4. is the basis for many examples in topological dynamics (e.g. see [Mk 72] and [M 76.1,78]). Note that for an arbitrary topological group T , $\beta(Tx_0)$ does not have to be a ttg; i.e., the action is in general not jointly continuous. However, there is a maximal compactification-flow $\beta_T(Tx_0)$ for Tx_0 in which Tx_0 is isomorphically embedded (see the beginning of section I.2.). Then $\beta_T(Tx_0)$ is the maximal a-a extension of $\mathfrak{X}$ which is one to one in the fiber of x_0 .

QUESTION

Does every nontrivial highly proximal extension admit a nontrivial a-a factor? I.e., let $\phi: \mathfrak{X} \to \mathfrak{Y}$ be hp and nontrivial. Do there exist a nontrivial a-a extension $\psi: \mathfrak{Z} \to \mathfrak{Y}$ and a homomorphism $\theta: \mathfrak{X} \to \mathfrak{Z}$ with $\phi = \psi \circ \theta$?

6.2. In section 2. we study hp extensions with emphasis on the topology. For that reason we gave a proof of 2.5. and 2.6. without using the action of S_T (compare [Sh 76] and [AG 77]).

The results except for 2.7. and 2.8. are standard; 2.7. is basically contained in [Ar 78] and 2.8. appeared in [AW 81]. With respect to 2.8. we remark that it was already known that a distal minimal ttg for $\mathbb{Z}$ with a 0-dimensional phase space is equicontinuous ([E 58]). In a stronger version:

> **THEOREM.** [MW 76] *If* T *is the direct product of a compactly generated separable group with a compact group and if* $\mathfrak{X}$ *is a minimal distal ttg with* 0*-dimensional phase space, then* $\mathfrak{X}$ *is equicontinuous.*

For more details on distality and homogeneity see [B 75/79] 2.11.7.-21..

QUESTIONS

a) Can 2.8. be proven without using the heavy tools (i.e., FST and the theorem that states that every homogeneous extremally disconnected CT_2 space is finite)?

b) Can we give a topological characterization of MHP ttgs in case T does not have the discrete topology?
Note that if the answer to the question in 6.1. is affirmative, we have:

$$\mathcal{X} \text{ is MHP iff } X \cong \beta_T(Tx_0) \text{ for every } x_0 \in X .$$

c) Do there exist nontrivial MHP ttgs which are point distal?

6.3. The main part of section 3. is devoted to the construction of "hp" shadow diagrams. The idea of constructing shadow diagrams stems from [V 70]. The intention is to change the homomorphism slightly, but in a canonical way, such that it has nicer properties and still reflects much of the original homomorphism.

Although those shadow diagrams can be found in [Sh 76], [AG 77] and [V 77] we also introduce them here. The proofs are somewhat shorter and the set-up is chosen similar to the one in [V 77] (especially see 3.6. and [V 77] page 819). Running through the section the following remarks occur:

(i) Theorem 3.1. slightly generalizes [Sh 76] 2.9. and [AG 77] lemma 1.1. (and the note before lemma 1.2.).

(ii) 3.8. and 3.9. can be found in [AG 77]. They form the basis for the study presented in chapter V..

(iii) In [V 77] 2.3.5. it is stated that (our) 3.12.c is true for strictly-quasi separable minimal ttgs (so not necessarily metric). However, this is not correct as the following example shows (T.S. WU)
Consider example 1.5.. As ϕ is highly proximal, clearly, its AG and EGS diagrams coincide. But for every $y \in Y$, $u \in J_y$ we have
$u \circ u\phi^{\leftarrow}(y) = u \circ \phi^{\leftarrow}(y) = u\phi^{\leftarrow}(y)$, hence $\bigcap \{u \circ u\phi^{\leftarrow}(y) \mid u \in J_y\} = \varnothing$
As $T = \mathbb{Z}$, $\mathcal{X}$ is strictly-quasi separable (I.1.7.).

(iv) Theorem 3.13. slightly generalizes [E 73] 6.4.; this generalization makes 3.14. easily accessible.

(v) It seems no proof of 3.16. has been published until now.

QUESTION

Can we give an internal characterization of ttgs for which the AG and EGS diagrams coincide? Note that together with an internal characterization of PI ttgs (III.5.7., 5.8.) this could give an internal characterization of HPI ttgs.

6.4. The forth section is meant to give some justification for the construction of hp shadow diagrams. We show that the hp lifting has much in common with the original homomorphism of minimal ttgs. Some of the preserved properties are preserved under more general circumstances, as is shown in 4.2. and 4.3.. In those theorems we extend [M 78] 2.1.. In the case of hp lifting much more can be done as a result of the irreducibility. So for instance in 4.16. we gave relativized versions of [Ar 78] prop 7., [AG 77] lemma I.3., theorem I.2. under fairly general conditions.
The results in this section are published in [AW 81], except for 4.2., 4.3., 4.8. through 4.12. which are not in the literature.
Note that in 4.17. openness is necessary:
Let ϕ be hp and nontrivial then ϕ^* is an isomorphism and so it is RIC and Bc, but clearly ϕ is neither RIC nor Bc.
Also there are examples of homomorphisms ϕ which are not weakly mixing, while ϕ^* is RIC and weakly mixing (cf. [M 78]).

QUESTION

a) What about the converse of 4.3.d ?

b) Characterize the homomorphisms $\phi : \mathcal{X} \to \mathcal{Y}$ with $\sigma \times \sigma[R_{\phi^*}] = R_\phi$.

6.5. The material in section 5. is the relativized version of a part of [AG 77]. It is contained in here for the sake of completeness and to facilitate the study in section V.4..

V

MAXIMALLY HIGHLY PROXIMAL GENERATORS

1. the circle operation extended
2. generators and quasifactors
3. some dynamical properties
4. the universal HPI ttg
5. remarks

This chapter is devoted to the study of a special kind of quasifactors of $\mathcal{M}$, namely the quasifactors that represent the MHP ttgs.
The techniques originate from the idea of J. AUSLANDER to extend the action of S_T on a ttg $\mathcal{X}$ to an action of 2^{S_T} on $2^{\mathcal{X}}$.
The first two sections are mainly spent on investigations of the techniques themselves. In section 1. we define the action of 2^{S_T} on 2^X (more or less) as an extension of the circle operation (II.3.), which results in a semigroup structure on 2^M. The idempotents in $(2^M, \circ)$ are the subsets of M that generate the MHP ttgs as quasifactors of $\mathcal{M}$.
In section 2. we study those MHP generators of $\mathcal{M}$ and the quasifactors thereof.
Several dynamical properties can be characterized in terms of the idempotents in $(2^M, \circ)$; in section 3. we do this, for example, for regularity and the Bronstein condition. In particular, we give a partial answer to the question whether or not an open Bc extension is a RIC extension. We show that this is the case if the map is regular.
In the forth section we construct the universal minimal HPI ttg for T. In doing so we construct idempotent sets in 2^M that generate interesting incontractible ttgs, that will be useful in chapter VI..
Almost all results of the sections 1., 2. and 3. appeared in [AW 81] as a result of the cooperative research with (and initiated by) J. AUSLANDER.

V.1. THE CIRCLE OPERATION EXTENDED

We introduce a semigroup structure on 2^{S_T} and for every ttg $\mathfrak{X}$ a semigroup action of 2^{S_T} on 2^X, which in a certain sense extends the circle operation (as discussed in II.3.). Anticipating on that we shall denote the operation under which 2^{S_T} is a semigroup as well as the semigroup action of 2^{S_T} on 2^X with "$\circ$".
Special attention will be given to (the internal form of) the idempotents in the subsemigroup $(2^M,\circ)$ of $(2^{S_T},\circ)$. For instance we show that an idempotent C in $(2^M,\circ)$ is fully determined by two components, an idempotent part $C\cap J$ and a group part $C\cap uM$ for some (every) $u\in C\cap J$.

Let $\mathfrak{X}$ be a ttg. Remember that the circle operation is defined as the extension to S_T of the action of T on 2^X. In that respect it may be useful to memorize that for every $A\subseteq X$ the map

$$\rho_A : S_T \to 2^X \text{ defined by } p \mapsto p\circ A \quad (p\in S_T)$$

is continuous; i.e., if $\{p_i\}_i$ is a net in S_T converging to p and if $A\subseteq X$ then $\{p_i\circ A\}_i$ converges to $p\circ A$ in 2^X (NB: $p\circ A := p\circ \overline{A}$).

Now let $R\subseteq S_T$ and $A\subseteq X$, then define a subset $R\circ A$ of X by

$$R\circ A := \bigcup\{r\circ A \mid r\in R\}.$$

1.1. **THEOREM.** *Let $\mathfrak{X}$ be a ttg, $A\subseteq X$ nonempty and $R\subseteq S_T$.*
   a) *If $R\in 2^{S_T}$ then $R\circ A\in 2^X$.*
   b) $\overline{R}\circ A = \overline{R\circ A}$.
   c) *The map $\tilde{\rho}_A : 2^{S_T}\to 2^X$, defined by $S\mapsto S\circ A$ for every $S\in 2^{S_T}$, is continuous.*

PROOF.

a) Let $\{x_i\}_i$ be a convergent net in $R\circ A$ and let $x = \lim x_i$ be its limit in X. Let $r_i\in R$ be such that $x_i\in r_i\circ A$ for all i. As S_T is compact, there is a subnet $\{r_j\}_j$ such that $r_j\to r$ for some $r\in S_T$. Then $r\in\overline{R} = R$ and so $r\circ A\subseteq R\circ A$. But

$$x = \lim x_j \in \lim_{2^X}(r_j\circ A)$$

and by continuity of ρ_A we have

$$\lim_{2^X}(r_j\circ A) = (\lim r_j)\circ A = r\circ A .$$

Hence $x \in r \circ A \subseteq R \circ A$ and $R \circ A$ is closed.

b) As $R \circ A \subseteq \overline{R} \circ A$ and $\overline{R} \circ A$ is closed (by a), it follows that $\overline{R \circ A} \subseteq \overline{R} \circ A$.
Let $x \in \overline{R} \circ A$, say $x \in r \circ A$ for some $r \in \overline{R}$. Then there exists a convergent net $\{r_i\}_i$ in R with $r = \lim r_i$. Hence, by continuity of ρ_A, $r \circ A = \lim_{2^X} r_i \circ A$. Let U be an open neighbourhood of x in X, then $<X, U>(= <U>^*)$ is an open neighbourhood of $r \circ A$ in 2^X. So $r_i \circ A \in <X, U>$ for some i, hence $r_i \circ A \cap U \neq \emptyset$ and, consequently, $R \circ A \cap U \neq \emptyset$. As U was chosen arbitrarily, this implies that $x \in \overline{R \circ A}$; hence $\overline{R} \circ A \subseteq \overline{R \circ A}$.

c) Let $A \subseteq X$, $R \in 2^{S_T}$ and let $<U_1, \ldots, U_n>$ be a neighbourhood of $R \circ A$ in 2^X. We shall construct a neighbourhood V of R in 2^{S_T} such that

$$S \circ A \in <U_1, \ldots, U_n> \text{ for all } S \in V .$$

As $R \circ A \cap U_i \neq \emptyset$ for $i \in \{1, \ldots, n\}$, we can find $r_i \in R$ such that $r_i \circ A \cap U_i \neq \emptyset$ for every $i \in \{1, \ldots, n\}$. As $\rho_A : S_T \to 2^X$ is continuous, there is a neighbourhood V_i of r_i in S_T such that $v \circ A \cap U_i \neq \emptyset$ for all $v \in V_i$ (for $<X, U_i>$ is a neighbourhood of $r_i \circ A$ in 2^X). Let

$$U = \bigcup \{U_i \mid i \in \{1, \ldots, n\}\} .$$

Then $R \circ A \subseteq U$; so, by continuity of ρ_A and by compactness of R, there is an open W in S_T with $R \subseteq W$ and $W \circ A \subseteq U$. Define

$$V := <W, V_1 \cap W, \ldots, V_n \cap W> .$$

Then V is a neighbourhood of R in S_T and $S \circ A \in <U_1, \ldots, U_n>$ for every $S \in V$. □

1.2. **LEMMA.** *Let* $\mathfrak{X}$ *be a ttg,* $A \subseteq X$ *and let* R *and* S *be subsets of* S_T. *Then* $S \circ (R \circ A) = (S \circ R) \circ A$.

PROOF. First suppose that $R \in 2^{S_T}$.
It is clear, that for each $t \in T$ we have

$$(t \circ R) \circ A = tR \circ A = t(R \circ A) = t \circ (R \circ A) .$$

As the mapping $p \mapsto p \circ R$ is continuous, it follows from 1.1.c that the mapping $p \mapsto (p \circ R) \circ A$ is continuous. Also the mapping $p \mapsto p \circ (R \circ A)$ is continuous. Since T is dense in S_T and as the mappings $p \mapsto (p \circ R) \circ A$ and $p \mapsto p \circ (R \circ A)$ coincide on the dense subset T, we have

$p\circ(R\circ A)=(p\circ R)\circ A$ for every $p\in S_T$. But then

$$(S\circ R)\circ A = \bigcup\{(s\circ R)\circ A \mid s\in S\} = \bigcup\{s\circ(R\circ A)\mid s\in S\} =$$
$$= S\circ(R\circ A).$$

Now suppose that $R\subseteq S_T$ is not necessarily closed.
As, by definition (II.3.), $p\circ R = p\circ\overline{R}$ for every $p\in S_T$, we have $S\circ R = S\circ\overline{R}$ and similarly $S\circ(R\circ A)=S\circ(\overline{R\circ A})$. So by 1.1.b,

$$S\circ(R\circ A)=S\circ(\overline{R\circ A})=S\circ(\overline{R}\circ A).$$

As $\overline{R}\in 2^{S_T}$, it follows that

$$S\circ(\overline{R}\circ A)=(S\circ\overline{R})\circ A = (S\circ R)\circ A \ ;$$

hence $S\circ(R\circ A)=(S\circ R)\circ A$, which proves the lemma. □

1.3. **THEOREM.** *With respect to the circle operation* 2^{S_T} *is a* CT$_2$ *semigroup with continuous right translations, in which* 2^M *is a closed subsemigroup.*

PROOF. The statement for 2^{S_T} follows immediately from 1.1. and 1.2..
For 2^M note that if $R\subseteq M$ and $S\subseteq M$ then $S\circ R\subseteq M$. □

1.4. It is obvious that 2^M contains idempotents under the circle operation ($(2^M,\circ)$ is a CT$_2$ semigroup!). We shall call them *idempotent sets in* $(2^M,\circ)$. A subset C of M will be called an *idempotent subset of* M if $C\circ C = C$. Some examples are:

(i) Every idempotent in M, considered as a singleton set, is an idempotent set in $(2^M,\circ)$.

(ii) The set M is an idempotent set in $(2^M,\circ)$.

An interesting collection of idempotent sets is formed as follows:

(iii) Let $\mathfrak{X}$ be a minimal ttg and let $x\in X$.
Then $M_x := \{p\in M\mid px = x\}$ is an idempotent subset in $(2^M,\circ)$.
For

$$(M_x\circ M_x).x = (M_x\circ M_x)\circ\{x\} = M_x\circ(M_x\circ\{x\}) = M_x\circ(M_x.x) =$$
$$= M_x\circ\{x\} = M_x.x = x\ ,$$

and so $M_x\circ M_x\subseteq\{p\in M\mid px = x\} = M_x$.
Let $v\in J_x$; then

$$M_x = M_x.v = M_x\circ\{v\}\subseteq M_x\circ M_x\ ,$$

hence $M_x \circ M_x = M_x$ and M_x is an idempotent set in $(2^M, \circ)$.
It is an open question, whether every idempotent subset in $(2^M, \circ)$ can be obtained in this way (for almost periodic idempotents see 2.1.).

Idempotent sets in $(2^M, \circ)$ give rise to interesting quasifactors of $\mathfrak{M}$ (see section 2. below). Therefore we shall study them now more closely.

1.5. **REMARK.** *Let* C *be a nonempty subset of* M.

a) *If* $C \circ C \subseteq C$, *then* $C \cap J \neq \emptyset$ *and* $C \circ C = C$, *i.e.,* C *is an idempotent subset of* M.

b) *Let* $u \in J$. *If* C *is an idempotent subset of* M, *then* $\overline{C}$ *and* $u \circ C$ *are idempotent sets in* $(2^M, \circ)$.

c) *Let* B_α *be an idempotent subset of* M *for all* $\alpha \in I$. *If* $B := \bigcap \{B_\alpha \mid \alpha \in I\} \neq \emptyset$, *then* B *is an idempotent subset of* M.

PROOF.

a) For every $c \in C$ we have

$$(c \circ C).(c \circ C) \subseteq (c \circ C) \circ (c \circ C) \subseteq (c \circ C) \circ (C \circ C) \subseteq (c \circ C) \circ C ,$$

so by 1.2.,

$$(c \circ C).(c \circ C) \subseteq (c \circ C) \circ C = c \circ (C \circ C) = c \circ C .$$

Then $c \circ C$ is a subsemigroup of M and clearly it is closed. Hence, by I.2.2.a, it follows that $c \circ C \cap J \neq \emptyset$. Since $c \circ C \subseteq C \circ C = C$, we have $C \cap J \neq \emptyset$, say $v \in C \cap J$. By I.2.2.b, $Cv = C$, so

$$C = Cv = C \circ \{v\} \subseteq C \circ C \subseteq C ,$$

and so C is an idempotent subset of M.

b) By definition, $\overline{C} \circ \overline{C} = \overline{C} \circ C$, and by 1.1.b, $\overline{C} \circ C = \overline{C \circ C}$. If C is an idempotent subset of M we have

$$\overline{C} \circ \overline{C} = \overline{C} \circ C = \overline{C \circ C} = \overline{C} ,$$

so $\overline{C}$ is an idempotent set in $(2^M, \circ)$.
Let $u \in J$, then by 1.2., we have $(u \circ C) \circ (u \circ C) = u \circ ((C \circ u) \circ C)$. As $C \circ u = Cu = C$, it follows that $(u \circ C) \circ (u \circ C) = u \circ (C \circ C)$. So if C is an idempotent subset of M we have

$$(u \circ C) \circ (u \circ C) = u \circ (C \circ C) = u \circ C ,$$

and $u \circ C$ turns out to be an idempotent set in $(2^M, \circ)$.

c) Let $B := \bigcap \{B_\alpha \mid \alpha \in I\} \neq \emptyset$, then for every $\alpha \in I$ we have

$$B \circ B \subseteq B_\alpha \circ B_\alpha = B_\alpha .$$

Hence $B \circ B \subseteq \bigcap \{B_\alpha \mid \alpha \in I\} = B$ and by a, it follows that B is an idempotent subset of M . □

1.6. **LEMMA.** *Let C and D be subsets of M and let $u \in J$.*

a) *If $C = u \circ C$ then $uC = C \cap uM$ and uC is $\mathfrak{F}(\mathfrak{M}, u)$-closed.*

b) $u(u \circ C \circ D) = u(u \circ C).u(u \circ D) = ((u \circ C) \cap uM).((u \circ D) \cap uM)$.

c) *If C is an idempotent subset of M and $u \in C \cap J$, then*

$$uC = C \cap uM = \overline{C} \cap uM = u\overline{C} = u(u \circ C)$$

and uC is an $\mathfrak{F}(\mathfrak{M}, u)$-closed subgroup of uM (which is contained in C).

d) *Let $K \subseteq J$, then $u(u \circ C) = u(K \circ C)$.*

PROOF.

a) As $uC \subseteq u \circ C = C$, we have $uC \subseteq C \cap uM$. On the other hand $C \cap uM = u(C \cap uM)$, so $C \cap uM \subseteq uC$. Hence $C \cap uM = uC$.
To show that uC is $\mathfrak{F}(\mathfrak{M}, u)$-closed, we have to prove that $uC = u(u \circ uC)$, which follows from the following sequence of equations and inclusions:

$$u(u \circ uC) = u(u \circ (C \cap uM)) \subseteq u(u \circ C) = uC = uuuC \subseteq u(u \circ uC) .$$

b) By a, $((u \circ C) \cap uM) = u(u \circ C)$ and $((u \circ D) \cap uM) = u(u \circ D)$, so

$$((u \circ C) \cap uM).((u \circ D) \cap uM) = u(u \circ C).u(u \circ D) \subseteq$$

$$\subseteq u(u \circ C \circ u \circ D) = u(u \circ C \circ D) .$$

Conversely, let $p \in u(u \circ C \circ D)$. Then $p = up$ and $p \in c \circ D$ for some $c = uc \in u \circ C$. For, there is an $r \in u \circ C$ with $p \in u(r \circ D) \subseteq ur \circ D$, and, clearly, $ur \in u(u \circ C)$. Then it follows that

$$(uc)^{-1}p = u(uc^{-1})p \in uc^{-1} \circ c \circ D = u \circ D ,$$

which implies that $(uc)^{-1}p \in u(u \circ D)$ and

$$p = uc.u(u \circ D) \subseteq u(u \circ C).u(u \circ D) .$$

Hence $u(u \circ C \circ D) \subseteq u(u \circ C).u(u \circ D)$ and so

$$u(u \circ C \circ D) = u(u \circ C).u(u \circ D).$$

c) Clearly,

$$uC \subseteq C \circ C \cap uM = C \cap uM \subseteq \overline{C} \cap uM = u(\overline{C} \cap uM) \subseteq u\overline{C}$$

and $u\overline{C} = u.u\overline{C} \subseteq u(u \circ \overline{C}) = u(u \circ C) \subseteq u(C \circ C) = uC$, which shows that the desired equalities hold.

As $u \circ C = u \circ (u \circ C)$, it follows from a and from III.2.3. that $u(u \circ C)$ is $\mathfrak{F}(\mathfrak{M}, u)$-closed in uM . But $uC = u(u \circ C)$, so uC is $\mathfrak{F}(\mathfrak{M}, u)$-closed in uM .

From b it follows that

$$u(u \circ C) = u(u \circ C \circ C) = u(u \circ C).u(u \circ C).$$

Hence $uC = uC.uC$ and so uC is an $\mathfrak{F}(\mathfrak{M}, u)$-closed subsemigroup of uM . By I.2.6., uC is a subgroup of uM .

d) By II.3.11.a, we have $u(v \circ C) = u(u \circ C)$ for every $v \in J$. But then

$$u(K \circ C) = \bigcup \{u(v \circ C) \mid v \in K\} = u(u \circ C).$$

☐

1.7. **THEOREM.** *Let C be an idempotent subset of M . Let $K = C \cap J$, $u \in J$ and $A = uC$. Then $C = KA = K \circ A$. In other words: C can be written as the product of its "idempotent part" and its "group part", and for a fixed u , this decomposition is unique.*

PROOF. Let $v \in K$ (K is nonempty by 1.5.a); then by II.3.11.b, we have $v \circ vC = v \circ A$. Hence

$$KA \subseteq K \circ A = \bigcup \{v \circ A \mid v \in K\} = \bigcup \{v \circ vC \mid v \in K\}.$$

But for every $v \in K$

$$v \circ vC \subseteq K \circ KC \subseteq C \circ C \circ C = C ,$$

so $KA \subseteq K \circ A \subseteq C$.

Conversely, if $c \in C$ and $w \in J$ with $wc = c$, then $w = c(uc)^{-1}$. By 1.6.c, vC is a $\mathfrak{F}(\mathfrak{M}, v)$-closed subgroup of vM for every $v \in K$; so $A = uvC$ is a $\mathfrak{F}(\mathfrak{M}, u)$-closed subgroup of uM . As $uc \in uC = A$, $(uc)^{-1} \in A$ and so $w = c(uc)^{-1} \in CA$. But

$$CA = CuC \subseteq C \circ u \circ C = C \circ C = C ,$$

so $w \in C$, hence $w \in C \cap J = K$ and $c = wuc \in KA$. Consequently, $C \subseteq KA$ and $C = KA = K \circ A$.
It is obvious that the way in which C can be written as the product of subsets of J and uM is unique. □

1.8. **REMARK.** *Let* $u \in J$ *and* F *a subgroup of* uM. *Then* $\mathrm{cl}_{\mathfrak{F}(\mathfrak{M},u)}F$ *is an* $\mathfrak{F}(\mathfrak{M},u)$*-closed subgroup of* uM *and* $u \circ F$ *is an idempotent set in* $(2^M, \circ)$.

PROOF. We shall prove that $u \circ F$ is an idempotent set in $(2^M, \circ)$. Hence, by 1.6.c, it follows that $u(u \circ F)$ is an $\mathfrak{F}(\mathfrak{M},u)$-closed subgroup of uM. As by III.2.3., $\mathrm{cl}_{\mathfrak{F}(\mathfrak{M},u)}F = u(u \circ F)$, this proves the corollary.
By II.3.11.c and by the assumption of F being a subgroup of uM, we have $f \circ F = u \circ fF = u \circ F$ for every $f \in F$; so

$$F \circ F = \bigcup \{f \circ F \mid f \in F\} = u \circ F .$$

But then it follows that

$$u \circ F \circ u \circ F = u \circ F \circ F = u \circ u \circ F = u \circ F$$

or, in other words, $u \circ F$ is an idempotent set in $(2^M, \circ)$. □

In theorem 1.7. a structure is given for the idempotent subsets of M (compare this with the structure of M itself given in I.2.2.e). It is not yet known whether or not every subset of M which has that structure is an idempotent subset; i.e., necessity of that structure for idempotent subsets of M is shown, but sufficiency is still an open question.
The remainder of this section will be devoted to this sufficiency problem.

1.9. **LEMMA.** *Let* $K \subseteq J$, $u \in J$ *and let* C *be an idempotent subset of* M.

a) $K \circ C = K'A = K' \circ A$ *for* $A = uC$ *and* $K' = (K \circ C) \cap J$.

b) *If* $u(u \circ K) \subseteq C$ *then* $u \circ K \circ C = K'A = K' \circ A$ *for* $A = uC$ *and* $K' = (u \circ K \circ C) \cap J$.

In particular, this applies to the idempotent subset $u \circ A$ *of* M, *for an* $\mathfrak{F}(\mathfrak{M},u)$*-closed subgroup* A *of* uM.

a′) $K \circ A = K'A = K' \circ A$ *for* $K' = (K \circ A) \cap J$.

b′) *If* $u(u \circ K) \subseteq A$ *then for* $K' = (u \circ K \circ A) \cap J$ *we have* $u \circ K \circ A = K'A = K' \circ A$.

PROOF.

a) Clearly,

$$K'A \subseteq K' \circ A = K' \circ uC \subseteq K \circ C \circ uC \subseteq K \circ C \circ u \circ C = K \circ C \ ,$$

so we only have to show that $K \circ C \subseteq K'A$. Let $p \in K \circ C$ and $v \in J$ with $vp = p$. As, by 1.6.c and 1.6.d,

$$up \in u(K \circ C) = u(u \circ C) = u \circ C = A \ ,$$

it follows by 1.6.c that $up^{-1} \in A$. So

$$v = p(up^{-1}) \in (K \circ C)A \subseteq K \circ C \circ uC \subseteq K \circ C \ ,$$

which implies that $v \in K'$ and $p = vup \in K'A$.

b) Clearly,

$$K'A \subseteq K' \circ A = K' \circ uC \subseteq u \circ K \circ C \circ uC \subseteq u \circ K \circ C \circ u \circ C = u \circ K \circ C \ ,$$

so we only have to show that $u \circ K \circ C \subseteq K'A$. Note that, by 1.6.b,

$$u(u \circ K \circ C) = u(u \circ K).u(u \circ C) \, .$$

By 1.6.c and by the assumption, we have

$$u(u \circ K \circ C) = u(u \circ K).u(u \circ C) \subseteq uC.uC = A.A = A \ .$$

From this the statement follows in a way similar to the proof of a. □

1.10. **REMARK.** *Let* $u \in J$ *and let* A *be an* $\mathfrak{F}(\mathcal{M},u)$*-closed subgroup of* uM *, such that* $u(u \circ J) \subseteq A$ *. Then* $J \circ A$ *and* $u \circ J \circ A$ *are idempotent subsets of* M *.*
In particular, if A *contains the Ellis group of the universal minimal point distal ttg with respect to a distal point, then* $J \circ A$ *and* $u \circ J \circ A$ *are idempotent subsets of* M *.*

PROOF. By 1.6.d, $u(J \circ A \circ J \circ A) = u(u \circ A \circ J \circ A)$, and by 1.6.b,

$$u(u \circ A \circ J \circ A) = u(u \circ A).u(u \circ J).u(u \circ A) \ ;$$

so by assumption, it follows that

$$u(J \circ A \circ J \circ A) = u(u \circ A).u(u \circ J).u(u \circ A) \subseteq A.A.A = A \ .$$

But then

$$J \circ A \circ J \circ A \subseteq J.u(J \circ A \circ J \circ A) \subseteq J.A \subseteq J \circ A \ ;$$

hence $J \circ A$ is an idempotent subset of M. By 1.5.b, it follows that $u \circ J \circ A$ is an idempotent subset too.
Let $B = \mathfrak{G}(\mathcal{X}, x_0)$, where $\mathcal{X}$ is the universal minimal point distal ttg for T and x_0 is a distal point in X. Then $Jx_0 = x_0$ and so

$$u(u \circ J)x_0 = u(u \circ Jx_0) = ux_0 = x_0 \ ,$$

which implies that $u(u \circ J) \subseteq B$. If A is an $\mathfrak{F}(\mathcal{M}, u)$-closed subgroup of uM such that $B \subseteq A$ then $u(u \circ J) \subseteq A$. Hence, by the above, $J \circ A$ and $u \circ J \circ A$ are idempotent subsets of M. □

1.11. **THEOREM.** *Let C be an almost periodic point in 2^M and let C have the form $C = K \circ A$ for some $K \subseteq J$ and some $\mathfrak{F}(\mathcal{M}, u)$-closed subgroup A of uM. Then C is an idempotent set in $(2^M, \circ)$ iff $C \cap J = C \circ C \cap J$.*

PROOF. If C is an idempotent subset of M, then clearly,

$$C \circ C \cap J = C \cap J \ .$$

Conversely, suppose that $C \circ C \cap J = C \cap J$; we have to show that $C \circ C = C$. Let $w \in J_C$; then $w \circ C \circ C = C \circ C$, and by 1.6.b,

$$w(w \circ C \circ C) = w(w \circ C).w(w \circ C) = wC.wC \ .$$

As $C = K \circ A$, it follows from 1.9.a′ that $C = K'A$ for $K' = C \cap J$. So $wC = wA$ and

$$w(w \circ C \circ C) = wA.wA = wA \ .$$

Let $p \in C \circ C$; then for $v \in J$ with $vp = p$ we have

$$v = p(wp)^{-1} \in C \circ C \circ wA \ .$$

But from II.3.11.b it follows readily that $C \circ wA = C \circ A$, so

$$C \circ wA = C \circ A = K \circ A \circ A = K \circ (A \circ A) = K \circ u \circ A = K \circ A = C \ .$$

Hence $v \in C \circ C \circ wA = C \circ C$, and by assumption, it follows that $v \in C \circ C \cap J = C \cap J$; so

$$p = v.wp \in (C \cap J).wA = K'.wA = K'A = C \ .$$

Consequently, $C \circ C \subseteq C$ and C is an idempotent subset of M. □

1.12. **THEOREM.** *Let* C *be an idempotent subset of* M , $u \in J$, $K \subseteq J$ *and* $K' = K \circ C \cap J$. *Then the following statements are equivalent:*

a) $K \circ C$ *is an idempotent subset of* M *;*

b) $v \circ C \circ K \cup K \circ K \subseteq K \circ C$ *for some* $v \in K$ *;*

c) $v \circ C \circ K \cup K \circ K \subseteq K'M$ *and* $u(u \circ K) \subseteq uC$ *for some* $v \in K$.

PROOF.

a $\Rightarrow$ b By assumption, $K \circ C \circ K \circ C \subseteq K \circ C$. Let $v \in K$ and $w \in C \cap J$; then

$$v \circ C \circ K = v \circ C \circ K \circ w \subseteq K \circ C \circ K \circ C \subseteq K \circ C$$

and $K \circ K = K \circ w \circ K \circ w \subseteq K \circ C \circ K \circ C \subseteq K \circ C$.

b $\Rightarrow$ c By 1.9.a, $K \circ C = K'A$ for $A = uC$; so

$$v \circ C \circ K \cup K \circ K \subseteq K \circ C = K'A \subseteq K'M \ .$$

By 1.6.d, $u(u \circ K) = u(K \circ K)$; so

$$u(u \circ K) \subseteq u(K \circ C) = u(K'A) = uA = A = uC \ .$$

c $\Rightarrow$ a We shall prove that $v \circ C \circ K \subseteq K \circ C$ and $K \circ K \subseteq K \circ C$. It then follows that

$$K \circ C \circ K \circ C = K \circ (v \circ C \circ K) \circ C \subseteq K \circ (K \circ C) \circ C = K \circ K \circ C \circ C \ ,$$

and so that

$$K \circ C \circ K \circ C \subseteq (K \circ K) \circ C \subseteq (K \circ C) \circ C = K \circ C \ .$$

Hence $K \circ C$ is an idempotent subset of M .

As, by 1.6.d, $u(K \circ K) = u(u \circ K)$, we have $u(K \circ K) \subseteq uC$. So

$$K \circ K \subseteq K'M \cap JuC = K'uC$$

and, by 1.9.a, $K \circ K \subseteq K \circ C$.

By 1.6.b,

$$u(u \circ C \circ K) = u(u \circ C).u(u \circ K) \subseteq u(u \circ C).uC \subseteq$$

$$\subseteq u(u \circ C \circ u \circ C) = u(u \circ C) \ .$$

Let $w \in C \cap J$ then $u(u \circ C) = uw(w \circ C)$; so by 1.6.c, it follows that

$$u(u \circ C) = uw(w \circ C) = uwC = uC \ .$$

As, by II.3.11.a, $u(v \circ C \circ K) = u(u \circ C \circ K)$, we have

$$u(v \circ C \circ K) = u(u \circ C \circ K) \subseteq u(u \circ C) = uC \ ,$$

so $v \circ C \circ K \subseteq JuC$. But then

$$v \circ C \circ K \subseteq K'M \cap JuC = K'uC = K \circ C \ ;$$

which proves the implication. □

The proof of the following remark is left as an easy exercise for the reader.

1.13. **REMARK.** *Let $u \in J$, $K \subseteq J$ and let A be an $\mathfrak{F}(\mathcal{M}, u)$-closed subgroup of uM . Define $C := u \circ K \circ A$ and $K' = C \cap J$. Consider the following statements:*

a) *C is an idempotent set in $(2^M, \circ)$;*

b) *$u \circ A \circ K \cup u \circ K \circ K \subseteq C$;*

c) *$u(u \circ K) \subseteq A$ and $u \circ K \circ A \circ K \subseteq K'M$.*

Then a *and* b *are equivalent and* c *implies* a *and* b .

If $A = uC$ then a, b *and* c *are equivalent.* □

V.2. GENERATORS AND QUASIFACTORS

In IV.3.8. we introduced the notion of MHP generator, which was defined to be an almost periodic point C in 2^M with $C \cap J \neq \varnothing$ and such that the collection $\{p \circ C \mid p \in M\}$ forms a partition of M , and which is characterized by the property that $\mathfrak{W}\mathfrak{F}(C, \mathcal{M})$ is an MHP ttg. We shall characterize the MHP generators as the almost periodic idempotent sets in $(2^M, \circ)$. We shall study the quasifactors of $\mathcal{M}$ generated by MHP generators and the quasifactors of MHP ttgs from that point of view. For instance we give a necessary and sufficient condition (in terms of idempotent subsets of M) for an MHP quasifactor of an MHP ttg to be a factor of that MHP ttg.

2.1. **THEOREM.** *Let C be an almost periodic point in 2^M , say $C = u \circ C$. Then C is an* MHP *generator iff C is an idempotent set in $(2^M, \circ)$.*

PROOF. Suppose that C is an MHP generator. As $C \cap J \neq \varnothing$, say $v \in C \cap J$, it follows that for every $c \in C$ we have $c = cv \in cC \subseteq c \circ C$. Hence $C \cap c \circ C \neq \varnothing$, and as $\{p \circ C \mid p \in M\}$ is a partition of M , $C = c \circ C$ for every $c \in C$. But then $C \circ C = \bigcup \{c \circ C \mid c \in C\} = C$

and C is an idempotent set in $(2^M, \circ)$.
Conversely, let C be an idempotent set in $(2^M, \circ)$. Then by 1.5.a, $C \cap J \neq \emptyset$. Define $\mathfrak{F} := \{c \circ C \mid c \in C\}$; then $\mathfrak{F}$ is partially ordered by inclusion. It is not difficult to show that, for every chain (under inclusion) $\{c_i \circ C\}_{i \in I}$ in $\mathfrak{F}$, the set $\bigcap \{c_i \circ C \mid i \in I\}$ is of the form $c \circ C$, with c a cluster point of $\{c_i\}_i$ in M (so, certainly, $c \in C$). By Zorn's lemma, the family $\mathfrak{F}$ contains a minimal member (under inclusion), say $C' = c' \circ C$ for some $c' \in C$. As C is an almost periodic element in 2^M, it follows that the orbit closures of C and C' coincide, i.e.,

$$\{p \circ C \mid p \in M\} = \{p \circ C' \mid p \in M\} .$$

So it is sufficient to show that $\{p \circ C' \mid p \in M\}$ forms a partition of M. As follows:
First note that

$$C' \circ C' = c' \circ C \circ c' \circ C \subseteq c' \circ C \circ C \circ C = c' \circ C = C' ,$$

so C' is an idempotent subset of M and $C' = c' \circ C \subseteq C \circ C = C$. Let $p \in C'$ then $p \circ C' = pc' \circ C$ and $pc' \in C'C \subseteq C \circ C = C$, so $p \circ C' \in \mathfrak{F}$. As C' is minimal in $\mathfrak{F}$, from the fact that $p \circ C' \subseteq C' \circ C' = C'$ it follows that $p \circ C' = C'$.
Next, consider p and q in M such that $p \circ C' \cap q \circ C' \neq \emptyset$, say $r \in p \circ C' \cap q \circ C'$. Then for a net $t_i \to p$ and for $p_i \in C'$ we have $r = \lim t_i p_i$ and so

$$r \circ C' = (\lim t_i p_i) \circ C' = \lim_{2^X} t_i p_i \circ C' = \lim_{2^X} t_i (p_i \circ C') .$$

As $p_i \in C'$, $p_i \circ C' = C'$ and so $r \circ C' = \lim_{2^X} t_i C' = p \circ C'$. Similarly, $r \circ C' = q \circ C'$ and so $p \circ C' = q \circ C'$. Hence $\{p \circ C' \mid p \in M\}$ is a partition of M if $\{p \circ C' \mid p \in M\}$ is a covering. But that is evident by the fact that $C' \cap J \neq \emptyset$ (1.5.a). □

2.2. **COROLLARY.** *The* MHP *ttgs are just the quasifactors of* M *generated by the almost periodic idempotent subsets of* M .

PROOF. Cf. IV.3.9.. □

So the MHP ttgs are fully determined by the idempotent subsets of M. This is similar to the characterization of the universal proximal extensions by the Ellis groups (III.2.10.). More of this similarity may be seen in V.3.9. in relation to III.1.6..

2.3. **REMARK.** *Let* C *be an almost periodic idempotent set in* $(2^M, \circ)$. *Then* $p \in q \circ C$ *iff* $p \circ C = q \circ C$ *and* $p \in C$ *iff* $p \circ C = C$.
In particular, for $u \in C \cap J$, *the Ellis group of* $\mathfrak{W}\mathfrak{F}(C, \mathcal{M})$ *with respect to* C *in* uM *is equal to* uC.

PROOF. As C is an almost periodic idempotent set in $(2^M, \circ)$ it follows that $C \cap J \neq \varnothing$ (1.5.a). So for every $p \in M$, $p \in pC \subseteq p \circ C$. Hence the first two statements follow from the fact that $\{p \circ C \mid p \in M\}$ is a partition of M. Let $u \in C \cap J$ and $a \in uM$. Then $u \circ C = C$ and, clearly, $a \circ C = u \circ C$ iff $a \in u \circ C$, so

$$\mathfrak{G}(\mathfrak{W}\mathfrak{F}(C, \mathcal{M}), C) = u \circ C \cap uM = uC .$$

□

Let $C \subseteq M$ be an almost periodic element of 2^M. Then we shall denote the ttg $\mathfrak{W}\mathfrak{F}(C, \mathcal{M})$ by $\mathcal{C}$. If no base point is specified, then we shall consider C to be the base point. A homomorphism $\phi: \mathcal{C} \to \mathcal{D}$ must be understood as an ambit morphism

$$\phi: (\mathfrak{W}\mathfrak{F}(C, \mathcal{M}), C) \to (\mathfrak{W}\mathfrak{F}(D, \mathcal{M}), D)$$

(unless stated otherwise).

2.4. **THEOREM.** *Let* $u \in J$ *and let* C *and* D *be* MHP *generators with* $u \in C \cap D$.

a) *The set* $p \circ C(up)^{-1}$ *is an* MHP *generator for all* $p \in M$.
b) *There is a homomorphism* $\phi: \mathcal{C} \to \mathcal{D}$ *iff* $C \subseteq D$.
c) *The ttgs (not ambits!)* $\mathcal{C}$ *and* $\mathcal{D}$ *are isomorphic iff* $C = a \circ Da^{-1}$ *for some* $a \in uM$.
d) *Let* $\phi: \mathcal{C} \to \mathcal{D}$ *be an ambit morphism, then* ϕ *is regular iff* $C = d \circ Cd^{-1}$ *for all* $d \in uD$. *In particular* $\mathcal{C}$ *is regular iff* $a \circ Ca^{-1} = C$ *for all* $a \in uM$.

PROOF.

a) Let $p \in M$ and note that $p \circ C(up)^{-1} = (p \circ C).(up)^{-1}$. As the map $\rho_{(up)^{-1}}: M \to M$ is an isomorphism (I.2.3.c), the collection $\{q \circ p \circ C(up)^{-1} \mid q \in M\}$ partitions M. Let $v \in J$ with $vp = p$. Then $v = p.(up)^{-1} \in p \circ C(up)^{-1}$; so $p \circ C(up)^{-1} \cap J \neq \varnothing$ and $p \circ C(up)^{-1}$ is an MHP generator.

b) Suppose that $C \subseteq D$, then $\phi: p \circ C \mapsto p \circ D : \mathcal{C} \to \mathcal{D}$ is well defined. For, let $p \circ C = q \circ C$. Then $p \circ C \subseteq p \circ D$ and $p \circ C = q \circ C \subseteq q \circ D$, so $p \circ D \cap q \circ D \neq \varnothing$; hence $p \circ D = q \circ D$.

Conversely, let $\phi: \mathcal{C} \to \mathfrak{D}$ be well defined. Let $c \in C$, then $C = c \circ C$ (2.3.) and so $D = \phi(C) = \phi(c \circ C) = c \circ D$. Hence, by 2.3., $c \in D$; consequently, $C \subseteq D$.

c) Suppose there is an isomorphism between $\mathcal{C}$ and $\mathfrak{D}$, say $\phi: \mathcal{C} \to \mathfrak{D}$ with $\phi(C) = a \circ D$ for some $a \in uM$. As $\rho_{a^{-1}}: \mathfrak{M} \to \mathfrak{M}$, defined by $\rho_{a^{-1}}(p) = pa^{-1}$, is an isomorphism of ttgs, it follows that $2^{\rho_{a^{-1}}}: 2^{\mathfrak{M}} \to 2^{\mathfrak{M}}$ is an isomorphism of ttgs. Hence

$$2^{\rho_{a^{-1}}}: (\mathfrak{D}, a \circ D) \to (\mathfrak{W}\mathfrak{F}(a \circ Da^{-1}, \mathfrak{M}), a \circ Da^{-1})$$

is an ambit isomorphism. But then $2^{\rho_{a^{-1}}} \circ \phi: (\mathcal{C}, C) \to (\mathfrak{F}, F)$ is an ambit isomorphism, where $F = a \circ Da^{-1}$. As F is an MHP generator (a) it follows from b, that $C = F$.

Conversely, let $C = a \circ Da^{-1}$ for some $a \in uM$. Then the map $2^{\rho_a}: (\mathcal{C}, C) \to (\mathfrak{D}, a \circ D)$ is an isomorphism of ambits. For $2^{\rho_a}: 2^{\mathfrak{M}} \to 2^{\mathfrak{M}}$ is an isomorphism and $2^{\rho_a}(C) = Ca = (a \circ Da^{-1}).a = a \circ D$.

d) Suppose that ϕ is a regular map and let $d \in uD$. Then, as $d \in uD \subseteq DD \subseteq D$, we have $\phi(d \circ C) = d \circ D \subseteq D \circ D = D = \phi(C)$; so $(C, d \circ C) \in JR_\phi$. Hence there is an isomorphism $\theta: (\mathcal{C}, C) \to (\mathcal{C}, d \circ C)$ (see the discussion just before I.2.15.). As

$$2^{\rho_{d^{-1}}}: (\mathcal{C}, d \circ C) \to (\mathfrak{W}\mathfrak{F}(d \circ Cd^{-1}, \mathfrak{M}), d \circ Cd^{-1})$$

is an isomorphism,

$$2^{\rho_{d^{-1}}} \circ \theta: (\mathcal{C}, C) \to (\mathfrak{W}\mathfrak{F}(d \circ Cd^{-1}, \mathfrak{M}), d \circ Cd^{-1})$$

is an isomorphism. Since by a, $d \circ Cd^{-1}$ is an MHP generator, it follows from b that $C = d \circ Cd^{-1}$.

Conversely, assume that $C = d \circ Cd^{-1}$ for every $d \in uD$. Let $p \circ C$ and $q \circ C$ in $\mathcal{C}$ with $(p \circ C, q \circ C) \in JR_\phi$, say $(p \circ C, q \circ C) = (vp \circ C, vq \circ C)$ for some $v \in J$. Then

$$(u \circ C, up^{-1}q \circ C) = up^{-1}(p \circ C, q \circ C) \in R_\phi,$$

so for $d = up^{-1}q$ we have

$$d \circ D = \phi(d \circ C) = \phi(u \circ C) = u \circ D = D$$

and $d \in D \cap uM = uD$. By assumption, it follows that $d \circ Cd^{-1} = C$. But then

$$2^{\rho_d}: (\mathcal{C}, C) \to (\mathfrak{W}\mathfrak{F}(C, \mathfrak{M}), d \circ C)$$

is an ambit isomorphism, and

$$2^{\rho_d}(vp \circ C) = vp \circ Cd = vp \circ (d \circ Cd^{-1})d = vpd \circ C = vpup^{-1}q \circ C = vq \circ C \ .$$

This shows that there exists a map $2^{\rho_d} : \mathcal{C} \to \mathcal{C}$, such that $p \circ C$ is mapped onto $q \circ C$; hence it follows that ϕ is regular. $\square$

In the remainder of this section we shall study quasifactors of MHP ttgs. For that we need some notation.
As we use the circle operation with respect to quasifactors of M as well as to quasifactors of quasifactors of $\mathfrak{M}$ it seems convenient to distinguish between them by denoting the action of S_T on 2^{2^M} by $\square$. So if $S \subseteq 2^M$ is a closed set in 2^M (with respect to the Vietoris topology) then $p \square S = \lim t_i S$ in 2^{2^M} for some (every) net $t_i \to p$.
A source of ambiguity is the fact that we shall consider a closed subset C of M both as a closed subset of M and as an element of 2^M . Let $D \subseteq S_T$ and let C be a closed subset of M . Then define

$D \circledcirc C := \{d \circ C \mid d \in D\} \subseteq 2^M$; compare this with:

$D \circ C = \bigcup \{d \circ C \mid d \in D\} \subseteq M$ and

$DC = \bigcup \{dc \mid d \in D \, , c \in C\} \subseteq M$.

If we consider C as an element of 2^M , then we can define a map $\rho_C : S_T \to 2^M$ by $p \mapsto p \circ C$; i.e., ρ_C is the right multiplication with C of elements of S_T (the evaluation mapping in C , induced by the action of S_T on 2^M). Then $D \circledcirc C = \rho_C[D]$.

2.5. **LEMMA.** *Let C be an almost periodic element of 2^M and let $D \subseteq S_T$ be a closed set.*

a) *$D \circledcirc C$ is a closed subset of 2^M , hence of $\mathfrak{QF}(C, \mathfrak{M})$.*

b) *$p \square (D \circledcirc C) = (p \circ D) \circledcirc C$ for every $p \in S_T$.*

c) *The almost periodic elements of $2^{\mathfrak{QF}(C,\mathfrak{M})}$ are just the subsets of $QF(C,\mathfrak{M})$ of the form $B \circledcirc C$, where B is an almost periodic element of 2^M .*
Hence the quasifactors of $\mathfrak{QF}(C,\mathfrak{M})$ are just the ttgs of the form $\mathfrak{QF}(B \circledcirc C, \mathfrak{QF}(C,\mathfrak{M}))$ for $B \in 2^M$ almost periodic.

PROOF.

a) As $\rho_C : S_T \to 2^M$ is continuous, it is a closed map. Hence it follows that $D \circledcirc C = \rho_C[D]$ is a closed subset of 2^M , hence of $QF(C,\mathfrak{M})$.

b) As $\rho_C : S_T \to 2^M$ is a homomorphism, also $2^{\rho_C} : 2^{S_T} \to 2^{2^M}$ is a homomorphism; so $\rho_C[p \circ D] = p \square \rho_C[D]$ and

$$(p \circ D) \circledcirc C = \rho_C[p \circ D] = p \square \rho_C[D] = p \square (D \circledcirc C) \ .$$

c) Let B be an almost periodic element of 2^M, say $B = v \circ B$ for some $v \in J$. Then by b,

$$B \circledcirc C = (v \circ B) \circledcirc C = v \,\square\, (B \circledcirc C)\,;$$

hence $B \circledcirc C$ is an almost periodic element of $2^{QF(C,\mathfrak{M})}$.
Conversely, let A be an almost periodic element of $2^{QF(C,\mathfrak{M})}$, say $A = w \,\square\, A$ for some $w \in J$. Let $B' = \{p \in M \mid p \circ C \in A\}$; then, as C is an almost periodic element of 2^M, we have $A = B' \circledcirc C$. Hence, by b, it follows that

$$A = w \,\square\, A = w \,\square\, (B' \circledcirc C) = (w \circ B') \circledcirc C$$

and, clearly, $w \circ B'$ is an almost periodic element of 2^M. □

2.6. **THEOREM.** *Let* C *be an* MHP *generator,* $\mathcal{C} = \mathfrak{QF}(C,\mathfrak{M})$ *and let* $u \in J$.

a) *Let* D *be an almost periodic element of* 2^M. *Then* $\mathfrak{QF}(D \circledcirc C, \mathcal{C})$ *is homeomorphic to* $\mathfrak{QF}(D \circ C, \mathfrak{M})$ *by the map* μ *defined by* $\mu(p \,\square\, (D \circledcirc C)) = p \circ D \circ C$ *for every* $p \in M$.

b) *The quasifactors of* $\mathcal{C}$ *are just the quasifactors of* $\mathfrak{M}$ *of the form* $\mathfrak{QF}(D \circ C, \mathfrak{M})$ *for* $D = u \circ D \in 2^M$ *(up to the isomorphism mentioned in* a*).*

PROOF.

a) Note that it is sufficient to prove that for every p and q in M we have $p \,\square\, (D \circledcirc C) = q \,\square\, (D \circledcirc C)$ iff $p \circ D \circ C = q \circ D \circ C$.
Suppose that $p \,\square\, (D \circledcirc C) = q \,\square\, (D \circledcirc C)$. Then by 2.5.b, we have $(p \circ D) \circledcirc C = (q \circ D) \circledcirc C$. Let $r \in p \circ D \circ C$; then $r \in s \circ C$ for some $s \in p \circ D$. As $s \circ C \in (p \circ D) \circledcirc C$, also $s \circ C \in (q \circ D) \circledcirc C$; so there is an $s' \in q \circ D$ with $s \circ C = s' \circ C$. But then

$$r \in s \circ C = s' \circ C \subseteq q \circ D \circ C\,,$$

and so $p \circ D \circ C \subseteq q \circ D \circ C$. Similarly, $q \circ D \circ C \subseteq p \circ D \circ C$; hence $p \circ D \circ C = q \circ D \circ C$.
On the other hand, suppose that $p \circ D \circ C = q \circ D \circ C$, and let $r \in p \circ D$. Then

$$r \circ C \subseteq p \circ D \circ C = q \circ D \circ C\,,$$

so $r \circ C \cap s \circ C \neq \varnothing$ for some $s \in q \circ D$. As C is an MHP generator it follows that $r \circ C = s \circ C$, which shows that $r \circ C = s \circ C \in (q \circ D) \circledcirc C$.

So $(p\circ D)\circledcirc C\subseteq (q\circ D)\circledcirc C$ and similarly $(q\circ D)\circledcirc C\subseteq (p\circ D)\circledcirc C$, hence

$$p\,\square\,(D\circledcirc C)=(p\circ D)\circledcirc C=(q\circ D)\circledcirc C=q\,\square\,(D\circledcirc C)\,.$$

b) From 2.5.c and 2.6.a it follows immediately that the quasifactors of $\mathcal{C}$ are just the quasifactors of $\mathcal{M}$ of the form $\mathfrak{QF}(D'\circ C,\mathcal{M})$ for $D'\in 2^M$ almost periodic (up to isomorphism). Clearly, the ttgs $\mathfrak{QF}(D'\circ C,\mathcal{M})$ and $\mathfrak{QF}(u\circ D'\circ C,\mathcal{M})$ are equal, and $D:=u\circ D'$ is such that $D=u\circ D$. $\square$

As every extension of an MHP ttg is open, it follows from IV.3.3. that every MHP factor of a minimal ttg $\mathcal{X}$ is an MHP quasifactor of $\mathcal{X}$.
We shall now be concerned with the converse in the case of $\mathcal{X}$ being an MHP ttg.

2.7. **THEOREM.** *Let* C *be a regular* MHP *generator (i.e.,* $\mathcal{C}$ *is regular). Let* $\mathcal{Y}$ *be a quasifactor of* $\mathcal{C}$*, say* $\mathcal{Y}=\mathfrak{QF}(D\circledcirc C,\mathcal{C})$ *with* $D=u\circ D\in 2^M$ *and suppose that* D *can be chosen to be an* MHP *generator. Then* $\mathcal{Y}$ *is a factor of* $\mathcal{C}$ *iff* $D\circ C$ *is an* MHP *generator.*

PROOF. If $D\circ C$ is an MHP generator, then by 2.4.b, there is an ambit morphism

$$\phi:(\mathcal{C},u\circ C)\to(\mathfrak{QF}(D\circ C,\mathcal{M}),D\circ C)\,.$$

For $u\circ C\subseteq u\circ D\circ C=D\circ C$ ($D\cap J\neq\varnothing$) and $u\circ C$ is an MHP generator (see 1.5.b and 2.1.). By 2.6.a, $\mathcal{Y}$ is isomorphic to $\mathfrak{QF}(D\circ C,\mathcal{M})$; so $\mathcal{Y}$ is a factor of $\mathcal{C}$.
Conversely, suppose that $\mathcal{Y}$ is a factor $\mathcal{C}$, so there is a homomorphism $\psi:\mathcal{C}\to\mathcal{Y}$ such that $\psi(u\circ C)=a\,\square\,(D\circledcirc C)$ for some $a\in uM$. As $u\circ C$ is an MHP generator we have $(u\circ C)\circledcirc(u\circ C)=\{u\circ C\}$, hence (identifying $\mathcal{Y}$ with $\mathfrak{QF}(D\circ C,\mathcal{M})$ by the homomorphism indicated in 2.6.a):

$$a\circ D\circ C=\psi(u\circ C)=\psi[(u\circ C)\circledcirc(u\circ C)]=(u\circ C)\circledcirc(a\circ D\circ C)\,.$$

But then for every $c\in u\circ C$ we have $a\circ D\circ C=c\circ a\circ D\circ C$ and so $a\circ D\circ C=C\circ a\circ D\circ C$; hence

$$D\circ C=a^{-1}\circ C\circ a\circ D\circ C\,.$$

As C is regular, $a^{-1}\circ Ca=C$; so

$$D\circ C=a^{-1}\circ C\circ a\circ D\circ C=C\circ D\circ C\,.$$

This implies that

$$D \circ C \circ D \circ C = D \circ (C \circ D \circ C) = D \circ D \circ C = D \circ C \ .$$

in other words, $D \circ C$ is an MHP generator. □

2.8. **THEOREM.** *Let* $\mathfrak{X}$ *be an* MHP *ttg, say* $\mathfrak{X} \cong \mathfrak{W}\mathfrak{F}(C, \mathfrak{M})$, *where* C *is an* MHP *generator with* $C = u \circ C$ *for some* $u \in J$. *Let* $\mathfrak{Y}$ *be an* MHP *ttg which is a quasifactor of* $\mathfrak{X}$. *Then* $\mathfrak{Y}$ *is a factor of* $\mathfrak{X}$ *iff* $\mathfrak{Y}$ *is homeomorphic to* $\mathfrak{W}\mathfrak{F}(D, \mathfrak{M})$ *for some* MHP *generator* D *with* $D = u \circ D$ *and* $C \subseteq D$.

PROOF. The "if"-part follows immediately from 2.4.b.
Conversely, let $\mathfrak{Y} \cong \mathfrak{W}\mathfrak{F}(D \circ C, \mathfrak{M})$ for some D with $D = u \circ D \in 2^M$ (2.6.b) and let $\phi: \mathfrak{X} \to \mathfrak{Y}$ be a homomorphism of minimal ttgs. Let $a \in uM$ be such that $\phi(C) = a \circ D \circ C$ and define

$$D' = u \circ \{p \in M \mid pa \circ D \circ C = a \circ D \circ C\} = u \circ M_{a \circ D \circ C} \ .$$

Then by 1.4.(iii) and 2.1., D' is an MHP generator and as $C \circledcirc C = \{C\}$, we have

$$a \circ D \circ C = \phi(C) = \phi[C \circledcirc C] = C \circ a \circ D \circ C \ ,$$

so $C \subseteq D'$. But, $\mathfrak{Y}^* = \mathfrak{W}\mathfrak{F}(D', \mathfrak{M})$, and so by the assumptions of $\mathfrak{Y}$ being an MHP ttg, it follows that $\mathfrak{Y} \cong \mathfrak{W}\mathfrak{F}(D', \mathfrak{M})$. □

V.3. SOME DYNAMICAL PROPERTIES

In this section we consider dynamical properties in relation to the theory developed in the previous sections. In particular, for two homomorphisms $\phi:\mathfrak{X}\to\mathfrak{Y}$ and $\psi:\mathfrak{Z}\to\mathfrak{Y}$ of minimal ttgs we give a criterion in terms of MHP generators that guarantees ϕ and ψ to satisfy the generalized Bronstein condition. As a result we prove that, in case the homomorphism under consideration is regular, an affirmative answer can be given to the question whether or not an open Bc extension is a RIC extension. Also we shall discuss disjointness from the point of view of MHP generators.

3.1. The situation we shall study comes down to the following:
Let $\phi:\mathfrak{X}\to\mathfrak{Y}$ and $\psi:\mathfrak{Z}\to\mathfrak{Y}$ be homomorphisms of minimal ttgs and let $\phi^*:\mathfrak{X}^*\to\mathfrak{Y}^*$ and $\psi^*:\mathfrak{Z}^*\to\mathfrak{Y}^*$ be the MHP liftings of ϕ and ψ (see IV.3.10.). To be more precise, fix $u\in J$, $y_0\in uY$, $x_0\in u\phi^{\leftarrow}(y_0)$ and $z_0\in u\psi^{\leftarrow}(y_0)$. Define the sets $C:=u\circ M_{x_0}=u\circ\{p\in M\mid px_0=x_0\}$, $D:=u\circ M_{y_0}$ and $F:=u\circ M_{z_0}$. Then C , D and F are MHP generators, $\mathfrak{X}^*=\mathfrak{C}$, $\mathfrak{Y}^*=\mathfrak{D}$, $\mathfrak{Z}^*=\mathfrak{F}$ and $\phi^*:\mathfrak{C}\to\mathfrak{D}$ and $\psi^*:\mathfrak{F}\to\mathfrak{D}$ are the MHP liftings of ϕ and ψ (note that $C\cup F\subseteq D$!).

$$\begin{tikzcd}
\mathfrak{C} \arrow[dr, "\phi^*"] \arrow[dd, "\chi_{\mathfrak{X}}"'] & & \mathfrak{F} \arrow[dl, "\psi^*"'] \arrow[dd, "\chi_{\mathfrak{Z}}"] \\
& \mathfrak{D} \arrow[dd, "\chi_{\mathfrak{Y}}"] & \\
\mathfrak{X} \arrow[dr, "\phi"'] & & \mathfrak{Z} \arrow[dl, "\psi"] \\
& \mathfrak{Y} &
\end{tikzcd}$$

3.2. **THEOREM.** *Let* ϕ *and* ψ *be homomorphisms as in* 3.1.. *Then with notation as in* 3.1. *we have:*

a) *the maps* ϕ^* *and* ψ^* *satisfy the generalized Bronstein condition iff* $D=C\circ uD\circ F$ *iff* $D=F\circ uD\circ C$ *;*

b) ϕ^* *satisfies the Bronstein condition iff* $D=C\circ uD\circ C$.

PROOF. Obviously, b follows from a; so we only have to prove a.
Suppose that ϕ^* and ψ^* satisfy gBc. Then by I.3.8.,

$$R_{\phi^*\psi^*}=\overline{T(\{C\}\times u\psi^{*\leftarrow}\phi^*(C))}\,.$$

As $u\psi^{*\leftarrow}\phi^*(C) = \{a \circ F \mid a \circ D = u \circ D \text{ and } a \in uM\}$ it follows that

$$u\psi^{*\leftarrow}\phi^*(C) = \{a \circ F \mid a \in (u \circ D \cap uM) = uD\} = uD \circledcirc F \ ,$$

and so

$$R_{\phi^*\psi^*} = \overline{T(\{C\} \times uD \circledcirc F)} \, .$$

Let $d \in D$. Then $(C, d \circ F) \in R_{\phi^*\psi^*}$, for $\phi^*(C) = D = d \circ D = \psi^*(d \circ F)$. So there is a net $\{t_i\}_i$ in T and there are $d_i \in uD$ such that

$$t_i(C, d_i \circ F) \to (C, d \circ F) \text{ in } R_{\phi^*\psi^*} \, .$$

Let $p = \lim t_i u \in M$ (after passing to a suitable subnet). Then

$$C = \lim t_i \circ C = \lim t_i(u \circ C) = \lim t_i u \circ C = (\lim t_i u) \circ C = p \circ C \ ,$$

and as $u \in C$ it follows that $p \in C$.
As $d_i = ud_i$, we have that $\lim t_i d_i = \lim t_i ud_i \in p \circ uD$; so it follows that

$$d \circ F = \lim t_i(d_i \circ F) = (\lim t_i d_i) \circ F \in (p \circ uD) \circledcirc F \ .$$

Hence $d \circ F \subseteq p \circ uD \circ F$ and so

$$d \in d \circ F \subseteq p \circ uD \circ F \subseteq C \circ uD \circ F \ .$$

As $d \in D$ was arbitrary it follows that $D \subseteq C \circ uD \circ F$. Clearly, $C \circ uD \circ F \subseteq D$ which implies $D = C \circ uD \circ F$.

Conversely, suppose $D = C \circ uD \circ F$ and let $(p \circ C, q \circ F) \in R_{\phi^*\psi^*}$, so $p \circ D = q \circ D$. Then, as $u \in C \circ uD \circ F$, we have

$$q = qu \in q \circ C \circ uD \circ F = q \circ D = p \circ D =$$

$$= p \circ C \circ uD \circ F = (p \circ C \circ uD) \circ F \ ,$$

say $q \in r \circ F$ for some $r \in p \circ C \circ uD = (p \circ C) \circ uD$.
Note that $q \in r \circ F \cap q \circ F$ so $r \circ F = q \circ F$.
Let $s \in p \circ C$ such that $r \in s \circ uD$; then $s \in s \circ C \cap p \circ C$ so $s \circ C = p \circ C$. Let $\{t_j\}_j$ be a net in T with $t_j \to s$ and let $d_j \in uD$ be such that $t_j d_j \to r$. Then $(C, d_j \circ F) = u(C, d_j \circ F)$ is almost periodic in $R_{\phi^*\psi^*}$ and

$$\lim t_j(C, d_j \circ F) = (\lim t_j C, \lim t_j d_j \circ F) = (s \circ C, r \circ F) = (p \circ C, q \circ F) \ ;$$

hence $(p \circ C, q \circ F)$ is the limit of a net in $JR_{\phi^*\psi^*}$. As $(p \circ C, q \circ F)$ was arbitrary in $R_{\phi^*\psi^*}$ it follows that $R_{\phi^*\psi^*}$ has a dense subset of almost periodic points; i.e., ϕ^* and ψ^* satisfy gBc.

So we proved that ϕ^* and ψ^* satisfy gBc iff $D = C \circ uD \circ F$. Interchanging the roles of C and F completes the proof. □

3.3. **THEOREM.** *Let ϕ and ψ be homomorphisms of minimal ttgs, and let ϕ be open. Then with notation as in* 3.1. *we have*

- a) *the maps ϕ and ψ satisfy the generalized Bronstein condition iff $Dx_0 = F \circ uDx_0$;*
- b) *ϕ is a* Bc *extension iff $Dx_0 = C \circ uDx_0 = J_{x_0} \circ uDx_0$.*

PROOF.

a) By IV.4.16.b, (ϕ,ψ) satisfies gBc iff (ϕ^*,ψ^*) satisfies gBc. So by 3.2.a, ϕ and ψ satisfy gBc iff $D = F \circ uD \circ C$. As $C = u \circ M_{x_0} \subseteq M_{x_0}$ we have $Cx_0 = x_0$; hence

$$Dx_0 = F \circ uD \circ Cx_0 = F \circ uDx_0 .$$

Conversely, suppose that $Dx_0 = F \circ uDx_0$. Since ϕ is open, it follows from I.3.9. that $R_{\phi\psi} = \overline{T(\phi^{\leftarrow}(y_0) \times \{z_0\})}$. So, in order to prove that (ϕ,ψ) satisfies gBc, it is enough to show that

$$\phi^{\leftarrow}(y_0) \times \{z_0\} \subseteq \overline{JR_{\phi\psi}} .$$

First note that

$$Dx_0 = (u \circ M_{y_0})x_0 = u \circ (M_{y_0}x_0) = u \circ \phi^{\leftarrow}(y_0)$$

and as ϕ is open this implies that $\phi^{\leftarrow}(y_0) = u \circ \phi^{\leftarrow}(y_0) = Dx_0$.
Let $x' \in \phi^{\leftarrow}(y_0)$, then $x' \in \phi^{\leftarrow}(y_0) = Dx_0 = F \circ uDx_0$, say $x' \in f \circ uDx_0$ for a certain $f \in F$. Let $\{t_i\}_i$ be a net in T with $f = \lim t_i$ and let $d_i \in uD$ be such that $x' = \lim t_i d_i x_0$. As $f \in F$ we have $fz_0 = z_0$ and

$$(x',z_0) = (x',fz_0) = \lim t_i (d_i x_0, z_0) .$$

Clearly, $(d_i x_0, z_0) \in JR_{\phi\psi}$ and so $t_i(d_i x_0, z_0) \in JR_{\phi\psi}$ for every i, hence $(x',z_0) \in \overline{JR_{\phi\psi}}$. As $x' \in \phi^{\leftarrow}(y_0)$ was arbitrary it follows that $\phi^{\leftarrow}(y_0) \times \{z_0\} \subseteq \overline{JR_{\phi\psi}}$, and so $R_{\phi\psi} = \overline{JR_{\phi\psi}}$.

b) By a and the proof of a, $Dx_0 = C \circ uDx_0 = \phi^{\leftarrow}(y_0)$, and obviously,

$$J_{x_0} \circ uDx_0 \subseteq \phi^{\leftarrow}(y_0) = Dx_0 .$$

Let $K = C \cap J$; then $K = (u \circ M_{x_0}) \cap J \subseteq M_{x_0} \cap J = J_{x_0}$. By 1.7., $C = K \circ uC$; and, as $uC \subseteq uD$, it follows that

$$C \circ uD = K \circ uC \circ uD = K \circ uD \subseteq J_{x_0} \circ uD \ .$$

Hence $Dx_0 = C \circ uDx_0 \subseteq J_{x_0} \circ uDx_0$ and so $Dx_0 = J_{x_0} \circ uDx_0$. ☐

By III.1.5. it follows that the characterization of gBc in terms of MHP generators gives rise to a characterization of RIC extensions in terms of MHP generators, as follows.

3.4. **THEOREM.** *Let* ϕ *be a homomorphism of minimal ttgs. Then, with notation as in* 3.1., ϕ^* *is a* RIC *extension iff* $D = C \circ uD$.

PROOF. By III.1.5., ϕ^* is a RIC extension iff (ϕ^*, θ) satisfies gBc for every homomorphism $\theta : \mathfrak{W} \to \mathfrak{Y}^*$. Suppose ϕ^* is a RIC extension. Define $B \subseteq M$ by $B := u \circ uD$. Then B is an MHP generator, and by 2.4.b, there is an ambit morphism $\theta : \mathfrak{B} \to \mathfrak{D}$. As (ϕ^*, θ) satisfies gBc it follows from 3.2. that

$$D = C \circ uD \circ B = C \circ uD \circ u \circ uD \ ;$$

hence $D = C \circ uD \circ uD = C \circ uD$.
If, conversely, $D = C \circ uD$, then for every MHP generator F with $F = u \circ F \subseteq D$ we have $D \subseteq D \circ F = C \circ uD \circ F$, so $C \circ uD \circ F = D$.
As $F \subseteq D$, there is an ambit morphism $\theta : \mathfrak{F} \to \mathfrak{D}$ (2.4.b), and so by 3.2., ϕ^* and $\theta = \theta^*$ satisfy gBc. Let $\psi : \mathfrak{Z} \to \mathfrak{Y}^*$ be a homomorphism of minimal ttgs and let $z_0 \in uZ$ be such that $\psi(z_0) = D$. Define $F = u \circ M_{z_0}$. Then F is an MHP generator with $F \subseteq D$, and the ambit morphism $\theta : \mathfrak{F} \to \mathfrak{D}$ is the MHP lifting of ψ (i.e., $\psi^* = \theta$). By the above ϕ^* and ψ^* satisfy gBc. As ϕ^* is open, it follows from IV.4.16.b, that ϕ^* and ψ satisfy gBc. As ψ was arbitrary, it follows from III.1.5. that ϕ^* is a RIC extension. ☐

3.5. **THEOREM.** *Let* $C = u \circ C$ *and* $D = u \circ D$ *be* MHP *generators such that* $C \subseteq D$ *and the map* $\phi^* : \mathfrak{C} \to \mathfrak{D}$ *is regular. Then*

a) $C \circ uD$ *is an* MHP *generator;*

b) $\phi^* = \theta^* \circ \psi^*$ *, where* ψ^* *is a* RIC *extension and* θ^* *is proximal;*

c) ϕ^* *is a* RIC *extension iff* ϕ^* *satisfies the Bronstein condition.*

PROOF.

a) By 2.4.d, we have $d \circ Cd^{-1} = C$ for all $d \in uD$. So

$$uD \circ C = \bigcup \{ d \circ C \mid d \in uD \} = \bigcup \{ Cd \mid d \in uD \} = C.uD \subseteq C \circ uD \ ,$$

which implies that

$$C \circ uD \circ C \circ uD = C \circ (uD \circ C) \circ uD \subseteq C \circ (C \circ uD) \circ uD = C \circ uD \ ,$$

so it follows that $C \circ uD$ is an MHP generator.

b) Define $F = C \circ uD$, then $F = u \circ F$ and F is an MHP generator (a). By 1.6.b, it follows that $uF = u(u \circ C).u(u \circ uD) = uC.uD$, and as $uC \subseteq uD$ we even have $uF = uCuD = uD$. As $uF = \mathfrak{G}(\mathfrak{F}, F)$ and $uD = \mathfrak{G}(\mathfrak{D}, D)$ it follows from I.2.13. that the ambit morphism $\theta^* : \mathfrak{F} \to \mathfrak{D}$ is proximal. (Note that $F = C \circ uD \subseteq D \circ uD = D$, so θ^* exists by 2.4.b.) Since $C \subseteq F$ and $C \circ uF = C \circ uD = F$, it follows from 2.4.b that the map $\psi^* : \mathcal{C} \to \mathfrak{F}$ exists; and by 3.4., it follows that ψ^* is a RIC extension.

c) If ϕ^* is a RIC extension, then ϕ^* is a Bc extension by III.1.9.. Suppose that ϕ^* is a Bc extension. Then, with notation as in b, θ^* as a factor of ϕ^* is a Bc extension. Hence, as θ^* is proximal, θ^* is an isomorphism and $F = D$, so $\phi^* = \psi^*$. But then ϕ^* is a RIC extension. □

3.6. **LEMMA.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs and let* $\phi^* : \mathcal{C} \to \mathfrak{D}$ *be the* MHP *lifting of* ϕ *as in* 3.1..

a) *If* ϕ *is regular then* ϕ^* *is regular.*

b) *If* ϕ *is distal then* ϕ *is regular iff* ϕ^* *is regular.*

PROOF.

a) Suppose ϕ is regular. We shall prove that $d \circ Cd^{-1} \subseteq C$ for every $d \in uD$. As uD is a group, it follows that $d \circ Cd^{-1} = C$ for every $d \in uD$ and so, by 2.4.d, that ϕ^* is regular.
Let $d \in uD$. As $uD = u(u \circ M_{y_0}) \subseteq uM_{y_0}$, it is clear that $(x_0, dx_0) \in JR_\phi$. Regularity of ϕ implies the existence of an isomorphism $\theta : \mathfrak{X} \to \mathfrak{X}$ such that $\theta(x_0) = dx_0$. Define $C' := u \circ M_{dx_0}$; then $\theta^* : \mathcal{C} \to \mathcal{C}'$ is the MHP lifting of θ and so θ^* is an isomorphism too. By 2.4.b, it follows that $C = C'$. As

$$(d \circ Cd^{-1})dx_0 = d \circ Cx_0 = dx_0 \ ,$$

we have that $d \circ Cd^{-1} \subseteq M_{dx_0}$ and so that

$$d \circ Cd^{-1} = u \circ d \circ Cd^{-1} \subseteq u \circ M_{dx_0} = C' = C \ .$$

b) Suppose that ϕ is a distal map and let ϕ^* be regular. Let $(x_1, x_2) \in R_\phi = JR_\phi$ (ϕ is distal!), say $(x_1, x_2) = v(x_1, x_2)$ for $v \in J$ and let $y_1 := \phi(x_1) = \phi(x_2)$. Then there is an $a \in vM$ such that

$x_1 = ax_0$ and so $y_1 = ay_0$. Let $b \in vM$ be such that $bx_0 = x_2$ and note that $y_1 = by_0$, so $ua^{-1}by_0 = y_0$, and $ua^{-1}b \in uD$; hence $ub^{-1}a \in uD$ and, by regularity, of ϕ^* $ub^{-1}a \circ Ca^{-1}b = C$. Define $\theta : X \to X$ by $\theta(px_0) = pa^{-1}bx_0$ for every $p \in M$. If θ is well defined then θ is a homomorphism of minimal ttgs such that

$$\theta(x_1) = \theta(ax_0) = aa^{-1}bx_0 = vbx_0 = bx_0 = x_2 ;$$

hence ϕ is regular.
Let p and q in M be such that $px_0 = qx_0$, so $py_0 = qy_0$. Then $up^{-1}qx_0 = x_0$, so $up^{-1}q \in C$. As $C = ub^{-1}a \circ Ca^{-1}b$ it follows that $ub^{-1}ap^{-1}qa^{-1}b \in C$ and so $upa^{-1}bx_0 = uqa^{-1}bx_0$, which implies that $pa^{-1}bx_0$ and $qa^{-1}bx_0$ are proximal. On the other hand, we have that

$$\phi(pa^{-1}bx_0) = pa^{-1}by_0 = py_0 = qy_0 = qa^{-1}by_0 = \phi(qa^{-1}bx_0) ;$$

so by distality of ϕ, $pa^{-1}bx_0$ and $qa^{-1}bx_0$ are distal. But then $pa^{-1}bx_0 = qa^{-1}bx_0$; hence it follows that θ is well defined, which completes the proof. □

By now we can give a partial answer to the question whether or not an open Bc extension is a RIC extension (see III.1.8.), which says that this indeed is the case if we put on the map the additional condition of being regular.

3.7. **THEOREM.** *Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a regular homomorphism of minimal ttgs. Then ϕ is open and satisfies the Bronstein condition iff ϕ is a* RIC *extension.*

PROOF. If ϕ is a RIC extension then we already know that ϕ is an open Bc extension (III.1.9.).
Suppose that ϕ is open and that ϕ is a Bc extension. Let $\phi^* : \mathfrak{X}^* \to \mathfrak{Y}^*$ be the MHP lifting of ϕ . Then by 3.6., ϕ^* is regular and, by IV.4.17.a, ϕ^* is a Bc extension. Hence by 3.5.c, ϕ^* is a RIC extension. As ϕ is open it follows from IV.4.17.c that ϕ is a RIC extension. □

3.8. **REMARK.** *Let $C = u \circ C$ and $D = u \circ D$ be* MHP *generators with $C \subseteq D$. From* 3.5.a *we know that $C \circ uD$ is an* MHP *generator if $\phi : \mathcal{C} \to \mathfrak{D}$ is regular. The converse of this statement is in general not true.*

PROOF. Let $\mathfrak{X}$ be a minimal distal ttg which is not regular (note that such a ttg exists [PW 70]). Then by 3.6.b, the MHP extension $\mathfrak{X}^*$ of $\mathfrak{X}$ is not regular. Let $x \in uX$ and define $F := u \circ M_x$. Then $\mathfrak{X}^* = \mathfrak{F}$ and the map

$\psi : \mathcal{X}^* \to \{\bigstar\}$ is not regular. In terms of MHP generators we can write ψ as the ambit morphism

$$\psi : (\mathfrak{WF}(F, \mathcal{M}), F) \to (\mathfrak{WF}(M, \mathcal{M}), M) .$$

As $\mathcal{X}$ is distal, x is a distal point and $J \subseteq M_x$. Hence $F = u \circ J \circ A$, for $A = uM_x$. So

$$F \circ uM = u \circ J \circ A \circ uM = u \circ J \circ uM = u \circ M = M .$$

So $F \circ uM = M$ while ψ is not regular! □

We shall now turn to a description of disjointness in terms of MHP generators. To that end consider the situation as sketched in 3.1. and, in particular, the upper half of the diagram. So let $C = u \circ C$, $D = u \circ D$ and $F = u \circ F$ be MHP generators with $C \cup F \subseteq D$ and let $\phi^* : \mathcal{C} \to \mathcal{D}$ and $\psi^* : \mathcal{F} \to \mathcal{D}$ be the canonical homomorphisms.

3.9. **THEOREM.** *With notation as above, the following statements are equivalent:*

a) $\phi^* \perp \psi^*$;

b) $R_{\phi^* \psi^*}$ *has a unique minimal subset and* (ϕ^*, ψ^*) *satisfies the generalized Bronstein condition;*

c) $C \circ F = D$ *(and also* $F \circ C = D$ *);*

d) $(p \circ C) \cap (q \circ F) \neq \emptyset$ *for all elements* p *and* q *of* M *with* $p \circ D = q \circ D$.

PROOF.

a ⇒ b Trivial.

b ⇒ c By 3.2.a, we know that $D = C \circ uD \circ F \; (= F \circ uD \circ C)$. By I.3.2., $R_{\phi^* \psi^*}$ has a unique minimal subset iff $\mathfrak{G}(\mathcal{D}, D) = \mathfrak{G}(\mathcal{C}, C).\mathfrak{G}(\mathcal{F}, F)$. Hence 2.3. implies that $uD = uC.uF \; (= uF.uC)$, and so we have

$$D = C \circ uD \circ F = C \circ (uCuF) \circ F \subseteq (C \circ uC) \circ (uF \circ F) =$$

$$= C \circ F \subseteq D \circ D = D .$$

Similarly one proves that $D = F \circ C$, so $D = C \circ F = F \circ C$.

c ⇒ d Suppose $C \circ F = D$ and let p and q in M be such that $p \circ D = q \circ D$. Then $p \circ C \circ F = q \circ C \circ F$, so $q \in p \circ C \circ F$ and there is an $r \in p \circ C$ with $q \in r \circ F$. As C and F are MHP generators it follows that $r \circ C = p \circ C$ and $q \circ F = r \circ F$; hence

$$r \in (r \circ C) \cap (r \circ F) = (p \circ C) \cap (q \circ F) ,$$

so $(p\circ C)\cap(q\circ F)\neq\varnothing$.

d $\Rightarrow$ a Let $(p\circ C,q\circ F)\in R_{\phi^*\psi^*}$; i.e., let p and q in M be such that $p\circ D=q\circ D$. Then there is an $r\in(p\circ C)\cap(q\circ F)$. As C and F are MHP generators it follows that $r\circ C=p\circ C$ and $r\circ F=q\circ F$, so

$$(p\circ C,q\circ F)=(r\circ C,r\circ F)=r(C,F)\,.$$

But this shows that $R_{\phi^*\psi^*}$ is the orbit closure of the almost periodic point $(C,F)\in R_{\phi^*\psi^*}$; hence $R_{\phi^*\psi^*}$ is minimal and $\phi^*\perp\psi^*$. □

3.10. **COROLLARY.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be minimal ttgs and let $x_0\in uX$ and $y_0\in uY$. Then $\mathfrak{X}\perp\mathfrak{Y}$ iff $M_{x_0}\circ M_{y_0}=M$.*

PROOF. Suppose $\mathfrak{X}\perp\mathfrak{Y}$; then (x_0,y_0) is an almost periodic point in $X\times Y$. Let $v\in J$ be such that $vx_0=x_0$ and $vy_0=y_0$. By 3.9., it follows that $v\circ M_{x_0}\circ v\circ M_{y_0}=M$. As

$$v\circ M_{x_0}\circ v\circ M_{y_0}=v\circ M_{x_0}\circ M_{y_0}\subseteq M_{x_0}\circ M_{y_0}\,,$$

we have $M\subseteq M_{x_0}\circ M_{y_0}$; hence $M=M_{x_0}\circ M_{y_0}$.

Suppose $M_{x_0}\circ M_{y_0}=M$ and remark that for every $u\in J$ the sets $u\circ M_{x_0}$ and $u\circ M_{y_0}$ are MHP generators. Let $(px_0,qy_0)\in X\times Y$ and note that $q\in p\circ M=M$. So $q\in p\circ M_{x_0}\circ M_{y_0}$ say $q\in r\circ M_{y_0}$ for certain $r\in p\circ M_{x_0}$. Then $q\circ M_{y_0}=r\circ M_{y_0}$ and $r\circ M_{x_0}=p\circ M_{x_0}$; hence

$$(px_0,qy_0)=((p\circ M_{x_0})x_0,(q\circ M_{y_0})y_0)=((r\circ M_{x_0})x_0,(r\circ M_{y_0})y_0)=(rx_0,ry_0)\,,$$

which implies that $X\times Y$ is the orbit closure of (ux_0,uy_0) , and so that $\mathfrak{X}\times\mathfrak{Y}$ is minimal. □

3.11. **REMARK.** *Let $\phi:\mathfrak{X}\to\mathfrak{Y}$ be an open homomorphism of minimal ttgs, and let $\psi:\mathfrak{Z}\to\mathfrak{Y}$ be a homomorphism of minimal ttgs with $\phi\perp\psi$. Then there is an* MHP *generator $B=u\circ B$ and a homomorphism $\xi:\mathfrak{B}\to\mathfrak{Z}$ such that $\psi\circ\xi$ is maximally disjoint from ϕ ; i.e., if $\phi\perp\psi\circ\xi\circ\eta$ then $\eta=id_{\mathfrak{B}}$* (see also I.3.1.c).

PROOF. Let $y_0\in Y$, $u\in J_{y_0}$ and $x_0\in u\phi^{\leftarrow}(y_0)$, $z_0\in u\psi^{\leftarrow}(y_0)$. Define $C:=u\circ M_{x_0}$, $D:=u\circ M_{y_0}$ and $F:=u\circ M_{z_0}$; then $\phi^*:\mathfrak{C}\to\mathfrak{D}$ and $\psi^*:\mathfrak{F}\to\mathfrak{D}$ are the MHP liftings of ϕ and ψ . Hence by IV.4.16.c, $\phi^*\perp\psi^*$, and so by 3.9., $C\circ F=D$.

Let

$$\mathcal{G} := \{A \mid A = u \circ A \subseteq M\,,\, A = A \circ A \subseteq F \text{ and } C \circ A = D\}$$

be the collection of all MHP generators A such that $\theta : \mathcal{A} \rightarrow \mathcal{F}$ exists and $\phi^* \perp \psi^* \circ \theta$. Clearly $\mathcal{G} \neq \varnothing$ and $\mathcal{G}$ is inductively ordered. So by Zorn's lemma, there is a minimal element $B \in \mathcal{G}$. Then the ambit morphism $\xi : (\mathcal{B}, B) \rightarrow (\mathcal{Z}, z_0)$ is well defined and the MHP lifting of ξ is just $\xi^* : \mathcal{B} \rightarrow \mathcal{F}$, while $(\psi \circ \xi)^* = \psi^* \circ \xi^*$. By construction, $\phi^* \perp \psi^* \circ \xi^*$, hence by IV.4.16.c, $\phi \perp \psi \circ \xi$.

Suppose $\phi \perp \psi \circ \xi \circ \eta$, then $\phi^* \perp \psi^* \circ \xi^* \circ \eta^*$. Let B' be the MHP generator such that the map η^* is defined as the ambit morphism $\eta^* : \mathcal{B}' \rightarrow \mathcal{B}$. Then $B' \subseteq B$ (2.4.b), so $B' \subseteq F$ and as $\phi^* \perp \psi^* \circ \xi^* \circ \eta^*$ it follows from 3.9. that $C \circ B' = D$. Hence, by minimality of B, it follows that $B' = B$ and so η^* turns out to be an isomorphism; hence η is an hp extension. As the codomain of η is an MHP ttg, it follows that η is an isomorphism, which proves that $\psi \circ \xi$ is maximally disjoint from ϕ. □

3.12. **REMARK.** *Let* $C = u \circ C$, $D = u \circ D$, $F = u \circ F$ *and* $H = u \circ H$ *be* MHP *generators such that* $C \cup D \cup F \subseteq H$. *Then the following statements are equivalent:*

a) $u \circ (C \cap D) \circ F = H$ *and* $C \circ D = H$;

b) $u \circ (F \cap C) \circ D = H$ *and* $F \circ C = H$;

c) $u \circ (D \cap F) \circ C = H$ *and* $D \circ F = H$.

PROOF. Consider the ambit morphisms $\phi : \mathcal{C} \rightarrow \mathcal{H}$, $\psi : \mathcal{D} \rightarrow \mathcal{H}$ and $\theta : \mathcal{F} \rightarrow \mathcal{H}$. We shall prove that

$$R_{\phi\psi\theta} = \{(p \circ C, q \circ D, r \circ F) \mid p \circ H = q \circ H = r \circ H\}$$

is minimal iff $u \circ (C \cap D) \circ F = H$ and $C \circ D = H$. As this statement is symmetric in ϕ, ψ and θ the remark follows.

Suppose that $R_{\phi\psi\theta}$ is minimal. Then, clearly, $R_{\phi\psi}$ is minimal and by 3.9., $C \circ D = H$. Define $\xi : \mathcal{R}_{\phi\psi} \rightarrow \mathcal{H}$ by $\xi(p \circ C, q \circ D) = p \circ H \ (= q \circ F)$ and let the MHP generator $B = u \circ B$ be defined as $B := u \circ (C \cap D)$. Then $B = u \circ \{p \in M \mid p(C,D) = (C,D)\}$ and the MHP lifting ξ^* of ξ is just the ambit morphism $\xi^* : \mathcal{B} \rightarrow \mathcal{H}$. As $R_{\phi\psi\theta} \cong R_{\xi\theta}$ it follows from the minimality of $R_{\phi\psi\theta}$ that $\xi \perp \theta$. Hence, as $\theta = \theta^*$, it follows that $\xi^* \perp \theta$ and so, by 3.9., that $B \circ F = H$; i.e., $u \circ (C \cap D) \circ F = H$.

Conversely, let $C \circ D = H$ and $u \circ (C \cap D) \circ F = H$. Then, by 3.9., $\phi \perp \psi$. As above, define the homomorphism $\xi : \mathcal{R}_{\phi\psi} \rightarrow \mathcal{H}$ of minimal ttgs.

Then, for $B := u \circ (C \cap D)$, we have $\xi^* : \mathcal{B} \to \mathcal{K}$ is the MHP lifting of ξ. So by 3.9. and the assumption, it follows that $\xi^* \perp \theta$. Since θ is open it follows from IV.4.16.c that $\xi \perp \theta$; hence $R_{\xi\theta}$ is minimal and clearly $R_{\xi\theta} \cong R_{\phi\psi\theta}$. This proves the remark. □

3.13. **NOTE.** *Let* $\phi : \mathcal{X} \to \mathcal{Y}$ *and* $\psi : \mathcal{Z} \to \mathcal{Y}$ *be homomorphisms of minimal ttgs such that* ψ *is maximally disjoint from* ϕ. *Let* $\xi : \mathcal{R}_{\phi\psi} \to \mathcal{Y}$ *be the induced homomorphism of minimal ttgs. If for some homomorphism* $\theta : \mathcal{W} \to \mathcal{Y}$ *of minimal ttgs* $\xi \perp \theta$, *then* θ *is an isomorphism.*

PROOF. Let $R_{\phi\psi\theta} := \{(x,z,w) \in X \times Z \times W \mid \phi(x) = \psi(z) = \theta(w)\}$.

$$\begin{tikzcd}
\mathcal{R}_{\phi\psi} \arrow[d] \arrow[rr] \arrow[ddrr, "\xi"'] & & \mathcal{Z} \arrow[dd, "\psi"] & & \mathcal{R}_{\psi\theta} \arrow[ll, dash, "\pi_2"'] \arrow[ddll, "\eta"] \arrow[d] \\
\mathcal{X} \arrow[drr, "\phi"'] & & & & \mathcal{W} \arrow[dll, "\theta"] \\
 & & \mathcal{Y} & &
\end{tikzcd}$$

Clearly, $R_{\phi\psi\theta} \cong R_{\xi\theta}$ and $R_{\phi\psi\theta} \cong R_{\phi\eta}$, where $\eta : R_{\theta\psi} \to Y$ is induced by θ and ψ. Hence, if $\theta \perp \eta$, then $R_{\phi\psi\theta}$ is minimal, so $R_{\phi\eta}$ is minimal and $\phi \perp \eta$. Since $\eta = \psi \circ \pi_2$ and ψ is maximally disjoint from ϕ, it follows that π_2 is an isomorphism. But then θ is an isomorphism. □

3.14. **COROLLARY.** *Let* $\mathcal{X}$, $\mathcal{Y}$ *and* $\mathcal{Z}$ *be minimal ttgs. Let* $\mathcal{Y}$ *be maximally disjoint from* $\mathcal{X}$, *then* $\mathcal{Z} \perp (\mathcal{X} \times \mathcal{Y})$ *iff* $\mathcal{Z} = \{\star\}$.

PROOF. Clearly, $\mathcal{Z} \perp (\mathcal{X} \times \mathcal{Y})$ if $\mathcal{Z} = \{\star\}$.
Suppose that $\mathcal{Z} \perp (\mathcal{X} \times \mathcal{Y})$, then by 3.13., the map $\theta : \mathcal{Z} \to \{\star\}$ is an isomorphism. □

V.4. THE UNIVERSAL HPI TTG

> In this section we shall construct the universal minimal HPI ttg for T. In fact, we construct the MHP generator by which it is generated as a quasifactor of $\mathfrak{M}$. The construction uses transfinite induction except for the case of T being locally compact σ-compact, where the smallest MHP generator that contains $u \circ J \circ G_\infty$ is the one that generates the universal minimal HPI ttg (4.9.b).

In order to facilitate reading and writing we shall fix $u \in J$ and denote the set uM by G (as many times before). In this section only, we shall understand an MHP generator C to be an idempotent subset of M such that $u \circ C = C$, hence $u \in C$.
Most of the techniques which we shall use were developed in section 1. and they are stated there more or less explicitly, in this respect we mention 1.1., 1.5., 1.6. and 1.7.. A lemma which is used frequently in the sequel is II.3.11.c; we shall repeat it here.

4.1. LEMMA. *Let H be an arbitrary subset of G and let $g \in G$; then $g \circ H = u \circ gH$. In particular, let A and B be subsets of G; then $u \circ A \circ B = u \circ AB$.*

PROOF. The first statement is II.3.11.c.
Let A and B be subsets of G; then

$$A \circ B = \bigcup \{a \circ B \mid a \in A\} = \bigcup \{u \circ aB \mid a \in A\} \subseteq u \circ AB ,$$

so $u \circ A \circ B \subseteq u \circ u \circ AB = u \circ AB \subseteq u \circ A \circ B$ and $u \circ A \circ B = u \circ AB$. □

We shall now define some " *incontractible* MHP *generators*":
Define a family $\mathcal{K}^*$ of subsets of J as follows

$$\mathcal{K}^* = \{K \subseteq J \mid u \circ K \circ G = M\} .$$

For every $K \in \mathcal{K}^*$ define $\mathcal{a}_K$ to be the smallest idempotent set in $(2^M, \circ)$ that contains $u \circ K$. Note that by 1.5.c, $\mathcal{a}_K$ exists. Also we know that $\mathcal{a}_K = u \circ \mathcal{a}_K$. For, clearly, $u \in u \circ K$, so

$$u \circ \mathcal{a}_K \subseteq (u \circ K) \circ \mathcal{a}_K \subseteq \mathcal{a}_K \circ \mathcal{a}_K = \mathcal{a}_K .$$

By 1.5.b, $u \circ \mathcal{a}_K$ is a closed idempotent subset of M, and as $u \circ K = u \circ (u \circ K)$, we have $u \circ K \subseteq u \circ \mathcal{a}_K$. So, by minimality of $\mathcal{a}_K$, it follows that $u \circ \mathfrak{a}_K = \mathfrak{a}_K$. By 2.1., $\mathfrak{a}_K$ is an MHP generator.

We call $\mathfrak{a}_K$ an *incontractible* MHP *generator*, because the quasifactor $\mathfrak{A}_K$ of $\mathfrak{M}$ generated by the MHP generator $\mathfrak{a}_K$ is an incontractible ttg. For, as $M = u \circ K \circ G \subseteq \mathfrak{a}_K \circ G$ we have that $\mathfrak{a}_K \circ (u \circ G) = M$, hence by 3.9.,

$$\mathfrak{W}\mathfrak{F}(\mathfrak{a}_K, \mathfrak{M}) \perp \mathfrak{W}\mathfrak{F}(u \circ G, \mathfrak{M})\,(= \mathcal{P}_T)\,.$$

Define A_K to be the smallest $\mathfrak{F}(\mathfrak{M}, u)$-closed subgroup of G that contains $u(u \circ K)$.

Note that $A_K \subseteq u\mathfrak{a}_K$. For by 1.6.c, $u\mathfrak{a}_K$ is an $\mathfrak{F}(\mathfrak{M}, u)$-closed subgroup of G and, clearly $u(u \circ K) \subseteq u\mathfrak{a}_K$; so, by minimality of A_K , we have $A_K \subseteq u\mathfrak{a}_K$.

It is not yet clear whether or not $A_K = u\mathfrak{a}_K$ for every $K \in \mathcal{K}^*$. However, for some specific kind of $K \in \mathcal{K}^*$ this indeed is the case, as is shown in the following remark. (Note that this definition differs from the one just before 4.3. and VI.2.12., which avoids the problems in 4.2..)

4.2. **REMARK.** *Let* $K \in \mathcal{K}^*$. *As* $\mathfrak{a}_K$ *is an* MHP *generator, it follows from* 1.7. *that* $\mathfrak{a}_K = K'.u\mathfrak{a}_K = K' \circ u\mathfrak{a}_K = u \circ K' \circ u\mathfrak{a}_K$ *for* $K' = \mathfrak{a}_K \cap J$. *Then*

a) $K' \in \mathcal{K}^*$ *and* $\mathfrak{a}_{K'} \subseteq \mathfrak{a}_K$ *;*

b) $\mathfrak{a}_{K'} = u \circ K' \circ A_{K'}$ *and* $A_{K'} = u\mathfrak{a}_{K'}$ *;*

c) $u\mathfrak{a}_K$ *is the* $\mathfrak{F}(\mathfrak{M}, u)$*-closed subgroup of* G *generated by* $A_K \cup A_{K'}$.

PROOF.

a) As $\mathfrak{a}_K = u \circ K' \circ u\mathfrak{a}_K$ we have by 4.1.,

$$\mathfrak{a}_K \circ G = u \circ K' \circ u\mathfrak{a}_K \circ G = u \circ K' \circ G\ .$$

Hence $M = u \circ K \circ G \subseteq \mathfrak{a}_K \circ G = u \circ K' \circ G$ and so $M = u \circ K' \circ G$; i.e., $K' \in \mathcal{K}^*$. Clearly, $u \circ K' \subseteq \mathfrak{a}_K$, so $\mathfrak{a}_{K'} \subseteq \mathfrak{a}_K$.

b) Obviously, $u \circ K' \circ A_{K'} \subseteq \mathfrak{a}_{K'}$ and $u \circ K' \subseteq u \circ K' \circ A_{K'}$. We shall prove that $u \circ K' \circ A_{K'}$ is an idempotent subset of M ; then it follows that $\mathfrak{a}_{K'} = u \circ K' \circ A_{K'}$.

First note that

$$u \circ K' \circ A_{K'} \circ u \circ K' \circ A_{K'} \subseteq \mathfrak{a}_{K'} \circ \mathfrak{a}_{K'} = \mathfrak{a}_{K'} \subseteq \mathfrak{a}_K = K'.u\mathfrak{a}_K \subseteq K'.uM\ .$$

On the other hand, by 1.6.b,

$$u(u \circ K' \circ A_{K'} \circ u \circ K' \circ A_{K'}) = u(u \circ K').u(u \circ A_{K'}).u(u \circ K').u(u \circ A_{K'}) \,,$$

hence

$$u(u \circ K' \circ A_{K'} \circ u \circ K' \circ A_{K'}) \subseteq A_{K'}.A_{K'}.A_{K'}.A_{K'} = A_{K'} \,.$$

But then

$$u \circ K' \circ A_{K'} \circ u \circ K' \circ A_{K'} \subseteq K'.uM \cap J.A_{K'} = K'.A_{K'} \,,$$

hence

$$u \circ K'.A_{K'} \subseteq u \circ K' \circ A_{K'} \subseteq u \circ K' \circ A_{K'} \circ u \circ K' \circ A_{K'} \subseteq u \circ K'.A_{K'} \,,$$

and so

$$u \circ K'A_{K'} = u \circ K' \circ A_{K'} = u \circ K' \circ A_{K'} \circ u \circ K' \circ A_{K'} \,.$$

This shows that $u \circ K' \circ A_{K'}$ is an MHP generator and that $\mathsf{a}_{K'} = u \circ K' \circ A_{K'}$. Also it is evident that

$$u\mathsf{a}_{K'} = u(u \circ K').u(u \circ A_{K'}) = A_{K'} \,.$$

c) As $\mathsf{a}_{K'} \cup A_K \subseteq \mathsf{a}_K$, it follows that $A_{K'} \cup A_K \subseteq u\mathsf{a}_K$ and so $[A_{K'} \cup A_K] \subseteq u\mathsf{a}_K$, where $[A_{K'} \cup A_K]$ denotes the $\mathfrak{F}(\mathfrak{M}, u)$-closed subgroup of G generated by $A_{K'} \cup A_K$. We shall prove that

$$\mathsf{a}_K = u \circ K' \circ [A_{K'} \cup A_K] \,;$$

it then follows that

$$u\mathsf{a}_K = u(u \circ K').[A_{K'} \cup A_K] = [A_{K'} \cup A_K] \,.$$

As $u \circ K' \cup [A_{K'} \cup A_K] \subseteq \mathsf{a}_K$ it follows that

$$u \circ K' \circ [A_{K'} \cup A_K] \subseteq \mathsf{a}_K \circ \mathsf{a}_K = \mathsf{a}_K \,.$$

Since $u \circ K \subseteq \mathsf{a}_K \subseteq K'.uM$ and $u \circ K \subseteq J.u(u \circ K) \subseteq J.A_K$ it follows that

$$u \circ K \subseteq K'.uM \cap J.A_K = K'.A_K \,;$$

hence

$$u \circ K \subseteq u \circ K'.A_K \subseteq u \circ K' \circ [A_{K'} \cup A_K] \,.$$

If $u \circ K' \circ [A_{K'} \cup A_K]$ is an MHP generator, it follows from the minimality of a_K that $\mathsf{a}_K = u \circ K' \circ [A_{K'} \cup A_K]$. As $u \circ K' \circ [A_{K'} \cup A_K] \subseteq \mathsf{a}_K$, it follows that

$$u\circ K'\circ[A_{K'}\cup A_K]\circ u\circ K'\circ[A_{K'}\cup A_K]\subseteq K'.uM\ .$$

and since $u(u\circ K'\circ[A_{K'}\cup A_K]\circ u\circ K'\circ[A_{K'}\cup A_K])=[A_{K'}\cup A_K]$, we have

$$u\circ K'\circ[A_{K'}\cup A_K]\circ u\circ K'\circ[A_{K'}\cup A_K]\subseteq$$
$$\subseteq K'.uM\cap J.[A_{K'}\cup A_K]=K'.[A_{K'}\cup A_K]\ .$$

Hence

$$u\circ K'\circ[A_{K'}\cup A_K]\circ u\circ K'\circ[A_{K'}\cup A_K]\subseteq u\circ K'\circ[A_{K'}\cup A_K]\ ,$$

which shows that $u\circ K'\circ[A_{K'}\cup A_K]$ is an MHP generator and so that $\mathfrak{a}_K=u\circ K'\circ[A_{K'}\cup A_K]$. ☐

Let $K\in\mathcal{K}^*$. For every ordinal α define the sets $\overset{\alpha}{\mathfrak{a}}_K$ and A_K^α inductively as follows (note the difference between this definition and the one just before 4.2.):

$\overset{0}{\mathfrak{a}}_K:=\mathfrak{a}_K$ and $A_K^0:=u\mathfrak{a}_K$.

If $\overset{\beta}{\mathfrak{a}}_K$ and A_K^β are defined, then we set $L:=\overset{\beta}{\mathfrak{a}}_K\cap J$; in 4.3. below we show that $L\in\mathcal{K}^*$. Define

$\overset{\beta+1}{\mathfrak{a}}_K:=\mathfrak{a}_L$, the smallest MHP generator that contains $u\circ L$ ($\mathfrak{a}_L$ exists by 2.1., 1.5.c and the almost periodicity of $u\circ L$); and

$A_K^{\beta+1}:=A_L$, the smallest $\mathfrak{F}(\mathcal{M},u)$-closed subgroup of G that contains $u(u\circ L)$.

If γ is a limit ordinal and if $\overset{\beta}{\mathfrak{a}}_K$ and A_K^β are defined for all $\beta<\gamma$, then define

$\overset{\gamma}{\mathfrak{a}}_K:=u\circ\bigcap\{\overset{\beta}{\mathfrak{a}}_K\mid\beta<\gamma\}$ and $A_K^\gamma:=\bigcap\{A_K^\beta\mid\beta<\gamma\}$.

4.3. **THEOREM.** *Let* $K\in\mathcal{K}^*$. *Then*

a) $A_K^0=[A_K^1\cup A_K]$, *the* $\mathfrak{F}(\mathcal{M},u)$*-closed subgroup of* G *generated by* $A_K^1\cup A_K$ *;*

b) *for every* $\alpha\geqslant 0$ *we have* $\overset{\alpha}{\mathfrak{a}}_K\cap J\in\mathcal{K}^*$ *;*

c) *for every ordinal* α *we have* $A_K^\alpha=u\overset{\alpha}{\mathfrak{a}}_K$ *;*

d) *for some nonlimit ordinal* ν , $\overset{\nu+1}{\mathfrak{a}}_K=\overset{\nu}{\mathfrak{a}}_K$ *and* $A_K^{\nu+1}=A_K^\nu$. (notation: $\overset{\infty}{\mathfrak{a}}_K:=\overset{\nu}{\mathfrak{a}}_K$ and $A_K^\infty:=A_K^\nu$.)

PROOF.

a) This is just 4.2.c, since it is clear that $A_K^1=A_{K'}$.

b) We shall prove this by transfinite induction.

For $\alpha=0$ the statement is proven in 4.2.a.

Suppose the statement is true for every ordinal $\beta\leqslant\alpha$. Let $L:=\overset{\alpha}{\mathfrak{a}}_K\cap J$;

then, by assumption, $L \in \mathcal{K}^*$ and by definition, $a_K^{\alpha+1} = a_L$. Set $L' := a_L \cap J$; then by 4.2.a, $L' \in \mathcal{K}^*$, so we have $a_K^{\alpha+1} \cap J \in \mathcal{K}^*$.
Let α be a limit ordinal and suppose the statement is true for every ordinal $\beta < \alpha$. Then $u \circ (a_K^{\beta} \cap J) \circ G = M$ and so $a_K^{\beta+1} \circ G = M$ for every $\beta < \alpha$. As $\{a_K^{\beta} \mid \beta < \alpha\}$ is a collection of closed sets in M , linearly ordered by inclusion, it follows that

$$\bigcap \{a_K^{\beta} \mid \beta < \alpha\} = \lim\nolimits_{2^M} \{a_K^{\beta} \mid \beta < \alpha\} .$$

By 1.1.c, we have

$$(\bigcap \{a_K^{\beta} \mid \beta < \alpha\}) \circ G = \lim\nolimits_{2^M} \{a_K^{\beta} \circ G \mid \beta < \alpha\} = M ,$$

so

$$a_K^{\alpha} \circ G = u \circ \bigcap \{a_K^{\beta} \mid \beta < \alpha\} \circ G = u \circ M = M .$$

Since, by 1.5.c and 1.5.b, a_K^{α} is an MHP generator it follows from 1.7. that $a_K^{\alpha} = (a_K^{\alpha} \cap J) \circ u a_K^{\alpha}$, which implies that

$$u \circ (a_K^{\alpha} \cap J) \circ G = u \circ (a_K^{\alpha} \cap J) \circ u a_K^{\alpha} \circ G = u \circ a_K^{\alpha} \circ G = a_K^{\alpha} \circ G = M .$$

Consequently, it follows that $a_K^{\alpha} \cap J \in \mathcal{K}^*$; so b is proven.

c) If $\alpha = 0$, then $A_K^0 = u a_K^0$ by definition.
Let α be an ordinal, then $a_K^{\alpha} = L.A_K^{\alpha}$ is an MHP generator, where $L := a_K^{\alpha} \cap J$. So, as in 4.2.b, it follows that $a_L = u \circ L \circ A_L$ and so that $A_L = u a_L$, hence $A_K^{\alpha+1} = u a_K^{\alpha+1}$.
If α is a limit ordinal, then it is an easy exercise to show that $A_K^{\alpha} = u a_K^{\alpha}$.

d) Note that the family $\{u \circ (a_K^{\alpha} \cap J) \mid \alpha \geqslant 1\}$ is linearly ordered by inclusion. As $u \circ (a_K^{\alpha} \cap J) \subseteq u \circ J$, there can be at most $|u \circ J|$ different elements in the family $\{u \circ (a_K^{\alpha} \cap J) \mid \alpha \geqslant 1\}$. But this means that $u \circ (a_K^{\alpha} \cap J) = u \circ (a_K^{\alpha+1} \cap J)$ for some ordinal α , hence $a_K^{\alpha+1} = a_K^{\alpha+2}$ and $A_K^{\alpha+1} = A_K^{\alpha+2}$. By construction, it follows that $a_K^{\beta} = a_K^{\alpha+1}$ and $A_K^{\beta} = A_K^{\alpha+1}$ for every $\beta \geqslant \alpha + 1$. □

In 4.3.d, we have seen that for every $K \in \mathcal{K}^*$ we can construct a kind of *minimal incontractible* MHP *generator* a_K^{∞} . Let $K^{\infty} = a_K^{\infty} \cap J$. Then a_K^{∞} is the MHP generator generated by the set $u \circ K^{\infty}$ and, clearly, $a_{K^{\infty}}^{\alpha} = a_{K^{\infty}} = a_K^{\infty}$ for every ordinal α ; so in this respect a_K^{∞} is minimal.

Let

$$\mathcal{K} := \{K \in \mathcal{K}^* \mid \mathfrak{a}_K = \mathfrak{a}_K^{\infty}\}$$

be the family of subsets of J that generate the minimal incontractible MHP generators.

4.4. **THEOREM.** *Let* $\mathcal{X}$ *be a minimal ttg. Then* $\mathcal{X}$ *is incontractible iff* $\mathcal{X}$ *is a factor of* $\mathcal{A}_K := \mathcal{W}\mathcal{F}(\mathfrak{a}_K, \mathcal{M})$ *for some* $K \in \mathcal{K}$.

PROOF. As discussed before 4.2., $\mathcal{A}_K$ is incontractible and so every factor of $\mathcal{A}_K$ is incontractible for every $K \in \mathcal{K}$.
Conversely, let $\mathcal{X}$ be incontractible. By IV.4.17.c, $\mathcal{X}^*$ is incontractible. Let $C = u \circ C$ be an MHP generator such that $\mathcal{X}^* \cong \mathcal{C}$. As $\mathcal{C}$ is incontractible, it follows that $C \circ G = M$. Let $K = C \cap J$; then, by 1.7., we have $C = u \circ K \circ uC$, hence

$$M = C \circ G = u \circ K \circ uC \circ G = u \circ K \circ uCG = u \circ K \circ G \quad \text{and} \quad K \in \mathcal{K}^* .$$

Construct the minimal MHP generator $\mathfrak{a}_K^{\infty}$ and let $L := \mathfrak{a}_K^{\infty} \cap J$. Then, clearly, $L \in \mathcal{K}$ and $\mathfrak{a}_L \subseteq \mathfrak{a}_K \subseteq C$. So, by 2.4.b, $\mathcal{C}$ is a factor of $\mathcal{A}_L$. □

We shall now discuss the construction in the special situation of $K = J$. Note that $J \in \mathcal{K}^*$, for $M = u \circ M = u \circ JG \subseteq u \circ J \circ G \subseteq M$. After a short discussion we shall formulate a lemma and a theorem for this situation, but those statements can easily be reformulated for the general case of $K \in \mathcal{K}^*$. This is a kind of lazyness intended to serve the clarity of the story.

Let α be an ordinal, then we denote $\mathfrak{a}_J^{\alpha}$ and A_J^{α} by $\mathfrak{a}_{\alpha}$ and A_{α}. So $\mathfrak{a}_0$ is the smallest MHP generator that contains $u \circ J$ and $A_0 = u\mathfrak{a}_0$. The sets $\mathfrak{a}_J^{\infty}$ and A_J^{∞} will be denoted by $\mathfrak{a}$ and A respectively. Note that in this case $A_0 = A_J^0$ equals A_J, the smallest $\mathfrak{F}(\mathcal{M}, u)$-closed subgroup of G that contains $u(u \circ J)$, which is clear from the observation that $\mathfrak{a}_0 = \mathfrak{a}_J = u \circ J \circ A_J$.
(By 1.10., $u \circ J \circ A_J$ is an idempotent set in $(2^M, \circ)$. As $u \circ J \cup A_J \subseteq \mathfrak{a}_J$, $u \circ J \circ A_J \subseteq \mathfrak{a}_J \circ \mathfrak{a}_J = \mathfrak{a}_J$. So by minimality of $\mathfrak{a}_J$, $u \circ J \circ A_J = \mathfrak{a}_J$. Clearly, $u\mathfrak{a}_J = A_J.A_J = A_J$.)

Define $\mathfrak{a}_{-1} := M$ and $A_{-1} := G$. Then $\mathfrak{a}_{-1}$ and A_{-1} behave in accordance with the construction. For $J = M \cap J = \mathfrak{a}_{-1} \cap J$ and so $\mathfrak{a}_0$ is the smallest MHP generator that contains $u \circ (\mathfrak{a}_{-1} \cap J) = u \circ J$; moreover,

$A_{-1} = u\mathcal{A}_{-1} = uM = G$, and A_0 is the smallest $\mathfrak{F}(\mathcal{M},u)$-closed subgroup of G that contains $u(u \circ J)$.
(In the sequel we consider -1 to be an ordinal preceding 0 .)

As in the preceding sections we shall denote the pointed ttgs $(\mathfrak{QF}(\mathcal{A}_\alpha,\mathcal{M}),\mathcal{A}_\alpha)$ by $\mathfrak{A}_\alpha$, so the map $\mathfrak{A}_\alpha \to \mathfrak{A}_\beta$ will be the canonical homomorphism from $\mathfrak{QF}(\mathcal{A}_\alpha,\mathcal{M})$ to $\mathfrak{QF}(\mathcal{A}_\beta,\mathcal{M})$ that carries $\mathcal{A}_\alpha$ over in $\mathcal{A}_\beta$ ($\alpha \geqslant \beta$). Note that $\mathfrak{A}_{-1}$ is the trivial ambit $(\{\star\},\star)$.

4.5. **LEMMA.**

a) *For all* $\alpha \geqslant -1$ *the map* $\mathfrak{A}_{\alpha+1} \to \mathfrak{A}_\alpha$ *is a* RIC *extension.*

b) *For every* $\alpha \geqslant \beta$ *the map* $\mathfrak{A}_\alpha \to \mathfrak{A}_\beta$ *is a* RIC *extension, hence* $\mathcal{A}_\alpha \circ A_\beta = \mathcal{A}_\beta$. *In particular,* $\mathcal{A} \circ A_\beta = \mathcal{A}_\beta$ *for every* $\beta \geqslant -1$.

PROOF.

a) By 3.4., we have to prove that $\mathcal{A}_{\alpha+1} \circ A_\alpha = \mathcal{A}_\alpha$.
As, by 1.7., $\mathcal{A}_\alpha = (\mathcal{A}_\alpha \cap J) \circ A_\alpha$, it follows from $u \circ (\mathcal{A}_\alpha \cap J) \subseteq \mathcal{A}_{\alpha+1}$ that

$$\mathcal{A}_\alpha = u \circ \mathcal{A}_\alpha = u \circ (\mathcal{A}_\alpha \cap J) \circ A_\alpha \subseteq \mathcal{A}_{\alpha+1} \circ A_\alpha \subseteq \mathcal{A}_\alpha \circ \mathcal{A}_\alpha = \mathcal{A}_\alpha \text{ ;}$$

so, indeed, $\mathcal{A}_{\alpha+1} \circ A_\alpha = \mathcal{A}_\alpha$ and $\mathfrak{A}_{\alpha+1} \to \mathfrak{A}_\alpha$ is a RIC extension.

b) As the composition as well as the inverse limit of RIC extensions is again a RIC extension (III.1.10.), it follows from a that $\mathfrak{A}_\alpha \to \mathfrak{A}_\beta$ is a RIC extension ($\alpha \geqslant \beta$). From 3.4., it follows that $\mathcal{A}_\alpha \circ A_\beta = \mathcal{A}_\beta$ if $\beta \leqslant \alpha$. So, in particular, $\mathcal{A} \circ A_\beta = \mathcal{A}_\beta$ for every ordinal β . □

4.6. **THEOREM.** *For every ordinal* $\alpha \geqslant -1$, $\mathcal{A} \circ \mathrm{H}(A_\alpha)$ *is an* MHP *generator and the* MHP *extension of the maximal almost periodic extension of* $\mathfrak{A}_\alpha$ *is* $\mathfrak{QF}(\mathcal{A} \circ \mathrm{H}(A_\alpha),\mathcal{M})$ *; and for every ordinal* $\beta \geqslant \alpha$ *the following equations hold:*

$$\mathcal{A}_\beta \circ \mathrm{H}(A_\alpha) = \mathcal{A} \circ A_\beta \mathrm{H}(A_\alpha) = \mathcal{A} \circ A_{\alpha+1}\mathrm{H}(A_\alpha) =$$

$$= \mathcal{A}_{\alpha+1} \circ \mathrm{H}(A_\alpha) = \mathcal{A} \circ \mathrm{H}(A_\alpha) .$$

In particular, $\mathcal{E}^* = \mathfrak{QF}(\mathcal{A} \circ \mathrm{H}(G),\mathcal{M}) = \mathfrak{QF}(\mathcal{A}_\alpha \circ \mathrm{H}(G),\mathcal{M})$ *for every* $\alpha \geqslant 0$.

PROOF. Let $(\mathcal{X},x)$ be the ambit with $x = ux$, such that $\theta : (\mathcal{X},x) \to \mathfrak{A}_\alpha$ is the maximal almost periodic extension of $\mathfrak{A}_\alpha$. Then $M_x = J_x . \mathfrak{G}(\mathcal{X},x)$ and $\mathcal{X}^* = \mathfrak{QF}(C,\mathcal{M})$, where $C = u \circ M_x$. As θ is an almost periodic map,

x is a θ-distal point. Hence

$$J_x = J_{\mathcal{A}_\alpha} = \{v \in J \mid v \circ \mathcal{A}_\alpha = \mathcal{A}_\alpha\} = \mathcal{A}_\alpha \cap J ,$$

so $u \circ (\mathcal{A}_\alpha \cap J) = u \circ J_x \subseteq u \circ M_x = C$, which shows that $\mathcal{A}_{\alpha+1} \subseteq C$. Hence $(\mathscr{X}, x)$ is a factor of $\mathfrak{A}_{\alpha+1}$; moreover, $(\mathscr{X}, x)$ is a factor of $\mathfrak{A}_\beta$ for every ordinal β with $\beta \geqslant \alpha+1$.
Consider the next diagram with $\beta \geqslant \alpha+1$.

$$\begin{tikzcd}
\mathfrak{A}_\beta \arrow[r, "\phi"] \arrow[d] \arrow[dr] & \mathfrak{A}_\alpha \\
(\mathscr{C}, C) \arrow[r, "\chi_{\mathscr{X}}"'] & (\mathscr{X}, x) \arrow[u, "\theta"']
\end{tikzcd}$$

Note that $\phi : \mathfrak{A}_\beta \to \mathfrak{A}_\alpha$ is a RIC extension (4.5.a) and that A_β and A_α are the Ellis groups of the ambits $\mathfrak{A}_\beta$ and $\mathfrak{A}_\alpha$. By III.3.13., it follows that $\mathfrak{S}(\mathscr{X}, x) = A_\beta \mathrm{H}(A_\alpha)$. As this is true for every $\beta \geqslant \alpha+1$, it follows that $A_\beta \mathrm{H}(A_\alpha) = A\, \mathrm{H}(A_\alpha)$ for every $\beta \geqslant \alpha+1$.
We may now conclude that $C = \mathcal{A}_{\alpha+1} \circ \mathrm{H}(A_\alpha)$. For

$$M_x = (\mathcal{A}_\alpha \cap J) A_{\alpha+1} \mathrm{H}(A_\alpha) ,$$

and so

$$C = u \circ M_x \subseteq u \circ (\mathcal{A}_\alpha \cap J) \circ A_{\alpha+1} \circ \mathrm{H}(A_\alpha) \subseteq \mathcal{A}_{\alpha+1} \circ \mathrm{H}(A_\alpha) .$$

But, on the other hand, $\mathcal{A}_{\alpha+1} \subseteq C$ and $u \circ \mathrm{H}(A_\alpha) \subseteq C$, so

$$C = u \circ M_x \subseteq \mathcal{A}_{\alpha+1} \circ \mathrm{H}(A_\alpha) \subseteq C \circ C = C .$$

By 4.5.b, we know that $\mathcal{A}_{\alpha+1} = \mathcal{A} \circ A_{\alpha+1}$. Hence (using 4.1.) it follows that

$$C = \mathcal{A}_{\alpha+1} \circ \mathrm{H}(A_\alpha) = \mathcal{A} \circ A_{\alpha+1} \circ \mathrm{H}(A_\alpha) = \mathcal{A} \circ A_{\alpha+1} \mathrm{H}(A_\alpha) .$$

By the above, $A_{\alpha+1} \mathrm{H}(A_\alpha) = A\, \mathrm{H}(A_\alpha) = A_\beta \mathrm{H}(A_\alpha)$ for every ordinal $\beta \geqslant \alpha+1$, hence

$$C = \mathcal{A} \circ A_{\alpha+1} \mathrm{H}(A_\alpha) = \mathcal{A} \circ A\, \mathrm{H}(A_\alpha) = \mathcal{A} \circ A_\beta \mathrm{H}(A_\alpha) \quad (\beta \geqslant \alpha+1) .$$

But this shows, by 4.5.b, that $C = \mathcal{A} \circ \mathrm{H}(A_\alpha) = \mathcal{A}_\beta \circ \mathrm{H}(A_\alpha)$ for every $\beta \geqslant \alpha+1$. Hence $\mathscr{X}^* \cong \mathfrak{W}\mathfrak{F}(\mathcal{A} \circ \mathrm{H}(A_\alpha), \mathfrak{M})$, $\mathcal{A} \circ \mathrm{H}(A_\alpha)$ is an MHP generator and the equations in the theorem hold.

In particular, this holds for $\alpha = -1$, and as $\mathfrak{A}_{-1}$ is the trivial ambit it follows that the maximal almost periodic extension of $\mathfrak{A}_{-1}$ is just $\mathfrak{G}$; hence

$$\mathfrak{G}^* = \mathfrak{WF}(\mathcal{A} \circ \mathrm{H}(A_{-1}), \mathfrak{M}) = \mathfrak{WF}(\mathcal{A} \circ \mathrm{H}(G), \mathfrak{M}) .$$

□

4.7. **THEOREM.** *For every ordinal* $\alpha \geqslant -1$, *the maximal* HPI *extension of* $\mathfrak{A}_\alpha$ *between* $\mathfrak{A}_{\alpha+1}$ *and* $\mathfrak{A}_\alpha$ *is*

$$(\mathfrak{WF}(\mathcal{A}_{\alpha+1} \circ (A_\alpha)_\infty, \mathfrak{M}), \mathcal{A}_{\alpha+1} \circ (A_\alpha)_\infty)$$

and $\mathcal{A}_{\alpha+1} \circ (A_\alpha)_\infty = \mathcal{A} \circ A_{\alpha+1}(A_\alpha)_\infty$.

As a result in between, we have that $\mathcal{A}_{\alpha+1} \circ \mathrm{H}_\beta(A_\alpha)$ *is an* MHP *generator for every ordinal* $\beta \geqslant 1$.

In particular, $\mathcal{A}_0 \circ G_\infty$ *is an* MHP *generator,* $\mathcal{A}_0 \circ G_\infty = \mathcal{A} \circ A_0 G_\infty$ *and* $\mathfrak{WF}(\mathcal{A}_0 \circ G_\infty, \mathfrak{M})$ *is an* HPI *ttg.*

PROOF. First we prove the following claim:

CLAIM:

a) Let F be an $\mathfrak{F}(\mathfrak{M}, u)$-closed subgroup of G such that $A_{\alpha+1} \subseteq F \subseteq A_\alpha$. Then $\mathcal{A}_{\alpha+1} \circ F$ is an MHP generator.

b) Let C be an MHP generator with $\mathcal{A}_{\alpha+1} \subseteq C \subseteq \mathcal{A}_\alpha$. Then $C = \mathcal{A}_{\alpha+1} \circ uC$; and, consequently, the map $\mathfrak{A}_{\alpha+1} \to \mathfrak{C}$ is a RIC extension.

PROOF (CLAIM):

a) By 1.6.b and the assumption, we have $u(\mathcal{A}_{\alpha+1} \circ F \circ \mathcal{A}_{\alpha+1} \circ F) \subseteq F$; and as $\mathcal{A}_{\alpha+1} \circ F \circ \mathcal{A}_{\alpha+1} \circ F \subseteq \mathcal{A}_\alpha$ it follows that

$$\mathcal{A}_{\alpha+1} \circ F \circ \mathcal{A}_{\alpha+1} \circ F \subseteq (\mathcal{A}_\alpha \cap J).F \subseteq (\mathcal{A}_\alpha \cap J) \circ F .$$

So, as $u \circ (\mathcal{A}_\alpha \cap J) \subseteq \mathcal{A}_{\alpha+1}$,

$$\mathcal{A}_{\alpha+1} \circ F \circ \mathcal{A}_{\alpha+1} \circ F = u \circ \mathcal{A}_{\alpha+1} \circ F \circ \mathcal{A}_{\alpha+1} \circ F \subseteq$$

$$\subseteq u \circ (\mathcal{A}_\alpha \cap J) \circ F \subseteq \mathcal{A}_{\alpha+1} \circ F$$

and $\mathcal{A}_{\alpha+1} \circ F$ turns out to be an MHP generator.

b) Clearly, $\mathcal{A}_{\alpha+1} \circ uC \subseteq C \circ C = C$.

By 1.7., we know that $C = u \circ (C \cap J) \circ uC$. As $C \cap J \subseteq \mathcal{A}_\alpha \cap J$ and $u \circ (\mathcal{A}_\alpha \cap J) \subseteq \mathcal{A}_{\alpha+1}$ we have

$$C = u\circ(C\cap J)\circ uC \subseteq u\circ(\mathcal{A}_\alpha\cap J)\circ uC \subseteq \mathcal{A}_{\alpha+1}\circ uC\ .$$

So $C = \mathcal{A}_{\alpha+1}\circ uC$; and by 3.4., the map $\mathfrak{A}_{\alpha+1}\to\mathfrak{C}$ is RIC. ☐ (CLAIM)

For every ordinal β, $\mathrm{H}_\beta(A_\alpha)$ is a normal subgroup of A_α ; so $A_{\alpha+1}\mathrm{H}_\beta(A_\alpha)$ is an $\mathfrak{T}(\mathfrak{M},u)$-closed subgroup of G between $A_{\alpha+1}$ and A_α. By 4.1. and claim a, $\mathcal{A}_{\alpha+1}\circ\mathrm{H}_\beta(A_\alpha)\ (=\mathcal{A}_{\alpha+1}\circ A_{\alpha+1}\mathrm{H}_\beta(A_\alpha))$ is an MHP generator.
In particular, $\mathcal{A}_{\alpha+1}\circ(A_\alpha)_\infty$ is an MHP generator (for example $\mathcal{A}_0\circ G_\infty$ $(\alpha=-1)$). Let

$$(\mathfrak{Z},z) = (\mathfrak{WF}(\mathcal{A}_{\alpha+1}\circ(A_\alpha)_\infty,\mathfrak{M}),\mathcal{A}_{\alpha+1}\circ(A_\alpha)_\infty)\ ;$$

then $\mathfrak{G}(\mathfrak{Z},z) = A_{\alpha+1}(A_\alpha)_\infty$.
By III.4.4.c, the map $\phi:(\mathfrak{Z},z)\to\mathfrak{A}_\alpha$ is a PI extension. We shall prove that every open map $\psi:(\mathfrak{Z},z)\to(\mathfrak{X},x)$ for which $\phi=\theta\circ\psi$, is a RIC extension. By IV.5.7., it then follows that ϕ is an HPI extension.
As such a ψ is open, $\mathfrak{X}$ is an MHP ttg (IV.3.9.). So there is an MHP generator C with $(\mathfrak{X},x)\cong\mathfrak{C}$ and $\mathcal{A}_{\alpha+1}\subseteq C\subseteq\mathcal{A}_\alpha$. By claim b, the map $\xi:\mathfrak{A}_{\alpha+1}\to\mathfrak{C}$ is a RIC extension; hence ψ as a factor of ξ is a RIC extension.
It is an easy exercise to show that ϕ is the maximal PI extension of $\mathfrak{A}_\alpha$ between $\mathfrak{A}_{\alpha+1}$ and $\mathfrak{A}_\alpha$; so, certainly, ϕ is the maximal HPI extension of $\mathfrak{A}_\alpha$ under $\mathfrak{A}_{\alpha+1}$. ☐

At the moment we know that $\mathfrak{WF}(\mathcal{A}_0\circ G_\infty,\mathfrak{M})$ is an HPI ttg. However, in a special situation we may conclude that $\mathfrak{WF}(\mathcal{A}_0\circ G_\infty,\mathfrak{M})$ is the universal minimal HPI ttg for T ; as follows:

4.8. **LEMMA.** *If* $\mathfrak{X}$ *is a metric minimal* HPI *ttg then* $\mathfrak{X}$ *is a factor of* $\mathfrak{WF}(\mathcal{A}_0\circ G_\infty,\mathfrak{M})$.

PROOF. By IV.5.13., we know that a metric minimal HPI ttg is point distal. So let $x\in X$ be a distal point. Then $J_x=J$, hence $u\circ J\subseteq u\circ M_x$. As $u\circ M_x$ is an MHP generator it follows that $\mathcal{A}_0\subseteq u\circ M_x$. As $\mathfrak{X}$ is a PI ttg it follows from III.4.4.c that $G_\infty\subseteq\mathfrak{G}(\mathfrak{X},x)$. Hence $G_\infty\subseteq u\circ M_x$ and

$$\mathcal{A}_0\circ G_\infty\subseteq u\circ M_x\circ u\circ M_x = u\circ M_x\ .$$

By 2.4.b, $\mathfrak{X}^*$ $(=\mathfrak{WF}(u\circ M_x,\mathfrak{M}))$ is a factor of $\mathfrak{WF}(\mathcal{A}_0\circ G_\infty,\mathfrak{M})$, so $\mathfrak{X}$ is a factor of $\mathfrak{WF}(\mathcal{A}_0\circ G_\infty,\mathfrak{M})$. ☐

4.9. **THEOREM.**

a) *If* $\mathfrak{X}$ *is a strictly-quasi separable minimal* HPI *ttg then* $\mathfrak{X}$ *is a factor of* $\mathfrak{WF}(\mathcal{A}_0 \circ G_\infty, \mathfrak{M})$.

b) *If* T *is a locally compact, σ-compact topological group then* $\mathfrak{WF}(\mathcal{A}_0 \circ G_\infty, \mathfrak{M})$ *is the universal minimal* HPI *ttg for* T.

PROOF.

a) If $\mathfrak{X}$ is strictly-quasi separable then $\mathfrak{X}$ is the inverse limit of metric minimal ttgs; say $\mathfrak{X} = \operatorname{inv\,lim} \mathfrak{X}_\alpha$, where $\mathfrak{X}_\alpha$ is a minimal metric ttg. As $\mathfrak{X}_\alpha$ is a factor of $\mathfrak{X}$ for every α it follows from IV.5.9. that every $\mathfrak{X}_\alpha$ is an HPI ttg. So by 4.8., every $\mathfrak{X}_\alpha$ is a factor of $\mathfrak{WF}(\mathcal{A}_0 \circ G_\infty, \mathfrak{M})$. But then $\mathfrak{X}$ is a factor of $\mathfrak{WF}(\mathcal{A}_0 \circ G_\infty, \mathfrak{M})$.

b) If T is locally compact, σ-compact, we know from I.1.7. that every minimal ttg is strictly-quasi separable. Hence every minimal HPI ttg for T is a factor of $\mathfrak{WF}(\mathcal{A}_0 \circ G_\infty, \mathfrak{M})$. As by 4.7., $\mathfrak{WF}(\mathcal{A}_0 \circ G_\infty, \mathfrak{M})$ is an HPI ttg itself, it follows that $\mathfrak{WF}(\mathcal{A}_0 \circ G_\infty, \mathfrak{M})$ is the universal minimal HPI ttg for T. ☐

Among others, the following remark is made in order to facilitate things to be done in chapter VI..

4.10. **REMARK.** *For every* $K \in \mathcal{K}^*$ *we have*

a) $\mathcal{A}_K \subseteq \mathcal{A}_0$ *and* $A_K \subseteq A_0$*;*

b) $\overset{\infty}{\mathcal{A}_K} \subseteq \mathcal{A}$ *and* $A_K^\infty \subseteq A$ *, hence for every* $K \in \mathcal{K}$ *it follows that* $\mathcal{A}_K \subseteq \mathcal{A}$ *and* $A_K \subseteq A$ *;*

c) $A\,\mathrm{H}(G) = A_0\mathrm{H}(G) = A_K\mathrm{H}(G)$ *is the Ellis group of* $\mathfrak{G}$ *and so it is a normal subgroup of* G *.*

PROOF.

a) As $K \subseteq J$; $u \circ K \subseteq u \circ J$, so, clearly, $\mathcal{A}_K \subseteq \mathcal{A}_0$ and

$$A_K \subseteq u\mathcal{A}_K \subseteq u\mathcal{A}_0 = A_0 .$$

b) Since $\mathcal{A}_K \subseteq \mathcal{A}_0$ it follows that for every ordinal $\alpha \geqslant 0$ we have $\overset{\alpha}{\mathcal{A}_K} \subseteq \mathcal{A}_\alpha$. Hence $\overset{\infty}{\mathcal{A}_K} \subseteq \mathcal{A}_\infty = \mathcal{A}$ and $A_K^\infty = u\overset{\infty}{\mathcal{A}_K} \subseteq u\mathcal{A} = A$ (4.3.c). If $K \in \mathcal{K}$ then $\overset{\infty}{\mathcal{A}_K} = \mathcal{A}_K$, so $\mathcal{A}_K \subseteq \mathcal{A}$ and $A_K \subseteq A$.

c) As $\mathfrak{G}$ is a factor of $\mathfrak{A}_0$, it is a factor of $\mathfrak{A}_K$ for every $K \in \mathcal{K}^*$ (by a and 2.4.b). So $\mathfrak{G}$ is the maximal almost periodic extension of $\{\star\}$ under $\mathfrak{A}_K$ for every $K \in \mathcal{K}^*$. As $\mathfrak{A}_K$ is an incontractible ttg it follows from III.3.11. that the Ellis group of $\mathfrak{G}$ equals $A_K\mathrm{H}(G)$ for every $K \in \mathcal{K}^*$.

This shows that $A_K \mathrm{H}(G) = A_0 \mathrm{H}(G)$ for every $K \in \mathcal{K}^*$. In particular, for $L = \mathcal{A} \cap J$ we have $A\,\mathrm{H}(G) = A_L \mathrm{H}(G) = A_0 \mathrm{H}(G)$ ($L \in \mathcal{K}$!). As $\mathcal{E}$ is a regular ttg (I.2.17.), it follows from I.2.15. that $A\,\mathrm{H}(G)$ is a normal subgroup of G . □

In 4.9. we have seen that $\mathfrak{W}\mathfrak{F}(\mathcal{A}_0 \circ G_\infty, \mathfrak{M})$ is the universal minimal HPI ttg in case T is locally compact, σ-compact. It is unlikely that this is true without the restriction on the phase group. But we can construct the universal minimal HPI ttg in general, in a way similar to the construction of the $\mathcal{A}_\alpha$'s.

Define

$\mathcal{C}_0 := \mathcal{A}_0$ and $C_0 = u\mathcal{C}_0 = A_0$.

Let α be an ordinal and suppose that $\mathcal{C}_\alpha$ and C_α are defined. Then define

$\mathcal{C}_{\alpha+1}$ to be the smallest MHP generator that contains the set $u \circ ((\mathcal{C}_\alpha \circ G_\infty) \cap J)$ and let $C_{\alpha+1} := u\mathcal{C}_{\alpha+1}$.

If β is a limit ordinal and if $\mathcal{C}_\alpha$ and C_α are defined for all $\alpha < \beta$, then define

$\mathcal{C}_\beta := u \circ \bigcap \{\mathcal{C}_\alpha \mid \alpha < \beta\}$ and $C_\beta = u\mathcal{C}_\beta$.

As the collection $\{\mathcal{C}_\alpha \mid \alpha \geqslant 0\}$ is a descending family of subsets of M , there is an ordinal ν such that $\mathcal{C}_\nu = \mathcal{C}_{\nu+1} = \mathcal{C}_\gamma$ for every $\gamma \geqslant \nu$. We shall denote this "smallest" $\mathcal{C}_\nu$ by $\mathcal{C}$ and C_ν by C .

4.11. **REMARK.** *For every ordinal* $\alpha \geqslant 0$ *we have*

a) $\mathcal{A}_\alpha \subseteq \mathcal{C}_\alpha$ *and* $A_\alpha \subseteq C_\alpha$ *; in particular,* $\mathcal{A} \subseteq \mathcal{C}$ *and* $A \subseteq C$ *;*

b) $C_\alpha \mathrm{H}(G) = A_\alpha \mathrm{H}(G) = A_0 \mathrm{H}(G) = C_0 \mathrm{H}(G) = C\,\mathrm{H}(G) = A\,\mathrm{H}(G)$ *;*

c) $\mathcal{C}_\alpha \circ G_\infty$ *is an MHP generator and* $u(\mathcal{C}_\alpha \circ G_\infty) = C_\alpha G_\infty$. *In particular,* $\mathcal{C} \circ G_\infty$ *is an MHP generator and* CG_∞ *is the Ellis group of* $\mathfrak{W}\mathfrak{F}(\mathcal{C} \circ G_\infty, \mathfrak{M})$ *with respect to* $\mathcal{C} \circ G_\infty$.

PROOF.

a) Obvious.

b) For every ordinal $\alpha \geqslant 0$ we have $A_\alpha \subseteq C_\alpha \subseteq C_0 = A_0$, so

$$A_\alpha \mathrm{H}(G) \subseteq C_\alpha \mathrm{H}(G) = C_0 \mathrm{H}(G) = A_0 \mathrm{H}(G)$$

and, by 4.10., it follows that

$$A\,\mathrm{H}(G) = A_\alpha \mathrm{H}(G) = C_\alpha \mathrm{H}(G) = C_0 \mathrm{H}(G) = A_0 \mathrm{H}(G) \, ,$$

while

$$\mathcal{A}\,\mathrm{H}(G) \subseteq \mathcal{C}\,\mathrm{H}(G) \subseteq \mathcal{C}_0\mathrm{H}(G) = \mathcal{A}\,\mathrm{H}(G)\,.$$

c) We shall prove this by transfinite induction.
As $\mathcal{C}_0 = \mathcal{A}_0$ and so $\mathcal{C}_0 \circ G_\infty = \mathcal{A}_0 \circ G_\infty$, it follows by 4.7. that $\mathcal{C}_0 \circ G_\infty$ is an MHP generator.
Let α be an ordinal and suppose that $\mathcal{C}_\alpha \circ G_\infty$ is an MHP generator. Then, by 1.7., $\mathcal{C}_\alpha \circ G_\infty = L.\mathcal{C}_\alpha G_\infty$ for $L = (\mathcal{C}_\alpha \circ G_\infty) \cap J$. So

$$\mathcal{C}_{\alpha+1} \circ G_\infty \circ \mathcal{C}_{\alpha+1} \circ G_\infty \subseteq \mathcal{C}_\alpha \circ G_\infty \circ \mathcal{C}_\alpha \circ G_\infty = \mathcal{C}_\alpha \circ G_\infty \subseteq L.G \ ,$$

and, by 1.6.b and by the normality of G_∞ ,

$$u(\mathcal{C}_{\alpha+1} \circ G_\infty \circ \mathcal{C}_{\alpha+1} \circ G_\infty) = \mathcal{C}_{\alpha+1} G_\infty \mathcal{C}_{\alpha+1} G_\infty = \mathcal{C}_{\alpha+1} G_\infty \,.$$

So it follows that

$$\mathcal{C}_{\alpha+1} \circ G_\infty \circ \mathcal{C}_{\alpha+1} \circ G_\infty \subseteq L.G \cap J.\mathcal{C}_{\alpha+1} G_\infty = L.\mathcal{C}_{\alpha+1} G_\infty \,.$$

Hence

$$\mathcal{C}_{\alpha+1} \circ G_\infty \circ \mathcal{C}_{\alpha+1} \circ G_\infty = u \circ \mathcal{C}_{\alpha+1} \circ G_\infty \circ \mathcal{C}_{\alpha+1} \circ G_\infty \subseteq u \circ L \circ \mathcal{C}_{\alpha+1} \circ G_\infty \ ,$$

and as $u \circ L \cup \mathcal{C}_{\alpha+1} \subseteq \mathcal{C}_{\alpha+1}$, it follows that

$$u \circ L \circ \mathcal{C}_{\alpha+1} \circ G_\infty \subseteq \mathcal{C}_{\alpha+1} \circ \mathcal{C}_{\alpha+1} \circ G_\infty = \mathcal{C}_{\alpha+1} \circ G_\infty \,.$$

This implies that $\mathcal{C}_{\alpha+1} \circ G_\infty$ is an idempotent subset of M , hence an MHP generator.
Let α be a limit ordinal and suppose that $\mathcal{C}_\beta \circ G_\infty$ is an MHP generator for every $\beta < \alpha$. Then by 1.5.c, 1.5.b and 2.1.,

$$D := u \circ \bigcap \{\mathcal{C}_\beta \circ G_\infty \mid \beta < \alpha\}$$

is an MHP generator. By 1.1.c, "right circling" with $u \circ G_\infty$ is continuous, so

$$\bigcap \{\mathcal{C}_\beta \circ G_\infty \mid \beta < \alpha\} = (\bigcap \{\mathcal{C}_\beta \mid \beta < \alpha\}) \circ G_\infty \ ,$$

hence

$$D = u \circ \bigcap \{\mathcal{C}_\beta \circ G_\infty \mid \beta < \alpha\} = u \circ (\bigcap \{\mathcal{C}_\beta \mid \beta < \alpha\}) \circ G_\infty = \mathcal{C}_\alpha \circ G_\infty \ ,$$

which implies that $\mathcal{C}_\alpha \circ G_\infty$ is an MHP generator.
The additional statements are obvious. □

4.12. **THEOREM.**

a) $\mathfrak{W}\mathfrak{F}(C_\alpha \circ G_\infty, \mathcal{M})$ *is an* HPI *ttg for every ordinal* $\alpha \geqslant 0$ *;*

b) $\mathfrak{W}\mathfrak{F}(C \circ G_\infty, \mathcal{M})$ *is the universal minimal* HPI *ttg for* T .

PROOF. First we shall prove that if D is an MHP generator such that $C_{\alpha+1} \circ G_\infty \subseteq D \subseteq C_\alpha \circ G_\infty$ then $D = C_{\alpha+1} \circ G_\infty \circ uD$, and so, by 3.4., that the ambit morphism

$$\eta : (\mathfrak{W}\mathfrak{F}(C_{\alpha+1} \circ G_\infty, \mathcal{M}), C_{\alpha+1} \circ G_\infty) \to \mathcal{Y}$$

is a RIC extension. As follows:
Obviously, $C_{\alpha+1} \circ G_\infty \circ uD \subseteq D \circ D = D$ (note that $G_\infty \subseteq uD$).
Conversely, $D \subseteq JuD$ and $D \subseteq C_\alpha \circ G_\infty \subseteq (C_\alpha \circ G_\infty \cap J).G$; hence

$$D \subseteq (C_\alpha \circ G_\infty \cap J).uD \subseteq (C_\alpha \circ G_\infty \cap J) \circ uD .$$

As $u \circ (C_\alpha \circ G_\infty \cap J) \subseteq C_{\alpha+1}$ and as $G_\infty \subseteq uD$, it follows that

$$D = u \circ D \subseteq u \circ (C_\alpha \circ G_\infty \cap J) \circ uD \subseteq C_{\alpha+1} \circ uD = C_{\alpha+1} \circ G_\infty \circ uD .$$

Hence $D = C_{\alpha+1} \circ G_\infty \circ uD$; and by 3.4., η is a RIC extension.

a) Since $(C_\alpha G_\infty)_\infty = G_\infty \subseteq C_{\alpha+1} G_\infty$, it follows that the map

$$\phi : (\mathfrak{W}\mathfrak{F}(C_{\alpha+1} \circ G_\infty, \mathcal{M}), C_{\alpha+1} \circ G_\infty) \to (\mathfrak{W}\mathfrak{F}(C_\alpha \circ G_\infty, \mathcal{M}), C_\alpha \circ G_\infty)$$

is a PI extension. Using the above (which is analogues to claim b in the proof of 4.7.) it follows, as in the proof of 4.7., that every open ψ , with $\phi = \theta \circ \psi$, is a RIC extension. So by IV.5.7., ϕ is an HPI extension.
As $C_0 \circ G_\infty = \mathfrak{a}_0 \circ G_\infty$, the ttg $\mathfrak{W}\mathfrak{F}(C_0 \circ G_\infty, \mathcal{M})$ is an HPI ttg (4.7.). So every $\mathfrak{W}\mathfrak{F}(C_\alpha \circ G_\infty, \mathcal{M})$ is an HPI ttg.

b) In particular, $\mathfrak{W}\mathfrak{F}(C \circ G_\infty, \mathcal{M})$ is an HPI ttg. Let

$$\xi : (\mathcal{X}, x) \to (\mathfrak{W}\mathfrak{F}(C \circ G_\infty, \mathcal{M}), C \circ G_\infty)$$

be an almost periodic extension. Then $\mathcal{X}$ is a PI ttg and $J_x = (C \circ G_\infty) \cap J$, so

$$u \circ (C \circ G_\infty \cap J) \circ G_\infty \subseteq u \circ J_x \circ G_\infty \subseteq u \circ M_x \circ M_x = u \circ M_x \subseteq C \circ G_\infty .$$

As C was "minimal" it follows that $u \circ M_x = C \circ G_\infty$, and so that ξ^* is an isomorphism. Hence ξ is an hp extension, so by almost periodicity of ξ , ξ is an isomorphism. This and the fact that $\mathfrak{W}\mathfrak{F}(C \circ G_\infty, \mathcal{M})$ is an MHP ttg and the existence of a universal HPI ttg (IV.5.14.) show that $\mathfrak{W}\mathfrak{F}(C \circ G_\infty, \mathcal{M})$ is the universal minimal HPI ttg. □

4.13. **REMARK.** *For all ordinals* α *and* β *with* $\alpha \geqslant \beta \geqslant 0$ *we have* $\mathcal{C}_\beta \circ G_\infty = \mathcal{C}_\alpha \circ C_\beta G_\infty$ *; in particular,* $\mathcal{C}_\alpha \circ G_\infty = \mathcal{C} \circ C_\alpha G_\infty$.

PROOF. By the proof of 4.12. the map

$$\phi : \mathfrak{W}\mathfrak{F}(\mathcal{C}_\alpha \circ G_\infty, \mathfrak{M}) \to \mathfrak{W}\mathfrak{F}(\mathcal{C}_\beta \circ G_\infty, \mathfrak{M})$$

is a RIC extension. By 3.4., it follows that

$$\mathcal{C}_\beta \circ G_\infty = \mathcal{C}_\alpha \circ G_\infty \circ C_\beta G_\infty = \mathcal{C}_\alpha \circ C_\beta G_\infty .$$

□

4.14. **REMARK.** *For every ordinal* $\alpha \geqslant 0$ *we have*

$$\mathcal{C} \circ \mathrm{H}(G) = \mathcal{C}_\alpha \circ \mathrm{H}(G) = \mathcal{A}_0 \circ \mathrm{H}(G) = \mathcal{A} \circ \mathrm{H}(G) .$$

In particular, $\mathfrak{S}^* = \mathfrak{W}\mathfrak{F}(\mathcal{A} \circ \mathrm{H}(G), \mathfrak{M}) = \mathfrak{W}\mathfrak{F}(\mathcal{C} \circ \mathrm{H}(G), \mathfrak{M})$.

PROOF. First note that $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{C}_\alpha \subseteq \mathcal{C}_0 = \mathcal{A}_0$, so

$$\mathcal{A} \circ \mathrm{H}(G) \subseteq \mathcal{C} \circ \mathrm{H}(G) \subseteq \mathcal{C}_\alpha \circ \mathrm{H}(G) \subseteq \mathcal{A}_0 \circ \mathrm{H}(G) .$$

As, by 4.5., $\mathcal{A}_0 = \mathcal{A} \circ A_0$ and by 4.10., $A\,\mathrm{H}(G) = A_0 \mathrm{H}(G)$, the following inclusions hold:

$$\mathcal{A}_0 \mathrm{H}(G) = \mathcal{A} \circ A_0 \circ \mathrm{H}(G) = \mathcal{A} \circ A_0 \mathrm{H}(G) = \mathcal{A} \circ A\,\mathrm{H}(G) \subseteq$$

$$\subseteq \mathcal{A} \circ A \circ \mathrm{H}(G) = \mathcal{A} \circ \mathrm{H}(G) .$$

But then $\mathcal{A} \circ \mathrm{H}(G) = \mathcal{C} \circ \mathrm{H}(G) = \mathcal{C}_\alpha \circ \mathrm{H}(G) = \mathcal{A}_0 \circ \mathrm{H}(G)$. □

V.5. REMARKS

5.1. In theorem 1.7. we have seen that an idempotent subset C of M can be written as $C = KA = K \circ A$ where $A = uC = u \circ C \cap uM$ and $K = C \cap J$. So C is the product of its idempotent part and its group part. The subsets of uM that can occur as group parts of idempotent sets in $(2^M, \circ)$ are already described as all $\mathfrak{F}(\mathfrak{M}, u)$-closed subgroups of uM

(1.6.c and 1.8.). But at the moment there is not a theory available that deals with possible structures on J . So we do not know what kind of subsets of J can occur as the idempotent parts of the idempotent sets in $(2^M, \circ)$.

QUESTIONS

a) Which subsets of J can occur as idempotent parts of idempotent sets in M . In particular (motivated by 1.8.), if F is an $\mathfrak{F}(\mathfrak{M}, u)$-closed subgroup, what are the sets $(v \circ F) \cap J$ for $v \in J$?

b) Let $K \subseteq J$. What do the sets $u \circ K$ and $u(u \circ K)$ look like? (see also section V.4. and 5.4.a).

5.2. In the sections 2. and 3. one of the problems (under the surface) is the question whether or not the "circling" of two MHP generators is again an MHP generator. One could extend that question to a :

QUESTIONS

a) Let $C = u \circ C$ and $D = u \circ D$ be MHP generators. What is the smallest MHP generator $F = u \circ F$ that contains C and D . In what situation do we have $F = (C \circ D)^n$ for some $n \in \mathbb{N}$. (i.e., $F = (C \circ D) \circ \cdots \circ (C \circ D)$ (n-times)).

b) Another, more elementary, question which is already stated in [AG 77] is whether or nor every quasifactor of $\mathfrak{M}$ is an MHP ttg.

5.3. We investigated a limited amount of dynamical properties in relation to MHP generators in section 3.. A lot of other problems could be stated in that respect, some of which are:

QUESTIONS

a) How do we characterize minimal weakly mixing ttgs in terms of MHP generators, and is there a relation of the sets $\mathfrak{C}_\alpha$ to weak mixing?

b) How do we characterize MHP generators that generate MHP ttgs which are prime up to high proximality? In other words: for what kind of MHP generator $C = u \circ C$ do we have $[C \cup \{p\}] = M$ for every $p \in M$, where $[C \cup \{p\}]$ denotes the smallest MHP generator that contains $C \cup \{p\}$ (i.e., what kind of MHP generator is "maximal").

c) Which minimal ttgs $\mathcal{X}$ satisfy the following property: if $\mathcal{X} \perp \mathfrak{Z}$, then $\mathfrak{Z} = \{\bigstar\}$. In other words: for what kind of ttg $\mathcal{X}$ is $\{\bigstar\}$ maximal disjoint from $\mathcal{X}$ (see also 3.14.).

5.4.

QUESTIONS

a) Let K be an arbitrary subset of J, what do a_K and A_K look like, and when is $a_K = u \circ K \circ A_K$ (see also 5.1.b)? Is a_K^{∞} regular?

b) Under what conditions is $a_\alpha \circ G_\infty$ an MHP generator for every ordinal α? When is $a \circ G_\infty = c \circ G_\infty$?

c) In several studies a specific kind of incontractible ttg is given much attention to, namely the kind of ttg $\mathcal{X}$ for which $uX = TuX$ (for instance, see [E 69], [EK 71], [EGS 76]). Note that if T is abelian then $TuX = uTX = uX$ for every minimal ttg $\mathcal{X}$. How are those ttgs related to our ttgs $\mathcal{A}_K$ for $K \in \mathcal{K}^*$, or better: what kind of MHP generators generate MHP extensions of those ttgs?

VI

DISJOINTNESS

In structure theory it is not only important to know how minimal ttgs are built up, but also how they are related to each other. A typical example of non-relation is disjointness. In this chapter we try to figure out (in rough lines) to what extent minimal ttgs are "classwise" non-related.
In section 1. we pay attention to the role quasifactors can play in this problem.
In the second section we change our point of view to classes of minimal ttgs that are in a certain sense consistent in their behavior towards disjointness; and we describe some of them with their relation to others. For instance in 2.13.a we show that $\mathbf{P}^{\perp} \cap \mathbf{PI} \subseteq \mathbf{D}^{\perp\perp}$, in words: every minimal incontractible PI ttg is disjoint from every minimal ttg without nontrivial uniformly almost periodic factors (compare [G 76] X.4.4.).
Section 3. deals with the question how those classes behave with respect to extensions, and we end the section with a picture of how the disjointness classes under view are related. In section 4. we apply some of the previous results to the problem to what extent disjointness is implied by the fact that the ttgs in question are relatively prime (i.e., do not admit a nontrivial common factor).
Most of the material in this chapter can be found in [Wo 79.1] and [Wo 79.2], but some results here are stronger by application of the results in chapter V.

VI.1. DISJOINTNESS AND QUASIFACTORS

In this section we establish some disjointness relations between factors and quasifactors of a minimal ttg.

1.1. **THEOREM.** *Let* $\phi:\mathscr{X}\to\mathscr{Y}$ *be a homomorphism of minimal ttgs.*

a) *Let* $\mathscr{Z}$ *be a nontrivial quasifactor of* $\mathscr{Y}$ *. Then* $\mathscr{Z}\not\perp\mathscr{X}$ *.*

b) *Let* $\mathscr{Z}$ *be a nontrivial quasifactor of* $\mathscr{X}$*. If* $2^{\phi}[\mathscr{Z}]\neq\{\star\}$ *then* $\mathscr{Z}\not\perp\mathscr{Y}$ *.*

In particular, it follows that a minimal ttg $\mathscr{X}$ *is not disjoint from its nontrivial quasifactors.*

PROOF.

a) Define $W:=\{(x,A)\in X\times Z \mid \phi(x)\in A\}$. Then, clearly, W is a nonempty closed invariant subset of $X\times Z$ and as $\mathscr{Z}$ is nontrivial $W\neq X\times Z$; hence $\mathscr{X}\times\mathscr{Z}$ is not minimal.

b) Define $W:=\{(y,A)\in Y\times Z \mid y\in\phi[A]\}$. Then W is a nonempty closed invariant subset of $Y\times Z$. As $2^{\phi}[\mathscr{Z}]\neq\{\star\}$, there is an $A\in Z$ with $\phi[A]\neq Y$ (so, as is easily seen, $\phi[A]\neq Y$ for every $A\in Z$). Hence $W\neq Y\times Z$ and $\mathscr{Y}\times\mathscr{Z}$ is not minimal. □

The conclusion of statement 1.1.b cannot hold for all nontrivial quasifactors of $\mathscr{X}$ without any further condition. For let $\mathscr{X}\perp\mathscr{Y}$ and let $\phi:\mathscr{X}\times\mathscr{Y}\to\mathscr{Y}$ be the projection. As the projection $\psi:\mathscr{X}\times\mathscr{Y}\to\mathscr{X}$ is open, $\mathscr{X}$ is a quasifactor of $\mathscr{X}\times\mathscr{Y}$ (II.3.3.c) and by assumption $\mathscr{X}\perp\mathscr{Y}$.
We shall now look for situations in which $\mathscr{Z}\not\perp\mathscr{Y}$ for certain (respectively all) nontrivial quasifactors of $\mathscr{X}$.

1.2. **REMARK.** *If* $\phi:\mathscr{X}\to\mathscr{Y}$ *is a highly proximal extension of minimal ttgs then* $\mathscr{Z}\not\perp\mathscr{Y}$ *for all nontrivial quasifactors* $\mathscr{Z}$ *of* $\mathscr{X}$*.*

PROOF. By IV.4.18., $\mathscr{Z}\perp\mathscr{Y}$ iff $\mathscr{Z}\perp\mathscr{X}$; but by 1.1., $\mathscr{Z}\not\perp\mathscr{X}$. □

1.3. **THEOREM.** *Let* $\phi:\mathscr{X}\to\mathscr{Y}$ *be an open homomorphism of minimal ttgs. Let* $\mathscr{Z}$ *be a nontrivial quasifactor of* $\mathscr{X}$ *such that* $\phi[X\setminus A]\neq Y$ *for some* $A\in Z$ *. Then* $\mathscr{Z}\not\perp\mathscr{Y}$ *.*

PROOF. Define $W:=\{(y,B)\in Y\times Z \mid \phi^{\leftarrow}(y)\subseteq B\}$. As $\phi[X\setminus A]\neq Y$ there is a $y_0\in Y$ with $\phi^{\leftarrow}(y_0)\subseteq A$; hence $W\neq\varnothing$. Also $W\neq Y\times Z$; for, equality would imply that $\phi^{\leftarrow}(y)\subseteq B$ for all $y\in Y$, so $X\subseteq B$ and

$\mathcal{Z}$ would be trivial. Clearly, W is invariant, and by openness of ϕ (i.e., continuity of $\phi_{\mathrm{ad}}: Y \to 2^X$), it follows that W is closed. So, $\mathcal{Y} \times \mathcal{Z}$ is not minimal. □

1.4. **THEOREM.** *Let* $\phi: \mathcal{X} \to \mathcal{Y}$ *be a proximal homomorphism of minimal ttgs. Let* $\mathcal{Z}$ *be a nontrivial quasifactor of* $\mathcal{X}$.

a) *If* $\mathcal{Z} \perp \mathcal{Y}$ *then* $u \circ uX \subseteq A$ *for every* $u \in J$ *and* $A \in uZ$ *(i.e., for every* $A = u \circ A \in Z$ *).*

b) *If either* $\mathcal{X}$ *or* $\mathcal{Z}$ *is incontractible then* $\mathcal{Z} \not\perp \mathcal{Y}$.

PROOF.

a) Suppose $\mathcal{Z} \perp \mathcal{Y}$. Then $\phi \times id_Z : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y} \times \mathcal{Z}$ is a proximal extension of a minimal ttg; so by I.1.23.c, $\mathcal{X} \times \mathcal{Z}$ has a unique minimal subset L. Define $W = \{(x,B) \in X \times Z \mid x \in B\}$. Then W is a nonempty closed and invariant subset of $X \times Z$, so $L \subseteq W$. Let $A = u \circ A \in Z$ and let $x \in X$. Then $(x,A) \in X \times Z$, hence

$$u(x,A) = (ux, u \circ A) \in L \subseteq W \quad \text{so} \quad ux \in u \circ A .$$

As $x \in X$ was arbitrary we have $uX \subseteq u \circ A$ and so $u \circ uX \subseteq u \circ u \circ A = u \circ A$.

b) If $\mathcal{X}$ or $\mathcal{Z}$ is incontractible, it follows from III.1.5.c that $X \times Z$ has a dense subset of almost periodic points. If $\mathcal{Z} \perp \mathcal{Y}$, then $\mathcal{X} \times \mathcal{Z}$ has a unique minimal subset and $\mathcal{X} \times \mathcal{Z}$ is minimal; which contradicts 1.1.. So, if $\mathcal{X}$ or $\mathcal{Z}$ is incontractible, $\mathcal{Z} \not\perp \mathcal{Y}$. □

1.5. **LEMMA.** *Let* $\phi: \mathcal{X} \to \mathcal{Y}$ *be a homomorphism of minimal ttgs. Let* $\mathcal{Z}$ *be a nontrivial quasifactor of* $\mathcal{X}$ *such that* $\mathcal{Z} \perp \mathcal{Y}$. *If* (A,B) *is a proximal pair in* $\mathcal{Z}$ *with* $A \neq B$ *then there is a proximal pair* $(x_1, x_2) \in R_\phi \cap A \times B$ *with* $x_1 \neq x_2$.

PROOF. Let I be a minimal left ideal in S_T such that $p \circ A = p \circ B$ for all $p \in I$ (I.2.7.c). Without loss of generality suppose there is an $x_1 \in A \setminus B$. Then $(\phi(x_1), B) \in Y \times Z$ and, as $\mathcal{Y} \times \mathcal{Z}$ is minimal, there is an idempotent $w \in I$ such that $w(\phi(x_1), B) = (\phi(x_1), B)$. Then we have $\phi(wx_1) = we\phi(x_1) = \phi(x_1)$; so $(x_1, wx_1) \in R_\phi$ and

$$wx_1 \in wA \subseteq w \circ A = w \circ B = B .$$

Hence $(x_1, wx_1) \in R_\phi \cap A \times B$. Clearly (x_1, wx_1) is a proximal pair, and $x_1 \notin B$ while $wx_1 \in B$, so $x_1 \neq wx_1$. □

1.6. **THEOREM.** *Let* $\phi:\mathscr{X}\to\mathscr{Y}$ *be a distal homomorphism of minimal ttgs. Let* $\mathscr{Z}$ *be a nontrivial quasifactor of* $\mathscr{X}$.

a) *If* $\mathscr{Z}\perp\mathscr{Y}$ *then* $\mathscr{Z}$ *is distal.*

b) *If* $\mathscr{X}$ *is disjoint from every minimal distal ttg (* $\mathscr{X}\in\mathbf{D}^{\perp}$ *) then* $\mathscr{Z}\not\perp\mathscr{Y}$.

PROOF.

a) By 1.5., there can be no proximal pairs in $\mathscr{Z}$, so $\mathscr{Z}$ is distal.

b) Suppose $\mathscr{Z}\perp\mathscr{Y}$. Then $\mathscr{Z}$ is distal (a). As $\mathscr{X}\in\mathbf{D}^{\perp}$ we have $\mathscr{X}\perp\mathscr{Z}$, which contradicts 1.1.. □

In section 3. w shall see other results with the flavor of 1.4. and 1.6. (cf. 3.7.).

The following characterization of disjointness in terms of quasifactors will be needed in the sequel (see also [AG 77] lemma II.4.).

1.7. **THEOREM.** *Let* $\mathscr{X}$ *and* $\mathscr{Y}$ *be minimal ttgs. Then* $\mathscr{X}\not\perp\mathscr{Y}$ *iff there is a nontrivial quasifactor* $\mathscr{Z}$ *of* $\mathscr{Y}$ *which is a factor of* $\mathscr{X}^{*}$ *(the* MHP *extension of* $\mathscr{X}$*).*

PROOF. Suppose there is a nontrivial quasifactor $\mathscr{Z}$ of $\mathscr{Y}$ and a surjective homomorphism $\phi:\mathscr{X}^{*}\to\mathscr{Z}$. Then $\phi\times id_{Y}:\mathscr{X}^{*}\times\mathscr{Y}\to\mathscr{Z}\times\mathscr{Y}$ is a surjective homomorphism. As, by 1.1., $\mathscr{Z}\times\mathscr{Y}$ is not minimal, $\mathscr{X}^{*}\times\mathscr{Y}$ cannot be minimal. Hence $\mathscr{X}^{*}\not\perp\mathscr{Y}$ and, by IV.4.18., $\mathscr{X}\not\perp\mathscr{Y}$.
Conversely, suppose that $\mathscr{X}\not\perp\mathscr{Y}$; then, by IV.4.18., $\mathscr{X}^{*}\not\perp\mathscr{Y}^{*}$. Let C and D be MHP generators with $C=u\circ C$ and $D=u\circ D$ such that $\mathscr{X}^{*}=\mathscr{C}$ and $\mathscr{Y}^{*}=\mathscr{D}$. Then, by V.3.9., we have that $C\circ D\neq M$. Hence, by V.2.6.b, $\mathscr{WF}(C\circ D,\mathscr{M})$ is a quasifactor of $\mathscr{Y}^{*}$ which (clearly) is nontrivial. Let $\chi_{\mathscr{Y}}:\mathscr{Y}^{*}\to\mathscr{Y}$ be the canonical MHP extension. Then, by irreducibility of $\chi_{\mathscr{Y}}$, we have that $\mathscr{Z}:=2^{\chi_{\mathscr{Y}}}[\mathscr{WF}(C\circ D,\mathscr{M})]$ is a nontrivial quasifactor of $\mathscr{Y}$. Obviously, $\psi:\mathscr{X}^{*}\to\mathscr{Z}$ defined by $\psi(p\circ C)=2^{\chi_{\mathscr{Y}}}(p\circ C\circ D)$ is a homomorphism of ttgs. □

VI.2. DISJOINTNESS CLASSES

In this section we study "disjointness classes" of minimal ttgs and we characterize them via quasifactors (2.3. through 2.7.). We also give some relations between those disjointness classes (e.g., $\mathbf{P}^{\perp} \cap \mathbf{PI} \subseteq \mathbf{D}^{\perp\perp}$ and $\mathbf{D}^{\perp} \cap \mathbf{PI} \subseteq \mathbf{P}^{\perp\perp}$ (2.13.)).

Let $\mathbf{K}$ be a set of minimal ttgs. Then $\mathbf{K}^{\perp}$ denotes the set of minimal ttgs $\mathcal{X}$ such that $\mathcal{X} \perp \mathcal{Y}$ for every $\mathcal{Y} \in \mathbf{K}$.

2.1. **REMARK.** *Let* $\mathbf{K}$, $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *be sets of minimal ttgs.*

a) $\mathbf{K}^{\perp}$ *is closed under factors, highly proximal extensions and inverse limits.*

b) *If* $\mathbf{K}_1 \subseteq \mathbf{K}_2$ *then* $\mathbf{K}_2^{\perp} \subseteq \mathbf{K}_1^{\perp}$.

c) $\mathbf{K} \subseteq \mathbf{K}^{\perp\perp}$ *and* $\mathbf{K}^{\perp} = \mathbf{K}^{\perp\perp\perp}$.

PROOF. For a cf. I.3.1.a, b and IV.4.18., b and c are obvious. ☐

Let $\mathbf{K}$ be a set of minimal ttgs. Define

$$[\mathbf{K}] = \{\mathcal{X} \mid \mathcal{X} \text{ is a minimal ttg and for some } \mathcal{Y} \in \mathbf{K}, \mathcal{X} \text{ is a factor of } \mathcal{Y}^*\}.$$

Evidently, $\mathbf{K} \subseteq [\mathbf{K}] = [[\mathbf{K}]]$ and $[\mathbf{K}]$ is closed under factors and hp extensions. Moreover, $[\mathbf{K}]$ is the smallest collection of minimal ttgs under these conditions.

2.2. **EXAMPLES.**

a) Let $\mathbf{K}$ be a set of minimal ttgs with a universal element, i.e., there is a $\mathcal{H} \in \mathbf{K}$ such that $\mathcal{H} \to \mathcal{Z}$ for every $\mathcal{Z} \in \mathbf{K}$. Then

$$[\mathbf{K}] = \{\mathcal{X} \mid \mathcal{X} \text{ is a factor of } \mathcal{H}^*\}.$$

To name a few:

(i) Let $\mathbf{E}$ be the collection of minimal uniformly almost periodic ttgs. Then $[\mathbf{E}] = \{\mathcal{X} \mid \mathcal{X} \text{ is a factor of } \mathcal{E}_T^*\}$.

(ii) Let $\mathbf{D}$ be the collection of minimal distal ttgs. Then $[\mathbf{D}] = \{\mathcal{X} \mid \mathcal{X} \text{ is a factor of } \mathcal{D}_T^*\}$.

(iii) Let $\mathbf{P}$, $\mathbf{PI}$, $\mathbf{HPI}$ be the collections of minimal proximal ttgs, minimal PI ttgs and minimal HPI ttgs respectively. Then $[\mathbf{P}] = \mathbf{P}$, $[\mathbf{PI}] = \mathbf{PI}$ and $[\mathbf{HPI}] = \mathbf{HPI}$.

(iv) Let F be an $\mathfrak{F}(\mathfrak{M},u)$-closed subgroup of G and let $\mathbf{M}(F)$ be the collection of minimal ttgs such that there is an $x \in X$ with $F \subseteq \mathfrak{G}(\mathfrak{X},ux)$. Then $[\mathbf{M}(F)] = \mathbf{M}(F)$ (cf. I.2.11. and I.2.13.b).

b) Let **WM** be the collection of minimal weakly mixing ttgs. Then $[\mathbf{WM}] = \mathbf{WM}$, for $\mathfrak{X}$ is weakly mixing iff $\mathfrak{X}^*$ is weakly mixing (IV.4.17.) and every factor of a weakly mixing minimal ttg is weakly mixing.

c) Let $\mathbf{K}$ be a set of minimal ttgs. Then $[\mathbf{K}^{\perp}] = \mathbf{K}^{\perp}$ (cf. 2.1.).

2.3. **THEOREM.** *Let* $\mathbf{K}$ *be a set of minimal ttgs. For a minimal ttg* $\mathfrak{X}$ *the following statements are equivalent:*

a) $\mathfrak{X} \in \mathbf{K}^{\perp}$;
b) $\mathfrak{X} \in [\mathbf{K}]^{\perp}$;
c) $\mathfrak{Z} \notin [\mathbf{K}]$ *for every nontrivial quasifactor* $\mathfrak{Z}$ *of* $\mathfrak{X}$.

PROOF.

b $\Rightarrow$ a Clear, as $\mathbf{K} \subseteq [\mathbf{K}]$.

a $\Rightarrow$ c Let $\mathfrak{X} \in \mathbf{K}^{\perp}$ and suppose that $\mathfrak{Z} \in [\mathbf{K}]$ for some quasifactor $\mathfrak{Z}$ of $\mathfrak{X}$. Then there is a $\mathfrak{Y} \in \mathbf{K}$ such that $\mathfrak{Z}$ is a factor of $\mathfrak{Y}^*$. As $\mathfrak{X} \in \mathbf{K}^{\perp}$, $\mathfrak{X} \perp \mathfrak{Y}$; hence $\mathfrak{X} \perp \mathfrak{Y}^*$ and so $\mathfrak{X} \perp \mathfrak{Z}$. But then $\mathfrak{Z}$ has to be trivial by 1.1..

c $\Rightarrow$ b Suppose $\mathfrak{X} \notin [\mathbf{K}]^{\perp}$, then there is a $\mathfrak{Y} \in [\mathbf{K}]$ with $\mathfrak{X} \not\perp \mathfrak{Y}$. By 1.7., there is a nontrivial quasifactor $\mathfrak{Z}$ of $\mathfrak{X}$ which is a factor of $\mathfrak{Y}^*$. As $\mathfrak{Y}^* \in [\mathbf{K}]$, also $\mathfrak{Z} \in [\mathbf{K}]$. □

2.4. **REMARK.** *Let* $\mathbf{K}$ *be a set of minimal ttgs containing a universal element* $\mathcal{H}$. *Let* C *be an* MHP *generator such that* $C = u \circ C$ *and* $\mathcal{H}^* = \mathcal{C}$. *For a minimal ttg* $\mathfrak{X}$ *the following statements are equivalent:*

a) $\mathfrak{X} \in \mathbf{K}^{\perp}$;
b) $\mathfrak{X} \perp \mathcal{H}$;
c) *No nontrivial quasifactor* $\mathfrak{Z}$ *of* $\mathfrak{X}$ *is a factor of* $\mathcal{H}^*$;
d) $Cx = X$ *for every* $x \in X$;
e) $Cx = X$ *for some* $x \in X$.

PROOF. The equivalence of b and c follows from 1.7., and clearly, a and b are equivalent.

b $\Rightarrow$ d Let $x \in X$ and define $\gamma := \rho_x : \mathfrak{M} \to \mathfrak{X}$. Let $F = u \circ \gamma^{\leftarrow}(x)$, then $\mathfrak{X}^* = \mathfrak{F}$. As $\mathfrak{X} \perp \mathcal{H}$, also $\mathfrak{X}^* \perp \mathcal{H}^*$ (IV.4.18.); hence, by V.3.9.c, $C \circ F = M$. But then $Cx = C \circ Fx = Mx = X$.

d $\Rightarrow$ e Trivial.

e $\Rightarrow$ b Suppose $Cx = X$ for some specific $x \in X$. Then we have $p \circ Cx = p \circ X = X$ for all $p \in M$. We shall prove $\mathcal{X} \perp \mathcal{H}^*$, from which it follows that $\mathcal{X} \perp \mathcal{H}$ (IV.4.18.). Let $(q \circ C, x') \in \mathcal{H}^* \times \mathcal{X}$. As $X = q \circ Cx = q \circ Cux$, we have $x' \in q \circ Cux$; so there is a net $\{t_i\}_i$ in T and there are $c_i \in C$ such that $t_i \to q$ and $t_i c_i ux \to x'$. As $C = c_i \circ C$ for every i we have

$$q \circ C = \lim t_i C = \lim t_i c_i \circ C = \lim t_i c_i (u \circ C) \; ;$$

so $(q \circ C, x') = \lim t_i c_i (C, ux)$. Hence $\mathcal{H}^* \times \mathcal{X} \subseteq \overline{T(C, ux)}$, and as (C, ux) is an almost periodic point, it follows that $\mathcal{H}^* \times \mathcal{X}$ is minimal. □

2.5. EXAMPLES *Let* $\mathcal{X}$ *be a minimal ttg.*

a) $\mathcal{X} \in \mathbf{P}^{\perp}$ *iff* $\mathcal{X}$ *does not have nontrivial proximal quasifactors iff* $u \circ Gx = X$ *for some (all)* $x \in X$.

b) $\mathcal{X} \in \mathbf{(H)PI}^{\perp}$ *iff* $\mathcal{X}$ *does not have nontrivial* (H)PI *quasifactors iff* $u \circ G_{\infty} x = X$ *(* $C \circ G_{\infty} x = X$ *) for some (all)* $x \in X$.

c) $\mathcal{X} \in \mathbf{WM}$ *iff* $\mathcal{X}$ *does not have nontrivial weakly mixing quasifactors.*

d) *Let* $\mathbf{K} = [\mathbf{K}]$ *(e.g.* $\mathbf{K}$ *is* $\mathbf{P}$ *or* $\mathbf{(H)PI}$*), then* $\mathcal{X} \in \mathbf{K}^{\perp\perp}$ *iff every nontrivial quasifactor of* $\mathcal{X}$ *has a nontrivial quasifactor in* $\mathbf{K}$.

e) *In particular, we have (because of* 2.1.c*) that* $\mathcal{X} \in \mathbf{K}^{\perp}$ *iff every nontrivial quasifactor of* $\mathcal{X}$ *has a nontrivial quasifactor in* $\mathbf{K}^{\perp}$.

2.6. THEOREM. *Let* $\mathbf{K}$ *be* $\mathbf{D}$ *or* $\mathbf{E}$ *and let* $\mathcal{H}$ *be the universal element in* $\mathbf{K}$. *For a minimal ttg* $\mathcal{X}$ *the following statements are equivalent:*

a) $\mathcal{X} \in \mathbf{K}^{\perp}$ *;*

b) $\mathcal{X} \perp \mathcal{H}$*;*

c) $\mathcal{X}$ *has no nontrivial quasifactors in* $[\mathbf{K}]$ *;*

d) $\mathcal{X}$ *has no nontrivial factors in* $\mathbf{K}$.

PROOF. The equivalence of a , b and c is just 2.4. (see also 2.2.a (i),(ii)).

c $\Rightarrow$ d Let $\mathcal{Y}$ be a nontrivial factor of $\mathcal{X}$ in $\mathbf{K}$. Then by IV.3.1., there is a quasifactor of $\mathcal{X}$ in $[\mathbf{K}]$ which obviously is nontrivial.

d $\Rightarrow$ b Suppose that $\mathcal{X} \not\perp \mathcal{H}$. Then, by 1.7. and II.3.7., $\mathcal{X}^*$ has a factor in $\mathbf{K}$. Hence, by I.4.1., $\mathcal{X}$ has a factor in $\mathbf{K}$. □

2.7. **COROLLARY.** *Let* $\mathcal{X}$ *be a minimal ttg.*

a) $\mathcal{X} \in \mathbf{E}^{\perp\perp}$ $(\mathbf{D}^{\perp\perp})$ *iff every nontrivial quasifactor of* $\mathcal{X}$ *has a nontrivial uniformly almost periodic (distal) factor.*

b) $\mathbf{D}^{\perp\perp} = \mathbf{E}^{\perp\perp}$, *hence* $\mathbf{D}^{\perp} = \mathbf{E}^{\perp}$.

PROOF.

a) Follows immediately from 2.3. and 2.6..

b) Follows from a and I.1.25.. □

2.8. **THEOREM.** *Let* $\phi : \mathcal{X} \to \mathcal{Y}$ *be a distal homomorphism of minimal ttgs. If* $\mathcal{Y} \in \mathbf{D}^{\perp\perp}$ *then* $\mathcal{X} \in \mathbf{D}^{\perp\perp}$. *In other words:* $\mathbf{D}^{\perp\perp}$ *is closed under distal extensions, hence it is closed under* HPI *extensions.*

PROOF. Suppose that $\mathcal{X} \notin \mathbf{D}^{\perp\perp}$ then there is a nontrivial quasifactor $\mathcal{Z}$ of $\mathcal{X}$ with $\mathcal{Z} \in \mathbf{D}^{\perp}$. As $\mathcal{Y} \in \mathbf{D}^{\perp\perp}$ it follows that $\mathcal{Z} \perp \mathcal{Y}$. Hence by 1.6.a, $\mathcal{Z}$ is distal, but this contradicts the assumption $\mathcal{Z} \in \mathbf{D}^{\perp}$. So $\mathcal{X} \in \mathbf{D}^{\perp\perp}$. As $\mathbf{D}^{\perp\perp}$ is closed under hp extensions and factors (2.1.), it follows that $\mathbf{D}^{\perp\perp}$ is even closed under HPI extensions. □

2.9. **COROLLARY.** $\mathbf{HPI}^{\perp\perp} = \mathbf{D}^{\perp\perp} = \mathbf{E}^{\perp\perp}$ *and* $\mathbf{HPI}^{\perp} = \mathbf{D}^{\perp} = \mathbf{E}^{\perp}$.

PROOF. As $\{\star\} \in \mathbf{D}^{\perp\perp}$ it follows from 2.8. that $\mathbf{HPI} \subseteq \mathbf{D}^{\perp\perp}$ and so that $\mathbf{HPI}^{\perp\perp} \subseteq \mathbf{D}^{\perp\perp}$. On the other hand, by FST, we know that $\mathbf{D} \subseteq \mathbf{HPI}$, so $\mathbf{D}^{\perp\perp} \subseteq \mathbf{HPI}^{\perp\perp}$; hence $\mathbf{HPI}^{\perp\perp} = \mathbf{D}^{\perp\perp} = \mathbf{E}^{\perp\perp}$. Consequently,

$$\mathbf{HPI}^{\perp} = \mathbf{HPI}^{\perp\perp\perp} = \mathbf{D}^{\perp\perp\perp} = \mathbf{D}^{\perp} = \mathbf{E}^{\perp}.$$

□

Let us first describe some fundamental relations between (disjointness) classes of minimal ttgs. For some of these, we need results which will be proved in chapter VII..

2.10. **THEOREM.**

a) $\mathbf{P} \subseteq \mathbf{WM} \subseteq \mathbf{WM}^{\perp\perp} \subseteq \mathbf{D}^{\perp}$;

b) $\mathbf{E} \subseteq \mathbf{D} \subseteq \mathbf{HPI} \subseteq \mathbf{D}^{\perp\perp} \subseteq \mathbf{P}^{\perp}$;

c) $\mathbf{PI}^{\perp} \subseteq \mathbf{P}^{\perp} \cap \mathbf{D}^{\perp} = \mathbf{P}^{\perp} \cap \mathbf{WM} = \mathbf{P}^{\perp} \cap \mathbf{WM}^{\perp\perp}$;

d) $\mathbf{P} \subseteq \mathbf{P}^{\perp\perp} \subseteq \mathbf{D}^{\perp} \cap \mathbf{PI}^{\perp\perp} \subseteq \mathbf{D}^{\perp}$;

e) $\mathbf{D}^{\perp\perp} \subseteq \mathbf{WM}^{\perp} \subseteq \mathbf{P}^{\perp} \cap \mathbf{PI}^{\perp\perp}$.

PROOF.

a) By I.3.10., every proximal minimal ttg is a weakly mixing ttg; i.e., $\mathbf{P} \subseteq \mathbf{WM}$. As a distal ergodic ttg is minimal (I.1.17.), a weakly mixing ttg does not admit nontrivial distal factors. (Otherwise, if $\mathcal{Y}$ were such a factor, $\mathcal{Y}\times\mathcal{Y}$ would be distal and ergodic, hence minimal.) Hence, $\mathbf{WM}\subseteq\mathbf{D}^{\perp}$ and so $\mathbf{WM}\subseteq\mathbf{WM}^{\perp\perp}\subseteq\mathbf{D}^{\perp\perp\perp}=\mathbf{D}^{\perp}$.

b) We know that $\mathbf{E}\subseteq\mathbf{D}$ and $\mathbf{D}\subseteq\mathbf{HPI}$ (FST). By 2.8., $\mathbf{HPI}\subseteq\mathbf{D}^{\perp\perp}$. In a we have seen that $\mathbf{P}\subseteq\mathbf{D}^{\perp}$; so by 2.1.b, $\mathbf{D}^{\perp\perp}\subseteq\mathbf{P}^{\perp}$.

c) As $\mathbf{P}\cup\mathbf{D}\subseteq\mathbf{PI}$ (FST), it follows from 2.1.b that $\mathbf{PI}^{\perp}\subseteq\mathbf{P}^{\perp}\cap\mathbf{D}^{\perp}$. By VII.3.11. and VI.2.6., we have $\mathbf{P}^{\perp}\cap\mathbf{E}^{\perp}\subseteq\mathbf{WM}$. Hence, by 2.7.b,

$$\mathbf{P}^{\perp}\cap\mathbf{D}^{\perp}\subseteq\mathbf{WM}\subseteq\mathbf{WM}^{\perp\perp}\subseteq\mathbf{D}^{\perp},$$

so $\mathbf{P}^{\perp}\cap\mathbf{D}^{\perp}=\mathbf{P}^{\perp}\cap\mathbf{WM}=\mathbf{P}^{\perp}\cap\mathbf{WM}^{\perp\perp}$.

d) Trivial from the fact that $\mathbf{P}\subseteq\mathbf{D}^{\perp}\cap\mathbf{PI}$.

e) As, by a, $\mathbf{P}\subseteq\mathbf{WM}\subseteq\mathbf{D}^{\perp}$ it follows from 2.1.b that $\mathbf{D}^{\perp\perp}\subseteq\mathbf{WM}^{\perp}\subseteq\mathbf{P}^{\perp}$. By c, $\mathbf{PI}^{\perp}\subseteq\mathbf{WM}$; so $\mathbf{WM}^{\perp}\subseteq\mathbf{PI}^{\perp\perp}$. □

2.11. EXAMPLE. *In general,* $\mathbf{D}^{\perp}\neq\mathbf{WM}$.

Consider the fourfold covering of the proximal circle, as presented in VIII.1.5. (also see I.4.7.). Then $\mathcal{Y}$ does not admit nontrivial uniformly almost periodic factors; so $\mathcal{Y}\in\mathbf{E}^{\perp}=\mathbf{D}^{\perp}$. But $Q_{\mathcal{Y}}\neq E_{\mathcal{Y}}$, whereas, if $\mathcal{Y}$ were weakly mixing, we should have

$$Q_{\mathcal{Y}}=\bigcap\{\overline{T\alpha}\mid\alpha\in\mathfrak{U}_Y\}=Y\times Y\ ,\ \text{ so }\ Q_{\mathcal{Y}}=E_{\mathcal{Y}}=Y\times Y\ .$$

In section V.3. we have seen that we can decide about disjointness by considering MHP generators. And from III.1.6. it follows that in case one of the ttgs involved is incontractible, we only need to consider the Ellis groups. So (III.1.6. in the absolute case):

> NOTE. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be minimal ttgs with Ellis groups* H *and* F *in* G *with respect to some* $x\in uX$ *and* $y\in uY$. *If* $\mathcal{X}\in\mathbf{P}^{\perp}$ *then* $\mathcal{X}\perp\mathcal{Y}$ *iff* $HF=G$.

For the following remember from V.4. (slightly changed):

> $\mathfrak{a}_0$ is the MHP generator generated by $u\circ J$ and $A_0=u\mathfrak{a}_0$.
> $\mathfrak{a}_K$ is the MHP generator generated by $u\circ K$ and $A_K=u\mathfrak{a}_K$, for every $K\in\mathcal{K}^*$ (i.e., the $\mathfrak{a}_K$'s are the incontractible MHP generators).
> For $K\in\mathcal{K}$, $\mathfrak{a}_K$ is a minimal incontractible MHP generator and A_K is the $\mathfrak{T}(\mathfrak{M},u)$-closed subgroup of G generated by $u(u\circ K)$.

Remember that $\mathscr{X} \in \mathbf{P}^{\perp}$ iff $\mathscr{X}$ is a factor of $\mathscr{A}_K$ for some $K \in \mathscr{K}$ (V.4.4.).

2.12. **THEOREM.** *Let* $\mathscr{X}$ *be a minimal ttg with Ellis group* H .

a) *The following statements are equivalent:*
   - (i) $\mathscr{X} \in \mathbf{D}^{\perp}$ *;*
   - (ii) $HA_0\mathrm{H}(G) = G$ *;*
   - (iii) $HA_K\mathrm{H}(G) = G$ *for every* $K \in \mathscr{K}^*$ *;*
   - (iv) $HA_K G_\infty = G$ *for every* $K \in \mathscr{K}^*$ *;*
   - (v) $HA_K G_\infty = G$ *for some* $K \in \mathscr{K}^*$.

b) $\mathscr{X} \in \mathbf{P}^{\perp\perp}$ *iff* $HA_K = G$ *for every* $K \in \mathscr{K}$.

PROOF.

a) The equivalence of (iii) and (iv) follows from III.2.13.c and, obviously, (v) follows from (iv). As $A_K \subseteq A_0$ and $G_\infty \subseteq \mathrm{H}(G)$, (v) implies (ii). By V.4.10., (iii) follows from (ii).
From 2.7.b and 2.6. we know that $\mathscr{X} \in \mathbf{D}^{\perp}$ iff $\mathscr{X} \perp \mathscr{E}$. Hence, by III.1.6. and V.4.10., we have $\mathscr{X} \in \mathbf{D}^{\perp}$ iff $HA_0\mathrm{H}(G) = G$.

b) As every incontractible minimal ttg is a factor of some $\mathscr{A}_K$, it follows that $\mathscr{X} \in \mathbf{P}^{\perp\perp}$ iff $\mathscr{X} \perp \mathscr{A}_K$ for every $K \in \mathscr{K}$. But $\mathscr{A}_K \in \mathbf{P}^{\perp}$, so $\mathscr{X} \perp \mathscr{A}_K$ iff $HA_K = G$. So $\mathscr{X} \in \mathbf{P}^{\perp\perp}$ iff $HA_K = G$ for every $K \in \mathscr{K}$. □

2.13. **THEOREM.**

a) $\mathbf{P}^{\perp} \cap \mathbf{PI} \subseteq \mathbf{D}^{\perp\perp}$ *, hence* $\mathbf{D}^{\perp} = (\mathbf{P}^{\perp} \cap \mathbf{PI})^{\perp}$.

b) $\mathbf{D}^{\perp} \cap \mathbf{PI} \subseteq \mathbf{P}^{\perp\perp}$ *, hence* $\mathbf{P}^{\perp} = (\mathbf{D}^{\perp} \cap \mathbf{PI})^{\perp}$.

PROOF.

a) Let $\mathscr{X} \in \mathbf{P}^{\perp} \cap \mathbf{PI}$ and let $\mathscr{Y} \in \mathbf{D}^{\perp}$. We shall prove that $\mathscr{X} \perp \mathscr{Y}$. Let H and F be the Ellis groups of $\mathscr{X}$ and $\mathscr{Y}$ respectively. As $\mathscr{X} \in \mathbf{P}^{\perp}$ it follows from V.4.4. that there is a $K \in \mathscr{K}$ such that $\mathscr{X}$ is a factor of $\mathscr{A}_K$, and so that $A_K \subseteq H$. As $\mathscr{X}$ is a PI ttg it follows from III.4.4. that $G_\infty \subseteq H$; hence $A_K G_\infty \subseteq H$. By 2.12.a, we know that $FA_K G_\infty = G$. So $G = FA_K G_\infty \subseteq FH$, which shows that $G = FH$. Hence, by III.1.6., $\mathscr{X} \perp \mathscr{Y}$, and consequently $\mathbf{P}^{\perp} \cap \mathbf{PI} \subseteq \mathbf{D}^{\perp\perp}$.
Therefore, by 2.1., $\mathbf{D}^{\perp} \subseteq (\mathbf{P}^{\perp} \cap \mathbf{PI})^{\perp}$. On the other hand, $\mathbf{D} \subseteq \mathbf{P}^{\perp} \cap \mathbf{PI}$; so $(\mathbf{P}^{\perp} \cap \mathbf{PI})^{\perp} \subseteq \mathbf{D}^{\perp}$, which proves statement a.

b) Let $\mathscr{X} \in \mathbf{D}^{\perp} \cap \mathbf{PI}$ and let H be the Ellis group of $\mathscr{X}$. Then by III.4.4., $G_\infty \subseteq H$. Let $K \in \mathscr{K}$. As $\mathscr{X} \in \mathbf{D}^{\perp}$, it follows from 2.12.a that $HA_K G_\infty = G$. Since G_∞ is a normal subgroup, $G = HG_\infty A_K$, and as $G_\infty \subseteq H$, we have $G = HA_K$. But then, by the incontractibility of $\mathscr{A}_K$, it

follows from III.1.6. that $\mathfrak{X} \perp \mathfrak{A}_K$. As K was arbitrary, $\mathfrak{X} \in \mathbf{P}^{\perp\perp}$ and consequently, $\mathbf{D}^{\perp} \cap \mathbf{PI} \subseteq \mathbf{P}^{\perp\perp}$.
Therefore, by 2.1., $\mathbf{P}^{\perp} \subseteq (\mathbf{D}^{\perp} \cap \mathbf{PI})^{\perp}$. On the other hand, $\mathbf{P} \subseteq \mathbf{D}^{\perp} \cap \mathbf{PI}$; so $(\mathbf{D}^{\perp} \cap \mathbf{PI})^{\perp} \subseteq \mathbf{P}^{\perp}$. □

In case the Ellis group is a normal subgroup, or (stronger) if one of the ttgs is regular, we can generalize 2.13. slightly. For that purpose let $\mathbf{A}$ be the collection of factors of $\mathfrak{A}_0 = \mathfrak{W}\mathfrak{F}(a_0, \mathfrak{M})$ and note that $\mathbf{D} \subseteq \mathbf{A} \subseteq \mathbf{P}^{\perp}$.

2.14. **REMARK.** *Let* $\mathfrak{X}$ *be a minimal ttg with Ellis group* H .

a) *If* HA_0 *is a group then* $\mathfrak{X} \in \mathbf{D}^{\perp} \cap \mathbf{PI}^{\perp\perp}$ *implies* $\mathfrak{X} \in \mathbf{A}^{\perp}$.

b) *If* H *is a normal subgroup in* G *then* $\mathfrak{X} \in \mathbf{A} \cap \mathbf{PI}^{\perp\perp}$ *implies* $\mathfrak{X} \in \mathbf{D}^{\perp\perp}$.

c) *If* $\mathfrak{X}$ *is a factor of a regular incontractible minimal ttg* $\mathfrak{X}'$ *then* $\mathfrak{X} \in \mathbf{PI}^{\perp\perp} (\cap \mathbf{P}^{\perp})$ *implies* $\mathfrak{X} \in \mathbf{D}^{\perp\perp}$.

d) *If* $\mathfrak{X} \in \mathbf{D}^{\perp} \cap \mathbf{PI}^{\perp\perp}$ *and* $\mathfrak{Y} \in \mathbf{P}^{\perp}$ *with* $\mathfrak{Y}$ *regular, then* $\mathfrak{X} \perp \mathfrak{Y}$.

PROOF.

a) First note that the fact that HA_0 is a group implies that $a_0 \circ H$ is an MHP generator and $a_0 \circ H = u \circ J \circ HA_0$ (apply V.1.10.).
As $\mathfrak{X} \in \mathbf{D}^{\perp}$ we know, by 2.12.a, that $HA_0 G_\infty = G$. So

$$a_0 \circ H \circ u \circ G_\infty = u \circ J \circ HA_0 G_\infty = u \circ J \circ G = u \circ M = M \ .$$

Hence, by V.3.9.,

$$\mathfrak{W}\mathfrak{F}(a_0 \circ H, \mathfrak{M}) \perp \mathfrak{W}\mathfrak{F}(u \circ G_\infty, \mathfrak{M}) \ ;$$

i.e., $\mathfrak{W}\mathfrak{F}(a_0 \circ H, \mathfrak{M}) \in \mathbf{PI}^{\perp}$. As $\mathfrak{X} \in \mathbf{PI}^{\perp\perp}$, $\mathfrak{X} \perp \mathfrak{W}\mathfrak{F}(a_0 \circ H, \mathfrak{M})$. So, by III.1.6. and the incontractibility of $\mathfrak{W}\mathfrak{F}(a_0 \circ H, \mathfrak{M})$, we have $H.HA_0 = G$; hence $HA_0 = G$. The incontractibility of $\mathfrak{A}_0$ and III.1.6. imply that $\mathfrak{X} \perp \mathfrak{A}_0$; i.e., $\mathfrak{X} \in \mathbf{A}^{\perp}$.

b) Let $\mathfrak{Y} \in \mathbf{D}^{\perp}$ and let F be the Ellis group of $\mathfrak{Y}$. Let $[FA_0]$ be the $\mathfrak{T}(\mathfrak{M}, u)$-closed subgroup of G generated by FA_0 . Note that $a_0 \circ [FA_0]$ is an MHP generator (V.1.10.).
As $\mathfrak{Y} \in \mathbf{D}^{\perp}$, we know, by 2.12.a, that $FA_0 G_\infty = G$; so $[FA_0] G_\infty = G$. Hence

$$a_0 \circ [FA_0] \circ u \circ G_\infty = a_0 \circ [FA_0] G_\infty = a_0 \circ G = M \ .$$

This shows that $\mathfrak{W}\mathfrak{F}(a_0 \circ [FA_0], \mathfrak{M}) \in \mathbf{PI}^{\perp}$. As $\mathfrak{X} \in \mathbf{PI}^{\perp\perp}$ it follows that

$\mathcal{X} \perp \mathfrak{WF}(a_0 \circ [FA_0], \mathcal{M})$ and so, by III.1.6. and the incontractibility of $\mathfrak{WF}(a_0 \circ [FA_0], \mathcal{M})$, we have $H.[FA_0] = G$. But H is a normal subgroup, so (using III.2.11.)

$$G = H.[FA_0] = [HFA_0] = [FHA_0] .$$

As $\mathcal{X} \in \mathbf{A}$, $A_0 \subseteq H$; hence $G = [FH] = FH$ (H is a normal subgroup). By III.1.6. and the fact that $\mathcal{X} \in \mathbf{A} \subseteq \mathbf{P}^{\perp}$ it follows that $\mathcal{X} \perp \mathcal{Y}$.

c) Let $\mathcal{X}'$ be a regular incontractible minimal ttg such that $\mathcal{X}$ is a factor of $\mathcal{X}'$. By V.3.6.a and IV.4.18., we may assume $\mathcal{X}'$ to be an MHP ttg; say generated by an MHP generator C such that $C = u \circ C$ and $uC \subseteq H$. As $\mathcal{X}'$ is incontractible, we can find a $K \in \mathcal{K}$ such that $K \subseteq C \cap J$. Then $a_K \subseteq C$ and $A_K \subseteq H$. Let $\mathcal{Y} \in \mathbf{D}^{\perp}$ and let F be the Ellis group of $\mathcal{Y}$. Then by 2.12.a, $FA_K G_\infty = G$. As

$$FA_K G_\infty = G_\infty A_K F = A_K G_\infty F = A_K F G_\infty = G ,$$

we have

$$M = u \circ K \circ G = u \circ K \circ A_K F G_\infty = u \circ K \circ A_K \circ F \circ G_\infty \subseteq C \circ F \circ G_\infty ,$$

so $M = C \circ F \circ G_\infty$. As $\mathcal{X}'$ is regular, $C \circ F$ is an MHP generator (V.2.4.d, and compare it with the proof of V.3.5.). Hence, by V.3.9., it follows that $\mathfrak{WF}(C \circ F, \mathcal{M}) \in \mathbf{PI}^{\perp}$. By assumption, $\mathcal{X} \in \mathbf{PI}^{\perp\perp}$, so $\mathcal{X} \perp \mathfrak{WF}(C \circ F, \mathcal{M})$. After noting that $\mathfrak{WF}(C \circ F, \mathcal{M}) \in \mathbf{P}^{\perp}$ and that $\mathfrak{WF}(C \circ F, \mathcal{M})$ has Ellis group uCF, it follows from III.1.6. that $H.uCF = G$. But $uC \subseteq H$, so $HF = G$. As $\mathcal{X} \in \mathbf{P}^{\perp}$ it follows that $\mathcal{X} \perp \mathcal{Y}$.

d) Without loss of generality $\mathcal{Y}$ is an MHP ttg, say generated by an MHP generator D with $D = u \circ D$ and $a \circ D = Da$ for every $a \in G$. As $\mathcal{X} \in \mathbf{D}^{\perp}$, we have $A_K H G_\infty = G = H A_K G_\infty$ for every $K \in \mathcal{K}$. Hence $D \circ H \circ G_\infty = M$; and as $D \circ H$ is an MHP generator, we have $\mathfrak{WF}(D \circ H, \mathcal{M}) \in \mathbf{PI}^{\perp}$. By assumption, $\mathcal{X} \perp \mathfrak{WF}(D \circ H, \mathcal{M})$. As $\mathcal{Y} \in \mathbf{P}^{\perp}$, $\mathfrak{WF}(D \circ H, \mathcal{M}) \in \mathbf{P}^{\perp}$; so by III.1.6. and by the fact that uDH is the Ellis group of $\mathfrak{WF}(D \circ H, \mathcal{M})$, it follows that $uDH.H = G$, so $uDH = G$. Hence $D \circ H = M$ and by V.3.9., $\mathcal{Y}$ is disjoint from the maximal proximal extension of $\mathcal{X}$; so $\mathcal{Y} \perp \mathcal{X}$. □

Another consequence of 2.12.a is the following:

2.15. **REMARK.** $\mathbf{PI}^{\perp} = \mathbf{P}^{\perp} \cap \mathbf{D}^{\perp}$.

PROOF. We already know that $\mathbf{PI}^{\perp} \subseteq \mathbf{P}^{\perp} \cap \mathbf{D}^{\perp}$ (2.10.c).
Conversely, let $\mathcal{X} \in \mathbf{P}^{\perp} \cap \mathbf{D}^{\perp}$; then also $\mathcal{X}^* \in \mathbf{P}^{\perp} \cap \mathbf{D}^{\perp}$. Let C be an MHP generator with $C = u \circ C$ and $\mathcal{X}^* = \mathcal{C}$, say $C = KH$ with $K \subseteq J$ and $H = uC$. As $\mathcal{X}^* \in \mathbf{P}^{\perp}$, $C \circ G = u \circ K \circ G = M$; and as $\mathcal{X}^* \in \mathbf{D}^{\perp}$, $HA_K G_{\infty} = G$. But $A_K \subseteq H$, so $HG_{\infty} = G$ and consequently, $M = C \circ G = C \circ H \circ G_{\infty} = C \circ G_{\infty}$, i.e., $\mathcal{X}^* \in \mathbf{PI}^{\perp}$. □

VI.3. CLASSES AND EXTENSIONS

> We continue the study of the relations between disjointness classes. But now we take a slightly different point of view. Let $\phi : \mathcal{X} \to \mathcal{Y}$ be a homomorphism of minimal ttgs, when is every minimal ttg which is disjoint from $\mathcal{Y}$ disjoint from $\mathcal{X}$ too?

The following is a variation on I.4.1. (for a stronger version see VII.4.9.).

3.1. **LEMMA.** *Consider the next commutative diagram of homomorphisms of minimal ttgs.*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\psi"] \arrow[d, "\phi"'] & \mathcal{Z} \arrow[d, "\eta"] \\
\mathcal{Y} \arrow[r, "\xi"'] \arrow[ur, dashed, "\theta"] & \mathcal{W}
\end{tikzcd}$$

Let η *be distal and* ϕ *weakly mixing. If* $\mathcal{Z}$ *is metric or if* $\mathcal{W} = \{\star\}$ *then there is a homomorphism* $\theta : \mathcal{Y} \to \mathcal{Z}$ *such that the diagram commutes.*

PROOF. As R_{ϕ} is ergodic, $\psi \times \psi[R_{\phi}]$ is ergodic and, clearly, $\psi \times \psi[R_{\phi}] \subseteq R_{\eta}$.
If $\mathcal{W} = \{\star\}$, $R_{\eta} = Z \times Z$ and $\psi \times \psi[R_{\phi}]$ is distal. Hence, by I.1.17., $\psi \times \psi[R_{\phi}]$ is minimal.
If $\mathcal{Z}$ is metric, $\psi \times \psi[R_{\phi}]$ is metric, hence point transitive (I.1.2.b). As R_{η} is pointwise almost periodic, $\psi \times \psi[R_{\phi}]$ is pointwise almost periodic, hence minimal.

Clearly, $\Delta_Z = \psi \times \psi[\Delta_X] \subseteq \psi \times \psi[R_\phi]$, so (in both cases) $\Delta_Z = \psi \times \psi[R_\phi]$ and $R_\phi \subseteq (\psi \times \psi)^{\leftarrow}[\Delta_Z] = R_\psi$. But then there is a homomorphism

$$\theta: \mathcal{Y} \cong \mathcal{X}/R_\phi \to \mathcal{Z} \cong \mathcal{X}/R_\psi .$$

□

3.2. **COROLLARY.** *Let* $\phi: \mathcal{X} \to \mathcal{Y}$ *be a weakly mixing homomorphism of minimal ttgs. Then* $\mathcal{Y} \in \mathbf{D}^\perp$ *iff* $\mathcal{X} \in \mathbf{D}^\perp$.

PROOF. If $\mathcal{X} \in \mathbf{D}^\perp$ then clearly, $\mathcal{Y} \in \mathbf{D}^\perp$.
Conversely, suppose that $\mathcal{Y} \in \mathbf{D}^\perp$ and let $\mathcal{Z}$ be a distal factor of $\mathcal{X}$. Then by 3.1., $\mathcal{Z}$ is a factor of $\mathcal{Y}$. Hence, by 2.6. and the fact that $\mathcal{Y} \in \mathbf{D}^\perp$, it follows that $\mathcal{Z}$ is trivial. So by 2.6., $\mathcal{X} \in \mathbf{D}^\perp$. □

For a minimal ttg $\mathcal{X}$ we shall denote $\{\mathcal{X}\}^\perp$ by $\mathcal{X}^\perp$.

3.3. **THEOREM.** *Let* $\phi: \mathcal{X} \to \mathcal{Y}$ *be a distal extension of minimal ttgs.*

a) *If* $\mathcal{X} \in \mathbf{D}^\perp$ *then* $\mathcal{X}^\perp = \mathcal{Y}^\perp$.

b) $\mathbf{D}^\perp \cap \mathcal{X}^\perp = \mathbf{D}^\perp \cap \mathcal{Y}^\perp$.

PROOF. In both cases the inclusion " $\subseteq$ " is obvious.
Let $\mathcal{Z}$ be a minimal ttg with $\mathcal{Z} \in \mathcal{Y}^\perp$ and suppose that $\mathcal{Z} \notin \mathcal{X}^\perp$. Without loss of generality we may assume that $\mathcal{Z}$ is an MHP ttg (IV.4.18.). By 1.7., there is a nontrivial quasifactor $\mathcal{W}$ of $\mathcal{X}$ which is a factor of $\mathcal{Z}$. As $\mathcal{Z} \in \mathcal{Y}^\perp$, also $\mathcal{W} \in \mathcal{Y}^\perp$. Hence, by 1.6.a, $\mathcal{W}$ is distal.

a) If $\mathcal{X} \in \mathbf{D}^\perp$ then $\mathcal{X} \perp \mathcal{W}$, which contradicts 1.1..

b) If $\mathcal{Z} \in \mathbf{D}^\perp$ then $\mathcal{W} \in \mathbf{D}^\perp$, contradicting the distallity of $\mathcal{W}$. □

3.4. **COROLLARY.** *Let* $\phi: \mathcal{X} \to \mathcal{Y}$ *be an HPI extension of minimal ttgs.*

a) *If* $\mathcal{X} \in \mathbf{D}^\perp$ *then* $\mathcal{X}^\perp = \mathcal{Y}^\perp$.

b) $\mathbf{D}^\perp \cap \mathcal{X}^\perp = \mathbf{D}^\perp \cap \mathcal{Y}^\perp$.

PROOF. Without loss of generality we assume that $\mathcal{X}$ and $\mathcal{Y}$ are MHP ttgs (IV.4.18., IV.5.1.). By IV.5.2., ϕ is strictly-HPI. Applying 3.3. to the almost periodic steps in the strictly-HPI tower for ϕ , IV.4.18. to the hp steps and I.3.1.b to the inverse limits, the corollary follows. □

3.5. **THEOREM.** *Let* $\phi: \mathcal{X} \to \mathcal{Y}$ *be a proximal extension of minimal ttgs.*

a) *If* $\mathcal{X} \in \mathbf{P}^\perp$ *then* $\mathcal{X}^\perp = \mathcal{Y}^\perp$.

b) $\mathbf{P}^\perp \cap \mathcal{X}^\perp = \mathbf{P}^\perp \cap \mathcal{Y}^\perp$.

PROOF. Clearly, $\mathscr{X}^{\perp} \subseteq \mathscr{Y}^{\perp}$. Let $\mathscr{Z}$ be a minimal ttg with $\mathscr{Z} \perp \mathscr{Y}$ such that $\mathscr{Z} \not\perp \mathscr{X}$ and without los of generality we may assume that $\mathscr{Z} = \mathscr{Z}^{*}$. Then, by 1.7., there exists a nontrivial quasifactor $\mathscr{W}$ of $\mathscr{X}$ which is a factor of $\mathscr{Z}$. As $\mathscr{Z} \perp \mathscr{Y}$ also $\mathscr{W} \perp \mathscr{Y}$.

a) If $\mathscr{X} \in \mathbf{P}^{\perp}$ then by 1.4.b, $\mathscr{W} \not\perp \mathscr{Y}$ which is a contradiction.

b) If $\mathscr{Z} \in \mathbf{P}^{\perp}$ then $\mathscr{W} \in \mathbf{P}^{\perp}$; hence, again by 1.4.b, $\mathscr{W} \not\perp \mathscr{Y}$. □

The proof of the next theorem is not similar to the proof of 3.4., although such seems to be logical at first sight. The reason is that we do not know whether for an incontractible ttg $\mathscr{X}$ and a PI extension $\phi : \mathscr{X} \to \mathscr{Y}$ there is a strictly-PI tower in $\mathbf{P}^{\perp}$ that factorizes over ϕ , which is necessary for application of 1.4..

3.6. **THEOREM.** *Let* $\phi : \mathscr{X} \to \mathscr{Y}$ *be a PI extension of minimal ttgs.*

a) *If* $\mathscr{X} \in \mathbf{PI}^{\perp}$ *then* $\mathscr{X}^{\perp} = \mathscr{Y}^{\perp}$.

b) $\mathbf{PI}^{\perp} \cap \mathscr{X}^{\perp} = \mathbf{PI}^{\perp} \cap \mathscr{Y}^{\perp}$.

PROOF. Let H and F be the Ellis groups of $\mathscr{X}$ and $\mathscr{Y}$ with respect to some $x_0 \in uX$ and $\phi(y_0) \in uY$ respectively. Remember that ϕ is a PI extension iff $F_\infty \subseteq H$ (III.4.4.); and note that always $\mathscr{X}^{\perp} \subseteq \mathscr{Y}^{\perp}$.

a) Let $\mathscr{Z} \in \mathscr{Y}^{\perp}$, and let L be the Ellis group of $\mathscr{Z}$. As $\mathscr{X} \in \mathbf{PI}^{\perp}$, clearly, $\mathscr{Y} \in \mathbf{PI}^{\perp} \subseteq \mathbf{P}^{\perp}$. Hence, by III.1.6., $LF = G$; so, by III.2.13.b, $LF_\infty = LG_\infty$. As $F_\infty \subseteq H$ we have $LG_\infty = LF_\infty \subseteq LH$ and so

$$LH = LuH \subseteq LG_\infty H \subseteq LHH = LH \ ; \text{ i.e., } LH = LG_\infty H \ .$$

Since $\mathscr{X} \in \mathbf{PI}^{\perp}$, also $\mathscr{X} \in \mathbf{P}^{\perp}$ and $\mathscr{X} \perp \mathscr{WF}(u \circ G_\infty , \mathscr{M})$; hence, by III.1.6., $HG_\infty = G_\infty H = G$. But then $LH = LG_\infty H = LG = G$. By III.1.6. and the incontractibility of $\mathscr{X}$, it follows that $\mathscr{X} \perp \mathscr{Z}$.

b) Let $\mathscr{Z}$ be a minimal ttg with Ellis group L such that $\mathscr{Z} \in \mathbf{PI}^{\perp}$ and $\mathscr{Z} \perp \mathscr{Y}$. Then $LF = G$ and so $LF_\infty = LG_\infty$. As $\mathscr{Z} \in \mathbf{PI}^{\perp}$, we have $LG_\infty = G$ so $G = LG_\infty = LF_\infty \subseteq LH$. Since $\mathscr{Z} \in \mathbf{P}^{\perp}$ it follows that $\mathscr{Z} \perp \mathscr{X}$. □

The next corollary is in the same spirit as 1.4. and 1.6..

3.7. **COROLLARY.** *Let* $\phi : \mathscr{X} \to \mathscr{Y}$ *be a homomorphism of minimal ttgs and let* $\mathscr{Z}$ *be a nontrivial quasifactor of* $\mathscr{X}$.

a) *If* $\mathscr{X} \in \mathbf{D}^{\perp}$ *and if* ϕ *is an* HPI *extension then* $\mathscr{Z} \not\perp \mathscr{Y}$.

b) *If* $\mathscr{X} \in \mathbf{PI}^{\perp}$ *and if* ϕ *is a* PI *extension then* $\mathscr{Z} \not\perp \mathscr{Y}$.

PROOF. Follows immediately from 3.4.a, 3.6.a and 1.1.. □

We shall now give a variation on 3.4. through 3.6., dealing with classes rather then with ttgs.

3.8. **THEOREM.** *Let* $\mathbf{K}$ *be a set of minimal ttgs.*

a) *If* $\mathbf{K} \subseteq \mathbf{D}^{\perp}$ *then* $\mathbf{K}^{\perp}$ *is closed under* HPI *extensions.*

b) *If* $\mathbf{K} \subseteq \mathbf{P}^{\perp}$ *then* $\mathbf{K}^{\perp}$ *is closed under proximal extensions.*

c) *If* $\mathbf{K} \subseteq \mathbf{PI}^{\perp}$ *then* $\mathbf{K}^{\perp}$ *is closed under* PI *extensions.*

PROOF.

a) Let $\mathcal{Y} \in \mathbf{K}^{\perp}$ and let $\phi: \mathcal{X} \to \mathcal{Y}$ be a HPI extension of minimal ttgs. Then $\mathbf{K} \subseteq \mathcal{Y}^{\perp} \cap \mathbf{D}^{\perp}$ and by 3.4.b,

$$\mathbf{K} \subseteq \mathcal{X}^{\perp} \cap \mathbf{D}^{\perp} = \mathcal{Y}^{\perp} \cap \mathbf{D}^{\perp} ,$$

hence $\mathcal{X} \in \mathbf{K}^{\perp}$. So $\mathbf{K}^{\perp}$ is closed under HPI extensions.

b and c are proven similarly using 3.5.b and 3.6.b instead of 3.4.b. □

3.9. **EXAMPLES.**

a) $\mathbf{P}^{\perp}$, $\mathbf{WM}^{\perp}$ *and* $\mathbf{D}^{\perp\perp}$ *are closed under* HPI *extensions.*

b) $\mathbf{D}^{\perp}$, $\mathbf{WM}^{\perp\perp}$ *and* $\mathbf{P}^{\perp\perp}$ *are closed under proximal extensions.*

c) $\mathbf{PI}^{\perp\perp}$ *is closed under* PI *extensions.*

3.10. **THEOREM.** *Let* $\mathbf{K}$ *be a set of minimal ttgs.*

a) $\mathbf{K}^{\perp}$ *is closed under* HPI *extensions within* $\mathbf{D}^{\perp}$. *(i.e., suppose that* $\mathcal{Y} \in \mathbf{K}^{\perp}$ *, let* $\phi: \mathcal{X} \to \mathcal{Y}$ *be an* HPI *extension of minimal ttgs and let* $\mathcal{X} \in \mathbf{D}^{\perp}$ *then* $\mathcal{X} \in \mathbf{K}^{\perp}$ *).*

b) $\mathbf{K}^{\perp}$ *is closed under proximal extensions within* $\mathbf{P}^{\perp}$.

c) $\mathbf{K}^{\perp}$ *is closed under* PI *extensions within* $\mathbf{PI}^{\perp}$.

PROOF.

a) Let $\mathcal{Y} \in \mathbf{K}^{\perp}$ and let $\phi: \mathcal{X} \to \mathcal{Y}$ be an HPI extension of minimal ttgs. If $\mathcal{X} \in \mathbf{D}^{\perp}$ then by 3.4.a, $\mathcal{X}^{\perp} = \mathcal{Y}^{\perp}$. As $\mathcal{Y} \in \mathbf{K}^{\perp}$, we have $\mathbf{K} \subseteq \mathcal{Y}^{\perp} = \mathcal{X}^{\perp}$ and so $\mathcal{X} \in \mathbf{K}^{\perp}$.

b and c are proven similarly. □

3.11. **EXAMPLES.**

a) $\mathbf{P}^{\perp\perp}$ *and* $\mathbf{WM}^{\perp\perp}$ *are closed under* HPI *extensions within* $\mathbf{D}^{\perp}$ *; hence, by* 3.9.b, *they are closed under* PI *extensions within* $\mathbf{D}^{\perp}$.

b) $\mathbf{D}^{\perp\perp}$ *and* $\mathbf{WM}^{\perp}$ *are closed under proximal extensions within* $\mathbf{P}^{\perp}$ *; so, by* 3.9.a, *they are closed under* PI *extensions within* $\mathbf{P}^{\perp}$.

3.12. **COROLLARY.**

a) $\mathbf{D}^{\perp} \cap \mathbf{PI} = \mathbf{P}^{\perp\perp} \cap \mathbf{PI} = \mathbf{WM}^{\perp\perp} \cap \mathbf{PI}$.

b) $\mathbf{P}^{\perp} \cap \mathbf{PI} = \mathbf{D}^{\perp\perp} \cap \mathbf{PI} = \mathbf{WM}^{\perp} \cap \mathbf{PI}$.

PROOF. Follows from 2.10. and 3.11. (and also from 2.10. and 2.13.). □

3.13. **REMARK.** *In case T does not admit nontrivial proximal minimal ttgs (T strongly amenable) the following relations hold:*

a) $\mathbf{D}^{\perp} = \mathbf{PI}^{\perp} = \mathbf{WM} = \mathbf{WM}^{\perp\perp}$*;*

b) $\mathbf{PI} \subseteq \mathbf{D}^{\perp\perp} = \mathbf{PI}^{\perp\perp} = \mathbf{WM}^{\perp}$.

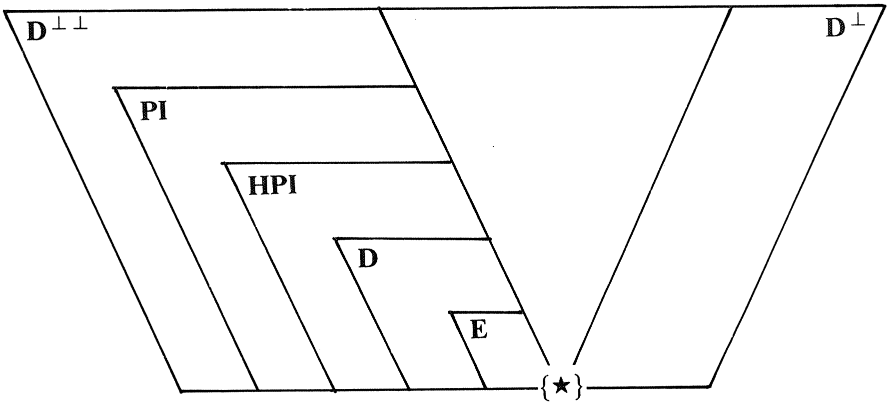
PROOF. As T is strongly amenable, every T-minimal ttg is in $\mathbf{P}^{\perp}$. So, by 2.10. and 2.13.a,

$$\mathbf{WM} = \mathbf{WM} \cap \mathbf{P}^{\perp} = \mathbf{P}^{\perp} \cap \mathbf{D}^{\perp} = \mathbf{D}^{\perp} = \mathbf{PI}^{\perp} ,$$

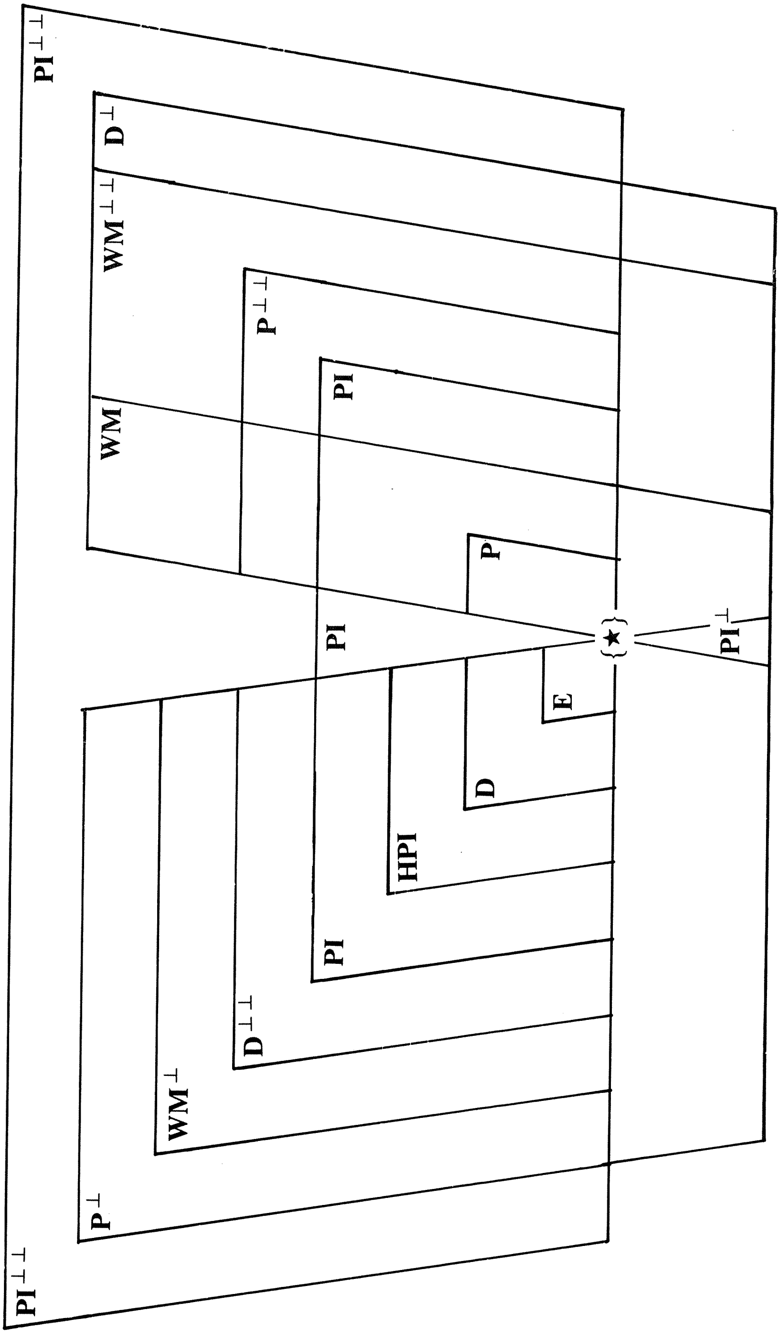
hence $\mathbf{D}^{\perp} = \mathbf{WM} \subseteq \mathbf{WM}^{\perp\perp} \subseteq \mathbf{D}^{\perp}$. But then $\mathbf{WM}^{\perp} = \mathbf{D}^{\perp\perp} = \mathbf{PI}^{\perp\perp}$. The inclusion $\mathbf{PI} \subseteq \mathbf{PI}^{\perp\perp} = \mathbf{D}^{\perp\perp}$ is trivial. □

Note that $\mathbf{D}^{\perp\perp} \neq \mathbf{PI}$ (see [G 80]).

In the following pictures we recapitulate the results of section 2. and 3. in the absolute case. First the case that T is strongly amenable:



T arbitrary:


$PI^{\perp\perp}$
$D^{\perp}$
$WM^{\perp\perp}$
$P^{\perp\perp}$
PI
WM
P
PI
$\{\star\}$
$PI^{\perp}$
E
D
HPI
PI
$D^{\perp\perp}$
$WM^{\perp}$
$P^{\perp}$
$PI^{\perp\perp}$

VI.4. DISJOINTNESS AND RELATIVE PRIMENESS

It is well known and easy to see that two disjoint ttgs are *relatively prime* (i.e., do not admit nontrivial common factors). In [F 67] the question is raised whether or not relative primeness is sufficient to imply disjointness. It turns out that even in the case that T is abelian the answer has to be in the negative [GW ?].
In this section we shall deal with the problem to what extent disjointness is implied by not having a common distal factor.

As we did in section 3., we shall use the notions introduced in V.4. without further notice.
If in the sequel we attach an Ellis group H to a minimal ttg $\mathcal{X}$, then we mean that there exists an $x \in uX$ such that $H = uM_x = \mathfrak{G}(\mathcal{X}, x)$.
Let E be the Ellis group of $\mathcal{E}$, the universal minimal uniformly almost periodic ttg for T (cf V.4.10.).
First we shall pay attention to the property of having a common distal factor.

4.1. **THEOREM.** *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be minimal ttgs with Ellis groups* H *and* F . *Suppose that* HFE *is a group. Then the following statements are equivalent:*

a) $\mathcal{X}$ *and* $\mathcal{Y}$ *have a nontrivial common distal factor;*
b) $HFE \neq G$;
c) $HFA_0\mathrm{H}(G) \neq G$;
d) $[HF]A_0G_\infty \neq G$;
e) $[HF]A_KG_\infty \neq G$ *for every* $K \in \mathcal{K}$.

Here $[HF]$ *denotes the* $\mathfrak{F}(\mathcal{M},u)$*-closed subgroup of* G *generated by* HF .

PROOF. First note that, by V.4.10., $E = A_0\mathrm{H}(G) = A_K\mathrm{H}(G)$ for every $K \in \mathcal{K}$. This shows the equivalence of b and c.
As HFE is a group, $HFE = [HF]E$; so

$$HFE = [HF]A_0\mathrm{H}(G) = [HF]A_K\mathrm{H}(G) \text{ for every } K \in \mathcal{K}.$$

Hence, by III.2.13.c,

$$HFE \neq G \text{ iff } [HF]A_0G_\infty \neq G \text{ iff } [HF]A_KG_\infty \neq G \text{ for every } K \in \mathcal{K}.$$

This reduces the proof of the theorem to showing the equivalence of a and b.

b $\Rightarrow$ a Let $L := HFE$. As L is a group, L is the Ellis group of $\mathfrak{A}(L)$. By III.1.15., $\mathfrak{A}(L)$ is a factor of $\mathfrak{A}(E)$. As $E_{\mathfrak{A}(E)} = P_{\mathfrak{A}(E)}$ it follows from I.4.3. that $E_{\mathfrak{A}(L)} = P_{\mathfrak{A}(L)}$; so $\mathfrak{A}(L)$ has a uniformly almost periodic factor $\mathfrak{Z}$ with Ellis group L . By the assumption of $L \neq G$, $\mathfrak{Z}$ is nontrivial. As $\mathfrak{A}(L)$ is a factor of both $\mathfrak{A}(H)$ and $\mathfrak{A}(F)$ (III.1.15.), it follows from I.4.1. that $\mathfrak{Z}$ is a common factor of $\mathfrak{X}$ and $\mathfrak{Y}$.

a $\Rightarrow$ b If $\mathfrak{X}$ and $\mathfrak{Y}$ have a nontrivial common distal factor, it follows from I.1.25. that $\mathfrak{X}$ and $\mathfrak{Y}$ have a nontrivial common uniformly almost periodic factor $\mathfrak{Z}$. Let N be the Ellis group of $\mathfrak{Z}$ such that $F \subseteq N$. As $\mathfrak{Z}$ is a factor of $\mathfrak{E}$, also $E \subseteq N$. Since $\mathfrak{Z}$ is a factor of $\mathfrak{X}$, there is a $g \in G$ such that $gHg^{-1} \subseteq N$. Hence $gHg^{-1}FE \subseteq N$.
Suppose $HFE = G$. As FE is a group (I.2.17. and I.2.15.), it follows from I.3.3.b that $Hg^{-1}FE = G$. But then $gHg^{-1}FE = G$ and $N = G$, which contradicts the nontriviality of $\mathfrak{Z}$. ☐

4.2. **THEOREM.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be minimal ttgs with Ellis groups H and F and suppose that HFE is a group. If $\mathfrak{X}$ or $\mathfrak{Y}$ is incontractible then the following statements are equivalent:*

a) *$\mathfrak{X}$ and $\mathfrak{Y}$ have a nontrivial common distal factor;*
b) $HF\mathrm{H}(G) \neq G$;
c) $HFG_\infty \neq G$.

PROOF. The equivalence of b and c is just III.2.13.c.
By the equivalence of 4.1.a and 4.1.b it is sufficient to prove that $HFE = HF\mathrm{H}(G)$. As follows:
Without loss of generality let $\mathfrak{X} \in \mathbf{P}^{\perp}$. Then, for some $K \in \mathcal{K}$, $A_K \subseteq H$; and so $H = HA_K$. By V.4.10., we have

$$H\mathrm{H}(G) = HA_K\mathrm{H}(G) = HE \ .$$

Hence, by normality of E and $\mathrm{H}(G)$,

$$HF\mathrm{H}(G) = H\mathrm{H}(G)F = HEF = HFE \ .$$

☐

4.3. **REMARK.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be minimal ttgs with Ellis groups H and F . In each of the following cases HFE is a group:*

a) *HF is a group;*
b) *$\mathfrak{X}$ or $\mathfrak{Y}$ has a regular maximal uniformly almost periodic factor;*
c) $\mathfrak{X}/E_{\mathfrak{X}} \perp \mathfrak{Y}/E_{\mathfrak{Y}}$.

PROOF.

a) If HF is a group it follows from the normality of E that HFE is a group.

b) By III.3.13. and I.2.15., we have that HE or FE is a normal subgroup, hence $HFE\ (=HEF)$ is a group.

c) As $\mathfrak{X}/E_{\mathfrak{X}} \in \mathbf{P}^{\perp}$ it follows from III.1.6. and III.3.13. that $HEFE = G$, so $HFE = G$ and HFE is a group. □

4.4. **COROLLARY.** *Let T be an abelian group. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be minimal ttgs for T with Ellis groups H and F. Then the following statements are equivalent:*

a) *$\mathfrak{X}$ and $\mathfrak{Y}$ have a nontrivial common distal factor;*
b) *$HF\mathrm{H}(G) \neq G$;*
c) *$HFG_{\infty} \neq G$.*

PROOF. Follows from 4.2., 4.3.b and I.2.16.. □

Now we turn to the problem to what extent disjointness is implied by relative primeness.

4.5. **THEOREM.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be minimal ttgs with Ellis groups H and F such that HF is a group and suppose that $\mathfrak{X} \in \mathbf{D}^{\perp\perp}$. Then $\mathfrak{X} \perp \mathfrak{Y}$ iff $\mathfrak{X}$ and $\mathfrak{Y}$ are relatively prime.*

PROOF. Clearly, the "only if"-part is true.
Suppose that $\mathfrak{X}$ and $\mathfrak{Y}$ are relatively prime. Then $\mathfrak{X}$ and $\mathfrak{Y}$ do not have a nontrivial common distal factor. So, by 4.3.a and 4.1., it follows that $HFE = G$. As $\mathfrak{G} \in \mathbf{P}^{\perp}$ and as HF is the Ellis group of $\mathfrak{A}(HF)$ it follows from III.1.6. that $\mathfrak{A}(HF) \perp \mathfrak{G}$; in other words, $\mathfrak{A}(HF) \in \mathbf{D}^{\perp} = \mathbf{E}^{\perp}$. As $\mathfrak{X} \in \mathbf{D}^{\perp\perp}$, $\mathfrak{X} \perp \mathfrak{A}(HF)$. Hence, by III.1.6. and the incontractibility of $\mathfrak{X}$ ($\mathbf{D}^{\perp\perp} \subseteq \mathbf{P}^{\perp}$) we have $H.HF = G$. So $HF = G$; and again by III.1.6. and the incontractibility of $\mathfrak{X}$, it follows that $\mathfrak{X} \perp \mathfrak{Y}$. □

The next theorem slightly generalizes [EGS 76] 4.3..

4.6. **THEOREM.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be minimal ttgs with Ellis groups H and F. Let $\mathfrak{X} \in \mathbf{P}^{\perp}$ and assume that HFE is a group (e.g. T abelian). If $G_{\infty} \subseteq HF$, then $\mathfrak{X} \perp \mathfrak{Y}$ iff $\mathfrak{X}$ and $\mathfrak{Y}$ are relatively prime.*

PROOF. Clearly the "only if"-part is true.
Suppose that $\mathfrak{X}$ and $\mathfrak{Y}$ are relatively prime. Then $\mathfrak{X}$ and $\mathfrak{Y}$ do not have a nontrivial common distal factor. As $\mathfrak{X} \in \mathbf{P}^{\perp}$ it follows from 4.2. that $HFG_{\infty} = G$. Since G_{∞} is normal in G, $G = HFG_{\infty} = HG_{\infty}F$. But $G_{\infty} \subseteq HF$; so

$$G = HG_\infty F \subseteq H.HF.F = HF\ .$$

Hence, by III.1.6., $\mathcal{X} \perp \mathcal{Y}$. □

4.7. **REMARK.** *Let* H *and* F *be* $\mathfrak{F}(\mathcal{M},u)$*-closed subgroups of* G *such that* $G_\infty \subseteq HF$. *Assume that* H *is the Ellis group of some incontractible minimal ttg and assume that* $\mathfrak{A}(H)$ *or* $\mathfrak{A}(F)$ *has a regular maximal uniformly almost periodic factor (those assumptions are satisfied if* T *is abelian). Then* $[HF] = G$ *implies* $HF = G$, *where* $[HF]$ *is the* $\mathfrak{F}(\mathcal{M},u)$*-closed subgroup of* G *generated by* HF (compare [E 81] 1.1.1.).

PROOF. If $[HF] = G$ then $[HF]E = G$. As by the assumption (and by 4.3.) it follows that HFE is a group, we have $HFE = G$. As H is the Ellis group of an incontractible minimal ttg, $HFE = HF\mathrm{H}(G)$. So, by III.2.13.c, $HFG_\infty = G$. By normality of G_∞ , $HFG_\infty = HG_\infty F = G$. Since $G_\infty \subseteq HF$, $G = HFG_\infty = HG_\infty F \subseteq HHFF = HF$. □

4.8. **THEOREM.** *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be minimal ttgs with Ellis groups* H *and* F . *Assume* $\mathcal{X}$ *to be incontractible and regular. If* $\mathcal{X}$ *or* $\mathcal{Y}$ *is in* $\mathbf{PI}^{\perp\perp}$, *then* $\mathcal{X} \perp \mathcal{Y}$ *iff* $\mathcal{X}$ *and* $\mathcal{Y}$ *are relatively prime.*

PROOF. Clearly the "only if"-part is true.
Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are relatively prime, then they do not have a non-trivial common distal factor. So by 4.2., $HFG_\infty = G$.
Let C be an MHP generator with $C = u \circ C$ and $uC \subseteq H$, such that $\mathcal{X}^* = \mathcal{C}$. By V.3.6.a, $\mathcal{X}^*$ is regular; so $C \circ F$ is an MHP generator. By IV.4.17., $\mathcal{X}^*$ is incontractible; so $\mathfrak{W}\mathfrak{F}(C \circ F, \mathcal{M})$ as a factor of $\mathcal{X}^*$ is incontractible. Note that $HF = u(C \circ F)$ is the Ellis group of the ttg $\mathfrak{W}\mathfrak{F}(C \circ F, \mathcal{M})$.
By III.1.6., it follows from $HFG_\infty = G$ that

$$\mathfrak{W}\mathfrak{F}(C \circ F, \mathcal{M}) \perp \mathfrak{W}\mathfrak{F}(u \circ G_\infty, \mathcal{M})\,.$$

As $\mathfrak{W}\mathfrak{F}(u \circ G_\infty, \mathcal{M})$ is the universal PI ttg, we have $\mathfrak{W}\mathfrak{F}(C \circ F, \mathcal{M}) \in \mathbf{PI}^{\perp}$. By assumption, $\mathcal{X}$ or $\mathcal{Y}$ is in $\mathbf{PI}^{\perp\perp}$, so $\mathfrak{W}\mathfrak{F}(C \circ F, \mathcal{M})$ is disjoint from $\mathcal{X}$ or $\mathcal{Y}$. By III.1.6. and the incontractibility of $\mathfrak{W}\mathfrak{F}(C \circ F, \mathcal{M})$ it follows that $H.HF = G$ or $F.HF = G$. In both cases, $HF = G$; hence $\mathcal{X} \perp \mathcal{Y}$□

VI.5. REMARKS

5.1. The role of quasifactors in disjointness problems is slightly touched at in [G 75] and more in [AG 77] (e.g. Theorem II.2. which was in fact the starting point for the study presented in this chapter). But there does not seem to be a detailed study in the literature except for [Wo 79.1]. In that paper a proof of 1.6.a is given, which is striking because of its length rather than its cleverness; so we replaced it by the proof J. AUSLANDER gave by proving 1.5..

QUESTIONS

a) (See 1.4. and 1.6.) Let $\phi: \mathscr{X} \to \mathscr{Y}$ be a proximal extension of minimal ttgs and let $\mathfrak{Z}$ be a nontrivial quasifactor of $\mathscr{X}$, such that $\mathfrak{Z} \perp \mathscr{Y}$. Is $\mathfrak{Z}$ proximal? is $\mathfrak{Z} \in \mathbf{P}^{\perp\perp}$?

b) Suppose $\phi: \mathscr{X} \to \mathscr{Y}$ is weakly mixing. Can we formulate a theorem in the spirit of 1.6.?

c) Let $\mathfrak{Z}$ be a nontrivial quasifactor of $\mathscr{X}$. When is $\mathfrak{Z} \in \mathscr{X}^{\perp\perp}$?
Note that the following statements are equivalent:
(i) $\mathscr{X}^{\perp} \subseteq \mathfrak{Z}^{\perp}$ for every quasifactor $\mathfrak{Z}$ of $\mathscr{X}$;
(ii) $\mathfrak{Z} \in \mathscr{X}^{\perp\perp}$ for every quasifactor $\mathfrak{Z}$ of $\mathscr{X}$;
(iii) $\mathscr{X} \not\perp \mathscr{Y}$ for every quasifactor $\mathscr{Y}$ of any quasifactor $\mathfrak{Z}$ of $\mathscr{X}$.

5.2. Disjointness classes (as studied in VI.2.) like $\mathbf{D}^{\perp}$, $\mathbf{WM}^{\perp}$, $\mathbf{E}^{\perp}$ and $\mathbf{PI}^{\perp}$ are treated in former papers [K 71], [Pe 73] and [S 71]. In those papers there are many restrictions on the ttgs. For instance [K 71] deals with strictly-quasi separable minimal ttgs for an abelian group T. In [Pe 70] it is proved that $\mathbf{D}^{\perp} = \mathbf{WM}$ for an abelian group T (cf. 3.13.); and [S 71] deals mainly with metric minimal ttgs.
However, since the deep results in [E 78] and [V 77] many of those restrictions became superfluous. Hence many results in VI.2. (and VI.3.) are generalizations of known results for special cases. Note that VI.2.8. was already in [AG 77].
In 5.5. and 5.6. below we shall look at some questions that could arise with respect to section III.2., namely the characterization of elements of [E] and the characterization of ttgs without proximal factors.

QUESTIONS

Are the following equations true?

a) $\mathbf{WM}^{\perp\perp} = \mathbf{D}^{\perp}$;

b) $\mathbf{PI}^{\perp\perp} \cap \mathbf{D}^{\perp} = \mathbf{P}^{\perp\perp}$;

c) $\mathbf{PI}^{\perp\perp} \cap \mathbf{D}^{\perp} = \mathbf{D}^{\perp\perp}$.

5.3. Questions about extensions and disjointness were formerly studied in [S 71], [W 74] and [AG 77], but none of the results mentioned in VI.3. seems to be in the literature (at least in the generality we give).

QUESTION

Let $\phi : \mathcal{X} \to \mathcal{Y}$ be weakly mixing. Do the following statements hold true? (compare 5.1.b, 3.4., 3.5. and 3.6.)

a) If $\mathcal{X} \in \mathbf{WM}^{\perp}$ then $\mathcal{X}^{\perp} = \mathcal{Y}^{\perp}$.

b) $\mathbf{WM}^{\perp} \cap \mathcal{X}^{\perp} = \mathbf{WM}^{\perp} \cap \mathcal{Y}^{\perp}$.

5.4. In the literature several times the question is considered whether or not relative primeness implies disjointness, and partial results are obtained ([K 71], [E 69], [P 72], [K 72], [EGS 76]). An example by A.W. KNAPP [Kn 68] shows that for uniformly almost periodic minimal ttgs one can construct counter examples (see [E 69] 18.11.); more counterexamples (even for $\mathbb{Z}$) can be found in [GW ?]. For a compilation of the known results see [B 75/79] section 3.19.. Many of the partial results obtained in the papers mentioned above are special cases of the results in our section 4.. The one that comes close to our result 4.6. is [EGS 76] 4.2., where minimal ttgs are considered such that the u-invariant part is T-invariant ($TuX = uX$ for some idempotent $u \in J$).

Note that the problem whether or not disjointness is implied by relative primeness can be restated for MHP ttgs as follows:
Let C and D be MHP generators with $C = u \circ C$ and $D = u \circ D$. Under what condition does $[C \cup D] = M$ imply $C \circ D = M$, where $[C \cup D]$ is the smallest MHP generator that contains both C and D .

A question we ran into implicitly in sections 2. and 4. is the following:
Let L be an Ellis group and let $\mathfrak{a}_K$ be an MHP generator as in V.4. and let $[\mathfrak{a}_K \cup u \circ L]$ be the smallest MHP generator that contains $\mathfrak{a}_K$ and $u \circ L$. Then $A_K L \subseteq u[\mathfrak{a}_K \cup u \circ L]$; but when is $[A_K L] = u[\mathfrak{a}_K \cup u \circ L]$?

5.5. In VI.2.6. we characterized the minimal ttgs in $\mathbf{D}^{\perp}$ as the minimal ttgs without distal factors. Does a similar result hold for $\mathbf{P}^{\perp}$?

> **REMARK.** *Let $\mathscr{X}$ be a regular minimal ttg. Then $\mathscr{X} \in \mathbf{P}^{\perp}$ iff $\mathscr{X}^*$ does not have nontrivial proximal factors.*

PROOF. The "only if"-part is trivial (1.1.)
Suppose $\mathscr{X}^*$ does not admit nontrivial proximal factors. Let $C = u \circ C$ be an MHP generator such that $\mathscr{X}^* = \mathscr{C}$. Then, as $\mathscr{X}^*$ is regular (V.3.6.), $C \circ G$ is an MHP generator. As $\mathscr{Q}\mathscr{F}(C \circ G, \mathscr{M})$ is a factor of $\mathscr{X}^*$ and as $\mathscr{Q}\mathscr{F}(C \circ G, \mathscr{M})$ is proximal, it follows from the assumption that $\mathscr{Q}\mathscr{F}(C \circ G, \mathscr{M})$ is trivial; hence $C \circ G = M$. But then, by V.3.9., $\mathscr{Q}\mathscr{F}(u \circ G, \mathscr{M}) \perp \mathscr{X}^*$, so $\mathscr{X}^* \in \mathbf{P}^{\perp}$; hence $\mathscr{X} \in \mathbf{P}^{\perp}$. □

The following theorem gives a necessary and sufficient condition for a minimal ttg to have a nontrivial proximal factor (T.S. WU, private communication).

> **THEOREM.** *A minimal ttg $\mathscr{X}$ has a nontrivial proximal factor iff there is a nontrivial u.s.c. equivariant map $\phi: \mathscr{X} \to 2^{\mathscr{X}}$ with*
> (i) $\phi(x) \cap \phi(x') \neq \varnothing$ *implies* $\phi(x) = \phi(x')$ *;*
> (ii) $\overline{\phi[X]} \subseteq 2^X$ *has a nontrivial proximal subttg.*

PROOF.

"$\Rightarrow$" Let $\psi: \mathscr{X} \to \mathscr{Y}$ be a homomorphism and let $\mathscr{Y}$ be proximal. Define $\phi: \mathscr{X} \to 2^{\mathscr{X}}$ by $\phi(x) = \psi^{\leftarrow}\psi(x)$. Then, by II.1.3.b, ϕ is u.s.c.. Clearly ϕ is nontrivial and it satisfies (i). Also ϕ satisfies (ii), for the representation $\mathscr{Y}'$ of $\mathscr{Y}$ in $\mathscr{X}$ is proximal and clearly $Y' \subseteq \overline{\phi[X]}$ (see IV.3.3.).

"$\Leftarrow$" Since ϕ is u.s.c., for every $A \in \overline{\phi[X]}$ we can find an $x \in X$ with $A \subseteq \phi(x)$. Define a relation R on $\overline{\phi[X]}$ by $(A, A') \in R$ iff $A \cup A' \subseteq \phi(x)$ for some $x \in X$. Then R is a T-invariant equivalence relation ((i)) which is closed (u.s.c.). Define $Y = \overline{\phi[X]}/R$ then $\mathscr{Y}$ is a ttg. Define $\psi: \mathscr{X} \to \mathscr{Y}$ by $\psi(x) := R[\phi(x)]$. Then ψ is a homomorphism, for equivariance is obvious. Let $\{x_i\}_i$ be a net converging to $x \in X$. Then

$$\lim \psi(x_i) = \lim R[\phi(x_i)] = R[\lim \phi(x_i)] .$$

By upper semi continuity, $\lim \phi(x_i) \subseteq \phi(x)$; so $R[\phi(x)] = R[\lim \phi(x_i)]$. But then

$$\lim \psi(x_i) = R[\lim \phi(x_i)] = R[\phi(x)] = \psi(x) ,$$

and ψ is continuous. Clearly ψ is a surjection and, as ϕ is nontrivial, $\mathcal{Y}$ is nontrivial. As $\mathcal{Y}$ is minimal, $\mathcal{Y}$ is the image of the nontrivial proximal subttg in $\overline{\phi[X]}$, so $\mathcal{Y}$ itself is proximal. □

5.6. The elements of **[E]** can be characterized as the locally almost periodic minimal ttgs, as follows:
In [MW 72] it is shown that a minimal ttg $\mathcal{X}$ is locally almost periodic iff $\mathcal{X}$ is proximal equicontinuous such that for every open U in X there is an $x \in X$ with $P_{\mathcal{X}}[x] = \{x' \in X \mid (x,x') \in P_{\mathcal{X}}\} \subseteq U$. So a minimal ttg $\mathcal{X}$ is locally almost periodic iff $\mathcal{X}$ is an hp extension of an uniformly almost periodic minimal ttg.
(For let $\mathcal{X}$ be locally almost periodic. Then there is an uniformly almost periodic ttg $\mathcal{Y}$ and a proximal map $\phi: \mathcal{X} \to \mathcal{Y}$. Let $U \subseteq X$ be open and let $x \in X$ be such that $P_{\mathcal{X}}[x] \subseteq U$, then $\phi^{\leftarrow}\phi(x) \subseteq P_{\mathcal{X}}[x] \subseteq U$; so ϕ is irreducible, hence highly proximal. Conversely, suppose that $\mathcal{X}$ is an hp extension of a uniformly almost periodic minimal ttg; say $\phi: \mathcal{X} \to \mathcal{Y}$, where ϕ is hp, and with $\mathcal{Y} \in \mathbf{E}$. Clearly, $\mathcal{X}$ is proximally equicontinuous. Let $U \subseteq X$ be open and let $y \in Y$ with $\phi^{\leftarrow}(y) \subseteq U$. Let $x \in \phi^{\leftarrow}(y)$; then $P_{\mathcal{X}}[x] = \phi^{\leftarrow}(y) \subseteq U$. Hence $\mathcal{X}$ is locally almost periodic.)
So clearly, **[E]** contains all locally almost periodic minimal ttgs. Conversely, note that, by the above, $\mathcal{E}^*$ is locally almost periodic, and that local almost periodicity is preserved under factors (use the characterization above and apply I.4.3.a,b and e). It follows that every element of **[E]** is locally almost periodic. Hence $\mathcal{E}^*$ is the universal minimal locally almost periodic ttg, and every element of **[E]** is an hp extension of a uniformly almost periodic ttg.
For a discussion of the relativized concept see [MW 80.1].

QUESTION

Does there exist a similar characterization for the elements of **[D]** ?

5.7. The material in chapter VI. could have been treated in a (more) relativized version, in the following way:
Let $\mathcal{X}$ be a minimal ttg and consider all extensions of $\mathcal{X}$. Then prove similar results as in this chapter, where $\mathcal{X}$ plays the role of the trivial ttg. For convenience it will be desirable to take for $\mathcal{X}$ an MHP ttg, as openness of maps will turn out to be needed many times. For example see [B 75/79] section 3.19.

VII

WEAK DISJOINTNESS

This chapter is almost entirely devoted to weak disjointness in relation to almost periodic factors, or rather to the equicontinuous structure relation. In doing so we profit from a decent additional measure structure on the fibers of a certain kind of homomorphism, which is, in fact, a relativization of the concept of invariant measure.
Therefore, the first section deals with the notion of Relatively Invariant Measure (RIM). Homomorphisms that admit such a RIM (RIM extensions) turn out to behave nicely with respect to the equicontinuous structure relation and weak mixing. As we are more interested in the properties of RIM extensions and their uses than in the technical background, we shall refrain from selfcontainedness and we shall refer to the literature for a few (technical) proofs. Most of the results in section 1. are well known and can be found for instance in [G 75.2], [M 78] or [VW ?], but we end the section with some new (although artificial) thoughts on a condition which is weaker than having a RIM.
In the second section we study the ergodic behavior inside the neighbourhood of a point (in its fiber with respect to a homomorphism). The main result is a generalization of [G 75.1] 1.1.; we prove that an open proximal homomorphism of minimal ttgs is weakly disjoint from every homomorphism of minimal ttgs with the same codomain.
As it turns out to be unsatisfactory to be stuck to choices of points and their fibers, we take a more global view in the third section. There the approach gives more results and we are able to generalize known results on weak

disjointness to situations without countability assumptions. For instance, we show that two homomorphisms ϕ and ψ of minimal ttgs are weakly disjoint if and only if their almost periodic factors are disjoint, provided that ϕ is an open RIM extension (compare [M 78] 1.9. and [P 72] 11.). Also, in 3.14. we generalize the far reaching result of W.A. VEECH in [V 77] 2.6.3., where he shows that under some conditions the product of an ergodic and a minimal ttg is again ergodic.
Most of the results in section 3. are already in [AMWW ?], they are obtained in cooperation with J. AUSLANDER, D.C. MCMAHON and T.S. WU.
In the forth section we generalize I.U. BRONSTEIN's characterization of PI extensions [B 77] (cf III.5.7.).

VII.1. RELATIVELY INVARIANT MEASURES

> In this section we briefly discuss the notion of relatively invariant measure. We only treat this material for the sake of definition and notation. So no new results are to be expected, just a glimpse at this part of the subject.
> For a more explicit treatment see [FG 78], [G 75.1], [G 75.2], [M 78], [MW 83] and [VW ?].

Let X be a CT_2 space and let $\mathfrak{M}(X)$ be the collection of regular Borel probability measures on X provided with the weak star topology; i.e., a net $\{\mu_i\}_i$ in $\mathfrak{M}(X)$ converges to $\mu \in \mathfrak{M}(X)$ iff $\int f\, d\mu_i$ converges to $\int f\, d\mu$ for all real valued continuous functions f on X. Then $\mathfrak{M}(X)$ is a CT_2 space in which X is embedded by the mapping $x \mapsto \delta_x$, where δ_x is the Dirac measure at x. Moreover, $\mathfrak{M}(X)$ is a convex space in which X is just the collection of extremal points, so by the Krein-Milman theorem $\mathfrak{M}(X) = \overline{\mathrm{co}X}$. Here $\mathrm{co}X$ denotes the convex hull of X as a subset of $\mathfrak{M}(X)$.
Let $\phi: \mathfrak{X} \to \mathfrak{Y}$ be a continuous map between CT_2 spaces. Then ϕ induces a continuous map $\mathfrak{M}(\phi): \mathfrak{M}(X) \to \mathfrak{M}(Y)$ which extends ϕ. Note, that $\mathfrak{M}(\phi)$ is surjective (injective) (homeomorphic) iff ϕ is.

1.1. If X is a metrizable CT_2 space, so is $\mathfrak{M}(X)$. For the space of real valued continuous functions on X endowed with the topology of uniform convergence is separable. Hence, $\mathfrak{M}(X)$ is first countable. As X is separable $\mathfrak{M}(X) = \overline{\mathrm{co}X}$ is separable. So $\mathfrak{M}(X)$ is CT_2 first countable and separable, hence metrizable.

Let $\mathfrak{X}$ be a ttg for T. For $t \in T$ and $\mu \in \mathfrak{M}(X)$ define $t\mu \in \mathfrak{M}(X)$ by $t\mu(A) = \mu(t^{-1}A)$; or, what amounts to the same, $\int f\, d(t\mu) = \int ft\, d\mu$, where $ft : X \to \mathbb{R}$ is defined by $ft(x) = f(tx)$. Also one could say $t\mu := \mathfrak{M}(\pi^t)(\mu)$, where $\pi^t := x \mapsto tx : X \to X$.
One can show (e.g. [VW ?]) that $(t,\mu) \mapsto t\mu : T \times \mathfrak{M}(X) \to \mathfrak{M}(X)$ is continuous. So $\mathfrak{M}(\mathfrak{X})$ is a ttg for T.
If $\phi : \mathfrak{X} \to \mathfrak{Y}$ is a homomorphism of ttgs, then $\mathfrak{M}(\phi) : \mathfrak{M}(\mathfrak{X}) \to \mathfrak{M}(\mathfrak{Y})$ is a homomorphism of ttgs.
By definition, $\mathfrak{X}$ has an invariant measure whenever $\mathfrak{M}(\mathfrak{X})$ has a fixed point; i.e., there is a $\mu \in \mathfrak{M}(X)$ with $\mu(tA) = \mu(A)$ for all $t \in T$ and every Borel set A in X.

1.2. A surjective homomorphism $\phi : \mathfrak{X} \to \mathfrak{Y}$ of ttgs is said to have a *relatively invariant measure* (ϕ has a RIM, ϕ is a RIM *extension*) if there exists a continuous homomorphism $\lambda : \mathfrak{Y} \to \mathfrak{M}(\mathfrak{X})$ of ttgs such that $\mathfrak{M}(\phi) \circ \lambda : \mathfrak{Y} \to \mathfrak{M}(\mathfrak{Y})$ is just the (Dirac) embedding. In other words: ϕ is a RIM extension iff for every $y \in Y$ there is a $\lambda_y \in \mathfrak{M}(X)$ with $\operatorname{supp}\lambda_y \subseteq \phi^{\leftarrow}(y)$ and the map $y \mapsto \lambda_y : \mathfrak{Y} \to \mathfrak{M}(\mathfrak{X})$ is a homomorphism of ttgs; this map λ is called a *section for* ϕ.
In particular, $\phi : \mathfrak{X} \to \{\star\}$ has a RIM iff $\mathfrak{X}$ has an invariant measure iff $\mathfrak{M}(\mathfrak{X})$ has a fixed point under the action of T.

RIM extensions of minimal ttgs turn out to behave nicely with respect to the interpolation of maximal almost periodic factors, as we shall see in 3.22..
We shall collect some information on RIM extensions. For the proof of 1.3. see [G 75.2].

1.3. **REMARK.** *Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ and $\psi : \mathfrak{Y} \to \mathfrak{Z}$ be homomorphisms of minimal ttgs.*

a) *If $\psi \circ \phi$ is a* RIM *extension then ψ is a* RIM *extension.*
b) *If ϕ and ψ are* RIM *extensions then $\psi \circ \phi$ is a* RIM *extension.*
c) *If ϕ is an almost periodic extension then ϕ has a unique section, say λ, and* $\operatorname{supp}\lambda_y = \phi^{\leftarrow}(y)$ *for all $y \in Y$.*
d) *If ϕ is a distal extension then ϕ has a* RIM, *which is not necessarily unique.* □

1.4. **LEMMA.** *Let* X *be a* CT_2 *space. The map* $\mathrm{supp}:\mathfrak{M}(X)\to 2^X$ *defined by* $\mu\mapsto\mathrm{supp}\,\mu$ *(support of* μ *) is lower semi continuous; i.e., if* $\mu_i\to\mu$ *in* $\mathfrak{M}(X)$ *then* $\mathrm{supp}\,\mu\subseteq S$ *for an arbitrary limit point* S *of the net* $\{\mathrm{supp}\,\mu_i\}_i$ *in* 2^X *.*

PROOF. Let $x\in\mathrm{supp}\,\mu$ and suppose $x\notin S$. Let U and V be open sets in X with $x\in U$, $S\subseteq V$ and $\overline{U}\cap\overline{V}=\varnothing$. Let $f:X\to[0,1]$ be a continuous map with $f[\overline{U}]=\{1\}$ and $f[\overline{V}]=\{0\}$. As $x\in\mathrm{supp}\,\mu$ it follows that $\mu[\overline{U}]>0$ and so $\int f\,d\mu\geqslant\mu[\overline{U}]>0$.
As, for a suitable subnet, $S=\lim_{2^X}\mathrm{supp}\,\mu_j$, $\mathrm{supp}\,\mu_j\subseteq V$ eventually; hence $\int f\,d\mu_i=0$ eventually. But $\int f\,d\mu_i\to\int f\,d\mu$, so $\int f\,d\mu=0$, which is in contradiction with the above. □

1.5. **REMARK.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a surjective* RIM *extension of ttgs with section* λ *. Then* ϕ *is open in all points* $x\in X$ *with* $x\in\mathrm{supp}\,\lambda_{\phi(x)}$ *. In particular, if* $\mathfrak{X}$ *is minimal then* $\mathrm{supp}\,\lambda_y\subseteq\bigcap\{u\circ\phi^{\leftarrow}(y)\mid u\in J_y\}$ *for all* $y\in Y$ *.*

PROOF. Suppose $x\in\mathrm{supp}\,\lambda_{\phi(x)}$ and let $U\in\mathcal{V}_x$. By lower semi continuity of the map $\mathrm{supp}:\mathfrak{M}(X)\to 2^X$, the set $\{\mu\in\mathfrak{M}(X)\mid U\cap\mathrm{supp}\,\mu\neq\varnothing\}$ is an open neighbourhood of $\lambda_{\phi(x)}$ in $\mathfrak{M}(X)$. As λ is continuous, there is a $V\in\mathcal{V}_{\phi(x)}$ such that $\lambda(V)\subseteq\{\mu\in\mathfrak{M}(X)\mid U\cap\mathrm{supp}\,\mu\neq\varnothing\}$, so $U\cap\mathrm{supp}\,\lambda_y\neq\varnothing$ for every $y\in V$. As $\mathrm{supp}\,\lambda_y\subseteq\phi^{\leftarrow}(y)$, this implies that $\phi(x)\in V\subseteq\phi[U]$. So $\phi[U]$ is a neighbourhood of $\phi(x)$; hence ϕ is open in x .
The second statement follows immediately from II.3.12.. □

1.6. **THEOREM.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a* RIM *extension of minimal ttgs with section* λ *. Then, for* $y_0\in Y$ *, the following statements are equivalent:*

a) $\mathrm{supp}\,\lambda_{y_0}=\phi^{\leftarrow}(y_0)$ *;*

b) *the map* $y\mapsto\mathrm{supp}\,\lambda_y:Y\to 2^X$ *is continuous in* y_0 *.*

In particular it follows that if X *is metric then there is a residual subset* $Y'\subseteq Y$ *with* $\mathrm{supp}\,\lambda_y=\phi^{\leftarrow}(y)$ *for all* $y\in Y'$ *.*

PROOF. (See also [G 75.2] 3.3..)

a $\Rightarrow$ b Let $\{y_i\}_i$ be a net in Y with $y_i\to y_0$. Let $p_i\in M$ be such that $y_i=p_iy_0$ and, after passing to a suitable subnet, let $q=\lim p_i\in M$. Then $qy_0=y_0$ and so, by continuity and equivariance of λ , $q\lambda_{y_0}=\lambda_{qy_0}=\lambda_{y_0}$. By 1.4., and after passing to a suitable subnet,

$$\operatorname{supp}\lambda_{y_0} = \operatorname{supp} q\lambda_{y_0} \subseteq \lim\nolimits_{2x} \operatorname{supp} p_i\lambda_{y_0} \subseteq \lim\nolimits_{2x} p_i \circ \phi^{\leftarrow}(y_0) = q \circ \phi^{\leftarrow}(y_0) .$$

As

$$\phi^{\leftarrow}(y_0) = \operatorname{supp}\lambda_{y_0} \subseteq q \circ \phi^{\leftarrow}(y_0) \subseteq \phi^{\leftarrow}(y_0) ,$$

it follows that $\operatorname{supp}\lambda_{y_0} = \lim_{2x} \operatorname{supp}\lambda_{y_i}$. Hence $y \mapsto \operatorname{supp}\lambda_y$ is continuous in y_0 .

b $\Rightarrow$ a Let $x \in \phi^{\leftarrow}(y_0)$ and let $x_0 \in \operatorname{supp}\lambda_{y_0}$. As X is minimal there is a net $\{t_i\}_i$ in T with $t_i x_0 \to x$. As $t_i x_0 \in \operatorname{supp}\lambda_{t_i y_0}$ and $t_i y_0 \to y_0$, it follows by the continuity assumption that $x \in \lim_{2x} \operatorname{supp}\lambda_{t_i y_0} = \operatorname{supp}\lambda_{y_0}$.

If X is metric, then the lower semi continuous map $y \mapsto \operatorname{supp}\lambda_y$ has a residual set of continuity points in Y ([Fo 51], compare II.1.3.e). ☐

1.7. **THEOREM.** *Let* $\phi: \mathcal{X} \to \mathcal{Y}$ *be a* RIM *extension of minimal ttgs. Then* $\phi': \mathcal{X}' \to \mathcal{Y}'$ *(in* AG(ϕ)*) is an open* RIM *extension, and* $\phi^*: \mathcal{X}^* \to \mathcal{Y}^*$ *(in* *(ϕ)*) can be written as* $\psi \circ \theta$ *with* θ *highly proximal and* ψ *a* RIM *extension.*

PROOF. Let λ be a section for ϕ . Then by 1.5.,

$$\operatorname{supp}\lambda_y \subseteq \bigcap \{v \circ \phi^{\leftarrow}(y) \mid v \in J_y\} \text{ for every } y \in Y .$$

First consider AG(ϕ), which is the right hand part of

$$\begin{tikzcd}
\mathcal{X}^* \arrow[rr, "\sigma'"] \arrow[dr, "\theta"] \arrow[dd, "\phi^*"'] & & \mathcal{X}' \arrow[r, "\sigma"] \arrow[dd, "\phi'"] & \mathcal{X} \arrow[dd, "\phi"] \\
& \mathcal{R}_{\phi'\tau'} \arrow[ur, "\xi"'] \arrow[dl, "\psi"] & & \\
\mathcal{Y}^* \arrow[rr, "\tau'"'] & & \mathcal{Y}' \arrow[r, "\tau"'] & \mathcal{Y}
\end{tikzcd}$$

By IV.3.4., $Y' = \{p \circ \phi^{\leftarrow}(y) \mid p \in M , y \in Y\}$, and so by II.3.11.e,

$$Y' = \{v \circ \phi^{\leftarrow}(y) \mid y \in Y , v \in J_y\} ,$$

whereas $X' = \{(x,y') \mid x \in y' \in Y'\}$, so

$$X' = \{(x, v \circ \phi^{\leftarrow}(y)) \mid y \in Y , v \in J_y , x \in v \circ \phi^{\leftarrow}(y)\} .$$

The map ϕ' is defined as the projection, so

$$\phi'^{\leftarrow}(\nu \circ \phi^{\leftarrow}(y)) = \nu \circ \phi^{\leftarrow}(y) \times \{\nu \circ \phi^{\leftarrow}(y)\} \subseteq X \times Y' .$$

For every $y' \in Y'$ define $\lambda'_{y'} := \lambda_{\tau(y')} \times \delta_{y'}$, or rather, for every $y \in Y$ and $\nu \in J_y$ let $\lambda'_{\nu \circ \phi^{\leftarrow}(y)} := \lambda_y \times \delta_{\nu \circ \phi^{\leftarrow}(y)}$. Clearly, $\lambda'_{\nu \circ \phi^{\leftarrow}(y)} \in \mathfrak{M}(X')$ and

$$\operatorname{supp} \lambda'_{\nu \circ \phi^{\leftarrow}(y)} = \operatorname{supp} \lambda_y \times \{\nu \circ \phi^{\leftarrow}(y)\} \subseteq \phi'^{\leftarrow}(\nu \circ \phi^{\leftarrow}(y)) .$$

As $((\lambda \circ \tau) \times \delta :) \nu \circ \phi^{\leftarrow}(y) \mapsto \lambda'_{\nu \circ \phi^{\leftarrow}(y)}$ is continuous and T-invariant, it follows that λ' is a section for ϕ', so ϕ' is an open RIM extension.
Consider the left hand part of the diagram above. As ϕ' is open, $\tau' \perp \phi'$ (IV.3.16.). So $\mathcal{R}_{\phi'\tau'}$ is minimal and there is a homomorphism $\theta : \mathcal{X}^* \to \mathcal{R}_{\phi'\tau'}$ which is hp, for $\sigma = \xi \circ \theta$ is hp. Let $y^* \in Y^*$ and $\tau'(y^*) = y' \in Y'$, then $\psi^{\leftarrow}(y^*) = \phi'^{\leftarrow}(y') \times \{y^*\} \subseteq R_{\phi'\tau'}$. Define $\lambda^*_{y^*} := \lambda'_{\tau'(y^*)} \times \delta_{y^*}$ and note that λ^* is a section for ψ. So ψ is a RIM extension and $\phi^* = \psi \circ \theta$. ☐

1.8. In [G 75.2] S. GLASNER has shown that every homomorphism $\phi : \mathcal{X} \to \mathcal{Y}$ is a RIM extension up to proximality; i.e., he constructed a diagram similar to the EGS and AG diagrams, which we shall call a G′ diagram, as follows

$$\begin{tikzcd}
\tilde{\mathcal{X}} \arrow[r, "\sigma"] \arrow[d, "\tilde{\phi}"'] & \mathcal{X} \arrow[d, "\phi"] \\
\tilde{\mathcal{Y}} \arrow[r, "\tau"'] & \mathcal{Y}
\end{tikzcd}$$

$\tilde{\mathcal{Y}}$ is a certain minimal subttg of $\mathfrak{M}(X)$, $\tau : \tilde{\mathcal{Y}} \to \mathcal{Y}$ is a proximal extension (even a strongly proximal extension, which we shall define below), and $\tilde{\mathcal{X}}$ is the unique minimal subttg of $\mathcal{R}_{\phi\tau}$. The projections are called σ and $\tilde{\phi}$.
It turns out that σ is (strongly) proximal and that $\tilde{\phi}$ is a RIM extension.
As the precise construction is not relevant for our purposes we shall not go into details on that. The interested reader may find it in [G 75.2] and [VW ?].

Let $\phi : \mathcal{X} \to \mathcal{Y}$ be a homomorphism of ttgs. Then ϕ is called *strongly proximal* if for every $\mu \in \mathfrak{M}(X)$, with $\mathfrak{M}(\phi)(\mu) = \delta_y$ for some $y \in Y$, there is a net $\{t_i\}_i$ in T such that $t_i \mu \to \delta_x$ for some $x \in X$.
In particular, a strongly proximal homomorphism is proximal. For let $x_1, x_2 \in \phi^{\leftarrow}(y)$, then $\mu := (\delta_{x_1} + \delta_{x_2})/2 \in \mathfrak{M}(X)$ and $\mathfrak{M}(\phi)(\mu) = \delta_y$. So there is a net $\{t_i\}_i$ in T and there is an $x \in X$ such that $t_i \mu \to \delta_x$.

Let $p = \lim t_i \in S_T$; then $t_i\mu \to (\delta_{px_1}+\delta_{px_2})/2$. So $(\delta_{px_1}+\delta_{px_2})/2 = \delta_x$ and $px_1 = px_2 = x$, which implies that x_1 and x_2 are proximal.

1.9. **REMARK.**

a) *Let* $\phi\colon \mathcal{X} \to \mathcal{Y}$ *and* $\psi\colon \mathcal{Y} \to \mathcal{Z}$ *be homomorphisms of minimal ttgs. Then* $\psi \circ \phi$ *is strongly proximal iff* ϕ *and* ψ *are strongly proximal.*

b) *A highly proximal extension* $\phi\colon \mathcal{X} \to \mathcal{Y}$ *of minimal ttgs is strongly proximal.*

c) *A* RIM *extension of minimal ttgs is strongly proximal iff it is an isomorphism.*

PROOF.

a) Straightforward.

b) Let $y \in Y$ and $\mu \in \mathfrak{M}(X)$ be such that $\mathfrak{M}(\phi)(\mu) = \delta_y$. Then $\operatorname{supp}\mu \subseteq \phi^{\leftarrow}(y)$. Let $u \in J_y$ and $x = ux \in \phi^{\leftarrow}(y)$. Then, by high proximality of ϕ , $\{x\} = u \circ \phi^{\leftarrow}(y)$; while, by 1.4., $\operatorname{supp} u\mu \subseteq u \circ \phi^{\leftarrow}(y) = \{x\}$. Hence $u\mu = \delta_x$, and ϕ is strongly proximal.

c) Let $\phi\colon \mathcal{X} \to \mathcal{Y}$ be a RIM extension of minimal ttgs with section λ . Then for every $y \in Y$, λ_y is a *minimal measure* ($\lambda_y \in \mathfrak{M}(X)$ is an almost periodic point), and $\mathfrak{M}(\phi)(\lambda_y) = \delta_y$. If ϕ is strongly proximal, there is a $\delta_{x'}$ in the orbit closure of λ_y , so $X = \overline{T\lambda_y}$; hence $\lambda_y = \delta_x$ for some $x \in \phi^{\leftarrow}(y)$. As λ is a homomorphism and $\mathcal{Y}$ is minimal, λ maps $\mathcal{Y}$ onto $\mathcal{X} \subseteq \mathfrak{M}(\mathcal{X})$. But then $|\phi^{\leftarrow}(y)| = 1$ for every $y \in Y$. □

Now we can extend the G′ diagram for ϕ to a diagram in which the associated RIM extension is even open. We shall refer to that diagram as a G diagram.

1.10. **THEOREM.** *Let* $\phi\colon \mathcal{X} \to \mathcal{Y}$ *be a homomorphism of minimal ttgs.*

$$\begin{tikzcd}
\mathcal{X}^{\#} \arrow[r, "\sigma"] \arrow[d, "\phi^{\#}"'] & \mathcal{X} \arrow[d, "\phi"] \\
\mathcal{Y}^{\#} \arrow[r, "\tau"'] & \mathcal{Y}
\end{tikzcd}$$

Then there is an open RIM *extension* $\phi^{\#} : \mathcal{X}^{\#} \to \mathcal{Y}^{\#}$ *of minimal ttgs, and there are strongly proximal extensions* $\sigma\colon \mathcal{X}^{\#} \to \mathcal{X}$ *and* $\tau\colon \mathcal{Y}^{\#} \to \mathcal{Y}$ *such that* $\phi \circ \sigma = \tau \circ \phi^{\#}$.

If $\mathcal{X}$ *is metric then* $\mathcal{X}^{\#}$ *and* $\mathcal{Y}^{\#}$ *can be chosen to be metric.*

PROOF. Consider the next diagram:

$$\begin{tikzcd}
\mathcal{X}^{\#} := \tilde{\mathcal{X}}' \arrow[r, "\sigma'"] \arrow[d, swap, "\phi^{\#} := \tilde{\phi}'"] & \tilde{\mathcal{X}} \arrow[r, "\sigma"] \arrow[d, swap, "\tilde{\phi}"] & \mathcal{X} \arrow[d, "\phi"] \\
\mathcal{Y}^{\#} := \tilde{\mathcal{Y}}' \arrow[r, swap, "\tau'"] & \tilde{\mathcal{Y}} \arrow[r, swap, "\tau"] & \mathcal{Y}
\end{tikzcd}$$

By 1.8., we can construct the right hand part of the diagram such that $\tilde{\phi}$ is a RIM extension, and such that σ and τ are strongly proximal.
The left hand part is AG($\tilde{\phi}$), so σ' and τ' are hp. Hence by 1.9., $\sigma\circ\sigma'$ and $\tau\circ\tau'$ are strongly proximal homomorphisms of minimal ttgs. By 1.7., $\tilde{\phi}'$ is an open RIM extension and clearly, $\phi\circ\sigma\circ\sigma' = \tau\circ\tau'\circ\tilde{\phi}'$.
If X is metric, $\mathfrak{M}(X)$ is metric (by 1.1.). Hence, $\tilde{Y}$ is metric and $\tilde{X}$ as a subset of $X \times \tilde{Y}$ is metric. But then $\tilde{Y}'$ and $\tilde{X}'$ are metric by IV.3.11.. □

Let $\mathcal{Y}$ be a minimal ttg. Completely analogues to the construction of the universal minimal (highly) proximal extension of $\mathcal{Y}$ (e.g. III.1.13.b) one can construct the universal minimal strongly proximal extension of $\mathcal{Y}$ (which will be denoted by $\mathfrak{A}_S(\mathcal{Y})$), as follows:
Let $\gamma: \mathcal{M} \to \mathcal{Y}$ be a homomorphism and construct the G(γ) diagram. Then $\tau: \mathcal{Y}^{\#} \to \mathcal{Y}$ is strongly proximal and $\gamma^{\#} : \mathcal{M}^{\#} = \mathcal{M} \to \mathcal{Y}^{\#}$ is a RIM extension. As every extension ψ of $\mathcal{Y}^{\#}$ is a factor of $\gamma^{\#}$, it follows by 1.3.a that ψ is a RIM extension. In particular, every strongly proximal extension of $\mathcal{Y}^{\#}$ is trivial (1.9.c).
If $\phi: \mathcal{X} \to \mathcal{Y}$ is a strongly proximal extension of minimal ttgs then it is easily checked that

$$\theta: (x,z) \mapsto \phi(x) = \tau(z) : \mathfrak{R}_{\phi\tau} \to \mathcal{Y}$$

is strongly proximal. As θ factorizes over τ , the unique minimal subttg of $\mathfrak{R}_{\phi\tau}$ is a strongly proximal extension of $\mathcal{Y}^{\#}$, so it is isomorphic to $\mathcal{Y}^{\#}$. This shows that $\mathcal{Y}^{\#}$ is the universal minimal strongly proximal extension of $\mathcal{Y}$.
The ttg $\mathfrak{A}_S(\{\star\})$ is the universal minimal strongly proximal ttg for T .
For the following theorem we refer to [G 76] or [VW ?].

1.11. **THEOREM.** *Let* T *be a topological group. Then the following statements are equivalent:*

a) T *is an amenable group;*

b) *Every minimal ttg for* T *has an invariant measure;*

c) *The minimal ttg* $\mathfrak{A}_S(\{\bigstar\})$ *for* T *is trivial; i.e.,* T *does not admit nontrivial strongly proximal minimal ttgs.* □

Clearly, a strongly amenable group T is amenable, but there are examples of amenable groups that are not strongly amenable [G 76] III.7..
Note that this shows that there do exist nontrivial proximal minimal ttgs that admit an invariant measure. So, in particular, an open RIM extension is in general not a RIC extension.
Also a RIC extension does not have to be a RIM extension, for [M 76.1] 2.2. provides an example of a minimal ttg that does not admit an invariant measure but which is incontractible. From this it follows that the notion of a RIM extension is not related to strong proximality in the same way as a RIC extension is to proximality and an open extension to high proximality; i.e.: One cannot characterize the RIM extensions as those homomorphisms that are disjoint from all strongly proximal extensions of its codomain.
We shall go into that in the following.

1.12. **THEOREM.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *and* $\psi: \mathfrak{Z} \to \mathfrak{Y}$ *be homomorphisms of minimal ttgs such that one of them is open. Let* ψ *be strongly proximal and let* ϕ *be such that there is a minimal measure* $\mu \in \mathfrak{M}(X)$ *and a* $y \in Y$ *with*
either $\operatorname{supp}\mu = \phi^{\leftarrow}(y)$
or $\operatorname{supp}\mu \subseteq \phi^{\leftarrow}(y)$ *and* $\bigcap\{\operatorname{supp} p\mu \mid p \in M_y\} \neq \varnothing$.
Then $\phi \perp \psi$.

PROOF. Let W be a minimal subset of $R_{\phi\psi}$ and define the homomorphisms $\pi_1: \mathfrak{W} \to \mathfrak{X}$ and $\pi_2: \mathfrak{W} \to \mathfrak{Z}$ as (restrictions of) the projections. Let $\mu \in \mathfrak{M}(X)$ and $y \in Y$ be as in the assumption. As μ is an almost periodic point in $\mathfrak{M}(X)$, we can find an almost periodic measure $\nu \in \mathfrak{M}(W)$ with $\mathfrak{M}(\pi_1)(\nu) = \mu$. Clearly,

$$\mathfrak{M}(\psi) \circ \mathfrak{M}(\pi_2)(\nu) = \mathfrak{M}(\phi) \circ \mathfrak{M}(\pi_1)(\nu) = \mathfrak{M}(\phi)(\mu) = \delta_y .$$

By strong proximality of ψ , there is a Dirac measure δ_z in the orbit closure of $\mathfrak{M}(\pi_2)(\nu)$. As $\nu \in \mathfrak{M}(W)$ is almost periodic, $\mathfrak{M}(\pi_2)(\nu)$ is almost periodic, hence $\mathfrak{M}(\pi_2)(\nu)$ is a Dirac measure, say $\mathfrak{M}(\pi_2)(\nu) = \delta_{z_0}$. Obviously,

$$z_0 \in \operatorname{supp} \delta_{z_0} = \operatorname{supp} \mathfrak{M}(\pi_2)(\nu) \subseteq \psi^{\leftarrow}(y)\,,$$

and for every $p \in M$ we have $\mathfrak{M}(\pi_2)(p\nu) = \delta_{pz_0}$. But then

$$\operatorname{supp} p\nu = \operatorname{supp} \mathfrak{M}(\pi_1)(p\nu) \times \operatorname{supp} \mathfrak{M}(\pi_2)(p\nu) = \operatorname{supp} p\mu \times \{pz_0\}$$

for all $p \in M$. As $p\nu \in \mathfrak{M}(W)$ it follows that $\operatorname{supp} p\nu \subseteq W$; hence $\operatorname{supp} p\mu \times \{pz_0\} \subseteq W$.
First suppose that $\operatorname{supp}\mu = \phi^{\leftarrow}(y)$ and let $q \in M$ be such that $q\mu = \mu$. Then by the above

$$\phi^{\leftarrow}(y) \times \{qz_0\} = \operatorname{supp}\mu \times \{qz_0\} = \operatorname{supp} q\mu \times \{qz_0\} \subseteq W\,.$$

As W is minimal, $\overline{T(\phi^{\leftarrow}(y) \times \{qz_0\})} = W$. By I.3.9. and the assumption that at least one of the maps ϕ and ψ is open, it follows that $R_{\phi\psi} = \overline{T(\phi^{\leftarrow}(y) \times \{qz_0\})} \subseteq W$. Hence $R_{\phi\psi}$ is minimal and $\phi \perp \psi$.
On the other hand, suppose that the second option is valid, say $x \in \bigcap \{\operatorname{supp} p\mu \mid p \in M_y\}$. Then for all $p \in M_y$ we may conclude that $(x, pz_0) \in \operatorname{supp} p\mu \times \{pz_0\} \subseteq W$. Hence $\{x\} \times \psi^{\leftarrow}(y) \subseteq W$ and similar to the above it follows that $R_{\phi\psi} = \overline{T(\{x\} \times \psi^{\leftarrow}(y))} \subseteq W$, which implies that $R_{\phi\psi}$ is minimal and $\phi \perp \psi$. □

1.13. **COROLLARY.** *If $\phi: \mathfrak{X} \to \mathfrak{Y}$ is an open* RIM *extension of minimal ttgs, then $\phi \perp \psi$ for every strongly proximal homomorphism $\psi: \mathfrak{Z} \to \mathfrak{Y}$ of minimal ttgs.*

PROOF. Let λ be a section for ϕ and let $y \in Y$. Then for all $p \in M_y$ we have $p\lambda_y = \lambda_{py} = \lambda_y$, so $\operatorname{supp}\lambda_y = \bigcap \{\operatorname{supp} p\lambda_y \mid p \in M_y\}$. Hence by 1.12. with the second option, the corollary follows. □

1.14. **THEOREM.** *Let $\phi: \mathfrak{X} \to \mathfrak{Y}$ be a homomorphism of minimal ttgs and let X be metrizable. Then ϕ is disjoint from every strongly proximal homomorphism $\psi: \mathfrak{Z} \to \mathfrak{Y}$ of minimal ttgs if and only if ϕ is open and a minimal measure $\mu \in \mathfrak{M}(X)$ exists with $\operatorname{supp}\mu = \phi^{\leftarrow}(y)$ for some $y \in Y$.*

PROOF. If ϕ is open and if some minimal measure $\mu \in \mathfrak{M}(X)$ exists with $\operatorname{supp}\mu = \phi^{\leftarrow}(y)$ for some $y \in Y$, then by 1.12., $\phi \perp \psi$ for every strongly proximal extension $\psi: \mathfrak{Z} \to \mathfrak{Y}$ of minimal ttgs.
Conversely, suppose that ϕ is disjoint from every strongly proximal homomorphism $\psi: \mathfrak{Z} \to \mathfrak{Y}$ of minimal ttgs. Then by 1.9.b, ϕ is disjoint from

every hp extension of $\mathcal{Y}$; hence by IV.3.16., ϕ is open. Construct G(ϕ):

$$\begin{tikzcd}
\mathcal{X}^{\#} \arrow[rr, "\sigma"] \arrow[dr, "\theta"] \arrow[dd, "\phi^{\#}"'] & & \mathcal{X} \arrow[dd, "\phi"] \\
& \mathcal{R}_{\phi\tau} \arrow[ur, "\pi_1"'] & \\
\mathcal{Y}^{\#} \arrow[ur, "\pi_2"'] \arrow[rr, "\tau"'] & & \mathcal{Y}
\end{tikzcd}$$

As ϕ is a homomorphism of metric minimal ttgs, $\phi^{\#} : \mathcal{X}^{\#} \to \mathcal{Y}^{\#}$ is a RIM extension of metric minimal ttgs. As ϕ is disjoint from every strongly proximal extension of $\mathcal{Y}$, $\phi \perp \tau$ and $R_{\phi\tau}$ is minimal. Hence there is a map $\theta : \mathcal{X}^{\#} \to \mathcal{R}_{\phi\tau}$ such that $\pi_1 \circ \theta = \sigma$ and $\pi_2 \circ \theta = \phi^{\#}$, where π_1 and π_2 are the projections, and so the diagram commutes. As $\phi^{\#}$ is a RIM extension of metric minimal ttgs, it follows by 1.3.a that $\pi_2 : \mathcal{R}_{\phi\tau} \to \mathcal{Y}^{\#}$ is a RIM extension of metric minimal ttgs, say with section λ . By 1.6., we can find a $y^{\#} \in Y^{\#}$ such that $\operatorname{supp} \lambda_{y^{\#}} = \pi_2^{\leftarrow}(y^{\#})$. Note that

$$\pi_2^{\leftarrow}(y^{\#}) = \phi^{\leftarrow}(y) \times \{y^{\#}\} ,$$

where $y := \tau(y^{\#})$. Define $\mu := \mathfrak{M}(\pi_1)(\lambda_{y^{\#}}) \in \mathfrak{M}(X)$. Then μ is a minimal measure (homomorphic image of the almost periodic point $y^{\#}$) and obviously, $\operatorname{supp} \mu = \pi_1(\operatorname{supp} \lambda_{y^{\#}})$, hence $\operatorname{supp} \mu = \phi^{\leftarrow}(y)$. □

1.15. Let $\phi : \mathcal{X} \to \mathcal{Y}$ be a homomorphism of minimal ttgs. Consider the diagram AG (ϕ).

$$\begin{tikzcd}
\mathcal{X}' \arrow[r, "\sigma"] \arrow[d, "\phi'"'] & \mathcal{X} \arrow[d, "\phi"] \\
\mathcal{Y}' \arrow[r, "\tau"'] & \mathcal{Y}
\end{tikzcd}$$

We shall call ϕ an RMM *extension* if $\sigma \times \sigma[R_{\phi'}] = R_{\phi}$ and ϕ' is disjoint from every strongly proximal extension $\psi : \mathcal{Z} \to \mathcal{Y}'$ of minimal ttgs.

Note that by IV.4.16. it follows that ϕ is an RMM extension iff ϕ^* (in *(ϕ)) is disjoint from every strongly proximal extension $\theta : \mathcal{W} \to \mathcal{Y}^*$ of minimal ttgs and $\sigma \times \sigma[R_{\phi^*}] = R_{\phi}$ (in *(ϕ)).

Moreover, an RMM extension ϕ is open iff ϕ is disjoint from every strongly proximal extension. In particular, RIC extensions and open RIM extensions are RMM. Also a Bc extension which is RIM or which has a minimal measure supported in a full fiber is an RMM extension.

1.16. Actually, in the proof of 1.14. we showed that for an RMM extension $\phi: \mathcal{X} \to \mathcal{Y}$ we can construct a $\flat$ diagram of homomorphisms of minimal ttgs,

$$\begin{tikzcd}
\mathcal{X}^\flat \arrow[rr, "\sigma^\flat"] \arrow[dr, "\sigma'"'] \arrow[ddd, "\phi^\flat"'] & & \mathcal{X} \arrow[ddd, "\phi"] \\
& \mathcal{X}' \arrow[ur, "\sigma"'] \arrow[d, "\phi'"] & \\
& \mathcal{Y}' \arrow[dr, "\tau"] & \\
\mathcal{Y}^\flat \arrow[ur, "\tau'"] \arrow[rr, "\tau^\flat"'] & & \mathcal{Y}
\end{tikzcd}$$

such that $\phi^\flat$ is an open RIM extension and $\sigma^\flat \times \sigma^\flat[R_{\phi^\flat}] = R_\phi$. As follows: Construct AG(ϕ). As ϕ is RMM, ϕ' is disjoint from every strongly proximal extension of $\mathcal{Y}'$ and $\sigma \times \sigma[R_{\phi'}] = R_\phi$. Then, as in the proof of 1.14., we take $\mathcal{Y}^\flat := \mathcal{Y}'^\#$ (in G(ϕ')) and $\mathcal{X}^\flat := \mathcal{R}_{\tau'\phi'}$, which is minimal as $\tau' \perp \phi'$. As $\phi^\flat$ is a factor of $\phi^\#$ (in G(ϕ')), $\phi^\flat$ is open and RIM. Moreover, $\sigma' \times \sigma'[R_{\phi^\flat}] = R_{\phi'}$, hence

$$\sigma^\flat \times \sigma^\flat[R_{\phi^\flat}] = \sigma \times \sigma[R_{\phi'}] = R_\phi .$$

In particular, we can apply IV.4.3. to this $\flat$ diagram, so to some extent we can transfer properties of open RIM extensions to RMM extensions (e.g. see 1.20.,3.16.,3.17. and 3.20. below).

In [M 78] D.C. MCMAHON developed a technique to investigate the equicontinuous structure relation in the case of RIM extensions. The most important results are 1.17. below ([M 78] corollary 1.4.) and its consequences (here) 1.18. and 1.19.. We shall merely state 1.17., as the techniques that lead to that result are not important for our purposes.

1.17. **THEOREM.** *Let* $\phi: \mathcal{X} \to \mathcal{Y}$ *be a homomorphism of minimal ttgs, and let* $\psi: \mathcal{Z} \to \mathcal{Y}$ *be a* RIM *extension with section* λ *(* $\mathcal{Z}$ *not necessarily minimal). Let* $x \in X$ *and let* U *be an open set in* Z *. Then*

$$E_\phi[x] \times (U \cap \operatorname{supp} \lambda_{\phi(x)}) \subseteq \overline{T(\{x\} \times U \cap R_{\phi\psi})} . \qquad \square$$

1.18. **THEOREM.** *Let* $\phi : \mathcal{X} \to \mathcal{Y}$ *be a* RIM *extension of minimal ttgs with section* λ. *Then for every* $x \in X$ *with* $x \in \operatorname{supp} \lambda_{\phi(x)}$ *we have the equality* $E_\phi[x] = Q_\phi[x]$. *In particular, if a minimal ttg* $\mathcal{X}$ *has an invariant measure then* $E_{\mathcal{X}} = Q_{\mathcal{X}}$.

PROOF. Let $x \in X$ be such that $x \in \operatorname{supp} \lambda_{\phi(x)}$. Let $\alpha \in \mathcal{U}_X$ be an index and let U be an open neighbourhood of x with $U \subseteq \alpha(x)$. Now we apply 1.17. to ϕ and ϕ, so

$$E_\phi[x] \times (U \cap \operatorname{supp} \lambda_{\phi(x)}) \subseteq \overline{T(\{x\} \times U \cap R_\phi)} \subseteq \overline{T\alpha \cap R_\phi}\,.$$

As $x \in U \cap \operatorname{supp} \lambda_{\phi(x)}$ it follows that $E_\phi[x] \times \{x\} \subseteq \overline{T\alpha \cap R_\phi}$. Since $\alpha \in \mathcal{U}_X$ was arbitrary, $E_\phi[x] \times \{x\} \subseteq Q_\phi$ and so $E_\phi[x] \subseteq Q_\phi[x]$, hence $E_\phi[x] = Q_\phi[x]$.

Now suppose that $\mathcal{X}$ is a minimal ttg which has an invariant measure μ. Then $\operatorname{supp} \mu = X$. (For let $U \subseteq X$ be open; then by minimality, $X \subseteq FU$ for some finite set $F \subseteq T$. As $\mu[fU] = \mu[U]$ for all $f \in F$, it follows that $\mu[U] \neq 0$.) So for every $x \in X$, $x \in \operatorname{supp} \mu$ and by the above, $E_{\mathcal{X}}[x] = Q_{\mathcal{X}}[x]$. But then $E_{\mathcal{X}} = Q_{\mathcal{X}}$. □

1.19. **COROLLARY.** *Let* $\phi : \mathcal{X} \to \mathcal{Y}$ *be a* RIM *extension of minimal ttgs. Then* $E_\phi = Q_\phi \circ P_\phi = P_\phi \circ Q_\phi = \{(x_1, x_2) \in R_\phi \mid (ux_1, ux_2) \in Q_\phi \text{ for some } u \in J\}$.

PROOF. Denote $\{(x_1, x_2) \in R_\phi \mid (ux_1, ux_2) \in Q_\phi \text{ for some } u \in J\}$ by S.

First note that by I.4.2., $S \subseteq Q_\phi \circ P_\phi = P_\phi \circ Q_\phi$.

Conversely, let $x \in X$ be such that $x \in \operatorname{supp} \lambda_{\phi(x)}$. Then by 1.18., we have $E_\phi[x] = Q_\phi[x]$. Let $(x_1, x_2) \in E_\phi$, and let $p \in M$ be such that $px_1 = x$. Then $(x, px_2) = (px_1, px_2) \in E_\phi$; hence $(x, px_2) \in Q_\phi$. Let $v \in J_{x_1}$; then

$$(x_1, vx_2) = vp^{-1}(x, px_2) \in \overline{TQ_\phi} = Q_\phi\,.$$

So $(x_1, x_2) \in S$, which shows that

$$E_\phi \subseteq S \subseteq Q_\phi \circ P_\phi = P_\phi \circ Q_\phi \subseteq E_\phi\,.$$

□

1.20. Let $\phi : \mathcal{X} \to \mathcal{Y}$ be an RMM extension of minimal ttgs and consider the $\mathfrak{b}$ diagram of ϕ.

$$\begin{tikzcd}
\mathcal{X}^{\mathfrak{b}} \arrow[rrr, "\sigma"] \arrow[d, "\phi^{\mathfrak{b}}"'] \arrow[dr] & & & \mathcal{X} \arrow[d, "\phi"] \arrow[dl] \\
& \mathcal{X}^{\mathfrak{b}}/E_{\phi^{\mathfrak{b}}} \arrow[r, "\xi"] \arrow[dl] & \mathcal{X}/E_{\phi} \arrow[dr] & \\
\mathcal{Y}^{\mathfrak{b}} \arrow[rrr, "\tau"'] & & & \mathcal{Y}
\end{tikzcd}$$

Then by 1.16., 1.19. and IV.4.3.c, it follows that $E_{\phi} = Q_{\phi} \circ P_{\phi}$. Hence by IV.4.3.e, we have $\sigma \times \sigma[E_{\phi^{\mathfrak{b}}}] = E_{\phi}$ and so, by IV.4.10., we know that the map $\xi : \mathcal{X}^{\mathfrak{b}}/E_{\phi^{\mathfrak{b}}} \to \mathcal{X}/E_{\phi}$ is proximal. In 3.22. we shall even show more, namely, $E_{\phi} = Q_{\phi}$ for RMM extensions.

VII.2. ERGODIC POINTS

In this section we consider the ergodic behavior inside the neighbourhood of a point. We use it to prove some results concerning the question whether or not the regionally proximal relation is an equivalence relation. In this context, we also discuss weak disjointness. In particular, we generalize a result of S. GLASNER [G 75.1] by proving that an open proximal homomorphism of minimal ttgs is weakly mixing (cf. 2.14. below). We also show that a RIM extension of metric minimal ttgs without nontrivial almost periodic factors is weakly disjoint from every homomorphism of minimal ttgs with the same codomain (2.13.).

2.1. Let $\phi : \mathcal{X} \to \mathcal{Y}$ be a homomorphism of minimal ttgs and let $n \in \mathbb{N}$ with $n \geqslant 2$. A point $x \in X$ is called a *ϕ-n-locally ergodic point* if for every open $W \subseteq X$ there exists a set U, open in $\phi^{\leftarrow}\phi(x)$, such that

(i) $E_{\phi}[x] \subseteq U$;

(ii) $T(V_1 \times \cdots \times V_n) \cap W^n \neq \emptyset$ for every choice of sets $V_i \subseteq U$ open in $\phi^{\leftarrow}\phi(x)$.

If for every W we can take U to be $\phi^{\leftarrow}\phi(x)$, then we call x a *ϕ-n-ergodic point.*

If x is a ϕ-n-(locally) ergodic point for all $n \in \mathbb{N}$ with $n \geqslant 2$, then x is called a *ϕ-(locally) ergodic point*.

If $\phi: \mathfrak{X} \to \{\star\}$ then we skip the prefix ϕ in the definitions above.

2.2. **REMARK.** *Let $\phi: \mathfrak{X} \to \mathfrak{Y}$ be a homomorphism of minimal ttgs.*

a) *If $x \in X$ is a ϕ-n-(locally) ergodic point, then tx is a ϕ-n-(locally) ergodic point for every $t \in T$.*

b) *If $x \in X$ is a ϕ-n-ergodic point, then every $x' \in \phi^{\leftarrow}\phi(x)$ is a ϕ-n-ergodic point.*

c) *If $x \in X$ is a ϕ-n-ergodic point, then it is a ϕ-n-locally ergodic point.*

d) *If $E_\phi = R_\phi$ then $x \in X$ is ϕ-n-ergodic iff it is ϕ-n-locally ergodic.*

PROOF. Straightforward. □

2.3. **EXAMPLE.**

a) *If $\phi: \mathfrak{X} \to \mathfrak{Y}$ is a proximal extension of minimal ttgs then every $x \in X$ is a ϕ-ergodic point.*

b) *If $\phi: \mathfrak{X} \to \mathfrak{Y}$ is such that $\kappa: \mathfrak{X} \to \mathfrak{X}/E_\phi$ is highly proximal (one could say that ϕ is a locally almost periodic map) then every $x \in X$ is a ϕ-locally ergodic point.*

PROOF.

a) Let $W \subseteq X$ be open. Let $x \in X$ and let $V_1, \ldots, V_n$ be open in $\phi^{\leftarrow}\phi(x)$. For every $i \in \{1, \ldots, n\}$ choose $x_i \in V_i$; then $\phi(x_i) = \phi(x)$. As ϕ is proximal $(x_1, \ldots, x_n)$ is proximal to $(x, \ldots, x)$ in X^n. As X and so the diagonal in X^n is minimal, $(x, \ldots, x) \in \overline{T(x_1, \ldots, x_n)}$. Let $t \in T$ be such that $tx \in W$; as $t(x, \ldots, x) \in \overline{T(x_1, \ldots, x_n)}$, $W^n \cap \overline{T(x_1, \ldots, x_n)} \neq \emptyset$. But then $T(V_1 \times \cdots \times V_n) \cap W^n \neq \emptyset$. Hence $x \in X$ is ϕ-ergodic.

b) Let $W \subseteq X$ be open and let $x \in X$. As $\kappa: \mathfrak{X} \to \mathfrak{X}/E_\phi$ is hp, there is a $t \in T$ with $tE_\phi[x] = t\kappa^{\leftarrow}\kappa(x) \subseteq W$. Define $U := t^{-1}W \cap \phi^{\leftarrow}\phi(x)$. Clearly, U satisfies the conditions (i) and (ii) of 2.1. for every $n \in \mathbb{N}$ with $n \geqslant 2$. So x is a ϕ-locally ergodic point. □

2.4. **THEOREM.** *Let* $\phi:\mathcal{X}\to\mathcal{Y}$ *be a homomorphism of minimal ttgs.*

a) *If* $x\in X$ *is a* ϕ*-2-locally ergodic point then* $Q_\phi[x]=E_\phi[x]$ *and* $E_\phi=Q_\phi\circ P_\phi=\{(x_1,x_2)\in R_\phi \mid u(x_1,x_2)\in Q_\phi \text{ for some } u\in J\}$.

b) *If* $x\in X$ *is a* ϕ*-2-ergodic point then* $Q_\phi[x]=\phi^{\leftarrow}\phi(x)$ *and* $R_\phi=Q_\phi\circ P_\phi=\{(x_1,x_2)\in R_\phi \mid u(x_1,x_2)\in Q_\phi \text{ for some } u\in J\}$.

PROOF.

b) (a)) For $x'\in\phi^{\leftarrow}\phi(x)$ ($x'\in E_\phi[x]$) we prove that $x'\in Q_\phi[x]$. For an arbitrary $\alpha\in\mathcal{U}_X$ let $\beta\in\mathcal{U}_X$ be such that $\beta^{-1}=\beta$ and $\beta\circ\beta\subseteq\alpha$, then $\beta(x)\times\beta(x)\subseteq\alpha$. Let $W:=\beta(x)$ and choose U for W as in the definition (2.1.) (in case b $U:=\phi^{\leftarrow}\phi(x)$). Then for every (basic) open neighbourhood $V\times V'$ of (x,x') in $U\times U$ we have $T(V\times V')\cap W\times W\neq\emptyset$, hence

$$\emptyset\neq V\times V'\cap T(\beta(x)\times\beta(x))\subseteq V\times V'\cap T\alpha=V\times V'\cap T\alpha\cap U\times U .$$

But then

$$(x,x')\in\overline{T\alpha\cap U\times U}\subseteq\overline{T\alpha\cap R_\phi} .$$

As $\alpha\in\mathcal{U}_X$ was arbitrary, it follows that

$$(x,x')\in\bigcap\{\overline{T\alpha\cap R_\phi}\mid\alpha\in\mathcal{U}_X\}=Q_\phi .$$

As for some $x\in X$ we have $E_\phi[x]=Q_\phi[x]$, it follows, as in the proof of 1.19., that

$$E_\phi=Q_\phi\circ P_\phi=\{(x_1,x_2)\in R_\phi\mid u(x_1,x_2)\in Q_\phi \text{ for some } u\in J\} .$$

For case b note that if $(x_1,x_2)\in R_\phi$ then

$$u(x_1,x_2)=up^{-1}(px_1,px_2)=up^{-1}(x,px_2)$$

for $p\in M$ with $px_1=x$. As we just proved that $Q_\phi[x]=\phi^{\leftarrow}\phi(x)$ it follows that

$$u(x_1,x_2)=up^{-1}(x,px_2)\in\overline{TQ_\phi}=Q_\phi ,$$

and $(x_1,x_2)\in E_\phi$. □

2.5. **THEOREM.** *Let* $\phi:\mathcal{X}\to\mathcal{Y}$ *be a homomorphism of minimal ttgs, such that* $\kappa:\mathcal{X}\to\mathcal{X}/E_\phi$ *is open. If there exists a* ϕ*-2-locally ergodic point* $x\in X$ *then* $E_\phi=Q_\phi$.

PROOF. Let $x\in X$ be a ϕ-2-locally ergodic point, then $E_\phi[x]=Q_\phi[x]$ by 2.4.a. Let $(x_1,x_2)\in E_\phi$ and define $z_0:=\kappa(x_1)=\kappa(x_2)$ and $z:=\kappa(x)$.

For a net $\{t_i\}_i$ in T with $t_i x \to x_1$ we have $t_i z \to z_0$. As κ is open we can find

$$x_2^i \in E_\phi[x] = \kappa^{\leftarrow}(z) = \kappa^{\leftarrow}\kappa(x) \text{ with } t_i x_2^i \to x_2 .$$

But $E_\phi[x] = Q_\phi[x]$, so $(x, x_2^i) \in Q_\phi$ and $(x_1, x_2) = \lim t_i(x, x_2^i) \in Q_\phi$. □

2.6. **COROLLARY.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *be an open homomorphism of minimal ttgs. If there exists an* $x \in X$ *which is* ϕ*-2-ergodic, then* $R_\phi = Q_\phi$.

PROOF. By 2.4. we have $R_\phi = E_\phi$, hence $\phi = \kappa: \mathfrak{X} \to \mathfrak{X}/E_\phi = \mathfrak{Y}$ is open. So, by 2.5., it follows that $E_\phi = Q_\phi$, hence $E_\phi = Q_\phi = R_\phi$. □

2.7. **COROLLARY.** *Let* $\mathfrak{X}$ *be minimal. Then* $Q_{\mathfrak{X}} = X \times X$ *iff there is a 2-ergodic point in* X *(i.e., iff every point in* X *is 2-ergodic).*

PROOF. If there is a 2-ergodic point in X then by 2.6., $Q_{\mathfrak{X}} = X \times X$. Conversely, let $W \subseteq X$ be open and let U and V in X be open. We have to show that $T(U \times V) \cap W \times W \neq \varnothing$. As follows: Since X is minimal, Δ_X is minimal. So by I.1.1.c, $T(W \times W)$ is a neighbourhood of Δ_X, and there is a $\beta \in \mathfrak{U}_X$ with $T\beta \subseteq T(W \times W)$. As $Q_{\mathfrak{X}} = X \times X$, we have $X \times X = \overline{T\beta}$, so $U \times V \subseteq \overline{T\beta}$ and $U \times V \cap T\beta \neq \varnothing$. Hence $U \times V \cap T(W \times W) \neq \varnothing$ and $T(U \times V) \cap W \times W \neq \varnothing$. □

In several situations (e.g. metrizability of the phase spaces) we can show that a ϕ-n-ergodic point is a point with some "dense proximality" in its fiber.
Let $\phi: \mathfrak{X} \to \mathfrak{Y}$ be a homomorphism of ttgs and let $n \in \mathbb{N}$ with $n \geqslant 2$. A point $x \in X$ is called a *P_ϕ^n-point* if

$$\{(x_1, \ldots, x_n) \in (\phi^{\leftarrow}\phi(x))^n \mid \overline{T(x_1, \ldots, x_n)} \cap \Delta_X^n \neq \varnothing\}$$

is dense in $(\phi^{\leftarrow}\phi(x))^n$ (Δ_X^n is the diagonal in X^n).
Clearly, if ϕ is proximal then every $x \in X$ is a P_ϕ^n-point for all $n \in \mathbb{N}$ with $n \geqslant 2$.

2.8. **THEOREM.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs and let* $n \in \mathbb{N}$ *with* $n \geqslant 2$.

a) *Every* P_ϕ^n*-point is a* ϕ*-*n*-ergodic point.*
b) *If there is a point* $x_0 \in X$ *which has a countable neighbourhood base* $\mathfrak{W}_{x_0}$, *then every* ϕ*-*n*-ergodic point is a* P_ϕ^n*-point.*

In particular, if $\mathfrak{X}$ *is a metric minimal ttg, then the* ϕ*-*n*-ergodic points are just the* P_ϕ^n*-points.*

PROOF.

a) Let $x \in X$ be a P_ϕ^n-point. Choose $W \subseteq X$ open and let $U_1, \ldots, U_n$ be open subsets of $\phi^{\leftarrow}\phi(x)$. We shall show that $T(U_1 \times \cdots \times U_n) \cap W^n \neq \varnothing$. Since $\mathfrak{X}$ is minimal, $\Delta_X^n \subseteq T(W^n)$. As $U_1 \times \cdots \times U_n$ is open in $(\phi^{\leftarrow}\phi(x))^n$ and as X is a P_ϕ^n-point, there is a point

$$(x_1, \ldots, x_n) \in U_1 \times \cdots \times U_n$$

such that $\Delta_X^n \cap \overline{T(x_1, \ldots, x_n)} \neq \varnothing$. So, by minimality of $\mathfrak{X}$,

$$\Delta_X^n \subseteq \overline{T(x_1, \ldots, x_n)} \subseteq \overline{T(U_1 \times \cdots \times U_n)}.$$

Hence $\Delta_X^n \subseteq \overline{T(U_1 \times \cdots \times U_n)} \cap TW^n$ and so

$$T(U_1 \times \cdots \times U_n) \cap W^n \neq \varnothing .$$

b) Let $x \in X$ be a ϕ-n-ergodic point. Choose $U \subseteq (\phi^{\leftarrow}\phi(x))^n$ open in $(\phi^{\leftarrow}\phi(x))^n$, and let $V_1, \ldots, V_n$ be open in $\phi^{\leftarrow}\phi(x)$ such that $V_1 \times \cdots \times V_n \subseteq U$. Let $\mathfrak{W}_{x_0} = \{W_\alpha \mid \alpha \in \mathbb{N}\}$ be the countable neighbourhood base for x_0 in X. For $\alpha \in \mathbb{N}$ define, inductively, $t_\alpha \in T$ and $V_1^\alpha, \ldots, V_n^\alpha$ open in $\phi^{\leftarrow}\phi(x)$ as follows:
As x is a ϕ-n-ergodic point, there is a $t_1 \in T$ with

$$t_1(V_1 \times \cdots \times V_n) \cap W_1^n \neq \varnothing .$$

Define $V_i^1 := V_i$.
Let $V_1^\alpha, \ldots, V_n^\alpha$, open in $\phi^{\leftarrow}\phi(x)$, be defined. Then there is a $t_\alpha \in T$ with

$$t_\alpha(V_1^\alpha \times \cdots \times V_n^\alpha) \cap W_\alpha^n \neq \varnothing .$$

Let $V_i^{\alpha+1} \neq \varnothing$ be open in $\phi^{\leftarrow}\phi(x)$ such that

$$V_i^{\alpha+1} \subseteq \overline{V_i^{\alpha+1}} \subseteq V_i^\alpha \cap t_\alpha^{-1} W_\alpha .$$

For all $i \in \{1, \ldots, n\}$ let

$$x_i \in \bigcap \{V_i^\alpha \mid \alpha \in \mathbb{N}\} \subseteq V_i \cap \bigcap \{t_\alpha^{-1} W_\alpha \mid \alpha \in \mathbb{N}\} .$$

Then $(x_1, \ldots, x_n) \in U$ and $t_\alpha(x_1, \ldots, x_n) \to (x_0, \ldots, x_0)$. Hence x is a P_ϕ^n-point. □

The following shows that there are situations in which lots of ϕ-locally ergodic points exist.

2.9. **LEMMA.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a* RIM *extension of minimal ttgs with section* λ . *Let* $\kappa : \mathfrak{X} \to \mathfrak{X}/E_\phi$ *be the quotient map and let* $x \in X$. *If* x *has a neighbourhood* V *in* $\phi^{\leftarrow}\phi(x)$ *such that*

(i) $E_\phi[x] \subseteq V$ *and* $E_\phi[V] \subseteq \operatorname{supp} \lambda_{\phi(x)}$ *;*

(ii) $\kappa' = \kappa|_{\phi^{\leftarrow}\phi(x)} : \phi^{\leftarrow}\phi(x) \to \kappa[\phi^{\leftarrow}\phi(x)]$ *is open in all points of a dense subset of* V *;*

then x *is a* ϕ*-locally ergodic point.*

PROOF. Let $W \subseteq X$ be open and let $n \in \mathbb{N}$ with $n \geqslant 2$. By I.1.4.a, $\kappa[W]^\circ \neq \varnothing$; so there is an open neighbourhood V^* of $\kappa(x)$ and a $t \in T$ with $tV^* \subseteq \kappa[W]^\circ$. Define $U := \kappa^{\leftarrow}[V^*] \cap V$, then U is an open neighbourhood of x in $\phi^{\leftarrow}\phi(x)$ with $E_\phi[x] \subseteq U$ and U has a dense set of points in which κ' is open. Let $V_1, \ldots, V_n$ be open in $\phi^{\leftarrow}\phi(x)$ with $V_i \subseteq U$. We shall show that $T(V_1 \times \cdots \times V_n) \cap W^n \neq \varnothing$ and so that x is a ϕ-n-locally ergodic point for all $n \in \mathbb{N}$ with $n \geqslant 2$. As the points of openness of κ' are dense in U , we can find $V'_i \subseteq V_i$ open in $\phi^{\leftarrow}\phi(x)$ such that $E_\phi[V'_i] = \kappa^{\leftarrow}\kappa[V'_i]$ is open in $\phi^{\leftarrow}\phi(x)$. Obviously,

$$E_\phi[V'_i] \subseteq E_\phi[U] \subseteq E_\phi[V] \subseteq \operatorname{supp} \lambda_{\phi(x)} .$$

Remember that for $m \in \mathbb{N}$ with $m \geqslant 2$, R^m_ϕ is defined by

$$R^m_\phi = \{(x_1, \ldots, x_m) \in X^m \mid \phi(x_1) = \phi(x_2) = \cdots = \phi(x_m)\} .$$

Let $\phi^m : \mathfrak{R}^m_\phi \to \mathfrak{Y}$ denote the obvious homomorphism. Define λ^m by $\lambda^m_{\phi(x)} = \lambda_{\phi(x)} \times \cdots \times \lambda_{\phi(x)}$ (m-times). Since the support of $\lambda^m_{\phi(x)}$ is included in R^m_ϕ , λ^m may be considered as a mapping from $\mathfrak{Y}$ into $\mathfrak{M}(R^m_\phi)$. Clearly, λ^m is a section for ϕ^m , so $\phi^m : \mathfrak{R}^m_\phi \to \mathfrak{Y}$ is a RIM extension (with section λ^m). As $V'_2 \times \cdots \times V'_n \subseteq \operatorname{supp} \lambda^{n-1}_{\phi(x)}$ and $V'_2 \times \cdots \times V'_n$ is open in $(\phi^{\leftarrow}\phi(x))^n$, it follows from 1.17. applied to ϕ and ϕ^{n-1} , that

$$E_\phi[V'_1] \times V'_2 \times \cdots \times V'_n \subseteq \overline{T(V'_1 \times V'_2 \times \cdots \times V'_n)} .$$

As the set $E_\phi[V'_1] \times V'_3 \times \cdots \times V'_n$ is an open subset of $(\phi^{\leftarrow}\phi(x))^n$ and since $E_\phi[V'_1] \times V'_3 \times \cdots \times V'_n \subseteq \operatorname{supp} \lambda^{n-1}_{\phi(x)}$, it follows from 1.17. and from what we have shown above, that

$$E_\phi[V'_1] \times E_\phi[V'_2] \times V'_3 \times \cdots \times V'_n \subseteq \overline{T(E_\phi[V'_1] \times V'_2 \times \cdots \times V'_n)} \subseteq$$
$$\subseteq \overline{T(V'_1 \times \cdots \times V'_n)} \subseteq \overline{T(V_1 \times \cdots \times V_n)} .$$

Proceeding this way, it follows that

$$E_\phi[V'_1]\times\cdots\times E_\phi[V'_n]\subseteq\overline{T(V'_1\times\cdots\times V'_n)}\subseteq\overline{T(V_1\times\cdots\times V_n)}\,.$$

Since $tE_\phi[V'_i]=E_\phi[tV'_i]=\kappa^{\leftarrow}\kappa[tV'_i]$ and $\kappa[tV'_i]\subseteq\kappa[tU]\subseteq tV^*\subseteq\kappa[W]^\circ$, it follows that $W\cap tE_\phi[V'_i]\neq\emptyset$ for $i\in\{1,\ldots,n\}$. Hence

$$\emptyset\neq W^n\cap t(E_\phi[V'_1]\times\cdots\times E_\phi[V'_n])\subseteq W^n\cap\overline{T(V_1\times\cdots\times V_n)}\,,$$

and so $T(V_1\times\cdots\times V_n)\cap W^n\neq\emptyset$, which completes the proof. □

The following lemma is taken from [V 70] (prop. 3.1.), to which we refer for the proof.

2.10. **LEMMA.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a homomorphism of minimal ttgs (it is sufficient to require* $\mathfrak{Y}$ *to be minimal and* X *to have a dense subset of almost periodic points). Let* $X_0\subseteq X$ *be a residual subset of* X. *Then there is a residual subset* $Y_0\subseteq Y$ *such that* $X_0\cap\phi^{\leftarrow}(y)$ *is residual in* $\phi^{\leftarrow}(y)$ *for all* $y\in Y_0$. □

2.11. **THEOREM.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a* RIM *extension of minimal ttgs.*

a) *If* X *is metrizable, then there is a residual set of* ϕ*-locally ergodic points.*

b) *If* $R_\phi=E_\phi$ *then every* $x\in X$ *with* $\operatorname{supp}\lambda_{\phi(x)}=\phi^{\leftarrow}\phi(x)$ *is a* ϕ*-ergodic point.*

PROOF.

a) Let $\kappa:X\to X/E_\phi$ be the quotient map. As X is metric it follows from II.1.3.e that there is a residual set $X_1\subseteq X$ in each point of which κ is open. Hence, in each point x of X_1 the map $\kappa':\phi^{\leftarrow}\phi(x)\to\kappa[\phi^{\leftarrow}\phi(x)]$ is open in x. By 1.6., there exists a residual set $X_2\subseteq X$ such that $\operatorname{supp}\lambda_{\phi(x)}=\phi^{\leftarrow}\phi(x)$ for every $x\in X_2$. (Note that the full original of a residual set in Y is a residual set in X, by IV.5.12.). Let $X_0=X_1\cap X_2$; then X_0 is residual. By 2.10., there is a residual set $Y_0\subseteq Y$ such that $X_0\cap\phi^{\leftarrow}(y)$ is residual in $\phi^{\leftarrow}(y)$ for every $y\in Y_0$. Let $x\in\phi^{\leftarrow}[Y_0]$; then κ is open in all points of $X_0\cap\phi^{\leftarrow}\phi(x)$, which is is a dense subset of $\phi^{\leftarrow}\phi(x)$. Also $\operatorname{supp}\lambda_{\phi(x)}=\phi^{\leftarrow}\phi(x)$. But then $\phi^{\leftarrow}\phi(x)$ is an open neighbourhood of x in $\phi^{\leftarrow}\phi(x)$ that satisfies the conditions in 2.9.. So by 2.9., x is a ϕ-locally ergodic point. As $\phi^{\leftarrow}[Y_0]$ is residual in X this proves a .

b) In this case κ and ϕ are identical. If for some $x\in X$ we have $\operatorname{supp}\lambda_{\phi(x)}=\phi^{\leftarrow}\phi(x)$ then, by 1.5., κ is open in every point of $\phi^{\leftarrow}\phi(x)$. So $\phi^{\leftarrow}\phi(x)$ is an open neighbourhood of x in $\phi^{\leftarrow}\phi(x)$ that satisfies the conditions in 2.9.. So by 2.9., x is a ϕ-locally ergodic point. But, since $E_\phi[x]=\phi^{\leftarrow}\phi(x)$, x is even a ϕ-ergodic point. □

Ergodic points can play a role in weak disjointness of homomorphisms of minimal ttgs as the following generalization of [G 75.1] Thm. 1.1. shows.

2.12. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *and* $\psi : \mathfrak{Z} \to \mathfrak{Y}$ *be homomorphisms of minimal ttgs and let one of them be open or suppose that* (ϕ,ψ) *satisfies the generalized Bronstein condition. If for every* $n \in \mathbb{N}$ *with* $n \geqslant 2$ *there exists a* ϕ-n*-ergodic point in* X *, then* ϕ *and* ψ *are weakly disjoint (* $\phi \;\underline{\,\cdot\,}\; \psi$ *).*

PROOF. Let $W = \overline{TW} \subseteq R_{\phi\psi}$ with $\operatorname{int}_{R_{\phi\psi}} W \neq \varnothing$. We shall show that $W = R_{\phi\psi}$, as follows: For $(x',z') \in R_{\phi\psi}$ and an arbitrary open neighbourhood O of (x',z') in $R_{\phi\psi}$ we shall prove that $O \cap W \neq \varnothing$ and so that $(x',z') \in \overline{W} = W$.

By the assumption and I.3.7., we can find open sets U_1 and V_1 in X and Z such that $\varnothing \neq U_1 \times V_1 \cap R_{\phi\psi} \subseteq O$ and $\phi[U_1] = \psi[V_1]$. Also we can find open sets U and V in X and Z with $\varnothing \neq U \times V \cap R_{\phi\psi} \subseteq W$ and $\phi[U] = \psi[V]$.

As $\mathfrak{Z}$ is minimal there are finitely many $t_1, \ldots, t_m$ in T such that $Z = \bigcup \{t_i V \mid i \in \{1, \cdots, m\}\}$. By assumption, X contains a ϕ-m-ergodic point, and so by 2.2.a, X has a dense set of ϕ-m-ergodic points. Let x be a ϕ-m-ergodic point in U_1 and let $z \in V_1$ be such that $\phi(x) = \psi(z)$, say $y = \phi(x) = \psi(z)$. As

$$\psi^{\leftarrow}(y) \subseteq Z \subseteq \{t_i V \mid i \in \{1, \ldots, m\}\} ,$$

we may renumerate (if necessary) the t_i's in such a way that for some $n \leqslant m$ we have $\psi^{\leftarrow}(y) \subseteq \bigcup \{t_i V \mid i \in \{1, \ldots, m\}\}$, while $\psi^{\leftarrow}(y) \cap t_i V \neq \varnothing$ for every $i \in \{1, \ldots, n\}$.

Suppose $n \geqslant 2$. Define

$$L := t_1 U \times \cdots \times t_n U \cap (\phi^{\leftarrow}(y))^n ,$$

then L is open in $(\phi^{\leftarrow}(y))^n$ and nonempty. For let $z_i \in V \cap t_i^{-1} \psi^{\leftarrow}(y)$ and let $x_i \in U$ be such that $\phi(x_i) = \psi(z_i)$, then $(t_1 x_1, \ldots, t_n x_n) \in L$. As x is ϕ-n-ergodic, $TL \cap (U_1)^n \neq \varnothing$. So for some $t \in T$ we have

$$t(t_1 U \times \cdots \times t_n U \cap (\phi^{\leftarrow}(y))^n) \cap (U_1)^n \neq \varnothing ;$$

i.e., we can find $x'_i \in U$ with $tt_i x'_i \in U_1 \cap \phi^{\leftarrow}(ty)$ for $i \in \{1, \ldots, n\}$. Let $z' \in V_1$ be such that $\psi(z') = \phi(tt_i x'_i) = ty$ for every $i \in \{1, \ldots, n\}$, then $t^{-1} z' \in \psi^{\leftarrow}(y)$. But then for some $i_0 \in \{1, \ldots, n\}$ we have $t^{-1} z' \in t_{i_0} V \cap \psi^{\leftarrow}(y)$ and $z' \in tt_{i_0} V$. Hence

$$(tt_{i_0} x'_{i_0}, z') \in tt_{i_0} U \times tt_{i_0} V \cap R_{\phi\psi} \subseteq T(U \times V \cap R_{\phi\psi}) \subseteq TW = W ,$$

and

$$(tt_{i_0}x'_{i_0}, z') \in U_1 \times V_1 \cap R_{\phi\psi} \subseteq O \ .$$

So $W \cap O \neq \varnothing$, which settles the case for $n \geqslant 2$.
Suppose that $n = 1$; i.e., $\psi^{\leftarrow}(y) \subseteq t_i V$. Then $t_1 U \cap \phi^{\leftarrow}(y) \neq \varnothing$ and by minimality of $\mathcal{X}$, we can find a $t \in T$ with $t(t_1 U \cap \phi^{\leftarrow}(y)) \cap U_1 \neq \varnothing$. Let $x'_1 \in U$ be such that $t_1 x'_1 \in \phi^{\leftarrow}(y)$ and $tt_1 x'_1 \in U_1$ and choose $z_1 \in V_1$ with $\phi(tt_1 x'_1) = \psi(z') = ty$. Then $t^{-1}z' \in \psi^{\leftarrow}(y) \subseteq t_1 V$, so $z' \in tt_1 V$ and

$$(tt_1 x'_1, z') \in T(U \times V \cap R_{\phi\psi}) \subseteq W \ ,$$

while

$$(tt_1 x'_1, z') \in U_1 \times V_1 \cap R_{\phi\psi} \subseteq O \ .$$

Hence $W \cap O \neq \varnothing$. □

2.13. **COROLLARY.** *Let* $\phi: \mathcal{X} \to \mathcal{Y}$ *be an open* RIM *extension of minimal ttgs with section* λ . *Suppose there is an* $x \in X$ *with* $\phi^{\leftarrow}\phi(x) = \operatorname{supp} \lambda_{\phi(x)}$ *(e.g.* X *is metrizable). Then the following statements are equivalent:*

a) $E_\phi = R_\phi$ *;*
b) $Q_\phi = R_\phi$ *;*
c) ϕ *is weakly mixing;*
d) $\phi \mathrel{\dot{-}} \psi$ *for every homomorphism* $\psi: \mathcal{Z} \to \mathcal{Y}$ *of minimal ttgs.*

In particular, if $\mathcal{X}$ *is minimal and has an invariant measure, then* $\mathcal{X}$ *is weakly mixing iff* $\mathcal{X}$ *is weakly disjoint from every minimal ttg iff* $E_{\mathcal{X}} = Q_{\mathcal{X}} = X \times X$.

PROOF. The implications d $\Rightarrow$ c $\Rightarrow$ b $\Rightarrow$ a are obvious (for c $\Rightarrow$ b see I.3.11.).

a $\Rightarrow$ d By 2.11.b, x is a ϕ-ergodic point and so x is ϕ-n-ergodic for every $n \in \mathbb{N}$ with $n \geqslant 2$. As ϕ is open it follows from 2.12. that $\phi \mathrel{\dot{-}} \psi$ for every homomorphism $\psi: \mathcal{Z} \to \mathcal{Y}$ of minimal ttgs.
If $\mathcal{X}$ is minimal and has an invariant measure μ , then $X = \operatorname{supp} \mu$, so we can apply the above equivalences to $\phi: \mathcal{X} \to \{\star\}$. □

2.14. **COROLLARY.** *Let* $\phi: \mathcal{X} \to \mathcal{Y}$ *and* $\psi: \mathcal{Z} \to \mathcal{Y}$ *be homomorphisms of minimal ttgs and let one of them be open. If* ϕ *is proximal, then* $\phi \mathrel{\dot{-}} \psi$. *In particular, an open proximal homomorphism of minimal ttgs is weakly mixing.*

PROOF. If ϕ is proximal, then every $x \in X$ is a ϕ-ergodic point by 2.3.a, and so every $x \in X$ is ϕ-n-ergodic for every $n \in \mathbb{N}$ with $n \geqslant 2$. The corollary follows from 2.12.. ☐

Looking at the ergodic behavior inside the neighbourhood of some specific point $x \in X$ turns out to be a little inconvenient. Too many times countability assumptions or openness are required to come to reasonable results. In the next section we shall "globalize" our efforts to prove stronger results for weak disjointness problems.

VII.3. WEAK DISJOINTNESS AND MAXIMALLY ALMOST PERIODIC FACTORS

> A central theme in this section is the question, how "unrelated are homomorphisms whose maximal almost periodic factors are disjoint (see [P 72], [K 72] and [EGS 76] 4.2.). So consider the next diagram of homomorphisms of minimal ttgs:

$$\begin{tikzcd}
\mathcal{X} \arrow[d, "\kappa_{\mathcal{X}}"'] \arrow[dr, "\phi"] & & \mathcal{Y} \arrow[dl, "\psi"'] \arrow[d, "\kappa_{\mathcal{Y}}"] \\
\mathcal{X}/E_\phi \arrow[r, "\theta_{\mathcal{X}}"'] & \mathcal{Z} & \mathcal{Y}/E_\psi \arrow[l, "\theta_{\mathcal{Y}}"]
\end{tikzcd}$$

> We shall prove that in several cases we have $\theta_{\mathcal{X}} \perp \theta_{\mathcal{Y}}$ iff $\phi \mathrel{\overset{\cdot}{\_}} \psi$.
> As a by-product we shall see that for an open RIM extension the regionally proximal relation is an equivalence relation.

We shall need the following remark on lifting of ergodicity.

3.1. **THEOREM.** *Let $\phi: \mathcal{X} \to \mathcal{Y}$ be a surjective proximal homomorphism of ttgs and let $\mathcal{X}$ have a dense subset of almost periodic points. Then $\mathcal{X}$ is ergodic iff $\mathcal{Y}$ is ergodic.*

PROOF. Clearly, if $\mathcal{X}$ is ergodic then $\mathcal{Y}$ is ergodic (I.1.3.e).
Conversely, suppose that $\mathcal{Y}$ is ergodic. Let $A \subseteq X$ with $A = \overline{TA}$ and $A^\circ \neq \emptyset$ and let $B := \overline{X \setminus A}$. Then $B = \overline{TB}$ and $X = A \cup B$.

As $\phi[A] \cup \phi[B] = \phi[X] = Y$, $\phi[A]$ or $\phi[B]$ must have a nonempty interior in Y , and so, by ergodicity of $\mathfrak{Y}$, $\phi[A] = Y$ or $\phi[B] = Y$.
Suppose that $\phi[A] = Y$. Let $x \in X$ be an almost periodic point. Then for some $a \in A$, $\phi(a) = \phi(x)$. As ϕ is proximal, a and x are proximal and by almost periodicity of x we have that $x \in \overline{Ta} \subseteq \overline{TA} = A$. Consequently, every almost periodic point in $\mathfrak{X}$ is in A , so $X = A$.
Suppose that $\phi[B] = Y$ then, similarly, it follows that $X = B$; which contradicts the assumption of $A^{\circ} \neq \varnothing$.
Hence $X = A$ and $\mathfrak{X}$ is ergodic. □

As we intent to relate weak disjointness of ϕ and ψ with the weak disjointness of their maximally almost periodic factors, we need to relate open sets in $R_{\phi\psi}$ with open sets in $R_{\theta_{\mathfrak{X}}\theta_{\mathfrak{Y}}}$. We shall do this in the following lemmas in a slightly generalized form.

3.2. **LEMMA.** *Consider the following commutative diagram of (surjective) homomorphisms of ttgs:*

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\kappa"] \arrow[dr, "\phi"'] & \mathfrak{X}' \arrow[d, "\theta"] & \mathfrak{Y} \arrow[dl, "\psi"] \\
& \mathfrak{Z} &
\end{tikzcd}$$

Let $\mathfrak{X}$ be minimal and suppose that one of the following conditions is satisfied

(i) *(ϕ,ψ) satisfies the generalized Bronstein condition;*

(ii) *ψ is open;*

(iii) *ϕ is open and Y has a dense set of points in which ψ is open;*

(iv) *ϕ is open and Y has a dense subset of almost periodic points.*

If W is a nonempty set which is open in $R_{\phi\psi}$, then:

a) *There exist open sets U and V in X and Y such that*

$$\varnothing \neq U \times V \cap R_{\phi\psi} \subseteq W \text{ and } \phi[U] = \psi[V] ;$$

b) *$\kappa \times id_Y[W]$ has a nonempty interior in $R_{\theta\psi}$.*

PROOF.

a) This is just I.3.7.; for case (iii) note that ψ is semi-open.

b) Let $W \subseteq R_{\phi\psi}$ be open and nonempty and let U and V be as in a. As ϕ is semi-open, $W' := \mathrm{int}_Z \phi[U]$ is nonempty, and, clearly, $W' \subseteq \mathrm{int}_Z(\phi \times \psi[W])$. Define $U' := U \cap \phi^{\leftarrow}[W']$; then U' is open and nonempty. As $\mathfrak{X}$ is minimal, $\kappa[U']$ has a nonempty interior in X'. Since

$$\varnothing \neq \theta[\kappa[U']^\circ] \subseteq \phi[U] = \psi[V]$$

it follows that $\kappa[U']^\circ \times V \cap R_{\theta\psi}$ is a nonempty open subset of $R_{\theta\psi}$, which is contained in $\kappa \times id_Y[W]$. □

3.3. We shall consider a diagram as in lemma 3.2., so $\mathfrak{X}$ and $\mathfrak{Z}$ are minimal, and we shall deal with the following question (in the case that $\mathfrak{X}' = \mathfrak{X}/E_\phi$):
Under what conditions do we have $\phi \mathrel{\underline{\cdot}} \psi$ if and only if $\theta \mathrel{\underline{\cdot}} \psi$.

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\kappa"] \arrow[dr, "\phi"'] & \mathfrak{X}/E_\phi \arrow[d, "\theta"] & \mathfrak{Y} \arrow[dl, "\psi"] \\
 & \mathfrak{Z} &
\end{tikzcd}$$

Clearly, $\phi \mathrel{\underline{\cdot}} \psi$ implies $\theta \mathrel{\underline{\cdot}} \psi$, as $\kappa \times id_Y[R_{\phi\psi}] = R_{\theta\psi}$. So the real problem is what we can say about the converse implication.

3.4. **LEMMA.** *Consider the diagram in* 3.3. *and let* ϕ *and* ψ *satisfy one of the conditions in lemma* 3.2.. *Suppose that for every nonempty (basic) open set* $U \times V \cap R_{\phi\psi}$ *in* $R_{\phi\psi}$ *there is an open set* $\tilde{U}$ *in* X *such that*

$$\tilde{U} = E_\phi[\tilde{U}] \text{ and } \varnothing \neq \tilde{U} \times V \cap R_{\phi\psi} \subseteq \overline{T(U \times V \cap R_{\phi\psi})}.$$

Then $\phi \mathrel{\underline{\cdot}} \psi$ *iff* $\theta \mathrel{\underline{\cdot}} \psi$.

PROOF. Suppose $\theta \mathrel{\underline{\cdot}} \psi$; i.e., suppose that $R_{\theta\psi}$ is ergodic. For an arbitrary nonempty (basic) open set $U \times V \cap R_{\phi\psi}$ in $R_{\phi\psi}$, we shall prove that $R_{\phi\psi} = \overline{T(U \times V \cap R_{\phi\psi})}$. Then it follows that $R_{\phi\psi}$ is ergodic.
Let $\tilde{U}$ be a nonempty open set in X as in the assumption, and note that $\kappa[\tilde{U}]$ is an open set in X/E_ϕ, because $\tilde{U} = E_\phi[\tilde{U}] = \kappa^{\leftarrow}\kappa[\tilde{U}]$. Hence $\kappa[\tilde{U}] \times V \cap R_{\theta\psi}$ is open in $R_{\theta\psi}$ and nonempty, for $\tilde{U} \times V \cap R_{\phi\psi}$ was supposed to be nonempty. As $R_{\theta\psi}$ is an ergodic set it follows that $T(\kappa[\tilde{U}] \times V \cap R_{\theta\psi})$ is dense in $R_{\theta\psi}$.
Let $U_1 \times V_1 \cap R_{\phi\psi}$ be an arbitrary nonempty (basic) open set in $R_{\phi\psi}$. Then by 3.2.b, $\kappa[U_1] \times V_1 \cap R_{\theta\psi}$ has a nonempty interior in $R_{\theta\psi}$. Hence, by ergodicity of $R_{\theta\psi}$, for some $t \in T$ we have

$$\kappa[U_1] \times V_1 \cap R_{\theta\psi} \cap t\kappa[\tilde{U}] \times tV \neq \varnothing .$$

Let $(x_1,y_1)\in U_1\times V_1\cap R_{\phi\psi}$ be such that

$$(\kappa(x_1),y_1)\in\kappa[U_1]\times V_1\cap R_{\theta\psi}\cap t\kappa[\tilde{U}]\times tV\,.$$

Then $t^{-1}x_1\in\kappa^{\leftarrow}\kappa[\tilde{U}]=E_\phi[\tilde{U}]=\tilde{U}$, so $(x_1,y_1)\in t(\tilde{U}\times V\cap R_{\phi\psi})$ and by assumption

$$(x_1,y_1)\in t(\tilde{U}\times V\cap R_{\phi\psi})\subseteq\overline{T(U\times V\cap R_{\phi\psi})}\,.$$

Hence $U_1\times V_1\cap\overline{T(U\times V\cap R_{\phi\psi})}\neq\varnothing$. As $U_1\times V_1$ is open, we have

$$U_1\times V_1\cap R_{\phi\psi}\cap T(U\times V\cap R_{\phi\psi})\neq\varnothing\,.$$

But $U_1\times V_1\cap R_{\phi\psi}$ was arbitrary, so it follows that $T(U\times V\cap R_{\phi\psi})$ is dense in $R_{\phi\psi}$, which proves the theorem. □

We shall look for situations in which 3.4. is applicable. For that purpose we need the following lemma.

3.5. **LEMMA.** *Consider the diagram in* 3.3. *and let* ϕ *and* ψ *satisfy one of the conditions in lemma* 3.2.. *Assume that for every nonempty (basic) open set* $U_1\times V_1\cap R_{\phi\psi}$ *in* $R_{\phi\psi}$ *there is a point* $(x,y)\in U_1\times V_1\cap R_{\phi\psi}$ *such that*

$$E_\phi[x]\times\{y\}\subseteq\overline{T(U_1\times V_1\cap R_{\phi\psi})}\,.$$

Then for every nonempty (basic) open set $U\times V\cap R_{\phi\psi}$ *in* $R_{\phi\psi}$ *we have*

$$\varnothing\neq\tilde{U}\times V\cap R_{\phi\psi}\subseteq\overline{T(U\times V\cap R_{\phi\psi})}\,,$$

where $\tilde{U}:=\kappa^{\leftarrow}[\kappa[U]^\circ]=E_\phi[\tilde{U}]$.
Consequently, under this assumption $\theta\mathrel{\underline{\cdot}}\psi$ *implies* $\phi\mathrel{\underline{\cdot}}\psi$.

PROOF. Let $W:=U\times V\cap R_{\phi\psi}$ be an arbitrary nonempty (basic) open set in $R_{\phi\psi}$. Define $\tilde{U}:=\kappa^{\leftarrow}[\kappa[U]^\circ]$ and note that $\tilde{U}=E_\phi[\tilde{U}]$ is nonempty and open in X. Let U' and V' be open sets for W as in 3.2.a. Then by 3.2.b, $\kappa[U']^\circ\times V'\cap R_{\theta\psi}\neq\varnothing$. Let $u\in U'$ with $\kappa(u)\in\kappa[U']^\circ$ and $v\in V'$ with $(u,v)\in R_{\phi\psi}$, then

$$(u,v)\in(U'\cap\kappa^{\leftarrow}[\kappa[U']^\circ])\times V'\cap R_{\phi\psi}\subseteq\kappa^{\leftarrow}[\kappa[U]^\circ]\times V\cap R_{\phi\psi}=$$
$$=\tilde{U}\times V\cap R_{\phi\psi}\,;$$

so $\tilde{U}\times V\cap R_{\phi\psi}$ is nonempty and open in $R_{\phi\psi}$.
In order to prove that $\tilde{U}\times V\cap R_{\phi\psi}\subseteq\overline{T(U\times V\cap R_{\phi\psi})}$ we have to show

that every (basic) open subset $U_1 \times V_1 \cap R_{\phi\psi}$ of $\tilde{U} \times V \cap R_{\phi\psi}$ has a nonempty intersection with $T(U \times V \cap R_{\phi\psi})$.
So let $U_1 \times V_1 \cap R_{\phi\psi}$ be a nonempty (basic) open set inside $\tilde{U} \times V \cap R_{\phi\psi}$; i.e., $U_1 \subseteq \tilde{U}$ and $V_1 \subseteq V$. By the assumption, we can find a point $(x,y) \in U_1 \times V_1 \cap R_{\phi\psi}$ such that $E_\phi[x] \times \{y\} \subseteq \overline{T(U_1 \times V_1 \cap R_{\phi\psi})}$. Then $x \in U_1 \subseteq \tilde{U} \subseteq \kappa^{\leftarrow}\kappa[U]$ and $y \in V$. Let $u' \in U$ with $\kappa(u') = \kappa(x)$; then $u' \in E_\phi[x]$. Since

$$(u',y) \in U \times V \cap R_{\phi\psi} \cap E_\phi[x] \times \{y\} \subseteq U \times V \cap R_{\phi\psi} \cap \overline{T(U_1 \times V_1 \cap R_{\phi\psi})}$$

and as $U \times V \cap R_{\phi\psi}$ is open, it follows that

$$U \times V \cap R_{\phi\psi} \cap T(U_1 \times V_1 \cap R_{\phi\psi}) \neq \emptyset .$$

But then

$$U_1 \times V_1 \cap R_{\phi\psi} \cap T(U \times V \cap R_{\phi\psi}) \neq \emptyset ,$$

which proves the lemma. □

We shall now consider two situations in which the assumptions of lemma 3.5. are satisfied.

3.6. **THEOREM.** *Consider the diagram in* 3.3. *with* ϕ *a* RIC *extension and let* (ϕ,ψ) *satisfy* gBc. *Then* $\phi \mathrel{\dot{-}} \psi$ *iff* $\theta \mathrel{\dot{-}} \psi$.

PROOF. We shall prove that if $(x,y) \in R_{\phi\psi}$ is an almost periodic point and if $U \times V \cap R_{\phi\psi}$ is a (basic) open neighbourhood of (x,y) in $R_{\phi\psi}$, then

$$E_\phi[x] \times \{y\} \subseteq \overline{T(U \times V \cap R_{\phi\psi})} .$$

Note that the assumption of (ϕ,ψ) satisfying gBc together with the above, gives that the assumptions of lemma 3.5. are satisfied. Hence it follows that $\phi \mathrel{\dot{-}} \psi$ iff $\theta \mathrel{\dot{-}} \psi$.
Let (x,y) be an almost periodic point in $R_{\phi\psi}$, say $(x,y) = u(x,y)$ for some $u \in J$. Let $U \times V \cap R_{\phi\psi}$ be a (basic) open neighbourhood of (x,y) in $R_{\phi\psi}$. As V is an open neighbourhood of $y = uy$ in Y, the set $V' := V \cap \overline{Ty}$ is a neighbourhood of y in $\overline{Ty}$. So by III.2.1.c, we can find an open set W in T which has the form $W = W(u)$, such that $W.y \subseteq V' \subseteq V$. Define $\mathcal{U} := [U, W] \cap u\phi^{\leftarrow}(z)$, where $z = \phi(x) = \psi(y)$. Then $\mathcal{U}$ is an $\mathfrak{T}(\mathfrak{X}, u)$-neighbourhood of x in $u\phi^{\leftarrow}(z)$ (III.2.2.). Let $x' \in \mathcal{U}$, then $x' = t^{-1}x_0 \in u\phi^{\leftarrow}(z)$ for some $t \in W$ and $x_0 \in U$. So

$$(x',y) = t^{-1}(x_0, ty) \in t^{-1}(U \times Wy) \cap R_{\phi\psi} \subseteq T(U \times V \cap R_{\phi\psi}) .$$

Hence $\boldsymbol{U}\times\{y\}\subseteq T(U\times V\cap R_{\phi\psi})$ and so

$$u\circ\boldsymbol{U}\times\{y\}=u\circ(\boldsymbol{U}\times\{y\})\subseteq\overline{T(U\times V\cap R_{\phi\psi})}\,.$$

By III.3.10.b, we know that $E_\phi[x]\subseteq u\circ\boldsymbol{U}$, so

$$E_\phi[x]\times\{y\}\subseteq u\circ\boldsymbol{U}\times\{y\}\subseteq\overline{T(U\times V\cap R_{\phi\psi})}\,,$$

which proves the theorem. □

3.7. Let $\psi:\mathcal{Y}\to\mathcal{Z}$ be a RIM extension and denote the collection of sections for ψ by $\Sigma(\psi)$. A point $y\in Y$ is called a *supprim point* if $y\in\operatorname{supp}\lambda_{\psi(y)}$ for some section $\lambda\in\Sigma(\psi)$.
Note that in the following cases the supprim points are dense in Y :

a) $\mathcal{Y}$ is minimal;
b) $\mathcal{Z}$ is minimal, $\operatorname{supp}\lambda_z=\psi^{\leftarrow}(z)$ for some $\lambda\in\Sigma(\psi)$ and some $z\in Z$, and either ψ is open or Y has a dense set of almost periodic points.

3.8. **THEOREM.** *Consider the diagram in* 3.3. *and let* ϕ *and* ψ *satisfy one of the conditions in lemma* 3.2.. *If* ψ *is a* RIM *extension and if* Y *has a dense set of supprim points, then* $\phi\mathrel{\underline{\,\cdot\,}}\psi$ *iff* $\theta\mathrel{\underline{\,\cdot\,}}\psi$.

PROOF. We shall prove that for every nonempty (basic) open set $U\times V\cap R_{\phi\psi}$ in $R_{\phi\psi}$ there is a point $(x,y)\in U\times V\cap R_{\phi\psi}$ such that $E_\phi[x]\times\{y\}\subseteq\overline{T(U\times V\cap R_{\phi\psi})}$. Then the theorem follows from 3.5..
Let $U\times V\cap R_{\phi\psi}$ be an arbitrary nonempty (basic) open set in $R_{\phi\psi}$. Then by 3.2., there are open sets U' and V' in X and Y such that $\phi[U']=\psi[V']$ and $\emptyset\neq U'\times V'\cap R_{\phi\psi}\subseteq U\times V\cap R_{\phi\psi}$. As the supprim points are dense in Y , there is a $\lambda\in\Sigma(\psi)$ and a $y\in V'$ with $y\in\operatorname{supp}\lambda_{\psi(y)}$. Let $x\in U'$ with $\phi(x)=\psi(y)$. Then by 1.17., we have

$$E_\phi[x]\times\{y\}\subseteq E_\phi[x]\times(V'\cap\operatorname{supp}\lambda_{\phi(x)})\subseteq\overline{T(\{x\}\times V'\cap R_{\phi\psi})}\,,$$

so $E_\phi[x]\times\{y\}\subseteq\overline{T(U'\times V'\cap R_{\phi\psi})}\subseteq\overline{T(U\times V\cap R_{\phi\psi})}$. □

Let $\phi:\mathcal{X}\to\mathcal{Y}$ be a homomorphism of minimal ttgs. We call ϕ a *totally weakly mixing extension* iff

$$R_\phi^n=\{(x_1,\ldots,x_n)\in X^n\mid\phi(x_1)=\cdots=\phi(x_n)\}$$

is ergodic for every $n\in\mathbb{N}$ with $n\geqslant 2$.

3.9. **LEMMA.** *Consider the diagram in* 3.3. *and let* ϕ *and* ψ *satisfy one of the conditions in lemma* 3.2. *(* $\mathfrak{X}$ *minimal). Let* ϕ *be a* RIM *extension with section* λ *and let* Y_0 *be a dense subset in* Y . *Then*

$$\bigcup\{\operatorname{supp}\lambda_z \times Y_0 \cap R_{\phi\psi} \mid z \in Z\}$$

is dense in $R_{\phi\psi}$.

PROOF. Let $U \times V \cap R_{\phi\psi}$ be a (basic) open set in $R_{\phi\psi}$. By 3.2.a, we may assume, without loss of generality, that $\phi[U] = \psi[V]$. Let $x \in U$ with $x \in \operatorname{supp}\lambda_{\phi(x)}$ and let $y \in V$ be such that $\psi(y) = \phi(x)$. As Y_0 is dense in Y , there is a net $\{y_i\}_i$ in Y_0 converging to y . Then $\{\psi(y_i)\}_i$ converges to $\phi(x)$, hence $\{\lambda_{\psi(y_i)}\}_i$ converges to $\lambda_{\phi(x)}$ in $\mathfrak{M}(X)$. By 1.4., it follows that $x \in \operatorname{supp}\lambda_{\phi(x)} \subseteq \lim_{2^X} \operatorname{supp}\lambda_{\psi(y_i)}$. As U is an open neighbourhood of x in X , there is a i_0 such that $U \cap \operatorname{supp}\lambda_{\psi(y_i)} \neq \emptyset$ for every $i \geqslant i_0$. So we can find an $i_1 \geqslant i_0$ with $y_1 := y_{i_1} \in V$ and a supprim point $x_1 \in U \cap \operatorname{supp}\lambda_{\psi(y_1)}$. Hence

$$(x_1, y_1) \in U \times V \cap R_{\phi\psi} \cap \operatorname{supp}\lambda_{\psi(y_1)} \times Y_0 .$$

□

3.10. **COROLLARY.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *be an open* RIM *extension of minimal ttgs with section* λ . *Then for every* $n \in \mathbb{N}$ *with* $n \geqslant 2$ *the canonical homomorphism* $\phi_n : \mathfrak{R}_\phi^n \to \mathfrak{Y}$ *is an open* RIM *extension with section* λ^n *and the supprim points are dense in* R_ϕ^n .

PROOF. Remember that λ^n is defined by $\lambda_y^n = \lambda_y \times \cdots \times \lambda_y$ (n-times) and note that $\operatorname{supp}\lambda_y^n = \operatorname{supp}\lambda_y \times \cdots \times \operatorname{supp}\lambda_y$ (n-times). Clearly, λ^n is a section for ϕ_n (cf. the proof of 2.9.) and the fact that ϕ_n is open is obvious from the observation that

$$\phi_n(U_1 \times \cdots \times U_n \cap R_\phi^n) = \bigcap_{i=1}^{n} \phi[U_i] .$$

As $\mathfrak{X}$ is minimal, the set X_0 of supprim points is dense in X . So by 3.9., applied to ϕ and ϕ , it follows that $\bigcup\{\operatorname{supp}\lambda_y \times \operatorname{supp}\lambda_y \mid y \in Y\}$ is dense in $R_\phi = R_\phi^2$. Suppose, the corollary is true for $n_0 \in \mathbb{N}$ ($n_0 \geqslant 2$); then apply 3.9. to ϕ_{n_0} and ϕ . It follows that

$$\bigcup\{\operatorname{supp}\lambda_y^{n_0} \times \operatorname{supp}\lambda_y \mid y \in Y\} = \bigcup\{\operatorname{supp}\lambda_y^{n_0+1} \mid y \in Y\}$$

is dense in $R_\phi^{n_0+1}$. By induction the corollary follows. □

3.11. **THEOREM.** *Let* $\phi : \mathscr{X} \to \mathscr{Y}$ *be a homomorphism of minimal ttgs. If* ϕ *is a* RIC *extension or an open* RIM *extension then the following statements are equivalent:*

a) $E_\phi = R_\phi$ *;*
b) ϕ *is totally weakly mixing;*
c) ϕ *is weakly mixing.*

PROOF.

b ⇒ c Trivial.

c ⇒ a If ϕ is weakly mixing then, by ergodicity of R_ϕ, it follows that $R_\phi = \overline{T\alpha \cap R_\phi}$ for every $\alpha \in \mathscr{U}_X$. Hence

$$R_\phi = \bigcap \{\overline{T\alpha \cap R_\phi} \mid \alpha \in \mathscr{U}_X\} = Q_\phi \subseteq E_\phi \subseteq R_\phi .$$

a ⇒ b If $E_\phi = R_\phi$ then $\theta : \mathscr{X}/E_\phi \to \mathscr{Y}$ is an isomorphism. Hence R_ϕ^n is ergodic iff $\theta \mathrel{\underline{\cdot}} \phi_n$, for $R_\phi^n \cong R_{\theta\phi_n}$.

Suppose ϕ is a RIC extension. By III.1.9., it follows that R_ϕ^n has a dense subset of almost periodic points for every $n \in \mathbb{N}$ with $n \geqslant 2$. Hence by III.1.5.b, it follows that (ϕ, ϕ_n) satisfies gBc for every $n \in \mathbb{N}$ with $n \geqslant 2$, where $\phi_1 := \phi$. As θ is an isomorphism, $\theta \perp \phi$; so it follows from 3.6., applied to ϕ and ϕ, that $\phi \mathrel{\underline{\cdot}} \phi$. In other words, it follows that R_ϕ^2 is ergodic. Assume that $R_\phi^{n_0}$ is ergodic for some $n_0 \geqslant 2$; then we may apply 3.6. to ϕ and ϕ_{n_0}. As $R_\phi^{n_0}$ is ergodic, $\theta \mathrel{\underline{\cdot}} \phi_{n_0}$ and so $\phi \mathrel{\underline{\cdot}} \phi_{n_0}$; i.e., $R_\phi^{n_0+1}$ is ergodic. This settles the case for RIC extensions.

Suppose ϕ is an open RIM extension. Then by 3.10., $\phi_n : \mathscr{R}_\phi^n \to \mathscr{Y}$ is an open RIM extension and R_ϕ^n has a dense set of supprim points for every $n \in \mathbb{N}$ with $n \geqslant 2$. Induction and application of 3.8. proves the case that ϕ is an open RIM extension. □

We shall now generalize 3.6., 3.8. and 3.11. to the "weaker" situations of ϕ being a Bc extension or an RMM extension. To that end we shall construct a kind of double diagram.

3.12. Consider the diagram in 3.3. with ϕ a Bc extension and let (ϕ, ψ) satisfy gBc. Then we can lift the left hand part of the diagram to the level of the universal minimal proximal extensions, as follows:

$$\begin{tikzcd}
\mathfrak{A}(\mathfrak{X}) \arrow[rr, "\phi'"] \arrow[dr, "\kappa'"] \arrow[dd, "\sigma"'] & & \mathfrak{A}(\mathfrak{Z}) \arrow[dd, "\eta"] & \mathfrak{Y}' \arrow[l, "\psi'"'] \arrow[dd, "\tau"] \\
& \mathfrak{A}(\mathfrak{X})/E_{\phi'} \arrow[ur, "\theta'"'] \arrow[dd, "\xi"] & & \\
\mathfrak{X} \arrow[rr, "\phi"] \arrow[dr, "\kappa"'] & & \mathfrak{Z} & \mathfrak{Y} \arrow[l, "\psi"] \\
& \mathfrak{X}/E_{\phi} \arrow[ur, "\theta"'] & &
\end{tikzcd}$$

For the exact construction, let $u \in J$, $z_0 = uz_0 \in Z$ and $x_0 = ux_0 \in \phi^{\leftarrow}(z_0)$. Let $K = \mathfrak{S}(\mathfrak{Z}, z_0)$ and $H = \mathfrak{S}(\mathfrak{X}, x_0)$ be the Ellis groups of $\mathfrak{Z}$ and $\mathfrak{X}$ in G with respect to the points z_0 and x_0. Then, by III.1.13.b,

$$\mathfrak{A}(\mathfrak{X}) := \mathfrak{WF}(u \circ H, \mathfrak{M}) \text{ and } \mathfrak{A}(\mathfrak{Z}) := \mathfrak{WF}(u \circ K, \mathfrak{M}) ;$$

$\sigma : \mathfrak{A}(\mathfrak{X}) \to \mathfrak{X}$ is defined by $\sigma(p \circ H) = px_0$ and $\eta : \mathfrak{A}(\mathfrak{Z}) \to \mathfrak{Z}$ by $\eta(p \circ K) = pz_0$. The induced RIC extension $\phi' : \mathfrak{A}(\mathfrak{X}) \to \mathfrak{A}(\mathfrak{Z})$ is defined by $\phi'(p \circ H) = p \circ K$ (III.1.15.). As ϕ satisfies Bc and η is proximal, we have $\sigma \times \sigma[R_{\phi'}] = R_{\phi}$ (IV.4.5.); as σ is proximal and $E_{\phi} = Q_{\phi}$ (III.3.9) it follows from IV.4.3. that $\sigma \times \sigma[E_{\phi'}] = E_{\phi}$. Hence by IV.4.10., $\xi : \mathfrak{A}(\mathfrak{X})/E_{\phi'} \to \mathfrak{X}/E_{\phi}$ is proximal. Define $Y' \subseteq Y \times \mathfrak{A}(Z)$ by $(y, p \circ K) \in Y'$ iff $y \in p \circ u\psi^{\leftarrow}(z_0)$, and let $\tau : Y' \to Y$ and $\psi' : Y' \to \mathfrak{A}(Z)$ be the projections.

3.13. **LEMMA.** *Consider the diagram in* 3.12. *(with the same notation).*

a) Y' *is closed (in* $Y \times \mathfrak{A}(Z)$ *),* T*-invariant and has a dense subset of almost periodic points. In particular* $\mathfrak{Y}'$ *is a ttg.*

b) $\tau : \mathfrak{Y}' \to \mathfrak{Y}$ *is a proximal surjection.*

PROOF. First note that Y' is well defined: Let $(y, p \circ K) \in Y'$ and let for certain $q \in M$, $p \circ K = q \circ K$. We have to show that $y \in q \circ u\psi^{\leftarrow}(z_0)$.
As $kz_0 = z_0$ for every $k \in K$, $k \circ u\psi^{\leftarrow}(z_0) = u \circ u\psi^{\leftarrow}(z_0)$ for every $k \in K$ (II.3.11.d); consequently, $K \circ u\psi^{\leftarrow}(z_0) = u \circ u\psi^{\leftarrow}(z_0)$. Since $p \in p \circ K = q \circ K$, it follows that

$$p \circ u\psi^{\leftarrow}(z_0) \subseteq q \circ K \circ u\psi^{\leftarrow}(z_0) = q \circ u \circ u\psi^{\leftarrow}(z_0) = q \circ u\psi^{\leftarrow}(z_0) .$$

Similarly, $q \circ u\psi^{\leftarrow}(z_0) \subseteq p \circ u\psi^{\leftarrow}(z_0)$; hence $p \circ u\psi^{\leftarrow}(z_0) = q \circ u\psi^{\leftarrow}(z_0)$, and $y \in q \circ u\psi^{\leftarrow}(z_0)$.

a) Clearly, Y' is T-invariant. Let $\{(y_i, p_i \circ K)\}_i$ be a net in Y' which converges in $Y \times \mathfrak{A}(Z)$, say $(y_i, p_i \circ K) \to (y, p \circ K)$. Then

$y = \lim y_i \subseteq \lim_{2^Y} p_i \circ u\psi^{\leftarrow}(z_0)$. For a suitable subnet let $q = \lim p_i$. Then $p \circ K = q \circ K$ and

$$y \in \lim_{2^Y} p_i \circ u\psi^{\leftarrow}(z_0) = (\lim p_i) \circ u\psi^{\leftarrow}(z_0) = q \circ u\psi^{\leftarrow}(z_0) .$$

So $(y, p \circ K) = (y, q \circ K) \in Y'$; hence Y' is closed and $\mathcal{Y}'$ is a ttg.
Let $(y, p \circ K) \in Y'$. We shall show that $(y, p \circ K)$ is the limit of a net in Y', consisting of almost periodic points in Y'. As $(y, p \circ K) \in Y'$, y is an element of $p \circ u\psi^{\leftarrow}(z_0)$. Let $\{t_i\}_i$ be a net in T with $p = \lim t_i$, then (after passing to a suitable subnet) there are $y_i \in u\psi^{\leftarrow}(z_0)$ such that $y = \lim t_i y_i$. So

$$(y, p \circ K) = \lim t_i (y_i, u \circ K) ,$$

while $(y_i, u \circ K) = u(y_i, u \circ K)$ is an almost periodic point in $Y \times \mathfrak{A}(Z)$; However, $y_i = uy_i \in u\psi^{\leftarrow}(z_0) \subseteq u \circ u\psi^{\leftarrow}(z_0)$, so $(y_i, u \circ K) \in Y'$.

b) First we shall show that τ is a surjection. Note that it is sufficient to show that $Y = \bigcup \{p \circ u\psi^{\leftarrow}(z_0) \mid p \in M\}$. Let $y \in Y$ and remark that Y as a factor of $R_{\phi\psi}$ has a dense subset of almost periodic points. Then $y = \lim y_i$ for almost periodic points $y_i \in Y$, say $y_i = v_i y_i$ with $v_i \in J$. Let $p_i \in M$ be such that $\psi(y_i) = p_i z_0$. Then $y_i = v_i p_i u p_i^{-1} y_i$ and $\psi(up_i^{-1} y_i) = up_i^{-1} p_i z_0 = z_0$, so $y_i \in v_i p_i u\psi^{\leftarrow}(z_0) \subseteq v_i p_i \circ u\psi^{\leftarrow}(z_0)$. After passing to a suitable subnet let $q = \lim v_i p_i \in M$, then

$$y = \lim y_i \in \lim_{2^Y} v_i p_i \circ u\psi^{\leftarrow}(z_0) = q \circ u\psi^{\leftarrow}(z_0) \subseteq \bigcup \{p \circ u\psi^{\leftarrow}(z_0) \mid p \in M\} .$$

Hence τ is surjective. Suppose $\tau(y_1, p_1 \circ K) = \tau(y_2, p_2 \circ K)$, so $y_1 = y_2$. Since $\eta \circ \psi' = \psi \circ \tau$, this implies that $\eta(p_1 \circ K) = \eta(p_2 \circ K)$ and, consequently, that $p_1 \circ K$ and $p_2 \circ K$ are proximal. But then $(y_1, p_1 \circ K)$ and $(y_2, p_2 \circ K)$ are proximal in Y'. □

3.14. **THEOREM.** *Consider the diagram in* 3.3.. *Let* ϕ *be a* Bc *extension and let* (ϕ,ψ) *satisfy the generalized Bronstein condition. Then* $\phi \mathrel{\underset{\cdot}{\perp}} \psi$ *iff* $\theta \mathrel{\underset{\cdot}{\perp}} \psi$.

PROOF. Consider the diagram in 3.12. and suppose $\theta \mathrel{\underset{\cdot}{\perp}} \psi$, i.e. $R_{\theta\psi}$ is ergodic. As (ϕ,ψ) satisfies gBc, Y has a dense subset of almost periodic points. Since θ is almost periodic and so RIC, it follows from III.1.5.b, that $R_{\theta\psi}$ has a dense subset of almost periodic points. With the same reasoning $R_{\theta'\psi'}$ has a dense subset of almost periodic points. By IV.4.5., the proximal map $\xi \times \tau : R_{\theta'\psi'} \to R_{\theta\psi}$ is a surjection. So, by 3.1., $R_{\theta'\psi'}$ is ergodic, i.e. $\theta' \mathrel{\underset{\cdot}{\perp}} \psi'$.

But ϕ' is a RIC extension and Y' has a dense subset of almost periodic points, so by III.1.5., (ϕ',ψ') satisfies gBc. Application of 3.6. to ϕ' and ψ' implies that $\phi' \mathrel{\dot\perp} \psi'$; i.e., $R_{\phi'\psi'}$ is ergodic. As η is proximal and as $R_{\phi\psi}$ has a dense subset of almost periodic points, it follows from IV.4.5. that $\sigma \times \tau : R_{\phi'\psi'} \to R_{\phi\psi}$ is a surjection, hence $R_{\phi\psi}$ is ergodic and $\phi \mathrel{\dot\perp} \psi$. □

3.15. Consider the diagram in 3.3. with $\mathfrak{Y}$ minimal and $\phi : \mathfrak{X} \to \mathfrak{Z}$ an open RMM extension of minimal ttgs. We shall lift the diagram to the following double diagram:

$$\begin{tikzcd}
\mathfrak{X}' \arrow[rr, "\phi'"] \arrow[dr, "\kappa'"] \arrow[dd, "\sigma"'] & & \mathfrak{A}_S(\mathfrak{Z}) \arrow[dd, "\eta"] & \mathfrak{A}_S(\mathfrak{Y}) \arrow[l, "\psi'"'] \arrow[dd, "\tau"] \\
& \mathfrak{X}'/E_{\phi'} \arrow[ur, "\theta'"'] \arrow[dd, "\xi"] & & \\
\mathfrak{X} \arrow[rr, "\phi"] \arrow[dr, "\kappa"'] & & \mathfrak{Z} & \mathfrak{Y} \arrow[l, "\psi"] \\
& \mathfrak{X}/E_{\phi} \arrow[ur, "\theta"'] & &
\end{tikzcd}$$

The right hand part is the lifting of ψ to the level of the universal minimal strongly proximal extensions; so $\psi' : \mathfrak{A}_S(\mathfrak{Z}) \to \mathfrak{A}_S(\mathfrak{Y})$ is an open RIM extension and η and τ are strongly proximal (cf. 1.10. and the remark after it). As ϕ is an open RMM extension, $\phi \perp \eta$ by 1.15.. Define $\mathfrak{X}' := \mathfrak{R}_{\phi\eta}$ and let ϕ' and σ be the projections, then σ is a proximal extension and ϕ' is an open RIM extension (also see 1.16.). Clearly, $\sigma \times \sigma[R_{\phi'}] = R_{\phi}$ and as $E_{\phi} = Q_{\phi} \circ P_{\phi}$ (1.20.), it follows from IV.4.3. and IV.4.10. that the map $\xi : \mathfrak{X}'/E_{\phi'} \to \mathfrak{X}/E_{\phi}$ is proximal.

3.16. **THEOREM.** *Consider the diagram in* 3.3. *with* $\mathfrak{Y}$ *minimal and let* $\phi : \mathfrak{X} \to \mathfrak{Z}$ *be an* RMM *extension of minimal ttgs. Suppose that either* (ϕ,ψ) *satisfies the generalized Bronstein condition or* ϕ *or* ψ *is open. Then* $\phi \mathrel{\dot\perp} \psi$ *iff* $\theta \mathrel{\dot\perp} \psi$.

PROOF. First we shall prove the theorem in case ϕ is an open RMM extension.

If ϕ is an open RMM extension, we can construct the diagram in 3.15.. Suppose $\theta \mathrel{\dot\perp} \psi$ and note that in the same way as in 3.14., $R_{\theta'\psi'}$ and $R_{\theta\psi}$ have a dense subset of almost periodic points. As in 3.14., it follows from 3.1. that $\theta' \mathrel{\dot\perp} \psi'$. As ψ' is an open RIM extension of minimal ttgs, we may apply 3.8. to conclude that $\phi' \mathrel{\dot\perp} \psi'$. We prove that $\sigma \times \tau[R_{\phi'\psi'}] = R_{\phi\psi}$, then $R_{\phi\psi}$ as a factor of an ergodic ttg is ergodic it self, and so $\phi \mathrel{\dot\perp} \psi$.

As ϕ is open it follows from I.3.9. that $R_{\phi\psi} = \overline{T(\phi^{\leftarrow}\psi(y) \times \{y\})}$ for every $y \in Y$. We shall show that

$$\phi^{\leftarrow}\psi(y) \times \{y\} \subseteq \sigma \times \tau[R_{\phi'\psi'}]$$

and so that $R_{\phi\psi} \subseteq \sigma \times \tau[R_{\phi'\psi'}]$.
Let $y \in Y$, $y' \in \mathfrak{A}_S(Y)$ with $\tau(y') = y$ and let $z' \in \mathfrak{A}_S(Z)$ be such that $z' = \psi'(y')$. As $\eta(z') = \psi(y)$, it follows from the fact that $X' = R_{\phi\eta}$ that $\phi^{\leftarrow}\psi(y) \times \{z'\} \subseteq X'$. Hence $(\phi^{\leftarrow}\psi(y) \times \{z'\}) \times \{y'\} \subseteq R_{\phi'\psi'}$ and so

$$\phi^{\leftarrow}\psi(y) \times \{y\} = \sigma \times \tau[(\phi^{\leftarrow}\psi(y) \times \{z'\}) \times \{y'\}] \subseteq \sigma \times \tau[R_{\phi'\psi'}] \subseteq R_{\phi\psi}.$$

Consequently, $\phi \mathrel{\dot{-}} \psi$, which settles the case for an open RMM extension ϕ.

Now let ϕ be an RMM extension and let ϕ or ψ be open or let (ϕ,ψ) satisfy gBc. We construct the double * diagram (cf. IV.3.10.):

$$\begin{tikzcd}
\mathfrak{X}^* \arrow[rr, "\phi^*"] \arrow[dr, "\kappa'"] \arrow[dd, "\sigma"'] & & \mathfrak{Z}^* \arrow[dd, "\eta"] & \mathfrak{Y}^* \arrow[l, "\psi^*"'] \arrow[dd, "\tau"] \\
& \mathfrak{X}^*/E_{\phi^*} \arrow[ur, "\theta'"'] \arrow[dd, "\xi", crossing over] & & \\
\mathfrak{X} \arrow[rr, "\phi"] \arrow[dr, "\kappa"'] & & \mathfrak{Z} & \mathfrak{Y} \arrow[l, "\psi"] \\
& \mathfrak{X}/E_\phi \arrow[ur, "\theta"'] & &
\end{tikzcd}$$

By the discussion in 1.15., ϕ^* is an open RMM extension. As, by the definition of RMM extension, $\sigma \times \sigma[R_{\phi^*}] = R_\phi$, and since, by 1.20., $E_\phi = Q_\phi \circ P_\phi$, it follows from IV.4.3. and IV.4.10. that $\xi: \mathfrak{X}^*/E_{\phi^*} \to \mathfrak{X}/E_\phi$ is a proximal extension. With the same reasoning as before, the map $\xi \times \tau: R_{\theta'\psi^*} \to R_{\theta\psi}$ is a proximal surjection between ttgs with dense subsets of almost periodic points. Suppose $\theta \mathrel{\dot{-}} \psi$; then by 3.1., $\theta' \mathrel{\dot{-}} \psi^*$. Hence by the first part of the proof, $\phi^* \mathrel{\dot{-}} \psi^*$. As (ϕ,ψ) satisfies gBc or ϕ or ψ is open, it follows from IV.4.16.c that $\phi \mathrel{\dot{-}} \psi$. □

3.17. **THEOREM.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs. If* ϕ *is a* Bc *extension or if* ϕ *is an* RMM *extension then* ϕ *is weakly mixing iff* $E_\phi = R_\phi$.

PROOF. Clearly, if ϕ is weakly mixing then $E_\phi = Q_\phi = R_\phi$ (see also 3.11.). Suppose that ϕ is a Bc extension with $E_\phi = R_\phi$. Then $\theta: \mathfrak{X}/E_\phi \to \mathfrak{Y}$ is an isomorphism; so $\theta \perp \phi$. By 3.14., it follows that $\phi \mathrel{\dot{-}} \phi$ and so that ϕ is weakly mixing.

Suppose that ϕ is an RMM extension with $E_\phi = R_\phi$. Then ϕ^* is an open RMM extension and so (in $*(\phi)$) $\sigma \times \sigma[R_{\phi^*}] = R_\phi$. As $E_\phi = Q_\phi \circ P_\phi$ (1.20.), it follows from IV.4.1.c that

$$R_{\phi^*} = (\sigma \times \sigma)^{\leftarrow}[R_\phi] = (\sigma \times \sigma)^{\leftarrow}[Q_\phi \circ P_\phi] = Q_{\phi^*} \circ P_{\phi^*} = E_{\phi^*}\,.$$

Similar to the Bc case above it follows from 3.16. that ϕ^* is weakly mixing. Hence it follows from IV.4.17. that ϕ is weakly mixing. □

3.18. Now we shall turn to what we announced in the abstract as the central theme of this section.

So consider the following diagram of homomorphisms of minimal ttgs.

$$\begin{tikzcd}
\mathfrak{X} \arrow[d, "\kappa_{\mathfrak{X}}"'] \arrow[drr, "\phi"] & & & & \mathfrak{Y} \arrow[d, "\kappa_{\mathfrak{Y}}"] \arrow[dll, "\psi"'] \\
\mathfrak{X}/E_\phi \arrow[rr, "\theta_{\mathfrak{X}}"'] & & \mathfrak{Z} & & \mathfrak{Y}/E_\psi \arrow[ll, "\theta_{\mathfrak{Y}}"]
\end{tikzcd}$$

We shall apply the results in 3.6., 3.8., 3.14. and 3.16. to show that in several cases $\theta_{\mathfrak{X}} \mathbin{\dot{\perp}} \theta_{\mathfrak{Y}}$ implies $\phi \mathbin{\dot{\perp}} \psi$.

Clearly, the equality $(\kappa_{\mathfrak{X}} \times \kappa_{\mathfrak{Y}})[R_{\phi\psi}] = R_{\theta_{\mathfrak{X}}\theta_{\mathfrak{Y}}}$ implies that the inverse implication is true.

First we shall show that $\theta_{\mathfrak{X}} \mathbin{\dot{\perp}} \theta_{\mathfrak{Y}}$ iff $\theta_{\mathfrak{X}} \perp \theta_{\mathfrak{Y}}$ (for a more general result see 4.5.). This is an immediate consequence of:

3.19. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Z}$ *and* $\psi : \mathfrak{Y} \to \mathfrak{Z}$ *be almost periodic extensions of minimal ttgs. Then* $\phi \mathbin{\dot{\perp}} \psi$ *iff* $\phi \perp \psi$.

PROOF. Let $\alpha_{\mathfrak{Z}} : \mathfrak{A}(\mathfrak{Z}) \to \mathfrak{Z}$ be the universal minimal almost periodic extension of $\mathfrak{Z}$ and let $\alpha_{\mathfrak{X}} : \mathfrak{A}(\mathfrak{Z}) \to \mathfrak{X}$ and $\alpha_{\mathfrak{Y}} : \mathfrak{A}(\mathfrak{Z}) \to \mathfrak{Y}$ be the almost periodic extensions such that $\alpha_{\mathfrak{Z}} = \phi \circ \alpha_{\mathfrak{X}} = \psi \circ \alpha_{\mathfrak{Y}}$.

$$\begin{tikzcd}
 & \mathfrak{A}(\mathfrak{Z}) \arrow[dl, "\alpha_{\mathfrak{X}}"'] \arrow[dr, "\alpha_{\mathfrak{Y}}"] \arrow[dd, "\alpha_{\mathfrak{Z}}"] & \\
\mathfrak{X} \arrow[dr, "\phi"'] & & \mathfrak{Y} \arrow[dl, "\psi"] \\
 & \mathfrak{Z} &
\end{tikzcd}$$

Since $\alpha_{\mathfrak{Z}}$ is almost periodic,

$$\Delta_{\mathfrak{A}(Z)} = Q_{\alpha_{\mathfrak{Z}}} = \bigcap \{\overline{T\xi \cap R_{\alpha_{\mathfrak{Z}}}} \mid \xi \in \mathfrak{U}_{\mathfrak{A}(Z)}\} .$$

As $\alpha_{\mathfrak{X}} \times \alpha_{\mathfrak{Y}} : R_{\alpha_{\mathfrak{Z}}} \to R_{\phi\psi}$ is a closed continuous surjection and as $\{\overline{T\xi \cap R_{\alpha_{\mathfrak{Z}}}} \mid \xi \in \mathfrak{U}_{\mathfrak{A}(Z)}\}$ is a collection of closed subsets of $R_{\alpha_{\mathfrak{Z}}}$ directed by inclusion, it follows that

$$\alpha_{\mathfrak{X}} \times \alpha_{\mathfrak{Y}} [\Delta_{\mathfrak{A}(Z)}] = \alpha_{\mathfrak{X}} \times \alpha_{\mathfrak{Y}} [\bigcap \{\overline{T\xi \cap R_{\alpha_{\mathfrak{Z}}}} \mid \xi \in \mathfrak{U}_{\mathfrak{A}(Z)}\}] =$$

$$= \bigcap \{\alpha_{\mathfrak{X}} \times \alpha_{\mathfrak{Y}} [\overline{T\xi \cap R_{\alpha_{\mathfrak{Z}}}}] \mid \xi \in \mathfrak{U}_{\mathfrak{A}(Z)}\} .$$

Applying 3.2.b to both sides of the diagram implies that for every $\xi \in \mathfrak{U}_{\mathfrak{A}(Z)}$ we have

$$\mathrm{int}_{R_{\phi\psi}} (\alpha_{\mathfrak{X}} \times \alpha_{\mathfrak{Y}} [\overline{T\xi \cap R_{\alpha_{\mathfrak{Z}}}}]) \neq \emptyset .$$

Suppose that $\phi \perp \psi$, then $R_{\phi\psi} = \alpha_{\mathfrak{X}} \times \alpha_{\mathfrak{Y}} [\overline{T\xi \cap R_{\alpha_{\mathfrak{Z}}}}]$ for every $\xi \in \mathfrak{U}_{\mathfrak{A}(Z)}$. Hence $\alpha_{\mathfrak{X}} \times \alpha_{\mathfrak{Y}} [\Delta_{\mathfrak{A}(Z)}] = R_{\phi\psi}$, and as $\Delta_{\mathfrak{A}(Z)}$ is minimal it follows that $R_{\phi\psi}$ is minimal; so $\phi \perp \psi$.
The converse is trivial. □

3.20. **THEOREM.** *Consider the diagram in* 3.18.. *In each of the following cases we have* $\phi \mathrel{\dot\perp} \psi$ *iff* $\theta_{\mathfrak{X}} \mathrel{\dot\perp} \theta_{\mathfrak{Y}}$ *(iff* $\theta_{\mathfrak{X}} \perp \theta_{\mathfrak{Y}}$ *).*

- a) (ϕ,ψ) *satisfies the generalized Bronstein condition and, in addition,*
  either ϕ *satisfies the Bronstein condition*
  or ϕ *is a* RIM *extension*
  or ϕ *is an* RMM *extension;*
- b) ψ *is open and* ϕ *is a* RIM *extension or an* RMM *extension;*
- c) ϕ *is an open* RMM *extension.*

PROOF. By 3.19., $\theta_{\mathfrak{X}} \mathrel{\dot\perp} \theta_{\mathfrak{Y}}$ iff $\theta_{\mathfrak{X}} \perp \theta_{\mathfrak{Y}}$. Clearly, $\phi \mathrel{\dot\perp} \psi$ implies $\theta_{\mathfrak{X}} \mathrel{\dot\perp} \theta_{\mathfrak{Y}}$. As $\theta_{\mathfrak{X}}$ and $\theta_{\mathfrak{Y}}$ are almost periodic extensions, they are open RIM extensions (1.3.c). So, by 3.8., $\theta_{\mathfrak{X}} \mathrel{\dot\perp} \theta_{\mathfrak{Y}}$ iff $\phi \mathrel{\dot\perp} \theta_{\mathfrak{Y}}$ and $\theta_{\mathfrak{X}} \mathrel{\dot\perp} \theta_{\mathfrak{Y}}$ iff $\theta_{\mathfrak{X}} \mathrel{\dot\perp} \psi$.

a) Suppose (ϕ,ψ) satisfies gBc. Let ϕ be a Bc map and let $\theta_{\mathfrak{X}} \mathrel{\dot\perp} \theta_{\mathfrak{Y}}$. Since we know already that $\theta_{\mathfrak{X}} \mathrel{\dot\perp} \psi$ it follows from 3.14. that $\phi \mathrel{\dot\perp} \psi$.
Let ϕ be a RIM extension. As $\mathfrak{X}$ is minimal, X has a dense set of supprim points. If $\theta_{\mathfrak{X}} \mathrel{\dot\perp} \theta_{\mathfrak{Y}}$ then by the above, $\phi \mathrel{\dot\perp} \theta_{\mathfrak{Y}}$. As ϕ and ψ satisfy one of the conditions in 3.2., it follows from 3.8. that $\phi \mathrel{\dot\perp} \psi$.
Let ϕ be a RMM extension and let $\theta_{\mathfrak{X}} \mathrel{\dot\perp} \theta_{\mathfrak{Y}}$. Then by the above $\theta_{\mathfrak{X}} \mathrel{\dot\perp} \psi$. So, by 3.16., it follows that $\phi \mathrel{\dot\perp} \psi$.

b) Let ψ be open and suppose that $\theta_{\mathscr{X}} \mathrel{\underline{\dot{-}}} \theta_{\mathscr{Y}}$.
If ϕ is a RIM extension (of minimal ttgs) then ϕ and ψ satisfy one of the conditions in lemma 3.2.. As by the above $\phi \mathrel{\underline{\dot{-}}} \theta_{\mathscr{Y}}$, it follows from 3.8. that $\phi \mathrel{\underline{\dot{-}}} \psi$.
If ϕ is an RMM extension then by 3.16., $\phi \mathrel{\underline{\dot{-}}} \psi$ iff $\theta_{\mathscr{X}} \mathrel{\underline{\dot{-}}} \psi$; but from the above we know that $\theta_{\mathscr{X}} \mathrel{\underline{\dot{-}}} \theta_{\mathscr{Y}}$ implies $\theta_{\mathscr{X}} \mathrel{\underline{\dot{-}}} \psi$.

c) If ϕ is an open RMM extension then by 3.16., $\phi \mathrel{\underline{\dot{-}}} \psi$ iff $\theta_{\mathscr{X}} \mathrel{\underline{\dot{-}}} \psi$.□

The following result is in fact a corollary of 3.5.. It forms a bridge between chapter VII. and chapter VIII..

3.21. **THEOREM.** *Let* $\phi : \mathscr{X} \to \mathscr{Y}$ *be a homomorphism of minimal ttgs. If* ϕ *and* ϕ *satisfy the conditions in* 3.5. *then*

$$E_\phi = Q_\phi = \bigcap \{ \operatorname{int}_{R_\phi}(\overline{T\alpha \cap R_\phi}) \mid \alpha \in \mathscr{U}_X \} .$$

PROOF. Let $\alpha \in \mathscr{U}_X$ be an arbitrary index and let $U \subseteq X$ be an open set such that $U \times U \subseteq \alpha$. Let $\kappa : \mathscr{X} \to \mathscr{X}/E_\phi$ be the quotient map. Define $\tilde{U} := \kappa^{\leftarrow}[\kappa[U]^\circ]$ and $U_0 := \tilde{U} \cap U$. Then U_0 is open and nonempty. By 3.5.,

$$\tilde{U} \times U \cap R_\phi \subseteq \overline{T(U \times U \cap R_\phi)} \subseteq \overline{T\alpha \cap R_\phi} ,$$

hence as $\tilde{U} = E_\phi[\tilde{U}]$

$$E_\phi[U_0] \times U_0 \cap R_\phi \subseteq \tilde{U} \times U \cap R_\phi \subseteq \overline{T\alpha \cap R_\phi} .$$

As $\tilde{U}$ is open, even

$$E_\phi[U_0] \times U_0 \cap R_\phi \subseteq \operatorname{int}_{R_\phi}(\overline{T\alpha \cap R_\phi}) .$$

If $x \in X$, then there is a $t \in T$ with $tx \in U_0$, and so

$$tE_\phi[x] = E_\phi[tx] \subseteq E_\phi[U_0] .$$

Hence

$$t(E_\phi[x] \times \{x\}) = E_\phi[tx] \times \{tx\} \subseteq E_\phi[U_0] \times U_0 \cap R_\phi \subseteq \operatorname{int}_{R_\phi}(\overline{T\alpha \cap R_\phi}) .$$

So

$$E_\phi[x] \times \{x\} \subseteq t^{-1}.\operatorname{int}_{R_\phi}(\overline{T\alpha \cap R_\phi}) = \operatorname{int}_{R_\phi}(\overline{T\alpha \cap R_\phi}) .$$

As $x \in X$ was arbitrary it follows that

$$E_\phi = \bigcup \{ E_\phi[x] \times \{x\} \mid x \in X \} \subseteq \operatorname{int}_{R_\phi}(\overline{T\alpha \cap R_\phi}) .$$

As $\alpha \in \mathfrak{U}_X$ was arbitrary

$$E_\phi \subseteq \bigcap \{\operatorname{int}_{R_\phi}(\overline{T\alpha \cap R_\phi} \mid \alpha \in \mathfrak{U}_X\} \subseteq \bigcap \{\overline{T\alpha \cap R_\phi} \mid \alpha \in \mathfrak{U}_X\} = Q_\phi . \qquad \square$$

3.22. **COROLLARY.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs.*

a) *If* ϕ *is a* RIC *extension or an open* RIM *extension then*

$$E_\phi = Q_\phi = \bigcap \{\operatorname{int}_{R_\phi}(\overline{T\alpha \cap R_\phi}) \mid \alpha \in \mathfrak{U}_X\} .$$

b) *If* ϕ *is an* RMM *extension then* $E_\phi = Q_\phi$.

PROOF.

a) In 3.6. and 3.8. we proved that if ϕ is a RIC extension or an open RIM extension then ϕ and ϕ satisfy the conditions in 3.5. (in both cases let ψ and ϕ be identical). The corollary follows from 3.21..

b) Let ϕ be an RMM extension; then by 1.16. we can construct a $\mathfrak{b}$ diagram

$$\begin{tikzcd}
\mathfrak{X}^{\mathfrak{b}} \arrow[r, "\sigma^{\mathfrak{b}}"] \arrow[d, "\phi^{\mathfrak{b}}"'] & \mathfrak{X} \arrow[d, "\phi"] \\
\mathfrak{Y}^{\mathfrak{b}} \arrow[r, "\tau^{\mathfrak{b}}"'] & \mathfrak{Y}
\end{tikzcd}$$

such that $\phi^{\mathfrak{b}}$ is an open RIM extension and $\sigma^{\mathfrak{b}} \times \sigma^{\mathfrak{b}}[R_{\phi^{\mathfrak{b}}}] = R_\phi$. So by a and IV.4.3.d, it follows that $E_\phi = Q_\phi$. □

We end this section with two observations on PI towers.

3.23. Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a homomorphism of minimal ttgs and construct the canonical PI tower for ϕ as in III.4.6. and III.4.7.. Then we have the next diagram of homomorphisms of minimal ttgs:

$$\begin{tikzcd}
\mathfrak{X}'_\infty \arrow[r, "\sigma'_\infty"] \arrow[d, "\phi'_\infty"'] & \mathfrak{X} \arrow[d, "\phi"] \\
\mathfrak{Y}'_\infty \arrow[r, "\tau'_\infty"'] & \mathfrak{Y}
\end{tikzcd}$$

where ϕ'_∞ is a RIC extension without nontrivial almost periodic factors,

σ'_∞ is proximal and τ'_∞ is a strictly-PI extension.
By 3.11., it follows that ϕ'_∞ is a weakly mixing homomorphism of minimal ttgs. So every homomorphism of minimal ttgs is a PI extension up to some weakly mixing junk in the top of the tower (cf. [V 77] 2.1.3.).

3.24. Similar to the construction of the canonical PI tower for ϕ, we can construct a canonical SPI tower for ϕ, using the G diagrams (1.10.). Then we get the following diagram of homomorphisms of minimal ttgs:

$$\begin{tikzcd}
\mathcal{X}^\#_\infty \arrow[r, "\sigma^\#_\infty"] \arrow[d, "\phi^\#_\infty"'] & \mathcal{X} \arrow[d, "\phi"] \\
\mathcal{Y}^\#_\infty \arrow[r, "\tau^\#_\infty"'] & \mathcal{Y}
\end{tikzcd}$$

where $\phi^\#_\infty$ is an open RIM extension without nontrivial almost periodic factors, $\sigma^\#_\infty$ is strongly proximal and $\tau^\#_\infty$ is a strictly-PI extension in which every proximal map is even strongly proximal.
Again by 3.11., we have that $\phi^\#_\infty$ is weakly mixing. So every homomorphism of minimal ttgs is an SPI extension up to some weakly mixing junk in the top of the SPI tower (cf. [M 80]).

VII.4. REMARKS

4.1. In section VII.1. we introduced RMM extensions in a somewhat artificial way. As strong proximality is a property between proximality and high proximality one should expect a natural notion between RIC and openness which is characterized similar to RIC and openness as in the definition and in IV.3.16. respectively, but then with respect to strong proximality. In the metric case 1.14. is such a decent characterization. In the nonmetric case such a characterization seems to be unknown.
A related problem is how to characterize universal strongly proximal extensions as quasifactors of $\mathcal{M}$. Clearly, $\mathfrak{A}_S(\mathcal{X})$ is an MHP ttg for every minimal ttg $\mathcal{X}$ (1.9.b). So the question could be "restated" as: what kind of

MHP generator generates "MSP" ttgs? Note that it must depend on the choice of the idempotents only. Because, for a ttg $\mathfrak{X}$ with Ellis group H and MHP generator $C = u \circ C = K.H$, where $K = C \cap J$, it is clear that $u \circ H \subseteq S \subseteq C$ if S is the MHP generator that generates $\mathfrak{A}_S(\mathfrak{X})$.

QUESTIONS

a) Characterize RMM extensions in the nonmetric case.
b) Characterize the universal strongly proximal extensions as quasifactors of $\mathfrak{M}$.
c) Let $\phi: \mathfrak{X} \to \mathfrak{Y}$ be a homomorphism of minimal ttgs with $\mathfrak{Y} = \mathfrak{A}_S(\mathfrak{Y})$. Then ϕ is a RIM extension, say with section λ . What can be said about this λ or about $\operatorname{supp}\lambda_y$ for $y \in Y$? Note that if $\operatorname{supp}\lambda_y = \phi^{\leftarrow}(y)$ for some $y \in Y$, then this answers question a too.
d) Let $\phi: \mathfrak{X} \to \mathfrak{Y}$ be a homomorphism of minimal ttgs. If ϕ is a RIM extension, is ϕ^* a RIM extension? If ϕ^* is a RIM extension and if ϕ is open, is ϕ a RIM extension?

4.2. The problem stated in the abstract of section VII.3. is attacked by many people; e.g., see [P 72], [K 72], [M 78] and [V 77]. The results in section VII.3. extend all of the known ones on that matter.

Also the problem whether or not $E_\phi = R_\phi$ implies weak mixing of the homomorphism is considered frequently in the literature. The strongest results until now are [V 77] 2.6.3., which answers the question in the affirmative for Bc extensions, and [M 78], where the question is answered in the affirmative for minimal ttgs with invariant measure as well as for some special other cases. Here we answered the question in the affirmative for RMM extensions.

Moreover we proved that an open RIM extension ϕ with $E_\phi = R_\phi$ is weakly mixing of countable order. The step to uncountable order is still open (see also [M 80]). Another "new" accomplishment in this chapter is the fact that for an RMM extension ϕ of minimal ttgs we have that $E_\phi = Q_\phi$. Until now the strongest result was that the equality holds true for an open RIM extension with $E_\phi = R_\phi$ ([MW 83]). In the absolute case it was already known that the equation holds for a minimal ttg $\mathfrak{X}$ supporting an invariant measure ([M 78]).

4.3. Note that by 3.17. and 1.11., it follows that for an amenable group T the collections $\mathbf{D}^{\perp}$ and **WM** coincide. What is more, it even follows that $\mathbf{SP}^{\perp} \cap \mathbf{D}^{\perp} \subseteq \mathbf{WM}$, where $\mathbf{SP}^{\perp}$ is the collection of minimal ttgs that are disjoint from every strongly proximal minimal ttg.

The following generalization of 3.19. as presented in 4.5. is suggested by J. AUSLANDER.

4.4. LEMMA. *Let $\phi: \mathcal{X} \to \mathcal{Y}$ be a surjective homomorphism of ttgs (not necessarily minimal). Let $X' \subseteq X$ be a closed invariant subset of X such that*

(i) $\phi[X'] = Y$,

(ii) $\phi|_{X'}: X' \to Y$ is open.

If $\mathcal{X}$ is ergodic then $X = Q_{\phi}[X']$.

PROOF. Let $x \in X$ and let $x' \in X'$ be such that $\phi(x') = \phi(x)$. As $\mathcal{X}$ is ergodic it follows that $x' \in \overline{T\alpha(x)}$ for every $\alpha \in \mathcal{U}_X$; so for every $\alpha \in \mathcal{U}_X$ we have $\alpha(x') \cap T\alpha(x) \neq \emptyset$. For $\alpha \in \mathcal{U}_X$ let $x_{\alpha} \in \alpha(x)$ and $t_{\alpha} \in T$ be such that $t_{\alpha}x_{\alpha} \in \alpha(x')$. Then, after passing to suitable subnets, $x_{\alpha} \to x$ and $t_{\alpha}x_{\alpha} \to x'$; so $t_{\alpha}\phi(x_{\alpha}) \to \phi(x')$. As $\phi|_{X'}$ is open, there are $x'_{\alpha} \in X'$ with $\phi(x'_{\alpha}) = \phi(x_{\alpha})$ such that $t_{\alpha}x'_{\alpha} \to x'$. Let for a suitable subnet $z = \lim x'_{\alpha}$. Then $z \in X'$ and $(x,z) \in Q_{\phi}$. Hence $x \in Q_{\phi}[z]$ and so $X \subseteq Q_{\phi}[X']$. □

4.5. THEOREM. *Let $\phi: \mathcal{X} \to \mathcal{Y}$ be an almost periodic extension with $\mathcal{X}$ ergodic and $\mathcal{Y}$ minimal. Then $\mathcal{X}$ is minimal.*

PROOF. Let X' be a minimal subset of X. As $\phi|_{X'}$ is almost periodic, $\phi|_{X'}$ is open. From 4.4. it follows that $X = Q_{\phi}[X']$. As ϕ is almost periodic, $Q_{\phi} = \Delta_X$, so $X = X'$. □

Note that 4.5., IV.4.13. and IV.2.2.c show that $\phi \mathrel{\underline{\perp}} \psi$ iff $\phi \perp \psi$ in case ϕ and ψ are HPI extensions which satisfy gBc.

As we promised in III.5.7., we shall now present a slight generalization of the characterization of PI extensions in [B 77].
A homomorphism $\phi: \mathcal{X} \to \mathcal{Y}$ of minimal ttgs is called a C′ *extension* if every ergodic subset of R_{ϕ} with a dense subset of almost periodic points is minimal. Note that a C′ extension is a C extension (cf. III.5.7.) and that a C extension of metric ttgs is a C′ extension (I.1.2.b).

4.6. **REMARK.**

a) *A weakly mixing C′ extension of minimal ttgs that satisfies the Bronstein condition is an isomorphism.*

b) *Let* ϕ *and* ψ *be homomorphisms of minimal ttgs such that* $\psi\circ\phi$ *is a C′ extension. Then* ϕ *is a C′ extension.*

c) *Let* $\{\phi_\alpha \mid \phi_\alpha : \mathfrak{X}_\alpha \to \mathfrak{Y},\ \alpha < \nu\}$ *be an inverse system of C′ extensions of minimal ttgs, and let* $\phi = \operatorname{inv\,lim} \phi_\alpha$. *Then* ϕ *is a C′ extension.*

PROOF.

a) Immediate.

b) Clear from the fact that $R_\phi \subseteq R_{\psi\circ\phi}$.

c) Let $\mathfrak{X} = \operatorname{inv\,lim} \mathfrak{X}_\alpha$ and let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be the inverse limit of the ϕ_α's. We denote by γ_α the canonical map $\gamma_\alpha : \mathfrak{X} \to \mathfrak{X}_\alpha$ such that $\phi = \phi_\alpha \circ \gamma_\alpha$. Let N be a closed invariant subset of R_ϕ with a dense subset of almost periodic points which is ergodic. Then $\gamma_\alpha \times \gamma_\alpha[N]$ is a C′ extension, $\gamma_\alpha \times \gamma_\alpha[N]$ is minimal. Clearly, $N = \operatorname{inv\,lim} \gamma_\alpha \times \gamma_\alpha[N]$, so by I.1.6., N is minimal. □

4.7. **LEMMA.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *and* $\psi : \mathfrak{Y} \to \mathfrak{Z}$ *be homomorphisms of minimal ttgs.*

a) *If* ψ *is a C′ extension and if* ϕ *is almost periodic then* $\psi\circ\phi$ *is a C′ extension.*

b) *If* ϕ *is a proximal extension then* ψ *is a C′ extension iff* $\psi\circ\phi$ *is a C′ extension.*

PROOF.

a) Let N be a closed invariant and ergodic subset of $R_{\psi\circ\phi}$ with a dense subset of almost periodic points. Then $\phi\times\phi[N]$ is an ergodic subset of R_ψ with a dense subset of almost periodic points. As ψ is a C′ extension, $\phi\times\phi[N]$ is minimal. By I.1.21., $\phi\times\phi : \mathfrak{X}\times\mathfrak{X} \to \mathfrak{Y}\times\mathfrak{Y}$ is almost periodic, so $\phi\times\phi|_N : \mathfrak{N} \to \phi\times\phi[\mathfrak{N}]$ is an almost periodic extension of a minimal ttg. Since N is ergodic it follows from 4.5. that N is minimal. Hence $\psi\circ\phi$ is a C′ extension.

b) Let ψ be a C′ extension and let N be an ergodic subset of $R_{\psi\circ\phi}$ with a dense subset of almost periodic points. As ψ is a C′ extension, $\phi\times\phi[N]$ is a minimal subset of R_ψ. The map $\phi\times\phi$ is proximal so $\phi\times\phi|_N : \mathfrak{N} \to \phi\times\phi[\mathfrak{N}]$ is a proximal extension of a minimal ttg. But then, by I.1.23.c, N has a unique minimal subset; hence N is a minimal subset of $R_{\psi\circ\phi}$. So $\psi\circ\phi$ is a C′ extension.

Conversely, let $\psi\circ\phi$ be a C′ extension. Let N be an ergodic subset of R_ψ

with a dense subset of almost periodic points. For every $n \in JN$ we can find a $n' \in JR_{\psi\circ\phi}$ such that $\phi \times \phi(n') = n$. Define

$$N' := \overline{\{tn' \mid t \in T , n \in JN\}} .$$

Then N' is a closed invariant subset with a dense subset of almost periodic points which is proximally mapped onto the ergodic subset N of R_ψ (by $\phi \times \phi$). Hence, by VII.3.1., N' is ergodic. As $\psi\circ\phi$ is a C′ extension, N' is minimal. So N is minimal; which shows that ψ is a C′ extension. □

4.8. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs. Then* ϕ *is a* C′ *extension iff* ϕ *is a* PI *extension.*

PROOF. Suppose ϕ is a C′ extension and construct the canonical PI tower for ϕ as in III.4.6. and III.4.7.. Then by VII.3.23., ϕ'_∞ is a weakly mixing RIC extension. As σ'_∞ is proximal, it follows from 4.7.b that $\phi\circ\sigma'_\infty$ is a C′ extension and so by 4.6.b, that ϕ'_∞ is a C′ extension. But then, by 4.6.a, ϕ'_∞ is an isomorphism; which shows that ϕ is a PI extension.
Conversely, let ϕ be a PI extension. Then there is a strictly-PI extension ψ and a proximal homomorphism θ such that $\psi = \phi\circ\theta$. By 4.7.b, it follows that we only have to show that ψ is a C′ extension. But it is obvious from 4.7.a, 4.7.b and 4.6.c that a strictly-PI extension is a C′ extension. □

We end this chapter with the next generalization of VI.3.1. (made possible by 4.5.).

4.9. **REMARK.** *Consider the following commutative diagram of homomorphisms of minimal ttgs:*

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\psi"] \arrow[d, "\phi"'] & \mathfrak{Z} \arrow[d, "\eta"] \\
\mathfrak{Y} \arrow[r, "\xi"'] \arrow[ur, dashed, "\theta"] & \mathfrak{W}
\end{tikzcd}$$

Let ϕ *be weakly mixing and* η *be distal. Then there is a homomorphism of minimal ttgs* $\theta : \mathfrak{Y} \to \mathfrak{Z}$ *such that the diagram commutes (so metrizability of* $\mathfrak{Z}$ *is not necessary).*

PROOF. We shall prove the remark for an almost periodic extension η . By FST the remark follows for a distal map η .

First note that, by I.1.21.b, the map $\eta \times \eta : R_\eta \to \Delta_W$ is almost periodic. As ϕ is weakly mixing, R_ϕ is ergodic. Hence $\psi \times \psi[R_\phi]$ is an ergodic subset of R_η. But $\eta \times \eta : \psi \times \psi[R_\phi] \to \Delta_W$ is an almost periodic extension of a minimal ttg (Δ_W). So, by 4.5., $\psi \times \psi[R_\phi]$ is minimal. Clearly, $\Delta_Z \subseteq \psi \times \psi[R_\phi]$; hence $\Delta_Z = \psi \times \psi[R_\phi]$ and $R_\phi \subseteq R_\psi$. This shows that there is a map $\theta : \mathfrak{Y} \cong \mathfrak{X}/R_\phi \to \mathfrak{Z} \cong \mathfrak{X}/R_\psi$. □

VIII

A VARIATION ON REGIONAL PROXIMALITY

1. sharp regional proximality
2. factors and lifting
3. transitivity and $Q^{\#}$
4. regional proximality of second order
5. remarks

In this final chapter we are interested in a sharp form of regional proximality, which in some cases implies the regionally proximal relation to be an equivalence relation.

In the first section we introduce sharp regional proximality, which is in fact regional proximality "in every direction". Also we give examples of extensions $\phi:\mathfrak{X}\to\mathfrak{Y}$ for which $E_\phi = Q_\phi = Q_\phi^{\#}$, where $Q_\phi^{\#}$ is the collection of sharp regionally proximal pairs for ϕ ; for instance: RIC extensions and open RIM extensions have that property.

The second section is devoted to the question whether or not $Q_\phi = Q_\phi^{\#}$ is preserved under factors and it is proved that this is the case if $E_\phi = Q_\phi$.

Transitivity problems are dealt with in the third section. In particular, we show that $Q_\phi = Q_\phi^{\#}$ implies that Q_ϕ is an equivalence relation in case ϕ is open or in case X is a metric space.

In the forth section the "vital part" of the equality $Q_\phi = Q_\phi^{\#}$ is used to give a necessary and sufficient condition for transitivity of the regionally proximal relation.

All results in this chapter are contained in [AMWW ?] and they result from joint research of J. AUSLANDER, D. C. MCMAHON, T. S. WU and the author.

VIII.1. SHARP REGIONAL PROXIMALITY

We shall discuss a special form of regional proximality, which could be paraphrased as regional proximality in every direction. The main objective in this section is to introduce sharp regional proximality and to give examples for which $Q = Q^{\#}$, i.e., examples for which every regionally proximal pair is sharply regionally proximal. In section VIII.3. we shall see the use of this in transitivity questions for Q .

1.1. Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a homomorphism of ttgs.
If $(x_1, x_2) \in Q_\phi$ then there are nets $\{(x_1^i, x_2^i)\}_i$ in R_ϕ and $\{t_i\}_i$ in T such that

$$(x_1^i, x_2^i) \to (x_1, x_2) \text{ and } t_i(x_1^i, x_2^i) \to (x, x) \text{ for some } x \in X .$$

In general, however, an arbitrary net $\{(x_1^i, x_2^i)\}_i$ that converges to the regionally proximal pair (x_1, x_2) is far from a net that "makes (x_1, x_2) regionally proximal" (see 1.5.).
We say that a pair $(x_1, x_2) \in R_\phi$ is *sharply relatively regionally proximal*, whenever for every net $\{(x_1^i, x_2^i)\}_i$ in R_ϕ that converges to (x_1, x_2) there are nets $\{(z_1^j, z_2^j)\}_j$ in R_ϕ arbitrarily close to $\{(x_1^i, x_2^i)\}_i$ (this will be explained below) having the following property:

$$(z_1^j, z_2^j) \to (x_1, x_2) \text{ and } t_j(z_1^j, z_2^j) \to (z, z)$$

for some net $\{t_j\}_j$ in T and some $z \in X$ (paraphrased: if (x_1, x_2) can be approximated from all directions in a regionally proximal way).
We say, that we can find nets (with a certain property) arbitrarily close to $\{(x_1^i, x_2^i)\}_i$ whenever for every net $\{U^i\}_i$ of neighbourhoods U^i of (x_1^i, x_2^i) in R_ϕ there are a subnet $\{U^j\}_j$ and a net $\{(z_1^j, z_2^j)\}_j$ (with that property) such that $(z_1^j, z_2^j) \in U^j$ for every j .
Denote the collection of sharply relatively regionally proximal pairs for ϕ by $Q_\phi^{\#}$.

1.2. **REMARK.** *Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a homomorphism of ttgs. Then*

$$Q_\phi^{\#} = \bigcap \{ \operatorname{int}_{R_\phi}(\overline{T\alpha \cap R_\phi}) \mid \alpha \in \mathfrak{U}_X \} .$$

PROOF. Let $(x_1, x_2) \in Q_\phi^{\#}$. Assume $(x_1, x_2) \notin \operatorname{int}_{R_\phi}(\overline{T\beta \cap R_\phi})$ for some $\beta \in \mathfrak{U}_X$; then we can find a net $\{(x_1^i, x_2^i)\}_i$ in $W := R_\phi \setminus (\overline{T\beta \cap R_\phi})$, which converges to (x_1, x_2) . Define $U^i := W$ for every i . Then there is

a net $\{(z_1^j, z_2^j)\}_j$ in W such that $(z_1^j, z_2^j) \to (x_1, x_2)$ and there is a net $\{t_j\}_j$ in T with $t_j(z_1^j, z_2^j) \to (z, z)$ for some $z \in X$. Hence $t_j(z_1^j, z_2^j) \in \beta \cap R_\phi$ eventually, and so $(z_1^j, z_2^j) \in T\beta \cap R_\phi$ eventually, which contradicts the fact that $(z_1^j, z_2^j) \in W$.
Conversely, let

$$(x_1, x_2) \in \bigcap \{\mathrm{int}_{R_\phi}(\overline{T\alpha \cap R_\phi}) \mid \alpha \in \mathcal{U}_X\}.$$

Let $\{(x_1^i, x_2^i)\}_{i \in I}$ be a net in R_ϕ which converges to (x_1, x_2) and let $\{U^i\}_{i \in I}$ be a net of open neighbourhoods U^i of (x_1^i, x_2^i). As for an open index $\alpha \in \mathcal{U}_X$ the set

$$\alpha(x_1) \times \alpha(x_2) \cap \mathrm{int}_{R_\phi}(\overline{T\alpha \cap R_\phi})$$

is a neighbourhood of (x_1, x_2) in R_ϕ, there is an $i(\alpha) \in I$ such that

$$U^i \cap \alpha(x_1) \times \alpha(x_2) \cap \mathrm{int}_{R_\phi}(\overline{T\alpha \cap R_\phi}) \neq \emptyset \quad \text{for every } i \geqslant i(\alpha),$$

and so for every $i \geqslant i(\alpha)$ there are $(z_1^i, z_2^i) \in R_\phi$ and $t_i \in T$ with

$$(z_1^i, z_2^i) \in U^i \cap \alpha(x_1) \times \alpha(x_2) \cap T\alpha \cap R_\phi \quad \text{and} \quad t_i(z_1^i, z_2^i) \in \alpha.$$

But then for a suitable directed subset $J \subseteq I \times \mathcal{U}_X$ there are nets $\{(z_1^j, z_2^j)\}_{j \in J}$ and $\{t_j\}_{j \in J}$ in R_ϕ and T such that

$$(z_1^j, z_2^j) \to (x_1, x_2), \quad t_j(z_1^j, z_2^j) \to (z, z) \quad \text{and} \quad (z_1^j, z_2^j) \in U_j$$

for some $z \in X$ and for $U^j := U^i$ whenever $j \in \{i\} \times \mathcal{U}_X$. □

1.3. **EXAMPLES.** *Let* $\phi: \mathcal{X} \to \mathcal{Y}$ *be a homomorphism of ttgs.*

a) $P_\phi \subseteq Q_\phi^\# \subseteq Q_\phi$*; so if* ϕ *is proximal,* $R_\phi = P_\phi = Q_\phi^\# = Q_\phi = E_\phi$.

b) *If* ϕ *is weakly mixing then* $R_\phi = Q_\phi^\# = Q_\phi = E_\phi$.

c) *If* ϕ *is almost periodic then* $\Delta_X = E_\phi = Q_\phi = Q_\phi^\# = P_\phi$.

PROOF.

a) Obviously, $T\alpha \cap R_\phi \subseteq \mathrm{int}_{R_\phi}(\overline{T\alpha \cap R_\phi})$ for every open $\alpha \in \mathcal{U}_X$, and so

$$\begin{aligned} P_\phi &= \bigcap \{T\alpha \cap R_\phi \mid \alpha \in \mathcal{U}_X\} = \bigcap \{T\alpha \cap R_\phi \mid \alpha \in \mathcal{U}_X,\ \alpha \text{ open}\} \subseteq \\ &\subseteq \bigcap \{\mathrm{int}_{R_\phi}(\overline{T\alpha \cap R_\phi}) \mid \alpha \in \mathcal{U}_X,\ \alpha \text{ open}\} = \\ &= \bigcap \{\mathrm{int}_{R_\phi}(\overline{T\alpha \cap R_\phi}) \mid \alpha \in \mathcal{U}_X\} = Q_\phi^\#. \end{aligned}$$

b) If ϕ is weakly mixing, R_ϕ is ergodic and so $\overline{T\alpha \cap R_\phi} = R_\phi$ for every $\alpha \in \mathfrak{U}_X$. Hence $\overline{T\alpha \cap R_\phi} = \mathrm{int}_{R_\phi}(\overline{T\alpha \cap R_\phi})$ for every $\alpha \in \mathfrak{U}_X$, and so $Q_\phi = Q_\phi^\# = R_\phi$.

c) If ϕ is almost periodic then $\Delta_X = Q_\phi$. As $\Delta_X \subseteq P_\phi \subseteq Q_\phi^\# \subseteq Q_\phi$, it follows that $\Delta_X = P_\phi = Q_\phi^\# = Q_\phi = E_\phi$. ☐

1.4. **EXAMPLES.** *Let* $\phi : \mathcal{X} \to \mathcal{Y}$ *be a homomorphism of minimal ttgs. In each of the following two cases we have* $E_\phi = Q_\phi = Q_\phi^\#$.

a) ϕ *is a* RIC *extension;*

b) ϕ *is an open* RIM *extension.*

PROOF. Cf. VII.3.22.. ☐

The following example shows that there are minimal ttgs for which $Q \neq Q^\#$.

Moreover, it shows that if ϕ and ψ are homomorphisms of minimal ttgs with $Q_\phi = Q_\phi^\#$ and $Q_\psi = Q_\psi^\#$ then $Q_{\psi\circ\phi}$ and $Q_{\psi\circ\phi}^\#$ may be different from each other.

1.5. **EXAMPLE.** *Let* $\mathcal{Y}$ *be the fourfold covering of the minimal proximal rotation* $\mathcal{X}$ (cf. I.4.7.). *Then* $Q_{\mathcal{Y}}^\# \neq Q_{\mathcal{Y}} \neq E_{\mathcal{Y}}$.

PROOF. Let T be the free group on two generators and let X , a and b be as in I.4.7.(i). Let Y be the circle and define the map $c : Y \to Y$ by $c(y) := y + \frac{1}{4}\alpha$ and $d : Y \to Y$ by $d(y) := \frac{1}{4}k + 4(y - \frac{1}{4}k)^2$ whenever $k \leqslant 4y < k+1$ ($k \in \{0,1,2,3\}$). Define the ttg $\mathcal{Y} := <T(c,d), Y>$ and let $\phi : \mathcal{Y} \to \mathcal{X}$ be defined as $\phi(y) = 4y \pmod 1$. Then $\mathcal{Y}$ (or better ϕ) is the fourfold covering of $\mathcal{X}$.

Note that $P_{\mathcal{X}} = Q_{\mathcal{X}}^\# = Q_{\mathcal{X}} = E_{\mathcal{X}} = X \times X$; and that ϕ is almost periodic, so that $P_\phi = Q_\phi^\# = Q_\phi = E_\phi = \Delta_Y$.

Obviously, $\mathcal{Y}$ does not admit nontrivial almost periodic factors, in other words $E_{\mathcal{Y}} = Y \times Y$. As c preserves distances, it is not difficult to see that $(y, y') \in Q_{\mathcal{Y}}$ iff the distance (mod 1) between y and y' is smaller then or equal to ¼. So $Q_{\mathcal{Y}} \neq E_{\mathcal{Y}}$.

If the distance between y and y' equals ¼, then we can approach (y, y') with pairs with a distance greater then ¼ (from the outside), which shows that $(y, y') \notin Q_{\mathcal{Y}}^\#$. So $Q_{\mathcal{Y}} \neq Q_{\mathcal{Y}}^\#$. ☐

An indication of the power of sharp regional proximality is given in the following theorem, which hints at regional proximality of second order as will be discussed in VIII.4. (1.6.b).

1.6. **THEOREM.** *Let* $\phi:\mathcal{X}\to\mathcal{Y}$ *be a homomorphism of minimal ttgs.*

a) *Let* $(x_1,x_2)\in R_\phi$. *If* $\overline{T(x_1,x_2)}\cap Q_\phi^\# \neq \emptyset$ *then we have* $(x_1,x_2)\in Q_\phi^\#$ *, and so* $\overline{T(x_1,x_2)}\subseteq \overline{Q_\phi^\#}\subseteq Q_\phi$.
In particular, if $Q_\phi = Q_\phi^\#$ *then* Q_ϕ *contains the orbit closures that have a nonempty intersection with* Q_ϕ.

b) *Let* $(x_1,x_2)\in Q_\phi^\#$ *and let* $\{(x_1^i,x_2^i)\}_i$ *be a net in* R_ϕ *converging to* (x_1,x_2). *Choose* $\{t_i\}_i$ *in* T *and (for a suitable subnet) let* $(z_1,z_2)=\lim t_i(x_1^i,x_2^i)$. *Then* $(z_1,z_2)\in Q_\phi$.

PROOF.

a) If $\overline{T(x_1,x_2)}\cap Q_\phi^\# \neq \emptyset$ then $\overline{T(x_1,x_2)}\cap \mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})\neq\emptyset$ for every $\alpha\in\mathcal{U}_X$, and so $T(x_1,x_2)\cap \mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})\neq\emptyset$. But then it follows that $(x_1,x_2)\in \mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})$ for every $\alpha\in\mathcal{U}_X$ and, consequently, $(x_1,x_2)\in Q_\phi^\#$.

b) Let $\alpha\in\mathcal{U}_X$. As $(x_1,x_2)\in \mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})$, there is an $i(\alpha)$ such that $(x_1^i,x_2^i)\in \mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})$ for every $i\geqslant i(\alpha)$. But then, also, $t_i(x_1^i,x_2^i)\in \mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})$ for every $i\geqslant i(\alpha)$ and so

$$(z_1,z_2)=\lim t_i(x_1^i,x_2^i)\in\overline{T\alpha\cap R_\phi}.$$

As α was arbitrary it follows that

$$(z_1,z_2)\in\bigcap\{\overline{T\alpha\cap R_\phi}\mid \alpha\in\mathcal{U}_X\}=Q_\phi.$$

□

1.7. **COROLLARY.** *Let* $\phi:\mathcal{X}\to\mathcal{Y}$ *be a homomorphism of minimal ttgs. If* $J.Q_\phi^\#\subseteq Q_\phi^\#$ *(e.g.* $Q_\phi^\#$ *is closed, in particular if* $Q_\phi=Q_\phi^\#$ *) then* $Q_\phi^\#\circ P_\phi = P_\phi\circ Q_\phi^\# = Q_\phi^\#$.

PROOF. Let $(x_1,x_2)\in P_\phi$ and $(x_2,x_3)\in Q_\phi^\#$. Let I be a minimal left ideal in S_T such that $px_1=px_2$ for every $p\in I$ and let $v\in J_{x_3}(I)$. Then

$$v(x_1,x_3)=(vx_1,x_3)=(vx_2,x_3)=v(x_2,x_3)\in J.Q_\phi^\#\subseteq Q_\phi^\#.$$

By 1.6.a, it follows that $(x_1,x_3)\in Q_\phi^\#$. Hence $Q_\phi^\#\circ P_\phi\subseteq Q_\phi^\#$.
Clearly, $Q_\phi^\#\subseteq Q_\phi^\#\circ P_\phi$, so $Q_\phi^\#\circ P_\phi = Q_\phi^\#$.
In a similar way it follows that $P_\phi\circ Q_\phi^\# = Q_\phi^\#$. □

1.8. **REMARK.** *Let* $\phi:\mathcal{X}\to\mathcal{Y}$ *be a* RIM *extension of minimal ttgs. If* $Q_\phi = Q_\phi^{\#}$ *then* $E_\phi = Q_\phi = Q_\phi^{\#}$.

PROOF. By VII.1.19., we know that $E_\phi = Q_\phi \circ P_\phi$, and so, by assumption, $E_\phi = Q_\phi^{\#} \circ P_\phi$. From 1.7. it follows that $Q_\phi^{\#} \circ P_\phi = Q_\phi^{\#} = Q_\phi$; so $E_\phi = Q_\phi = Q_\phi^{\#}$. □

The following theorem reflects the way we proved VII.3.22. using VII.3.5.. But first we need a lemma.

1.9. **LEMMA.** *Let* $\phi:\mathcal{X}\to\mathcal{Y}$ *be a homomorphism of minimal ttgs and let* $\kappa:\mathcal{X}\to\mathcal{X}/E_\phi$ *be the quotient map and* $\theta:\mathcal{X}/E_\phi\to\mathcal{Y}$ *the maximal almost periodic factor of* ϕ . *Denote the collection of nonempty open sets in* X/E_ϕ *by* $\mathcal{O}$. *Then*

$$E_\phi = \bigcap\{T(\kappa^{\leftarrow}[U]\times\kappa^{\leftarrow}[U]\cap R_\phi) \mid U\in\mathcal{O}\} =$$

$$= \bigcap\{\overline{T(\kappa^{\leftarrow}[U]\times\kappa^{\leftarrow}[U]\cap R_\phi)} \mid U\in\mathcal{O}\} .$$

PROOF. Let $U\in\mathcal{O}$ and $(x_1,x_2)\in E_\phi$. Then for some $t\in T$ we have $t\kappa(x_1) = t\kappa(x_2)\in U$ and so

$$(x_1,x_2)\in\kappa^{\leftarrow}[t^{-1}U]\times\kappa^{\leftarrow}[t^{-1}U]\cap R_\phi \subseteq T(\kappa^{\leftarrow}[U]\times\kappa^{\leftarrow}[U]\cap R_\phi) .$$

Hence

$$E_\phi \subseteq \bigcap\{T(\kappa^{\leftarrow}[U]\times\kappa^{\leftarrow}[U]\cap R_\phi) \mid U\in\mathcal{O}\} \subseteq$$

$$\subseteq \bigcap\{\overline{T(\kappa^{\leftarrow}[U]\times\kappa^{\leftarrow}[U]\cap R_\phi)} \mid u\in\mathcal{O}\} .$$

On the other hand,

$$\kappa\times\kappa[\bigcap\{\overline{T(\kappa^{\leftarrow}[U]\times\kappa^{\leftarrow}[U]\cap R_\phi)} \mid U\in\mathcal{O}\}] \subseteq$$

$$\subseteq \bigcap\{\overline{T(\kappa\times\kappa(\kappa^{\leftarrow}[U]\times\kappa^{\leftarrow}[U]\cap R_\phi)} \mid U\in\mathcal{O}\} \subseteq$$

$$\subseteq \bigcap\{\overline{T(U\times U\cap R_\phi)} \mid U\in\mathcal{O}\} = Q_\theta = \Delta_{X/E_\phi} .$$

So $\bigcap\{\overline{T(\kappa^{\leftarrow}[U]\times\kappa^{\leftarrow}[U]\cap R_\phi)} \mid U\in\mathcal{O}\} \subseteq (\kappa\times\kappa)^{\leftarrow}[\Delta_{X/E_\phi}] = E_\phi$. □

1.10. **THEOREM.** *Let* $\phi:\mathcal{X}\to\mathcal{Y}$ *be a homomorphism of minimal ttgs and let* $\kappa:\mathcal{X}\to\mathcal{X}/E_\phi$ *be the quotient map and* $\theta:\mathcal{X}/E_\phi\to\mathcal{Y}$ *the maximal almost periodic factor of* ϕ. *Then the following statements are equivalent:*

a) $E_\phi=Q_\phi=Q_\phi^\#$;

b) *for every* $\alpha\in\mathcal{U}_X$ *there is a nonempty open set* V *in* X *such that* $V=E_\phi[V]$ *and* $V\times V\cap R_\phi\subseteq\overline{T\alpha\cap R_\phi}$;

c) *for every open set* U *in* X *there is a nonempty open set* V *in* X *such that* $V=E_\phi[V]$ *and* $V\times V\cap R_\phi\subseteq\overline{T(U\times U\cap R_\phi)}$.

PROOF.

b $\Rightarrow$ c As $\mathcal{X}$ is minimal, $T(U\times U)$ is an open set containing the diagonal for every open U in X. Hence $\alpha:=T(U\times U)\in\mathcal{U}_X$.

c $\Rightarrow$ b For every $\alpha\in\mathcal{U}_X$ there is a $\beta\in\mathcal{U}_X$ with $\beta=\beta^{-1}$ and $\beta^2\subseteq\alpha$. Then $\beta(x)\times\beta(x)\cap R_\phi\subseteq\alpha\cap R_\phi$ for every $x\in X$ and so there is a nonempty open U in X with $T(U\times U\cap R_\phi)\subseteq T\alpha\cap R_\phi$.

b $\Rightarrow$ a Let $\alpha\in\mathcal{U}_X$. By assumption, there is a nonempty open set V in X with $V=E_\phi[V]=\kappa^{\leftarrow}\kappa[V]$ and $V\times V\cap R_\phi\subseteq\overline{T\alpha\cap R_\phi}$. As $\kappa[V]$ is open in X/E_ϕ it follows from 1.9. that

$$E_\phi\subseteq T(\kappa^{\leftarrow}\kappa[V]\times\kappa^{\leftarrow}\kappa[V]\cap R_\phi)=T(V\times V\cap R_\phi).$$

So $E_\phi\subseteq T(V\times V\cap R_\phi)\subseteq T.\overline{T\alpha\cap R_\phi}=\overline{T\alpha\cap R_\phi}$ and as $T(V\times V\cap R_\phi)$ is an open set in R_ϕ, $E_\phi\subseteq\operatorname{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})$. As $\alpha\in\mathcal{U}_X$ was arbitrary, it follows that $E_\phi\subseteq Q_\phi^\#\subseteq Q_\phi\subseteq E_\phi$.

a $\Rightarrow$ b Let $\mathcal{V}$ be the collection of nonempty open sets V in X with $V=E_\phi[V]$. Suppose there is an $\alpha\in\mathcal{U}_X$ with

$$V\times V\cap R_\phi\cap(X\times X\setminus\overline{T\alpha\cap R_\phi})\neq\emptyset$$

for every $V\in\mathcal{V}$. Define

$$\mathcal{K}(V)=\overline{T(V\times V\cap R_\phi)}\setminus\operatorname{int}_{R_\phi}(\overline{T\alpha\cap R_\phi}),$$

then $\mathcal{K}(V)$ is closed and nonempty for every $V\in\mathcal{V}$. As $\mathcal{V}$ is closed under finite intersections and invariant under T, it follows that $\{\mathcal{K}(V)\mid V\in\mathcal{V}\}$ has the finite intersection property. Hence

$$H:=\bigcap\{\mathcal{K}(V)\mid V\in\mathcal{V}\}\neq\emptyset.$$

By 1.9., $H\subseteq E_\phi$ and by construction $H\cap Q_\phi^\#=\emptyset$, which contradicts assumption a. □

1.11. **THEOREM.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *be a* Bc *extension of minimal ttgs. Then* $E_\phi = Q_\phi = Q_\phi^{\#}$.

PROOF. First we shall show that ϕ and ϕ satisfy the conditions of lemma VII.3.5. (compare the proof of VII.3.6.).
Let $U_1 \times U_2 \cap R_\phi$ be a nonempty (basic) open set in R_ϕ and let $(x_1, x_2) \in U_1 \times U_2 \cap R_\phi$ be an almost periodic point; say $(x_1, x_2) = u(x_1, x_2)$ for some $u \in J$. We shall show that

$$E_\phi[x_1] \times \{x_2\} \subseteq \overline{T(U_1 \times U_2 \cap R_\phi)}\,.$$

Let V be an open set in T with $V = V(u)$ and $Vx_2 \subseteq U_2$ (III.2.1.c). Define $\mathcal{U} := [U_1, V] \cap u\phi^{\leftarrow}\phi(x_1)$, then $\mathcal{U}$ is an $\mathfrak{T}(\mathfrak{X}, u)$-neighbourhood of x_1 in $u\phi^{\leftarrow}\phi(x_1)$. Consider an arbitrary $x' \in \mathcal{U}$; say $x' = t^{-1}z$ for some $t \in V$ and $z \in U_1$. Then $(x', x_2) = t^{-1}(z, tx_2) \in T(U_1 \times U_2)$, so $(x', x_2) \in T(U_1 \times U_2 \cap R_\phi)$. Hence

$$\mathcal{U} \times \{x_2\} \subseteq T(U_1 \times U_2 \cap R_\phi)\,.$$

By III.3.10.a, $E_\phi[x_1] \subseteq J_{x_2} \circ \mathcal{U}$, so

$$E_\phi[x_1] \times \{x_2\} \subseteq J_{x_2} \circ \mathcal{U} \times \{x_2\} = J_{x_2} \circ (\mathcal{U} \times \{x_2\}) \subseteq \overline{T(U_1 \times U_2 \cap R_\phi)}\,.$$

Therefore ϕ and ϕ satisfy the conditions of lemma VII.3.5..
Let U be a nonempty open set in X . By VII.3.5., there is a nonempty open set $\tilde{U}$ with $\tilde{U} = E_\phi[\tilde{U}]$ such that

$$\varnothing \neq \tilde{U} \times U \cap R_\phi \subseteq \overline{T(U \times U \cap R_\phi)}\,.$$

Again by VII.3.5. and the facts that $\phi[\tilde{U}] = \phi[\tilde{U} \cap U]$ and $\tilde{U} = \kappa^{\leftarrow}[\kappa[\tilde{U} \cap U]^\circ]$, it follows that

$$\varnothing \neq \tilde{U} \times \tilde{U} \cap R_\phi \subseteq \overline{T(\tilde{U} \times U \cap R_\phi)}\,.$$

Hence $\tilde{U} \times \tilde{U} \cap R_\phi \subseteq \overline{T(U \times U \cap R_\phi)}$ and the theorem follows from 1.10..□

VIII.2. FACTORS AND LIFTING

Let $\phi:\mathcal{X}\to\mathcal{Y}$ be a homomorphism of minimal ttgs and let $\theta:\mathcal{Z}\to\mathcal{Y}$ be a factor of ϕ. By I.4.3., $E_\phi=Q_\phi$ implies $E_\theta=Q_\theta$. We shall see that $E_\phi=Q_\phi=Q_\phi^\#$ implies $E_\theta=Q_\theta=Q_\theta^\#$ too. Also we shall study the lifting of sharp regional proximality in shadow diagrams.

2.1. **THEOREM.** *Consider the following diagram of homomorphisms of minimal ttgs.*

$$\begin{tikzcd}
\mathcal{X}' \arrow[r, "\sigma"] \arrow[d, "\phi'"'] & \mathcal{X} \arrow[d, "\phi"] \\
\mathcal{Y}' \arrow[r, "\tau"'] & \mathcal{Y}
\end{tikzcd}$$

Let σ *be proximal and suppose that* $\sigma\times\sigma[R_{\phi'}]=R_\phi$. *Then*

a) $\sigma\times\sigma[Q_{\phi'}^\#\cap JR_{\phi'}]\subseteq Q_\phi^\#$;

b) $Q_{\phi'}=Q_{\phi'}^\#$ *implies* $Q_\phi=Q_\phi^\#$; *in particular,* $E_{\phi'}=Q_{\phi'}=Q_{\phi'}^\#$ *implies* $E_\phi=Q_\phi=Q_\phi^\#$.

PROOF.

a) Let $(z_1,z_2)\in Q_{\phi'}^\#\cap JR_{\phi'}$, then

$$(x_1,x_2):=\sigma\times\sigma(z_1,z_2)\subseteq\sigma\times\sigma[Q_{\phi'}]\subseteq Q_\phi .$$

Suppose that $(x_1,x_2)\notin Q_\phi^\#$. Then there is an index $\alpha\in\mathcal{U}_X$ such that $(x_1,x_2)\notin\operatorname{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})$. And so there is a net $\{(x_1^i,x_2^i)\}_i$ converging to (x_1,x_2) with $(x_1^i,x_2^i)\notin\overline{T\alpha\cap R_\phi}$ for every i. Let $(z_1^i,z_2^i)\in R_{\phi'}$ be such that $\sigma\times\sigma(z_1^i,z_2^i)=(x_1^i,x_2^i)$ and, after passing to a suitable subnet, let $(\bar z_1,\bar z_2)=\lim(z_1^i,z_2^i)$. Then

$$\sigma\times\sigma(\bar z_1,\bar z_2)=(x_1,x_2)=\sigma\times\sigma(z_1,z_2),$$

and as $\sigma\times\sigma:\mathcal{X}'\times\mathcal{X}'\to\mathcal{X}\times\mathcal{X}$ is proximal (I.1.21.b), it follows that $(\bar z_1,\bar z_2)$ and (z_1,z_2) are proximal in $R_{\phi'}$. However, (z_1,z_2) is an almost periodic point, so $(z_1,z_2)\in\overline{T(\bar z_1,\bar z_2)}$. As $(z_1,z_2)\in Q_{\phi'}^\#$ it follows from 1.6.a that $(\bar z_1,\bar z_2)\in Q_{\phi'}^\#$.
Let $\beta\in\mathcal{U}_{X'}$ be such that $\sigma\times\sigma[\beta]\subseteq\alpha$, then $\sigma\times\sigma[\overline{T\beta\cap R_{\phi'}}]\subseteq\overline{T\alpha\cap R_\phi}$.

Since

$$(\bar{z}_1,\bar{z}_2)\in Q^{\#}_{\phi'} \subseteq \mathrm{int}_{R_{\phi'}}(\overline{T\beta\cap R_{\phi'}})\ ,$$

we know that $(z_1^i, z_2^i)\in \overline{T\beta\cap R_{\phi'}}$ for i large enough. But then

$$(x_1^i, x_2^i) = \sigma\times\sigma(z_1^i, z_2^i)\in\sigma\times\sigma[\overline{T\beta\cap R_{\phi'}}]\subseteq \overline{T\alpha\cap R_{\phi}}$$

for i large enough, which contradicts the choice of the net $\{(x_1^i, x_2^i)\}_i$. Hence $(x_1,x_2)\in Q^{\#}_{\phi}$.

b) Note that by IV.4.2.b, we have $\sigma\times\sigma[Q_{\phi'}]=Q_\phi$; so it follows that $\sigma\times\sigma[JQ_{\phi'}]=JQ_\phi$. If $Q_{\phi'}=Q^{\#}_{\phi'}$ then $JQ_{\phi'}=Q^{\#}_{\phi'}\cap JR_{\phi'}$ and so, by a, it follows that

$$JQ_\phi\subseteq\sigma\times\sigma[Q^{\#}_{\phi'}\cap JR_{\phi'}]\subseteq Q^{\#}_{\phi}\ .$$

If $(x_1,x_2)\in Q_\phi$ then $\overline{T(x_1,x_2)}$ contains an almost periodic point; hence $\overline{T(x_1,x_2)}\cap JQ_\phi\neq\emptyset$ and so $\overline{T(x_1,x_2)}\cap Q^{\#}_{\phi}\neq\emptyset$. Hence by 1.6.a, it follows that $(x_1,x_2)\in Q^{\#}_{\phi}$. Consequently, $Q_\phi=Q^{\#}_{\phi}$.
Suppose $E_{\phi'}=Q_{\phi'}=Q^{\#}_{\phi'}$; then by IV.4.3.d, $E_\phi=Q_\phi$ and by the above $Q_\phi=Q^{\#}_{\phi}$. □

Theorem 2.1. enables us to give an alternative proof of 1.11. as follows.

2.2. **COROLLARY.** *Let* $\phi:\mathcal{X}\to\mathcal{Y}$ *be a homomorphism of minimal ttgs. If* ϕ *is an* RMM *extension or if* ϕ *satisfies the Bronstein condition, then* $E_\phi=Q_\phi=Q^{\#}_{\phi}$.

PROOF. If ϕ is an RMM extension then, by VII.1.16., we can construct a diagram as in 2.1. such that ϕ' is an open RIM extension. Hence by 1.4.b, $E_{\phi'}=Q_{\phi'}=Q^{\#}_{\phi'}$ and by 2.1., we may conclude that $E_\phi=Q_\phi=Q^{\#}_{\phi}$.
If ϕ is a Bc extension then, by IV.4.5., EGS(ϕ) is a diagram which satisfies the assumptions in 2.1., such that ϕ' is a RIC extension. Again by 1.4. and 2.1., it follows that $E_\phi=Q_\phi=Q^{\#}_{\phi}$. □

In IV.4.8., IV.4.16. and IV.4.17. we have shown that highly proximal lifting of homomorphisms of minimal ttgs preserves many decent properties of those homomorphisms. In addition to this, we show:

2.3. **THEOREM.** *Consider the following diagram of homomorphisms of minimal ttgs:*

$$\begin{tikzcd}
X' \arrow[rrr, "\sigma"] \arrow[dr, "\kappa'"] \arrow[dd, "\phi'"'] & & & X \arrow[dl, "\kappa"'] \arrow[dd, "\phi"] \\
& X'/E_{\phi'} \arrow[r, "\xi"'] \arrow[dl, "\theta'"] & X/E_{\phi} \arrow[dr, "\theta"] & \\
Y' \arrow[rrr, "\tau"'] & & & Y
\end{tikzcd}$$

Assume that ϕ' *is open,* σ *is highly proximal and* $\sigma\times\sigma[R_{\phi'}]=R_{\phi}$. *Then* $E_{\phi'}=Q_{\phi'}=Q_{\phi'}^{\#}$ *iff* $E_{\phi}=Q_{\phi}=Q_{\phi}^{\#}$.

PROOF. By 2.1.b, it follows that $E_{\phi}=Q_{\phi}=Q_{\phi}^{\#}$ if $E_{\phi'}=Q_{\phi'}=Q_{\phi'}^{\#}$. Conversely suppose that $E_{\phi}=Q_{\phi}=Q_{\phi}^{\#}$. Remember that the openness of ϕ' implies that $\sigma' := \sigma\times\sigma|_{R_{\phi'}}: R_{\phi'}\to R_{\phi}$ is an irreducible map (IV.4.13.). Let W be a nonempty open set in X', which by IV.2.1., without loss of generality may be chosen such that it is of the form $W=\sigma^{\leftarrow}\sigma[W]$; hence $\sigma[W]$ is an open set in X. We want to find a nonempty open set U in X' such that

$$U=E_{\phi'}[U] \text{ and } U\times U\cap R_{\phi'}\subseteq \overline{T(W\times W\cap R_{\phi'})},$$

which proves the theorem by 1.10.c.

As $E_{\phi}=Q_{\phi}=Q_{\phi}^{\#}$ and $\sigma[W]$ is open in X, by 1.10.c, we can find a nonempty open set V in X such that

$$V=E_{\phi}[V] \text{ and } V\times V\cap R_{\phi}\subseteq \overline{T(\sigma[W]\times\sigma[W]\cap R_{\phi})}.$$

Define an open set U in X' by $U:=\sigma^{\leftarrow}[V]$. Then

$$U=\sigma^{\leftarrow}[V]=\sigma^{\leftarrow}[\kappa^{\leftarrow}[\kappa[V]]]=\kappa'^{\leftarrow}[\xi^{\leftarrow}[\kappa[V]]]=\kappa'^{\leftarrow}[\kappa'[\sigma^{\leftarrow}[V]]]=\kappa'^{\leftarrow}\kappa'[U].$$

The proof is finished if we show that $U\times U\cap R_{\phi'}\subseteq\overline{T(W\times W\cap R_{\phi'})}$. We shall show that every nonempty open subset U' of $U\times U\cap R_{\phi'}$ intersects $T(W\times W\cap R_{\phi'})$, which implies that every element of $U\times U\cap R_{\phi'}$ is in the closure of $T(W\times W\cap R_{\phi'})$.

So let U' be open and nonempty in $U\times U\cap R_{\phi'}$. As $\sigma': R_{\phi'}\to R_{\phi}$ is irreducible, by IV.2.1., we can find a nonempty open set $V'\subseteq U'$ such that $V'=\sigma'^{\leftarrow}\sigma'[V']$. Note that $\sigma'[V']$ is open and that

$$\sigma'[V']\subseteq\sigma\times\sigma[U\times U\cap R_{\phi'}]\subseteq V\times V\cap R_{\phi}\subseteq\overline{T(\sigma[W]\times\sigma[W]\cap R_{\phi})},$$

so $\sigma'[V']\cap T(\sigma[W]\times\sigma[W]\cap R_\phi)\neq\emptyset$. As $V'=\sigma'^{\leftarrow}\sigma'[V']$ it follows that $V'\cap T(W\times W\cap R_{\phi'})\neq\emptyset$, hence that $U'\cap T(W\times W\cap R_{\phi'})\neq\emptyset$. This concludes the proof. □

2.4. **COROLLARY.** *Let* $\phi:\mathcal{X}\to\mathcal{Y}$ *be an open homomorphism of minimal ttgs and let* $\phi^*:\mathcal{X}^*\to\mathcal{Y}^*$ *be the* MHP *lifting of* ϕ . *Then* $E_\phi=Q_\phi=Q_\phi^\#$ *iff* $E_{\phi^*}=Q_{\phi^*}=Q_{\phi^*}^\#$.

PROOF. If ϕ is open then $*(\phi)$ is a diagram as in 2.3. (IV.4.7.). □

2.5. Consider the following diagram consisting of homomorphisms of minimal ttgs.

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & \mathcal{Y} \\
& \mathcal{Z} \arrow[ur, "\theta"'] &
\end{tikzcd}$$

In the remainder of this section we shall deal with the question: does $Q_\phi=Q_\phi^\#$ imply $Q_\theta=Q_\theta^\#$?

2.6. **THEOREM.** *Consider the diagram in 2.5.. If* ψ *is open then* $Q_\phi=Q_\phi^\#$ *implies* $Q_\theta=Q_\theta^\#$. *In particular, if* ψ *is open then* $E_\phi=Q_\phi=Q_\phi^\#$ *implies* $E_\theta=Q_\theta=Q_\theta^\#$.

PROOF. If ψ is open then $\psi\times\psi|_{R_\phi}:R_\phi\to R_\theta$ is an open homomorphism of ttgs. For $\psi\times\psi:X\times X\to Z\times Z$ is open and $R_\phi=(\psi\times\psi)^{\leftarrow}[R_\theta]$. Let $\alpha\in\mathcal{U}_Z$; then there is a $\beta\in\mathcal{U}_X$ such that $\psi\times\psi[\beta]\subseteq\alpha$, hence

$$\overline{T.\psi\times\psi[\beta\cap R_\phi]}\subseteq\overline{T\alpha\cap R_\theta}\,.$$

By I.4.3.b, $Q_\theta=\psi\times\psi[Q_\phi]$ and so, assuming that $Q_\phi=Q_\phi^\#$,

$$Q_\theta=\psi\times\psi[Q_\phi^\#]\subseteq\psi\times\psi[\mathrm{int}_{R_\phi}(\overline{T\beta\cap R_\phi})]\,.$$

As $\psi\times\psi|_{R_\phi}$ is open

$$Q_\theta\subseteq\mathrm{int}_{R_\theta}(\psi\times\psi[\overline{T\beta\cap R_\phi}])=\mathrm{int}_{R_\theta}(\overline{T\psi\times\psi[\beta\cap R_\phi]})\,.$$

Hence it follows that

$$Q_\theta \subseteq \mathrm{int}_{R_\theta}(\overline{T\psi \times \psi[\beta \cap R_\phi]}) \subseteq \mathrm{int}_{R_\theta}(\overline{T\alpha \cap R_\theta}) .$$

As $\alpha \in \mathcal{U}_Z$ was arbitrary, it follows that $Q_\theta \subseteq Q_\theta^{\#}$; so $Q_\theta = Q_\theta^{\#}$.
If $E_\phi = Q_\phi$ then, by I.4.3., $E_\theta = Q_\theta$. ☐

2.7. **THEOREM.** *Consider the diagram in* 2.5..

a) *If* ϕ *is open and if* ψ *is highly proximal then* $E_\phi = Q_\phi = Q_\phi^{\#}$ *iff* $E_\theta = Q_\theta = Q_\theta^{\#}$.

b) *If* ψ *is proximal then* $Q_\phi = Q_\phi^{\#}$ *implies* $Q_\theta = Q_\theta^{\#}$.

c) *If* $\mathcal{X} = \mathcal{X}^*$ *then* $Q_\phi = Q_\phi^{\#}$ *implies* $Q_\theta = Q_\theta^{\#}$.

PROOF.

a) As the diagram of 2.5. is a special case of the diagram in 2.3. and as the assumption guarantees that the assumptions in 2.3. are satisfied, a follows immediately from 2.3..

b) In the same way b is a special case of 2.1..

c) Let $\chi_{\mathcal{Z}} : \mathcal{Z}^* \to \mathcal{Z}$ be the MHP extension of $\mathcal{Z}$ and let $\psi^* : \mathcal{X}^* \to \mathcal{Z}^*$ be the MHP lifting of ψ . Then $\phi = \theta \circ \chi_{\mathcal{Z}} \circ \psi^*$. As ψ^* is open it follows from 2.6. that $Q_\phi = Q_\phi^{\#}$ implies $Q_{\theta \circ \chi_{\mathcal{Z}}} = Q_{\theta \circ \chi_{\mathcal{Z}}}^{\#}$. Hence by b, we know that $Q_\theta = Q_\theta^{\#}$. ☐

2.8. **THEOREM.** *Consider the diagram in* 2.5.. *If* $Q_\phi = (\psi \times \psi)^{\leftarrow}[Q_\theta]$ *then* $Q_\phi = Q_\phi^{\#}$ *implies* $Q_\theta = Q_\theta^{\#}$.

PROOF. Let $\beta \in \mathcal{U}_Z$ and let $\alpha \in \mathcal{U}_X$ be such that $\psi \times \psi[\alpha] \subseteq \beta$. Then

$$\psi \times \psi[\overline{T\alpha \cap R_\phi}] \subseteq \overline{T\psi \times \psi[\alpha] \cap R_\theta} \subseteq \overline{T\beta \cap R_\theta} .$$

Suppose $Q_\phi = Q_\phi^{\#}$ then

$$Q_\phi \subseteq \mathrm{int}_{R_\phi}(\overline{T\alpha \cap R_\phi}) = R_\phi \setminus \mathrm{cl}_{R_\phi}(R_\phi \setminus (\overline{T\alpha \cap R_\phi})) .$$

As $Q_\phi = (\psi \times \psi)^{\leftarrow} Q_\theta = (\psi \times \psi)^{\leftarrow}(\psi \times \psi)[Q_\phi]$ it follows that

$$Q_\theta = \psi \times \psi[Q_\phi] \subseteq \psi \times \psi[R_\phi] \setminus \psi \times \psi[\mathrm{cl}_{R_\phi}(R_\phi \setminus (\overline{T\alpha \cap R_\phi}))] \subseteq$$

$$\subseteq R_\theta \setminus \mathrm{cl}_{R_\theta}(R_\theta \setminus \psi \times \psi[\overline{T\alpha \cap R_\phi}] =$$

$$= \mathrm{int}_{R_\theta}(R_\theta \setminus (R_\theta \setminus \psi \times \psi[\overline{T\alpha \cap R_\phi}])) = \mathrm{int}_{R_\theta}(\psi \times \psi[\overline{T\alpha \cap R_\phi}]) \subseteq$$

$$\subseteq \mathrm{int}_{R_\theta}(\overline{T\psi \times \psi[\alpha] \cap R_\theta}) \subseteq \mathrm{int}_{R_\theta}(\overline{T\beta \cap R_\theta}) .$$

As β was arbitrary this shows that $Q_\theta \subseteq Q_\theta^{\#}$. ☐

2.9. **REMARK.** *Consider the diagram in* 2.5.. *If* $E_\phi = Q_\phi$ *and if* $R_\psi \subseteq Q_\phi$ *then* $Q_\phi = (\psi \times \psi)^{\leftarrow}[Q_\theta]$.

PROOF. Note that $\psi \times \psi[Q_\phi] = Q_\theta$ (I.4.3.b), hence $Q_\phi \subseteq (\psi \times \psi)^{\leftarrow}[Q_\theta]$. Let $(x_1, x_2) \in (\psi \times \psi)^{\leftarrow}[Q_\theta]$. Then, by I.4.3.b, there is a $(z_1, z_2) \in Q_\phi$ such that $\psi \times \psi(z_1, z_2) = \psi \times \psi(x_1, x_2)$. But then $(x_1, z_1) \in R_\psi$ and also $(x_2, z_2) \in R_\psi$. Hence

$$(x_1, x_2) \in R_\psi \circ Q_\phi \circ R_\psi \subseteq Q_\phi^3 ,$$

and so $(x_1, x_2) \in E_\phi = Q_\phi$. □

By now we are able to prove the main result of this section.

2.10. **THEOREM.** *Consider the diagram in* 2.5.. *If* $E_\phi = Q_\phi = Q_\phi^\#$ *then* $E_\theta = Q_\theta = Q_\theta^\#$.

PROOF. Note that $E_\phi = Q_\phi$ implies that $E_\theta = Q_\theta$ (I.4.3.).
Now consider the following diagram of homomorphisms of minimal ttgs.

$$\begin{tikzcd}
\mathcal{X} \arrow[rrrr, "\psi"] \arrow[rrd, "\xi"] \arrow[rdd, "\kappa"'] \arrow[rrddd, bend right, "\phi"'] & & & & \mathcal{Z} \arrow[ldd, "\lambda"] \arrow[llddd, bend left, "\theta"] \\
 & & \mathcal{W} \arrow[rru, "\pi_2"'] \arrow[ld, "\pi_1"'] \arrow[dd, "\chi"] & & \\
 & \mathcal{X}/Q_\phi \arrow[rr, "\mu"] \arrow[rd, "\alpha"'] & & \mathcal{Z}/Q_\theta \arrow[ld, "\beta"] & \\
 & & \mathcal{Y} & &
\end{tikzcd}$$

Let $\kappa : \mathcal{X} \to \mathcal{X}/Q_\phi$ and $\lambda : \mathcal{Z} \to \mathcal{Z}/Q_\theta$ be the quotient maps. Since $\psi \times \psi[Q_\phi] = Q_\theta$ there exists a unique homomorphism $\mu : \mathcal{X}/Q_\phi \to \mathcal{Y}/Q_\theta$ such that $\lambda \circ \psi = \mu \circ \kappa$. As $\alpha = \beta \circ \mu$, μ is almost periodic. Let $x \in uX$, $z := \psi(x)$ and note that $(\kappa(x), z) \in R_{\mu\lambda}$. Define $W := \overline{T(\kappa(x), z)}$, then W is a minimal subset of $R_{\mu\lambda}$ (for $J_x \subseteq J_{\kappa(x)} \cap J_z$) and W projects onto X/Q_ϕ and Z by π_1 and π_2 respectively. It is an elementary exercise to show that π_2 is an almost periodic map (μ is almost periodic!), so π_2 is open. Define $\chi : \mathcal{W} \to \mathcal{Y}$ by $\chi = \alpha \circ \pi_1$ and let $\xi : \mathcal{X} \to \mathcal{W}$ be defined by $\xi(x) = (\kappa(x), z)$. Then $\phi = \chi \circ \xi$. As, clearly, $R_\xi \subseteq R_\kappa = Q_\phi$ it follows from 2.9. that $Q_\phi = (\xi \times \xi)^{\leftarrow}[Q_\chi]$. Hence by 2.8., we know that $Q_\chi = Q_\chi^\#$. As $\chi = \theta \circ \pi_2$ and π_2 is open it follows from 2.6. that $Q_\theta = Q_\theta^\#$, which proves the theorem. □

VIII.3. TRANSITIVITY AND $Q^{\#}$

> In general the regionally proximal relation is not an equivalence relation. However, there are conditions that imply transitivity of the regionally proximal relation, for instance the Bronstein condition and "open RIM". In all these cases the equicontinuous structure relation turns out to be the sharply regionally proximal relation. From that one could conjecture that $Q_\phi = Q_\phi^{\#}$ implies transitivity of Q_ϕ. In this section we shall see it does in case ϕ is open or if the ttgs in question are metric. One also could conjecture the converse: transitivity of Q_ϕ implies $Q_\phi = Q_\phi^{\#}$. However, we don't have evidence for that.

First we introduce some notation:
Let $\phi : \mathscr{X} \to \mathscr{Y}$ be a homomorphism of minimal ttgs. Let $(x_1, x_2) \in R_\phi$ and $p \in S_T$. Then define

$$p\star(x_1, x_2) := \bigcap \{ p \circ V \mid V \text{ is a neighbourhood of } (x_1, x_2) \text{ in } R_\phi \} .$$

Clearly, $p\star(x_1, x_2) = \bigcap \{ p \circ (U_1 \times U_2 \cap R_\phi) \mid U_i \in \mathcal{V}_{x_i} \}$ (remember that we denote the neighbourhood system of x in X by $\mathcal{V}_x$).
Note that there is some ambiguity in the notation as we do not specify the map. As we use it only in the situation of one specific homomorphism ϕ and never with respect to $X \times X$, no serious problem will arise.

3.1. **THEOREM.** *Let $\phi : \mathscr{X} \to \mathscr{Y}$ be a homomorphism of ttgs (not necessarily minimal), let $(x_1, x_2) \in R_\phi$. Then $(x_1, x_2) \in Q_\phi$ iff there is a minimal left ideal I in S_T with $p\star(x_1, x_2) \cap \Delta_X \neq \emptyset$ for every $p \in I$.*

PROOF. Let $(x_1, x_2) \in Q_\phi$. Then there are nets $\{(x_1^i, x_2^i)\}_i$ and $\{t_i\}_i$ in R_ϕ and T such that $(x_1^i, x_2^i) \to (x_1, x_2)$ and $t_i(x_1^i, x_2^i) \to (x, x)$ for some $x \in X$. Without loss of generality we may assume that the net $\{t_i\}_i$ converges to some $p \in S_T$. Let V be a neighbourhood of (x_1, x_2) in R_ϕ. Then there is an i_0 such that $(x_1^i, x_2^i) \in V$ for every $i \geqslant i_0$. Hence

$$(x, x) = \lim \{ t_i(x_1^i, x_2^i) \mid i \geqslant i_0 \} \in \lim t_i \overline{V} = p \circ V .$$

As V was arbitrary, $(x, x) \in p\star(x_1, x_2)$ and so $p\star(x_1, x_2) \cap \Delta_X \neq \emptyset$.
Conversely, suppose that for some $p \in S_T$ we have $p\star(x_1, x_2) \cap \Delta_X \neq \emptyset$, say $(x, x) \in p\star(x_1, x_2)$. For $\alpha \in \mathscr{U}_X$, $p \circ (\alpha(x_1) \times \alpha(x_2) \cap R_\phi) \in 2^{R_\phi}$ and $<(\alpha \cap R_\phi)^\circ, R_\phi>$ is a neighbourhood of $p \circ (\alpha(x_1) \times \alpha(x_2) \cap R_\phi)$ in 2^{R_ϕ}.
Let $\{t_i\}_i$ be a net in T with $t_i \to p$ in S_T. Then

$$t_i \overline{(\alpha(x_1) \times \alpha(x_2) \cap R_\phi)} \to p \circ (\alpha(x_1) \times \alpha(x_2) \cap R_\phi) \text{ in } 2^{R_\phi} .$$

So there is an i_α such that

$$t_{i_\alpha}\overline{(\alpha(x_1)\times\alpha(x_2)\cap R_\phi)}\cap(\alpha\cap R_\phi)^\circ\neq\varnothing\ .$$

Hence $t_{i_\alpha}(\alpha(x_1)\times\alpha(x_2)\cap R_\phi)\cap\alpha\cap R_\phi\neq\varnothing$ and we can find $t_\alpha:=t_{i_\alpha}$ in T and $(x_1^\alpha,x_2^\alpha)\in\alpha(x_1)\times\alpha(x_2)\cap R_\phi$ such that $t_\alpha(x_1^\alpha,x_2^\alpha)\in\alpha\cap R_\phi$. Doing this for every $\alpha\in\mathfrak{U}_X$, we obtain nets $\{t_\alpha\}_{\alpha\in\mathfrak{U}_X}$ in T and $\{(x_1^\alpha,x_2^\alpha)\}_{\alpha\in\mathfrak{U}_X}$ in R_ϕ such that, after taking a suitable subnet,

$$(x_1^\alpha,x_2^\alpha)\to(x_1,x_2)\ \text{ and }\ t_\alpha(x_1^\alpha,x_2^\alpha)\to(x',xprime)\ .$$

for some $x'\in X$. Consequently, $(x_1,x_2)\in Q_\phi$. By now we have shown

$$(x_1,x_2)\in Q_\phi\ \text{ iff }\ p\star(x_1,x_2)\cap\Delta_X\neq\varnothing\ \text{ for some }\ p\in S_T\ ,$$

hence the "if"-part of the theorem is proved.
Let $(x_1,x_2)\in Q_\phi$ and define

$$S:=\{p\in S_T\mid p\star(x_1,x_2)\cap\Delta_X\neq\varnothing\}\ .$$

By the above, $S\neq\varnothing$ and, clearly, S is T-invariant. We shall show that S is closed. Then S contains a minimal left ideal, proving the theorem. For each neighbourhood V of (x_1,x_2) in R_ϕ the mapping $p\mapsto p\circ V$ is continuous, hence the mapping

$$\Psi:p\mapsto\bigcap\{\overline{p\circ V}\mid V\ \text{neighbourhood of }(x_1,x_2)\text{ in }R_\phi\}:S_T\to 2^{R_\phi}$$

is upper semi continuous. Since Δ_X is closed and as S is the full original under Ψ of the closed subset $\{A\in 2^{R_\phi}\mid A\cap\Delta_X\neq\varnothing\}$ of 2^{R_ϕ}, it follows that S is closed. □

3.2. **REMARK.** *Let $\phi:\mathfrak{X}\to\mathfrak{Y}$ be a homomorphism of ttgs and let $(x_1,x_2)\in R_\phi$.*

a) *If $(x_1,x_2)\in Q_\phi^\#$, then $p\star(x_1,x_2)\subseteq Q_\phi$ for every $p\in S_T$.*

b) *If $p\star(x_1,x_2)\cap Q_\phi^\#\neq\varnothing$ for some $p\in S_T$, then $(x_1,x_2)\in Q_\phi$.*

PROOF.

a) Let $\alpha\in\mathfrak{U}_X$, then $(x_1,x_2)\in\mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})$. So there are open neighbourhoods $U_1\in\mathfrak{V}_{x_1}$ and $U_2\in\mathfrak{V}_{x_2}$ such that

$$(x_1,x_2)\in U_1\times U_2\cap R_\phi\subseteq\mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})\ .$$

For every $p\in S_T$ it follows that

$$p\star(x_1,x_2)\subseteq p\circ(U_1\times U_2\cap R_\phi)\subseteq\overline{T.\,\mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})}\subseteq\overline{T\alpha\cap R_\phi}\ .$$

As α was arbitrary, $p\star(x_1,x_2)\subseteq Q_\phi$ for every $p\in S_T$.

b) Suppose $p\star(x_1,x_2)\cap Q_\phi^\# \neq \emptyset$. Let $\{t_i\}_i$ be a net in T with $t_i\to p$ and let $\alpha,\beta\in \mathfrak{U}_X$ be such that $\beta\subseteq\alpha$. Then

$$p\circ(\beta(x_1)\times\beta(x_2)\cap R_\phi)\cap \mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})\neq\emptyset ,$$

so $<\mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi}),R_\phi>$ is an open neighbourhood of the element $p\circ(\beta(x_1)\times\beta(x_2)\cap R_\phi)$ of 2^{R_ϕ}. From

$$t_i(\overline{\beta(x_1)\times\beta(x_2)\cap R_\phi})\to p\circ(\beta(x_1)\times\beta(x_2)\cap R_\phi) ,$$

it follows that

$$t_i(\overline{\beta(x_1)\times\beta(x_2)\cap R_\phi})\cap \mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})\neq\emptyset .$$

But then $\beta(x_1)\times\beta(x_2)\cap\overline{T\alpha\cap R_\phi}\neq\emptyset$, and as is easily seen, this implies $(x_1,x_2)\in Q_\phi$. □

3.3. LEMMA. *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a homomorphism of ttgs and suppose that* $Q_\phi=Q_\phi^\#$. *Let* $(x,y)\in Q_\phi$ *and* $(y,z)\in Q_\phi$. *If* ϕ *is open in* $x\in X$, *then* $(x,z)\in Q_\phi$.

PROOF. By 3.1., we can find a minimal left ideal I in S_T, $p\in I$ and a $z'\in X$ such that $(z',z')\in p\star(y,z)$. Let $\alpha\in\mathfrak{U}_X$ and let $U_x\subseteq\alpha(x)$, $U_y\subseteq\alpha(y)$ and $U_z\subseteq\alpha(z)$ be open neighbourhoods of x, y and z in X, such that

$$U_x\times U_y\cap R_\phi\subseteq \mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})$$

(for U_z no further conditions). As ϕ is open in x, we may assume that U_y is such that $\phi[U_y]\subseteq\phi[U_x]$. Since

$$(z',z')\in p\star(y,z)\subseteq p\circ(U_y\times U_z\cap R_\phi) ,$$

we can find nets $\{t_i\}_i$ in T and $\{(y_i,z_i)\}_i$ in $U_y\times U_z\cap R_\phi$ such that $p=\lim t_i$ and $(z',z')=\lim t_i(y_i,z_i)$. Let $x_i\in U_x$ be such that $\phi(x_i)=\phi(y_i)$. Then, for every i,

$$(x_i,y_i)\in U_x\times U_y\cap R_\phi \text{ and } (x_i,z_i)\in U_x\times U_z\cap R_\phi .$$

Let $x'_\alpha:=\lim t_i x_i$ (after passing to a suitable subnet). Then

$$(x'_\alpha,z')=\lim t_i(x_i,y_i)\in p\circ(U_x\times U_y\cap R_\phi)\subseteq p\circ(\overline{T\alpha\cap R_\phi})\subseteq\overline{T\alpha\cap R_\phi}$$

and

$$(x'_\alpha,z')=\lim t_i(x_i,z_i)\in p\circ(U_x\times U_z\cap R_\phi)\subseteq p\circ(\alpha(x)\times\alpha(z)\cap R_\phi) .$$

So for every $\alpha \in \mathfrak{U}_X$ we can define in this way an element $x'_\alpha \in X$. Let $x' = \lim x'_\beta$ (after passing to a suitable subnet). Then

$$(x',z') = \lim(x'_\beta, z') \in \overline{T\alpha \cap R_\phi} \text{ for every } \alpha \in \mathfrak{U}_X ;$$

hence $(x',z') \in Q_\phi = Q_\phi^\#$. And

$$(x',z') = \lim(x'_\beta, z') \in p \circ (\alpha(x) \times \alpha(z) \cap R_\phi) \text{ for every } \alpha \in \mathfrak{U}_X .$$

As $p_\star(x,z) = \bigcap \{p \circ (\alpha(x) \times \alpha(z) \cap R_\phi) \mid \alpha \in \mathfrak{U}_X\}$, it follows that $(x',z') \in p_\star(x,z)$ and so that $p_\star(x,z) \cap Q_\phi^\# \neq \varnothing$. By 3.2.b, it follows that $(x,z) \in Q_\phi$. □

3.4. **THEOREM.** *Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a homomorphism of minimal ttgs, such that ϕ is open in some point $x \in X$. Then $Q_\phi = Q_\phi^\#$ implies $E_\phi = Q_\phi$.*

PROOF. Let $(x_1,x_2) \in Q_\phi$ and $(x_2,x_3) \in Q_\phi$ and let $p \in M$ be such that $x = px_1$. Then $(x, px_2) = p(x_1,x_2) \in Q_\phi$ and $(px_2, px_3) \in Q_\phi$; so, by 3.3., it follows that $(x, px_3) \in Q_\phi$. Let $v \in J_{x_1}$, then

$$(x_1, vx_3) = vp^{-1}(x, px_3) \in Q_\phi .$$

As $(vx_3, x_3) \in P_\phi$ we have $(x_1,x_3) \in P_\phi \circ Q_\phi$. So, by 1.7., $(x_1,x_3) \in Q_\phi$. Hence $Q_\phi \circ Q_\phi \subseteq Q_\phi$ and Q_ϕ is an equivalence relation. □

3.5. **COROLLARY.**

a) *If $\phi : \mathfrak{X} \to \mathfrak{Y}$ is a* RIM *extension or if ϕ is a homomorphism of metric minimal ttgs, then $Q_\phi = Q_\phi^\#$ implies $E_\phi = Q_\phi = Q_\phi^\#$.*

b) *If $\mathfrak{X}$ is a minimal ttg then $Q_\mathfrak{X} = Q_\mathfrak{X}^\#$ implies $E_\mathfrak{X} = Q_\mathfrak{X} = Q_\mathfrak{X}^\#$.*

c) *Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a homomorphism of minimal ttgs and let $\phi = \theta \circ \psi$ (as in 2.5.). If ϕ is open in some point $x \in X$, then $Q_\phi = Q_\phi^\#$ implies $Q_\theta = Q_\theta^\#$.*

PROOF.

a) By VII.1.5. and II.1.3.e, this follows immediately from 3.4..

b) As $\phi : \mathfrak{X} \to \{\star\}$ is open, the statement is obvious from 3.4..

c) By 3.4., $Q_\phi = Q_\phi^\#$ implies $E_\phi = Q_\phi = Q_\phi^\#$. But then by 2.10., we know that $E_\theta = Q_\theta = Q_\theta^\#$; in particular, $Q_\theta = Q_\theta^\#$. □

It is not known whether or not $Q_\phi = Q_\phi^\#$ implies $E_\phi = Q_\phi$ without further restrictions on ϕ. We shall now give some other conditions on ϕ that are sufficient to deduce $E_\phi = Q_\phi$ from $Q_\phi = Q_\phi^\#$.

3.6. **THEOREM.** *Consider the next diagram consisting of homomorphisms of minimal ttgs:*

$$\begin{tikzcd}
\mathfrak{X} \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & \mathfrak{Y} \\
& \mathfrak{Z} \arrow[ur, "\theta"'] &
\end{tikzcd}$$

Suppose that ψ *is proximal. In each of the following two cases we have* $Q_\phi = Q_\phi^\#$ *implies* $E_\phi = Q_\phi = Q_\phi^\#$.

a) θ *is open;*

b) $E_\theta = Q_\theta \circ P_\theta$ *; e.g.,* θ *is a* RIM *extension.*

PROOF. As ψ is proximal, $Q_\phi = Q_\phi^\#$ implies $Q_\theta = Q_\theta^\#$ (2.7.b). Hence, in both cases a and b, it follows that $E_\theta = Q_\theta$ (cf. 3.4. and 1.7. respectively). As ψ is proximal and as, by I.4.3.,

$$\psi \times \psi[E_\phi] = E_\theta = Q_\theta = \psi \times \psi[Q_\phi] \,,$$

it follows that $E_\phi \subseteq P_\phi \circ Q_\phi \circ P_\phi$. But, by 1.7., this gives

$$E_\phi \subseteq P_\phi \circ Q_\phi \circ P_\phi = P_\phi \circ Q_\phi^\# \circ P_\phi = Q_\phi^\# \ .$$

3.7. **THEOREM.** *Let* $\phi : \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of minimal ttgs and let* $\phi = \theta \circ \psi$. *Suppose* ψ *is open,* $R_\psi \subseteq Q_\phi$ *and let* $E_\theta = Q_\theta \circ P_\theta$. *Then* $Q_\phi = Q_\phi^\#$ *implies* $E_\phi = Q_\phi = Q_\phi^\#$.

PROOF. As ψ is open, $Q_\phi = Q_\phi^\#$ implies $Q_\theta = Q_\theta^\#$ by 2.6.. Hence, by 1.7., it follows that

$$E_\theta = Q_\theta \circ P_\theta = Q_\theta^\# \circ P_\theta = Q_\theta^\# = Q_\theta \,.$$

Also, by the openness of ψ we have that $\psi \times \psi : R_\phi \to R_\theta$ is an open map. We shall show that $Q_\phi = (\psi \times \psi)^{\leftarrow}[Q_\theta]$, hence that Q_ϕ is an equivalence relation.

Let $(x_1, x_2) \in (\psi \times \psi)^{\leftarrow}[Q_\theta]$; then $(z_1, z_2) := \psi \times \psi(x_1, x_2) \in Q_\theta$. So there are nets $\{(z_1^i, z_2^i)\}_i$ in R_θ and $\{t_i\}_i$ in T such that $(z_1^i, z_2^i) \to (z_1, z_2)$ and $t_i(z_1^i, z_2^i) \to (z_1, z_1)$. As $(x_1, x_2) \in (\psi \times \psi)^{\leftarrow}(z_1, z_2)$ and as the map $\psi \times \psi : R_\phi \to R_\theta$ is open, we can find (x_1^i, x_2^i) in R_ϕ such that $\psi \times \psi(x_1^i, x_2^i) = (z_1^i, z_2^i)$ and $(x_1^i, x_2^i) \to (x_1, x_2)$. After passing to a suitable subnet let $(\bar{x}_1, \bar{x}_2) = \lim t_i(x_1^i, x_2^i)$. Then

$$\psi(\bar{x}_1)=\lim t_i\psi(x_1^i)=\lim t_i z_1^i=z_1=\lim t_i z_2^i=\lim t_i\psi(x_2^i)=\psi(\bar{x}_2)\ ,$$

hence $(\bar{x}_1,\bar{x}_2)\in R_\psi\subseteq R_\phi$ and therefore $(\bar{x}_1,\bar{x}_2)\in Q_\phi=Q_\phi^\#$. By 1.6.b, it follows that $(x_1,x_2)\in Q_\phi$. Consequently, $(\psi\times\psi)^\leftarrow[Q_\theta]\subseteq Q_\phi$ and as, clearly, $Q_\phi\subseteq(\psi\times\psi)^\leftarrow[Q_\theta]$, it follows that $Q_\phi=(\psi\times\psi)^\leftarrow[Q_\theta]$. □

3.8. For the last results in this section remember that for a homomorphism $\phi:\mathfrak{X}\to\mathfrak{Y}$ of minimal ttgs the relation Q_ϕ^* is defined by

$$Q_\phi^*:=\bigcap\{\overline{T\alpha\cap JR_\phi}\mid\alpha\in\mathfrak{U}_X\}\ ,$$

i.e., Q_ϕ^* is the collection of regionally proximal pairs that can (regionally proximal) be reached by nets consisting of almost periodic pairs. Also remember that for $x\in X$ and $u\in J$,

$$Q_\phi^*[x]=\bigcup_{v\in J_x}[\bigcap\{v\circ U\mid U\in\mathfrak{N}_{ux}^\phi\}]$$

where $\mathfrak{N}_{ux}^\phi$ denotes the $\mathfrak{F}(\mathfrak{X},u)$-neighbourhood system of ux in $u\phi^\leftarrow\phi(x)$ (III.3.7.).
In particular, $uQ_\phi^*[x]=\mathrm{H}(F)x$, where $F=\mathfrak{G}(\mathfrak{Y},\phi(ux))\subseteq uM$ is the Ellis group of $\mathfrak{Y}$ with respect to $\phi(ux)$ (III.3.4.,III.2.15.b).

3.9. **LEMMA.** *Let $\phi:\mathfrak{X}\to\mathfrak{Y}$ be a homomorphism of minimal ttgs and let $u\in J$. Suppose $(x_1,x_2)=u(x_1,x_2)\in Q_\phi^\#$, then*

$$uQ_\phi^*[x_1]\times uQ_\phi^*[x_2]=\mathrm{H}(F)x_1\times\mathrm{H}(F)x_2\subseteq Q_\phi\ .$$

where $F=\mathfrak{G}(\mathfrak{Y},u\phi(x_1))\subseteq uM$ is the Ellis group of $\mathfrak{Y}$.

PROOF. Let $L^u[x_i]:=\bigcap\{u\circ U\mid U\in\mathfrak{N}_{ux_i}^\phi\}$ for $i=1,2$; and note that, by III.3.4. and III.3.1.,

$$uQ_\phi^*[x_i]=\mathrm{H}(F)x_i=uL^u[x_i]\ .$$

We shall prove that $(u\circ L^u[x_1])\times L^u[x_2]\subseteq Q_\phi$ and so it follows that

$$uQ_\phi^*[x_1]\times uQ_\phi^*[x_2]=\mathrm{H}(F)x_1\times\mathrm{H}(F)x_2=uL^u[x_1]\times uL^u[x_2]\subseteq$$

$$\subseteq u((u\circ L^u[x_1])\times L^u[x_2])\subseteq Q_\phi\ .$$

Let $\alpha\in\mathfrak{U}_X$; then $(x_1,x_2)\in\mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})$. So there are open neighbourhoods U and V of x_1 and x_2 in X such that

$$(x_1,x_2)\in U\times V\cap R_\phi\subseteq\overline{T\alpha\cap R_\phi}\ .$$

As $(x_1,x_2)=u(x_1,x_2)$ we can find an open set W in T with $W=W(u)$ such that $Wx_1\subseteq U$ (see III.2.1.c). Define $\mathcal{U}\in\mathfrak{N}^{\phi}_{x_2}$ by $\mathcal{U}:=[V,W]\cap u\phi^{\leftarrow}\phi(x_2)$. Then

$$\{x_1\}\times\mathcal{U}\subseteq T(U\times V\cap R_\phi)\subseteq\overline{T\alpha\cap R_\phi}\,.$$

Let $x'_2\in\mathcal{U}$ and note that $(x_1,x'_2)=u(x_1,x'_2)$. In the same way as above we can find a $\mathcal{V}\in\mathfrak{N}^{\phi}_{x_1}$ such that $\mathcal{V}\times\{x'_2\}\subseteq T(U\times V\cap R_\phi)$. Hence

$$u\circ\mathcal{V}\times\{x'_2\}=u\circ(\mathcal{V}\times\{x'_2\})\subseteq\overline{T(U\times V\cap R_\phi)}\subseteq\overline{T\alpha\cap R_\phi}$$

and as $L^u[x_1]\subseteq u\circ\mathcal{V}$, it follows that $L^u[x_1]\times\{x'_2\}\subseteq\overline{T\alpha\cap R_\phi}$. Since $x'_2\in\mathcal{U}$ was arbitrary, $L^u[x_1]\times\mathcal{U}\subseteq\overline{T\alpha\cap R_\phi}$. Hence, as $L^u[x_2]\subseteq u\circ\mathcal{U}$, it follows that

$$u\circ L^u[x_1]\times L^u[x_2]\subseteq u\circ L^u[x_1]\times u\circ\mathcal{U}=u\circ(L^u[x_1]\times\mathcal{U})\subseteq\overline{T\alpha\cap R_\phi}\,.$$

As $\alpha\in\mathcal{U}_X$ was arbitrary: $u\circ L^u[x_1]\times L^u[x_2]\subseteq Q_\phi$. □

3.10. **THEOREM.** *Let $\phi:\mathcal{X}\to\mathcal{Y}$ be a homomorphism of minimal ttgs.*

a) *If $Q_\phi=Q^{\#}_\phi$ then $Q_\phi\circ Q^{*}_\phi=Q^{*}_\phi\circ Q_\phi=Q_\phi$.*

b) *If $Q_\phi=Q^{\#}_\phi$ and if for some $x\in X$ and some $u\in J$ we have $uQ_\phi[x]=uQ^{*}_\phi[x]$ then $E_\phi=Q_\phi$.*

PROOF.

a) Let $(x_1,x_2)\in Q^{*}_\phi$ and $(x_2,x_3)\in Q_\phi$. Then $u(x_1,x_2)\in Q^{*}_\phi$ and $u(x_2,x_3)\in Q_\phi=Q^{\#}_\phi$. As $ux_1\in uQ^{*}_\phi[ux_2]$ it follows from 3.9. that

$$(ux_1,ux_3)\in uQ^{*}_\phi[ux_2]\times\{ux_3\}\subseteq Q_\phi=Q^{\#}_\phi\,.$$

Hence, by 1.6., $(x_1,x_3)\in Q_\phi$ and $Q_\phi\circ Q^{*}_\phi\subseteq Q_\phi$, and so $Q_\phi\circ Q^{*}_\phi=Q_\phi$. Similarly, $Q^{*}_\phi\circ Q_\phi=Q_\phi$.

b) Let $(x_1,x_2)\in Q_\phi$ and $(x_2,x_3)\in Q_\phi$ and let $p\in M$ be such that $px_2=x$. Then $(px_1,x)=(px_1,px_2)\in Q_\phi$, and so we have $upx_1\in uQ_\phi[x]=uQ^{*}_\phi[x]$. As $(x,px_3)\in JQ_\phi=JQ^{\#}_\phi$ it follows by 3.9. that

$$(upx_1,upx_3)\in uQ^{*}_\phi[x]\times uQ^{*}_\phi[px_3]\subseteq Q_\phi=Q^{\#}_\phi\,,$$

and so, by 1.6., it follows that $(x_1,x_3)\in Q_\phi$. Consequently, Q_ϕ is an equivalence relation. □

VIII.4. REGIONAL PROXIMALITY OF SECOND ORDER

> Let $\mathcal{X}$ be a ttg. It is not difficult to see that a pair $(x_1, x_2) \in X \times X$ is regionally proximal if we can find suitable pairs in the neighbourhood of (x_1, x_2) such that after suitable T-translations they tend to a proximal pair. If we could find pairs in the neighbourhood of (x_1, x_2) that after suitable T-translations tend to a regionally proximal pair, we could say that the pair (x_1, x_2) is regionally regionally proximal. We call it *regionally proximal of second order.*

Let $\mathcal{X}$ be a ttg and let $A \subseteq X$. Then define

$$D(A, \mathcal{X}) := \bigcup \{p \star A \mid p \in S_T\},$$

where $p \star A$ is defined as

$$p \star A := \bigcap \{p \circ V \mid A \subseteq V \text{ and } V \text{ open in } X\}.$$

Remark that the $\star$ defined in section 3. is in full accordance with this definition, after noting that $p \star a := p \star \{a\}$.

4.1. **REMARK.** *Let $\mathcal{X}$ be a ttg and let $A \subseteq X$. Then*
a) *$D(A, \mathcal{X})$ is T-invariant;*
b) *$D(A, \mathcal{X}) = D(tA, \mathcal{X})$ for every $t \in T$;*
c) *if A is closed then $D(A, \mathcal{X}) = \bigcup \{D(\{a\}, \mathcal{X}) \mid a \in A\}$;*
d) *if A is closed then $D(A, \mathcal{X})$ is closed.*

PROOF.

a) Let $x \in D(A, \mathcal{X})$ and let $p \in S_T$ be such that $x \in p \star A$. Then $x \in p \circ V$ for every open V in X with $A \subseteq V$. Hence $tx \in tp \circ V$ for such V and $tx \in tp \star A \subseteq D(A, \mathcal{X})$.

b) Note that $p \circ V = pt^{-1} \circ tV$ for every $V \subseteq X$, $p \in S_T$ and $t \in T$. As

$$\{W \mid W \subseteq X \text{ open}, tA \subseteq W\} = \{tV \mid V \subseteq X \text{ open}, A \subseteq V\}$$

for every $t \in T$, it follows that $p \star A = pt^{-1} \star tA$.

c) Obviously, $D(\{a\}, \mathcal{X}) \subseteq D(A, \mathcal{X})$ for every $a \in A$.
Conversely, let $x \in D(A, \mathcal{X})$ and let $p \in S_T$ be such that $x \in p \star A$. Let $\alpha \in \mathcal{U}_X$ be an open index. Then there are $a_1, \ldots, a_n$ in A such that

$$V_\alpha := \bigcup \{\alpha(a_i) \mid i \in \{1, \ldots, n\}\}$$

is an open neighbourhood of A (in X). So $x \in p \circ V_\alpha$ and as

$$p \circ V_\alpha = \bigcup \{p \circ \alpha(a_i) \mid i \in \{1, \ldots, n\}\},$$

we can find $a_\alpha \in \{a_i \mid i \in \{1, \ldots, n\}\}$ such that $x \in p \circ \alpha(a_\alpha)$. In this way we obtain a point a_α in A for every open index $\alpha \in \mathcal{U}_X$. Let $a := \lim\{a_\alpha \mid \alpha \in I\}$ for a suitable subnet $I \subseteq \mathcal{U}_X$. We shall prove that $x \in p \star \{a\}$.

Let $V \subseteq X$ be open and let $\{a\} \subseteq V$. Then there are β and γ in I such that $\beta(a) \subseteq V$ and $\gamma \circ \gamma \subseteq \beta$. Let $\delta \in I$ with $\delta \subseteq \gamma$ such that $a_\delta \in \gamma(a)$. Then

$$x \in p \circ \delta(a_\delta) \text{ and } \delta(a_\delta) \subseteq \gamma(a_\delta) \subseteq \gamma(\gamma(a)) \subseteq \beta(a),$$

so $x \in p \circ \delta(a_\delta) \subseteq p \circ \beta(a) \subseteq p \circ V$; hence $x \in p \star \{a\}$. As $a \in \overline{A} = A$ it follows that $D(A, \mathcal{X}) \subseteq \bigcup\{D(\{a\}, \mathcal{X}) \mid a \in A\}$.

d) Let $\{x_i\}_i$ be a convergent net in $D(A, \mathcal{X})$ and let $x = \lim x_i$. By c, we may find nets $\{a_i\}_i$ and $\{p_i\}_i$ in A and S_T such that $x_i \in p_i \star \{a_i\}$. Let $p = \lim p_i$ and $a = \lim a_i$ after passing to suitable subnets. We shall prove that $x \in p \star \{a\}$.

Let $V \subseteq X$ be open with $\{a\} \subseteq V$. Then $\{a_i\} \subseteq V$ for all $i \geqslant i(V)$. Hence

$$x_i \in p_i \star \{a_i\} \subseteq p_i \circ V \text{ for all } i \geqslant i(V).$$

But then it follows that

$$x = \lim x_i \in \lim_{2^X}(p_i \circ V) = p \circ V.$$

As V was arbitrary, it follows that $x \in p \star \{a\}$, hence $x \in D(A, \mathcal{X})$. □

The proof of the following remark is straightforward and will be omitted

4.2. **REMARK.** *For a ttg* $\mathcal{X}$, $x \in X$ *and* $a \in X$ *the following statements are equivalent:*

a) $x \in p \star a$ *for some* $p \in S_T$, *in other words,* $x \in D(\{a\}, \mathcal{X})$;

b) *for every* $V_a \in \mathcal{V}_a$, *and every* $V_x \in \mathcal{V}_x$ *there is a* $t \in T$ *such that* $tV_a \cap V_x \neq \varnothing$;

c) *there is a net* $\{a_i\}_i$ *in* X *with* $a_i \to a$ *and there are* t_i *in* T *with* $x = \lim t_i a_i$;

d) $a \in q \star x$ *for some* $q \in S_T$, *in other words,* $a \in D(\{x\}, \mathcal{X})$. □

4.3. **EXAMPLES.** *Let* $\mathfrak{X}$ *be a ttg and let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a homomorphism of ttgs. Then*

a) $D(\Delta_X, \mathfrak{X}\times\mathfrak{X}) = Q_{\mathfrak{X}}$*;*

b) $D(\Delta_X, \mathfrak{R}_\phi) = Q_\phi$*;*

c) $D(E_\phi, \mathfrak{R}_\phi) = E_\phi$ *and so* $D(Q_\phi, \mathfrak{R}_\phi) \subseteq E_\phi$*;*

d) $D(Q_\phi^{\#}, \mathfrak{R}_\phi) = Q_\phi$*, hence* $Q_\phi = Q_\phi^{\#}$ *implies* $D(Q_\phi, \mathfrak{R}_\phi) = Q_\phi$*.*

PROOF.

a) Follows immediately from b.

b) Using 4.1.c and 4.2. this follows easily from the proof of 3.1..

c) Let $\theta:\mathfrak{X}/E_\phi \to \phi[\mathfrak{X}]$ be the maximal almost periodic factor of ϕ and let $\kappa:\mathfrak{X}\to\mathfrak{X}/E_\phi$ be the quotient map. Then it is easily seen that

$$\kappa\times\kappa[D(E_\phi,\mathfrak{R}_\phi)] \subseteq D(\Delta_{X/E_\phi}, \kappa\times\kappa[\mathfrak{R}_\phi]) \subseteq Q_\theta .$$

As θ is an almost periodic extension, $\kappa\times\kappa[D(E_\phi,\mathfrak{R}_\phi)] \subseteq \Delta_{X/E_\phi}$; hence $D(E_\phi,\mathfrak{R}_\phi)\subseteq E_\phi$. The converse inclusion is obvious.

d) Clearly, $Q_\phi = D(\Delta_X,\mathfrak{R}_\phi) \subseteq D(Q_\phi^{\#},\mathfrak{R}_\phi)$.
Conversely, as $Q_\phi^{\#} \subseteq \mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi})$ for every $\alpha\in\mathfrak{U}_X$, we have

$$p\star Q_\phi^{\#} \subseteq p\circ \mathrm{int}_{R_\phi}(\overline{T\alpha\cap R_\phi}) \subseteq p\circ\overline{T\alpha\cap R_\phi} \subseteq \overline{T\alpha\cap R_\phi} \quad (\alpha\in\mathfrak{U}_X).$$

So $p\star Q_\phi^{\#}\subseteq Q_\phi$ and $D(Q_\phi^{\#},\mathfrak{R}_\phi)\subseteq Q_\phi$. □

The next theorem as well as its proof resemble 3.3. and 3.4..

4.4. **THEOREM.** *Let* $\phi:\mathfrak{X}\to\mathfrak{Y}$ *be a homomorphism of ttgs. If for every* $x_1\in X$ *there is an* $x\in X$ *with* $\overline{Tx}\cap\overline{Tx_1}\neq\emptyset$ *, such that* $\phi(x)$ *is an almost periodic point and* ϕ *is open in* x *, then* $E_\phi = Q_\phi$ *iff* $D(Q_\phi,\mathfrak{R}_\phi) = Q_\phi$*.*

PROOF. If $E_\phi = Q_\phi$ then, by 4.3., it follows that $D(Q_\phi,\mathfrak{R}_\phi)=Q_\phi$.
Conversely, suppose that $D(Q_\phi,\mathfrak{R}_\phi)=Q_\phi$. Let $(x_1,x_2)\in Q_\phi$ and $(x_2,x_3)\in Q_\phi$, and assume ϕ is open in x_1. We shall prove that $(x_1,x_3)\in Q_\phi$.
Let $\{(x_2^i,x_3^i)\}_i$ and $\{t_i\}_i$ be nets in R_ϕ and T such that

$$(x_2^i,x_3^i)\to(x_2,x_3) \text{ and } t_i(x_2^i,x_3^i)\to(w,w) \text{ for some } w\in X .$$

As $\phi(x_2^i)\to\phi(x_2)=\phi(x_1)$ and as ϕ is open in x_1, there are $z_i\in\phi^{\leftarrow}\phi(x_2^i)$ such that $z_i\to x_1$. Define $z=\lim t_i z_i$ (after passing to a suitable subnet). Then

$$(z_i, x_2^i) \to (x_1, x_2) \text{ and } t_i(z_i, x_2^i) \to (z, w).$$

As $(x_1, x_2) \in Q_\phi$ it follows that

$$(z, w) \in p\star(x_1, x_2) \subseteq D(\{(x_1, x_2)\}, \mathfrak{R}_\phi) \subseteq D(Q_\phi, \mathfrak{R}_\phi) = Q_\phi,$$

where $p = \lim t_i \in S_T$ (after passing to a suitable subnet). As

$$(z_i, x_3^i) \to (x_1, x_3) \text{ and } t_i(z_i, x_3^i) \to (z, w),$$

it follows that

$$(x_1, x_3) \in q\star(z, w) \subseteq D(\{(z, w)\}, \mathfrak{R}_\phi) \subseteq D(Q_\phi, \mathfrak{R}_\phi) = Q_\phi,$$

where $q = \lim t_i^{-1} \in S_T$ (after passing to a suitable subnet).
Now assume that ϕ is not open in x_1. By assumption, we may find $x \in X$ such that $\overline{Tx} \cap \overline{Tx_1} \neq \emptyset$ and ϕ is open in x, while $\phi(x) \in Y$ is an almost periodic point. For an almost periodic point $z \in \overline{Tx} \cap \overline{Tx_1}$ let I and K be minimal left ideals in S_T such that $z = px$ and $z = qx_1$ for some $p \in I$ and some $q \in K$. Let $v \in J_{\phi(x)}(I)$. Then $vx = vp^{-1}qx_1$, and

$$(vx, vp^{-1}qx_2) = vp^{-1}q(x_1, x_2) \in Q_\phi \text{ and } (vp^{-1}qx_2, vp^{-1}qx_3) \in Q_\phi.$$

As $(x, vx) \in P_\phi$, we have $(x, vp^{-1}qx_2) \in Q_\phi \circ P_\phi$ and it is easily seen that $Q_\phi \circ P_\phi \subseteq D(Q_\phi, \mathfrak{R}_\phi) = Q_\phi$. By the above, $(x, vp^{-1}qx_3) \in Q_\phi$ and so

$$(vp^{-1}qx_1, vp^{-1}qx_3) = (vx, vp^{-1}qx_3) = v(x, vp^{-1}qx_3) \in Q_\phi.$$

But then

$$(x_1, x_3) \in D(\{(vp^{-1}qx_1, vp^{-1}qx_3)\}, \mathfrak{R}_\phi) \subseteq D(Q_\phi, \mathfrak{R}_\phi) = Q_\phi,$$

which shows the transitivity of Q_ϕ. □

4.5. **COROLLARY.** *Let* $\phi: \mathfrak{X} \to \mathfrak{Y}$ *be a homomorphism of ttgs.*

a) *If* ϕ *is open then* $E_\phi = Q_\phi$ *iff* $D(Q_\phi, \mathfrak{R}_\phi) = Q_\phi$. *In particular, for every ttg* $\mathfrak{X}$ *we have* $E_{\mathfrak{X}} = Q_{\mathfrak{X}}$ *iff* $D(Q_{\mathfrak{X}}, \mathfrak{X} \times \mathfrak{X}) = Q_{\mathfrak{X}}$.

b) *If* $\mathfrak{X}$ *is a metric ergodic ttg and if* $\mathfrak{Y}$ *is minimal, then* $E_\phi = Q_\phi$ *iff* $D(Q_\phi, \mathfrak{R}_\phi) = Q_\phi$.

PROOF.

a) This follows immediately from the first part of the proof of 4.4..

b) If X is metric, there is a residual set of points in which ϕ is open, also there is a residual set of transitive points. As $\mathfrak{Y}$ is minimal, the assumptions of 4.4. are satisfied. □

VIII.5. REMARKS

In this final section we shall mention an other variation on regional proximality. This variation is closely related to what is called "Ellis' trick" in [G 76], namely, that open sets in the regular topology on the phase space X of a minimal ttg $\mathfrak{X}$ do have some thickness in the $\mathfrak{F}(\mathfrak{X},u)$-topology. For a more detailed treatment of this other variation on regional proximality we refer to [V 77] and [VW ?].
We also consider the regional proximal relation for special kinds of incontractible minimal ttgs.

Let $\phi:\mathfrak{X}\to\mathfrak{Y}$ be a homomorphism of minimal ttgs. Define

$$U_\phi[x]:=\bigcap\{\overline{(T\alpha)(x)\cap\phi^{\leftarrow}\phi(x)}\mid\alpha\in\mathfrak{U}_X\},$$

where $(T\alpha)(x)=\{x'\in X\mid(x,x')\in T\alpha\}$.
In other words: $x'\in U_\phi[x]$ iff there are nets $\{x'_i\}_i$ in $\phi^{\leftarrow}\phi(x)$ and $\{t_i\}_i$ in T such that

$$x'_i\to x' \quad\text{and}\quad t_i(x,x'_i)\to(x,x)\,;$$

i.e., the "regionally proximal-making net" may be chosen to be constant in x. Define

$$U_\phi:=\{(x,x')\in R_\phi\mid x'\in U_\phi[x]\}.$$

If $\phi:\mathfrak{X}\to\{\star\}$, then we write $U_{\mathfrak{X}}[x]$ and $U_{\mathfrak{X}}$.
Note that this a-symmetric defined notion has a counterpart in the notion of $\mathrm{SRP}(\phi^{\leftarrow}\phi(x),x)$, see III.5.8..

Clearly, $P_\phi\subseteq U_\phi\subseteq Q_\phi$; but in [V 77] W.A. VEECH has shown that in several cases one can say more:

5.1. **THEOREM.** ([V 77] 2.7.5.) *Let $\phi:\mathfrak{X}\to\mathfrak{Y}$ be a homomorphism of minimal ttgs. If for every $y\in Y$ and $u\in J_y$ the set $u\phi^{\leftarrow}(y)$ is dense in $\phi^{\leftarrow}(y)$ (e.g., if ϕ is distal), then $U_\phi=Q_\phi=E_\phi$.* □

In the absolute case even more is true ([V 77] 2.7.6., also see [VW ?]):

5.2. **THEOREM.** *If $\mathfrak{X}$ is a minimal ttg that satisfies the Bronstein condition (i.e., $X\times X$ has a dense subset of almost periodic points) then $U_{\mathfrak{X}}=Q_{\mathfrak{X}}=E_{\mathfrak{X}}$.* □

In the proofs of 5.1. and 5.2. the following set turns out to be of vital importance. For a homomorphism $\phi : \mathfrak{X} \to \mathfrak{Y}$ of minimal ttgs and for $y \in Y$ and $u \in J_y$ define

$$\Sigma_1(y) := \{x \in \phi^{\leftarrow}(y) \mid \operatorname{int}_{(u\phi^{\leftarrow}(y), \mathfrak{F}(\mathfrak{X}, u))}(u(u \circ U)) \neq \emptyset \text{ for every } U \in \mathcal{V}_x^{\phi}\} ,$$

where $\mathcal{V}_x^{\phi} := \mathcal{V}_x \cap \phi^{\leftarrow}(y)$. One can show that $\Sigma_1(y)$ is a closed subset of $\phi^{\leftarrow}(y)$ (easily) and that $\Sigma_1(y) \neq \emptyset$ ([V 77] 2.7.2.).
The following theorem is the basis for 5.2., it can be found in [VW ?] (and without proof in [V 77] 2.7.6.).

5.3. **THEOREM.** *Let* $\mathfrak{X}$ *be a minimal ttg. Then* $\Sigma_1(\star) = X$ *, where* $\star$ *is the only element of* $\{\star\}$ *, the trivial ttg.* ☐

5.4. **QUESTIONS.**

a) Does 5.2. hold in the relativized case? I.e.: If $\phi : \mathfrak{X} \to \mathfrak{Y}$ is an open Bc extension of minimal ttgs, is $U_\phi = Q_\phi = E_\phi$?

b) Is there any relation between U_ϕ and $Q_\phi^{\#}$? For instance: Does $U_\phi = Q_\phi$ imply $Q_\phi = Q_\phi^{\#}$?

We end this section with some remarks on $E_{\mathfrak{X}}[x]$ for an incontractible minimal ttg $\mathfrak{X}$.

5.5. **REMARK.** *Let* $\mathfrak{X}$ *be a ttg and let* $A \subseteq X$ *be nonempty, then for every* $u \in J$ *we have* $E_{\mathfrak{X}}[\overline{A}] = E_{\mathfrak{X}}[u \circ A]$.

PROOF. Let $\kappa : \mathfrak{X} \to \mathfrak{X}/E_{\mathfrak{X}}$ be the quotient map. Then

$$E_{\mathfrak{X}}[u \circ A] = E_{\mathfrak{X}}[u \circ \overline{A}] = \kappa^{\leftarrow}\kappa[u \circ \overline{A}] = \kappa^{\leftarrow}(u \circ \kappa[\overline{A}]) .$$

As $\kappa[\overline{A}] \in 2^{\mathfrak{X}/E_{\mathfrak{X}}}$ and as, by II.2.7., $2^{\mathfrak{X}/E_{\mathfrak{X}}}$ is uniformly almost periodic, it follows that $\kappa[\overline{A}] = u \circ \kappa[\overline{A}]$ for every $u \in J$. Hence

$$E_{\mathfrak{X}}[u \circ A] = \kappa^{\leftarrow}(u \circ \kappa[\overline{A}]) = \kappa^{\leftarrow}\kappa[\overline{A}] = E_{\mathfrak{X}}[\overline{A}] .$$

☐

5.6. **THEOREM.** *Let* $\mathfrak{X}$ *be a minimal ttg.*

a) *Let* $\mathfrak{X}$ *satisfy the Bronstein condition and let* $x' \in X$ *be arbitrary. Then for every nonempty open* U *in* X *there is an* $x \in U$ *with* $E_{\mathfrak{X}}[x] \subseteq J_{x'} \circ U$.

b) *Let* $\mathfrak{X}$ *be incontractible and let* $u \in J$. *Then for every nonempty open* U *in* X *there is an* $x \in U$ *with* $E_{\mathfrak{X}}[x] \subseteq u \circ U$.

PROOF. Let $u \in J$. For $U \subseteq X$ nonempty and open let V be a nonempty open set in X with $V \subseteq \overline{V} \subseteq U$. By 5.3., we know that $u(u \circ V)$ has a nonempty $\mathfrak{F}(\mathcal{X},u)$-interior W in uX . Let $\tilde{x} \in W$ and note that $\tilde{x} = u\tilde{x} \in W \subseteq u \circ V$. So, by 5.5., there is an $x \in \overline{V} \subseteq U$ such that $E_{\mathcal{X}}[x] = E_{\mathcal{X}}[\tilde{x}]$.

a) By III.3.10.a, we have

$$E_{\mathcal{X}}[\tilde{x}] \subseteq J_{x'} \circ W \subseteq J_{x'} \circ u(u \circ V) \subseteq J_{x'} \circ V \subseteq J_{x'} \circ U \ .$$

So by the above, $E_{\mathcal{X}}[x] = E_{\mathcal{X}}[\tilde{x}] \subseteq J_{x'} \circ U$ for some $x \in U$.

b) Similarly, b follows from III.3.10.b. □

5.7. **THEOREM.** *Let $\mathcal{X}$ be a minimal ttg and assume that the quotient map $\kappa : \mathcal{X} \to \mathcal{X}/E_{\mathcal{X}}$ is open.*

a) *If $\mathcal{X}$ satisfies the Bronstein condition then for every $x' \in X$ we have for U nonempty and open in X that $E_{\mathcal{X}}[\overline{U}] = Q_{\mathcal{X}}[\overline{U}] = \overline{J_{x'} \circ U}$.*

b) *If $\mathcal{X}$ is incontractible then for $u \in J$ and for every nonempty open U in X we have that $E_{\mathcal{X}}[\overline{U}] = Q_{\mathcal{X}}[\overline{U}] = u \circ U$.*

PROOF.

a) Let $x \in \overline{U}$ and let $V \in \mathcal{V}_x$. By 5.6.a, there is an $x_V \in U \cap V$ such that $E_{\mathcal{X}}[x_V] \subseteq J_{x'} \circ (U \cap V)$, so $E_{\mathcal{X}}[x_V] \subseteq J_{x'} \circ U$. As κ is open, $E_{\mathcal{X}}[x] = \lim_{2x} E_{\mathcal{X}}[x_V]$ and so $E_{\mathcal{X}}[x] \subseteq \overline{J_{x'} \circ U}$. Hence $E_{\mathcal{X}}[\overline{U}] \subseteq \overline{J_{x'} \circ U}$. As, by 5.5., $E_{\mathcal{X}}[\overline{U}] = E_{\mathcal{X}}[u \circ U]$ for every $u \in J$, we have:

$$w \circ U \subseteq E_{\mathcal{X}}[w \circ U] = E_{\mathcal{X}}[\overline{U}] \text{ for every } w \in J_{x'} \ .$$

Hence $E_{\mathcal{X}}[\overline{U}] \subseteq \overline{J_{x'} \circ U} \subseteq \overline{E_{\mathcal{X}}[\overline{U}]} = E_{\mathcal{X}}[\overline{U}]$.

b) Similar to the above one proves, using 5.6.b, that $E_{\mathcal{X}}[\overline{U}] = \overline{u \circ U}$. But $u \circ U$ is closed, so $E_{\mathcal{X}}[\overline{U}] = u \circ U$. □

5.8. **COROLLARY.** *If $\mathcal{X}$ is distal then for every nonempty open U in X we have $E_{\mathcal{X}}[\overline{U}] = u \circ U$.* □

5.9. **COROLLARY.** *Let $\mathcal{X}$ be incontractible and assume that $\kappa : \mathcal{X} \to \mathcal{X}/E_{\mathcal{X}}$ is open. Then for every $u \in J$ we have $E_{\mathcal{X}}[x] = Q_{\mathcal{X}}[x] = u \star x$.*

PROOF. It is not difficult to see that $u \star x \subseteq Q_{\mathcal{X}}[x] = E_{\mathcal{X}}[x]$.
Conversely, by 5.7.b,

$$Q_{\mathcal{X}}[x] \subseteq \bigcap \{ Q_{\mathcal{X}}[\overline{U}] \mid U \in \mathcal{V}_x \} = \bigcap \{ u \circ U \mid U \in \mathcal{V}_x \} = u \star x \ .$$

□

5.10. **QUESTIONS.**

a) If $\mathfrak{X}$ satisfies the Bronstein condition and if $x' \in X$, do we have for every $x \in X$ that $Q_{\mathfrak{X}}[x] = \bigcup \{w \star x \mid w \in J_{x'}\}$?

b) Can we relativize 5.9.? I.e., if $\phi: \mathfrak{X} \to \mathfrak{Y}$ is a RIC extension of minimal ttgs such that $\kappa: \mathfrak{X} \to \mathfrak{X}/E_\phi$ is open, do we have $E_\phi[x] = Q_\phi[x] = u \star x$ for every $x \in X$ and every $u \in J_{\phi(x)}$? If so, then one can prove that $E_\phi = Q_\phi = U_\phi$.

In 5.9. we have the restriction of κ being open. The following remark deals with a situation in which κ is not necessarily open.

5.11. **THEOREM.** *Let $\mathfrak{X}$ be an incontractible minimal ttg. If $x \in X$ is such that uTx is dense in X for some $u \in J$, then there is a $q \in M$ with $Q_{\mathfrak{X}}[x] = q \star x$.*

PROOF. Let $x \in X$ and let $u \in J$ be such that uTx is dense in X , and let $U \in \mathfrak{V}_x$. Then, by 5.3., $u(u \circ U)$ has a nonempty $\mathfrak{J}(\mathfrak{X},u)$-interior in uX . By III.2.4., there is an $x' \in uX$, a continuous pseudometric σ and an $\epsilon > 0$ such that

$$U(x',\sigma,\epsilon) \cap uX \subseteq u(u \circ U) .$$

As $U(x',\sigma,\epsilon)$ is open in X , there is a $t \in T$ with $utx \in U(x',\sigma,\epsilon)$. But then, by III.3.10.b, $Q_{\mathfrak{X}}[utx] \subseteq u \circ U$. Since $(t^{-1}utx, x) \in P_{\mathfrak{X}}$ and $E_{\mathfrak{X}} = Q_{\mathfrak{X}} = Q_{\mathfrak{X}} \circ P_{\mathfrak{X}}$, it follows that

$$Q_{\mathfrak{X}}[x] = Q_{\mathfrak{X}}[t^{-1}utx] \subseteq t^{-1}u \circ U .$$

So we proved that for every $\alpha \in \mathfrak{U}_X$, there is a $p_\alpha \in M$ with $Q_{\mathfrak{X}}[x] \subseteq p_\alpha \circ \alpha(x)$. As for every $\beta \subseteq \alpha$ we have:

$$Q_{\mathfrak{X}}[x] \subseteq p_\beta \circ \beta(x) \subseteq p_\beta \circ \alpha(x) ,$$

it follows that $Q_{\mathfrak{X}}[x] \subseteq q \circ \alpha(x)$, where $q = \lim p_\beta$ for a suitable subnet of the p_β's with $\beta \subseteq \alpha$. Hence $Q_{\mathfrak{X}}[x] \subseteq q \star x$.
Conversely, if $x' \in q \star x$ then it is easily seen that $(qx, x') \in Q_{\mathfrak{X}}$. So, if $Q_{\mathfrak{X}}[x] \subseteq q \star x$, then $Q_{\mathfrak{X}}[x] = Q_{\mathfrak{X}}[qx]$. However, it is not difficult to see that $q \star x \subseteq Q_{\mathfrak{X}}[qx]$, so

$$Q_{\mathfrak{X}}[x] = Q_{\mathfrak{X}}[qx] = q \star x .$$

□

5.12. **QUESTION.** Do we really need the assumption of uTx being dense in X in the above?

REFERENCES

[Ak 78] ARKHANGEL'SKII, A. V., Structure and classification of topological spaces and cardinal invariants, *Russian Math. Surveys* **33**:6, 33-96 (1978).

[Ar 78] ARMSTRONG, T. S., Gleason spaces and topological dynamics, *Indiana Math. J.* **27**, 283-292 (1978).

[A 60] AUSLANDER, J., On the proximal relation in topological dynamics, *Proc. Amer. Math. Soc.* **11**, 890-895 (1966).

[A 66] AUSLANDER, J., Regular minimal sets (I), *Trans. Amer. Math. Soc.* **123**, 469-479 (1966).

[AG 77] AUSLANDER, J. and S. GLASNER, Distal and highly proximal extensions of minimal flows, *Indiana Math. J.* **26**, 731-749 (1977).

[AMWW ?]AUSLANDER, J., D. C. MCMAHON, J. C. S. P. VAN DER WOUDE and T.S. WU, Weak disjointness and the equicontinuous structure relation, to appear in *Ergod. Th. & Dynam. Sys.*.

[AW 81] AUSLANDER, J. and J. C. S. P. VAN DER WOUDE, Maximally highly proximal generators of minimal flows, *Ergod. Th. & Dynam. Sys.* **1**, 389-412 (1981).

[Bi 27] BIRKHOFF, G. D., *Dynamical systems,* Amer. Math. Soc. Colloq. Publ., vol **9**, Amer. Math. Soc., Providence, R.I., 1927.

[B 73] BRONSTEIN, I. U., Stable and equicontinuous extensions of minimal sets, *Math. Issled.* **8**, 3-11 (1973). (Russian)

[B 75/79] BRONSTEIN, I. U., *Extensions of minimal transformation groups,* Sijthoff & Noordhoff, Alphen aan den Rijn, 1979. (Russian edition: (1975))

[B 77] BRONSTEIN, I. U., A characteristic property of PD-extensions, *Bull. Akad. Stiince RSS Moldoven* **3**, 11-15 (1977).

[Cr 81] CHRISTENSEN, J. P. R., Joint continuity of separately continuous functions, *Proc. Amer. Math. Soc.* **82**, 455-461 (1981).

[D 80] DEKKING, F. M., *Combinatorial and statistical properties of sequences generated by substitutions,* Ph.D. thesis, Kath. Univ. Nijmegen 1980.

[Du 66] DUGUNDJI, J., *Topology,* Allyn and Bacon Inc, Boston, 1966.

[E 57] ELLIS, R., Locally compact transformation groups, *Duke Math. J.* **24**, 119-125 (1957).

[E 58] ELLIS, R., Distal transformation groups, *Pacific J. Math.* **8**, 401-405 (1958).

[E 60] ELLIS, R., A semigroup associated with a transformation group, *Trans. Amer. Math. Soc.* **94**, 272-281 (1960).

[E 65] ELLIS, R., The construction of minimal discrete flows, *Amer. J. Math.* **87**, 564-574 (1965).

[E 67] ELLIS, R., Group-like extensions of minimal sets, *Trans. Amer. Math. Soc.* **127**, 125-135 (1967).

[E 69] ELLIS, R., *Lectures on topological dynamics,* Benjamin, New York, 1969.

[E 73] ELLIS, R., The Veech structure theorem, *Trans. Amer. Math. Soc.* **186**, 203-218 (1973).

[E 78] ELLIS, R., The Furstenberg structure theorem, *Pacific J. Math.* **76**, 345-349 (1978).

[E 81] ELLIS, R., Some problems in topological dynamics, *Rep.* **81-144**, Univ. of Minnesota, MN, 1981.

[EGS 75] ELLIS, R., S. GLASNER and L. SHAPIRO, Proximal isometric ($\mathcal{P}\mathcal{I}$-) flows, *Advances in Math.* **17**, 213-260 (1975).

[EGS 76] ELLIS, R., S. GLASNER and L. SHAPIRO, Algebraic equivalents of flow disjointness, *Illinois J. Math.* **20**, 354-360 (1976).

[EK 71] ELLIS, R. and H.B. KEYNES, A characterization of the equicontinuous structure relation, *Trans. Amer. Math. Soc.* **161**, 171-184 (1971).

[F 63] FURSTENBERG, H., The structure of distal flows, *Amer. J. Math.* **85**, 477-515 (1963).

[F 67] FURSTENBERG, H,. Disjointness in ergodic theory, minimal sets, and a problem in diophantine approximation, *Math. Syst. Theory* **1**, 1-49 (1967).

[FG 78] FURSTENBERG, H. and S. GLASNER, On the existence of isometric extensions, *Amer. J. Math.* **100**, 1185-1212 (1978).

[FKS 73] FURSTENBERG, H., H. B. KEYNES and L. SHAPIRO, Prime flows in topological dynamics, *Isreal J. Math.* **14**, 26-38 (1973).

[Fo 51] FORT Jr., M. K., Points of continuity of semi-continuous functions, *Public. Mathem. Debrecen* **2**, 100-102 (1951).

[G 74] GLASNER, S., Topological dynamics and group theory, *Trans. Amer. Math. Soc.* **187**, 327-334 (1974).

[G 75.1] GLASNER, S., Compressibility properties in topological dynamics, *Amer. J. Math.* **97**, 148-171 (1975).

[G 75.2] GLASNER, S., Relatively invariant measures, *Pacific J. Math.* **58**, 393-410 (1975).

[G 76] GLASNER, S., *Proximal flows,* Lecture Notes in Math. **517**, Springer Verlag, New York 1976.

[G 79] GLASNER, S., Almost periodic sets and measures on the torus, *Israel J. Math.* **32**, 161-172 (1979).

[G 80] GLASNER, S., Minimal skew products, *Trans. Amer. Math. Soc.* **260**, 509-514 (1980).

[GH 55] GOTTSCHALK, W. H. and G. A. HEDLUND, *Topological dynamics,* Amer. Math. Colloq. Publ., vol. **36**, Amer. Math. Soc., Providence, R.I., 1955.

[GJ 60] GILLMAN, L. and M. JERISON, *Rings of continuous functions,* Van Nostrand, Princeton, N.J. 1960.

[GW 79] GLASNER, S. and B. WEISS, On the construction of minimal skew products, *Israel J. Math.* **34**, 321-336 (1979).

[GW 81] GLASNER, S. and B. WEISS, A weakly mixing upside-down tower of isometric extensions, *Ergod. Th. & Dynam. Sys.* **1**, 151-157 (1981).

[GW ?] GLASNER, S. and B. WEISS, Minimal transformations with no common factor need not be disjoint, to appear.

[H 98] HADAMARD, J., Les surfaces a courbures opposées et leurs géodesiques, *J. Math. Pures Appl. Ser. V, vol* **4**, 27-73 (1898).

[IM ?] IHRIG, E. and D. C. MCMAHON, On distal and point distal flows of finite codimension, to appear.

[J 82] JUNCO, A. DEL, Prime systems in ergodic theory and topological dynamics, *Rep.* **82-23**, Technische Hogeschool, Delft 1982.

[K 71] KEYNES, H. B., The structure of weakly mixing minimal transformation groups, *Illinois J. Math.* **15**, 475-489 (1971).

[K 72] KEYNES, H. B., Disjointness in transformation groups, *Proc. Amer. Math. Soc.* **36**, 253-259 (1972).

[Kl 55] KELLEY, J. L., *General topology,* D. Van Nostrand Co, New York, 1955.

[Kn 68] KNAPP, A. W., Functions behaving like almost automorphic functions, in *Topological dynamics* (Sympos., Colorado,1967), Benjamin, New York 1968.

[Ko 75] KOO, S.-C., Recursive properties of transformation groups in hyperspaces, *Math. Systems Theory* **9**, 75-82 (1975).

[LV 80] LUDESCHER, H. and J. DE VRIES, A sufficient condition for the existence of a *G*-compactification, *Nederl. Akad. Wetensch. Proc. Ser. A* **83**, 263-268 (1980).

[M 76.1] MCMAHON, D. C., Weak mixing and a note on a structure theorem for minimal transformation groups, *Illinois J. Math.* **20**, 186-197 (1976).

[M 76.2] MCMAHON, D. C., On the role of an abelian phase group in relativized problems in topological dynamics, *Pacific J. Math.* **64**, 493-504 (1976).

[M 78] MCMAHON, D. C., Relativized weak disjointness and relatively invariant measures, *Trans. Amer. Math. Soc.* **236**, 225-237 (1978).

[M 80] MCMAHON, D. C., Relativized weak mixing of uncountable order, *Canad. J. Math.* **32**, 559-566 (1980).

[MN 80] MCMAHON, D. C. and L. J. NACHMAN, An intrinsic characterization for PI-flows, *Pacific J. Math.* **89**, 391-403 (1980).

[MW 72] MCMAHON, D. C. and T. S. WU, On weak mixing and local almost periodicity, *Duke Math. J.* **39**, 333-343 (1972).

[MW 74] MCMAHON, D. C. and T. S. WU, On proximal and distal extensions of minimal sets, *Bull. Inst. Math. Acad. Sinica* **2**, 93-107 (1974).

[MW 76] MCMAHON, D. C. and T. S. WU, On the connectedness of homomorphisms in topological dynamics, *Trans. Amer. Math. Soc.* **217**, 257-270 (1976).

[MW 80.1]MCMAHON, D. C. and T. S. WU, Notes on topological dynamics IV: relative equicontinuity and its variations, *Bull. Inst. Math. Acad. Sinica* **8**, 277-281 (1980).

[MW 80.2]MCMAHON, D. C. and T. S. WU, Notes on topological dynamics V: equicontinuous structure relations of minimal transformation groups, *Bull. Inst. Math. Acad. Sinica* **8**, 283-294 (1980).

[MW 81] MCMAHON, D. C. and T. S. WU, Distal homomorphisms of nonmetric minimal flows, *Proc. Amer. Math. Soc.* **82**, 283-287 (1981).

[MW 83] MCMAHON, D. C. and T. S. WU, Homomorphisms of minimal flows and generalizations of weak mixing, *Pacific J. Math.* **104**, 401-416 (1983).

[Mi 51] MICHAEL, E., Topologies on spaces of subsets, *Trans. Amer. Math. Soc.* **71**, 152-182 (1951).

[Mk 72] MARKLEY, N., *F*-Minimal sets, *Trans. Amer. Math. Soc.* **163**, 85-100 (1972).

[Mo 21] MORSE, M., A one-to-one representation of geodesics on a surface of negative curvature, *Amer. J. Math.* **43**, 33-51 (1921).

[Mo 66] MORSE, M., *Symbolic dynamics,* notes by R. Oldenburger, Inst. for Advanced Study, Princeton, N.J. 1966.

[Mt 71] MARTIN, J. C., Substitution minimal flows, *Amer. J. Math.* **93**, 503-526 (1971).

[P 71] PELEG, R., Some extensions of weakly mixing flows, *Israel J. Math.* **9**, 330-336 (1971).

[P 72] PELEG, R., Weak disjointness of transformation groups, *Proc. Amer. Math. Soc.* **33**, 165-170 (1972).

[PW 70] PARRY, W. and P. WALTERS, Minimal skew product homeomorphisms and coalescence, *Compositio Math.* **22**, 283-288 (1970).

[Pe 70] PETERSEN, K. E., Disjointness and weak mixing of minimal sets, *Proc. Amer. Math. Soc.* **24**, 278-280 (1970).

[R ?] REES, M., *On the structure of a minimal distal transformation groups with manifolds as phase spaces,* Ph.D. thesis, Univ. of Warwick, Coventry, England.

[S 70] SHAPIRO, L., Proximality in minimal transformation groups, *Proc. Amer. Math. Soc.* **26**, 521-525 (1970).

[S 71] SHAPIRO, L., Distal and proximal extensions of minimal flows, *Math. Systems Theory* **5**, 76-88 (1971).

[Sh 74] SHOENFELD, P. S., *Regular homomorphisms of minimal sets,* Ph.D. thesis, Univ. of Maryland, 1974.

[Sh 76] SHOENFELD, P. S., Highly proximal and generalized almost finite extensions of minimal sets, *Pacific J. Math.* **66**, 265-280 (1976).

[T 79] TROALLIC, J.-P., Espaces fonctionels et théorèmes de I. Namioka, *Bull. Soc. Math. France* **107**, 127-137 (1979).

[V 70] VEECH, W. A., Point-distal flows, *Amer. J. Math.* **92**, 205-242 (1970).

[V 77] VEECH, W. A., Topological dynamics, *Bull. Amer. Math. Soc.* **83**, 775-830 (1977).

[dV 75] VRIES, J. DE, *Topological transformation groups* I *: a categorical approach,* Mathematical Centre Tracts Nr. **65**, Mathematisch Centrum, Amsterdam 1975.

[VW ?] VRIES, J. DE and J. C. S. P. VAN DER WOUDE, *Topological dynamics, an introduction, with emphasis on the structure of minimal sets,* to appear in the series Mathematical Centre Tracts, Mathematisch Centrum, Amsterdam 198?.

[W 67] WU, T. S., Two homomorphic but non-isomorphic minimal sets, *Michigan Math. J.* **14**, 401-404 (1967).

[W 74] WU, T. S., Notes on topological dynamics I: relative disjointness, *Bull. Inst. Math. Acad. Sinica* **2**, 343-356 (1974).

[Wa 74] WALKER, R. C., *The Stone-Cech compactification,* Erg. der Math. Band **83**, Springer Verlag, New York 1974.

[Wi 70] WILANSKI, A., *Topology for analysis,* Xerox College Publishing, Toronto 1970.

[Wo 79.1] WOUDE, J. C. S. P. VAN DER, Disjointness and quasifactors in topological dynamics, in *Topological structures* II, *part* 2, Mathematical Centre Tracts Nr. **116**, Mathematisch Centrum, Amsterdam 1979.

[Wo 79.2] WOUDE, J. C. S. P. VAN DER, Disjointness and classes of minimal transformation groups, *Rep.* **ZW 125**, Mathematisch Centrum, Amsterdam 1979.

SUBJECT INDEX

NOTATION AND SYMBOLS

A_K, A_K^α, A_α	Ellis groups related to $K \in \mathcal{K}^*$	170, 172, 174
$A_{\mathcal{Y}}$	Ellis group of $\mathcal{A}(\mathcal{Y})$	90
$\mathrm{AG}(\phi)$	AG shadow diagram	112
$\mathrm{AG}(\phi, \psi)$	double AG shadow diagram	128
a_K, $\overset{\alpha}{a}_K$, a_α	MHP generators related to $K \in \mathcal{K}^*$	169, 172, 174
$\mathcal{A}_K$, $\mathcal{A}_\alpha$	ambits related to $K \in \mathcal{K}^*$	174, 175
$\mathcal{A}(\mathcal{Y})$	univ. min. unif. almost periodic ext.	90
$\mathfrak{A}(H)$	univ. min. prox. extension	69
$\mathfrak{A}_S(\mathcal{Y})$	univ. min. strongly prox. ext.	219
$\mathfrak{A}(\mathfrak{Z})$, $\mathfrak{A}(Z)$	univ. min. prox. extension	30
$\mathfrak{A}(\phi)$	max. RIC shadow diagram	70
bT	Bohr compactification of T, phase space of $\mathcal{E}$	11
$\beta_T X$	equivariant T-compactification of X	16
$\mathrm{cl}_{\mathfrak{T}}$	closure operator for τ-topology	72
C_α, c_α	Ellis group and MHP gen. for max. HPI ttg	180
$\chi_{\mathcal{Y}} : \mathcal{Y}^* \to \mathcal{Y}$	max. hp ext. of $\mathcal{Y}$	109
$D(A, \mathcal{X})$	prolongational limit	277
$\mathbf{D}$	collection of all min. distal ttgs	190
$\mathcal{D}$, $\mathcal{D}_T$	univ. min. distal ttg (for T)	23
$E_{\mathcal{X}}$, E_ϕ	(relative) equicont. struct. relation	13
$E(x)$, $E(x, \phi, u)$		77
$E_\alpha(x)$, $E_\infty(x)$		78
$E(X)$, $E(\mathcal{X})$	enveloping semigroup	7
$\mathrm{EGS}(\phi)$	EGS shadow diagram	67

$\mathrm{Reg}(\phi)$	regularizer of ϕ	24
R_ϕ	(kernel) relation induced by ϕ	5
R_ϕ^n	fibered n-power	52
$R_{\phi\psi}$	fibered product	25
ρ_p , ρ_x	right multiplication by p, x	19
S_T , $\mathfrak{S}_T$	universal ambit	17
$\mathrm{SRP}(K,y)$	strongly regionally prox. relation w.r.t. K	100
$\Sigma_1(y)$	strong points for τ-topology	282
$\Sigma(\psi)$	supprim points for ψ	239
T_d	phase group T with discrete topology	2
Tx , $\overline{Tx}$	orbit (closure) of x	3
$U(x,\sigma,\epsilon)$	base element for τ-topology	72
U_ϕ	strongly regionally prox. relation for ϕ	281
$\mathcal{U}_X$	unique uniformity for X	3
$\mathcal{U}^*$	"hyper" uniformity induced by $\mathcal{U}$	41
$V(u)$	open syndetic subset of T	71
$\mathcal{V}_x$	neighbourhood filter for x	9
WM	collection of all min. weakly mixing ttgs	192
$\{\star\}$	trivial one point ttg	5
$*(\phi)$	$*$-diagram (max. open)	116
$*(\phi,\psi)$	double $*$-diagram	128
2^X	hyper space of X	41
2_ϕ^X, $2_\phi^{\mathcal{X}}$	relativized hyper space (ttg)	52
$2^{\mathcal{X}}$	hyper ttg of $\mathcal{X}$	43
2^ϕ	hyper homomorphism induced by ϕ	41
$(2^M,\circ)$	semigroup 2^M with circle operation	143
$\mathcal{X}:=<T,X,\pi>$	ttg	2, 3
$(\mathcal{X},x)$	ambit	16
$\mathcal{X}/R$	quotient ttg	5
$\phi:\mathcal{X}\rightarrow\mathcal{Y}$	homomorphism	5

$\mathbf{K}^{\perp}$	min. ttgs disjoint from every member of **K**	190
$<U>, <U>^*$	subbase elements for the Vietoris topology	41
$<U_1, \ldots, U_n>$	base element for the Vietoris topology	41
$[HF]$	Ellisgroup generated by $H \cup F$	204
$[U, V]$	base element for the τ-topology	72
$[\mathbf{K}]$	min. ttgs closed under factors and hp ext.	190
ϕ_{ad}	preimage map	41
$\phi^{\mathfrak{A}}$	max. RIC lifting of ϕ ($\mathfrak{A}(\phi)$)	70
$\phi_{\infty}, X_{\infty}$	top of the PI-tower	96
$\phi^*, \mathfrak{X}^*$	max. open lifting ($^*(\phi)$)	116
$\phi^{\#}, \mathfrak{X}^{\#}$	open RIM lifting (G-diagram)	218, 219

$\sim$	$v \sim v'$	equivalence of idempotents	19
$\perp$	$\phi \perp \psi$	(relative) disjointness	25
$\not\perp$	$\mathfrak{X} \not\perp \mathfrak{Y}$	(relative) nondisjointness	25
$\overset{\cdot}{-}$	$\phi \overset{\cdot}{-} \psi$	(relative) weak disjointness	30
$\circ$	$p \circ A , D \circ C$	circle operation	43, 141
$\square$	$q \square D$	second order circle operation	155
©	r © C	univ. ambit action on $2^{\mathfrak{X}}$	155
$\star$	$p \star (x,y), p \star A$	star operation	270, 277

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13 H.A. Lauwerier. *Asymptotic expansions.* 1966, out of print; replaced by MCT 54.

14 H.A. Lauwerier. *Calculus of variations in mathematical physics.* 1966.

15 R. Doornbos. *Slippage tests.* 1966.

16 J.W. de Bakker. *Formal definition of programming languages with an application to the definition of ALGOL 60.* 1967.

17 R.P. van de Riet. *Formula manipulation in ALGOL 60, part 1.* 1968.

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27 European Meeting 1968. *Selected statistical papers, part II.* 1968.

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37 J.W. de Bakker, G.A. Blaauw, A.J.W. Duijvestijn, E.W. Dijkstra, P.J. van der Houwen, G.A.M. Kamsteeg-Kemper, F.E.J. Kruseman Aretz, W.L. van der Poel, J.P. Schaap-Kruseman, M.V. Wilkes, G. Zoutendijk. *MC-25 Informatica Symposium.* 1971.

38 W.A. Verloren van Themaat. *Automatic analysis of Dutch compound words.* 1972.

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47 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 2: the compiler.* 1973.

48 F.E.J. Kruseman Aretz, P.J.W. ten Hagen, H.L. Oudshoorn. *An ALGOL 60 compiler in ALGOL 60, text of the MC-compiler for the EL-X8.* 1973.

49 H. Kok. *Connected orderable spaces.* 1974.

50 A. van Wijngaarden, B.J. Mailloux, J.E.L. Peck, C.H.A. Koster, M. Sintzoff, C.H. Lindsey, L.G.L.T. Meertens, R.G. Fisker (eds.). *Revised report on the algorithmic language ALGOL 68.* 1976.

51 A. Hordijk. *Dynamic programming and Markov potential theory.* 1974.

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53 M.J. Faber. *Metrizability in generalized ordered spaces.* 1974.

54 H.A. Lauwerier. *Asymptotic analysis, part 1.* 1974.

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