Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

The CWI is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

CWI Tract 24

Filters and ultrafilters over definable subsets of admissible ordinals

J.C.M. Baeten

Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

1980 Mathematics Subject Classification: 03D60, 03E45, 03E55, 04A20.
ISBN 90 6196 301 X

Printed in the Netherlands

CONTENTS

ACKNOWLEDGEMENTS

The present text is an almost unaltered version of my Ph.D. thesis, written at the University of Minnesota under the guidance of professor Wayne Richter. I am very grateful for all the help he has given me.

Also, this work benefited greatly from many discussions with my friends Evangelos Kranakis and Iain Phillips.

Furthermore, I would like to express my appreciation to the University of Minnesota, where I was employed from 1978 to 1983, and where I had a fellowship in the summer of 1981, and to the Centre for Mathematics and Computer Science, where I was employed in 1984 and 1985, and where this work is published.

Finally, I dedicate this work to my wife Jeanne.

Without her unending support and encouragement, it would never have been possible.

CHAPTER 0. INTRODUCTION

This thesis deals with definability theory. Definability theory is the result of the confluence and common development of recursion theory and axiomatic set theory.

Recursion theory developed in the 1930's as an attempt to give a rigorous meaning to the notion of a mechanically or algorithmically computable function. Such a function is, in a natural sense, more constructive and less complex than an arbitrary function. Work of Church, Kleene and Turing showed that there are several equivalent characterizations of the class of recursive functions. Subsequently, their work was further developed, extended and generalized. In particular, several suggestions were given to do recursion theory on an ordinal larger than ω (the ordinal of the natural numbers).

Axiomatic set theory developed in the first decade of the century, after the Russell paradox established the inconsistency of naive set theory. Naive set theory starts from the idea, that any collection of objects forms a set, and that such a collection can be given by a property, or more precisely, by a formula in a formal language, the language of set theory, which is the language of predicate logic with equality, enriched with a binary relation $\in$, the element relation. Thus, naive set theory expresses

the idea, that if $\phi(x)$ is any formula of this language, then $\{x : \phi(x)\}$ (the collection of all x such that $\phi(x)$) is a set. The inconsistency arises, if we consider $\{x : x \notin x\}$.
Therefore, axiomatic set theory uses the so-called *cumulative hierarchy* to build up sets from the bottom. We start from $\emptyset$ (the empty set), have levels indexed by the ordinal numbers, and form the next level by taking subsets of sets in the previous level. Now, Zermelo-Fraenkel set theory (ZF) takes *all* subsets of a given set, uses the power set operation to go from one level to the next, but does not specify the power set operation, does not say what constitutes a subset of a given set.
Then, we get the levels V_α of the cumulative hierarchy, and the universe of set theory $V = \bigcup\{V_\alpha : \alpha \text{ an ordinal}\}$.
Then, Gödel defined and used the *constructible universe* L in the 1930's. He uses a hierarchy as in the construction of V, but restricts the power set operation, taking only subsets of a given set that are definable, i.e. given by a formula of set theory, thus going back to the idea of naive set theory. Then, we obtain the levels L_α (α an ordinal) of the constructible hierarchy.
Now, Takeuti discovered in the early 1960's that a set is a recursive subset of an ordinal α just in case it is definable by a restricted formula over L_α, so that recursion theory over ordinals becomes the same as definability theory over the

constructible hierarchy. Thus, the link between recursion theory and axiomatic set theory was forged.

Central to definability theory is the notion of an admissible ordinal. This notion arises out of the notion of a recursive ordinal in the work of Church and Kleene. This work can be seen as the recursive counterpart to the classical theory of ordinals; the least nonrecursive ordinal ω_1^c is the recursive analogue of ω_1, the least uncountable ordinal. In the same way, an admissible ordinal is the recursive analogue of a regular cardinal. To be a little bit more precise, an ordinal κ is a regular cardinal if no sequence of ordinals of length less than κ can be cofinal in κ (i.e. can have sup κ), and an ordinal κ is admissible if no sequence of ordinals of length less than κ, that is definable by a restricted formula over L_κ, can be cofinal in κ.

The first admissible ordinal is ω, and the second is ω_1^c.

The important advance made possible by the definition of admissible ordinal is that it allows one to study recursion on important ordinals (like ω_1^c) which are not cardinals, but countable.

Kripke and Platek introduced admissible ordinals in the 1960's, and Barwise [1975] clearly establishes the importance of the notions of admissibility and definability.

Then, we can study large cardinals (cardinals that cannot be shown to exist in ZF) by studying their recursive analogues.

Thus, Richter & Aczel [1974] study recursively inaccessible, recursively Mahlo and reflecting ordinals, and in Kranakis [1980] we find recursive analogues of indescribable, weakly compact, Ramsey and Erdős cardinals (also see Phillips [1983] for some of these). Kaufmann [1981] started the study of recursive analogues of measurable cardinals, with which this thesis is mainly concerned. Work on this subject is also done in Kranakis [1982b], Kaufmann & Kranakis [1984] and Phillips [1983].

Measures were first studied by Lebesgue in connection with the real line. It was soon shown, using the axiom of choice, that not all sets of real numbers can be Lebesgue-measurable, and Ulam and Tarski showed in the 1930's that the property of having a total measure on a set is a property of the cardinal of that set. A cardinal admitting a total measure, or equivalently, a complete nonprincipal ultrafilter, was called a measurable cardinal, and it soon turned out that measurable cardinals, if they exist, must be very big, much bigger than any cardinals studied until then. One of the theorems about these cardinals says, that if κ is measurable, P is a property expressible by a Π^2_1 formula, and κ has this property, then κ is the κth cardinal with this property. Thus, the property of measurability cannot be expressed by a Π^2_1 formula, is Π^2_1-indescribable, and consequently it is very difficult to imagine a process which builds up from smaller ordinals to give the first measurable cardinal. Results like these have led many people to believe that measurable cardinals should not exist, but so far, much work in this direction has led to many results, but not to a proof of non-existence. In the meantime, the class of measurable cardinals has become the most studied and most intriguing class of large cardinals.

This monograph studies recursive analogues of measurable cardinals, using techniques from definability theory and admissibility theory on the constructible hierarchy.

We will see that there are different possibilities to pick recursive analogues, that some properties of measurable cardinals still hold, such as the existence of end extensions, that other properties do not hold, such as the equivalence between the existence of ultrafilters and the existence of normal ultrafilters, and that in general we have more differentiating and refined notions. Thus, an analogue of Fodor's theorem, proved in chapter II, immediately leads to certain definability questions that have no meaning in the classical case. Also, we will see that these recursive analogues can be shown to exist in ZF, so without assuming any large cardinal axioms.

Recursive analogues of measurable cardinals are ordinals, mostly countable, that have filters, that

are complete ultrafilters, or normal ultrafilters, only on a Boolean algebra of definable subsets, not on the whole power set. These so-called *definable* filters and ultrafilters are defined in chapter I. In chapter II, we first look at definable filters, define an analogue of the co-finite filter on ω, and use it to relate the existence of definable filters to admissibility.
In the second half of chapter II, we study definable normal filters, look at definable closed unbounded and stationary sets, and find the surprising result that in this setting, closed unbounded sets never form a normal filter. In chapter III, we discuss definable ultrafilters and definable normal ultrafilters. In the first section we relate their existence to the existence of certain end extensions, and in the second section we prove an extension theorem: on a countable ordinal, we can extend a definable filter to a definable ultrafilter, and extend a definable normal filter to a definable normal ultrafilter. This is of course completely contrary to the classical case.
Another difference we find is that the existence of a definable normal ultrafilter is not equivalent to the existence of a definable ultrafilter.
Finally, in chapter IV we see that definable ultrafilters cannot really be too definable, so e.g. there is no definable normal filter for which the membership relation is first order definable.

CHAPTER I. PRELIMINARIES AND NOTATION

§1. Set theory

1.1 Lower case Greek letters α, β, γ, η, κ, λ, μ, ν, ξ, ρ, σ stand for ordinal numbers; ω is the least infinite ordinal and ω_1 is the least uncountable ordinal.

Lower case Latin letters n, m, k, l stand for non-negative integers.

IMPORTANT: Throughout this thesis, κ is an ordinal such that $\omega\kappa = \kappa$, and n an integer with $n>0$.

1.2 Capital Latin letters X,Y, Z, A, B, C,... stand for sets.

Our set-theoretic notation is standard. We mention:

$X-Y = \{x \in X : x \notin Y\}$;

${}^{X}Y = \{f : f:X \longrightarrow Y\}$;

$\mathcal{P}X = \{Y : Y \subseteq X\}$, the power set of X;

f: $\subseteq X \longrightarrow Y$ means that $\mathrm{dom}(f) \subseteq X$ and $\mathrm{ran}(f) \subseteq Y$;

id is the identity function,

$f^{-1}(Z) = \{x \in X : f(x) \in Z\}$, and

$f|X$ is the function f restricted to the set X.

1.3 Let α be an ordinal and $X \subseteq \alpha$.

X is *bounded* in α if $\exists\beta<\alpha \quad X \subseteq \beta$.

X is *cofinal* or *unbounded* in α if X is not bounded in α.

X is *closed* in α if $\forall\beta<\alpha \quad \sup(X \cap \beta) \in X$.

If $\langle X_\beta : \beta<\alpha\rangle$ is a sequence of subsets of α, then we define their *diagonal intersection* $\underset{\beta<\alpha}{\Delta} X_\beta$ by:

$0 \in \underset{\beta<\alpha}{\Delta} X_\beta$ iff $\forall\beta<\alpha \;\; 0 \in X_\beta$, and if $\xi>0$, $\xi<\alpha$, then

$\xi \in \underset{\beta<\alpha}{\Delta} X_\beta$ iff $\forall\beta<\xi \;\; \xi \in X_\beta$.

$f\colon \underline{(}\alpha \longrightarrow \alpha$ is *regressive* if $\forall\beta \in \mathrm{dom}(f) \;\; f(\beta)<\beta$.

$f\colon X \xrightarrow{\text{cf}} \alpha$ means that $\mathrm{ran}(f)$ is cofinal in α.

§2. The constructible hierarchy

2.1 The *Lévy hierarchy* of classes of formulas of set theory (i.e. in the language $\{\in\}$) is defined as follows:

$\Sigma_0 = \Pi_0$ = the set of all formulas with only bounded quantification (where the bounded quantifiers are $\forall x \in y$ and $\exists x \in y$),

and for $m<\omega$

$\Sigma_{m+1} = \{\exists x_1 \exists x_2 \ldots \exists x_k \phi : k<\omega,\ \phi \in \Pi_m\}$, and

$\Pi_{m+1} = \{\forall x_1 \forall x_2 \ldots \forall x_k \phi : k<\omega,\ \phi \in \Sigma_m\}$.

$\Sigma_\omega = \bigcup_{m<\omega} \Sigma_m$.

Some other classes of formulas are defined as follows ($m<\omega$):

$\mathbb{C}_m = \{\phi \wedge \psi : \phi \in \Sigma_m,\ \psi \in \Pi_m\}$;

$\mathbb{D}_m = \{\phi \vee \psi : \phi \in \Sigma_m,\ \psi \in \Pi_m\}$;

$\mathbb{B}_m$ = the set of Boolean combinations of Σ_m formulas, i.e. the closure of Σ_m under $\neg$, $\wedge$, $\vee$.

$\Pi^1_1 = \{\forall X_1 \forall X_2 \ldots \forall X_k \phi : \phi \in \Sigma_\omega,\ k<\omega$, the X_i are second-order variables$\}$.

Letters ϕ, ψ, θ will stand for formulas, and letters Φ, Ψ for a class of formulas. $\neg\Phi = \{\neg\phi : \phi \in \Phi\}$.

2.2 If $\mathcal{M} = \langle M,E\rangle$ is a structure for the language of set theory (i.e. M is a set and E a binary relation on M), $A \subseteq M^n$, and $N \subseteq M$, then we say $A \in \Phi^{\mathcal{M}}N$, A is $\Phi^{\mathcal{M}}N$, or A is Φ-*definable on* $\mathcal{M}$ *with parameters from* N if there is a $\phi \in \Phi$ and constants $a_1,\dots,a_k \in N$ such that for all $x_1,\dots,x_n \in M$:

$\langle x_1,\dots,x_n\rangle \in A \iff \mathcal{M} \models \phi(x_1,\dots,x_n,a_1,\dots,a_k)$.

If N = M, we write $\Phi\mathcal{M}$ or even ΦM for $\Phi^{\mathcal{M}}M$.

Also we define $\Delta_m{}^{\mathcal{M}}N = \Sigma_m{}^{\mathcal{M}}N \cap \Pi_m{}^{\mathcal{M}}N$, for $m<\omega$.

If ϕ is a formula with parameters from N, we say $\phi \in \Phi^{\mathcal{M}}N$ if $\{\langle x_1,\dots,x_n\rangle \in M^n : \mathcal{M} \models \phi(\vec{x})\} \in \Phi^{\mathcal{M}}N$. Likewise for $\Phi\mathcal{M}$, ΦM.

If Φ is a class of formulas or Δ_m for some $m<\omega$, we write $f: \subseteq M \xrightarrow{\Phi} M$ if f (as a binary relation) is ΦM.

$Ord^{\mathcal{M}} = \{a \in M : \mathcal{M} \models$ "a is an ordinal"$\}$.

2.3 Gödel's *constructible hierarchy* is defined as follows:

$L_0 = \emptyset$,

$L_{\alpha+1} = \mathcal{P}L_\alpha \cap \Sigma_\omega L_\alpha$,

$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$, if λ is a limitordinal, and

$L = \bigcup\{L_\alpha : \alpha \text{ an ordinal}\}$.

We often write L_α for $\langle L_\alpha, \in\rangle$.

Certain drawbacks of this construction led Jensen to define a new hierarchy $\langle J_\alpha : \alpha \text{ an ordinal}\rangle$ such that again $L = \bigcup\{J_\alpha : \alpha \text{ an ordinal}\}$ which leads to the so-called *fine-structure theory* (see e.g. Devlin [1974]). The only result we need from fine-structure theory is part of the Σ_n-*uniformization theorem,* which says that every $\Sigma_n J_\alpha$ relation can be uniformized by a $\Sigma_n J_\alpha$ function, i.e.

$\forall R \in J_\alpha{}^m \cap \Sigma_n J_\alpha \quad \exists f \in \Sigma_n J_\alpha \quad (f\colon J_\alpha{}^{m-1} \longrightarrow J_\alpha$ & $\mathrm{dom}(f) = \mathrm{dom}(R)$ &

& $\forall \vec{x}\ [\exists y\ R(\vec{x},y) \Longleftrightarrow R(\vec{x},f(\vec{x}))] \quad (m>0)$.

Our assumption that for an ordinal κ we always have $\omega\kappa=\kappa$ ensures that $J_\kappa=L_\kappa$ and so that the Σ_n-uniformization theorem holds on L_κ.

§3. Recursive analogues of cardinals

3.1 If M is a structure for set theory, we say

$M \models \Sigma_n$-collection if for all formulas $\phi \in \Sigma_n M$ we have

$M \models \forall a\ (\forall x\in a\ \exists \vec{y}\ \phi \rightarrow \exists b\ \forall x\in a\ \exists \vec{y}\in b\ \phi)$.

We say $M \models X$-Σ_n-collection if the above only holds for all $\phi\in\Sigma_n{}^M X$.

Definition: κ is Σ_n-*admissible* if $L_\kappa \models \Sigma_n$-collection.

We shall need the following theorems.

Theorem (e.g. Kranakis [1980], Kaufmann & Kranakis [1984])

If κ is Σ_n-admissible, then $\Sigma_n L_\kappa$ and $\Pi_n L_\kappa$ are closed under bounded quantification.

Theorem (e.g. Kranakis [1980], from Devlin [1974])

If κ is Σ_n-admissible, then the Σ_n-*recursion theorem* holds on L_κ, i.e. if $G \in \Sigma_n L_\kappa$ is m+2-ary, then there is a unique m+1-ary $\Sigma_n L_\kappa$ function f such that $\forall \vec{x} \in L_\kappa\ \forall\alpha<\kappa \quad f(x,\alpha) = G(x,\alpha,\{\langle\beta,f(\vec{x},\beta)\rangle : \beta<\alpha\})$.

3.2 Some more definitions:

Definition (see Devlin [1974])

$L_\kappa \models \Delta_n$-separation $\Longleftrightarrow \neg\exists f\colon \alpha \xrightarrow{\text{onto},\ \Delta_n} \kappa$ for some $\alpha<\kappa$.

Definition (see Richter & Aczel [1974])

Let Φ be a set of formulas and $X \subseteq \kappa$.

κ is Φ-*reflecting on* X if for all $\phi \in \Phi L_\kappa$ $\quad L_\kappa \models \phi \implies \exists \alpha \in X \quad L_\alpha \models \phi$.

Definition: Let M, N be structures for set theory, and $m<\omega$.

$M \prec_m N$, M is a Σ_m-*substructure of* N, if $M \subseteq N$ and for all $\phi \in \Sigma_m$ (and hence for all $\phi \in \mathbb{B}_m$) and $\vec{a} \in M$ we have

$M \models \phi(\vec{a}) \iff N \models \phi(\vec{a})$.

Definition: $S^m_\kappa = \{\alpha<\kappa : L_\alpha \prec_m L_\kappa\}$. Kranakis [1980] shows that S^m_κ is defined by a Π_m formula (without parameters, and uniformly in κ).

3.3 Recursive analogues of partition cardinals are studied by Kranakis [1982a] and Phillips [1983]. We will use two of their notions.

Definition: $\kappa \xrightarrow{\Delta_n} (cf)^1_{<\kappa}$ if for all $\lambda<\kappa$ and all f: $\kappa \xrightarrow{\Delta_n} \lambda$ there is an $\alpha<\lambda$ such that $f^{-1}(\{\alpha\})$ is cofinal in κ.

Definition: $\subseteq\kappa \xrightarrow{\Pi_n} (cf)^1_{<\kappa}$ if for all $\lambda<\kappa$ and all f: $\subseteq\kappa \xrightarrow{\Pi_n} \lambda$, if dom(f) is cofinal in κ, then there is an $\alpha<\lambda$ such that $f^{-1}(\{\alpha\})$ is cofinal in κ.

3.4 If we want to discuss recursive analogues of measurable cardinals, we need the notions of an end extension and a filter:

Definition: Let $M=\langle M,E\rangle$ and $N=\langle N,F\rangle$ be structures for set theory.

$M \subseteq_e N$, N is an *end extension* of M, if $M \subseteq N$ and

$\forall a \in M \; \forall b \in N \; (bFa \to b \in M)$.

Definition: $\mathcal{M} \subseteq_e^{blunt} \mathcal{N}$, $\mathcal{N}$ is a *blunt* end extension of $\mathcal{M}$, if $Ord^{\mathcal{N}}-Ord^{\mathcal{M}}$ has a minimal element.

Definition: $\mathcal{M} \prec_{m,e} \mathcal{N}$ if $\mathcal{M} \subseteq_e \mathcal{N}$ and $\mathcal{M} \prec_m \mathcal{N}$.

Definition: a set $\mathcal{F} \subseteq \mathcal{P}X$ is a *filter* on X if

i. $X \in \mathcal{F}$,

ii. if $Y \in \mathcal{F}$ and $Y \subseteq Z \subseteq X$, then $Z \in \mathcal{F}$,

iii. if $Y,Z \in \mathcal{F}$, then $Y \cap Z \in \mathcal{F}$;

$\mathcal{F}$ is *proper* if $\emptyset \notin \mathcal{F}$ and $\mathcal{F} \neq \{X\}$;

$\mathcal{F}$ is *nonprincipal* if $\forall x \in X \quad X-\{x\} \in \mathcal{F}$.

3.5 Finally we define the filters we will study in this thesis:

Definition: Let $\mathcal{F}$ be a proper nonprincipal filter on κ and let Φ be a set of formulas or $\Phi = \Delta_m$.

i. We say $\mathcal{F}$ is a *Φ-filter on* κ if

$\forall\lambda<\kappa \ \forall\langle X_\alpha : \alpha<\lambda\rangle \in \Phi L_\kappa \cap {}^\lambda\mathcal{F} \quad \bigcap_{\alpha<\lambda} X_\alpha \in \mathcal{F}$.

ii. $\mathcal{F}$ is a *Φ-normal filter on* κ if

$\forall\langle X_\alpha : \alpha<\kappa\rangle \in \Phi L_\kappa \cap {}^\kappa\mathcal{F} \quad \Delta_{\alpha<\kappa} X_\alpha \in \mathcal{F}$.

iii. $\mathcal{F}$ is a *Φ-ultrafilter on* κ if $\mathcal{F}$ is a Φ-filter on κ and

$\forall X \in \Phi L_\kappa \cap \mathcal{P}\kappa \quad X \in \mathcal{F}$ or $\kappa-X \in \mathcal{F}$.

iv. $\mathcal{F}$ is a *Φ-normal ultrafilter on* κ if $\mathcal{F}$ is a Φ-normal filter and a Φ-ultrafilter on κ.

CHAPTER II. FILTERS AND NORMAL FILTERS

In this chapter we investigate filters, as defined in I.3.5. We establish some basic properties, and consider the similarities and differences with filters in the classical sense. Some results are improvements of results in § 5 of Kaufmann & Kranakis [1984].

§1. Filters

First we look at Δ_n- and Π_n-filters. We define a Π_n-filter $\mathcal{H}$, which is minimal in the sense that it is included in every Δ_n- and Π_n-filter. In 1.4 we characterize those ordinals κ that have a Δ_n- or Π_n-filter in terms of admissibility. In the remainder of the paragraph we consider the problem of the Σ_n-filter. It is not known whether there are ordinals that have a Δ_n-filter but not a Σ_n-filter. This problem relates to others questions, as the question in Kaufmann [1981] and question 3.26 in Kaufmann & Kranakis [1984]. This relationship is explained in III.2. Although I cannot solve the problem, some suggestions are given that might help to solve it.

1.1 Definition

$\mathcal{H} = \{X \subseteq \kappa : \kappa\text{-}X \text{ is bounded in } \kappa\}$.

For all κ, this is a nonprincipal proper filter on κ. We will find out, when it is a Π_n- respectively a Δ_n-filter.

1.2 Lemma

If $\mathcal{F}$ is a Δ_1-filter, then $\mathcal{H} \subseteq \mathcal{F}$.

Proof

Let $\lambda<\kappa$. Then $\kappa-\lambda = \bigcap\{\kappa-\{\alpha\}: \alpha<\lambda\} \in F$, and if $X \in H$, there is a $\lambda<\kappa$ such that $\kappa-\lambda \subseteq X$.

We need a lemma from Kranakis [1982a] for theorem 1.4:

1.3 Lemma

The following are equivalent:

i. κ is Σ_{n+1}-admissible

ii. $\kappa \xrightarrow{\Delta_n} (cf)^1_{<\kappa}$

Now we can characterize those ordinals κ that have a Δ_n-filter or a Π_n-filter. Also see Phillips [1983], III.1.2.a.

1.4 Theorem

The following are equivalent:

i. κ is Σ_{n+1}-admissible

ii. there is a Δ_n-filter on κ

iii. there is a Π_n-filter on κ

Proof

iii $\rightarrow$ ii: immediate.

ii $\rightarrow$ i: This improves Kaufmann & Kranakis [1984], 5.1 and 5.2. Let F be a Δ_n-filter on κ. To show κ is Σ_{n+1}-admissible, we use 1.3, so suppose, for a contradiction, that $\lambda<\kappa$, $f: \kappa \xrightarrow{\Delta_n} \lambda$, but for each $\alpha<\lambda$ we have that $f^{-1}(\{\alpha\})$ is bounded in κ. Then for each $\alpha<\lambda$ $\kappa - f^{-1}(\{\alpha\}) \in H \subseteq F$ (by 1.2), so

$\emptyset = \bigcap\{\kappa - f^{-1}(\{\alpha\}) : \alpha<\lambda\} \in F$, a contradiction.

i $\to$ iii: We show H is a Π_n-filter on κ.

Let $\lambda<\kappa$ and $\langle X_\alpha : \alpha<\lambda\rangle \in \Pi_n L_\kappa \cap {}^\lambda H$. We have to show

$\bigcap\{X_\alpha : \alpha<\lambda\} \in H$. Take $\phi \in \Pi_n L_\kappa$ such that

$\xi \in X_\alpha \iff L_\kappa \models \phi(\alpha,\xi)$ (for $\alpha<\lambda$, $\xi<\kappa$).

Then by definition of H $L_\kappa \models \forall\alpha<\lambda\ \exists\beta\ \forall\xi\geq\beta\ \phi(\alpha,\xi)$.

Since κ is Σ_{n+1}-admissible, there is a $\gamma<\kappa$ such that

$L_\kappa \models \forall\alpha<\lambda\ \exists\beta<\gamma\ \forall\xi\geq\beta\ \phi(\alpha,\xi)$, so

$L_\kappa \models \forall\alpha<\lambda\ \forall\xi\geq\gamma\ \phi(\alpha,\xi)$, or

$L_\kappa \models \forall\xi\geq\gamma\ (\forall\alpha<\lambda\ \phi(\alpha,\xi))$, which means $\bigcap\{X_\alpha : \alpha<\lambda\} \in H$.

Note that it follows from the theorem that H is a Π_n-filter on κ iff κ is Σ_{n+1}-admissible.

Now we turn to Σ_n-filters. It is obvious by 1.4 that if there is a Σ_n-filter on κ, then κ is Σ_{n+1}-admissible.

To prove theorem 1.8, we need to borrow a result from III.1, and we also need a lemma from Phillips [1983].

1.5 <u>Lemma</u> (from III.1. 9)

If κ is Σ_{n+2}-admissible, then $\{\alpha<\kappa :$ there is a Σ_n-filter on $\alpha\}$ is cofinal in κ.

1.6 Lemma (Phillips [1983], II.2.5)

The following are equivalent:

i. κ is Σ_{n+2}-admissible

ii. $\underline{(}\kappa \xrightarrow{\Pi_n} (cf)^1_{<\kappa}$

1.7 Lemma

The following are equivalent:

i. $\mathcal{H}$ is a Σ_n-filter on κ

ii. κ is Σ_{n+2}-admissible

Proof

ii $\rightarrow$ i: by the proof of 1.4

i $\rightarrow$ ii: To show κ is Σ_{n+2}-admissible, we use 1.6, so let $f\colon \underline{(}\kappa \xrightarrow{\Pi_n} \lambda$ for some $\lambda<\kappa$ and suppose that for all $\alpha<\lambda$ $f^{-1}(\{\alpha\})$ is bounded in κ. We have to show that dom(f) is bounded in κ.

But look, $\langle \kappa - f^{-1}(\{\alpha\}) : \alpha<\lambda \rangle \in \Sigma_n L_\kappa \cap {}^\lambda\mathcal{H}$, so $\kappa - \mathrm{dom}(f) = \bigcap\{\kappa - f^{-1}(\{\alpha\}) : \alpha<\lambda\} \in \mathcal{H}$, which means dom(f) is bounded in κ.

1.8 Theorem

Let κ be the least ordinal that has a Σ_n-filter.

Then $\mathcal{H}$ is not a Σ_n-filter on κ.

Proof

Combine 1.5 and 1.7.

1.8 shows, that the filter $\mathcal{H}$ cannot help us to characterize those

κ that have a Σ_n-filter. If κ is the least ordinal that has a Σ_n-filter, then $\mathcal{H}$ is not closed under $\Sigma_n L_\kappa$ intersections on κ. Therefore, any Σ_n-filter will contain some extra $\Sigma_n L_\kappa$ sets. We will show in III.2.14 that these $\Sigma_n L_\kappa$ sets we are committed to must be of a certain form. This leads us to define a filter $\mathcal{D}$, slightly larger than $\mathcal{H}$, which is a good candidate for a Σ_n-filter (see 1.14). First of all, we have the following characterizations of Δ_n- and Π_n-filters.

1.9 <u>Theorem</u>

Let F be a nonprincipal proper filter on κ.

a. The following are equivalent:

i. F is a Δ_n-filter on κ

ii. $\forall\lambda<\kappa\ \forall f:\subseteq \kappa \xrightarrow{\Delta_n} \lambda\ (\kappa\text{-dom}(f) \notin F \Rightarrow \exists\alpha<\lambda\ \kappa\text{-}f^{-1}(\{\alpha\}) \notin F)$.

b. The following are equivalent:

i. F is a Π_n-filter on κ

ii. $\forall\lambda<\kappa\ \forall f:\subseteq \kappa \xrightarrow{\Sigma_n} \lambda\ (\kappa\text{-dom}(f) \notin F \Rightarrow \exists\alpha<\lambda\ \kappa\text{-}f^{-1}(\{\alpha\}) \notin F)$.

<u>Proof</u>

a. i → ii: if $\lambda<\kappa$ and $f:\subseteq \kappa \xrightarrow{\Delta_n} \lambda$, then $\kappa\text{-dom}(f) = \bigcap_{\alpha<\lambda} (\kappa\text{-}f^{-1}(\{\alpha\}))$.

ii → i: we will first prove two claims:

<u>Claim 1</u> $\mathcal{H} \subseteq F$

<u>Proof</u> Let $\lambda<\kappa$. Define $f:\subseteq \kappa \xrightarrow{\Delta_n} \lambda$ by $f = \text{id}\restriction\lambda$. Then $\kappa\text{-}f^{-1}(\{\alpha\}) = \kappa\text{-}\{\alpha\} \in F$ for $\alpha<\lambda$, since F is nonprincipal. Thus $\kappa\text{-}\lambda = \kappa\text{-dom}(f) \in F$, whence $\mathcal{H} \subseteq F$. ⌑

<u>Claim 2</u> κ is Σ_{n+1}-admissible.

Proof We use 1.3, so suppose, for a contradiction, that $\lambda<\kappa$ and $f:\kappa \xrightarrow{\Delta_n} \lambda$, but for each $\alpha<\lambda$ $f^{-1}(\{\alpha\})$ is bounded in κ. Then, for each $\alpha<\lambda$, $\kappa-f^{-1}(\{\alpha\}) \in H \subseteq F$, so $\emptyset=\kappa-\mathrm{dom}(f) \in F$, which contradicts the fact that F is proper. ◻

Now we can show that F is a Δ_n-filter, so let $\lambda<\kappa$ and $\langle X_\alpha : \alpha<\lambda \rangle \in \Delta_n L_\kappa \cap {}^\lambda F$

Define $f: \subseteq \kappa \to \lambda$ by $f(\xi) \simeq \alpha \iff \xi \notin X_\alpha$ and $\xi \in \bigcap_{\beta<\alpha} X_\beta$ $(\alpha<\lambda)$

Since κ is Σ_n-admissible (by claim 2), we find f is $\Delta_n L_\kappa$.

If $\alpha<\lambda$ then $\kappa-f^{-1}(\{\alpha\}) \supseteq X_\alpha \in F$, so $\bigcap_{\alpha<\lambda} X_\alpha=\kappa-\mathrm{dom}(f) \in F$.

b. i $\to$ ii: as in a.

ii $\to$ i: let $\lambda<\kappa$ and $\langle X_\alpha:\alpha<\lambda\rangle \in \Pi_n L_\kappa \cap {}^\lambda F$. Define a $\Sigma_n L_\kappa$ relation R by $R(\xi,\alpha) \iff \alpha<\lambda$ and $\xi\notin X_\alpha$. By the Σ_n-uniformization theorem there is a $\Sigma_n L_\kappa$ function $f:\subseteq \kappa \to \lambda$ such that $\mathrm{dom}(f) = \mathrm{dom}(R) = \kappa-\bigcap_{\alpha<\lambda} X_\alpha$ and $\forall\xi\in\mathrm{dom}(f)\ R(\xi,f(\xi))$. Then for each $\alpha<\lambda$ $\kappa-f^{-1}(\{\alpha\}) \supseteq X_\alpha \in F$ so $\bigcap_{\alpha<\lambda} X_\alpha=\kappa-\mathrm{dom}(f) \in F$.

1.10 Remark

Result 1.9 leads us to consider the following property for a filter F:

$*$: $\forall\lambda<\kappa\ \forall f: \subseteq \kappa \xrightarrow{\Pi_n} \lambda\ (\kappa-\mathrm{dom}(f) \notin F \implies \exists\alpha<\lambda\ \kappa-f^{-1}(\{\alpha\}) \notin F)$.

As in 1.9, it is easy to show that F has property $*$, if F is a Σ_n-filter on κ. However, the converse does not necessarily hold. In the case of normal filters, we can define a similar property, and then III.1.6 shows that the converse does not hold.

1.11 <u>Lemma</u>

If $\mathcal{F}$ is a Π_n-ultrafilter on κ, then $\mathcal{F}$ has property $*$.

<u>Proof</u>

Let $\lambda<\kappa$, $f: \underline{(}\kappa \xrightarrow{\Pi_n} \lambda$ and suppose for a contradiction that $\kappa\text{-dom}(f) \notin \mathcal{F}$ but $\forall\alpha<\lambda \quad \kappa\text{-}f^{-1}(\{\alpha\}) \in \mathcal{F}$. Then we have $\text{dom}(f) \in \mathcal{F}$, since $\mathcal{F}$ is a Π_n-ultrafilter and dom(f) is $\Pi_n L_\kappa$ ($\xi \in \text{dom}(f) \iff \exists\alpha<\lambda\ f(\xi)=\alpha$, use that κ is Σ_n-admissible by 1.4).

Likewise, we have $\text{dom}(f)-f^{-1}(\{\alpha\})$ is $\Pi_n L_\kappa$ for $\alpha<\lambda$.

But then $\langle \text{dom}(f)-f^{-1}(\{\alpha\}) : \alpha<\lambda\rangle \in \Pi_n L_\kappa \cap {}^\lambda\mathcal{F}$, so

$\emptyset = \bigcap_{\alpha<\lambda}(\text{dom}(f)-f^{-1}(\{\alpha\}) \in \mathcal{F}$, contradiction.

1.12 <u>Definition</u>

$\mathcal{D}' = \{X \subseteq \kappa : X \in \Sigma_n L_\kappa \ \&\ \forall Y \supseteq X \ (Y \in \Delta_n L_\kappa \to Y \in \mathcal{H})\}$.

$\mathcal{D} = \{Z \subseteq \kappa : \exists X \subseteq Z \ (X \in \mathcal{D}')\}$.

1.13 <u>Lemma</u>

Let κ be Σ_n-admissible. Let $X \in \mathcal{D} \cap \Pi_n L_\kappa$. Then $X \in \mathcal{H}$.

<u>Proof</u>

Suppose $X \notin \mathcal{H}$, then κ-X is cofinal in κ and $\Sigma_n L_\kappa$. By a well-known fact (see e.g. Kaufmann & Kranakis [*].) there is a $Y \subseteq \kappa\text{-}X$ such that Y is cofinal in κ and $\Delta_n L_\kappa$. Thus κ-Y is $\Delta_n L_\kappa$, $\kappa\text{-}Y \supseteq X$ and $\kappa\text{-}Y \notin \mathcal{H}$, so $X \notin \mathcal{D}$.

1.1 4 Lemma

Let κ be Σ_{n+1}-admissible. Then $\mathcal{D}$ is a Π_n-filter on κ.

Proof

First note $\mathcal{H} \subseteq \mathcal{D}$, so $\mathcal{D}$ is nonprincipal. Now let $Z_1, Z_2 \in \mathcal{D}$. Take $X_1, X_2 \in \mathcal{D}$ such that $X_1 \subseteq Z_1$, $X_2 \subseteq Z_2$ and X_1, X_2 are $\Sigma_n L_\kappa$. Suppose $Y \supseteq X_1 \cap X_2$ and $Y \in \Delta_n L_\kappa$. We'll show $Y \in \mathcal{H}$, which gives that $\mathcal{D}$ is a nonprincipal proper filter on κ.

Define $Y' = Y \cup (\kappa - X_1)$, then $Y' \supseteq X_2$, so $Y' \in \mathcal{D}$. Also Y' is $\Pi_n L_\kappa$, so by 1.13 $Y' \in \mathcal{H}$. Therefore, we can take $\lambda < \kappa$ such that $\{\alpha < \kappa : \lambda \leq \alpha\} \subseteq Y'$. But then $Y \supseteq X_1 - \lambda \in \mathcal{D}$ (the last fact is easy to check), so $Y \in \mathcal{H}$.

To show $\mathcal{D}$ is a Π_n-filter, take $\lambda < \kappa$ and $\langle X_\alpha : \alpha < \lambda \rangle \in \Pi_n L_\kappa \cap {}^\lambda\mathcal{D}$. If $\alpha < \lambda$, $X_\alpha \in \mathcal{D} \cap \Pi_n L_\kappa$, so by 1.13 $X_\alpha \in \mathcal{H}$. By 1.4, $\mathcal{H}$ is a Π_n-filter, so $\bigcap_{\alpha<\lambda} X_\alpha \in \mathcal{H} \subseteq \mathcal{D}$.

1.15 Theorem

Let κ be Σ_{n+1}-admissible. Then $\mathcal{D}$ has property * of 1. 10.

Proof

Let $\lambda < \kappa$ and $f : (\kappa \xrightarrow{\Pi_n} \lambda$ and suppose for each $\alpha < \lambda$ we have $\kappa - f^{-1}(\{\alpha\}) \in \mathcal{D}$. Let $Y \supseteq \kappa - \mathrm{dom}(f)$ and Y be $\Delta_n L_\kappa$. We have to show that $Y \in \mathcal{H}$.

Define $\langle Y_\beta : \beta < \lambda \rangle \in \Pi_n L_\kappa$ by

$\xi \in Y_\beta \iff \xi \in Y$ or $\exists \alpha < \lambda \ (\alpha \neq \beta \ \& \ f(\xi) = \alpha)$.

Claim 1: $Y_\beta \supseteq (\kappa - f^{-1}(\{\beta\}))$ for $\beta < \lambda$.

Proof: $f(\xi) \neq \beta \implies \xi \notin \mathrm{dom}(f)$ or $\exists \alpha < \lambda \ (\alpha \neq \beta \ \& \ f(\xi) = \alpha)$

$\Longrightarrow\ \xi \in Y$ or $\exists\alpha<\lambda\ (\alpha\neq\beta\ \&\ f(\xi)=\alpha)$

$\Longrightarrow\ \xi \in Y_\beta$. ◻

By the claim $Y_\beta \in \mathcal{D}$, so $Y_\beta \in \mathcal{H}$ by 1.13.

Since $\mathcal{H}$ is a Π_n-filter, we have $\bigcap_{\beta<\lambda} Y_\beta \in \mathcal{H}$.

The proof is finished if we show

Claim 2: $\bigcap_{\beta<\lambda} Y_\beta = Y$

Proof: Obviously $\bigcap_{\beta<\lambda} Y_\beta \supseteq Y$. Conversely, let $\xi \in \bigcap_{\beta<\lambda} Y_\beta$.

Then $\forall\beta<\lambda\ (\xi \in Y$ or $\exists\alpha<\lambda\ (\alpha\neq\beta\ \&\ f(\xi)=\alpha))$, so

$\xi \in Y$ or $\forall\beta<\lambda\ \exists\alpha<\lambda\ (\alpha\neq\beta\ \&\ f(\xi)=\alpha)$.

But the second alternative cannot happen, so $\xi \in Y$. ◻

1.16 Corollary

Let κ be Σ_{n+1}-admissible. Then

$\forall\lambda<\kappa\ \forall f: (\kappa \xrightarrow{\mathcal{D}_n} \lambda\ (\kappa\text{-dom}(f) \notin \mathcal{D} \Longrightarrow \exists\alpha<\lambda\ \kappa-f^{-1}(\{\alpha\}) \notin \mathcal{D})$.

Proof

Let $\lambda<\kappa$ and $f: (\kappa \xrightarrow{\mathcal{D}_n} \lambda$ and suppose for each $\alpha<\lambda$ $\kappa-f^{-1}(\{\alpha\}) \in \mathcal{D}$. Take $\phi \in \Sigma_n L_\kappa$ and $\psi \in \Pi_n L_\kappa$ so that

$f(\xi)=\alpha \iff L_\kappa \models \phi(\xi,\alpha) \vee \psi(\xi,\alpha)$.

Now for $\alpha<\lambda$ $X_\alpha = \{\xi<\kappa : L_\kappa \models \neg\phi(\xi,\alpha)\} \supseteq \kappa-f^{-1}(\{\alpha\}) \in \mathcal{D}$, and X_α is $\Pi_n L_\kappa$, so $X_\alpha \in \mathcal{H}$ by 1.13. Then by 1.4 $\bigcap_{\alpha<\lambda} X_\alpha \in \mathcal{H}$, so we can take $\sigma<\kappa$ so that $\{\gamma<\kappa : \sigma\leq\gamma\} \subseteq \bigcap_{\alpha<\lambda} X_\alpha$, or $L_\kappa \models \forall\xi\geq\sigma\ \forall\alpha<\lambda\ \neg\phi(\xi,\alpha)$. Then $g=f\restriction\{\gamma<\kappa : \sigma\leq\gamma\}$ is $\Pi_n L_\kappa$ and it is easy to see that $\forall\alpha<\lambda\ \kappa-g^{-1}(\{\alpha\}) \in \mathcal{D}$. Then by 1.15 $\kappa\text{-dom}(g) \in \mathcal{D}$, so $(\kappa\text{-dom}(g))-\sigma \in \mathcal{D}$ and $\kappa\text{-dom}(f) \in \mathcal{D}$.

1.17 Notes

i. We think that under certain circumstances $\mathcal{D}$ is a Σ_n-filter, even a $\mathbb{C}_n$-filter, although probably not for each Σ_{n+1}-admissible.

ii. Phillips [1983], III.3.1, shows that if there is a $\mathbb{D}_n$- or $\mathbb{B}_n$-filter on κ, then κ is a limit of Σ_{n+1}-admissibles.

§2. Normal filters

The most well-known (classical) normal filter is the closed unbounded filter on a regular cardinal. This leads us to study definable closed unbounded sets, and sets which are stationary with respect to these c.u.b.'s. Surprisingly, we find in 2.18 that in this setting, closed unbounded sets never form a normal filter. We do however in 2.9 derive a recursive analogue of Fodor's theorem.

2.1 Definition

Let $X \subseteq \kappa$, Φ a set of formulas or $\Phi=\Delta_n$.

i. X is a Φ-*cub* if X is closed unbounded and ΦL_κ.

ii. X is Φ-*stationary* if for all Φ-cubs C we have $X \cap C \neq \emptyset$.

Note: if X is Φ-stationary, X does not need to be ΦL_κ-definable.

For theorem 2.4 we need a lemma from Kranakis [1982a]:

2.2 Lemma

The following are equivalent:

i. κ is Σ_n-admissible

ii. κ is Π_{n+1}-reflecting on S_κ^{n-1}.

The next theorem shows that on a Σ_n-admissible ordinal, cubsets are closed under "Σ_n"-normal intersections, as one would expect. For later reference, we isolate a lemma used in its proof. This is 2.3.

2.3 Lemma

Let C be $\Sigma_n L_\kappa$ and closed in κ. Let $\sigma \in S_\kappa^{n-1}$ and

$L_\sigma \models$ "C is unbounded". Then $\sigma \in C$.

Proof

Take $\phi \in \Sigma_n L_\kappa$ defining C (i.e. $\xi \in C \iff L_\kappa \models \phi(\xi)$) such that

$L_\sigma \models \forall\alpha\ \exists\xi>\alpha\ \phi(\xi)$. This means

$\forall\alpha<\sigma\ \exists\xi<\sigma\ (\xi>\alpha\ \&\ L_\sigma \models \phi(\xi))$. But since $\sigma \in S_\kappa^{n-1}$ it follows that

$\forall\alpha<\sigma\ \exists\xi<\sigma\ (\xi>\alpha\ \&\ L_\kappa \models \phi(\xi))$, so

$\forall\alpha<\sigma\ \exists\xi<\sigma\ (\xi>\alpha\ \&\ \xi \in C)$.

This formula says that C is unbounded in σ, so since C is closed in κ we have $\sigma \in C$.

2.4 Theorem

Let κ be Σ_n-admissible, $\langle C_\beta : \beta<\kappa\rangle \in \Sigma_n L_\kappa$ and C_β is cub for $\beta<\kappa$. Then $\underset{\beta<\kappa}{\Delta} C_\beta$ is a Σ_n-cub.

Proof

Take $\langle C_\beta : \beta<\kappa\rangle$ as stated, and take $\phi \in \Sigma_n L_\kappa$ such that

$\xi \in C_\beta \iff L_\kappa \models \phi(\beta,\xi)$. It is not hard to see that

$\underset{\beta<\kappa}{\Delta} C_\beta$ is closed, and, using the fact that κ is Σ_n-admissible, that $\underset{\beta<\kappa}{\Delta} C_\beta$ is $\Sigma_n L_\kappa$. So all that remains is to show that $\underset{\beta<\kappa}{\Delta} C_\beta$ is unbounded. Fix $\mu<\kappa$. We'll find a $\sigma \in \underset{\beta<\kappa}{\Delta} C_\beta - \mu$.

Since each C_β is unbounded, we have

$L_\kappa \models \forall\beta\ \forall\alpha\ \exists\xi>\alpha\ \phi(\beta,\xi)$. This sentence is $\Pi_{n+1} L_\kappa$, so using 2.2 there is a $\sigma \in S_\kappa^{n-1}$, $\sigma>\mu$ with

$L_\sigma \models \forall\beta\ \forall\alpha\ \exists\xi>\alpha\ \phi(\beta,\xi)$. This means

$\forall\beta<\sigma \quad L_\sigma \models$ "C_β is unbounded". Therefore, by 2.3,

$\forall\beta<\sigma \quad \sigma \in C_\beta$, which means $\sigma \in \underset{\beta<\kappa}{\Delta} C_\beta$.

2.5 Example

Let κ be Σ_n-admissible, but less than the least Σ_{n+1}-admissible. Then $L_\kappa \not\models \Sigma_{n+1}$-collection, and from this it follows that there is a $\lambda<\kappa$ and an f: $\lambda \xrightarrow{cf,\Sigma_{n+1}} \kappa$ (see Devlin [1974]).

Simpson [1970] showed that this implies that there is a $\lambda<\kappa$ and an f: $\lambda \xrightarrow{cf,\Pi_n} \kappa$ (for a proof, see Phillips [1983], II.2.3).

Now let λ_0 be the least λ for which such an f exists.

Claim 1: $\lambda_0 = \omega$.

Proof: Suppose not, so $\lambda_0>\omega$. Then there is no $\mu<\lambda_0$ and a g: $\mu \xrightarrow{cf,\Sigma_{n+1}} \lambda_0$, for if there was, $f \circ g$: $\mu \xrightarrow{cf,\Sigma_{n+1}} \kappa$, which contradicts the choice of λ_0. But this means that λ_0 is Σ_{n+1}-admissible, and that contradicts the choice of κ.

Therefore, we have f: $\omega \xrightarrow{cf,\Pi_n} \kappa$. ◻

Claim 2: we can assume that f is increasing.

Proof: if f is not increasing, define f' by:

$f'(n) = \xi \iff \forall m\leq n \;\; f(m)\leq\xi \;\;\&\;\; \exists m\leq n \;\; f(m) = \xi$ (for $n<\omega$).

Then also f': $\omega \xrightarrow{cf,\Pi_n} \kappa$, and f' is increasing (to see f' is $\Pi_n L_\kappa$, use I.3.1). ◻

Now define C, D $\subseteq \kappa$ by:

$\xi \in C \iff \lim(\xi) \;\&\; \exists n,m<\omega \; (f(n) = \xi + m)$, and

$D = \{\xi + 1 : \xi \in C\}$.

Again by I.3.1, C and D are $\Pi_n L_\kappa$. Since ran(f) is cofinal in κ, C and D are cofinal in κ; since the order type of C and D is ω, we trivially have that C and D are closed in κ.

Thus C and D are Π_n-cubs, but $C \cap D = \emptyset$.

2.4 and 2.5 give, that on a Σ_n-admissible ordinal, Σ_n-cubsets "behave as" unrestricted cubsets on a regular cardinal, but Π_n-cubsets do not. One might think, that 2.4 shows that the Σ_n-cubsets form a definable normal filter, but that is not the case, as 2.18 shows. 2.8 gives, how much we can say in this direction.

2.6 Definition

$F_n = \{X \subseteq \kappa : \exists C \subseteq X \quad C \text{ is a } \Sigma_n\text{-cub}\}.$

2.7 Examples

i. If κ is Σ_n-admissible, then $S_\kappa^{n-1} \in F_n$.

ii. $Cd^\kappa \in F_2 \iff L_\kappa \models \text{Pow}$

(Here $Cd^\kappa = \{\alpha<\kappa : L_\kappa \models \text{"}\alpha \text{ is a cardinal"}\}$ and Pow is the power set axiom).

Proof: Kranakis [1982a].

2.8 Lemma

Let κ be Σ_{n+1}-admissible, $\langle A_\alpha : \alpha<\kappa\rangle \in \Pi_n L_\kappa \cap {}^\kappa F_{n-1}$.

Then $\Delta_{\alpha<\kappa} A_\alpha \in F_{n+1}$.

Proof

Let $\langle A_\alpha : \alpha<\kappa\rangle$ be as stated, and take $\phi \in \Pi_n L_\kappa$ so that $\beta \in A_\alpha \iff L_\kappa \models \phi(\alpha,\beta)$. Fix $\alpha<\kappa$. Since $A_\alpha \in F_{n-1}$, there is a Σ_{n-1}-cub $C \subseteq A_\alpha$, so there is a $\theta \in \Sigma_{n-1}L_\kappa$ with $\beta \in C \iff L_\kappa \models \theta(\beta)$. We will also use the letter θ for an effective (Gödel) code of θ.

Now $L_\kappa \models \psi(\alpha,\theta)$, where $\psi(\alpha,\theta)$ is $\Pi_n L_\kappa$, equivalent to:

"$\forall\lambda\ [(\forall\delta<\lambda\ \exists\gamma<\lambda\ (\delta<\gamma\ \&\ \theta(\gamma))) \to \theta(\lambda)]\ \&$

$\&\ \forall\delta\ \exists\gamma>\delta\ \theta(\gamma)\ \&\ \forall\beta\ (\theta(\beta) \to \phi(\alpha,\beta))$".

Thus $L_\kappa \models \forall\alpha\ \exists\theta \in \Sigma_{n-1}L_\kappa\ \psi(\alpha,\theta)$ and by the Σ_{n+1}-uniformization theorem there is a function f: $\kappa \xrightarrow{\Sigma_{n+1}} \Sigma_{n-1}L_\kappa$ so that $L_\kappa \models \forall\alpha\ \psi(\alpha,f(\alpha))$.

Define $\xi \in C_\alpha \iff L_\kappa \models \theta(\xi)$, where $\theta=f(\alpha)$, then C_α is cub, $C_\alpha \subseteq A_\alpha$ and $\langle C_\alpha : \alpha<\kappa\rangle \in \Sigma_{n+1}L_\kappa$, since f is $\Sigma_{n+1}L_\kappa$.

Then by 2.3, using the Σ_{n+1}-admissibility of κ, $\Delta_{\alpha<\kappa} C_\alpha$ is a Σ_{n+1}-cub. But $\Delta_{\alpha<\kappa} C_\alpha \subseteq \Delta_{\alpha<\kappa} A_\alpha$, so $\Delta_{\alpha<\kappa} A_\alpha \in F_{n+1}$.

Notice that the definition of the function f in the proof of 2.8 increases the complexity, so that a diagonal intersection from F_{n-1} can only be put in F_{n+1}. It is shown in 2.18, that it is impossible to get every intersection in F_{n-1}, but it is an open question whether 2.8 can be improved to get the intersection in F_n. The following theorem 2.9 gives a recursive analogue of Fodor's theorem (see e.g. Jech [1978]).

2.9 Theorem

Let κ be Σ_{n+1}-admissible, $f:\subseteq \kappa \xrightarrow{\Sigma_n} \kappa$ regressive and dom(f) Σ_{n+1}-stationary. Then there is an $\alpha<\kappa$ such that $f^{-1}(\{\alpha\})$ is Σ_{n-1}-stationary.

Proof

Suppose not, then $\kappa-f^{-1}(\{\alpha\}) \in \mathcal{F}_{n-1}$ for each $\alpha<\kappa$. Also $\langle\kappa-f^{-1}(\{\alpha\}) : \alpha<\kappa\rangle \in \Pi_n L_\kappa$, so by lemma 2.7 we have that $\Delta_{\alpha<\kappa}(\kappa-f^{-1}(\{\alpha\})) \in \mathcal{F}_{n+1}$. But since f is regressive, $\Delta_{\alpha<\kappa}(\kappa-f^{-1}(\{\alpha\})) = \kappa$-dom(f), contradicting the fact that dom(f) is Σ_{n+1}-stationary.

In Fodor's theorem (2.9) we again have that complexity is increased by two quantifier switches. 2.20 gives, that we cannot do without any increase. Again it is open whether a lesser increase is sufficient.

Our next theorem (2.11) extends 2.2 and gives a characterization of Σ_n-stationary sets. For later reference, we first give a lemma used in its proof.

2.10 Lemma

Let $\phi \in \Pi_{m+2}L_\kappa$ and $\{\alpha \in S^m_\kappa : L_\alpha \models \phi\}$ be cofinal in κ. Then $L_\kappa \models \phi$.

Proof

Let ϕ be as stated. Write ϕ as $\forall\xi\ \exists\eta\ \psi(\xi,\eta)$, with $\psi \in \Pi_m L_\kappa$. Let $\xi_0<\kappa$. Since $\{\alpha \in S^m_\kappa : L_\alpha \models \phi\}$ is cofinal in κ, we can take an $\alpha \in S^m_\kappa$ with $\alpha>\xi_0$ and $L_\alpha \models \phi$, or $L_\alpha \models \forall\xi\ \exists\eta\ \psi(\xi,\eta)$.

Therefore, there is an $\eta_0<\alpha$ with $L_\alpha \models \psi(\xi_0,\eta_0)$.

Then, since $\alpha \in S^m_\kappa$, $L_\kappa \models \psi(\xi_0,\eta_0)$, so

$L_\kappa \models \exists\eta\ \psi(\xi_0,\eta)$. Finally, since $\xi_0<\kappa$ was chosen arbitrarily,

$L_\kappa \models \forall\xi\ \exists\eta\ \psi(\xi,\eta)$, so $L_\kappa \models \phi$.

2.11 Theorem

Let κ be Σ_n-admissible, $X \subseteq \kappa$.

κ is Π_{n+1}-reflecting on $S^{n-1}_\kappa \cap X \iff X$ is Σ_n-stationary.

Proof

$\Rightarrow$: Let C be a Σ_n-cub and $\phi \in \Sigma_n L_\kappa$ such that

$\xi \in C \iff L_\kappa \models \phi(\xi)$. Then $L_\kappa \models \forall\alpha\ \exists\xi>\alpha\ \phi(\xi)$, so by assumption

there is a $\sigma \in S^{n-1}_\kappa \cap X$ with $L_\sigma \models \forall\alpha\ \exists\xi>\alpha\ \phi(\xi)$.

By 2.3, $\sigma \in C$. Therefore, $C \cap X \neq \emptyset$.

$\Leftarrow$: Let $\phi \in \Pi_{n+1} L_\kappa$ and $L_\kappa \models \phi$. Put $C = \{\alpha \in S^{n-1}_\kappa : L_\alpha \models \phi\}$.

Since κ is Σ_n-admissible, we have by 2.2 that C is unbounded

in κ. Since S^{n-1}_κ is $\Pi_{n-1} L_\kappa$, we have that C is $\Pi_{n-1} L_\kappa$.

To show C is closed, let $\beta<\kappa$ be such that $\beta = \sup(C \cap \beta)$.

Since $C \subseteq S^{n-1}_\kappa$, and S^{n-1}_κ is closed, $\beta \in S^{n-1}_\kappa$.

It is easily seen that $S^{n-1}_\kappa \cap \beta \subseteq S^{n-1}_\beta$, so $\{\alpha \in S^{n-1}_\beta : L_\alpha \models \phi\}$

is cofinal in β. Then by 2.10 $L_\beta \models \phi$, so we have $\beta \in C$, and

C is closed.

We've shown that C is a Π_{n-1}-cub, so since X is Σ_n-stationary,

$C \cap X \neq \emptyset$ and so there is a $\sigma \in S^{n-1}_\kappa \cap X$ with $L_\sigma \models \phi$.

2.12 Corollary

$X \in F_n \iff \kappa$ is not Π_{n+1}-reflecting on $S^{n-1}_\kappa - X$.

The next corollary was first stated by Wimmers for n=1 and extended by Kranakis [*] to the general case (n≥1). However, the proof given here is much simpler than theirs.

2.13 Corollary

If κ is Σ_n-admissible, then each Σ_n-cub contains a Π_{n-1}-cub.

Proof

Let C be a Σ_n-cub and let $\phi \in \Sigma_n L_\kappa$ so that $\xi \in C \iff L_\kappa \models \phi(\xi)$.
Define $D = \{\alpha \in S^{n-1}_\kappa : L_\alpha \models \forall\beta\ \exists\xi>\beta\ \phi(\xi)\}$.
By 2.2, D is unbounded in κ, and by 2.10, D is closed.
Thus D is a Π_{n-1}-cub. By 2.3, $D \subseteq C$.

The following result improves a result of Kaufmann & Kranakis [1984], 5.3.

2.14 Theorem

Let F be a Π_n-normal filter on κ. Then $F_{n+1} \subseteq F$ (so each Π_n-normal filter contains all Σ_{n+1}-cubs).

Proof

Note κ is Σ_{n+1}-admissible by 1.4. Let $X \in F_{n+1}$. By 2.13 there is a Π_n-cub $C \subseteq X$. For $\alpha<\kappa$, define
$\xi \in X_\alpha \iff \exists\gamma<\xi\ (\gamma>\alpha\ \&\ \gamma \in C)$. Then $\langle X_\alpha : \alpha<\kappa\rangle \in \Pi_n L_\kappa$

and since C is unbounded, $X_\alpha \in \mathcal{H}$ for each $\alpha<\kappa$. Thus $\Delta_{\alpha<\kappa} X_\alpha \in \mathcal{F}$. But if $\xi \in \Delta_{\alpha<\kappa} X_\alpha$, then $\forall\alpha<\xi \;\; \xi \in X_\alpha$, so $\forall\alpha<\xi \; \exists\gamma<\xi \; (\gamma>\alpha \;\&\; \gamma \in C)$, which means that C is unbounded in ξ, so $\xi \in C$ by closedness. So we have $\Delta_{\alpha<\kappa} X_\alpha \subseteq C$, whence $X \in \mathcal{F}$.

2.1 5 Theorem

Let $\mathcal{F}$ be a Δ_1-normal filter on κ with $S^n_\kappa \in \mathcal{F}$.
Let $\phi \in \Pi_{n+3} L_\kappa$ and $L_\kappa \models \phi$. Then $\{\alpha \in S^n_\kappa : L_\alpha \models \phi\} \in \mathcal{F}$.

Proof

Write ϕ as $\forall\xi \; \psi(\xi)$ with $\psi \in \Sigma_{n+2} L_\kappa$. Suppose $X = \{\alpha \in S^n_\kappa : L_\alpha \models \forall\xi \; \psi(\xi)\} \notin \mathcal{F}$. Define, for $\xi<\kappa$, $X_\xi = \{\alpha<\kappa : L_\alpha \models \psi(\xi)\}$. Then $\langle X_\xi : \xi<\kappa\rangle \in \Delta_1 L_\kappa$ and $S^n_\kappa \cap \Delta_{\xi<\kappa} X_\xi = X$, so since $S^n_\kappa \in \mathcal{F}$, we can take $\xi_0<\kappa$ with $X_{\xi_0} \notin \mathcal{F}$. But then it is easy to see that $S^n_\kappa - X_{\xi_0}$ is cofinal in κ. Also $S^n_\kappa - X_{\xi_0} = \{\alpha \in S^n_\kappa : L_\alpha \models \neg\psi(\xi_0)\}$.
By 2.10, $L_\kappa \models \neg\, \psi(\xi_0)$, a contradiction.

2.16 Note

By 2.14, any Π_n-normal filter contains S^n_κ, so 2.15 applies to any Π_n-normal filter.

2.17 Corollary

If there is a Δ_1-normal filter on κ containing S^n_κ, then κ is Π_{n+3}-reflecting on S^n_κ, so in particular κ is Σ_{n+1}-admissible and a limit of Σ_{n+1}-admissibles.

Proof

It follows immediately from 2.15 that κ is Π_{n+3}-reflecting on S^n_κ. Then by 2.2 κ is Σ_{n+1}-admissible. To show that κ is a limit of Σ_{n+1}-admissibles, use the fact that there is a Π_{n+3} sentence ϕ such that for any ordinal α,

$$L_\alpha \models \phi \iff \alpha \text{ is } \Sigma_{n+1}\text{-admissible}$$

(this follows from characterization 2.2, see Kranakis [1980], II.2.5.c; this sentence is also used in 2.20).

2.18 Corollary

F_{n+1} is never a Δ_1-normal filter on κ.

Proof

2.7.i gives that $S^n_\kappa \in F_{n+1}$. Suppose that F_{n+1} is a Δ_1-normal filter on κ. Then by 2.15 and 2.17 $\{\alpha \in S^n_\kappa : \alpha \text{ is } \Sigma_{n+1}\text{-admissible}\} \in F_{n+1}$. We'll show κ is Π_{n+2}-reflecting on $\{\alpha \in S^n_\kappa : \alpha \text{ is not } \Sigma_{n+1}\text{-admissible}\}$, thus getting a contradiction with 2.12.

So take $\phi \in \Pi_{n+2}L_\kappa$ with $L_\kappa \models \phi$. Since κ is Σ_{n+1}-admissible (by 2.17), κ is Π_{n+2}-reflecting on S^n_κ (by 2.2), so we can take $\alpha \in S^n_\kappa$ with $L_\alpha \models \phi$. Define $f: \omega \longrightarrow S^n_\kappa$ as follows:

$$f(0)=\alpha$$

$$f(m+1)= \text{the least } \beta \text{ such that } \beta>f(m) \ \& \ \beta \in S^n_\kappa \ \& \ L_\beta \models \phi.$$

(for $m<\omega$). (Notice that such β always exists since $\{\beta \in S^n_\kappa : L_\beta \models \phi\}$ is cofinal in κ).

Since Σ_{n+1}-recursion holds on κ (see Devlin [1974], thm. 18), we find that f is $\Sigma_{n+1}L_\kappa$. Put $\gamma = \bigcup_{m<\omega} f(m)$.

Claim 1: $\gamma<\kappa$.

Proof: Since κ is Σ_{n+1}-admissible, there is no $f: \omega \xrightarrow{cf, \Sigma_{n+1}} \kappa$ (Kranakis [1980], II.1.6.a, from Devlin [1974], thm. 40). Therefore, ran(f) is bounded in κ. ◻

Claim 2: $\gamma \in S^n_\kappa$ and $L_\gamma \models \phi$.

Proof: by 2.10. ◻

Claim 3: γ is not Σ_{n+1}-admissible.

Proof: Let ψ be a $\Sigma_{n+1}L_\gamma$-formula such that $L_\gamma \models \psi(m,\beta) \iff$

$\iff \exists\alpha_0,\ldots,\alpha_m\ [\alpha=\alpha_0<\alpha_1<\ldots<\alpha_m=\beta<\gamma\ \&\ \forall i\leq m\ (\alpha_i \in S^n_\gamma\ \&\ L_{\alpha_i} \models \phi)]$.

Then $L_\gamma \models \forall m<\omega\ \exists\beta\ \psi(m,\beta)$, but

$L_\gamma \models \forall\delta\ \exists m<\omega\ \forall\beta<\delta\ \neg\psi(m,\beta)$. ◻

Combining the claims gives that κ is Π_{n+2}-reflecting on $\{\alpha \in S^n_\kappa : \alpha \text{ is not } \Sigma_{n+1}\text{-admissible}\}$.

2.19 Remark

If there is a Π_n-normal filter on κ, then $N = \bigcap\{F : F \text{ is a } \Pi_n\text{-normal filter on } \kappa\}$ is the "least" Π_n-normal filter on κ, and will play the role H plays for the Π_n-filters. We found $F_{n+1} \subsetneq N$ by 2.14 plus 2.18.

2.20 Example

There is a Π_{n+3} sentence ϕ such that for any ordinal α, $L_\alpha \models \phi \iff \alpha$ is Σ_{n+1}-admissible (see 2.17).

Thus we can take $\psi \in \Sigma_{n+2}$ such that

$L_\alpha \models \forall\xi\ \psi(\xi) \iff \alpha$ is Σ_{n+1}-admissible. Now let κ be Σ_{n+1}-admissible.

Define $f: \underset{\sim}{(}\kappa \xrightarrow{\Delta_1} \kappa$ by

$f(\alpha) \simeq \beta \iff \beta<\alpha\ \&\ L_\alpha \models \neg\psi(\beta)\ \&\ \forall\gamma<\beta\ \psi(\gamma)$.

Then f is regressive and $\mathrm{dom}(f) = \{\alpha<\kappa : \alpha>0\ \&\ \alpha$ is not Σ_{n+1}-admissible$\}$. We saw in 2.18 that κ is Π_{n+2}-reflecting on $\mathrm{dom}(f)$, so by 2.12 $\mathrm{dom}(f)$ is Σ_{n+1}-stationary.

Now fix $\beta<\kappa$. If $f^{-1}(\{\beta\})$ were Σ_{n+1}-stationary, then $S^n_\kappa \cap f^{-1}(\{\beta\})$ is cofinal in κ, so $\{\alpha \in S^n_\kappa : L_\alpha \models \neg\psi(\beta)\}$ is cofinal in κ.

But then by 2.10 $L_\kappa \models \neg\psi(\beta)$, which contradicts the Σ_{n+1}-admissibility of κ.

Therefore we must have that for all $\beta<\kappa$, $f^{-1}(\{\beta\})$ is not Σ_{n+1}-stationary. This shows that in Fodor's theorem 2.9 we cannot do without any increase in complexity.

Lastly we'll state normal analogues of 1.9:

2.21 <u>Proposition</u>

Let F be a nonprincipal proper filter on κ.

a. The following are equivalent:

  i. F is a Δ_n-normal filter on κ.

  ii. for all regressive $f: \underset{\sim}{(}\kappa \xrightarrow{\Delta_n} \kappa$ $(\kappa\text{-dom}(f) \notin F \Rightarrow$

  $\Rightarrow \exists\alpha<\kappa\ \ \kappa - f^{-1}(\{\alpha\}) \notin F)$.

b. The following are equivalent:

  i. F is a Π_n-normal filter on κ.

ii. for all regressive $f: \underline{(}\kappa \xrightarrow{\Sigma_n} \kappa$ $(\kappa\text{-dom}(f) \notin F \Rightarrow$

$\Rightarrow \exists\alpha<\kappa \;\; \kappa - f^{-1}(\{\alpha\}) \notin F)$.

Proof

As the proof of 1.9, using diagonal intersections instead of regular intersections.

CHAPTER III. ULTRAFILTERS

In this chapter we discuss Φ-ultrafilters and Φ-normal ultrafilters. In §1 we review some basic facts, in particular the connections with Σ_n-end extensions. This is based on work by Kaufmann [1981], Kranakis [1982b] and Kaufmann & Kranakis [1984].
In §2 we prove our main extension theorem (2.1 and 2.2), which says that on a *countable* ordinal, Φ-(normal) filters can be extended to Φ-(normal) ultrafilters (under easy conditions on Φ). The rest of the paragraph mainly deals with consequences of these theorems, and also gives some improvements of chapter II.

§1. Basic facts

We define "ultrapowers", give a Łoś-type theorem, and give methods to go from ultrafilter to ultrapower and back. In 1.8, we give a correct version and correct proof of a result of Kaufmann & Kranakis [1984].

1.1 Theorem (Kaufmann [1981], thm. 1; Kranakis [1982b], thm. 2.4)
The following are equivalent:

i. there is a Δ_n-ultrafilter on κ

ii. there is a Π_n-ultrafilter on κ

iii. L_κ has a Σ_{n+1}-end extension

Proof

Since the proof uses constructions we will use more often, I will give it here:

ii → i is immediate;

i → iii: If F is a Δ_n-ultrafilter on κ, define $M(F) = \langle M,E\rangle$ as follows: M consists of equivalenceclasses [f] of functions $f: \kappa \xrightarrow{\Delta_n} L_\kappa$ under the equivalencerelation given by

$f \sim g \iff \{\xi<\kappa : f(\xi)=g(\xi)\} \in F$, and

$[f]\ E\ [g] \iff \{\xi<\kappa : f(\xi) \in g(\xi)\} \in F$.

Then $L_\kappa \prec_{n+1,e} M(F)$ is a consequence of a Łoś-type theorem: for all $\phi \in \mathbb{B}_{n-1}$ and $[f_1],\ldots,[f_n] \in M$ we have

$M(F) \models \phi([f_1],\ldots,[f_n]) \iff \{\xi<\kappa : L_\kappa \models \phi(f_1(\xi),\ldots,f_n(\xi))\} \in F$.

ii → iii: If F is a Π_n-ultrafilter on κ, we define Ult$F$ = $\langle M,E\rangle$, where M consists of equivalenceclasses of functions $f: \subseteq\kappa \xrightarrow{\Sigma_n} L_\kappa$ with dom(f) $\in F$, $\sim$ and E are as before, and the Łoś theorem now holds for all $\phi \in \mathbb{B}_n$.

iii → ii: If $L_\kappa \prec_{n+1,e} M$ and $c \in \mathrm{Ord}^M - \kappa$ (such c always exists), define a Π_n-ultrafilter on κ by: $F(M,c) = \{X \subseteq \kappa :$ there is a $\phi \in \not{c}_n L_\kappa$ such that $\forall\xi<\kappa\ (L_\kappa \models \phi(\xi) \Rightarrow \xi \in X)$ and $M \models \phi(c)\}$.

1.2 <u>Theorem</u> (Kranakis [1982b], thm. 3.3)

The following are equivalent:

i. there is a Δ_n-normal ultrafilter on κ

ii. there is a Π_n-normal ultrafilter on κ

iii. L_κ has a blunt Σ_{n+1}-end extension

<u>Proof</u>

If M is a blunt end extension of L_κ, then, maybe after first doing a transitive collapse on the well-founded part of M, we can assume that $\kappa \in M$. Then $F(M,\kappa)$ is a Π_n-normal ultrafilter on κ (this is easy to check).

On the other hand, if F is a Δ_n-normal ultrafilter on κ, then $M(F)$ is blunt; and if F is a Π_n-normal ultrafilter on κ, then Ult$F$ is blunt. In each case we have that the minimal element of the new ordinals is the equivalence class of $id\restriction\kappa$. This follows from characterization II.2.21.

For theorems 1.6 and 1.8, we need lemma 1.3. The idea for 1.3 came from Kaufmann & Kranakis [1984], 2.5.

1.3 Lemma

i. Let F be a Σ_n-filter on κ, $\lambda<\kappa$ and $\langle X_\alpha : \alpha<\lambda\rangle \in {}^\lambda F \cap \Pi_n L_\kappa$. Then $\kappa-\bigcap_{\alpha<\lambda} X_\alpha \notin F$.

ii. Let F be a Σ_n-normal filter on κ, and $\langle X_\alpha : \alpha<\kappa\rangle \in {}^\kappa F \cap \Pi_n L_\kappa$. Then $\kappa-\Delta_{\alpha<\kappa} X_\alpha \notin F$.

Proof

As the proof is similar in both cases, we will only prove (i).

So let $\langle X_\alpha : \alpha<\lambda\rangle \in {}^\lambda F \cap \Pi_n L_\kappa$, put $X = \bigcap_{\alpha<\lambda} X_\alpha$, and suppose $\kappa-X \in F$.

Now define a $\Sigma_n L_\kappa$ relation R on κ^2 by:

$R(\xi,\alpha) \iff \alpha<\lambda \ \& \ \xi \notin X_\alpha$. Then $dom(R) = \kappa-X$.

By the Σ_n-uniformization theorem, there is a $\Sigma_n L_\kappa$ function $f: (\kappa \to \lambda$ with $dom(f) = \kappa-X$ and $\forall\xi \in dom(f) \ \ R(\xi,f(\xi)))$.

Put $Y_\alpha = (\kappa-X)-f^{-1}(\{\alpha\})$, for $\alpha<\lambda$. Then

$\xi \in Y_\alpha \iff \exists\beta<\lambda\ (\beta\neq\alpha\ \&\ f(\xi) = \beta)$, so $\langle Y_\alpha : \alpha<\lambda\rangle \in \Sigma_n L_\kappa$.
Now $Y_\alpha \supseteq X_\alpha - X \in F$, so, since F is a Σ_n-filter, $\emptyset = \bigcap_{\alpha<\lambda} Y_\alpha \in F$, contradiction.

1.4 Corollary (Kranakis [1982b], 4.4)

i. Each Σ_n-ultrafilter is a Π_n-ultrafilter

ii. Each Σ_n-normal ultrafilter is a Π_n-normal ultrafilter.

1.5 Corollary

Let F be a Σ_n-normal filter on κ, and $X \in F_{n+1}$ (as defined in II.2.6). Then $\kappa - X \notin F$.

Proof

If $X \in F_{n+1}$, then there is a sequence $\langle X_\alpha : \alpha<\kappa\rangle \in {}^\kappa H \cap \Pi_n L_\kappa$ such that $\Delta_{\alpha<\kappa} X_\alpha = X$. This follows from II.2.14: if C is a Π_n-cub with $C \subseteq X$, let $\xi \in X_\alpha \iff \exists\gamma<\xi\ (\gamma>\alpha\ \&\ \gamma \in C) \vee \xi \in X$.
The proof is done if we note that $H \subseteq F$ (II.1.2).

1.6 Theorem (Kranakis [1982b], 4.7)

The following are equivalent:

i. there is a Σ_n-ultrafilter on κ

ii. there is a ϕ_n-ultrafilter on κ

iii. L_κ has a Σ_{n+1}-end extension satisfying κ-Σ_n-collection

Proof

It follows easily from 1.4, that if F is a Σ_n-ultrafilter on κ, then F is a ϕ_n-ultrafilter. Next, if F is a Σ_n-ultrafilter, then

$\mathrm{Ult}\mathcal{F} \models \kappa\text{-}\Sigma_n$-collection, and if $\mathcal{M}$ is such that

$L_\kappa \prec_{n+1,e} \mathcal{M} \models \kappa\text{-}\Sigma_n$-collection, and $c \in \mathrm{Ord}^{\mathcal{M}}-\kappa$, then $\mathcal{F}(\mathcal{M},c)$ is a Σ_n-ultrafilter on κ.

1.7 Theorem (Kranakis [1982b], 4.7)

The following are equivalent:

i. there is a Σ_n-normal ultrafilter on κ

ii. there is a $\not{\mathbb{C}}_n$-normal ultrafilter on κ

iii. L_κ has a blunt Σ_{n+1}-end extension satisfying $(\kappa+1)\text{-}\Sigma_n$-collection.

Proof

As 1.6.

Kaufmann & Kranakis [1984], 5.10 states:

If $\mathcal{F}$ is a Σ_n-normal filter on κ, then $\{\beta \in S^n_\kappa : \beta$ has a Π_n-normal ultrafilter$\} \in \mathcal{F}$.

However, their proof uses the unproven assumption $S^n_\kappa \in \mathcal{F}$.

It is not clear whether this can be proved in general (for a proof under the assumption that κ is countable, see §2), so we cannot use it. Then a slightly weaker version of this theorem still holds, which was first observed by I.Phillips. A version of his result is given here.

1.8 Theorem

Let there be a Σ_n-normal filter on κ.

Then $\{\beta \in S^n_\kappa : \beta$ has a Π_n-normal ultrafilter$\}$ is cofinal in κ.

Proof

Let F be a Σ_n-normal filter on κ. Note $S_\kappa^{n-1} \in F$ by II.2.14 (since F is a Π_{n-1}-normal filter) and $\kappa - S_\kappa^n \notin F$ by 1.5.

Define $I = \{\beta<\kappa$: for all Π_n-formulas ϕ with parameters from L_β we have $L_\kappa \models \phi(\beta) \Rightarrow \exists\gamma<\beta \;\; L_\beta \models \phi(\gamma)\}$ (this is related to the invisibility of β on κ, see Kranakis [1980] or Phillips [1983]).

Claim 1: $I \in F$.

Proof: Enumerate the Π_n formulas with parameters from L_κ and one free variable in a sequence $<\phi_\delta : \delta<\kappa>$. Clearly, the function $\delta \mapsto \phi_\delta$ can be chosen in $\Sigma_1 L_\kappa$, so $S = \{\alpha<\kappa : \phi_\delta \in L_\alpha$ for all ϕ_δ with parameters from $L_\alpha\} \in F$.

Then define $<T_\delta : \delta<\kappa> \in \Sigma_n L_\kappa$ by:

$\beta \in T_\delta \iff L_\kappa \models \phi_\delta(\beta) \rightarrow \exists\gamma<\beta \;\; L_\beta \models \phi_\delta(\gamma)$.

Then $\Delta_{\delta<\kappa} T_\delta \cap S \subseteq I$, so we are done if we show $T_\delta \in F$ for $\delta<\kappa$.

So fix $\delta<\kappa$. If $L_\kappa \models \forall\xi \neg \phi_\delta(\xi)$, we are done.

Otherwise, take $\beta_0<\kappa$ such that $L_\kappa \models \phi_\delta(\beta_0)$.

Now if $\beta>\beta_0$ and $\beta \in S_\kappa^{n-1}$, then $L_\beta \models \phi_\delta(\beta_0)$ (for ϕ_δ is Π_n), so $\exists\gamma<\beta \;\; L_\beta \models \phi_\delta(\gamma)$. Therefore $T_\delta \supseteq S_\kappa^{n-1} - (\beta_0+1)$, so $T_\delta \in F$. ⌑

Claim 2: $I \cap S_\kappa^n$ is cofinal in κ.

Proof: Suppose not, then there is a $\beta_0<\kappa$ such that $I - \beta_0 \subseteq (\kappa - S_\kappa^n) - \beta_0$. But then $\kappa - S_\kappa^n \in F$, contradiction. ⌑

Our proof is finished if we show

$\beta \in I \cap S_\kappa^n \Rightarrow \beta$ has a Π_n-normal ultrafilter,

or, equivalently (by 1.2)

$\beta \in I \cap S_\kappa^n \Rightarrow \beta$ has a blunt Σ_{n+1}-end extension.

So fix $\beta \in I \cap S_\kappa^n$. Let Φ be the set of all finite conjunctions of

Π_{n+1} formulas with parameters from L_β such that $L_\beta \models \phi$.

Of course $\Phi \in L_\kappa$.

Claim 3: $\forall\phi \in \Phi\ \exists\xi>\beta\ (\xi \in S_\kappa^{n-1}\ \&\ L_\xi \models \phi)$.

Proof: suppose not, so take $\phi \in \Phi$ such that

$L_\kappa \models \forall\xi>\beta\ (\psi(\xi) \rightarrow L_\xi \models\neg\phi)$, (*)

where ψ is a Π_{n-1}-formula such that $L_\kappa \models \psi(\xi) \iff L_\xi \prec_{n-1} L_\kappa$.

(*) is a Π_n-formula, so by definition of I there is a $\gamma_0<\beta$ such that

$L_\beta \models \forall\xi>\gamma_0\ (\psi(\xi) \rightarrow L_\xi \models\neg\phi)$.

But since $\beta \in S_\kappa^n$ we have

$L_\kappa \models \forall\xi>\gamma_0\ (\psi(\xi) \rightarrow L_\xi \models\neg\phi)$, which contradicts

$L_\kappa \models \psi(\beta)\ \&\ L_\beta \models \phi$. ⌑

Now for each $\phi \in \Phi$, let $\alpha(\phi)$ = the least $\alpha>\beta$ such that $\alpha \in S_\kappa^{n-1}$ and $L_\alpha \models \phi$. Since $\phi \mapsto \alpha(\phi)$ is $\Sigma_n L_\kappa$, and κ is Σ_{n+1}-admissible (by II.1.4), we have $\alpha = \sup\{\alpha(\phi) : \phi \in \Phi\} < \kappa$.

Since S_κ^{n-1} is closed, $L_\alpha \prec_{n-1} L_\kappa$. Our proof is finished if we show

Claim 4: $L_\beta \prec_{n+1} L_\alpha$.

Proof: let ϕ be a Π_{n+1} formula with parameters from L_β such that $L_\beta \models \phi$ but $L_\alpha \models\neg\phi$. Write $\neg\phi$ as $\exists x\ \psi(x)$ for some $\psi \in \Pi_n L_\beta$.

Take $u \in L_\alpha$ with $L_\alpha \models \psi(u)$. By definition of α, there is a $\theta \in \Phi$ with $u \in L_{\alpha(\theta)}$. But now $\theta\ \&\ \phi \in \Phi$ and $\gamma = \alpha(\theta\ \&\ \phi) \geq \alpha(\theta)$ and $L_\gamma \models \theta\ \&\ \phi$. Since $L_\gamma \prec_{n-1} L_\alpha$ and $\psi(u)$ is Π_n we must have $L_\gamma \models \psi(u)$, but this contradicts $L_\gamma \models \forall x\neg\psi(x)$. ⌑

1.9 Corollary

If κ is Σ_{n+2}-admissible, then

$\{\alpha<\kappa : \alpha$ has a Σ_n-normal ultrafilter$\}$ is cofinal in κ.

Proof:

If κ is Σ_{n+2}-admissible, then S_κ^{n+1} is cofinal in κ (Kranakis [1980] or [1982a]), and if $\alpha \in S_\kappa^{n+1}$, then $L_\alpha \prec_{n+1,e}^{blunt} L_\kappa \models \Sigma_n$-collection, so by 1.7 we have that $\mathcal{F}(L_\kappa,\alpha)$ is a Σ_n-normal ultrafilter on α.

1.10 Example

By 1.8, there is an ordinal κ that has a Π_n-normal ultrafilter, but no Σ_n-normal filter. If $\mathcal{F}$ is a Π_n-normal ultrafilter on such a κ, then $\mathcal{F}$ has the property (*) that for all regressive

$f: (\kappa \xrightarrow{\Pi_n} \kappa \ (\kappa\text{-dom}(f) \notin \mathcal{F} \Rightarrow \exists\alpha<\kappa \ \ \kappa\text{-}f^{-1}(\{\alpha\}) \notin \mathcal{F})$,

by II.1.11 and II.2.21, but $\mathcal{F}$ is not a Σ_n-normal filter.

§2. Extending filters to ultrafilters

In this paragraph we prove two extension theorems: on a countable ordinal, we can extend each Φ-filter to a Φ-ultrafilter (2.2) and each Φ-normal filter to a Φ-normal ultrafilter (2.1), if Φ satisfies some easy conditions (which are satisfied by Δ_n, Σ_n and Π_n).

As a corollary, 2.4 follows, a result that was known before. In fact, the basic idea for 2.1 and 2.2 comes from Kranakis' proof of 2.4 (see note 2.5).

2.6 through 2.17 deal with consequences of 2.2, and the rest of the paragraph with consequences of 2.1.

IMPORTANT NOTE: Throughout this paragraph, we assume that κ is a *countable* ordinal.

2.1 Theorem

Let Φ be a set of $\in$-formulas, or $\Phi=\Delta_n$, such that ΦL_κ is closed under disjunction and bounded universal quantification, and $\{\kappa-\{\alpha\} : \alpha<\kappa\} \subseteq \Phi L_\kappa$.

Let G be a Φ-normal filter on κ.

Let $X \subseteq \kappa$ be such that $\kappa-X \in \Phi L_\kappa$ and $\kappa-X \notin G$.

Then there is a Φ-normal ultrafilter F on κ with $X \in F$ and $G \cap \Phi L_\kappa \subseteq F$.

Proof

Let Φ, $\mathcal{G}$, and X be as in the statement of the theorem. Enumerate all $\phi \in \Phi L_\kappa$ with two free variables in a sequence $\langle \phi_m : 1 \leq m < \omega \rangle$ such that each formula occurs infinitely many times in the list (note that this is the only place where we use the countability of κ). By induction, we will define a sequence $\langle Z_m : m<\omega \rangle$ of sets such that $\kappa \supseteq Z_0 \supseteq Z_1 \supseteq \ldots$ and $\kappa - Z_m \in \Phi L_\kappa - \mathcal{G}$ for $m<\omega$.

Put $Z_0 = X$. Now suppose $Z_0 \supseteq \ldots \supseteq Z_m$ have been defined, and $\kappa - Z_m \in \Phi L_\kappa - \mathcal{G}$. To define Z_{m+1} we look at ϕ_{m+1}.

For $\alpha<\kappa$ define $X_\alpha = \{\xi<\kappa : L_\kappa \models \phi_{m+1}(\xi,\alpha)\}$. Note that $\langle X_\alpha : \alpha<\kappa \rangle \in \Phi L_\kappa$.

case 1: $Z_m \cap \underset{\alpha<\kappa}{\Delta} X_\alpha \neq \emptyset$. Put $Z_{m+1} = Z_m$. Then $\kappa - Z_{m+1} \in \Phi L_\kappa - \mathcal{G}$ and $Z_m \supseteq Z_{m+1}$ are obvious.

case 2: $Z_m \cap \underset{\alpha<\kappa}{\Delta} X_\alpha = \emptyset$.

claim: $\exists \alpha<\kappa \quad (\kappa - Z_m) \cup X_\alpha \notin \mathcal{G}$.

proof: otherwise $\langle (\kappa - Z_m) \cup X_\alpha : \alpha<\kappa \rangle \in {}^{\kappa}\mathcal{G} \cap \Phi L_\kappa$, so $\kappa - Z_m = (\kappa - Z_m) \cup \underset{\alpha<\kappa}{\Delta} X_\alpha = \underset{\alpha<\kappa}{\Delta}((\kappa - Z_m) \cup X_\alpha) \in \mathcal{G}$, contradiction. □

In this case we put $Z_{m+1} = Z_m - X_\alpha$, with α as in the claim. Then $Z_m \supseteq Z_{m+1}$, and $\kappa - Z_{m+1} = \kappa - (Z_m - X_\alpha) = (\kappa - Z_m) \cup X_\alpha \in \Phi L_\kappa - \mathcal{G}$.

Next we will define $\mathcal{F}$. First we define two subsets of $\mathcal{F}$ by:

$\mathcal{F}_1 = \{A \in \neg\Phi L_\kappa : \exists m<\omega \quad Z_m \subseteq A\}$;

$\mathcal{F}_2 = \{B \in \Phi L_\kappa : \forall m<\omega \quad Z_m \cap B \neq \emptyset\}$.

Then we define $\mathcal{F}$ by:

$\mathcal{F} = \{X \subseteq \kappa : \exists A \in \mathcal{F}_1 \ \exists B \in \mathcal{F}_2 \ (A \subseteq X \vee B \subseteq X \vee A \cap B \subseteq X)\}$.

<u>Claim 1</u>: A_1 , $A_2 \in F_1 \Rightarrow A_1 \cap A_2 \in F_1$.

<u>Proof</u>: Since ΦL_κ is closed under disjunction, $\neg\Phi L_\kappa$ is closed under conjunction, so $A_1 \cap A_2 \in \neg\Phi L_\kappa$.

By definition of F_1, there are m_1 , $m_2<\omega$ such that $Z_{m_1} \subseteq A_1$, $Z_{m_2} \subseteq A_2$. Take $m_0 = \max\{m_1, m_2\}$, then $Z_{m_0} \subseteq A_1 \cap A_2$. ⌑

<u>Claim 2</u>: B_1, $B_2 \in F_2 \Rightarrow B_1 \cap B_2 \in F_2$.

<u>Proof</u>: This is a simple case of claim 7. ⌑

<u>Claim 3</u>: $\kappa \in F$, $\emptyset \notin F$.

<u>Proof</u>: Since $X \in \neg\Phi L_\kappa$ and $Z_0 = X$, we have $X \in F_1$. Since $X \subseteq \kappa$, $\kappa \in F$. Suppose $A \in F_1$, $B \in F_2$. Then $A \neq \emptyset$, since all the $Z_m \neq \emptyset$ (for $\kappa - Z_m \notin G$), and also $B \neq \emptyset$, which is obvious from the definition of F_2. If m is such that $Z_m \subseteq A$, then $Z_m \cap B \neq \emptyset$, so $A \cap B \neq \emptyset$. Therefore $\emptyset \notin F$. ⌑

It follows from Claim 1 - 3 that F is a proper filter on κ.

<u>Claim 4</u>: If $B \in \Phi L_\kappa - F$, then $\kappa - B \in F$.

<u>Proof</u>: Let $B \in \Phi L_\kappa - F$. Then $B \notin F_2$, so there is an $m<\omega$ with $Z_m \cap B = \emptyset$. But that means $Z_m \subseteq \kappa - B$, so $\kappa - B \in F_1 \subseteq F$. ⌑

<u>Claim 5</u>: $X \in F$ and $G \cap \Phi L_\kappa \subseteq F$.

<u>Proof</u>: $X = Z_0 \in F_1 \subseteq F$. If $B \in \Phi L_\kappa$, but $B \notin F$, then, as we saw in claim 4, there is an $m<\omega$ sith $Z_m \subseteq \kappa - B$. But that means $B \subseteq \kappa - Z_m \notin G$, so $B \notin G$. Therefore $G \cap \Phi L_\kappa \subseteq F$. ⌑

<u>Claim 6</u>: F is nonprincipal.

<u>Proof</u>: G is nonprincipal, so $\{\kappa - \{\alpha\} : \alpha<\kappa\} \subseteq G \cap \Phi L_\kappa \subseteq F$. ⌑

<u>Claim 7</u>: If $\langle X_\alpha : \alpha<\kappa\rangle \in \Phi L_\kappa \cap {}^\kappa F$, then $\Delta_{\alpha<\kappa} X_\alpha \in F$.

<u>Proof</u>: Suppose $\langle X_\alpha : \alpha<\kappa\rangle \in \Phi L_\kappa$, but $\Delta_{\alpha<\kappa} X_\alpha \notin F$. By assumption

$\underset{\alpha<\kappa}{\Delta} X_\alpha \in \Phi L_\kappa$, so there is an $m_0<\omega$ such that $Z_{m_0} \subseteq \kappa - \underset{\alpha<\kappa}{\Delta} X_\alpha$.

Take $\phi \in \Phi L_\kappa$ such that $\xi \in X_\alpha \iff L_\kappa \models \phi(\xi,\alpha)$. Since this ϕ occurs infinitely times in the list $\langle\phi_m : 1\leq m<\omega\rangle$, it occurs with index $k+1>m_0$. Then $Z_k \subseteq Z_{m_0} \subseteq \kappa - \underset{\alpha<\kappa}{\Delta} X_\alpha$, so there is an $\alpha<\kappa$ with $Z_{k+1} = Z_k - X_\alpha$. Thus $Z_{k+1} \cap X_\alpha = \emptyset$, so $X_\alpha \notin F$. ◻

We have proved that F is a Φ-normal ultrafilter on κ with $X \in F$ and $G \cap \Phi L_\kappa \subseteq F$.

2.2 <u>Theorem</u>

Let Φ be a set of $\in$-formulas, or $\Phi=\Delta_n$, such that ΦL_κ is closed under disjunction and bounded universal quantification, and $\{\kappa-\{\alpha\} : \alpha<\kappa\} \subseteq \Phi L_\kappa$. Let G be a Φ-filter on κ.

Let $X \subseteq \kappa$ be such that $\kappa - X \in \Phi L_\kappa - G$.

Then there is a Φ-ultrafilter on κ with $X \in F$ and $G \cap \Phi L_\kappa \subseteq F$.

<u>Proof</u>

Enumerate all pairs $\langle\phi,\lambda\rangle$, with $\phi \in \Phi L_\kappa$, having two free variables, and $\lambda<\kappa$, in a sequence $\langle\langle\phi_m,\lambda_m\rangle : 1\leq m<\omega\rangle$, such that each pair $\langle\phi,\lambda\rangle$ occurs infinitely many times in the list.

Again, we define a descending sequence $\langle Z_m : m<\omega\rangle$ such that $\kappa - Z_m \in \Phi L_\kappa - G$ for $m<\omega$. We put $Z_0 = X$, and if $Z_0 \supseteq \ldots \supseteq Z_m$ are defined, we set $X_\alpha = \{\xi<\kappa : L_\kappa \models \phi_{m+1}(\xi,\alpha)\}$ for $\alpha<\lambda_{m+1}$.

If $Z_m \cap \bigcap_{\alpha<\lambda_{m+1}} X_\alpha \neq \emptyset$, we put $Z_{m+1} = Z_m$, and otherwise we put $Z_{m+1} = Z_m - X_\alpha$, where $\alpha<\lambda_{m+1}$ is such that $(\kappa - Z_m) \cup X_\alpha \notin G$.

We define F as in 2.1, and we will have that F is a Φ-ultrafilter on κ with $X \in F$ and $G \cap \Phi L_\kappa \subseteq F$.

2.3 Note

The assumption that κ is countable is necessary in 2.1 and 2.2, for there is a (real) normal filter on ω_1, namely the closed unbounded filter, but L_{ω_1} has no Σ_2-end extension, so by 1.1 there is no Δ_1-ultrafilter on L_{ω_1}. (See Kranakis [1982b], 2.10.)

Now we will consider consequences of 2.2. In 2.4, we take $\Phi = \Delta_n$ and $\Phi = \Pi_n$; in 2.6 we take $\Phi = \Sigma_n$. In 2.7 and 2.8 we see what happens when $\Phi = \Delta_n$ and $\mathcal{G} = \mathcal{H}$ (as defined in II.2.1); in 2.9 we have the case that $\Phi = \Pi_n$ or $\Phi = \Sigma_n$ and $\mathcal{G} = \mathcal{H}$. This leads us again to consider the difference between the Π_n-filters $\mathcal{H}$ and $\mathcal{D}$ (as defined in II.1.12). We do this in 2.11 to 2.17; 2.13 states a theorem for the Π_n-case, while 2.14 - 2.17 consider the Σ_n-case.

2.4 Corollary

The following are equivalent:

i. κ is Σ_{n+1}-admissible

ii. there is a Δ_n-ultrafilter on κ

iii. there is a Π_n-ultrafilter on κ

Proof

By II.1.4, any of the statements (i), (ii), (iii) implies that κ is Σ_{n+1}-admissible. Then, by I.3.1, $\Sigma_n L_\kappa$ is closed under bounded universal quantification, so we can take $\Phi = \Sigma_n$, Π_n or Δ_n in 2.1 or 2.2. The corollary then follows from II.1.4.

2.5 Note

The equivalence of i and ii in 2.4 was first proved by Kaufmann [1981] and the equivalence with iii was shown by Kranakis [1982b], using a construction like the one in 2.2. The basic idea for this construction comes from MacDowell & Specker [1961].

2.6 Corollary

The following are equivalent:

i. there is a Σ_n-filter on κ

ii. there is a Σ_n-ultrafilter on κ

iii. there is a $¢_n$-filter on κ

iv. there is a $¢_n$-ultrafilter on κ

Proof

i $\Longleftrightarrow$ ii and iii $\Longleftrightarrow$ iv by 2.2; iv $\Longrightarrow$ ii is immediate and ii $\Longrightarrow$ iv by 1.4.

2.7 Theorem

The following are equivalent:

i. κ is Σ_{n+1}-admissible

ii. $\bigcap\{\mathcal{F} : \mathcal{F}$ is a Δ_n-ultrafilter on $\kappa\} = \mathcal{H}$

Proof

ii $\rightarrow$ i: use II.1.4.

i $\rightarrow$ ii: If κ is Σ_{n+1}-admissible, then $\mathcal{H}$ is a Δ_n-filter on κ. Let $X \in \Delta_n L_\kappa$, and $X \notin \mathcal{H}$, then by 2.2 there is a Δ_n-ultrafilter $\mathcal{F}$ on κ with $\kappa - X \in \mathcal{F}$, so $X \notin \mathcal{F}$. By II.1.2 $\mathcal{H} \subseteq \mathcal{F}$ for each Δ_n-ultrafilter $\mathcal{F}$ on κ. Finally, observing that if $\mathcal{F}$ is a Δ_n-(ultra)filter on κ, then so is $\{X \subseteq \kappa : \exists Y \subseteq X \ \ (Y \in \Delta_n L_\kappa \cap \mathcal{F})\}$, gives ii.

2.8 Corollary

Let κ be Σ_{n+1}-admissible, $X \in \Delta_n L_\kappa$.

The following are equivalent:

i. there is a Δ_n-ultrafilter F on κ with $X \in F$

ii. X is cofinal in κ

Proof

X is cofinal in $\kappa \iff \kappa - X \notin H$.

Theorem 2.7 tells us, that whenever H is a Δ_n-filter, it is the intersection of the Δ_n-ultrafilters. Theorem 2.9 will show that this situation also occurs in the Σ_n-case: whenever H is a Σ_n-filter, it is the intersection of the Σ_n-ultrafilters (using II.1.7). However, 2.9 also shows that this is not the case for Π_n: H is a Π_n-filter iff κ is Σ_{n+1}-admissible, but H is the intersection of the Π_n-ultrafilters iff κ is Σ_{n+2}-admissible.

2.9 Theorem

The following are equivalent:

i. κ is Σ_{n+2}-admissible

ii. $\bigcap\{F : F \text{ is a } \Pi_n\text{-ultrafilter on } \kappa\} = H$

iii. $\bigcap\{F : F \text{ is a } \Sigma_n\text{-ultrafilter on } \kappa\} = H$

Proof

i $\rightarrow$ iii: since each Δ_{n+1}-ultrafilter is a Σ_n-ultrafilter, this follows from 2.7.

iii $\rightarrow$ ii: each Σ_n-ultrafilter is a Π_n-ultrafilter (by 1.4).

ii → i: Suppose κ is not Σ_{n+2}-admissible. Then by II.1.6, there is a f: $\underset{\sim}{\subset}\kappa \xrightarrow{\Pi_n} \lambda$, for some $\lambda<\kappa$, such that dom(f) is cofinal in κ, but each $f^{-1}(\{\alpha\})$ for $\alpha<\lambda$ is bounded. Now let F be a Π_n-ultrafilter on κ. If dom(f) $\in F$, then $\langle \mathrm{dom}(f)-f^{-1}(\{\alpha\}) : \alpha<\lambda\rangle \in \Pi_n L_\kappa \cap {}^\lambda F$ (for $\xi \in \mathrm{dom}(f)-f^{-1}(\{\alpha\}) \iff \exists\beta<\lambda\ (\beta\neq\alpha\ \&\ f(\xi) = \beta)$), so $\emptyset = \bigcap_{\alpha<\lambda}(\mathrm{dom}(f)-f^{-1}(\{\alpha\})) \in F$, contradiction.

Therefore we must have κ-dom(f) $\in F$, and since F was chosen arbitrarily, $(\kappa\text{-dom}(f)) \in \bigcap\{F : F \text{ is a } \Pi_n\text{-ultrafilter on } \kappa\}-H$.

2.10 Notes

i. It follows from 2.9, that the X in theorem 2.1 or 2.2 cannot always be chosen in ΦL_κ.

ii. If κ is Σ_{n+2}-admissible and $X \in \mathbb{B}_n L_\kappa$, then we have:
X is cofinal in $\kappa \iff$ there is a Φ_n-ultrafilter F on κ with $X \in F$.
(this follows from 2.9.iii, since each Σ_n-ultrafilter is a Φ_n-ultrafilter).

2.9 raises the following question. If κ is Σ_{n+1}-admissible, but not Σ_{n+2}-admissible, how large can the intersection of the Π_n-ultrafilters be, and how large can the intersection of the Σ_n-ultrafilters be (when and if they exist)?
This problem is considered in 2.13 and 2.14. It is necessary though, to throw away some unwanted sets that might stray in. That is formulated in 2.11 and 2.12.

2.11 <u>Definition</u>

Let $\mathcal{G} \subseteq \mathcal{P}\kappa$. Define $\mathcal{G}^{min} = \{X \subseteq \kappa : \exists Y \in \mathcal{G} \cap ¢_n L_\kappa \;\; Y \subseteq X\}$.

2.12 <u>Lemma</u>

Let $\mathcal{G}$ be a Π_n-filter (respectively a Σ_n-filter, $¢_n$-filter, Π_n-normal filter, Σ_n-normal filter, $¢_n$-normal filter) on κ.
Then $\mathcal{G}^{min}$ is a Π_n-filter (respectively a Σ_n-filter, $¢_n$-filter, Π_n-normal filter, Σ_n-normal filter, $¢_n$-normal filter) on κ.
<u>Proof</u>: easy.

2.13 <u>Theorem</u>

Let κ be Σ_{n+1}-admissible. Then
$\bigcap\{\mathcal{F} : \mathcal{F}$ is a Π_n-ultrafilter on $\kappa\}^{min} \subseteq \mathcal{D}$
(where $\mathcal{D}$ is as defined in II.1.12).

<u>Proof</u>

Put $\mathcal{G} = \bigcap\{\mathcal{F} : \mathcal{F}$ is a Π_n-ultrafilter on $\kappa\}$. Then $\mathcal{G}$ is a Π_n-filter on κ. By definition 2.11, it is enough to show $\mathcal{G} \cap ¢_n L_\kappa \subseteq \mathcal{D}$.
If $X \in \Pi_n L_\kappa$ and $\kappa - X$ is cofinal in κ, then by 2.2 there is a Π_n-ultrafilter $\mathcal{F}$ on κ with $\kappa - X \in \mathcal{F}$ (using the Π_n-filter $\mathcal{H}$, see II.1.4). Therefore $\mathcal{G} \cap \Pi_n L_\kappa = \mathcal{H} \cap \Pi_n L_\kappa = \mathcal{D} \cap \Pi_n L_\kappa$ (II.1.13).
Now let $X \in \Sigma_n L_\kappa$. If $\kappa - X$ contains a cofinal $\Delta_n L_\kappa$ set Y, then there is a Π_n-ultrafilter $\mathcal{F}$ on κ with $Y \in \mathcal{F}$, so also $\kappa - X \in \mathcal{F}$.
Therefore, if $X \in \Sigma_n L_\kappa \cap \mathcal{G}$, then $\kappa - X$ contains no cofinal $\Delta_n L_\kappa$ set, which means $X \in \mathcal{D}$ by II.1.11. Then $\mathcal{G} \cap ¢_n L_\kappa \subseteq \mathcal{D}$ follows.

The following theorem was suggested by I. Phillips.

2.14 Theorem

Let there be a Σ_n-filter on κ. Then

$\bigcap\{F : F \text{ is a } \Sigma_n\text{-ultrafilter on } \kappa\}^{\min} \subseteq \mathcal{D}$.

Proof

Take $X \in \Pi_n L_\kappa$, and suppose $\kappa - X$ is cofinal in κ. Take $\phi \in \Pi_n L_\kappa$ with $\alpha \in X \iff L_\kappa \models \phi(\alpha)$. Since there is a Σ_n-ultrafilter on κ, there is an M such that $L_\kappa \prec_{n+1,e} M \models \kappa\text{-}\Sigma_n$-collection by 1.6. We have $L_\kappa \models \forall\alpha\ \exists\beta>\alpha\ \neg\phi(\beta)$, so $M \models \forall\alpha\ \exists\beta>\alpha\ \neg\phi(\beta)$. Therefore, we can take $\beta \in \mathrm{Ord}^M - \kappa$ with $M \models \neg\phi(\beta)$. But then $\kappa - X \in F(M,\beta)$, a Σ_n-ultrafilter. Therefore we have

$\bigcap\{F : F \text{ is a } \Sigma_n\text{-ultrafilter on } \kappa\} \cap \Pi_n L_\kappa = H \cap \Pi_n L_\kappa = \mathcal{D} \cap \Pi_n L_\kappa$.

We finish the proof as in 2.13.

2.15 Corollary

Let there be a Σ_n-filter on κ. Let $\lambda<\kappa$ and $\langle X_\alpha : \alpha<\lambda\rangle \in {}^\lambda H \cap \Sigma_n L_\kappa$. Then $\bigcap_{\alpha<\lambda} X_\alpha \in \mathcal{D}$.

2.16 Example

We do not necessarily have equality in 2.13 or 2.14, even if κ is not Σ_{n+2}-admissible. For if there is a Π_n-normal filter on κ (which occurs below the least Σ_{n+2}-admissible), then $\kappa - S^n_\kappa \in \mathcal{D} - \bigcap\{F : F \text{ is a } \Pi_n\text{-ultrafilter on } \kappa\}$.

For we have $\kappa\text{-}S^n_\kappa \notin \bigcap\{F : F \text{ is a } \Pi_n\text{-ultrafilter on } \kappa\}$,
because, if there is a Π_n-normal filter on κ, it contains S^n_κ by II.2.14, so by 2.2 there is a Π_n-normal ultrafilter on κ containing S^n_κ.
On the other hand, we have $\kappa\text{-}S^n_\kappa \in \mathcal{D}$, because S^n_κ does not contain a cofinal $\Delta_n L_\kappa$ subset (if X were a cofinal $\Delta_n L_\kappa$ subset of S^n_κ, we'd have $\xi \in S^n_\kappa \iff \exists\alpha \in X\ (\alpha>\xi \;\&\; \xi \in S^n_\alpha)$,
and S^n_κ would be $\Delta_n L_\kappa$, a contradiction).

Theorem 2.17 gives an explicit description of a subset of $\mathcal{P}\kappa$, which is a Σ_n-filter on κ whenever one exists. In fact, this Σ_n-filter is also a $\cancel{c}_n$-filter, and it is the least Σ_n-filter and $\cancel{c}_n$-filter on κ, by which I mean that it is included in every Σ_n-filter and every $\cancel{c}_n$-filter on κ.
Therefore, it plays the same role for the Σ_n- and $\cancel{c}_n$-filters as $\mathcal{H}$ plays for the Δ_n- and Π_n-filters.

2.17 <u>Theorem</u>

Let there be a Σ_n-filter on κ. Then
$\bigcap\{F : F \text{ is a } \Sigma_n\text{-ultrafilter on } \kappa\}^{min}$ is the least Σ_n-filter and least $\cancel{c}_n$-filter on κ.

<u>Proof</u>

Put $\mathcal{G} = \bigcap\{F : F \text{ is a } \Sigma_n\text{-ultrafilter on } \kappa\}$. Since each Σ_n-ultrafilter is a $\cancel{c}_n$-ultrafilter, $\mathcal{G}$ is a $\cancel{c}_n$-filter on κ, so $\mathcal{G}^{min}$ is a $\cancel{c}_n$-filter

on κ by 2.12. Now let K be any Σ_n-filter on κ and $X \in G^{min}$.
We'll show $X \in K$. Take $Y \subseteq X$ with $Y \in G \cap ¢_n L_\kappa$. Write $Y = S \cap P$, with $S \in \Sigma_n L_\kappa$ and $P \in \Pi_n L_\kappa$. Then $P \in G \cap \Pi_n L_\kappa = H \cap \Pi_n L_\kappa$, so $P \in K$.
Suppose $S \notin K$. Applying theorem 2.2, there is a Σ_n-ultrafilter F on κ with $\kappa - S \in F$, so $S \notin G$, contradiction.
Therefore $S \in K$, and $X \supseteq Y = S \cap P \in K$.
Thus G^{min} is a subset of each Σ_n-filter on κ, so certainly a subset of each $¢_n$-filter on κ.

Now we turn to normal filters. We first prove lemma 2.18, which is based on Kaufmann & Kranakis [1984], 2.2. This allows us to get 2.19, an analogue of 2.4 for the normal case.

2.18 Lemma

Let F be a Δ_n-filter (respectively a Δ_n-normal filter, Δ_n-ultrafilter, Δ_n-normal ultrafilter) on κ.
Then there is a Π_n-filter (respectively a Π_n-normal filter, Π_n-ultrafilter, Π_n-normal ultrafilter) F^* on κ such that $F \cap \Delta_n L_\kappa \subseteq F^*$.

Proof

Take F to be a Δ_n-normal ultrafilter on κ (the proof of this case will include the proof of all other cases).
We first define two subsets of F^*:

$F_1 = \{A \in \Sigma_n L_\kappa : \exists X \in \Delta_n L_\kappa \cap F \;\; X \subseteq A\}$, and

$F_2 = \{B \in \Pi_n L_\kappa : \forall X \in \Delta_n L_\kappa \; (B \subseteq X \rightarrow X \in F)\}$.

Then we define $\mathcal{F}^* = \{X \subseteq \kappa : \exists A \in \mathcal{F}_1\ \exists B \in \mathcal{F}_2\ \ A \cap B \subseteq X\}$.

(Compare this construction with the definition of $\mathcal{F}$ in 2.1.)

Note that $\mathcal{F} \cap \Delta_n L_\kappa \subseteq \mathcal{F}_1 \cap \mathcal{F}_2 \subseteq \mathcal{F}^*$.

We will prove that $\mathcal{F}^*$ is a Π_n-normal ultrafilter on κ in a series of claims.

<u>Claim 1</u>: $A_1, A_2 \in \mathcal{F}_1 \Rightarrow A_1 \cap A_2 \in \mathcal{F}_1$, and

$B_1, B_2 \in \mathcal{F}_2 \Rightarrow B_1 \cap B_2 \in \mathcal{F}_2$.

<u>Proof</u>: The first statement follows easily from the definition of $\mathcal{F}_1$, and the second is a simple case of claim 4. ⌑

<u>Claim 2</u>: $\kappa \in \mathcal{F}^*$, $\emptyset \notin \mathcal{F}^*$, and $\mathcal{F}^*$ is nonprincipal.

<u>Proof</u>: $\kappa \in \mathcal{F}_1 \cap \mathcal{F}_2$ and, for each $\alpha<\kappa$, $\kappa-\{\alpha\} \in \mathcal{F}_1 \cap \mathcal{F}_2$, so $\kappa \in \mathcal{F}^*$ and $\mathcal{F}^*$ is nonprincipal.

Now take $X \in \mathcal{F}^*$. Then there are $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$ with $A \cap B \subseteq X$, and so there is a $D \subseteq A$ with $D \in \mathcal{F} \cap \Delta_n L_\kappa$. If $A \cap B = \emptyset$, then $\kappa-D \supseteq B$ and $\kappa-D \in \Delta_n L_\kappa - \mathcal{F}$, which is a contradiction. Therefore $A \cap B \neq \emptyset$, and so $X \neq \emptyset$. ⌑

<u>Claim 3</u>: $X \in \Pi_n L_\kappa - \mathcal{F}^* \Rightarrow \kappa-X \in \mathcal{F}^*$.

<u>Proof</u>: Suppose $X \in \Pi_n L_\kappa - \mathcal{F}^*$. Then $X \notin \mathcal{F}_2$, so there is a $D \supseteq X$, $D \in \Delta_n L_\kappa - \mathcal{F}$. Since $\mathcal{F}$ is a Δ_n-ultrafilter, $\kappa-D \in \mathcal{F}$, but then $\kappa-D \subseteq \kappa-X$ and so $\kappa-X \in \mathcal{F}_1 \subseteq \mathcal{F}^*$. ⌑

<u>Claim 4</u>: $\langle X_\alpha : \alpha<\kappa\rangle \in {}^\kappa\mathcal{F}^* \cap \Pi_n L_\kappa \Rightarrow \underset{\alpha<\kappa}{\Delta} X_\alpha \in \mathcal{F}^*$.

<u>Proof</u>: Suppose not, so $\langle X_\alpha : \alpha<\kappa\rangle \in {}^\kappa\mathcal{F}^* \cap \Pi_n L_\kappa$, but $X = \underset{\alpha<\kappa}{\Delta} X_\alpha \notin \mathcal{F}^*$. Then $X \notin \mathcal{F}_2$, so there is a $D \supseteq X$, $D \in \Delta_n L_\kappa - \mathcal{F}$.

Let $R(\xi,\alpha)$ be the $\Sigma_n L_\kappa$ relation defined by

$R(\xi,\alpha) \iff \alpha<\xi \ \& \ \xi \notin X_\alpha$, and let $f \in \Sigma_n L_\kappa$ be a Σ_n-uniformization of R. Then $f\restriction(\kappa-D)$ is a $\Delta_n L_\kappa$ relation, so $\langle S_\alpha : \alpha<\kappa\rangle \in \Delta_n L_\kappa$, if $S_\alpha = \{\xi<\kappa : \xi \notin D \ \& \ f(\xi) = \alpha\}$.

Now note that $\kappa-S_\alpha \supseteq X_\alpha$, for $\alpha<\kappa$, so $\kappa-S_\alpha \in F$.

Since F is a Δ_n-normal filter, $D = \underset{\alpha<\kappa}{\Delta}(\kappa-S_\alpha) \in F$, contradiction. ◻

The conjunction of these four claims yields the required result.

2.19 Theorem

The following are equivalent:

i. there is a Δ_n-normal filter on κ

ii. there is a Π_n-normal filter on κ

iii. there is a Δ_n-normal ultrafilter on κ

iv. there is a Π_n-normal ultrafilter on κ

Proof: 2.1 plus 2.18

2.20 Theorem

The following are equivalent:

i. there is a Σ_n-normal filter on κ

ii. there is a ϕ_n-normal filter on κ

iii. there is a Σ_n-normal ultrafilter on κ

iv. there is a ϕ_n-normal ultrafilter on κ

Proof: like 2.5 from 2.1.

2.21 Note

By 1.8, any of the statements in 2.20 is stronger than any of the statements in 2.19.

Now we'll state some more analogues of 2.7 - 2.17. Note though that not everything goes through in the normal case, because of special properties of the filter H (e.g. $H^* = H$, and $X \notin H \Longleftrightarrow \kappa - X$ is cofinal).

If Φ is the symbol Δ, Π or Σ, then we'll abbreviate $\bigcap\{F : F \text{ is a } \Phi_n\text{-normal ultrafilter on } \kappa\}$ by N_Φ.

2.22 Proposition

Let there be a Δ_n-normal filter on κ.

i. N_Δ is the least Δ_n-normal filter on κ.

ii. $N_\Pi \cap \Pi_n L_\kappa \subseteq (N_\Delta)^*$, in fact $N_\Pi \cap \Pi_n L_\kappa$ is a subset of every Π_n-normal filter on κ.

Proof

i. Obviously N_Δ is a Δ_n-normal filter on κ. Let G be any Δ_n-normal filter on κ. We'll show $N_\Delta \subseteq G$.

Note that if F is a Δ_n-normal ultrafilter on κ, then so is $\{X \subseteq \kappa : \exists Y \subseteq X \;\; Y \in F \cap \Delta_n L_\kappa\}$. Therefore it is enough to show $N_\Delta \cap \Delta_n L_\kappa \subseteq G$. So take $X \in N_\Delta \cap \Delta_n L_\kappa$. If $X \notin G$, then by 2.1 there is a Δ_n-normal ultrafilter F on κ with $\kappa - X \in F$, which is a contradiction. Therefore $X \in G$.

ii. First note that $(N_\Delta)^*$ is a Π_n-normal filter by 2.18.

If $X \in \Pi_n L_\kappa$ and $X \notin G$, with G an arbitrary Π_n-normal filter on κ, then by 2.1 there is a Π_n-normal ultrafilter F on κ with $\kappa - X \in F$, so $X \notin N_\Pi$.

2.23 Proposition

Let there be a Σ_n-normal filter on κ. Then N_Σ is a $\not{C}_n$-normal filter, and $N_\Sigma \cap \Sigma_n L_\kappa$ is a subset of each Σ_n-normal filter on κ.

Proof

Use 1.4 for the first statement, and 2.1 for the second (like in 2.22.ii).

2.24 Note

Let there be a Σ_n-normal filter on κ. Then $\{\alpha \in S^n_\kappa : \alpha$ has a Π_n-normal ultrafilter$\} \in N_\Sigma$ follows from 2.23 and 1.8.

The following theorem improves II.2.14 and II.2.15, and comes from Kaufmann & Kranakis [1984].

2.25 Theorem

i. Let F be a Δ_n-normal filter on κ. Then $\{X \in \Delta_n L_\kappa : \kappa$ is not Π^1_1-reflecting on $S^n_\kappa - X\} \subseteq F$.

ii. Let F be a Π_n-normal filter on κ. Then

$\{X \in \Pi_n L_\kappa : \kappa$ is not Π^1_1-reflecting on $S^n_\kappa - X\} \subseteq F$.

Proof

i. Let $X \in \Delta_n L_\kappa$ and let κ be not Π^1_1-reflecting on $S^n_\kappa - X$. Let F be a Δ_n-normal filter on κ. If $X \notin F$, then by 2.1 there is a Δ_n-normal ultrafilter G on κ with $X \notin G$. If $\phi \in \Sigma_n L_\kappa$ (or $\Pi_n L_\kappa$) defines X, then by the Łoś theorem $M(G) \models \neg\phi(\kappa)$. But by Kaufmann & Kranakis [1984], 4.5, that means that κ is Π^1_1-reflecting on $S^n_\kappa - X$, contradiction.

ii. Like i, using Ult$G$ instead of $M(G)$.

We finish by mentioning the following corollary of 2.2, which uses a theorem of Phillips [1983], III.3.10.

2.26 Theorem

Let F be a Π_n-filter on κ with $S^n_\kappa \in F$.

Then either (i) there is a Π_n-normal ultrafilter on κ

or (ii) κ is Σ_{n+2}-admissible.

Proof

Extend F to a Π_n-ultrafilter G by 2.2, then $S^n_\kappa \in G$. If we put $M = $ Ult$G$, we get that there is an $\alpha \in \mathrm{Ord}^M - \kappa$ such that $M \models \psi(\alpha)$, if $\psi \in \Pi_n$ defines S^n_κ. By the above-mentioned result of Phillips it follows that either $\kappa \in M$ (so $F(M,\kappa)$ is a Π_n-normal ultrafilter) or κ is Σ_{n+2}-admissible.

CHAPTER IV. DEFINABLE FILTERS

In this chapter we will investigate whether we can require that the relation "$X \in F$", where $X \in \Phi L_\kappa$ and F is a Φ-filter, is definable over L_κ. This question has connections with the definability of the homogeneous set for definable partition relations, see Kranakis & Phillips [*]. §1 contains the necessary preliminaries.

§1. Preliminaries

In this paragraph we give all definitions of concepts we will use in §2. First of all, we will formulate when a Φ-filter is Ψ-definable.

1.1 Definition

Let Φ be a set of formulas, or $\Phi=\Delta_n$.

We let $\sigma(\Phi)$ be the *Boolean algebra generated by* Φ, i.e.

i. $\Phi L_\kappa \subseteq \sigma(\Phi)L_\kappa$, and

ii. if $X,Y \in \sigma(\Phi)L_\kappa$, then $\kappa-X$, $X \cup Y$ and $X \cap Y \in \sigma(\Phi)L_\kappa$.

1.2 Examples

$\sigma(\Delta_n) = \Delta_n$; $\sigma(\Pi_n) = \sigma(\Sigma_n) = \mathbb{B}_n$.

1.3 Definition

i) Let Φ be a set of formulas, and F a Φ-filter on κ.

Let Ψ be a set of formulas, or $\Psi=\Delta_m$ (for some m).

If $\phi \in \sigma(\Phi)L_\kappa$, a formula with parameters from L_κ, we will also use the letter ϕ for an effective Gödel code of ϕ, and so we can define a subset R of L_κ by:

$$R(\phi) \iff \text{"}\phi \text{ codes a } \sigma(\Phi)L_\kappa \text{ formula"} \;\&\; \{\xi<\kappa : L_\kappa \models \phi(\xi)\} \in F.$$

We say F is Ψ-*definable* if $R \in \Psi L_\kappa$.

ii) In case $\Phi=\Delta_n$, we have to be a little more careful.

In that case, we look at pairs of $\Sigma_n L_\kappa$ and $\Pi_n L_\kappa$ formulas, and say that a Δ_n-filter F is Ψ-definable if there is a Ψ-definable subset R of $\Sigma_n L_\kappa \times \Pi_n L_\kappa$ such that:

$\forall A \in \Delta_n L_\kappa \ \forall\phi \in \Sigma_n L_\kappa \ \forall\psi \in \Pi_n L_\kappa$

if $A = \{\xi<\kappa : L_\kappa \models \phi(\xi)\} = \{\xi<\kappa : L_\kappa \models \psi(\xi)\}$,

then $R(\phi,\psi) \iff A \in F$.

1.4 Example

Let κ be Σ_{n+1}-admissible. Then H is a Σ_{n+1}-definable Δ_n-filter and a Σ_{n+2}-definable Π_n-filter on κ, since

$\{\xi<\kappa : L_\kappa \models \phi(\xi)\} \in H \iff L_\kappa \models \exists\eta \ \forall\xi\geq\eta \ \phi(\xi)$.

(In the first case, we define $R(\phi,\psi) \iff$

"ϕ codes a $\Sigma_n L_\kappa$ formula and ψ codes a $\Pi_n L_\kappa$ formula" &

& $L_\kappa \models \exists\eta \ \forall\xi\geq\eta \ \psi(\xi)$).

1.5 Definition

$T = \langle\kappa,<_T\rangle$ is a Σ_n^κ-*tree* if

i. $<_T$ is $\Sigma_n L_\kappa$,

ii. $\{<\xi,\alpha> : \xi \in T_\alpha\} \in \Sigma_n L_\kappa$, where T_α is the α^{th} level of T.

iii. $\forall\alpha<\kappa\ (\emptyset \neq T_\alpha \in L_\kappa)$.

1.6 Definition

κ has the Σ_n*-tree property* iff every Σ_n^κ-tree has a branch of length κ. For more information about the Σ_n-tree property, and connections with the (classical) tree property, see Kranakis [1980],[1982a], [1982b].

1.7 Definition

$\kappa \xrightarrow{\Sigma_n} (\kappa)_2^2$ iff for all h: $[\kappa]^2 \xrightarrow{\Sigma_n} 2$ there is an $I \subseteq \kappa$ of type κ which is homogeneous for h.

Here $[\kappa]^2 = \{<\xi,\eta> : \xi<\eta<\kappa\}$ and I is *homogeneous* means that $\exists i<2\ \forall\xi,\eta \in I\ (\xi<\eta \rightarrow h(\xi,\eta) = i)$.

1.8 Definition

If Φ is a set of formulas, or $\Phi=\Delta_n$, then $\kappa \xrightarrow{\Sigma_n} (\kappa - \Phi)_2^2$ means that each h: $[\kappa]^2 \xrightarrow{\Sigma_n} 2$ has a homogeneous set of type κ in ΦL_κ. For more information about definitions 1.7 and 1.8, and connections with weakly compact cardinals, see Kranakis [1980], Phillips [1983] or Kranakis & Phillips [*].

§2. Definable filters

In the first section of this paragraph we prove two theorems (2.10 and 2.9) which say that a Φ-ultrafilter cannot be Φ-definable, and we also show that the definability of a filter is related to the definability of a branch in a Σ_n^κ-tree and the definability of a homogeneous set for a definable partition relation.

2.1 Lemma (Kranakis [1982b])

If there is a Π_n-ultrafilter on κ, then κ has the Σ_n-tree property.

From the proof of 2.1 we can obtain:

2.2 Lemma

Let $n,m\geq 1$ and let $\Phi=\Delta_m$, Σ_m or Π_m. If there is a Φ-definable Π_n-ultrafilter on κ, then every Σ_n^κ-tree has a Φ-definable branch of length κ.

Proof

Let $\langle\kappa,<_T\rangle$ be a Σ_n^κ-tree. Define $B = \{\alpha<\kappa : \{\beta<\kappa : \alpha<_T\beta\} \in F\}$.
Since F is Φ-definable, B is Φ-definable on L_κ.
It remains to be shown that B is a branch of length κ.
First of all, if $\gamma,\delta \in B$, then $\{\beta<\kappa : \gamma<_T\beta\} \in F$ and

$\{\beta<\kappa : \delta<_T\beta\} \in F$, so $\{\beta<\kappa : \gamma<_T\beta\} \cap \{\beta<\kappa : \delta<_T\beta\} \neq \emptyset$.

Take β in this intersection, then $\gamma<_T\beta$ and $\delta<_T\beta$, so γ and δ are $<_T$-comparable. Then all we need is to show that $B \cap T_\alpha \neq \emptyset$ for each $\alpha<\kappa$.

Assume on the contrary that $B \cap T_\alpha = \emptyset$ for some $\alpha<\kappa$, then

$\forall\beta \in T_\alpha \quad \{\xi<\kappa : \beta<_T\xi\} \notin F$, so

$\forall\beta \in T_\alpha \quad \kappa-\{\xi<\kappa : \beta<_T\xi\} \in F$.

But $T_\alpha \in L_\kappa$, and F is a Π_n-ultrafilter, so

$\bigcap_{\beta \in T_\alpha} (\kappa-\{\xi<\kappa : \beta<_T\xi\}) = \bigcup_{\gamma\leq\alpha} T_\gamma \in F$.

However, κ is Σ_{n+1}-admissible, so $L_\kappa \models \Sigma_n$-collection, from which it follows that $\bigcup_{\gamma\leq\alpha} T_\gamma$ is bounded in κ, and cannot be a member of F (by II.1.2). Thus we found a contradiction.

2.3 <u>Lemma</u> (Kranakis [1980], [1982a])

If $L_\kappa \models$ Pow (which means that L_κ satisfies the power set axiom) and κ has the Σ_n-tree property, then $\kappa \xrightarrow{\Sigma_n} (\kappa)^2_2$.

Again, we can adapt the proof of 2.3 to get:

2.4 <u>Lemma</u>

Let $n,m\geq 1$ and let $\Phi=\Delta_m$, Σ_m or Π_m.

Define Ψ by:

i. $\Psi = \Phi$ if $\Sigma_n L_\kappa \subseteq \Phi L_\kappa$

ii. $\Psi = \Sigma_n$ if $\Phi L_\kappa \subseteq \Sigma_n L_\kappa$.

If $L_\kappa \models$ Pow and every Σ_n^κ-tree has a Φ-definable branch, then $\kappa \xrightarrow{\Sigma_n} (\kappa - \Psi)^2_2$.

Proof

Let $h: [\kappa]^2 \xrightarrow{\Sigma_n} 2$. First we will define a Σ_n^κ-tree.

Define $G: \kappa \times L_\kappa \xrightarrow{\Sigma_n} L_\kappa$ by

$$G(\alpha,g) = \begin{cases} \{\beta<\alpha : \exists u\ (g(\beta)=u\ \&\ \forall\gamma<\beta\ (\gamma\in u \rightarrow h(\gamma,\beta)=h(\gamma,\alpha)))\} & \\ & \text{if } g \text{ is a function with domain } \alpha; \\ \{1\} & \text{otherwise.} \end{cases}$$

Since $L_\kappa \models \Sigma_n$-separation, we have $G(\alpha,g) \in L_\kappa$ for each $\alpha, g \in L_\kappa$.

It is easy to see that G is $\Sigma_n L_\kappa$.

By the Σ_n-recursion theorem there is an $f: \kappa \xrightarrow{\Sigma_n} L_\kappa$ with

$f(\alpha) = G(\alpha, f{\restriction}\alpha)$ for $\alpha<\kappa$.

Then we put $\beta <_T \alpha \iff \beta \in f(\alpha)$.

Then it can be shown that $\mathcal{T} = \langle\kappa, <_T\rangle$ is a Σ_n^κ-tree.

(In this proof, the assumption $L_\kappa \models$ Pow is needed to show that $T_\lambda \in L_\kappa$ if λ is a limit ordinal.)

By assumption, we have that $\mathcal{T}$ has a branch B of length κ, which is ΦL_κ.

Now define $g: B \longrightarrow 2$ by $g(\alpha) = h(\alpha,\beta)$, where $\beta \in B$ and $\alpha <_T \beta$.

Now, both $g^{-1}(\{0\})$ and $g^{-1}(\{1\})$ are homogeneous for h and ΨL_κ-definable, and at least one of both has type κ.

This completes the proof.

Finally we need two lemma's from Kranakis & Phillips [*]:

2.5 Lemma (Kranakis & Phillips [*], 5.7, or Kranakis [§], 2.9)

There is no κ with $\kappa \xrightarrow{\Sigma_n} (\kappa - \Sigma_n)^2_2$.

2.6 Lemma (Kranakis & Phillips [*], 5.8)

There is no κ with $L_\kappa \models$ Pow and $\kappa \xrightarrow{\Sigma_n} (\kappa - \Sigma_{n+1})^2_2$.

2.7 Corollary

There is no κ with $L_\kappa \models$ Pow and a Σ_{n+1}-definable Π_n-ultrafilter.

2.8 Lemma

Let $n \geq 1$.

Let $\Phi = \Delta_n, \Sigma_n, \Pi_n$ or $\not{c}_n$.

If there is a Φ-definable Φ-ultrafilter on κ, then $L_\kappa \models$ Pow.

Proof

Suppose F is a Φ-definable Φ-ultrafilter on κ, but $L_\kappa \models \neg$Pow. Then there is a $\lambda<\kappa$ with $P\lambda \cap L_\kappa \notin L_\kappa$, and it follows that there is a G: $\kappa \xrightarrow{1-1, \Delta_1} ({}^\lambda 2) \cap L_\kappa$.

Then define $\langle X_\alpha : \alpha<\lambda\rangle$ by $X_\alpha = \{\xi<\kappa : G(\xi)(\alpha) = i\}$, where $i<2$ is such that $X_\alpha \in F$. Since for each $\alpha<\lambda$ either $\{\xi<\kappa : G(\xi)(\alpha)=0\}$ or $\{\xi<\kappa : G(\xi)(\alpha)=1\}$ is in F, we have $\langle X_\alpha : \alpha<\lambda\rangle \in \Phi L_\kappa \cap {}^\lambda F$, so $\bigcap_{\alpha<\lambda} X_\alpha \in F$. But we must have that $\bigcap_{\alpha<\lambda} X_\alpha$ contains only one element, and that gives a contradiction.

2.9 Corollary

There is no κ which has a Π_n-definable Π_n-ultrafilter.

Proof

2.7 plus 2.8.

2.10 Corollary

There is no κ which has a $\not{c}_n$-definable Σ_n-ultrafilter.

Proof

2.7, 2.8 and III.1.4.

Now we'll see what we can say in the positive direction.

2.11 Theorem

Let κ be countable and Σ_{n+1}-admissible, but less than the least ordinal with Δ_{n+2}-separation. Then there is a Δ_{n+3}-definable Δ_n-ultrafilter and a $\mathbb{B}_{n+3}$-definable Π_n-ultrafilter on κ.

Proof

By assumption, there is a function $g: \omega \xrightarrow{\Delta_{n+2},\ \text{onto}} L_\kappa$.
We will use the construction of III.2.2 to extend H to a Π_n-ultrafilter. Using g, it is easy to give a $\Delta_{n+2}L_\kappa$ enumeration $\langle\langle\phi_m,\lambda_m\rangle : 1 \leq m < \omega\rangle$ of all $\phi \in \Pi_n L_\kappa$ such that $\bigcap_{\alpha<\lambda_m}\{\xi<\kappa : L_\kappa \models \phi(\xi,\alpha)\} = \emptyset$. Note that it is enough to take only those $\langle\phi,\lambda\rangle$ in III.2.2 which give empty intersection, for if $X = \bigcap_{\alpha<\lambda}\{\xi<\kappa : L_\kappa \models \phi(\xi,\alpha)\} \notin F$, where F is the ultra-

filter to be defined, then $(\kappa - X) \cap \bigcap_{\alpha<\lambda} \{\xi<\kappa : L_\kappa \models \phi(\xi,\alpha)\} = \emptyset$, and this collection will appear in the enumeration.

Therefore, one of the $\{\xi<\kappa : L_\kappa \models \phi(\xi,\alpha)\}$ will be excluded from F.

We take $Z_0 = \kappa$, and then it is proved in III.2.2 that

$$L_\kappa \models \forall m<\omega\ \exists\beta\ [\exists\phi_1,\ldots,\phi_m\ \exists\lambda_1,\ldots,\lambda_m\ (\text{"}\phi_i \text{ and } \lambda_i \text{ are right"}\ \&\ \beta = \langle\beta_1,\ldots,\beta_m\rangle\ \&\ \forall i\leq m\ (\beta_i<\lambda_i\ \&\ \forall\gamma\ \exists\delta>\gamma\ \neg\phi_i(\delta,\beta_i)))],$$

so by the Σ_{n+2}-uniformization theorem we can define a $\Sigma_{n+2}L_\kappa$ sequence $\langle\beta_m : 1\leq m<\omega\rangle$ such that defining $Z_0 = \kappa$ and $Z_{m+1} = Z_m - \{\xi<\kappa : L_\kappa \models \phi_{m+1}(\xi,\beta_{m+1})\}$ gives a correct sequence as in III.2.2.

It follows that $\langle Z_m : m<\omega\rangle \in \Sigma_{n+2}L_\kappa$.

Defining the Π_n-ultrafilter F as in III.2.2 has the following result: $X \in F \cap \Pi_n L_\kappa \iff \forall m<\omega\ \exists\delta\ (\delta \in Z_m\ \&\ \delta \in X)$, so that F is a $\mathbb{B}_{n+3}$-definable Π_n-ultrafilter.

It follows immediately that $F \cap \Delta_n L_\kappa$ gives a Δ_{n+3}-definable Δ_n-ultrafilter.

Finally we state a theorem for the normal case:

2.12 <u>Theorem</u>

There is no Σ_ω-definable Δ_1-normal filter.

<u>Proof</u>

Let $n<\omega$ be given.

Let ϕ be the following first-order sentence

(note that again we let ψ, χ, etc. stand for codes of themselves):

$$\phi \equiv \quad \exists\theta \in \Sigma_n\ \exists a\ \forall\psi,\psi_1,\psi_2 \in \Sigma_1\ \forall b,b_1,b_2\ \forall\chi,\chi_1,\chi_2 \in \Pi_1\ \forall c,c_1,c_2$$

$$[\forall\xi\ [(\psi(\xi,b) \leftrightarrow \chi(\xi,c))\ \&\ (\psi_1(\xi,b_1) \leftrightarrow \chi_1(\xi,c_1))\ \&$$

$$\forall\gamma\ (\psi_2(\xi,\gamma,b_2) \leftrightarrow \chi_2(\xi,\gamma,c_2))]] \rightarrow$$

$$\rightarrow [\text{(i)}\ \ (\forall\xi\ \psi(\xi,b) \rightarrow \theta(\psi,\chi,a))\ \&$$

$$\text{(ii)}\ \ (\forall\xi\ \neg\psi(\xi,b) \rightarrow \neg\theta(\psi,\chi,a))\ \&$$

$$\text{(iii)}\ \ (\theta(\psi,\chi,a)\ \&\ \forall\xi\ (\psi(\xi,b) \rightarrow \psi_1(\xi,b_1)) \rightarrow \theta(\psi_1,\chi_1,a))\ \&$$

$$\text{(iv)}\ \ (\theta(\psi,\chi,a)\ \&\ \theta(\psi_1,\chi_1,a) \rightarrow \theta(\psi\&\psi_1,\chi\&\chi_1,a))\ \&$$

$$\text{(v)}\ \ \forall\eta\ (\forall\xi\neq\eta\ \psi(\xi,b) \rightarrow \theta(\psi,\chi,a))\ \&$$

$$\text{(vi)}\ \ (\forall\gamma\ \theta(\psi_2(\xi,\gamma,b_2),\chi_2(\xi,\gamma,c_2)) \rightarrow$$

$$\rightarrow \theta(\forall\gamma<\xi\ \psi_2(\xi,\gamma,b_2),\forall\gamma<\xi\ \chi_2(\xi,\gamma,c_2),a))].$$

Note that then we have for each α:

$L_\alpha \models \phi \iff$ there is a Σ_n-definable Δ_1-normal filter on α.

For the formula θ in ϕ defines a Δ_1-normal filter F, given by

$X \in F \iff \exists Y \in \Delta_1 L_\alpha$ ($Y \subseteq X$ & "θ holds of Y").

Then (i) says $\alpha \in F$; (ii) $\emptyset \notin F$; (iii) $X \in F$ & $X \subseteq Y \rightarrow Y \in F$; (iv) $X,Y \in F \rightarrow X \cap Y \in F$ (so F is a filter); (v) says that F is nonprincipal and (vi) that F is Δ_1-normal.

If we assume that there is an α with $L_\alpha \models \phi$, we let κ be the least such α. It follows that κ is countable (use the Löwenheim-Skolem theorem plus the condensation lemma), so by III.2.19 and III.1.2 L_κ has a blunt Σ_2-end extension M. Note that since $\kappa \in M$, we have $L_\kappa \in M$ (for $L_\kappa \models \forall x\, \exists y$ ("x is an ordinal" $\rightarrow$ "$y = L_x$"), so M satisfies the same sentence, since it is Π_2).

Therefore $M \models \phi^{L_\kappa}$, so $M \models \exists\alpha\ (\phi^{L_\alpha})$, and $L_\kappa \models \exists\alpha\ (\phi^{L_\alpha})$.

But that means $\exists\alpha<\kappa\ \ L_\alpha \models \phi$, a contradiction.

INDEX OF SYMBOLS AND NOTATION

INDEX (sequel)

REFERENCES

J. Barwise [1975], *Admissible sets and structures*, Springer-Verlag, 1975.

K. Devlin [1974], *An introduction to the fine structure of the constructible hierarchy (results of Ronald Jensen)*, in: J.E.Fenstad & P.G.Hinman, eds., Generalized Recursion Theory, North-Holland 1974, pp. 123 - 163.

T. Jech [1978], *Set theory*, Academic Press, 1978.

M. Kaufmann [1981], *On existence of Σ_n end extensions*, in: M.Lerman, J.H.Schmerl & R.I.Soare, eds., Logic Year 1979/80, University of Connecticut, Springer Lecture Notes 859, 1981, pp. 92 - 103.

M. Kaufmann & E. Kranakis [1984], *Definable ultrapowers and ultrafilters over admissible ordinals*, Zeitschrift für mathematische Logik und Grundlagen der Mathematik, 1984, pp. 99 - 118.

M. Kaufmann & E. Kranakis [*], *Comparing and counting end extensions of constructible sets*, to appear.

E. Kranakis [1980], *Recursive analogues of large cardinals*, Ph.D. Thesis, University of Minnesota, 1980.

E. Kranakis [1982a], *Reflection and partition properties of admissible ordinals*, Annals of Mathematical Logic, 1982, pp. 213 - 242.

E. Kranakis [1982b], *Definable ultrafilters and end extensions of constructible sets,* Zeitschrift für mathematische Logik und Grundlagen der Mathematik, 1982, pp. 395 - 412.

E. Kranakis [*], personal communication.

E. Kranakis [§], *Stepping up lemmas in definable partitions,* to appear.

E. Kranakis & I. Phillips [*], *Partitions and homogeneous sets for admissible ordinals,* to appear.

R. MacDowell & E. Specker [1961], *Modelle der Arithmetik,* in: Infinitistic Methods, Pergamon Press, London 1961, pp. 257 - 263.

I. Phillips [1983], *Definability theory for* Σ_n *admissible ordinals, with particular reference to partition relations and end extensions,* Oxford D.Phil. Thesis, 1983.

W. Richter & P. Aczel [1974], *Inductive definitions and reflecting properties of admissible ordinals,* in: J.E. Fenstad & P.G.Hinman, eds., Generalized Recursion Theory, North-Holland 1974, pp. 301 - 381.

S. Simpson [1970], Σ_2 *admissibility,* unpublished notes, 1970.

MATHEMATICAL CENTRE TRACTS

1 T. van der Walt. *Fixed and almost fixed points.* 1963.

2 A.R. Bloemena. *Sampling from a graph.* 1964.

3 G. de Leve. *Generalized Markovian decision processes, part I: model and method.* 1964.

4 G. de Leve. *Generalized Markovian decision processes, part II: probabilistic background.* 1964.

5 G. de Leve, H.C. Tijms, P.J. Weeda. *Generalized Markovian decision processes, applications.* 1970.

6 M.A. Maurice. *Compact ordered spaces.* 1964.

7 W.R. van Zwet. *Convex transformations of random variables.* 1964.

8 J.A. Zonneveld. *Automatic numerical integration.* 1964.

9 P.C. Baayen. *Universal morphisms.* 1964.

10 E.M. de Jager. *Applications of distributions in mathematical physics.* 1964.

11 A.B. Paalman-de Miranda. *Topological semigroups.* 1964.

12 J.A.Th.M. van Berckel, H. Brandt Corstius, R.J. Mokken, A. van Wijngaarden. *Formal properties of newspaper Dutch.* 1965.

13 H.A. Lauwerier. *Asymptotic expansions.* 1966, out of print; replaced by MCT 54.

14 H.A. Lauwerier. *Calculus of variations in mathematical physics.* 1966.

15 R. Doornbos. *Slippage tests.* 1966.

16 J.W. de Bakker. *Formal definition of programming languages with an application to the definition of ALGOL 60.* 1967.

17 R.P. van de Riet. *Formula manipulation in ALGOL 60, part 1.* 1968.

18 R.P. van de Riet. *Formula manipulation in ALGOL 60, part 2.* 1968.

19 J. van der Slot. *Some properties related to compactness.* 1968.

20 P.J. van der Houwen. *Finite difference methods for solving partial differential equations.* 1968.

21 E. Wattel. *The compactness operator in set theory and topology.* 1968.

22 T.J. Dekker. *ALGOL 60 procedures in numerical algebra, part 1.* 1968.

23 T.J. Dekker, W. Hoffmann. *ALGOL 60 procedures in numerical algebra, part 2.* 1968.

24 J.W. de Bakker. *Recursive procedures.* 1971.

25 E.R. Paërl. *Representations of the Lorentz group and projective geometry.* 1969.

26 European Meeting 1968. *Selected statistical papers, part I.* 1968.

27 European Meeting 1968. *Selected statistical papers, part II.* 1968.

28 J. Oosterhoff. *Combination of one-sided statistical tests.* 1969.

29 J. Verhoeff. *Error detecting decimal codes.* 1969.

30 H. Brandt Corstius. *Exercises in computational linguistics.* 1970.

31 W. Molenaar. *Approximations to the Poisson, binomial and hypergeometric distribution functions.* 1970.

32 L. de Haan. *On regular variation and its application to the weak convergence of sample extremes.* 1970.

33 F.W. Steutel. *Preservation of infinite divisibility under mixing and related topics.* 1970.

34 I. Juhász, A. Verbeek, N.S. Kroonenberg. *Cardinal functions in topology.* 1971.

35 M.H. van Emden. *An analysis of complexity.* 1971.

36 J. Grasman. *On the birth of boundary layers.* 1971.

37 J.W. de Bakker, G.A. Blaauw, A.J.W. Duijvestijn, E.W. Dijkstra, P.J. van der Houwen, G.A.M. Kamsteeg-Kemper, F.E.J. Kruseman Aretz, W.L. van der Poel, J.P. Schaap-Kruseman, M.V. Wilkes, G. Zoutendijk. *MC-25 Informatica Symposium.* 1971.

38 W.A. Verloren van Themaat. *Automatic analysis of Dutch compound words.* 1972.

39 H. Bavinck. *Jacobi series and approximation.* 1972.

40 H.C. Tijms. *Analysis of (s,S) inventory models.* 1972.

41 A. Verbeek. *Superextensions of topological spaces.* 1972.

42 W. Vervaat. *Success epochs in Bernoulli trials (with applications in number theory).* 1972.

43 F.H. Ruymgaart. *Asymptotic theory of rank tests for independence.* 1973.

44 H. Bart. *Meromorphic operator valued functions.* 1973.

45 A.A. Balkema. *Monotone transformations and limit laws.* 1973.

46 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 1: the language.* 1973.

47 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 2: the compiler.* 1973.

48 F.E.J. Kruseman Aretz, P.J.W. ten Hagen, H.L. Oudshoorn. *An ALGOL 60 compiler in ALGOL 60, text of the MC-compiler for the EL-X8.* 1973.

49 H. Kok. *Connected orderable spaces.* 1974.

50 A. van Wijngaarden, B.J. Mailloux, J.E.L. Peck, C.H.A. Koster, M. Sintzoff, C.H. Lindsey, L.G.L.T. Meertens, R.G. Fisker (eds.). *Revised report on the algorithmic language ALGOL 68.* 1976.

51 A. Hordijk. *Dynamic programming and Markov potential theory.* 1974.

52 P.C. Baayen (ed.). *Topological structures.* 1974.

53 M.J. Faber. *Metrizability in generalized ordered spaces.* 1974.

54 H.A. Lauwerier. *Asymptotic analysis, part 1.* 1974.

55 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 1: theory of designs, finite geometry and coding theory.* 1974.

56 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 2: graph theory, foundations, partitions and combinatorial geometry.* 1974.

57 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 3: combinatorial group theory.* 1974.

58 W. Albers. *Asymptotic expansions and the deficiency concept in statistics.* 1975.

59 J.L. Mijnheer. *Sample path properties of stable processes.* 1975.

60 F. Göbel. *Queueing models involving buffers.* 1975.

63 J.W. de Bakker (ed.). *Foundations of computer science.* 1975.

64 W.J. de Schipper. *Symmetric closed categories.* 1975.

65 J. de Vries. *Topological transformation groups, 1: a categorical approach.* 1975.

66 H.G.J. Pijls. *Logically convex algebras in spectral theory and eigenfunction expansions.* 1976.

68 P.P.N. de Groen. *Singularly perturbed differential operators of second order.* 1976.

69 J.K. Lenstra. *Sequencing by enumerative methods.* 1977.

70 W.P. de Roever, Jr. *Recursive program schemes: semantics and proof theory.* 1976.

71 J.A.E.E. van Nunen. *Contracting Markov decision processes.* 1976.

72 J.K.M. Jansen. *Simple periodic and non-periodic Lamé functions and their applications in the theory of conical waveguides.* 1977.

73 D.M.R. Leivant. *Absoluteness of intuitionistic logic.* 1979.

74 H.J.J. te Riele. *A theoretical and computational study of generalized aliquot sequences.* 1976.

75 A.E. Brouwer. *Treelike spaces and related connected topological spaces.* 1977.

76 M. Rem. *Associons and the closure statement.* 1976.

77 W.C.M. Kallenberg. *Asymptotic optimality of likelihood ratio tests in exponential families.* 1978.

78 E. de Jonge, A.C.M. van Rooij. *Introduction to Riesz spaces.* 1977.

79 M.C.A. van Zuijlen. *Emperical distributions and rank statistics.* 1977.

80 P.W. Hemker. *A numerical study of stiff two-point boundary problems.* 1977.

81 K.R. Apt, J.W. de Bakker (eds.). *Foundations of computer science II, part 1.* 1976.

82 K.R. Apt, J.W. de Bakker (eds.). *Foundations of computer science II, part 2.* 1976.

83 L.S. van Benthem Jutting. *Checking Landau's "Grundlagen" in the AUTOMATH system.* 1979.

84 H.L.L. Busard. *The translation of the elements of Euclid from the Arabic into Latin by Hermann of Carinthia (?), books vii-xii.* 1977.

85 J. van Mill. *Supercompactness and Wallman spaces.* 1977.

86 S.G. van der Meulen, M. Veldhorst. *Torrix I, a programming system for operations on vectors and matrices over arbitrary fields and of variable size.* 1978.

88 A. Schrijver. *Matroids and linking systems.* 1977.

89 J.W. de Roever. *Complex Fourier transformation and analytic functionals with unbounded carriers.* 1978.

90 L.P.J. Groenewegen. *Characterization of optimal strategies in dynamic games.* 1981.

91 J.M. Geysel. *Transcendence in fields of positive characteristic.* 1979.

92 P.J. Weeda. *Finite generalized Markov programming.* 1979.

93 H.C. Tijms, J. Wessels (eds.). *Markov decision theory.* 1977.

94 A. Bijlsma. *Simultaneous approximations in transcendental number theory.* 1978.

95 K.M. van Hee. *Bayesian control of Markov chains.* 1978.

96 P.M.B. Vitányi. *Lindenmayer systems: structure, languages, and growth functions.* 1980.

97 A. Federgruen. *Markovian control problems; functional equations and algorithms.* 1984.

98 R. Geel. *Singular perturbations of hyperbolic type.* 1978.

99 J.K. Lenstra, A.H.G. Rinnooy Kan, P. van Emde Boas (eds.). *Interfaces between computer science and operations research.* 1978.

100 P.C. Baayen, D. van Dulst, J. Oosterhoff (eds.). *Proceedings bicentennial congress of the Wiskundig Genootschap, part 1.* 1979.

101 P.C. Baayen, D. van Dulst, J. Oosterhoff (eds.). *Proceedings bicentennial congress of the Wiskundig Genootschap, part 2.* 1979.

102 D. van Dulst. *Reflexive and superreflexive Banach spaces.* 1978.

103 K. van Harn. *Classifying infinitely divisible distributions by functional equations.* 1978.

104 J.M. van Wouwe. *Go-spaces and generalizations of metrizability.* 1979.

105 R. Helmers. *Edgeworth expansions for linear combinations of order statistics.* 1982.

106 A. Schrijver (ed.). *Packing and covering in combinatorics.* 1979.

107 C. den Heijer. *The numerical solution of nonlinear operator equations by imbedding methods.* 1979.

108 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science III, part 1.* 1979.

109 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science III, part 2.* 1979.

110 J.C. van Vliet. *ALGOL 68 transput, part I: historical review and discussion of the implementation model.* 1979.

111 J.C. van Vliet. *ALGOL 68 transput, part II: an implementation model.* 1979.

112 H.C.P. Berbee. *Random walks with stationary increments and renewal theory.* 1979.

113 T.A.B. Snijders. *Asymptotic optimality theory for testing problems with restricted alternatives.* 1979.

114 A.J.E.M. Janssen. *Application of the Wigner distribution to harmonic analysis of generalized stochastic processes.* 1979.

115 P.C. Baayen, J. van Mill (eds.). *Topological structures II, part 1.* 1979.

116 P.C. Baayen, J. van Mill (eds.). *Topological structures II, part 2.* 1979.

117 P.J.M. Kallenberg. *Branching processes with continuous state space.* 1979.

118 P. Groeneboom. *Large deviations and asymptotic efficiencies.* 1980.

119 F.J. Peters. *Sparse matrices and substructures, with a novel implementation of finite element algorithms.* 1980.

120 W.P.M. de Ruyter. *On the asymptotic analysis of large-scale ocean circulation.* 1980.

121 W.H. Haemers. *Eigenvalue techniques in design and graph theory.* 1980.

122 J.C.P. Bus. *Numerical solution of systems of nonlinear equations.* 1980.

123 I. Yuhász. *Cardinal functions in topology - ten years later.* 1980.

124 R.D. Gill. *Censoring and stochastic integrals.* 1980.

125 R. Eising. *2-D systems, an algebraic approach.* 1980.

126 G. van der Hoek. *Reduction methods in nonlinear programming.* 1980.

127 J.W. Klop. *Combinatory reduction systems.* 1980.

128 A.J.J. Talman. *Variable dimension fixed point algorithms and triangulations.* 1980.

129 G. van der Laan. *Simplicial fixed point algorithms.* 1980.

130 P.J.W. ten Hagen, T. Hagen, P. Klint, H. Noot, H.J. Sint, A.H. Veen. *ILP: intermediate language for pictures.* 1980.

131 R.J.R. Back. *Correctness preserving program refinements: proof theory and applications.* 1980.

132 H.M. Mulder. *The interval function of a graph.* 1980.

133 C.A.J. Klaassen. *Statistical performance of location estimators.* 1981.

134 J.C. van Vliet, H. Wupper (eds.). *Proceedings international conference on ALGOL 68.* 1981.

135 J.A.G. Groenendijk, T.M.V. Janssen, M.J.B. Stokhof (eds.). *Formal methods in the study of language, part I.* 1981.

136 J.A.G. Groenendijk, T.M.V. Janssen, M.J.B. Stokhof (eds.). *Formal methods in the study of language, part II.* 1981.

137 J. Telgen. *Redundancy and linear programs.* 1981.

138 H.A. Lauwerier. *Mathematical models of epidemics.* 1981.

139 J. van der Wal. *Stochastic dynamic programming, successive approximations and nearly optimal strategies for Markov decision processes and Markov games.* 1981.

140 J.H. van Geldrop. *A mathematical theory of pure exchange economies without the no-critical-point hypothesis.* 1981.

141 G.E. Welters. *Abel-Jacobi isogenies for certain types of Fano threefolds.* 1981.

142 H.R. Bennett, D.J. Lutzer (eds.). *Topology and order structures, part 1.* 1981.

143 J.M. Schumacher. *Dynamic feedback in finite- and infinite-dimensional linear systems.* 1981.

144 P. Eijgenraam. *The solution of initial value problems using interval arithmetic; formulation and analysis of an algorithm.* 1981.

145 A.J. Brentjes. *Multi-dimensional continued fraction algorithms.* 1981.

146 C.V.M. van der Mee. *Semigroup and factorization methods in transport theory.* 1981.

147 H.H. Tigelaar. *Identification and informative sample size.* 1982.

148 L.C.M. Kallenberg. *Linear programming and finite Markovian control problems.* 1983.

149 C.B. Huijsmans, M.A. Kaashoek, W.A.J. Luxemburg, W.K. Vietsch (eds.). *From A to Z, proceedings of a symposium in honour of A.C. Zaanen.* 1982.

150 M. Veldhorst. *An analysis of sparse matrix storage schemes.* 1982.

151 R.J.M.M. Does. *Higher order asymptotics for simple linear rank statistics.* 1982.

152 G.F. van der Hoeven. *Projections of lawless sequences.* 1982.

153 J.P.C. Blanc. *Application of the theory of boundary value problems in the analysis of a queueing model with paired services.* 1982.

154 H.W. Lenstra, Jr., R. Tijdeman (eds.). *Computational methods in number theory, part I.* 1982.

155 H.W. Lenstra, Jr., R. Tijdeman (eds.). *Computational methods in number theory, part II.* 1982.

156 P.M.G. Apers. *Query processing and data allocation in distributed database systems.* 1983.

157 H.A.W.M. Kneppers. *The covariant classification of two-dimensional smooth commutative formal groups over an algebraically closed field of positive characteristic.* 1983.

158 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science IV, distributed systems, part 1.* 1983.

159 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science IV, distributed systems, part 2.* 1983.

160 A. Rezus. *Abstract AUTOMATH.* 1983.

161 G.F. Helminck. *Eisenstein series on the metaplectic group, an algebraic approach.* 1983.

162 J.J. Dik. *Tests for preference.* 1983.

163 H. Schippers. *Multiple grid methods for equations of the second kind with applications in fluid mechanics.* 1983.

164 F.A. van der Duyn Schouten. *Markov decision processes with continuous time parameter.* 1983.

165 P.C.T. van der Hoeven. *On point processes.* 1983.

166 H.B.M. Jonkers. *Abstraction, specification and implementation techniques, with an application to garbage collection.* 1983.

167 W.H.M. Zijm. *Nonnegative matrices in dynamic programming.* 1983.

168 J.H. Evertse. *Upper bounds for the numbers of solutions of diophantine equations.* 1983.

169 H.R. Bennett, D.J. Lutzer (eds.). *Topology and order structures, part 2.* 1983.

CWI TRACTS

1 D.H.J. Epema. *Surfaces with canonical hyperplane sections.* 1984.

2 J.J. Dijkstra. *Fake topological Hilbert spaces and characterizations of dimension in terms of negligibility.* 1984.

3 A.J. van der Schaft. *System theoretic descriptions of physical systems.* 1984.

4 J. Koene. *Minimal cost flow in processing networks, a primal approach.* 1984.

5 B. Hoogenboom. *Intertwining functions on compact Lie groups.* 1984.

6 A.P.W. Böhm. *Dataflow computation.* 1984.

7 A. Blokhuis. *Few-distance sets.* 1984.

8 M.H. van Hoorn. *Algorithms and approximations for queueing systems.* 1984.

9 C.P.J. Koymans. *Models of the lambda calculus.* 1984.

10 C.G. van der Laan, N.M. Temme. *Calculation of special functions: the gamma function, the exponential integrals and error-like functions.* 1984.

11 N.M. van Dijk. *Controlled Markov processes; time-discretization.* 1984.

12 W.H. Hundsdorfer. *The numerical solution of nonlinear stiff initial value problems: an analysis of one step methods.* 1985.

13 D. Grune. *On the design of ALEPH.* 1985.

14 J.G.F. Thiemann. *Analytic spaces and dynamic programming: a measure theoretic approach.* 1985.

15 F.J. van der Linden. *Euclidean rings with two infinite primes.* 1985.

16 R.J.P. Groothuizen. *Mixed elliptic-hyperbolic partial differential operators: a case-study in Fourier integral operators.* 1985.

17 H.M.M. ten Eikelder. *Symmetries for dynamical and Hamiltonian systems.* 1985.

18 A.D.M. Kester. *Some large deviation results in statistics.* 1985.

19 T.M.V. Janssen. *Foundations and applications of Montague grammer, part 1: Philosophy, framework, computer science.* 1986.

20 B.F. Schriever. *Order dependence.* 1986.

21 D.P. van der Vecht. *Inequalities for stopped Brownian motion.* 1986.

22 J.C.S.P. van der Woude. *Topological dynamix.* 1986.

23 A.F. Monna. *Methods, concepts and ideas in mathematics: aspects of an evolution.* 1986.

24 J.C.M. Baeten. *Filters and ultrafilters over definable subsets of admissible ordinals.* 1986.

25 A.W.J. Kolen. *Tree network and planar rectilinear location theory.* 1986.