Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

The CWI is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

CWI Tract 38

On Banach algebras, renewal measures and regenerative processes

J.B.G. Frenk

Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

1980 Mathematics Subject Classification: 60K05, 13J05.
ISBN 90 6196 321 4

ACKNOWLEDGEMENTS

This monograph on Banach algebras and Renewal theory is an extended and completely rewritten version of my Ph.D. thesis, which appeared in 1983.

Although the original Ph.D. thesis was accepted for publication as a C.W.I.-tract, I did have the feeling that using the same techniques most of the results could be improved and so I decided to rewrite and extend it. This revision was partly undertaken during my stay as a visiting scholar at the IEOR department of the University of California (Berkeley) and was supported by a grant from the Netherlands Organization for the Advancement of Pure Research (ZWO) and by a Fulbright scholarship.

Among the people who deserve my thanks are my teacher Prof.Dr.Ir. J.W. Cohen for showing me as a student the beauty of mathematics and my thesis advisor Prof.Dr. L.F.M. de Haan who suggested this research topic and introduced me to the field of regular variation and its connection with renewal theory. Special thanks deserves Prof.Dr. D. van Dulst without whose help the key results in Chapter 1 never would have been written down. His suggestions greatly improved the contents of Chapter 1.

Finally, I would also like to thank my wife Jikke-Marie for her patience and support, Mrs. E. Baselmans-Weijers for the excellent typing of the manuscript and the Mathematical Centre in Amsterdam for the opportunity to publish this monograph in their series C.W.I.-tracts.

Hans Frenk

CONTENTS

CHAPTER 1. COMMUTATIVE BANACH ALGEBRAS

0. Introduction

This chapter is divided into three sections.

In section 1 an introduction to the theory of commutative Banach algebras will be given. The main result (Theorem 1.1.21) is the one-to-one correspondence between the set of all maximal ideals in a commutative Banach algebra V and the set of all homomorphisms L: $V \to \mathbb{C}$. All results in this section are known and can be found in the literature on this subject (cf. [NAI], [HIL], [RUD-1], [RUD-2], [RIC], [GEL]).

In section 2 attention will be paid to the Banach algebra $V(\psi)$ of (weighted) complex valued summable sequences on the nonnegative integers and to some subalgebras of $V(\psi)$. Most of the results in this section are known (cf. [GRÜ], [ROG-2], [CHO]). However, the purpose of this section is to present simplified proofs and at the same time unify the proof techniques for the different possible cases.

In section 3 the Banach algebra $S(\psi)$ of (weighted) complex measures, concentrated on the positive halfline, and some subalgebras of $S(\psi)$ are considered. Among the most important and new results is a characterization of the space of homomorphisms on $S(\psi)$ (Theorem 1.3.2) and the implication of this result (Theorem 1.3.4). These results fill an important gap in the reasoning of Rogozin (cf. [ROG-3], [ROG-5]) who derived similar results for a special case. The rest of the proofs and results are straightforward generalizations of the ideas used in section 2.

Finally, we like to mention that we will use the theorems in sections 2 and 3 to derive asymptotic results for the renewal sequence (Chapter 2), respectively the renewal measure (Chapter 3) in case the expectation of the waiting time distribution is finite.

1. General properties

DEFINITION 1.1.1. (cf. [DUN]). A nonempty set V is called a *complex algebra* if V is a *ring* as well as a *vectorspace* over the complex field $\mathbb{C}$ with the property that for every $x,y \in V$ and $\alpha \in \mathbb{C}$ the equality $\alpha(xy) = (\alpha x)y = x(\alpha y)$ holds.
If in addition V possesses a complete norm $\|\cdot\|$ satisfying $\|xy\| \leq \|x\|\|y\|$ for all $x,y \in V$ we call V a *Banach algebra*.

REMARK 1.1.2.

1. Since V is a vectorspace there exists a unique element θ (the zero element) such that $\theta + x = x$ for all $x \in V$.
   Also $0x = (1-1)x = x - x = \theta$ for all $x \in V$, where 0 denotes the zero element of $\mathbb{C}$.
2. The multiplicative norminequality implies that the mapping $(x,y) \to xy$ is jointly continuous.
3. The existence of an element e, which satisfies $xe = ex = x$ for all $x \in V$ and $\|e\| = 1$, is assumed. This element is called the unit of V. (These conditions are not restrictive (cf. [HIL]).)
4. The operation of multiplication in V is assumed to be commutative, i.e. $xy = yx$ for all $x,y \in V$.

DEFINITION 1.1.3. An element $x \in V$ is called *invertible* if there exists an element $y \in V$ satisfying $yx = e$. (This element y is unique and is denoted by x^{-1}.)

LEMMA 1.1.4. *Suppose* V *is a commutative Banach algebra with unit.*

1. *For every* $x \in V$ *with* $\|x\| < 1$ *the element* $e - x$ *is invertible and*

$$(e-x)^{-1} = \sum_{n=0}^{\infty} x^n ,$$

   where $x^0 := e$.

2. *For every* $h \in V$ *and every invertible* $x \in V$ *with* $\|h\| \leq (\|x^{-1}\|)^{-1}$ *the element* $x + h$ *is invertible and it satisfies*

$$\|(x+h)^{-1} - x^{-1} + x^{-2}h\| \leq \frac{(\|x^{-1}\|)^3(\|h\|)^2}{1 - \|x^{-1}h\|} \qquad (1)$$

PROOF. Since $\|x\| < 1$ and $\|x^n\| \leq \|x\|^n$ for every $n \in \mathbb{N}$ we obtain that $s_k := \sum_{n=0}^k x^n$ is a Cauchy sequence. Hence

$$\lim_{k\to\infty} s_k = \sum_{n=0}^{\infty} x^n$$

belongs to V. But $(e-x)s_k = e - x^{k+1}$ and so by the continuity of the multiplication and the uniqueness of the inverse element we have

$$(e-x)^{-1} = \sum_{n=0}^{\infty} x^n .$$

This proves the first part of the lemma.
A proof of the second part is given as follows. Since $x + h = x(e + x^{-1}h)$ and by assumption $\|x^{-1}h\| < 1$ we obtain applying the first part that $x + h$ is invertible. Also

$$\|(x+h)^{-1} - x^{-1} + x^{-2}h\| \leq \|(e + x^{-1}h)^{-1} - e + x^{-1}h\|\|x^{-1}\| =$$

$$= \|\sum_{n=2}^{\infty} (-1)^n(x^{-1}h)^n\|\|x^{-1}\| \leq \frac{(\|x^{-1}\|)^3(\|h\|)^2}{1 - \|x^{-1}h\|} . \qquad \square$$

REMARK 1.1.5. By Lemma 1.1.4 the set G of all invertible elements is open and the mapping $x \to x^{-1}$ (defined on G) is continuous.

DEFINITION 1.1.6. Suppose V is a commutative Banach algebra with unit. For $x \in V$, denote by $\sigma_V(x)$ the set of all $\lambda \in \mathbb{C}$ such that $x - \lambda e$ is not invertible in V. This set $\sigma_V(x)$ is called the *spectrum* of x and the complements of $\sigma_V(x)$ in $\mathbb{C}$ (=: $\rho_V(x)$) the *resolvent* of x.

LEMMA 1.1.7. *Let* Φ *be a bounded linear functional on a Banach algebra* V *with unit. Take* $x \in V$ *and define* $f(\lambda) := \Phi((x-\lambda e)^{-1})$ *for all* $\lambda \in \rho_V(x)$. *Then* f *is analytic on the resolvent of* x *and* $\lim_{|\lambda|\to\infty} f(\lambda) = 0$.

PROOF. First note that the resolvent is not empty since $e - \lambda^{-1}x$ is invertible for $|\lambda| > \|x\|$ and hence $x - \lambda e \in G$. This observation implies $\{\lambda \in \mathbb{C}: |\lambda| > \|x\|\} \subseteq \rho_V(x)$. Taking $\lambda \in \rho_V(x)$ and applying (1) with $(\lambda-\mu)e$ replacing h and $x - \lambda e$ replacing x yields

$$\|(x-\mu e)^{-1} - (x-\lambda e)^{-1} + (\lambda-\mu)(x-\lambda e)^{-2}\| = O(|\lambda-\mu|^2)$$

for $|\lambda-\mu|$ sufficiently small. Hence

$$\lim_{\mu\to\lambda} \left\| \frac{(x-\mu e)^{-1} - (x-\lambda e)^{-1}}{\mu - \lambda} - (x-\lambda e)^{-2} \right\| = 0 \ . \tag{2}$$

By the definition of f and the continuity and linearity of Φ we obtain from (2) that

$$\lim_{\mu\to\lambda} \frac{f(\mu) - f(\lambda)}{\mu - \lambda} = \Phi((x-\lambda e)^{-2}) \ .$$

So f is analytic on the resolvent of x. Also

$$\lambda f(\lambda) = \Phi(\lambda(x-\lambda e)^{-1}) = \Phi((\lambda^{-1}x-e)^{-1})$$

and this implies by the continuity of the mapping $x \to x^{-1}$ that $\lim_{|\lambda|\to\infty} \lambda f(\lambda) = -\Phi(e)$. Hence $\lim_{|\lambda|\to\infty} f(\lambda) = 0$. □

THEOREM 1.1.8. *For every* $x \in V$ *the set* $\sigma_V(x)$ *is nonempty and compact.*

PROOF. Suppose $\sigma_V(x)$ is an empty set. Then it follows from Lemma 1.1.7 that for every $\Phi \in V^*$ the function $f: \rho_V(x) \to \mathbb{C}$ defined by $f(\lambda) := \Phi((x-\lambda e)^{-1})$ is entire (cf. [TIT], [KOD]).
Since $\lim_{|\lambda|\to\infty} f(\lambda) = 0$, Liouville's theorem implies that $\Phi((x-\lambda e)^{-1}) = 0$ for all $\lambda \in \mathbb{C}$ and $\Phi \in V^*$. Thus, by the Hahn-Banach theorem,

$$(x-\lambda e)^{-1} = \theta \quad \text{for all } \lambda \in \mathbb{C} \ ,$$

an obvious contradiction, and so $\sigma_V(x)$ is not empty.
Since $\sigma_V(x)$ is bounded and the complement of the set G of invertible elements is closed, the compactness of $\sigma_V(x)$ follows easily. □

Before proving a key theorem in the general theory of Banach algebras we need the following definition.

DEFINITION 1.1.9. Let V be a Banach algebra and L a complex-valued functional (not identically zero) on V.

1. L is called an *(algebra) homomorphism* if
   (i) L is a linear functional, i.e.

$$L(\alpha x+\beta y) = \alpha L(x) + \beta L(y) \quad \text{for all } x,y \in V \text{ and } \alpha,\beta \in \mathbb{C} \ .$$

(ii) L is a multiplicative functional, i.e.

$$L(xy) = L(x)L(y) \quad \text{for all } x,y \in V .$$

2. L is called an *isometric isomorphism* if
   (i) L is an (algebra) homomorphism.
   (ii) L is onto, i.e. $L(V) = \mathbb{C}$.
   (iii) L is an isometry, i.e. $|L(x)| = \|x\|$ for all $x \in V$.

THEOREM 1.1.10 (Gelfand-Mazur). *Suppose* V *is a commutative Banach algebra with unit in which every nonzero element is invertible. Then* V *is isomorphic to the complex field* $\mathbb{C}$ *(notation:* $V \cong C$*).*

PROOF. Fix $x \neq \theta$ and choose $\lambda \in \sigma_V(x)$. (This is possible by Theorem 1.1.8.) Then $x - \lambda e$ is not invertible, and so, by assumption, $x - \lambda e = \theta$ or equivalently $x = \lambda e$. This proves that $\sigma_V(x)$ consists of precisely the element λ, which we denote by $\lambda(x)$. Define also $\lambda(\theta) := 0$. It is now obvious that the mapping $L: V \to \mathbb{C}$ defined by $L(x) = \lambda(x)$ is an isometric isomorphism. □

DEFINITION 1.1.11.

1. A subset I of the commutative Banach algebra V is called an *ideal* if
   (i) I is a linear subspace of V.
   (ii) $xy \in I$ for all $x \in I$ and $y \in V$.
2. An ideal I is said to be *proper* if $I \neq V$.
3. An ideal I is said to be *maximal* if it is a proper ideal and there exists no proper ideal I' such that $I \subset I'$ and $I \neq I'$.

REMARK 1.1.12. It is easy to verify that $e \notin I$ if I is a proper ideal.

In order to prove that every proper ideal is contained in a maximal ideal we need the next result.

LEMMA 1.1.13 (Zorn's lemma). *If in a partially ordered nonempty set* X *every linearly ordered subset has an upperbound in* X *then* X *contains a maximal element.*

PROOF. Cf. [HIL], [NAI]. □

<u>LEMMA 1.1.14</u>. *Every proper ideal is contained in a maximal ideal.*

<u>PROOF</u>. The set of all proper ideals which contain a given proper ideal is nonempty and partially ordered by inclusion. Since every linearly ordered subset K has the upperbound $\cup\{I: I \in K\}$ and $\cup\{I: I \in K\}$ is an ideal with $e \notin \cup\{I: I \in K\}$ (hence $\cup\{I: I \in K\}$ is a proper ideal) and $\cup\{I: I \in K\}$ contains the given one we can apply Lemma 1.1.13. This yields the stated result.

□

The following theorem is very helpful in identifying whether an element is invertible or not.

<u>THEOREM 1.1.15</u>. *An element* $x \in V$ *is invertible if and only if* x *does not belong to any maximal ideal.*

<u>PROOF</u>. Suppose $x \in V$ is invertible and belongs to a maximal ideal I_M. Then $e = xx^{-1} \in I_M$ and so $I_M = V$. This proves that an invertible element does not belong to a maximal ideal.
Conversely, suppose the element $x \in V$ is not invertible and consider the set $Vx := \{yx: y \in V\}$. Then Vx is an ideal (use the commutative property of the multiplication) and $e \notin Vx$ (use x is not invertible).
Hence Vx is a proper ideal with $x = ex \in Vx$ and by Lemma 1.1.14 we arrive at the desired result.

□

The Gelfand-Mazur theorem states that a Banach algebra V can be identified with $\mathbb{C}$ in case every nonzero element in V is invertible. Therefore we like to reduce every Banach algebra V into a Banach algebra V' with the above property. This reduction can be carried out as follows.

Suppose I is a proper ideal in the complex algebra V with unit e and consider for fixed $x \in V$ the coset $\Pi(x) := x + I := \{x + y: y \in I\}$. Since I is a linear subspace of V we obtain for $x_1 - x_2 \notin I$ that $\Pi(x_1) \cap \Pi(x_2) = \emptyset$ and for $x_1 - x_2 \in I$ that $\Pi(x_1) = \Pi(x_2)$. The set of all cosets is then denoted by V/I (V modulo I) and one defines in this set in a consistent way (since I is an ideal) as follows the operations of multiplication, addition and scalar multiplication (cf. [RUD-1], [RUD-2]):

$$\Pi(x+y) := \Pi(x) + \Pi(y), \quad x,y \in V$$

$$\Pi(\lambda x) := \lambda\Pi(x), \quad x \in V, \lambda \in \mathbb{C}$$

$$\Pi(xy) := \Pi(x)\Pi(y), \quad x,y \in V .$$

This makes V/I a *complex algebra* with unit $\Pi(e)$.

LEMMA 1.1.16. *Let* V *be a commutative complex algebra with unit* e *and* I *a proper ideal. Then every nonzero element in* V/I *is invertible if and only if* I *is a maximal ideal.*

PROOF. Suppose I is a maximal ideal and consider the set V/I. In this set the unit is $\Pi(e)$ and so we have to prove for every nonzero element $\Pi(x)$ (in V/I) the existence of an element $\bar{v} \in V$ such that $\Pi(\bar{v})\Pi(x) = \Pi(e)$.
Take now an arbitrary nonzero element $\Pi(x) \in V/I$ and define

$$I' := \{vx + y\colon v \in V,\ y \in I\} \supset I \ .$$

Then I' is an ideal. Also I' contains I properly since $x = ex + \theta \in I'$ and $x \notin I$. This implies by the maximality of I that $I' = V$ and so there exists some $\bar{v} \in V$ such that $\bar{v}x + y = e$. Hence $\Pi(\bar{v})\Pi(x) = \Pi(\bar{v}x) = \Pi(e)$ and this proves the first part of this lemma.
If I is not a maximal ideal there exists by Lemma 1.1.14 a maximal ideal I_M containing I properly and so we can find some $v_0 \in V$ such that $v_0 \in I_M$ and $v_0 \notin I$. Hence $\Pi(v_0)$ is a nonzero element in V/I. Note also, since I_M is a maximal ideal, that $\Pi(v)\Pi(v_0) = \Pi(vv_0) \neq \Pi(e)$ for every $v \in V$. Thus we have found a nonzero element $\Pi(v_0)$ (in V/I) which is not invertible in V/I. □

In the preceding notes we looked at the algebraic properties of ideals and quotient spaces. We now pay attention to the topological properties.

LEMMA 1.1.17. *Every maximal ideal* I_M *is closed.*

PROOF. It is easily verified that the closure of an ideal is again an ideal. Since by Remark 1.1.5 the nonempty set G of invertible elements is open and by Theorem 1.1.15 every invertible element does not belong to any proper ideal we obtain that the closure of a proper ideal is again a proper ideal. By this observation and the maximality of I_M we get that the closure of I_M equals I_M and so I_M is closed.

Consider the quotient space V/I with I a proper and closed ideal and V a commutative Banach algebra. Define now:

$$\|\Pi(x)\|_q := \inf\ \{\|x+y\|\colon y \in I\} \ . \qquad (3)$$

It follows that this defines a norm (the so-called quotient norm) on V/I which satisfies the following property.

LEMMA 1.1.18. *If* V *is a commutative Banach algebra and* I *is a proper and closed ideal, then* V/I *with norm* $\|\cdot\|_q$ *is a commutative Banach algebra.*

PROOF. Cf. [RUD-2]. □

Combining the algebraic and topological properties the next important theorems hold.

THEOREM 1.1.19. *If* I_M *is a maximal ideal in a commutative Banach algebra* V *with unit, then* V/I_M *is isometrically isomorphic to the complex field* $\mathbb{C}$. *Also there exists for every* $x \in V$ *a unique number* $x(I_M) \in \mathbb{C}$ *such that*

$$x = x(I_M)e + y , \quad y \in I_M . \tag{4}$$

The correspondence $x \to x(I_M)$ *has the following properties for every* $x,y \in V$ *and* $\alpha \in \mathbb{C}$

$$(x+y)(I_M) = x(I_M) + y(I_M) \tag{5}$$

$$(\alpha x)(I_M) = \alpha x(I_M) \tag{6}$$

$$(xy)(I_M) = x(I_M)y(I_M) \tag{7}$$

$$e(I_M) = 1 \tag{8}$$

$$x(I_M) = 0 \Leftrightarrow x \in I_M \tag{9}$$

$$x(I_M) \in \sigma_V(x) \tag{10}$$

$$|x(I_M)| \leq \|x\| . \tag{11}$$

PROOF. Since by Lemmas 1.1.17 and 1.1.18 V/I_M is a commutative Banach algebra and every nonzero element $\Pi(x) \in V/I_M$ is invertible by Lemma 1.1.16 we can apply the Gelfand-Mazur theorem (Theorem 1.1.10). Hence for every $x \in V$ there exists some complex number $x(I_M)$ such that $\Pi(x) = x(I_M)\Pi(e)$. The complex number $x(I_M)$ is unique since $\Pi(e) \neq I_M$ (zero element in V/I_M). Obviously, by the above observation we can find for every $x \in V$ and corresponding complex number $x(I_M)$ some $y \in I_M$ such that

$$x = x(I_M)e + y \ . \qquad (12)$$

Using this representation we easily obtain (5) up to (9) and so we only have to prove (10) and (11).
In order to prove (10) we note that the element $x - x(I_M)e$ belongs to I_M. Hence $x - x(I_M)e$ is not invertible by Theorem 1.1.15 or equivalently $x(I_M) \in \sigma_V(x)$. Also, $|x(I_M)| \leq \sup\{|\lambda|: \lambda \in \sigma_V(x)\}$ and since for all $\lambda \in \mathbb{C}$ with $|\lambda| > \|x\|$ the element $x - \lambda e$ is invertible, i.e. $\lambda \in \rho_V(x)$, we obtain $|x(I_M)| \leq \|x\|$. □

REMARK 1.1.20. In the sequel we will use homomorphism instead of (algebra) homomorphism.

THEOREM 1.1.21. *Let* V *be a commutative Banach algebra with unit. Then for every homomorphism* $L: V \to \mathbb{C}$ *the set* $L^{\leftarrow}(0) := \{x \in V: L(x) = 0\}$ *is a maximal ideal.*
Conversely, for every maximal ideal I_M *there exists a unique homomorphism* L *such that* $L^{\leftarrow}(0) = I_M$.

PROOF. Since L is a homomorphism the set $L^{\leftarrow}(0)$ is an ideal and its codimension equals one. This implies directly that $L^{\leftarrow}(0)$ is a maximal ideal.
Conversely, suppose the set I_M is a maximal ideal in V. Then by (12) there exists for every $x \in V$ a unique complex number $x(I_M)$ such that $x = x(I_M)e + y$ for some $y \in I_M$.
From (5) up to (8) it is easy to see that the mapping $L: x \to x(I_M)$ defines a homomorphism. Also by (9), $L^{\leftarrow}(0) = I_M$ and hence we have found a homomorphism satisfying the given property.
On the other hand this homomorphism is unique since every $x \in V$ has by (12) a unique representation and $L(e) = 1$ for every homomorphism L on V. □

REMARK 1.1.22. By Theorem 1.1.21 we obtain immediately that there exists a one-to-one mapping T of the set $\Delta(V)$ of all homomorphisms $L: V \to C$ onto the set of all maximal ideals in V. (Take $T: L \to L^{\leftarrow}(0)$.)

LEMMA 1.1.23. *Suppose* V *is a commutative Banach algebra with unit. Then for every* $x \in V$

$$\sigma_V(x) = \{L(x): L \in \Delta(V)\} \ .$$

PROOF. Suppose $x \in V$ fixed and let $L: V \to \mathbb{C}$ be a homomorphism. Then $x - L(x)e$ belongs to the set $L^{\leftarrow}(0)$ and this set is a maximal ideal by Theorem 1.1.21. Hence $x - L(x)e$ is not invertible by Theorem 1.1.15 and this means by definition $L(x) \in \sigma_V(x)$. So we have proved

$$\{L(x)\colon L \in \Delta(V)\} \subseteq \sigma_V(x) .$$

Take now an arbitrary $\lambda \in \sigma_V(x)$. Then by definition $x - \lambda e$ is not invertible and this implies by Theorem 1.1.15 that $x - \lambda e$ belongs to some maximal ideal I_M.
By Theorem 1.1.21 we can find a unique homomorphism $L: V \to C$ such that $L^{\leftarrow}(0) = I_M$. For this homomorphism we get $L(x) = \lambda$ and so

$$\sigma_V(x) \subseteq \{L(x)\colon L \in \Delta(V)\} .$$ □

LEMMA 1.1.24. *Suppose* V *is a commutative Banach algebra with unit and let* $L: V \to \mathbb{C}$ *be a homomorphism. Then the operator norm of* L *equals one.*

PROOF. By Lemma 1.1.23 we have $L(x) \in \sigma_V(x)$ for every $x \in V$. Since $x - \lambda e$ is invertible for all $\lambda \in \mathbb{C}$ with $|\lambda| > \|x\|$, i.e. $\lambda \notin \sigma_V(x)$, we obtain $|L(x)| \leq \|x\|$.
Also $L(e) = 1$, $\|e\| = 1$, and this implies together with the above inequality that the operator norm

$$\|L\| := \sup \{\frac{L(x)}{\|x\|} : x \in V,\ \|x\| \neq 0\}$$

equals one. □

In order to discuss the next theorems we need the following observations. A Banach algebra V with unit contains every polynomial

$$\tilde{\Lambda}(x) = a_0 e + a_1 x + \dots + a_k x^k ; \quad x \in V,\ k \in \mathbb{N}$$

and in general every function

$$\tilde{\Lambda}(x) = \sum_{n=0}^{\infty} a_n x^n ,$$

where $\Lambda: \mathbb{C} \to \mathbb{C}$, defined by $\Lambda(z) = \sum_0^\infty a_n z^n$, is an entire function. (Use the multiplicative norm inequality in V, the absolute convergence of the series $\sum_{n=0}^\infty a_n z^n$ for every $z \in \mathbb{C}$ and the completeness of V.)

These examples are special cases of Theorem 1.1.28 as can be seen from Theorem 1.1.30. However, before mentioning these theorems, we introduce some well-known definitions.

DEFINITION 1.1.25 (cf. [RUD-2]).

1. A *curve* in $\mathbb{C}$ is a continuous mapping γ of a compact interval $[\alpha,\beta] \subset \mathbb{R}$ $(\alpha < \beta)$ into $\mathbb{C}$. We call $[\alpha,\beta]$ the *parameter interval* of γ and denote the range of γ by γ^*, i.e. $\gamma^* := \{\gamma(t): t \in [\alpha,\beta]\}$.
2. A *path* is a piecewise continuously differentiable curve in $\mathbb{C}$.

It is now possible to define the integral of a function $\Lambda: \mathbb{C} \to V$ over the path γ^* in case Λ is continuous on an open set containing γ^*.

DEFINITION 1.1.26. Let $\Lambda: \mathbb{C} \to V$ be continuous on an open set containing the path γ^*. Then

$$\int_{\gamma^*} \Lambda(\lambda)d\lambda := \int_{\alpha}^{\beta} \Lambda(\gamma(t))\gamma'(t)dt$$

where γ is the piecewise continuously differentiable curve with parameter interval $[\alpha,\beta]$ belonging to γ^*.

REMARK 1.1.27.

1. This definition does not depend on the parametrization of γ^*. (Use the substitution theorem.) Since the integral $\int_{\alpha}^{\beta} \Lambda(\gamma(t))\gamma'(t)dt$ is defined in a similar way as the classical Riemann integral, i.e. as the limit of a Cauchy sequence $\{c_k\}$ of which every element c_k has the form

$$\sum_{n=0}^{k} \Lambda(\gamma(t_n))\gamma'(t_n)(t_{n+1}-t_n) \ ; \quad \alpha = t_0 < t_1 < \ldots < t_k = \beta \ ,$$

we obtain by the completeness of V that $\int_{\gamma^*} \Lambda(\lambda)d\lambda$ belongs to V.

THEOREM 1.1.28. *Let* $x \in V$ *be fixed and suppose*

1. $\Lambda: D \to \mathbb{C}$ *is analytic in an open set* D *containing the compact spectrum* $\sigma_V(x) = \{L(x): L \in \Delta(V)\}$.
2. Γ *is a collection of finitely many paths* $\gamma_1^*,\ldots,\gamma_n^*$ *in* D *with*

$$\gamma_i^* \cap \sigma_V(x) = \emptyset \qquad (i = 1,\dots,n)$$

and

$$\mathrm{Ind}_{\Gamma^*}(z) := \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k^*} \frac{d\lambda}{\lambda - z} = \begin{cases} 1 & \text{if } z \in \sigma_V(x) \\ 0 & \text{if } z \notin D \end{cases}$$

where

$$\Gamma^* = \bigcup_{k=1}^{n} \gamma_k^* .$$

If we define

$$\tilde{\Lambda}(x) := \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\gamma_k^*} \Lambda(\lambda)(\lambda e - x)^{-1}\, d\lambda$$

$$:= \frac{1}{2\pi i} \int_{\Gamma^*} \Lambda(\lambda)(\lambda e - x)^{-1}\, d\lambda \tag{13}$$

then

1. $\tilde{\Lambda}(x) \in V$
2. $L(\tilde{\Lambda}(x)) = \Lambda(L(x))$ *for every homomorphism* L *on* V.

<u>PROOF</u>. Note that $\lambda \to (\lambda e - x)^{-1}$ is a continuous function on the open set $\rho_V(x)$. Hence by Remark 1.1.27 and condition (1) the first result follows immediately.

Observe also by the definition of a contour integral and the continuity of every $L \in \Delta(V)$ (Lemma 1.1.24) that

$$L(\tilde{\Lambda}(x)) = \frac{1}{2\pi i} \int_{\Gamma^*} \Lambda(\lambda) L((\lambda e - x)^{-1})\, d\lambda = \frac{1}{2\pi i} \int_{\Gamma^*} \Lambda(\lambda)(\lambda - L(x))^{-1}\, d\lambda .$$

Hence, since the Cauchy formula holds (cf. [RUD-2]), we get

$$L(\tilde{\Lambda}(x)) = \Lambda(L(x)) .$$ □

<u>REMARK 1.1.29</u>.

1. In order for the Cauchy formula to hold we assumed the existence of a contour Γ^* satisfying condition (2) of Theorem 1.1.28. However, this condition is not restrictive since the first condition in Theorem 1.1.28 implies the existence of such a contour Γ^*. (A constructive proof of this

result can be found in [RUD-2].) Finally, we like to mention that we will always use in (13) (unless stated otherwise) a contour Γ^* constructed along the lines of the proof of Theorem 13.5 of [RUD-2].

2. By Lemma 1.1.23 and Theorem 1.1.28 it is easy to see that $\sigma_V(\tilde{\Lambda}(x)) = \Lambda(\sigma_V(x))$ for every Λ satisfying the conditions of Theorem 1.1.28.

THEOREM 1.1.30. *Let $x \in V$ be fixed and Λ_x the collection of all analytic functions Λ on some open set D containing $\sigma_V(x)$. Then the mapping $\Lambda \to \tilde{\Lambda}(x)$ satisfies the following properties:*

1. $\Lambda(z) \equiv 1 \to \tilde{\Lambda}(x) = e$
2. $\Lambda(z) \equiv z \to \tilde{\Lambda}(x) = x$
3. $\alpha_1\Lambda_1(z) + \alpha_2\Lambda_2(z) \to \alpha_1\tilde{\Lambda}_1(x) + \alpha_2\tilde{\Lambda}_2(x) \quad (\alpha_1,\alpha_2 \in \mathbb{C})$
4. $\Lambda(z) = \Lambda_1(z)\Lambda_2(z) \to \tilde{\Lambda}(x) = \tilde{\Lambda}_1(x)\tilde{\Lambda}_2(x)$
5. *If the sequence $\{\Lambda_n\}$ converges uniformly to Λ in every compact subset of D then the sequence $\{\Lambda_n(x)\}$ converges in norm to $\Lambda(x)$.*

PROOF. We will only prove (1), (2) and (4), since (3) and (5) are obvious. For the proof of (1) and (2) we note that the functions Λ are entire and so by Cauchy's formula we can take for the contour in (13) the positively oriented circle $\Gamma^* := \{\lambda \in \mathbb{C}: |\lambda| = \|x\| + 1\}$. On this circle $(\lambda e-x)^{-1}$ has the expansion $\sum_{n=0}^{\infty} \lambda^{-n-1} x^n$ (Lemma 1.1.4) and so substituting this in (13) and using Cauchy's formula yields the desired results (1) and (2).
In order to prove (4) we have to show that the product of $\tilde{\Lambda}_1(x)$ and $\tilde{\Lambda}_2(x)$ equals

$$\frac{1}{2\pi i} \int_{\Gamma^*} (\lambda e-x)^{-1} \Lambda_1(\lambda)\Lambda_2(\lambda)d\lambda .$$

By the construction of Γ^* (see Remark 1.1.29) it is easy to verify that $\sigma_V(x) \subset \text{int}(\Gamma^*)$ and $\text{cl}(\text{int}(\Gamma^*)) \subset D$. Applying again Theorem 13.5 of [RUD-2] with $K = \text{cl}(\text{int}(\Gamma^*))$ and $\Omega = D$, we can construct a contour Γ_1^* with

$$\Gamma_1^* \subset D - \text{cl}(\text{int}(\Gamma^*)) \quad \text{and} \quad \Lambda_1(x) = \frac{1}{2\pi i} \int_{\Gamma_1^*} \Lambda_1(\lambda)(\lambda e-x)^{-1} d\lambda .$$

Hence the product of $\Lambda_1(x)$ and $\Lambda_2(x)$ equals

$$- \frac{1}{4\pi^2} \int_{\Gamma^*} \int_{\Gamma_1^*} \Lambda_1(\lambda)\Lambda_2(\mu)(\lambda e-x)^{-1}(\mu e-x)^{-1}\, d\lambda\, d\mu \ .$$

Since

$$(\lambda e-x)^{-1} - (\mu e-x)^{-1} = (\lambda e-x)^{-1}(\mu e-x)^{-1}((\mu e-x) - (\lambda e-x)) =$$
$$= (\mu-\lambda)(\lambda e-x)^{-1}(\mu e-x)^{-1}$$

this implies that the product $\tilde{\Lambda}_1(x)\tilde{\Lambda}_2(x)$ equals the sum of

$$\frac{1}{2\pi i} \int_{\Gamma_1^*} \Lambda_1(\lambda)(\lambda e-x)^{-1} \frac{1}{2\pi i} \int_{\Gamma^*} \frac{\Lambda_2(\mu)}{\mu - \lambda}\, d\mu\, d\lambda$$

and

$$\frac{1}{2\pi i} \int_{\Gamma^*} \Lambda_2(\mu)(\mu e-x)^{-1} \frac{1}{2\pi i} \int_{\Gamma_1^*} \frac{\Lambda_1(\lambda)}{\lambda - \mu}\, d\lambda\, d\mu \ .$$

The first term in this sum equals zero (use $\Lambda_2(\mu)/(\mu-\lambda)$ is analytic on cl(int Γ^*) for every $\lambda \in \Gamma_1^*$), while the second by Cauchy's formula equals

$$\frac{1}{2\pi i} \int_{\Gamma^*} \Lambda_2(\mu)\Lambda_1(\mu)(\mu e-x)^{-1}\, d\mu \ . \qquad \square$$

Finally, we conclude this section with the following remark.

<u>REMARK 1.1.31</u>. If $0 \notin \sigma_V(x)$ ($\Leftrightarrow x^{-1}$ exists) then by (1) and (4) of Theorem 1.1.30 we obtain $x^{-1} = \tilde{\Lambda}(x)$ with $\tilde{\Lambda}(z) = 1/z$.

2. The Banach algebra of complex-valued sequences on the nonnegative integers

Let Ψ be the set of all functions $\psi: \mathbb{N} \to [0,\infty)$ satisfying $\psi(0) = 1$ and $0 < \psi(n+m) \leq \psi(n)\psi(m)$ for all $m,n \in \mathbb{N}$. It is known (cf. [HIL]) that each such ψ has the property

$$\lim_{n\to\infty} (\psi(n))^{1/n} = \inf_{n\geq 1} (\psi(n))^{1/n}$$

with $0 \leq \inf_{n\geq 1} (\psi(n))^{1/n} < \infty$.

For each $\psi \in \Psi$ let $V(\psi)$ be the set of all complex-valued sequences $\{x(n)\}_{n=0}^{\infty}$ for which

$$\|x\|_{\psi} := \sum_{n=0}^{\infty} \psi(n)|x(n)| < \infty .$$

Define in $V(\psi)$ as follows the operations of addition, scalar multiplication and multiplication:

$$(x+y)(n) := x(n) + y(n) \qquad \text{for all } n \in \mathbb{N}$$

$$(\alpha x)(n) := \alpha x(n) \qquad \text{for all } \alpha \in \mathbb{C},\ n \in \mathbb{N}$$

$$(x * y)(n) := \sum_{k=0}^{n} x(n-k)y(k) \qquad \text{for all } n \in \mathbb{N} .$$

Then the following result holds.

THEOREM 1.2.1 (cf. [RUD-1]). *The space* $V(\psi)$ *with norm* $\|\cdot\|_{\psi}$ *is a commutative Banach algebra (with unit) for every* $\psi \in \Psi$.

PROOF. The only properties of a commutative Banach algebra, which need verification, are the completeness and the (multiplicative) norm inequality. We start with the completeness.
Since $V(\psi)$ can be identified with the space $L^1(\nu)$ with ν a positive measure on the nonnegative integers defined by $\nu(\{n\}) = \psi(n)$ and $L^1(\nu)$ (cf. [RUD-2]) is a complete space, we obtain the completeness of $V(\psi)$.
The multiplicative norm inequality follows immediately by the observation that

$$\sum_{n=0}^{\infty} \psi(n)|(x*y)(n)| \leq \sum_{n=0}^{\infty}\sum_{k=0}^{n} \psi(n-k)\psi(k)|y(n-k)x(k)| =$$

$$= \sum_{k=0}^{\infty} \psi(k)|x(k)| \cdot \sum_{n=k}^{\infty} \psi(n-k)|y(n-k)| . \qquad \square$$

REMARK 1.2.2. The unit in $V(\psi)$ is given by $e = \{e(n)\}_{n=0}^{\infty}$ with $e(0) = 1$ and $e(n) = 0$ for all $n \geqq 1$. Also, since $\psi(0) = 1$, we obtain $\|e\|_{\psi} = 1$.

THEOREM 1.2.3 (cf. [RUD-1]). *Let* $\psi \in \Psi$ *and suppose*

$$C := \{z \in \mathbb{C}: |z| \leq \inf_{n\geq 1} (\psi(n))^{1/n}\} .$$

Then $L: V(\psi) \to \mathbb{C}$ *is a homomorphism if and only if there exists some* $z \in C$ *such that* $L(x) = \sum_{n=0}^{\infty} x(n)z^n$ *for all* $x \in V(\psi)$.

PROOF. Obviously for all $z \in C$ and $x \in V(\psi)$ we obtain $\sum_{n=0}^{\infty} |x(n)|z^n \leq \|x\|_{\psi}$ and so the mapping $L: V(\psi) \to \mathbb{C}$ with $L(x) = \sum_{n=0}^{\infty} x(n)z^n$ is well defined. This mapping is clearly a homomorphism for all $z \in C$.
It remains to prove that every homomorphism has this form. This can be done as follows. Consider the real valued sequence $x_0 = (0,1,0,\dots)$. By the definition of multiplication in $V(\psi)$ the real valued sequence x_0^{n*} for $n \geq 1$ equals 1 in the $(n+1)$th component and 0 in the other components. Also, $x_0^{0*} := e$. Hence every complex valued sequence x in $V(\psi)$, where $x = \{x(n)\}_{n=0}^{\infty}$, has the representation $x = \sum_{n=0}^{\infty} x(n)x_0^{n*}$ and this implies that

$$\|x - \sum_{k=0}^{n} x(k)x_0^{k*}\|_{\psi} \to 0 \qquad (n \to \infty) .$$

Thus by Lemma 1.1.24 we obtain for every homomorphism $L: V(\psi) \to \mathbb{C}$ that

$$L(x) = \lim_{n\to\infty} L\left(\sum_{k=0}^{n} x(k)x_0^{k*}\right) = \sum_{k=0}^{\infty} x(k)(L(x_0))^k .$$

We now have to prove that $L(x_0) \in C$. Since the operator norm of L equals one (Lemma 1.1.24) we get for every $n \in \mathbb{N}$ that

$$|L(x_0)|^n = |L(x_0^{n*})| \leq \|x_0^{n*}\|_{\psi} = \psi(n) .$$

This implies $|L(x_0)| \leq \inf_{n\geq 1} (\psi(n))^{1/n}$ and thus we have proved the desired result.
□

THEOREM 1.2.4. *If* $x \in V(\psi)$ *for some* $\psi \in \Psi$ *and* $\sum_{n=0}^{\infty} x(n)z^n \neq 0$ *for all* $z \in C$ *then* x *is invertible in* $V(\psi)$.

PROOF. The result follows immediately by applying Theorems 1.1.21, 1.1.15 and 1.2.3.
□

REMARK 1.2.5. It is easy to verify by the compactness of C and the continuity of the function $\hat{x}(z) = \sum_{n=0}^{\infty} x(n)z^n$ that the condition $\sum_{n=0}^{\infty} x(n)z^n \neq 0$ for all $z \in C$ is equivalent with the condition

$$\inf \left\{ \left| \sum_{n=0}^{\infty} x(n)z^n \right| : z \in C \right\} > 0 .$$

We now introduce the following class of commutative Banach algebras (with unit), which are connected with $V(\psi)$.

Let S denote the set of all positive functions $\mu: \mathbb{N} \to [0,\infty)$ for which

$$\sup_{n\geq 0} \frac{\mu^{2*}(n)}{\mu(n)} < \infty .$$

This class contains the so-called subexponential and other related sequences, discussed in [EMB-1].

For all $\psi \in \Psi$ and $\mu \in S$ we define

$$V(\psi,\mu) := \left\{ x \in V(\psi) : \bar{P}_\mu(x) := \sup_{n\geq 0} \frac{|x(n)|}{\mu(n)} < \infty \right\} ,$$

$$V^0(\psi,\mu) := \left\{ x \in V(\psi,\mu) : \lim_{n\to\infty} \frac{|x(n)|}{\mu(n)} = 0 \right\} .$$

Clearly, $V(\psi,\mu)$ is a vectorspace and it is easy to verify that

$$\|x\|_{\psi,\mu} := \|x\|_\psi + M\bar{P}_\mu(x) \qquad (x \in \bar{V}(\psi,\mu))$$

with

$$1 \leq M := \sup_{n\geq 0} \frac{\mu^{2*}(n)}{\mu(n)} < \infty$$

is a norm on $V(\psi,\mu)$.

THEOREM 1.2.6. $V(\psi,\mu)$ *with the above norm is a commutative Banach algebra (with unit) for every* $\psi \in \Psi$ *and every* $\mu \in S$ *and* $V^0(\psi,\mu)$ *is a closed subalgebra, also with unit.*

PROOF. Fix $\psi \in \Psi$, $\mu \in S$ and let $x,y \in V(\psi,\mu)$ be arbitrary. Then for every $n \in \mathbb{N}$ we have

$$|(x * y)(n)| \leq \sum_{k=0}^{n} |x(n-k)||y(k)| \leq \bar{P}_\mu(x)\bar{P}_\mu(y) \sum_{k=0}^{n} \mu(n-k)\mu(k) .$$

Hence

$$\frac{|(x * y)(n)|}{\mu(n)} \leq M\bar{P}_{\mu}(x)\bar{P}_{\mu}(y) \qquad \text{for every } n \in \mathbb{N}$$

and this implies $\bar{P}_{\mu}(xy) \leq M\bar{P}_{\mu}(x)\bar{P}_{\mu}(y)$.
Using this inequality it is immediately clear that $\|xy\|_{\psi,\mu} \leq \|x\|_{\psi,\mu}\|y\|_{\psi,\mu}$.
The completeness can be proven as follows. Let $V(\mu)$ be the set of all complex-valued sequences x for which

$$\|x\|_{\mu} := \sup_{n\geq 0} \frac{|x(n)|}{\mu(n)} < \infty .$$

This set is isomorphic to the Banach space ℓ^{∞} of bounded sequences and so it is a complete space.
Now $\{x^N\}_{N=0}^{\infty}$ is a Cauchy sequence in $V(\psi,\mu)$ if and only if $\{x^N\}_{N=0}^{\infty}$ is a Cauchy sequence in $V(\mu)$ and $V(\psi)$. Since $V(\mu)$ and $V(\psi)$ with norms respectively $\|\cdot\|_{\mu}$ and $\|\cdot\|_{\psi}$ are complete spaces, we can find elements $x_1^{\infty} \in V(\psi)$ and $x_2^{\infty} \in V(\mu)$ such that

$$\|x^N - x_1^{\infty}\|_{\psi} \to 0 \quad (N \to \infty) \qquad \text{and} \qquad \|x^N - x_2^{\infty}\|_{\mu} \to 0 \quad (N \to \infty) .$$

It is clear that $x^N(n) \to x_i^{\infty}(n)$ $(i = 1,2)$ as $N \uparrow \infty$ for every $n \in \mathbb{N}$ and so $x_1^{\infty} = x_2^{\infty}$. This implies

$$x_1^{\infty} \in V(\psi) \cap V(\mu) = V(\psi,\mu) \qquad \text{and} \qquad \lim_{N\to\infty} \|x^N - x_1^{\infty}\|_{\psi,\mu} = 0 .$$

It is easy to verify that $V^0(\psi,\mu)$ is a closed subspace of $V(\psi,\mu)$ and so we only have to verify that $V^0(\psi,\mu)$ is closed under multiplication.
Let $x_1,x_2 \in V^0(\psi,\mu)$. Then clearly

$$x_{i,n} := 1_{[0,n]}x_i \in V^0(\psi,\mu) \qquad (i = 1,2)$$

and

$$x_{1,n} * x_{2,n} \in V^0(\psi,\mu) \qquad \text{for every } n \in \mathbb{N} .$$

Also, $\lim_{n\to\infty} x_{i,n} = x_i$ $(i = 1,2)$ in $V^0(\psi,\mu)$ and so by the joint continuity of $*$ and the closedness of $V^0(\psi,\mu)$ we obtain

$$\lim_{n\to\infty} x_{1,n} * x_{2,n} = x_1 * x_2 \in V^0(\psi,\mu) . \qquad \square$$

REMARK 1.2.7. The only deficiency in Theorem 1.2.6 is that $\|e\|_{\psi,\mu} \neq 1$, where e is the (discrete) Dirac measure at zero. However, one can renorm $V(\psi,\mu)$ with an equivalent norm $\|\cdot\|^*_{\psi,\mu}$ such that $\|e\|^*_{\psi,\mu} = 1$. This can be carried out as follows:
Give any element $x \in V(\psi,\mu)$ the norm of the bounded linear transformation $T_x(y) = x * y$. Then

$$\|x\|^*_{\psi,\mu} := \sup_{y\neq\theta} \left\{\frac{\|x * y\|_{\psi,\mu}}{\|y\|_{\psi,\mu}}\right\}$$

and by the definition of $\|x\|^*_{\psi,\mu}$ and the multiplicative norm inequality one obtains

$$\frac{\|x\|_{\psi,\mu}}{\|e\|_{\psi,\mu}} \leqq \|x\|^*_{\psi,\mu} \leqq \|x\|_{\psi,\mu} .$$

Obviously, $\|e\|^*_{\psi,\mu} = 1$ and by the above inequality the different norms are equivalent. (In the sequel we assume that the equivalent norm $\|\cdot\|^*_{\psi,\mu}$ is used.)

Like in the case of $V(\psi)$ we are interested in the representation of every homomorphism on $V(\psi,\mu)$. It turns out that solving this problem is easy on the Banach subalgebra $V^0(\psi,\mu)$. However, before stating the result, we need the following observation.

REMARK 1.2.8. For every positive sequence $\{\mu(n)\}_{n=0}^{\infty}$ it is obvious that

$$\mu(m)\mu(n) \leqq \sum_{k=0}^{n+m} \mu(k)\mu(n+m-k) \qquad \text{for all } m,n \in \mathbb{N} .$$

This implies, in case $\mu \in S$, that

$$\mu(m)\mu(n) \leqq M\mu(n+m) \qquad \text{for all } m,n \in \mathbb{N} .$$

Putting $f(n) := \ln \frac{M}{\mu(n)}$ we see that $f(n)$ is a subadditive function. This proves the existence of $\lim_{n\to\infty} \frac{f(n)}{n}$ (cf. [HIL]). Moreover, since $\lim_{n\to\infty} \frac{f(n)}{n} < \infty$, we have $\lim_{n\to\infty} (\mu(n))^{1/n} > 0$.

THEOREM 1.2.9. *Let $\psi \in \Psi$ and $\mu \in S$. Then the following conditions are equivalent:*

1. $\lim_{n\to\infty} (\psi(n)\mu(n))^{1/n} \geq 1$ *(the limit exists by the preceding remark and is* $< \infty$*).*

2. *The mapping* $L\colon V^0(\psi,\mu) \to \mathbb{C}$ *is a homomorphism if and only if there exists some* $z \in C$ *such that* $L(x) = \sum_{n=0}^{\infty} x(n)z^n$ *for all* $x \in V^0(\psi,\mu)$.

PROOF. Suppose $\lim_{n\to\infty} (\psi(n)\mu(n))^{1/n} \geq 1$. Note first that $V^0(\psi,\mu)$ is a commutative Banach algebra with unit.
Since the proof of (2) is very similar to the proof of Theorem 1.2.3 we will only discuss the differences.
Observe that

$$\bar{P}_\mu\left(x - \sum_{k=0}^{n} x(k)x_0^{k*}\right) = \sup_{k\geq n+1} \frac{|x(k)|}{\mu(k)} \to 0 \qquad (n \to \infty) ,$$

for all $x \in V^0(\psi,\mu)$, and so

$$\lim_{n\to\infty} \|x - \sum_{k=0}^{n} x(k)x_0^{k*}\|_{\psi,\mu} = 0 ,$$

or equivalently

$$\lim_{n\to\infty} \|x - \sum_{k=0}^{n} x(k)x_0^{k*}\|_{\psi,\mu}^{*} = 0 .$$

This implies

$$L(x) = \sum_{k=0}^{\infty} x(k)(L(x_0))^k \qquad \text{for every } x \in V^0(\psi,\mu) .$$

It remains to prove that $\lambda := L(x_0) \in C$. Suppose

$$|\lambda| > \omega_0 := \inf_{n\geq 1} (\psi(n))^{1/n} .$$

In that case, define

$$\rho := \frac{|\lambda| + \omega_0}{2} \qquad (\Rightarrow \rho < |\lambda|)$$

and

$$\tilde{x}(n) := \frac{\rho^n}{\lambda^n \psi(n)} \qquad \text{for all } n \in \mathbb{N} .$$

Obviously, $\tilde{x} \in V(\psi)$ and $\lim_{n\to\infty} \frac{|\tilde{x}(n)|}{\mu(n)} = 0$. (Use $\lim_{n\to\infty} (\psi(n)\mu(n))^{1/n} \geqq 1$ and $\rho < |\lambda|$.) Hence $\tilde{x} \in V^0(\psi,\mu)$.

On the other hand,

$$L(\tilde{x}) = \lim_{n\to\infty} \sum_{k=0}^{n} \tilde{x}(k)\lambda^k = \lim_{n\to\infty} \sum_{k=0}^{n} \rho^k(\psi(k))^{-1}$$

and this tends to infinity since

$$\rho > \inf_{n\geqq 1} (\psi(n))^{1/n} = \lim_{n\to\infty} (\psi(n))^{1/n} .$$

Now we have obtained a contradiction since $|L(\tilde{x})| \leqq \|\tilde{x}\|_{\psi,\mu} < \infty$.

The converse can be proved as follows. Suppose

$$\lim_{n\to\infty} (\psi(n)\mu(n))^{1/n} = a , \qquad 0 < a < 1 .$$

Then there exists some $n_0 \in \mathbb{N}$ such that

$$(\psi(n)\mu(n))^{1/n} \leqq \frac{a+1}{2} < 1 \qquad \text{for all } n \geqq n_0 .$$

Hence

$$\sum_{k=n_0}^{\infty} |x(k)|\left(\left(\frac{1}{a+1} + \tfrac{1}{2}\right)\omega_0\right)^k \leqq C \sum_{k=n_0}^{\infty} \mu(k)\psi(k)\left(\frac{1}{a+1} + \tfrac{1}{2}\right)^k$$

$$\leqq C \sum_{k=n_0}^{\infty} \left(\tfrac{1}{2} + \frac{a+1}{4}\right)^k < \infty$$

for some constant C, depending on $x \in V^0(\psi,\mu)$. This implies that

$$L(x) := \sum_{k=0}^{\infty} x(k)\left(\frac{1}{a+1} + \tfrac{1}{2}\right)^k \omega_0^k$$

is a well defined mapping on $V^0(\psi,\mu)$. Clearly L is a homomorphism and we have obtained a contradiction since $\frac{1}{a+1} + \frac{1}{2} > 1$ or, equivalently, $(\frac{1}{a+1} + \frac{1}{2})\omega_0 > \omega_0$. □

For the proof of the next result we need the following simple lemma.

<u>LEMMA 1.2.10</u>. *Let* $\mu \in S$ *and* $\psi \in \Psi$. *Put* $x_n = 1_{[0,n]}x$ $(n = 1,2,\ldots)$ *for every sequence* $x = \{x(n)\}_{n=0}^{\infty}$. *Then*

$$\lim_{n\to\infty} \sup_{p\in\mathbb{N}} \frac{(\mu-\mu_n)^{2*}(p)}{\mu(p)} = 0$$

implies that

$$\lim_{n\to\infty} \|(x-x_n)^{2*}\|_{\psi,\mu} = 0 \qquad \textit{for every } x \in V(\psi,\mu) \ .$$

PROOF. Let $x \in V(\psi,\mu)$ be given. Trivially, $\lim_{n\to\infty} \|x-x_n\|_\psi = 0$, so all that must be proved is that

$$\lim_{n\to\infty} \bar{P}_\mu((x-x_n)^{2*}) = 0 \ .$$

If $M = \bar{P}_\mu(x)$, then for all $p,n \in \mathbb{N}$ we have $|x-x_n|(p) \leqq M(\mu-\mu_n)(p)$ and therefore

$$|(x-x_n)^{2*}|(p) \leqq (|x-x_n|)^{2*}(p) \leqq M^2(\mu-\mu_n)^{2*}(p) \ .$$

Dividing by $\mu(p)$ and using the assumption yields

$$\lim_{n\to\infty} \bar{P}_\mu((x-x_n)^{2*}) = 0 \ . \qquad \square$$

REMARK 1.2.11.

1. Let

$$a(n) := \sup_{p\in\mathbb{N}} \frac{(\mu-\mu_n)^{2*}(p)}{\mu(p)} \ .$$

Then $\mu \in S$ is equivalent with $a(0) < \infty$. Since $\{a(n)\}_{n=0}^\infty$ is a nonincreasing sequence, $\lim_{n\to\infty} a(n)$ exists (and is finite) in case $\mu \in S$.

2. Observe that

$$\sup_{p\in\mathbb{N}} \frac{(\mu-\mu_n)^{2*}(p)}{\mu(p)} = \sup_{p\geqq 2n+2} \frac{(\mu-\mu_n)^{2*}(p)}{\mu(p)}$$

since

$$\mathrm{supp}\ ((\mu-\mu_n)^{2*}) \subseteq [2n+2,\infty) \cap \mathbb{N} \ .$$

We now prove an analogue of Theorem 1.2.9 for the subalgebra $V(\psi,\mu)$.

THEOREM 1.2.12. *Let* $\psi \in \Psi$ *and* $\mu \in S$. *Suppose also*

$$\limsup_{n\to\infty}\ \sup_{p\in\mathbb{N}} \frac{(\mu-\mu_n)^{2*}(p)}{\mu(p)} = 0 .$$

Then the following conditions are equivalent.

1. $\lim_{n\to\infty} (\psi(n)\mu(n))^{1/n} \geq 1$.

2. *The mapping* $L\colon V(\psi,\mu) \to \mathbb{C}$ *is a homomorphism if and only if there exists some* $z \in C$ *such that* $L(x) = \sum_{n=0}^{\infty} x(n)z^n$ *for all* $x \in V(\psi,\mu)$.

PROOF. Suppose $\lim_{n\to\infty} (\mu(n)\psi(n))^{1/n} \geq 1$ and let $L\colon V(\psi,\mu) \to \mathbb{C}$ be given. Put $\tilde{L} := L/V^0(\psi,\mu)$. Then $\tilde{L}$ is a homomorphism on $V^0(\psi,\mu)$ and so by Theorem 1.2.9 there exists some $z \in C$ such that $\tilde{L}(x) = \sum_{p=0}^{\infty} x(p)z^p$ for all $x \in V^0(\psi,\mu)$. This implies in particular $\tilde{L}(x_n) = \sum_{p=0}^{\infty} x_n(p)z^p$ for every $n \in \mathbb{N}$. (Remember: $x_n := 1_{[0,n]}x$.)
On the other hand, by lemma 1.2.10 we obtain $(L(x) - L(x_n))^2 = L((x-x_n)^{2*}) \to 0$ $(n \to \infty)$ and so $\lim_{n\to\infty} L(x_n) = L(x)$. This implies (since $x_n \in V^0(\psi,\mu)$ for every $n \in \mathbb{N}$)

$$L(x) = \lim_{n\to\infty} L(x_n) = \lim_{n\to\infty} \tilde{L}(x_n) = \lim_{n\to\infty} \sum_{p=0}^{\infty} x_n(p)z^p = \sum_{p=0}^{\infty} x(p)z^p .$$

The other result can be proved in a similar way as Theorem 1.2.9. □

We shall now derive a result similar to Theorems 1.2.9 and 1.2.12 for yet another subalgebra of $V(\psi)$ and for a special choice of $\psi \in \Psi$.

DEFINITION 1.2.13. For $r \geq 1$ let $S(r)$ denote the set of probability measures $\mu\colon \mathbb{N} \to (0,1]$ satisfying

(i) $$\hat{\mu}(r) := \sum_{n=0}^{\infty} r^n\mu(n) < \infty$$

(ii) $$\lim_{p\to\infty} \frac{\mu^{2*}(p)}{\mu(p)} = 2\hat{\mu}(r)$$

(iii) $$\lim_{n\to\infty} \frac{\mu(n)}{\mu(n+1)} = r .$$

REMARK 1.2.14.

1. The members of $S(r)$ are called r-*subexponential* (cf. [EMB-1]). In the special case $r = 1$ we call them *subexponential* instead of 1-subexponential.

2. It is possible to show that the probability measure μ with $\mu(n) = Cr^n h(n)$ and h satisfying

(i) h positive,

(ii) $h \in L^1(0,\infty)$,

(iii) $\lim\limits_{n\to\infty} \frac{h(n)}{h(n+1)} = 1$,

(iv) $\overline{\lim\limits_{n\to\infty}} \sup\limits_{a\leq t\leq 1} \frac{h([nt])}{h(n)} < \infty$ for some $0 < a < 1$,

belongs to $S(r)$. (cf. Theorem 2.1.18.)
In particular, the probability measure μ with

$$\mu(n) := C\cdot r^n(1+n)^{-\beta}(\ln(e+n))^\gamma , \qquad \beta > 1, \gamma \in \mathbb{R}$$

are r-subexponential sequences.

3. An equivalent set of conditions characterizing $S(r)$ is:

(ii') $\lim\limits_{n\to\infty} \frac{\mu^{2*}(n)}{\mu(n)}$ exists

(iii') $\lim\limits_{n\to\infty} \frac{\mu(n)}{\mu(n+1)} = r$.

(cf. [CHO]).

In the next lemma we discuss another set of conditions characterizing $S(r)$ (cf. [EMB-1]).

LEMMA 1.2.15. *Let* $\mu\colon \mathbb{N} \to (0,1]$ *be a probability measure satisfying*

(i) $\hat{\mu}(r) := \sum\limits_{n=0}^{\infty} \mu(n)r^n < \infty$; $r \geq 1$

(ii) $\lim\limits_{p\to\infty} \frac{\mu^{2*}(p)}{\mu(p)} = 2\hat{\mu}(r)$

(iii) $\liminf\limits_{n\to\infty} \frac{\mu(n)}{\mu(n+1)} \geq r$.

Then necessarily $\lim_{n\to\infty} \frac{\mu(n)}{\mu(n+1)} = r$ *(or equivalently* $\mu \in S(r)$*).*

PROOF. We only have to verify that $\lim_{n\to\infty} \frac{\mu(n)}{\mu(n+1)} = r$ for $r = 1$, since the general case $r > 1$ can easily be reduced to the case $r = 1$ by putting

$$\tilde{\mu}(n) := \frac{r^n \mu(n)}{\hat{\mu}(r)} .$$

The proof for $r = 1$ can now be carried out as follows. Since by definition $\mu(n) > 0$ for every $n \in \mathbb{N}$, one easily obtains

$$\mu(n-1) \leqq \frac{1}{2\mu(1)} \left(\mu^{2*}(n) - 2 \sum_{k\neq 1}^{m} \mu(n-k)\mu(k)\right)$$

for every $m \leqq [\frac{n}{2}]$.
Hency by (iii) and (ii) it follows

$$\limsup_{n\to\infty} \frac{\mu(n-1)}{\mu(n)} \leq \frac{1}{\mu(1)} \left(1 - \sum_{k\neq 1}^{m} \mu(k)\right) \qquad \text{for every } m \in \mathbb{N} .$$

Letting $m \uparrow \infty$ finally yields

$$\limsup_{n\to\infty} \frac{\mu(n-1)}{\mu(n)} \leqq 1$$

and so

$$\lim_{n\to\infty} \frac{\mu(n-1)}{\mu(n)} = 1 .$$ □

For simplicity we write $\psi_r(n) := r^n$ and we define for all $\mu \in S(r)$ and $a \in \mathbb{C}$

$$\underset{\sim}{V}^a(\psi_r,\mu) := \{x \in V(\psi_r,\mu) : \lim_{n\to\infty} \frac{x(n)}{\mu(n)} = a\}$$

$$\underset{\sim}{V}(\psi_r,\mu) := \bigcup_{a\in\mathbb{C}} \underset{\sim}{V}^a(\psi_r,\mu) .$$

LEMMA 1.2.16. $\underset{\sim}{V}(\psi_r,\mu)$ *is a closed subalgebra of* $V(\psi_r,\mu)$ *for all* $r \geqq 1$ *and* $\mu \in S(r)$.

PROOF. We omit the proof since a proof for a similar Banach algebra in a more general setting will be given in the next section. □

LEMMA 1.2.17. *If* $\mu \in S(r)$ *for some* $r \geq 1$, *then*

$$\lim_{n\to\infty} \sup_{p\in\mathbb{N}} \frac{(\mu-\mu_n)^{2*}(p)}{\mu(p)} = 0 .$$

PROOF. Let $\varepsilon > 0$ be arbitrary. Since $\mu \in S(r)$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} r^n \mu(n) \leq \varepsilon . \qquad (1)$$

An elementary computation shows for all $p \geq 2n_0 + 2$ that

$$(\mu-\mu_{n_0})^{2*}(p) = \mu^{2*}(p) - 2 \sum_{k=0}^{n_0} \mu(p-k)\mu(k) .$$

This implies using (1) and Definition 1.2.13 that

$$\frac{(\mu-\mu_{n_0})^{2*}(p)}{\mu(p)} \leq 2\varepsilon$$

for all $p \in \mathbb{N}$ with $p \geq p_0 = p(n_0)$.
Since $(\mu-\mu_m)^{2*} \leq (\mu-\mu_n)^{2*}$ whenever $m \geq n$ it follows that for $n \geq \max(n_0, \frac{p_0}{2})$ we have

$$\sup_{p\in\mathbb{N}} \frac{(\mu-\mu_n)^{2*}(p)}{\mu(p)} = \sup_{p\geq 2n+2} \frac{(\mu-\mu_n)^{2*}(p)}{\mu(p)} \leq 2\varepsilon . \qquad \square$$

REMARK 1.2.18. We do not know whether the conclusion of Lemma 1.2.17 holds for every $\mu \in S$ (see also Remark 1.2.11).

LEMMA 1.2.19. *If* $r \geq 1$ *and* $\mu \in S(r)$, *then* $\lim_{n\to\infty} (\mu(n))^{1/n} = r^{-1}$. *(Hence the conclusion of Theorem 1.2.9 holds with* $\psi = \psi_r$ *and* $\mu \in S(r)$.*)*

PROOF. Setting $h(n) = -\ln(\mu(n))$ it follows from $\lim_{n\to\infty} \frac{\mu(n)}{\mu(n+1)} = r$ that

$$\lim_{n\to\infty} h(n+1) - h(n) = \ln r .$$

Hence, in particular, h is bounded on each finite interval, so Lemma 1.12 of [SEN] implies that $\lim_{n\to\infty} \frac{h(n)}{n} = \ln r$, i.e. $\lim_{n\to\infty} (\mu(n))^{1/n} = r^{-1}$. $\square$

We are now prepared for the proof of the announced analogue of Theorems 1.2.9 and 1.2.12 for the subalgebra $\underset{\sim}{V}(\psi_r,\mu)$ of $V(\psi_r)$ $(\mu \in S(r))$.

<u>THEOREM 1.2.20</u>. *Let* $\mu \in S(r)$ *for some* $r \geq 1$. *Then the mapping* $L\colon \underset{\sim}{V}(\psi_r,\mu) \to \mathbb{C}$ *is a homomorphism if and only if there exists some* $z \in C$ *such that* $L(x) = \sum_{n=0}^{\infty} x(n)z^n$ *for all* $x \in \underset{\sim}{V}(\psi_r,\mu)$.

<u>PROOF</u>. Let $L\colon \underset{\sim}{V}(\psi_r,\mu) \to \mathbb{C}$ be given and put $\tilde{L} := L/V^0(\psi_r,\mu)$. (Clearly, $V^0(\psi_r,\mu) \subseteq \underset{\sim}{V}^0(\psi_r,\mu) \subseteq \underset{\sim}{V}(\psi_r,\mu)$.) By Lemma 1.2.19 we obtain

$$\lim_{n\to\infty} (\psi_r(n)\mu(n))^{1/n} = 1$$

and so applying Theorem 1.2.9 yields the existence of some $z \in C$ such that

$$L(x) = \sum_{n=0}^{\infty} x(n)z^n \quad \text{for all } x \in V^0(\psi_r,\mu) .$$

On the other hand, we know by Lemmas 1.2.10 and 1.2.17 that $\lim_{n\to\infty} L(x_n) = L(x)$ for every $x \in \underset{\sim}{V}(\psi_r,\mu)$.
Since $x_n \in V^0(\psi_r,\mu)$ for every $n \in \mathbb{N}$ we finally obtain that

$$L(x) = \lim_{n\to\infty} L(x_n) = \lim_{n\to\infty} \tilde{L}(x_n) = \lim_{n\to\infty} \sum_{p=0}^{\infty} x_n(p)z^p =$$

$$= \sum_{p=0}^{\infty} x(p)z^p \quad \text{for every } x \in \underset{\sim}{V}(\psi_r,\mu) .$$

Clearly a mapping of this form defines a homomorphism. □

From the general theory of commutative Banach algebras (section 1) the results stated in Theorems 1.2.9, 1.2.12 and 1.2.20 have the following corollaries. (We do not repeat the assumptions here.)

(A) $x \in V^0(\psi,\mu)$ and x invertible in $V(\psi) \to x^{-1} \in V^0(\psi,\mu)$ (Theorem 1.2.9).

(B) $x \in V(\psi,\mu)$ and x invertible in $V(\psi) \to x^{-1} \in V(\psi,\mu)$ (Theorem 1.2.12).

(C) $x \in \underset{\sim}{V}(\psi_r,\mu)$ and x invertible in $V(\psi) \to x^{-1} \in \underset{\sim}{V}(\psi_r,\mu)$ (Theorem 1.2.20).

These corollaries are special cases of the following results, which will be stated without proof. (Proofs of similar results in a more general setting will be given in the next section.)

Before stating these results we like to remind the reader of the following well-known facts (see also Theorem 1.1.28). Suppose V is a commutative Banach algebra with unit e and Λ is an analytic function on an open set D containing the compact spectrum $\sigma_V(x)$. If Γ is a contour surrounding $\sigma_V(x)$ in D ([RUD-2]) then the formula

$$\tilde{\Lambda}(x) := \frac{1}{2\pi i} \int_\Gamma \Lambda(\lambda)(\lambda e - x)^{-1}\, d\lambda$$

defines an element of V.

Moreover, for every homomorphism $L\colon V \to \mathbb{C}$ we have

$$L(\tilde{\Lambda}(x)) = \Lambda(L(x)) \ .$$

In particular

$$\sigma_V(\tilde{\Lambda}(x)) = \Lambda(\sigma_V(x)) \ .$$

THEOREM 1.2.21. *Suppose* $\psi \in \Psi$, $x \in V(\psi)$ *and* Λ *is analytic on an open set* D *containing* $\sigma_{V(\psi)}(x)$.

1. *If* $\mu \in S$ *and* $\lim_{n\to\infty} (\psi(n)\mu(n))^{1/n} \geq 1$ *then* $x \in V^0(\psi,\mu)$ *implies* $\tilde{\Lambda}(x) \in V^0(\psi,\mu)$.

2. *If, in addition,* $\lim_{n\to\infty} \sup_{p\in\mathbb{N}} \frac{(\mu-\mu_n)^{2*}(p)}{\mu(p)} = 0$ *then* $x \in V(\psi,\mu)$ *implies* $\tilde{\Lambda}(x) \in V(\psi,\mu)$.

We now sharpen en generalize (C).

THEOREM 1.2.22. *Let* $\mu \in S(r)$ *for some* $r \geq 1$ *and let* $x \in \underset{\sim}{V}(\psi_r,\mu)$. *If* Λ *is analytic on an open set* D *containing* $\sigma_{V(\psi)}(x)$ *then* $\tilde{\Lambda}(x) \in \underset{\sim}{V}(\psi_r,\mu)$.
More precisely, if $x \in \underset{\sim}{V}^b(\psi_r,\mu)$ *then* $\tilde{\Lambda}(x) \in \underset{\sim}{V}^d(\psi_r,\mu)$ *with* $d = b\Lambda'(\sum_{n=0}^{\infty} r^n\mu(n))$.
(Λ' denotes the derivative of Λ.)

Finally, we mention the following class of Banach algebras. Let ST be the set of probability measures $\mu\colon \mathbb{N} \to (0,1]$ satisfying

$$M := \sup_{n\geq 0} \frac{\mu^{2*}([n,\infty))}{\mu([n,\infty))} < \infty \ .$$

For all $\psi \in \Psi$ and $\mu \in ST$ we now define

$$VT(\psi,\mu) := \left\{ x \in V(\psi):\ P_\mu(x) := \sup_{n\geq 0} \frac{\sum_{k=n}^{\infty} |x(k)|}{\mu([n,\infty))} < \infty \right\}$$

and

$$VT^0(\psi,\mu) := \left\{ x \in VT(\psi,\mu):\ \lim_{n\to\infty} \frac{\sum_{k=n}^{\infty} |x(k)|}{\mu([n,\infty))} = 0 \right\} .$$

Clearly, $VT(\psi,\mu)$ is a vector space.

It is also easy to verify that $\|x\|_{\psi,\mu} := \|x\|_\psi + MP_\mu(x)$ is a norm on $VT(\psi,\mu)$.

Moreover, if the multiplication is given by the convolution operation $*$, $VT(\psi,\mu)$ is a Banach algebra (with unit) and $VT^0(\psi,\mu)$ is a closed subalgebra of $VT(\psi,\mu)$.

Now we mention without proof the following results. (The proofs of these results follow the same lines as previous proofs.)

THEOREM 1.2.23. *Let* $\psi \in \Psi$ *and* $\mu \in ST$. *Then the following conditions are equivalent:*

1. $\lim_{n\to\infty} (\psi(n)\mu([n,\infty)))^{1/n} \geq 1$. *(Note that this limit always exists.)*

2. *The mapping* $L: VT^0(\psi,\mu) \to \mathbb{C}$ *is a homomorphism if and only if there exists some* $z \in C$ *such that* $L(x) = \sum_{n=0}^{\infty} x(n)z^n$ *for all* $x \in VT^0(\psi,\mu)$.

THEOREM 1.2.24. *Let* $\psi \in \Psi$ *and* $\mu \in ST$. *Suppose also*

$$\lim_{n\to\infty} \sup_{p\in\mathbb{N}} \frac{(\mu-\mu_n)^{2*}([p,\infty))}{\mu([p,\infty))} = 0 .$$

Then the following conditions are equivalent:

1. $\lim_{n\to\infty} (\psi(n)\mu([n,\infty)))^{1/n} \geq 1$.

2. *The mapping* $L: VT(\psi,\mu) \to \mathbb{C}$ *is a homomorphism if and only if there exists some* $z \in C$ *such that* $L(x) = \sum_{n=0}^{\infty} x(n)z^n$ *for all* $x \in VT(\psi,\mu)$.

At last we introduce the following set of probability measures.

DEFINITION 1.2.25. For $r \geq 1$ let ST(r) denote the set of probability measures $\mu: \mathbb{N} \to (0,1]$ satisfying

(i) $$\hat{\mu}(r) := \sum_{n=0}^{\infty} r^n \mu(n) < \infty$$

(ii) $$\lim_{p\to\infty} \frac{\mu^{2*}([n,\infty))}{\mu([n,\infty))} = 2\hat{\mu}(r)$$

(iii) $$\lim_{n\to\infty} \frac{\mu([n,\infty))}{\mu([n+1,\infty))} = r \ .$$

If we define for all $\mu \in ST(r)$, $r \geq 1$ and $a \in \mathbb{C}$,

$$\underset{\sim}{VT}^a(\psi_r,\mu) := \left\{x \in VT(\psi_r,\mu) : \lim_{n\to\infty} \frac{\sum_{k=n}^{\infty} x(k)}{\mu([n,\infty))} = a\right\}$$

and

$$\underset{\sim}{VT}(\mu_r,\mu) := \bigcup_{a\in\mathbb{C}} \underset{\sim}{VT}^a(\psi_r,\mu)$$

then it is easy to prove that $\underset{\sim}{VT}(\psi_r,\mu)$ (remember $\psi_r(n) = r^n$, $n \geq 0$) is a commutative Banach algebra.

Before mentioning the next result we like to make the following remark.

REMARK 1.2.26. The above definition and results are only interesting for $r = 1$, since for $r > 1$ one can easily prove that $S(r) = ST(r)$ and $\underset{\sim}{VT}(\psi_r,\mu) = \underset{\sim}{V}(\psi_r,\mu)$.

THEOREM 1.2.27. *Let* $\mu \in ST(r)$ *for some* $r \geq 1$. *Then the mapping* $L: \underset{\sim}{VT}(\psi_r,\mu) \to \mathbb{C}$ *is a homomorphism if and only if there exists some* $z \in C$ *such that* $L(x) = \sum_{n=0}^{\infty} x(n)z^n$ *for all* $x \in \underset{\sim}{VT}(\psi_r,\mu)$.

REMARK 1.2.28. Most of the results in this section already appeared in [CHO] and [GRÜ]. However, the purpose of this section was to unify the proofs for the O-o and limit-results.

Also we have stated the results and given the proofs since it is relatively easy to *identify every homomorphism* in the Banach algebra of complex valued sequences (in contrast to the more general Banach algebra in the next section).

Finally we like to remark that the important results (A), (B), (C) (similar results also hold for the Banach algebras $VT(\psi,\mu)$, $VT^0(\psi,\mu)$ and $\underset{\sim}{VT}(\psi_r,\mu)$!) can also be derived from the more general setting in the next section.

3. The Banach algebra of complex measures concentrated at $[0,\infty)$

Let Ψ be the set of all (Borel) measurable functions ψ: $[0,\infty) \to (0,\infty)$ satisfying

$$\psi(0) = 1 \quad \text{and} \quad 0 < \psi(x+y) \leqq \psi(x)\psi(y) \quad \text{for all } x,y \geqq 0 .$$

It is known (cf. [HIL]) that each such ψ has the following properties:

$$\psi \text{ and } \psi^{-1} \text{ are bounded on } (\varepsilon,\varepsilon^{-1}) \quad \text{for every } 0 < \varepsilon < 1$$

and

$$\inf_{x>0} \frac{\ln \psi(x)}{x} = \lim_{x\to\infty} \frac{\ln \psi(x)}{x} .$$

For each $\psi \in \Psi$ let $S(\psi)$ be the set of all complex measures ν on $[0,\infty)$ for which

$$\|\nu\|_\psi := \int_0^\infty \psi(x)\,|\nu|(dx) < \infty .$$

The term 'measure' is to be taken here in the Bourbaki sense. Alternatively, a measure ν is a complex valued set function on the bounded Borel sets which is countably additive, and therefore of bounded variation, on $\mathcal{B}([0,a])$ (= the Borel sets in $[0,a]$) for every $a \geqq 0$. The variation $|\nu|$ can of course be extended to a countably additive (generally non-finite) set function on $\mathcal{B}([0,\infty))$. It is a well-known fact (cf. [HIL]) that with the usual addition and scalar multiplication and with convolution as the product operation, $S(\psi)$, with norm $\|\cdot\|_\psi$ is a commutative Banach algebra.

Our main goal in this section is to derive Theorem 1.3.4, an essential tool in later developments. For this we must first prove an integral representation theorem (Theorem 1.3.2) for the homomorphisms of the algebra $S(\psi)$.

Special cases of this are treated in [HIL]. [ŠRE] presents a general representation, but only in the case of bounded measures without weight functions. Our approach is essentially Šreier's. We start with the following simple result.

PROPOSITION 1.3.1. *If*

$$T := \left\{ \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} : \lambda \text{ a nonnegative measure on } [0,\infty) \right\}$$

and if, for every $\psi \in \Psi$ *and* $\mu \in T$,

$$S_\mu(\psi) := \{\nu \in S(\psi) : \nu << \mu\} ,$$

then

(i) $S_\mu(\psi)$ *is a commutative Banach algebra with unit;*

(ii) $S(\psi) = \bigcup_{\mu \in T} S_\mu(\psi)$.

PROOF. Let $\psi \in \Psi$ and $\nu \in S(\psi)$ be arbitrary. Then obviously

$$\mu := \sum_{n=0}^{\infty} \frac{|\nu|^n}{n!} \in T \quad \text{and} \quad \nu << \mu .$$

Hence, $\nu \in S_\mu(\psi)$ and (ii) is proved.

To prove (i), let $\mu = \sum_{n=0}^{\infty} \lambda^n/n! \in T$ be fixed. It is easily checked that for every bounded Borel set B

$$(\mu * \mu)(B) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda^{n+k}(B)}{n!\, k!} ,$$

from which it follows that

$$\mu * \mu << \mu \quad \text{and} \quad \mu << \mu * \mu .$$

Hence for every pair $\nu_1, \nu_2 \in S_\mu(\psi)$ we have $\nu_1 * \nu_2 << \mu * \mu << \mu$, so $\nu_1 * \nu_2 \in S_\mu(\psi)$. Since evidently $\nu_1 + \nu_2$ and $\alpha\nu_1$ ($\alpha \in \mathbb{C}$) also belong to $S_\mu(\psi)$, we have thus shown that $S_\mu(\psi)$ is an algebra. Also $S_\mu(\psi)$ contains the unit $e =: \lambda^0$ of $S(\psi)$, the Dirac measure at 0.

We shall now show that $S_\mu(\psi)$ is complete by proving that it is isometrically isomorphic to $L^1(\mu)$. Indeed, let $\nu \in S_\mu(\psi)$ be given. Define

$$\rho_\nu(B) := \int_B \psi(x)\nu(dx) \qquad (B \text{ Borel}).$$

Since

$$\|\nu\|_\psi = \int_0^\infty \psi(x)\,|\nu|(dx) < \infty ,$$

the measure ρ_ν has finite variation $|\rho_\nu|([0,\infty)) = \|\nu\|_\psi$. Also, since $\nu << \mu$, we have $\rho_\nu << \mu$. Therefore the Radon-Nikodym derivative

$$h_\nu := \frac{d\rho_\nu}{d\mu} \in L^1(\mu)$$

exists and $\|\nu\|_\psi = \|h_\nu\|_1$. The map

$$S_\mu(\psi) \ni \nu \xrightarrow{\Phi_\mu} h_\nu \in L^1(\mu)$$

is therefore a linear isometry. Since for every $h \in L^1(\mu)$ the measure $\nu := (h/\psi)\mu$ is in $S_\mu(\psi)$ and $h = h_\nu$, Φ_μ is also surjective. This proves (i). □

Our next result gives an integral representation for the homomorphisms of the algebra $S(\psi)$ (cf. [SRE]).

<u>THEOREM 1.3.2</u>. *Let* T *be the set of measures defined in Proposition 1.3.1 and let* $\psi \in \Psi$ *be fixed. Suppose that* $\{g_\mu : \mu \in T\}$ *is a collection of Borel measurable complex functions on* $[0,\infty)$ *satisfying the following properties, for all* $\mu \in T$*:*

(i) $g_\mu(0) = 1$ *and* $|g_\mu| \leqq \psi$ μ *a.e.*

(ii) $g_\mu(x+y) = g_\mu(x)g_\mu(y)$ $\mu \times \mu$ *a.e.*

(iii) $\mu_1 << \mu_2 \Rightarrow g_{\mu_1} = g_{\mu_2}$ μ_1 *a.e.*

Then the formulas (one for each $\mu \in T$*)*

$$\langle L,\nu\rangle = \int_0^\infty g_\mu(x)\nu(dx) \qquad (\nu \in S_\mu(\psi)) \tag{1}$$

defines a homomorphism L *on* $S(\psi) = \bigcup_{\mu\in T} S_\mu(\psi)$. *Conversely, for each homomorphism* L *on* $S(\psi)$ *there exists a collection* $\{g_\mu : \mu \in T\}$ *of complex Borel measurable functions on* $[0,\infty)$ *satisfying* (i), (ii) *and* (iii) *above such that* (1) *holds. Moreover, the collection* $\{g_\mu : \mu \in T\}$ *is unique in the sense that each* g_μ *is determined* μ *a.e.*

PROOF. Let L be a homomorphism of $S(\psi) = \bigcup_{\mu\in T} S_\mu(\psi)$. For each $\mu \in T$, set $L_\mu := L|_{S_\mu(\psi)}$. In the proof of Proposition 1.3.1 we have defined a surjective isometry $\Phi_\mu: S_\mu(\psi) \to L^1(\mu)$. Since the adjoint Φ_μ^* is also surjective (cf. [LUE], [KAN]) and since $L^1(\mu)^*$ may be identified with $L^\infty(\mu)$, there exists an $f_\mu \in L^\infty(\mu)$ such that $L_\mu = \Phi_\mu^* f_\mu$. Observe also that $\|f_\mu\|_\infty = 1$ since $\|L_\mu\| = 1$ and Φ_μ^* is isometric. Thus, with $h_\nu = \Phi_\mu \nu$ and $g_\mu := \psi \cdot f_\mu$, we have

$$<L_\mu,\nu> = <\Phi_\mu^* f_\mu,\nu> = <f_\mu,\Phi_\mu \nu> = \int_0^\infty f_\mu(x)h_\nu(x)\mu(dx) =$$

$$= \int_0^\infty f_\mu(x)\psi(x)\nu(dx) = \int_0^\infty g_\mu(x)\nu(dx) \qquad (\nu \in S_\mu(\psi)) \ . \tag{2}$$

We shall now prove the properties (i), (ii) and (iii) for the functions g_μ $(:= \psi \cdot f_\mu)$. (i) is immediate since $\|f_\mu\|_\infty = 1$ and $1 = <L_\mu,e> = g_\mu(0)$. To prove (ii), let $\nu \in S_\mu(\psi)$ and $B_1,B_2 \in \mathcal{B}([0,\infty))$ be arbitrary. Then

$$<L_\mu,(1_{B_1}\nu) * (1_{B_2}\nu)> = <L_\mu,1_{B_1}\nu> \cdot <L_\mu,1_{B_2}\nu> \tag{3}$$

since L_μ is a homomorphism. Applying (2) to both members of (3) yields

$$\iint_{B_1\times B_2} g_\mu(x+y)\nu(dx)\nu(dy) = \iint_{B_1\times B_2} g_\mu(x)g_\mu(y)\nu(dx)\nu(dy)$$

and therefore, more generally,

$$\iint_B g_\mu(x+y)\nu(dx)\nu(dy) = \iint_B g_\mu(x)g_\mu(y)\nu(dx)\nu(dy)$$

for all $B \in \mathcal{B}([0,\infty) \times [0,\infty))$. We conclude from this that

$$g_\mu(x+y) = g_\mu(x)g_\mu(y) \qquad \nu \times \nu \quad \text{a.e.} \tag{4}$$

Since the function ψ is bounded on $(\varepsilon,\varepsilon^{-1})$ whenever $0 < \varepsilon < 1$ and since $\psi(0) = 1$, the measures $1_{\{0\}\cup(1/n,n)}\mu$ belong to $S_\mu(\psi)$ for every $n \in \mathbb{N}$. Applying (4) to $\nu = 1_{\{0\}\cup(1/n,n)}\mu$ and letting $n \to \infty$, yields (ii).
For the proof of (iii), let $\mu_1,\mu_2 \in T$ with $\mu_1 << \mu_2$ be given. Then $S_{\mu_1}(\psi) \subset S_{\mu_2}(\psi)$ and so $<L_{\mu_1},\nu> = <L_{\mu_2},\nu>$ for all $\nu \in S_{\mu_1}(\psi)$. Thus, by (2),

$$\int_0^\infty g_{\mu_1}(x)\nu(dx) = \int_0^\infty g_{\mu_2}(x)\nu(dx) \qquad \text{for all } \nu \in S_{\mu_1}(\psi) .$$

Taking $\nu = 1_{\{0\}\cup(1/n,n)}\cdot\mu_1$ and arguing as in the proof of (ii), we get that $g_{\mu_1} = g_{\mu_2}$ μ_1 a.e.
This completes the proof of the second part of the theorem. That, conversely, given a collection $\{g_\mu : \mu \in T\}$, the formulas (1) define a homomorphism, is an easy matter which we leave to the reader. Finally, to see that g_μ is μ a.e. determined for each $\mu \in T$, observe that this is so for f_μ, since Φ_μ^* is injective. □

As an illustration of the preceding theorem we mention the following well-known result (cf. [HIL]).

PROPOSITION 1.3.3. *Let* $\psi \in \Psi$ *be fixed and let* $L(\psi) := \{\nu \in S(\psi): \nu << \ell\}$, *where* ℓ *is the Lebesgue measure. Then*

(i) $L(\psi)$ *is a commutative Banach algebra (without unit);*

(ii) *there is a* 1-1 *correspondence between the set of homomorphisms* L *on* $L(\psi)$ *and the halfplane* $\{c \in \mathbb{C}: \mathrm{Re}\, c \le \inf_{x>0} \frac{\ln \psi(x)}{x}\}$, *given by*

$$\langle L,\nu\rangle = \int_0^\infty \exp(cx)\nu(dx) \qquad (\nu \in L(\psi)) . \tag{5}$$

PROOF. Set

$$\mu := \sum_{n=0}^\infty \frac{\ell^n}{n!} .$$

Then $\mu \in T$. Since ℓ is translation invariant, we have $\ell * \ell << \ell$, hence

$$\sum_{n=1}^\infty \frac{\ell^n}{n!} << \ell .$$

It easily follows from this that $S_\mu(\psi) = [e] + L(\psi)$ (e is the Dirac measure at 0 and [e] denotes the span of e) and that $L(\psi)$ is a closed subalgebra of $S_\mu(\psi)$. Thus (i) is a consequence of Proposition 1.3.1.
For the proof of (ii), note that the homomorphisms of $L(\psi)$ are the restrictions of those of $S_\mu(\psi)$, except for the one with kernel $L(\psi)$. From Theorem 1.3.2 we know that for every homomorphism L of $S_\mu(\psi)$ there exists a function

$g = g_\mu$ with $g(0) = 1$ and

$$|g| \leqq \psi \quad \mu \text{ a.e.} \quad \text{and} \quad g(x+y) = g(x)g(y) \quad \mu \times \mu \text{ a.e.} \tag{6}$$

such that

$$\langle L,\nu\rangle = \int_0^\infty g(x)\nu(dx) \qquad (\nu \in S_\mu(\psi)) .$$

Since $g(0) = \psi(0) = 1$ and $\sum_{n=1}^\infty \ell^n/n! \ll \ell$, (6) is equivalent to

$$|g| \leqq \psi \quad \ell \text{ a.e.} \quad \text{and} \quad g(x+y) = g(x)g(y) \quad \ell \times \ell \text{ a.e.} \tag{7}$$

Now it is clear that the function $g = 1_{\{0\}}$ corresponds to the trivial homomorphism on $L(\psi)$. We may therefore assume that $\ell(\{g \neq 0\}) > 0$. We then have

$$\langle L,\nu_0\rangle = \int_0^\infty g(x)\nu_0(dx) \neq 0 \qquad \text{for some } \nu_0 \in L(\psi) .$$

For every $x \in [0,\infty)$ let us denote by e_x the Dirac measure at x. It is not difficult to show that $e_x * \nu_0 \in L(\psi)$ for all $x \geqq 0$ ($\|e_x * \nu_0\|_\psi < \infty$ follows from $\psi(x+y) \leqq \psi(x)\psi(y)$ $\forall x,y$) and that the function

$$[0,\infty) \ni x \to e_x * \nu_0 \in S_\mu(\psi)$$

is continuous (see [HIL]). It follows that

$$x \to \langle L,e_x*\nu_0\rangle = \int_0^\infty g(y)(e_x*\nu_0)(dy) = \int_0^\infty g(x+y)\nu_0(dy)$$

is continuous. From the fact that $g(x+y) = g(x)g(y)$ $\ell \times \ell$ a.e., we infer that

$$\int_0^\infty g(x+y)\nu_0(dy) = g(x)\int_0^\infty g(y)\nu_0(dy) = g(x)\ \langle L,\nu_0\rangle$$

for ℓ a.a.x. Since $\langle L,\nu_0\rangle \neq 0$ it follows that g coincides ℓ a.e. with a continuous function. Observe next that changing g on an ℓ-null set does not effect (7). This is clear from the proof of Theorem 1.3.2 and can also be seen directly in this case, using $\ell * \ell \ll \ell$. We may therefore assume that g is continuous. But then $g(x+y) = g(x)g(y)$ holds everywhere and it follows that g is of the form $g(x) = \exp(cx)$ for some $c \in \mathbb{C}$ (this is well known and at any rate easy to prove). The inequality $|g| \leq \psi$ ℓ a.e. finally implies that

$$\mathrm{Re}\, c \leq \lim_{x\to\infty} \frac{\ln \psi(x)}{x} = \inf_{x>0} \frac{\ln \psi(x)}{x} .$$

Conversely, it is obvious that for every c with $\mathrm{Re}\ c \leq \inf_{x>0} \frac{\ln \psi(x)}{x}$ (5) defines a homomorphism on $L(\psi)$. It is also evident that the correspondence is 1-1. □

We now come to the principal result of this section.

<u>THEOREM 1.3.4</u>. *Let* $\psi_1, \psi_2 \in \Psi$ *satisfy*

$$\psi_1 \leq \psi_2 \qquad (\textit{hence } S(\psi_1) \supset S(\psi_2))$$

and

$$\lim_{x\to\infty} \frac{\ln \psi_1(x)}{x} = \lim_{x\to\infty} \frac{\ln \psi_2(x)}{x} > -\infty . \tag{8}$$

Let $\Delta(S(\psi_i))$ *(i = 1,2) denote the set of all homomorphisms on* $S(\psi_i)$. *Then every* $L_2 \in \Delta(S(\psi_2))$ *is the restriction to* $S(\psi_2)$ *of a unique* $L_1 \in \Delta(S(\psi_1))$.

<u>PROOF</u>. Let $L_2 \in \Delta(S(\psi_2))$ be arbitrary. By Theorem 1.3.2 there exists a collection $\{g_\mu : \mu \in T\}$ of functions g_μ satisfying

$$g_\mu(0) = 1 \quad \text{and} \quad |g_\mu| \leq \psi_2 \qquad \mu \ \text{a.e.}$$

$$g_\mu(x+y) = g_\mu(x) g_\mu(y) \qquad \mu \times \mu \ \text{a.e.}$$

$$\mu_1 << \mu_2 \Rightarrow g_{\mu_1} = g_{\mu_2} \qquad \mu_1 \ \text{a.e.}$$

so that L_2 is given by the formulas (one for each $\mu \in T$)

$$\langle L_2, \nu\rangle = \int_0^\infty g_\mu(x)\nu(dx) \qquad (\nu \in S_\mu(\psi_2)) \tag{9}$$

Suppose that we have shown that, for each $\mu \in T$,

$$|g_\mu| \leq \psi_1 \quad \mu \ \text{a.e.} \tag{10}$$

Then the formulas

$$\langle L_1,\nu\rangle = \int_0^\infty g_\mu(x)\nu(dx) \qquad (\nu \in S_\mu(\psi_1)) \tag{11}$$

clearly define the required extension of L_2, by the other half of Theorem 1.3.2.

Before proving (10) we make a preliminary observation. Let us fix $\mu \in T$ and $\psi \in \Psi$ and let us defined

$$\mu_n := 1_{\{0\}\cup[\frac{1}{n},\infty)}\mu \qquad \text{for every } n \in \mathbb{N} .$$

Then $S_{\mu_n}(\psi) := \{\nu \in S(\psi): \nu << \mu_n\}$ is a closed subalgebra of $S_\mu(\psi)$ for every $n \in \mathbb{N}$. (Use that supp $\mu_n * \mu_n \subset \{0\} \cup [\frac{1}{n},\infty)$, so that $\mu_n * \mu_n << \mu_n$.) As in the proof of Theorem 1.3.2 one shows that every homomorphism on $S_{\mu_n}(\psi)$ is given by the formula

$$\nu \to \int_0^\infty g_{\mu_n}(x)\nu(dx) \qquad (\nu \in S_{\mu_n}(\psi)) , \tag{12}$$

where g_{μ_n} is μ_n a.e. determined and satisfies

$$g_{\mu_n}(0) = 1 , \quad |g_{\mu_n}| \leq \psi \qquad \mu_n \text{ a.e.}$$

$$g_{\mu_n}(x+y) = g_{\mu_n}(x)g_{\mu_n}(y) \quad \mu_n \times \mu_n \text{ a.e.}$$

Conversely, for every such g_{μ_n} formula (12) defines a homomorphism on $S_{\mu_n}(\psi)$.

We now prove (10). Again fix $\mu \in T$ and let $\varepsilon > 0$ and $n \in \mathbb{N}$ be arbitrary. By (8) we have

$$\exp(-\varepsilon x) \cdot \psi_2(x) \leq \psi_1(x) \quad \text{for sufficiently large } x.$$

This implies, together with the fact that ψ_2/ψ_1 is bounded on (δ,δ^{-1}) whenever $0 < \delta < 1$, that for each $\nu \in S_{\mu_n}(\psi_1)$ the measure $H\nu$, defined by

$$(H\nu)(B) = \int_B \exp(-\varepsilon x)\nu(dx) ,$$

belongs to $S_{\mu_n}(\psi_2)$. Moreover, H: $S_{\mu_n}(\psi_1) \to S_{\mu_n}(\psi_2)$ is clearly an algebra homomorphism. Now let us consider the homomorphism $L_2 \circ H$ on $S_{\mu_n}(\psi_1)$. It is easily verified that $L_2 \circ H$ is given by

$$(L_2 \circ H)(\nu) = \int_0^\infty g_\mu(x)\exp(-\varepsilon x)\nu(dx) \qquad (\nu \in S_{\mu_n}(\psi_1)) \ .$$

As observed in the second paragraph of this proof we may now conclude that

$$|g_\mu(x)| \ \exp(-\varepsilon x) \leq \psi_1 \quad \mu_n \text{ a.e.} \tag{13}$$

Since $\varepsilon > 0$ and $n \in \mathbb{N}$ were arbitrary, (10) now follows from (13) by letting $\varepsilon \downarrow 0$ and $n \to \infty$.
Finally, the uniqueness of the extension L_1 is clear from the fact that $S_\mu(\psi_2)$ is dense in $S_\mu(\psi_1)$: for every $\nu \in S_\mu(\psi_1)$ we have

$$\nu_n := 1_{\{0\}\cup[\frac{1}{n},n]} \ \nu \in S_\mu(\psi_2) \qquad \text{for all } n \in \mathbb{N}$$

and

$$\lim_{n\to\infty} \|\nu_n - \nu\|_{\psi_1} = 0 \ .$$

□

REMARK 1.3.5. If

$$\lim_{x\to\infty} \frac{\ln \psi_1(x)}{x} = \lim_{x\to\infty} \frac{\ln \psi_2(x)}{x} = -\infty \ ,$$

Theorem 1.3.4 is obvious since in that case both sets $\Delta(S(\psi_i))$ $(i = 1,2)$ consist of only the element $L: \nu \to \nu(\{0\})$. This can be seen as follows: Define for every fixed $\xi > 0$ the measure e_ξ by

$$e_\xi(B) = \begin{cases} 1 & \text{if } \xi \in B \\ 0 & \text{if } \xi \notin B \end{cases} \ , \qquad B \in \mathcal{B}([0,\infty))$$

and consider an arbitrary $L \in \Delta(S(\psi))$, where $\lim\limits_{x\to\infty} \frac{\ln \psi(x)}{x} = -\infty$. Then

$$|L(e_1)| = |(L(e_1^n))^{1/n}| = |L(e_n)|^{1/n} \leq (\psi(n))^{1/n}$$

for every $n \geq 1$ and hence $L(e_1) = 0$. (Use $\lim\limits_{n\to\infty} (\psi(n))^{1/n} = 0$.) Applying Theorem 4.18.1 of [HIL] yields $L(\nu) = \nu(\{0\})$ for all $\nu \in S(\psi)$.

We now introduce the following class of commutative Banach algebras (with unit) which are connected with $S(\psi)$.

Let SMT denote the set of all Borel probability measures on $[0,\infty)$ for which

$$\sup_{t\geq 0} \frac{(\mu*\mu)([t,\infty))}{\mu([t,\infty))} < \infty \ .$$

This set contains the so-called subexponential and other related distributions discussed in [ATH], [EMB-2], [EMB-3] and [TEU]. We now define for all $\psi \in \Psi$ and $\mu \in$ SMT

$$ST(\psi,\mu) := \left\{\nu \in S(\psi) \ : \ P_\mu(\nu) \ := \sup_{t\geq 0} \frac{|\nu|([t,\infty))}{\mu([t,\infty))} < \infty\right\}$$

$$ST^0(\psi,\mu) := \left\{\nu \in ST(\psi,\mu) : \lim_{t\to\infty} \frac{|\nu|([t,\infty))}{\mu([t,\infty))} = 0\right\} \ .$$

Clearly $ST(\psi,\mu)$ is a vector space and it is easy to verify that

$$\|\nu\|_{\psi,\mu} := \|\nu\|_\psi + MP_\mu(\nu) \qquad (\nu \in ST(\psi,\mu))$$

with $M := \sup_{t\geq 0} \frac{(\mu*\mu)([t,\infty))}{\mu([t,\infty))}$ is a norm on $ST(\psi,\mu)$.

<u>PROPOSITION 1.3.6</u>. $ST(\psi,\mu)$ *with the above norm is a commutative Banach algebra with unit for every* $\psi \in \Psi$ *and every* $\mu \in$ SMT *and* $ST^0(\psi,\mu)$ *is a closed subalgebra, also with unit.*

<u>PROOF</u>. Fix $\psi \in \Psi$ and $\mu \in$ SMT and let $\nu_1,\nu_2 \in ST(\psi,\mu)$ be arbitrary. Then for every $t \geq 0$ we have

$$\begin{aligned}|\nu_1*\nu_2|([t,\infty)) &\leq \int_0^t |\nu_2|([t-x,\infty))|\nu_1|(dx) + |\nu_2|([0,\infty))|\nu_1|([t,\infty)) \\ &\leq P_\mu(\nu_2)\left[\int_0^t \mu([t-x,\infty))|\nu_1|(dx) + \mu([0,\infty))|\nu_1|([t,\infty))\right] \\ &= P_\mu(\nu_2)(|\nu_1|*\mu)([t,\infty)) \ .\end{aligned}$$

Repeating this argument and dividing by $\mu([t,\infty))$ we find

$$\frac{|\nu_1*\nu_2|([t,\infty))}{\mu([t,\infty))} \leq P_\mu(\nu_1)P_\mu(\nu_2)\,\frac{(\mu*\mu)([t,\infty))}{\mu([t,\infty))} \ ,$$

so

$$P_\mu(\nu_1*\nu_2) \leq MP_\mu(\nu_1)P_\mu(\nu_2) \ .$$

This implies that

$$\|\nu_1*\nu_2\|_{\psi,\mu} = \|\nu_1*\nu_2\|_\psi + MP_\mu(\nu_1*\nu_2) \leq$$
$$\leq \|\nu_1\|_\psi \|\nu_2\|_\psi + M^2P_\mu(\nu_1)P_\mu(\nu_2) \leq$$
$$\leq (\|\nu_1\|_\psi + MP_\mu(\nu_1))(\|\nu_2\|_\psi + MP_\mu(\nu_2)) =$$
$$= \|\nu_1\|_{\psi,\mu}\|\nu_2\|_{\psi,\mu} .$$

Hence $ST(\psi,\mu)$ is a subalgebra.
Now let (ν_n) be a Cauchy sequence. Then in particular (ν_n) is Cauchy for $\|\cdot\|_\psi$ and therefore converges for $\|\cdot\|_\psi$ to some $\nu \in S(\psi)$. Since (ν_n) is also Cauchy for $P_\mu(\cdot)$, hence a fortiori for the variation norm, (ν_n) converges in variation to a limit which obviously coincides with ν. Consequently, for every $t \geq 0$ and for every $\varepsilon > 0$ we have

$$\frac{|\nu_n-\nu|([t,\infty))}{\mu([t,\infty))} = \lim_{m\to\infty} \frac{|\nu_n-\nu_m|([t,\infty))}{\mu([t,\infty))} \leq \sup_{m\geq n} P_\mu(\nu_n-\nu_m) < \varepsilon$$

for sufficiently large n. Hence $\|\nu_n-\nu\|_{\psi,\mu} \to 0$ as $n \to \infty$ and the completeness of $ST(\psi,\mu)$ is proved.
Since it is trivial that $ST^0(\psi,\mu)$ is a closed linear subspace of $ST(\psi,\mu)$ containing e, it remains to prove that

$$ST^0(\psi,\mu) * ST^0(\psi,\mu) \subset ST^0(\psi,\mu) .$$

Let $\nu_1,\nu_2 \in ST^0(\psi,\mu)$ be given. Then clearly

$$\nu_{i,n} := 1_{[0,n]}\nu_i \in ST^0(\psi,\mu) \qquad (1 = 1,2)$$

and

$$\nu_{1,n} * \nu_{2,n} \in ST^0(\psi,\mu) \qquad \text{for every } n \in \mathbb{N} .$$

Also, $\lim_{n\to\infty} \nu_{i,n} = \nu_i$ (i = 1,2) and therefore

$$\nu_1 * \nu_2 = \lim_{n\to\infty} \nu_{1,n} * \nu_{2,n} \in ST^0(\psi,\mu)$$

by the joint continuity of $*$. □

<u>REMARK 1.3.7.</u>

(i) If $\psi \equiv 1$ and μ is a probability measure then $\mu \in SMT$ is necessary for $ST(\psi,\mu)$ to be an algebra. Indeed, since $\mu \in ST(\psi,\mu)$, we must have $\mu*\mu \in ST(\psi,\mu)$, i.e. $\mu \in SMT$.

(ii) For a positive measure μ

$$\mu([t,\infty))\mu([s,\infty)) \leqq (\mu*\mu)([t+s,\infty))$$

holds for all t and s. If we assume that $\mu \in$ SMT, i.e.

$$1 \leqq M := \sup_{t\geqq 0} \frac{(\mu*\mu)([t,\infty))}{\mu([t,\infty))} < \infty ,$$

then it follows that

$$\mu([t,\infty))\mu([s,\infty)) \leqq M\mu([t+s,\infty)) \quad \text{for all } t,s \geqq 0 .$$

Putting $f(t) := \frac{M}{\mu([t,\infty))}$ we see that $\ln f(t)$ is a subadditive function. This proves the existence of $\lim_{t\to\infty} \frac{\ln \mu([t,\infty))}{t}$ (see [HIL]). Moreover, since $\lim_{t\to\infty} \frac{\ln f(t)}{t} < \infty$, we have

$$\lim_{t\to\infty} \frac{\ln \mu([t,\infty))}{t} > -\infty .$$

THEOREM 1.3.8. *Let $\psi \in \Psi$ and $\mu \in$ SMT. Then the following conditions are equivalent.*

1. $\lim_{t\to\infty} \frac{\ln \psi(t)\mu([t,\infty))}{t} \geqq 0$.

2. *Every* $L \in \Delta(ST^0(\psi,\mu))$ *is the restriction of a unique* $\tilde{L} \in \Delta(S(\psi))$.

PROOF. We shall define a function $\bar{\psi} \in \Psi$ with the following properties:

(i) $\bar{\psi} \geqq \psi$,

(ii) $\lim_{t\to\infty} \frac{\ln \bar{\psi}(t)}{t} = \lim_{t\to\infty} \frac{\ln \psi(t)}{t} \quad (> -\infty)$,

(iii) $S(\bar{\psi}) \subset ST^0(\psi,\mu) \quad (\subset S(\psi))$.

Assuming for the moment that we have such $\bar{\psi}$, let us show how to complete the proof. For any $L \in \Delta(ST^0(\psi,\mu))$ the restriction $L|_{S(\bar{\psi})}$ has a unique extension $\tilde{L} \in \Delta(S(\psi))$, by Theorem 1.3.4. Since $S(\bar{\psi})$ is dense in $ST^0(\psi,\mu)$ for $\|\cdot\|_{\psi,\mu}$ (hint: take $\nu_n := 1_{\{0\}\cup(1/n,n)}\nu$), and therefore also for $\|\cdot\|_{\psi}$, we have

$$\tilde{L}|_{ST^0(\psi,\mu)} = L .$$

We now construct $\bar{\psi} \in \Psi$ with the above properties. Setting

$$\alpha := \lim_{t\to\infty} \frac{\ln(\psi(t)\cdot\mu([t,\infty)))}{t} ,$$

we have

$$\lim_{t\to\infty} \frac{\ln\left[\frac{\exp(\alpha t)}{\psi(t)\mu([t,\infty))}\right]}{t} = 0 .$$

Put

$$h(t) := \ln\left[\frac{\exp(\alpha t)}{\psi(t)\mu([t,\infty))}\right] .$$

Since $\lim_{t\to\infty} \frac{h(t)}{t} = 0$, there exists an increasing sequence $(t_n)_{n=0}^{\infty}$ with $\lim_{n\to\infty} t_n = \infty$ such that $|h(t)| \leqq 2^{-n}t$ whenever $t > t_n$ $(n = 0,1,\ldots)$.
We now define

$$\bar{h}(t) := \begin{cases} 2^{-n}t & \text{if } t_n < t \leqq t_{n+1} \\ \max(t, \sup_{0\leqq t\leqq t_0} h(t)) & \text{if } 0 \leqq t \leqq t_0 . \end{cases}$$

The following properties of $\bar{h}$ are evident:

$$h \leqq \bar{h} \tag{14}$$

$$\bar{h}(t+s) \leqq \bar{h}(t) + \bar{h}(s) \quad \text{for all } t,s \geqq 0 \tag{15}$$

$$\lim_{t\to\infty} \frac{\bar{h}(t)}{t} = 0 . \tag{16}$$

Now we set $\psi_0(t) = \exp(\bar{h}(t))$ and $\bar{\psi} := \psi_0\psi$ and verify that $\bar{\psi}$ has the required properties. (i) is obvious and (ii) follows from (16). Furthermore, (16) implies that $\psi_0 \in \Psi$, hence $\bar{\psi} = \psi_0\cdot\psi \in \Psi$. Finally we use the assumption $\alpha \geqq 0$ to show that (iii) holds. For all measures ν and $t \in [0,\infty)$ we have

$$\frac{|\nu|([t,\infty))}{\mu([t,\infty))} \leqq \int_t^\infty \mu([x,\infty))^{-1}|\nu|(dx) \leqq \int_t^\infty \psi(x)\cdot\frac{\exp(\alpha x)}{\psi(x)\mu([x,\infty))}|\nu|(dx) \leqq$$

$$\leqq \int_t^\infty \psi(x)\psi_0(x)|\nu|(dx) = \int_t^\infty \bar{\psi}(x)|\nu|(dx) .$$

Hence $\nu \in S(\bar{\psi})$ implies $\nu \in ST^0(\psi,\mu)$.
The converse can be proved by contradiction as follows: Suppose

$$\lim_{t\to\infty} \frac{\ln \psi(t)\mu([t,\infty))}{t} = a < 0$$

and let $\nu \in ST^0(\psi,\mu)$ be given. Then there exists some t_0 such that

$$\psi(t)\mu([t,\infty)) \leq \exp(\frac{at}{2}) \quad \text{for every } t \geq t_0 .$$

If $\omega_0 \geq 0$, where

$$\omega_0 := \lim_{t\to\infty} \frac{\ln \psi(t)}{t} = \inf_{t>0} \frac{\ln \psi(t)}{t}$$

consider then the integral

$$\int_{t_0}^{\infty} \int_{t_0}^{t} \exp((\omega_0 - \frac{a}{4})x)dx|\nu|(dt) .$$

By Fubini's theorem this integral equals

$$\int_{t_0}^{\infty} |\nu|([x,\infty))\exp((\omega_0 - \frac{a}{4})x)dx$$

and so, using the definition of ω_0 and $\nu \in ST^0(\psi,\mu)$ we can find some $M > 0$ such that

$$\int_{t_0}^{\infty} |\nu|([x,\infty))\exp((\omega_0 - \frac{a}{4})x)dx \leq M \int_{t_0}^{\infty} \mu([x,\infty))\psi(x)\exp(-\frac{ax}{4})dx \leq$$

$$\leq M \int_{t_0}^{\infty} \exp(\frac{ax}{4})dx < \infty .$$

Hence

$$\int_{t_0}^{\infty} \int_{t_0}^{t} \exp((\omega_0 - \frac{a}{4})x)dx|\nu|(dt)$$

is finite and since $|\nu|([0,\infty)) < \infty$ ($\omega_0 \geq 0!$) this implies

$$\int_{0}^{\infty} \exp((\omega_0 - \frac{a}{4})x)|\nu|(dx) < \infty .$$

If $\omega_0 < 0$, define $\tilde{\nu}(dx) := \exp(\omega_0 x)\nu(dx)$. Then obviously $\tilde{\nu} \in ST^0(\psi,\tilde{\mu})$, where $\psi \equiv 1$ and $\tilde{\mu}([t,\infty)) := \exp(\omega_0 t)\mu([t,\infty))$ for all $t \geq 0$. By our previous result

$$\int_0^\infty \exp((\omega_0 - \tfrac{a}{4})x)\,|\nu|(dx) = \int_0^\infty \exp(-\tfrac{ax}{4})\,|\tilde{\nu}|(dx) < \infty$$

and so, combining the two cases $\omega_0 \geqq 0$, $\omega_0 < 0$, finally yields that

$$\int_0^\infty \exp((\omega_0 - \tfrac{a}{4})x)\,|\nu|(dx)$$

is finite for every $\nu \in S^0(\psi,\mu)$.
This implies that the mapping L: $ST^0(\psi,\mu) \to \mathbb{C}$ given by

$$L(\nu) = \int_0^\infty \exp((\omega_0 - \tfrac{a}{4})x)\,\nu(dx)$$

is well-defined and clearly a homomorphism. Hence by our assumption, there exists a unique homomorphism $\tilde{L} \in \Delta(S(\psi))$ such that

$$\tilde{L}(\nu) = \int_0^\infty \exp((\omega_0 - \tfrac{a}{4})x)\,\nu(dx) \qquad \text{for every } \nu \in ST^0(\psi,\mu)$$

and so

$$\tilde{L}(\nu) = \int_0^\infty \exp(\omega_0 - \tfrac{a}{4})x)\,\nu(dx) < \infty \qquad \text{for every } \nu \in S(\psi).$$

(Use $\|\nu-\nu_n\|_\psi \to 0 \quad (n\to\infty)$, where $\nu_n = 1_{[0,n]}\nu \in ST^0(\psi,\mu)$.)
However, if

$$\nu(dx) = ((\psi(x))^{-1}\,1_{[0,1]}(x) + \exp(-(\omega_0 - \tfrac{a}{8})x)\,1_{[1,\infty)}(x))(dx)$$

we have $\nu \in S(\psi)$ and $_0\int^\infty \exp((\omega_0 - \frac{a}{4})x)\nu(dx) = \infty$. This gives a contradiction and so

$$\lim_{t\to\infty} \frac{\ln \psi(t)\mu([t,\infty))}{t} \geqq 0 \; . \qquad \square$$

For the proof of the next result we need the following simple lemma:

<u>LEMMA 1.3.9</u>. *Let* $\mu \in$ SMT *and* $\psi \in \Psi$. *Put* $\nu_n := 1_{[0,n]}\nu$ *(n = 1,2,...) for every measure* μ. *Then*

$$\lim_{n\to\infty} \sup_{t\geq 0} \frac{(\mu-\mu_n)^2([t,\infty))}{\mu([t,\infty))} = 0 \tag{17}$$

implies that

$$\lim_{n\to\infty} \|(\nu-\nu_n)^2\|_{\psi,\mu} = 0 \qquad \text{for all } \nu \in ST(\psi,\mu) \tag{18}$$

PROOF. Let $\nu \in ST(\psi,\mu)$ be given. Trivially, $\lim_{n\to\infty} \|\nu-\nu_n\|_\psi = 0$, so all that must be proved is that $\lim_{n\to\infty} P_\mu((\nu-\nu_n)^2) = 0$. If $M = P_\mu(\nu)$, then for all $t \geq 0$ and all $n \in \mathbb{N}$ we have

$$|\nu-\nu_n|([t,\infty)) \leq M(\mu-\mu_n)([t,\infty))$$

and therefore

$$|(\nu-\nu_n)^2|([t,\infty)) \leq |\nu-\nu_n|^2([t,\infty)) \leq M^2(\mu-\mu_n)^2([t,\infty)) .$$

Dividing by $\mu([t,\infty))$ and using (17), we get (18). □

REMARK 1.3.10. Observe that

$$\sup_{t\geq 0} \frac{(\mu-\mu_n)^2([t,\infty))}{\mu([t,\infty))} = \sup_{t\geq 2n} \frac{(\mu-\mu_n)^2([t,\infty))}{\mu([t,\infty))}$$

since $\text{supp}(\mu-\mu_n)^2 \subset [2n,\infty)$.

We now prove an analogue of Theorem 1.3.8 for the subalgebra $ST(\psi,\mu)$.

THEOREM 1.3.11. *Let* $\mu \in SMT$ *and* $\psi \in \Psi$ *and suppose that*

$$\lim_{n\to\infty} \sup_{t\geq 0} \frac{(\mu-\mu_n)^2([t,\infty))}{\mu([t,\infty))} = 0 .$$

Then the following conditions are equivalent:

1. $\lim_{t\to\infty} \frac{\ln \psi(t)\mu([t,\infty))}{t} \geq 0.$

2. *Every* $L \in \Delta(ST(\psi,\mu))$ *is the restriction of a unique* $\tilde{L} \in \Delta(S(\psi))$.

PROOF. Let $L \in \Delta(ST)\psi,\mu))$ be given and put $L_0 := L|_{ST^0(\psi,\mu)}$. By (1) and Theorem 1.3.8, L_0 has a unique extension $\tilde{L} \in \Delta(S(\psi))$. It remains to show that $\tilde{L}|_{ST(\psi,\mu)} = L$. By our assumption on μ and Lemma 1.3.9 we have

$$\lim_{n\to\infty} \|(\nu-\nu_n)^2\|_{\psi,\mu} = 0 \qquad \text{for every } \nu \in ST(\psi,\mu) .$$

Therefore, given $\nu \in ST(\psi,\mu)$, we have

$$L((\nu-\nu_n)^2) = (L(\nu-\nu_n))^2 \to 0 \qquad \text{as } n \to \infty ,$$

so $\lim_{n\to\infty} L(\nu_n) = L(\nu)$. Since $\nu_n \in ST^0(\psi,\mu)$ we also have $L(\nu_n) = L_0(\nu_n) = \tilde{L}(\nu_n)$ for all $n \in \mathbb{N}$. It now follows from $\lim_{n\to\infty} \|\nu-\nu_n\|_\psi = 0$ that

$$\tilde{L}(\nu) = \lim_{n\to\infty} \tilde{L}(\nu_n) = \lim_{n\to\infty} L(\nu_n) = L(\nu) .$$

The converse can be proved in a similar way as in Theorem 1.3.8, so we omit it. □

We shall now derive a result similar to Theorems 1.3.8 and 1.3.11 for yet another subalgebra of $S(\psi)$, and for a special choice of $\psi \in \Psi$. Some preparations are needed first.

DEFINITION 1.3.12. For $c \geq 0$ let $SMT(c)$ denote the set of Borel probability measures on $[0,\infty)$ satisfying

(i) $$\hat{\mu}(-c) := \int_0^\infty \exp(cx)\mu(dx) < \infty ,$$

(ii) $$\lim_{t\to\infty} \frac{(\mu*\mu)([t,\infty))}{\mu([t,\infty))} = 2\hat{\mu}(-c) ,$$

(iii) $$\lim_{t\to\infty} \frac{\mu([t,\infty))}{\mu([t+y,\infty))} = \exp(cy) \qquad \text{for all } y \geq 0 .$$

REMARK 1.3.13.

a) For $c = 0$ the members of $SMT(c)$ are called subexponential distributions (cf. [ATH], [EMB-2]). It can be proved that for each $c \geq 0$ and each $\beta > 1$ the measure μ defined by $\mu([t,\infty)) = (1+t)^{-\beta}\exp(-ct)$ belongs to $SMT(c)$.

b) An equivalent set of conditions characterizing $SMT(c)$ is

(ii)' $$\lim_{t\to\infty} \frac{(\mu*\mu)([t,\infty))}{\mu([t,\infty))} \text{ exists and is finite, and}$$

(iii) $\lim_{t\to\infty} \frac{\mu([t,\infty))}{\mu([t+y,\infty))} = \exp(cy)$ for all $y \geq 0$

(see [CHO], [TEU], [EMB-3]).

c) The convergence of (iii) is uniform on compacta in y (cf. [SEN]).

For simplicity we write $\psi_c(x) := \exp(cx)$ $(c \geq 0)$ and we define, for all $\mu \in SMT(c)$ and $a \in \mathbb{C}$,

$$\underset{\sim}{ST}^a(\psi_c,\mu) := \left\{\nu \in ST(\psi_c,\mu) : \lim_{t\to\infty} \frac{\nu([t,\infty))}{\mu([t,\infty))} = a\right\}$$

$$\underset{\sim}{ST}(\psi_c,\mu) := \bigcup_{a\in\mathbb{C}} \underset{\sim}{S}^a(\psi_c,\mu) .$$

<u>PROPOSITION 1.3.14</u>. $\underset{\sim}{ST}(\psi_c,\mu)$ *is a closed subalgebra of* $ST(\psi_c,\mu)$ *for all* $c \geq 0$ *and* $\mu \in SMT(c)$.

<u>PROOF</u>. Fix $c \geq 0$ and $\mu \in SMT(c)$. Observe that $\mu \in \underset{\sim}{ST}^1(\psi_c,\mu)$ by Definition 1.3.12. Clearly also $\underset{\sim}{ST}(\psi_c,\mu)$ is a vector space. Hence, given $\nu_1,\nu_2 \in \underset{\sim}{ST}(\psi_c,\mu)$, there are constants $a_1,a_2 \in \mathbb{C}$ and measures $\rho_1,\rho_2 \in \underset{\sim}{ST}^0(\psi_c,\mu)$ such that

$$\nu_1 = a_1\mu + \rho_1 , \quad \nu_2 = a_2\mu + \rho_2 .$$

Thus

$$\nu_1 * \nu_2 = (a_1\mu+\rho_1) * (a_2\mu+\rho_2) =$$

$$= a_1a_2\mu * \mu + a_1\mu * \rho_2 + a_2\mu * \rho_1 + \rho_1 * \rho_2 . \tag{19}$$

To prove that $\underset{\sim}{ST}(\psi_c,\mu)$ is an algebra we must show that

$$\lim_{t\to\infty} \frac{(\nu_1*\nu_2)([t,\infty))}{\mu([t,\infty))}$$

exists and is finite. First of all we have by Definition 1.3.12 that

$$\lim_{t\to\infty} \frac{a_1a_2(\mu*\mu)([t,\infty))}{\mu([t,\infty))} = 2a_1a_2\hat{\mu}(-c) . \tag{20}$$

Next we will show that

$$\lim_{t\to\infty} \frac{(\mu*\rho_1)([t,\infty))}{\mu([t,\infty))} = \hat{\rho}_1(-c) \qquad \left(:= \int_0^\infty \exp(cx)\rho_1(dx)\right) \tag{21}$$

Let $\varepsilon > 0$ be arbitrary. Since $\rho_1 \in \underset{\sim}{ST}^0(\psi_c,\mu)$, there exists a constant $M = M(\varepsilon)$ such that

$$\frac{|\rho_1([y,\infty))|}{\mu([y,\infty))} \leq \varepsilon \qquad \text{for all } y \geq M(\varepsilon) .$$

This implies that

$$\left|\int_0^{t-y} \rho_1([t-x,\infty))\mu(dx)\right| < \varepsilon(\mu*\mu)([t,\infty))$$

whenever $t \geq y \geq M(\varepsilon)$, and thus Definition 1.3.12 implies that

$$0 \leq \overline{\lim_{t\to\infty}} \frac{\left|\int_0^{t-y} \rho_1([t-x,\infty))\mu(dx)\right|}{\mu([t,\infty))} \leq 2\varepsilon\hat{\mu}(-c) \tag{22}$$

for all $y \geq M(\varepsilon)$. Since $\rho_1 \in S(\psi_c)$ there exists a constant $y(\varepsilon) \geq M(\varepsilon)$ such that

$$0 \leq \exp(cy(\varepsilon))|\rho_1|([y(\varepsilon),\infty)) < \int_{y(\varepsilon)}^{\infty} \exp(cx)|\rho_1|(dx) \leq \varepsilon . \tag{23}$$

We may assume that $\lim_{\varepsilon\to 0} y(\varepsilon) = \infty$.
Now for $t \geq y(\varepsilon)$ we have, by the Fubini theorem, that

$$\int_{t-y(\varepsilon)}^{t} \rho_1([t-x,\infty))\mu(dx) = \int_0^{y(\varepsilon)} \mu([t-x,\infty))\rho_1(dx) +$$

$$+ \mu([t-y(\varepsilon),\infty))\rho_1([y(\varepsilon),\infty)) - \mu([t,\infty))\rho_1([0,\infty)) .$$

Using Definition 1.3.12 we get

$$\lim_{t\to\infty} \frac{\int_{t-y(\varepsilon)}^{t} \rho_1([t-x,\infty))\mu(dx)}{\mu([t,\infty))} =$$

$$= \int_0^{y(\varepsilon)} \exp(cx)\rho_1(dx) + \exp(cy(\varepsilon))\rho_1([y(\varepsilon),\infty)) - \rho_1([0,\infty)) . \tag{24}$$

Combining (22), (23) and (24) and letting $\varepsilon \to 0$ yields

$$\lim_{t\to\infty} \frac{\int_0^t \rho_1([t-x,\infty))\mu(dx)}{\mu([t,\infty))} = \hat{\rho}_1(-c) - \rho_1([0,\infty))$$

and this in turn implies

$$\lim_{t\to\infty} \frac{(\mu*\rho_1)([t,\infty))}{\mu([t,\infty))} = \hat{\rho}_1(-c) \ . \tag{25}$$

Replacing ρ_1 by ρ_2 in this argument gives

$$\lim_{t\to\infty} \frac{(\mu*\rho_2)([t,\infty))}{\mu([t,\infty))} = \hat{\rho}_2(-c) \tag{26}$$

and, by similar reasoning,

$$\lim_{t\to\infty} \frac{(\rho_1*\rho_2)([t,\infty))}{\mu([t,\infty))} = 0 \ . \tag{27}$$

By (19), we conclude from (20), (25), (26) and (27) that

$$\lim_{t\to\infty} \frac{(\nu_1*\nu_2)([t,\infty))}{\mu([t,\infty))} = 2a_1a_2\hat{\mu}(-c) + a_2\hat{\rho}_1(-c) + a_1\hat{\rho}_2(-c) =$$
$$= a_1\hat{\nu}_2(-c) + a_2\hat{\nu}_1(-c) \ .$$

Thus $\underset{\sim}{\mathrm{ST}}(\psi_c,\mu)$ is an algebra, and obviously closed in $\mathrm{ST}(\psi_c,\mu)$. □

We need two more preparatory lemmas.

LEMMA 1.3.15. *If* $\mu \in \mathrm{SMT}(c)$ *for some* $c \geq 0$, *then*

$$\lim_{n\to\infty} \sup_{t\geq 0} \frac{(\mu-\mu_n)^2([t,\infty))}{\mu([t,\infty))} = 0 \ .$$

PROOF. Let $\varepsilon > 0$ be arbitrary. Since $\mu \in \mathrm{SMT}(c)$ there exists an $n_0 = n_0(\varepsilon)$ such that

$$\int_{n_0}^{\infty} \exp(cx)\mu(dx) < \varepsilon \ . \tag{28}$$

An elementary computation shows that for all $t \geq 2n_0$ we have

$$(\mu-\mu_{n_0})^2([t,\infty)) = (\mu*\mu)([t,\infty)) - 2\int_0^{n_0} \mu([t-x,\infty))\mu(dx) \ .$$

This implies, using (28) and Definition 1.3.12, that

$$\frac{(\mu-\mu_{n_0})^2([t,\infty))}{\mu([t,\infty))} < 2\varepsilon \qquad \text{for } t \geqq t_0 = t(n_0) .$$

Since $(\mu-\mu_m)^2 \leqq (\mu-\mu_n)^2$ whenever $m \geqq n$, it follows that for $n \geqq \max(n_0, \frac{1}{2}t_0)$ we have (cf. Remark 1.3.10)

$$\sup_{t\geqq 0} \frac{(\mu-\mu_n)^2([t,\infty))}{\mu([t,\infty))} = \sup_{t\geqq 2n} \frac{(\mu-\mu_n)^2([t,\infty))}{\mu([t,\infty))} \leqq 2\varepsilon . \qquad \square$$

REMARK 1.3.16. For any $\mu \in$ SMT let

$$a_n := \sup_{t\geqq 0} \frac{(\mu-\mu_n)^2([t,\infty))}{\mu([t,\infty))} \qquad (n = 0,1,\ldots) .$$

It is easy to see that $a_{n+1} \leqq a_n$ for all $n \in \mathbb{N}$. Since $a_0 < \infty$ by the definition of SMT, $\lim_{n\to\infty} a_n$ exists. We do not know if necessarily $\lim_{n\to\infty} a_n = 0$, i.e. whether the conclusion of Lemma 1.3.15 holds for all $\mu \in$ SMT.

LEMMA 1.3.17. *If* $c \geqq 0$ *and* $\mu \in$ SMT(c), *then*

$$\lim_{n\to\infty} \frac{\ln \mu([t,\infty))}{t} = -c .$$

Hence the conclusion of Theorem 1.3.8 holds with $\psi = \psi_c$ *and* $\mu \in$ SMT(c).

PROOF. Setting $h(t) := -\ln \mu([t,\infty))$, it follows from

$$\lim_{t\to\infty} \frac{\mu([t,\infty))}{\mu([t+x,\infty))} = \exp cx$$

that

$$\lim_{t\to\infty} \{h(t+x) - h(t)\} = cx \quad \text{for all } x \geqq 0 .$$

Hence, in particular, h is bounded on each finite interval so Lemma 1.12 of [SEN] implies that

$$\lim_{t\to\infty} \frac{h(t)}{t} = c , \quad \text{i.e.} \quad \lim_{t\to\infty} \frac{\ln \mu([t,\infty))}{t} = -c . \qquad \square$$

We are now prepared for the proof of the announced analogue of Theorems 1.3.8 and 1.3.11 for the subalgebra $\underset{\sim}{\mathrm{ST}}(\psi_c,\mu)$ of $S(\psi_c)$ ($\mu \in$ SMT(c)).

THEOREM 1.3.18. *Let* $\mu \in SMT(c)$ *for some* $c \geq 0$. *Then every* $L \in \Delta(\underset{\sim}{ST}(\psi_c,\mu))$ *is the restriction of a unique* $\tilde{L} \in \Delta(S(\psi_c))$.

PROOF. Let $L \in \Delta(\underset{\sim}{ST}(\psi_c,\mu))$ be given. Observe that

$$ST^0(\psi_c,\mu) \subset \underset{\sim}{ST}^0(\psi_c,\mu) \subset \underset{\sim}{ST}(\psi_c,\mu) \subset S(\psi_c) \ .$$

By Theorem 1.3.8 and Lemma 1.3.17 the restriction $L_0 := L|_{ST^0(\psi_c,\mu)}$ has a unique extension $\tilde{L} \in \Delta(S(\psi_c))$. It remains to be shown that $\tilde{L}|_{\underset{\sim}{ST}(\psi_c,\mu)} = L$. However, by Lemmas 1.3.15 and 1.3.9 we know that

$$\lim_{n\to\infty} \|(\nu-\nu_n)^2\|_{\psi_c,\mu} = 0 \qquad \text{for every } \nu \in \underset{\sim}{ST}(\psi_c,\mu) \ .$$

Since $\nu_n \in ST^0(\psi_c,\mu)$ for all $n \in \mathbb{N}$, we conclude that

$$L(\nu) = \lim_{n\to\infty} L(\nu_n) = \lim_{n\to\infty} \tilde{L}(\nu_n) = \tilde{L}(\nu) \qquad \text{for all } \nu \in \underset{\sim}{ST}(\psi_c,\mu) \ . \quad \square$$

REMARK 1.3.19. Let V be a commutative Banach algebra with unit e. It is well-known (cf. Lemma 1.1.23) that the spectrum $\sigma_V(x) := \{\lambda \in \mathbb{C}: \lambda e - x \text{ not invertible}\}$ of an element $x \in V$ equals the set $\{L(x): L \in \Delta(V)\}$. If W is a subalgebra of V that contains e and is a Banach algebra (possibly for a norm other than that of V) then in general we have $\sigma_V(x) \subsetneq \sigma_W(x)$ for $x \in W$ (σ_V, respectively σ_W denotes the spectrum of x taken with respect to V, respectively W.) If, however, every $L \in \Delta(W)$ is the restriction of an $\tilde{L} \in \Delta(V)$, unique or not, then clearly $\sigma_V(x) = \sigma_W(x)$ for every $x \in W$. In particular, if x^{-1} exists in V, then $x^{-1} \in W$. (Observe that inverses are unique.) Hence Theorems 1.3.8, 1.3.11 and 1.3.18 have the following corollaries (we do not repeat the assumptions here):

(A) $\nu \in ST^0(\psi,\mu)$ and ν invertible in $S(\psi)$ $\Rightarrow \nu^{-1} \in ST^0(\psi,\mu)$ (Theorem 1.3.8)

(B) $\nu \in ST(\psi,\mu)$ and ν invertible in $S(\psi)$ $\Rightarrow \nu^{-1} \in ST(\psi,\mu)$ (Theorem 1.3.11)

(C) $\nu \in \underset{\sim}{ST}(\psi_c,\mu)$ and ν invertible in $S(\psi_c)$ $\Rightarrow \nu^{-1} \in \underset{\sim}{ST}(\psi_c,\mu)$ (Theorem 1.3.18)

These corollaries are special cases of more general results as we shall now show. Let us first recall some well-known facts (cf. Theorem 1.1.28). Suppose that V is a commutative Banach algebra with unit e, that

$x \in V$ and that Λ is an analytic function on an open set D containing the (compact) spectrum $\sigma_V(x)$. If Γ is a contour which surrounds $\sigma_V(x)$ in D (cf. [RUD-2] for this terminology), then the formula

$$\tilde{\Lambda}(x) := \frac{1}{2\pi i} \int_\Gamma \Lambda(\lambda)(\lambda e - x)^{-1}\, d\lambda \tag{29}$$

defines an element Γ of V which is independent of the choice of Γ. Moreover, for every $L \in \Delta(V)$ we have

$$L(\tilde{\Lambda}(x)) = \Lambda(L(x)) \ . \tag{30}$$

In particular, by Remark 1.3.19, $\sigma_V(\tilde{\Lambda}(x)) = \Lambda(\sigma_V(x))$.

If $x \in V$ is invertible, i.e. $0 \notin \sigma_V(x)$, then the function $\Lambda(\lambda) := \lambda^{-1}$ is analytic on a neighbourhood of $\sigma_V(x)$ and it can easily be shown (Remark 1.1.31) that for this choice of Λ we have $\tilde{\Lambda}(x) = x^{-1}$. Thus (A) and (B) above are special cases of the next result.

THEOREM 1.3.20. *Suppose that* $\psi \in \Psi$, $\nu \in S(\psi)$ *and that* Λ *is analytic on an open set containing* $\sigma(\nu) = \sigma_{S(\psi)}(\nu)$.

(i) *If* $\mu \in$ SMT *and* $\lim\limits_{t\to\infty} \dfrac{\ln(\psi(t)\mu([t,\infty)))}{t} \geq 0$,

then $\nu \in ST^0(\psi,\mu)$ *implies* $\tilde{\Lambda}(\nu) \in ST^0(\psi,\mu)$

(ii) *If, in addition,* $\lim\limits_{n\to\infty} \sup\limits_{t\geq 0} \dfrac{(\mu-\mu_n)^2([t,\infty))}{\mu([t,\infty))} = 0$,

then $\nu \in ST(\psi,\mu)$ *implies* $\tilde{\Lambda}(\nu) \in ST(\psi,\mu)$.

PROOF. Since by Theorems 1.3.8 and 1.3.11 and Remark 1.3.19 we have

$$\sigma_{ST^0(\psi,\mu)}(\nu) = \sigma_{S(\psi)}(\nu) \quad \text{(in case (i))}$$

and

$$\sigma_{ST(\psi,\mu)}(\nu) = \sigma_{S(\psi)}(\nu) \quad \text{(in case (ii))},$$

the formula (29) defines an element of $ST^0(\psi,\mu)$, respectively $ST(\psi,\mu)$. □

We now generalize and at the same time sharpen (C).

<u>THEOREM 1.3.21</u>. *Let* $\mu \in \mathrm{SMT}(c)$ *for some* $c \geq 0$ *and let* $\nu \in \underset{\sim}{\mathrm{ST}}(\psi_c,\mu)$. *If* Λ *is analytic on an open set containing* $\sigma_{S(\psi_c)}(\nu)$, *then* $\widetilde{\Lambda}(\nu) \in \underset{\sim}{\mathrm{ST}}(\psi_c,\mu)$. *More precisely, if* $\nu \in \underset{\sim}{\mathrm{ST}}^b(\psi_c,\mu)$ *then* $\widetilde{\Lambda}(\nu) \in \underset{\sim}{\mathrm{ST}}^d(\psi_c,\mu)$ *with* $d = b\Lambda'({}_0\int^\infty \exp(cx)\,\nu(dx))$ *(*Λ' *denotes the derivatives of* Λ*.)*

<u>PROOF</u>. The first statement follows exactly as in the previous proof, this time by Theorem 1.3.18. What remains to be proved is that

$$\lim_{t\to\infty} \frac{(\widetilde{\Lambda}(\nu))([t,\infty))}{\mu([t,\infty))} = b\Lambda'\left(\int_0^\infty \exp(cx)\nu(dx)\right) . \tag{31}$$

(Observe that $\Lambda'({}_0\int^\infty \exp(cx)\nu(dx))$ makes sense: $\nu \to {}_0\int^\infty \exp(cx)\nu(dx)$ is an element of $\Delta(S(\psi_c))$, so ${}_0\int^\infty \exp(cx)\nu(dx) \in \sigma_{S(\psi_c)}(\nu)$.) For simplicity we write

$$\rho := \frac{1}{2\pi i} \int_\Gamma \Lambda(\lambda)(\lambda e-\nu)^{-1}\, d\lambda \qquad (= \widetilde{\Lambda}(\nu)) \tag{32}$$

$$\rho_1 := \frac{1}{2\pi i} \int_\Gamma \Lambda'(\lambda)(\lambda e-\nu)^{-1}\, d\lambda \quad (= \widetilde{\Lambda}'(\nu)) . \tag{33}$$

Note that also $\rho_1 \in \underset{\sim}{\mathrm{ST}}(\psi_c,\mu)$. Taking the Laplace transforms of ρ and ρ_1 (which exist for Re $\lambda \leq c$ and are analytic for Re $\lambda < c$) we find, using (30), that

$$\int_0^\infty \exp(\lambda x)\rho(dx) = \Lambda\left(\int_0^\infty \exp(\lambda x)\nu(dx)\right) \tag{34}$$

$$\int_0^\infty \exp(\lambda x)\rho_1(dx) = \Lambda'\left(\int_0^\infty \exp(\lambda x)\nu(dx)\right) . \tag{35}$$

With these notations (31) becomes

$$\lim_{t\to\infty} \frac{\rho([t,\infty))}{\mu([t,\infty))} = b\Lambda'\left(\int_0^\infty \exp(cx)\nu(dx)\right) . \tag{36}$$

Before proving (36) we observe that (34) implies that for all λ with Re $\lambda < c$,

$$\int_0^\infty \exp(\lambda x)x\rho(dx) = \frac{d}{d\lambda}\left(\int_0^\infty \exp(\lambda x)\rho(dx)\right) =$$

$$= \frac{d}{d\lambda}\left(\Lambda\left(\int_0^\infty \exp(\lambda x)\nu(dx)\right)\right) =$$

$$= \int_0^\infty \exp(\lambda x)x\nu(dx) \cdot \Lambda'\left(\int_0^\infty \exp(\lambda x)\nu(dx)\right) =$$

$$= \int_0^\infty \exp(\lambda x)x\nu(dx) \cdot \int_0^\infty \exp(\lambda x)\rho_1(dx) =$$

$$= \int_0^\infty \exp(\lambda x)(\tilde{\nu}*\rho_1)(dx)$$

with $\tilde{\nu}(dx) := x\nu(dx)$. Since a measure is uniquely determined by its Laplace transform (see Chapter 3) it follows that

$$x\rho(dx) = (\tilde{\nu}*\rho_1)(dx) \quad . \tag{37}$$

Now we prove (36). Fix $a > 0$. For all $t \geq 2a$ we have, by (37),

$$\rho([t,\infty)) = \int_t^\infty \frac{1}{x}(\rho_1*\tilde{\nu})(dx) = \int_0^\infty \frac{1}{x} 1_{\{x\geq t\}}(\rho_1*\tilde{\nu})(dx) =$$

$$= \int_0^\infty\int_0^\infty \frac{y}{x+y} 1_{\{x+y\geq t\}}\rho_1(dx)\nu(dy) =$$

$$= \int_0^a\int_{t-y}^\infty \frac{y}{x+y}\rho_1(dx)\nu(dy) + \int_0^a\int_{t-x}^\infty \frac{y}{x+y}\nu(dy)\rho_1(dx) +$$

$$+ \int_a^{t-a}\int_{t-y}^\infty \frac{y}{x+y}\rho_1(dx)\nu(dy) + \int_{t-a}^\infty\int_a^\infty \frac{y}{x+y}\rho_1(dx)\nu(dy) \quad .$$

We shall separately estimate the last four integrals. Let us call them I_1, I_2, I_3, I_4, respectively. We have

$$|I_1| \leq \frac{a}{t}|\rho_1|([t-a,\infty))|\nu|([0,a])$$

and therefore, using that $\rho_1 \in \underset{\sim}{ST}(\psi_c,\mu)$ and $\mu \in SMT(c)$,

$$\lim_{t\to\infty} \frac{I_1}{\mu([t,\infty))} = 0 \ . \tag{38}$$

Since $\nu \in \underset{\sim}{ST}^b(\psi_c,\mu)$ and $\mu \in SMT(c)$, it follows that

$$\lim_{t\to\infty} \frac{I_2}{\mu([t,\infty))} = b \int_0^a \exp(cx)\rho_1(dx) \ . \tag{39}$$

For I_3 we have

$$|I_3| \leq \int_a^{t-a} |\rho_1|([t-y,\infty))|\nu|(dy) \ . \tag{40}$$

Since $|\rho_1|, |\nu| \in ST(\psi_c,\mu)$, Lemma 1.3.15 implies that for a given $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\sup_{t\geq 0} \frac{(|\rho_1|-|\rho_1|_{n_0}) * (|\nu|-|\nu|_{n_0})([t,\infty))}{\mu([t,\infty))} < \varepsilon \ .$$

Now it follows from (40) that for $a \geq n$ (and $t \geq 2a$) we have

$$|I_3| \leq (|\rho_1| - |\rho_1|_n) * (|\nu|-|\nu|_n)([t,\infty)) \ .$$

Hence

$$\lim_{a\to\infty} \overline{\lim_{t\to\infty}} \frac{I_3}{\mu([t,\infty))} = 0 \ . \tag{41}$$

Finally, since $\mu \in SMT(c)$ and $\nu \in \underset{\sim}{ST}(\psi_c,\mu)$ we get

$$\overline{\lim_{t\to\infty}} \frac{|I_4|}{\mu([t,\infty))} \leq P_\mu(\nu)\exp(ca)|\rho_1|([a,\infty)) \ ,$$

so

$$\lim_{a\to\infty} \overline{\lim_{t\to\infty}} \frac{|I_4|}{\mu([t,\infty))} = 0 \tag{42}$$

since $\rho_1 \in S(\psi_c)$. Combining (38), (39), (40) and (42) we see by first choosing a large and then letting $t \to \infty$, that (36) holds. □

The previous results will be used to give very short proofs of first order renewal theorems. In order to prove second order renewal theorems we need analogous results for a different class of Banach algebras which we shall now discuss.

DEFINITION 1.3.22. SM denotes the set of all Lebesgue measurable functions $m: [0,\infty) \to (0,\infty)$ satisfying

(i) m and $\frac{1}{m}$ are locally bounded on $[0,\infty)$;

(ii) $$\sup_{t\geq 0} \frac{(m*m)(t)}{m(t)} := \sup_{t\geq 0} \frac{{}_0\int^t m(t-y)m(y)\,dy}{m(t)} < \infty ;$$

(iii) $$\overline{\lim_{t\to\infty}} \sup_{|x|\leq a} \frac{m(t)}{m(t+x)} < \infty$$ for some (hence for every) $a > 0$.

REMARK 1.3.23.

(a) If m is non-increasing then for every $a > 0$ and $t \geq a$ we have

$$m(t-a)\int_a^t m(x)\,dx \leq \int_a^t m(t-x)m(x)\,dx \leq \int_0^t m(t-x)m(x)\,dx .$$

From this it easily follows that (ii) implies (iii). In this case also (i) holds.

(b) It follows from (i) and (iii) that

$$\sup_{t\geq 0} \sup_{|x|\leq a} \frac{m(t)}{m(t+x)} < \infty \quad \text{for every } a > 0 .$$

Let $m \in SM$ and $\psi \in \Psi$ be arbitrary. Furthermore, let us fix $h > 0$ once and for all and let us write $A := [0,h]$. For any measure ν on $[0,\infty)$ we define

$$\bar{P}_m(\nu) := \sup_{t\geq 0} \frac{|\nu|(A+t)}{m(t)} .$$

We are interested in the following spaces of measures:

$$S(\psi,m) := \{\nu \in S(\psi) : \bar{P}_m(\nu) < \infty\}$$

$$S^0(\psi,m) := \{\nu \in S(\psi) : \lim_{t\to\infty} \frac{|\nu|(A+t)}{m(t)} = 0\} .$$

Using Remark 1.3.23(b) one easily checks that these sets do not depend on the choice of h.

Our first goal is to prove that a norm can be defined on $S(\psi,m)$ so that it becomes a Banach algebra.

<u>LEMMA 1.3.24</u>. *There exists a constant* $M > 0$ *such that*

$$\bar{P}_m(\nu_1*\nu_2) \leq M\bar{P}_m(\nu_1)\bar{P}_m(\nu_2) \quad \textit{for all } \nu_1,\nu_2 \in S(\psi,m) \ . \tag{43}$$

<u>PROOF</u>. Let $\nu_1,\nu_2 \in S(\psi,m)$ be given. For every $t \geq 0$ we have

$$|\nu_1*\nu_2|(A+t) \leq (|\nu_1|*|\nu_2|)(A+t) =$$

$$= \int_0^t |\nu_1|(A+t-x)|\nu_2|(dx) + \int_t^{t+h} |\nu_1|([0,t+h-x])|\nu_2|(dx) \leq$$

$$\bar{P}_m(\nu_1)\Big(\int_0^t m(t-x)|\nu_2|(dx) + m(0)|\nu_2|(A+t)\Big) \ . \tag{44}$$

Set $K := \sup_{t\geq 0}\ \sup_{|x|\leq h} \frac{m(t)}{m(t+x)}$. Then

$$\int_0^t m(t-x)|\nu_2|(dx) = \sum_{k=0}^{[t/h]-1} \int_{kh}^{(k+1)h} m(t-x)|\nu_2|(dx) + \int_{h[t/h]}^{t} m(t-x)|\nu_2|(dx) \leq$$

$$\leq K \sum_{k=0}^{[t/h]-1} m(t-kh)|\nu_2|(A+kh) + Km(0)|\nu_2|([h[t/h],t]) <$$

$$< K\bar{P}_m(\nu_2)\Big(\sum_{k=0}^{[t/h]-1} m(t-kh)m(kh) + Km(0)m(t)\Big) \ . \tag{45}$$

Also

$$\sum_{k=0}^{[t/h]-1} m(t-kh)m(kh) \leq \frac{K^2}{h} \int_0^t m(t-x)m(x)dx \ . \tag{46}$$

Combining (44), (45) and (46) we find

$$|\nu_1*\nu_2|(A+t) \leq \frac{K^3}{h}\bar{P}_m(\nu_1)\bar{P}_m(\nu_2) \int_0^t m(t-x)m(x)dx + (K^2+1)m(0)m(t)\bar{P}_m(\nu_1)\bar{P}_m(\nu_2) \ .$$

Dividing by $m(t)$ and using Definition 1.3.22(ii), we get (43). □

We now define

$$\|\nu\|_{\psi,m} := \|\nu\|_{\psi} + M\bar{P}_m(\nu) \qquad (\nu \in S(\psi,m))$$

where M is a constant satisfying (43). Clearly $\|\cdot\|_{\psi,m}$ is a norm on the vector space $S(\psi,m)$. Moreover, we have

PROPOSITION 1.3.25. $S(\psi,m)$ *with the above norm is a commutative Banach algebra with unit for every* $\psi \in \Psi$ *and* $m \in SM$ *and* $S^0(\psi,m)$ *is a closed subalgebra, also with unit.*

PROOF. The inequality

$$\|\nu_1 * \nu_2\|_{\psi,m} \leq \|\nu_1\|_{\psi,m} \cdot \|\nu_2\|_{\psi,m}$$

is a consequence of Lemma 1.3.24. The completeness of $S(\psi,m)$ and the fact that $S^0(\psi,m)$ is closed are shown almost as in Proposition 1.3.6. Observe that we do not have $\|e\|_{\psi,m} = 1$. However, it is well known that an equivalent norm $\|\cdot\|$ exists such that $S(\psi,m)$ is a Banach algebra with $\|e\| = 1$ (cf. [HIL]). □

REMARK 1.3.26. Since

$$\sup_{t\geq 0} \sup_{|x|\leq \frac{1}{2}} \frac{m(t)}{m(t+x)} < \infty ,$$

it is easily verified that there exists a constant $K_1 > 0$ such that $m(x)m(y) \leq K_1 m(x+y)$ whenever $\min(x,y) \leq \frac{1}{2}$. Now suppose $x,y > \frac{1}{2}$. Then for the same reason we have, for some constant K_2 that $m(x)m(y) \leq K_2 m(z)m(x+y-z)$ whenever $z \in [x-\frac{1}{2},x+\frac{1}{2}]$. Hence

$$m(x)m(y) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} m(x)m(y)dz \leq K_2 \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} m(x+y-z)m(z)dz \leq$$

$$\leq K_2 \int_0^{x+y} m(x+y-z)m(z)dz .$$

From Definition 1.3.22(ii) we now conclude that there is a constant $K_3 > 0$ such that $m(x)m(y) \leq K_3 m(x+y)$ whenever $x,y > \frac{1}{2}$. Combining both cases, we find a constant K such that $m(x)m(y) \leq Km(x+y)$ for all x and y. Thus K/m is a submultiplicative function, so $-\ln m + \ln K$ is subadditive. The theory of subadditive functions now shows that $\lim_{t\to\infty} \frac{\ln m(t)}{t}$ exists and $-\infty < \lim_{t\to\infty} \frac{\ln m(t)}{t} \leq \infty$.

The next result should be compared with Theorem 1.3.8. The proof is virtually identical.

<u>THEOREM 1.3.27</u>. *Let* $\psi \in \Psi$ *and* $m \in SM$. *Then the following conditions are equivalent:*

1. $\lim_{t\to\infty} \frac{\ln \psi(t)m(t)}{t} \geqq 0$ *(limit exists by preceding remark!)*

2. *Every* $L \in \Delta(S^0(\psi,m))$ *is the restriction of a unique* $\tilde{L} \in \Delta(S(\psi))$.

We now aim for an analogue of Theorem 1.3.11. The next lemma should be compared to Lemma 1.3.9.

<u>LEMMA 1.3.28</u>. *Let* $\psi \in \Psi$ *and* $m \in SM$. *If*

$$\lim_{n\to\infty} \sup_{t\geqq 2n} \frac{\int_n^{t-n} m(t-x)m(x)dx}{m(t)} = 0 ,$$

then

$$\lim_{n\to\infty} \|(\nu-\nu_n)^2\|_{\psi,m} = 0 \qquad \text{for every } \nu \in S(\psi,m).$$

<u>PROOF</u>. Since obviously $\lim_{n\to\infty} \|(\nu-\nu_n)^2\|_\psi = 0$, we only have to show that

$$\lim_{n\to\infty} \sup_{t\geqq 0} \frac{|(\nu-\nu_n)^2|(A+t)}{m(t)} = 0 .$$

Since $\operatorname{supp} |\nu-\nu_n|^2 \subset [2n,\infty)$ we have, also using Remark 1.3.23(b), that

$$\sup_{t\geqq 0} \frac{|(\nu-\nu_n)^2|(A+t)}{m(t)} \leqq \sup_{t\geqq 0} \frac{|\nu-\nu_n|^2(A+t)}{m(t)} \leqq C \sup_{t\geqq 2n} \frac{|\nu-\nu_n|^2(A+t)}{m(t)} ,$$

for some constant $C > 0$. Now fix $n \in \mathbb{N}$ and let $t \geqq 2n$ be arbitrary. Then

$$|\nu-\nu_n|^2(A+t) \leqq \int_n^{t-n} |\nu|([t-x,t+h-x])|\nu|(dx) + |\nu|([t-n,t-n+h])|\nu|([n,n+h]) .$$

Arguing as in the proof of Lemma 1.3.24 we see that there exists a constant M such that

$$\frac{|\nu-\nu_n|^2(A+t)}{m(t)} \leq M\left[\int_n^{t-n} \frac{m(t-x)m(x)dx}{m(t)} + \frac{m(t-n)m(n)}{m(t)}\right] .$$

By assumption the first term on the right hand side is small for large n. Also we have, for suitable constants $K_1, K_2 > 0$ that

$$m(t-n)m(n) = \int_n^{n+1} m(t-n)m(n)dx \leq K_1 \int_n^{n+1} m(t-x)m(x)dx \leq$$

$$\leq K_2 \int_n^{n+1} m(t+1-x)m(x)dx \leq K_2 \int_n^{t+1-n} m(t+1-x)m(x)dx ,$$

so $\frac{m(t-n)m(n)}{m(t)}$ is also small for large n, again by the assumption. This completes the proof. □

THEOREM 1.3.29. *Let* $\psi \in \Psi$, $m \in SM$ *and suppose*

$$\lim_{n\to\infty} \sup_{t\geq 2n} \frac{\int_n^{t-n} m(t-x)m(x)dx}{m(t)} = 0 .$$

Then the following conditions are equivalent:

1. $\lim_{t\to\infty} \frac{\ln \psi(t)m(t)}{t} \geq 0$;

2. *Every* $L \in \Delta(S(\psi,m))$ *is the restriction of a unique* $\tilde{L} \in \Delta(S(\psi))$.

PROOF. Replace $ST^0(\psi,\mu)$ and $ST(\psi,\mu)$ by $S^0(\psi,m)$ respectively $S(\psi,m)$ in the proof of Theorem 1.3.11 and use Lemma 1.3.28 instead of Lemma 1.3.9. This proves the first part. The second part ($2 \to 1$) can be proved almost identical as the corresponding part in Theorem 1.3.8. □

Finally, we derive an analogue of Theorem 1.3.18.

DEFINITION 1.3.30. For each $c \in \mathbb{R}$ let SM(c) be the set of all Lebesgue measurable functions m: $[0,\infty) \to (0,\infty)$ satisfying

(i) m and $\frac{1}{m}$ are locally bounded on $[0,\infty)$;

(ii) $\lim_{t\to\infty} \frac{m(t)}{m(t+y)} = \exp(cy)$ for all $y \in [0,\infty)$;

(iii) $\hat{m}(-c) := \int_0^\infty \exp(ct)m(t)dt < \infty$;

(iv) $\lim_{t\to\infty} \frac{(m*m)(t)}{m(t)} = 2\hat{m}(-c) < \infty$.

<u>REMARK 1.3.31.</u>

(a) Definition 1.3.30(ii) implies that $\lim_{t\to\infty} (\ln m(t) - \ln m(t+y)) = cy$, or $\lim_{t\to\infty} (h(t) - h(t+y)) = 0$ for all y, where $h(t) := \ln m(t) - ct$. Applying now Lemma 1.1 of [SEN] yields that $\lim_{t\to\infty} \frac{m(t)}{m(t+y)}$ holds uniformly on compacta (in y). Hence $SM(c) \subset SM$ for every $c \in \mathbb{R}$.

(b) Replacing $\mu([t,\infty))$ by $m(t)$ in the argument given in the proof of Lemma 1.3.17 yields $\lim_{t\to\infty} \frac{\ln m(t)}{t} = -c$ whenever $m \in SM(c)$ (c arbitrary).

(c) It is possible to show that $m(t) := \exp(-ct)h(t)$ belongs to $SM(c)$ for every measurable function h satisfying

(i) $\forall y \geq 0 \;\lim_{t\to\infty} \frac{h(t)}{h(t+y)} = 1$;

(ii) $\overline{\lim_{t\to\infty}} \sup_{a\leq x\leq 1} \frac{h(tx)}{h(t)} < \infty$ for some $0 < a < 1$;

(iii) h and $\frac{1}{h}$ locally bounded on $[0,\infty)$;

(iv) $h \in L^1[0,\infty)$.

In particular the functions $\exp(-ct)(1+t)^{-\beta}[\log(e+t)]^\gamma$ belong to $SM(c)$ for all $\beta > 1$ and $\gamma \in \mathbb{R}$.

We now define for every $c \in \mathbb{R}$ and $a \in \mathbb{C}$

$$\underset{\sim}{S}^a(\psi_c,m) := \left\{\nu \in S(\psi_c,m) : \lim_{t\to\infty} \frac{\nu(A+t)}{m(t)} = a\right\}$$

and

$$\underset{\sim}{S}(\psi_c,m) := \bigcup_{a\in\mathbb{C}} \underset{\sim}{S}^a(\psi_c,m)$$

(Recall that $\psi_c(x) = \exp(cx)$.)

PROPOSITION 1.3.32. $\underset{\sim}{S}(\psi_c,m)$ *is a commutative Banach algebra with unit for every* $c \in \mathbb{R}$ *and* $m \in SM(c)$.

PROOF. The only fact that needs verification is that $\underset{\sim}{S}(\psi_c,m)$ is closed with respect to $*$. We show, more precisely, that for any $\nu_1,\nu_2 \in S(\psi_c,m)$ we have

$$\lim_{t\to\infty} \frac{\nu_i(A+t)}{m(t)} = a_i \ (i = 1,2) \Rightarrow \lim_{t\to\infty} \frac{(\nu_1*\nu_2)(A+t)}{m(t)} = a_1\hat{\nu}_2(-c) + a_2\hat{\nu}_1(-c) \ . \tag{47}$$

Before proving (47), let us note that for any $\varepsilon > 0$ there exists a $b_0 = b_0(\varepsilon)$ such that

$$\overline{\lim_{t\to\infty}} \frac{\int_b^{t/2} m(t-y)m(y)dy}{m(t)} < \varepsilon \quad \text{whenever } b \geq b_0 \ . \tag{48}$$

Indeed, choose b_0 so that

$$\int_{b_0}^{\infty} \exp(cy)m(y)dy < \varepsilon \tag{49}$$

and write, for every $t > 2b_0$,

$$\frac{\int_0^{t/2} m(t-y)m(y)dy}{m(t)} = \int_0^{b_0} \frac{m(t-y)}{m(t)} \cdot m(y)dy + \frac{\int_{b_0}^{t/2} m(t-y)m(y)dy}{m(t)} \ . \tag{50}$$

We have

$$\lim_{t\to\infty} \frac{\int_0^{t/2} m(t-y)m(y)dy}{m(t)} = \tfrac{1}{2} \lim_{t\to\infty} \frac{(m*m)(t)}{m(t)} = \hat{m}(-c) \ , \tag{51}$$

by Definition 1.3.30(iv). Also, by Definition 1.3.30(ii) and Remark 1.3.31(a)

$$\lim_{t\to\infty} \int_0^{b_0} \frac{m(t-y)}{m(t)} \cdot m(y)dy = \int_0^{b_0} \exp(cy)m(y)dy \ . \tag{52}$$

Evidently (49), (50), (51) and (52) imply that b_0 satisfies (48).
We now prove (47). Let us take $h = 1$, for simplicity. We have

$$(\nu_1 * \nu_2)(A+t) = \int_0^{t/2} \nu_1(A+t-y)\nu_2(dy) + \int_0^{t/2} \nu_2(A+t-y)\nu_1(dy) + $$
$$+ \int_{t/2}^{t/2+1} \int_{t/2}^{t+1-x} \nu_2(dy)\nu_1(dx) \ . \qquad (53)$$

For $t > 2b_0$ we rewrite the first term on the right as

$$\int_0^{t/2} \nu_1(a+t-y)\nu_2(dy) = \int_0^{b_0} \nu_1(A+t-y)\nu_2(dy) + \int_{b_0}^{t/2} \nu_1(A+t-y)\nu_2(dy) \ .$$

Using that $\nu_1, \nu_2 \in S(\psi_c, m)$ one easily checks that for some constant M we have

$$\frac{\left| \int_{b_0}^{t/2} \nu_1(A+t-y)\nu_2(dy) \right|}{m(t)} \leqq M \frac{\int_{b_0}^{t/2} m(t-y)m(y)dy}{m(t)} ,$$

so that, by (36),

$$\overline{\lim_{t\to\infty}} \frac{\left| \int_{b}^{t/2} \nu_1(A+t-y)\nu_2(dy) \right|}{m(t)} < M\varepsilon \qquad \text{whenever } b \geqq b_0 \ . \qquad (54)$$

Also, since $\lim_{t\to\infty} \frac{m(t)}{m(t+y)} = \exp(cy)$ uniformly on compacta, and since $\nu_1 \in \underset{\sim}{S}^{a_1}(\psi_c, m)$, we find that for all $b > 0$,

$$\lim_{t\to\infty} \frac{\int_0^b \nu_1(A+t-y)\nu_2(dy)}{m(t)} = a_1 \int_0^b \exp(cy)\nu_2(dy) \ . \qquad (55)$$

Taking ε small and b large we infer from (54) and (55) that

$$\lim_{t\to\infty} \frac{\int_0^{t/2} \nu_1(A+t-y)\nu_2(dy)}{m(t)} = a_1\hat{\nu}_2(-c) \ . \qquad (56)$$

Similarly, interchanging ν_1 and ν_2,

$$\lim_{t\to\infty} \frac{\int_0^{t/2} \nu_2(A+t-y)\nu_1(dy)}{m(t)} = a_2\hat{\nu}_1(-c) \ . \qquad (57)$$

Finally, using $\bar{P}_m(\nu_i) < \infty$ $(i = 1,2)$ and again Remark 1.3.31(a) we see that for some constant K we have

$$\frac{1}{m(t)} \left| \int_{t/2}^{t/2+1} \int_{t/2}^{t+1-x} \nu_2(dy)\nu_1(dx) \right| \le$$

$$\le \frac{1}{m(t)} |\nu_1|([\tfrac{t}{2},\tfrac{t}{2}+1])|\nu_2|([\tfrac{t}{2},\tfrac{t}{2}+1]) \le$$

$$\le K \frac{\int_{t/2-1}^{t/2} m(t-y)m(y)dy}{m(t)} \to 0 \quad \text{as } t \to \infty \text{ (by (48))}. \tag{58}$$

Combining (56), (57) and (58) with (53) yields the desired result. □

The next result should be compared with Theorem 1.3.18.

THEOREM 1.3.33. *Let* $m \in SM(c)$. *Then every* $L \in \Delta(\underset{\sim}{S}(\psi_c,m))$ *is the restriction of a unique* $\tilde{L} \in \Delta(S(\psi_c))$.

PROOF. From $m \in SM(c)$ it follows that

$$\lim_{n\to\infty} \sup_{t\ge 2n} \frac{\int_n^{t-n} m(t-x)m(x)dx}{m(t)} = 0$$

(compare (48) in the proof of Proposition 1.3.32). Hence by Lemma 1.3.28 we have

$$\lim_{n\to\infty} \|(\nu-\nu_n)^2\|_{\psi,m} = 0 .$$

Observe also that $\lim_{t\to\infty} \frac{\ln m(t)}{t} = -c$, by Remark 1.3.31(b), so that

$$\lim_{t\to\infty} \frac{\ln(\psi_c(t)m(t))}{t} = 0 .$$

Now argue as in the proof of Theorem 1.3.18, replacing μ by m everywhere. □

REMARK 1.3.34. It is also possible to prove an analogue of Theorem 1.3.21 in this case. In particular, if $m \in SM(c)$ and $\nu \in \underset{\sim}{S}^b(\psi_c,m)$ has an inverse ν^{-1} in $S(\psi_c)$ (hence in $\underset{\sim}{S}(\psi_c,m)$), then $\nu^{-1} \in \underset{\sim}{S}^d(\psi_c,m)$, where $d = -b(\hat{m}(-c))^{-2}$ (take $\Lambda(z) = z^{-1}$).

As noted in Remark 1.2.28 the results of Section 2 can also be derived from the more general results of this section. This can be seen as follows. Obviously,

$$V(\psi) = S_{\mu}(\psi) \tag{59}$$

where

$$\mu := \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \quad \text{and } \lambda \text{ denotes the counting measure on the nonnegative integers} \tag{$*$}$$

In order to verify the above observation it is now sufficient to prove the next result.

LEMMA 1.3.35. *If* $\nu \in S_{\mu}(\psi)$ $(\subset S(\psi))$ *with* μ *as in* $(*)$ *and* ν *is invertible in* $S(\psi)$, *then* $\nu^{-1} \in S_{\mu}(\psi)$.

PROOF. Since $\nu^{-1} \in S(\psi)$ we can decompose ν^{-1} into an absolutely continuous, an atomic and a non-atomic singular part. hence

$$e = \nu * (\nu^{-1})_a + \nu * ((\nu^{-1})_{ac} + (\nu^{-1})_s)$$

and using $\nu * ((\nu^{-1})_{ac} + (\nu^{-1})_s)$ is non-atomic, this implies

$$\nu * ((\nu^{-1})_{ac} + (\nu^{-1})_s) = \Theta .$$

The unicity of the Laplace transform (remember $\varphi_\lambda \in \Delta(S(\psi))$ and $\varphi_\lambda(\nu) \neq 0$!) now yields $(\nu^{-1})_{ac} + (\nu^{-1})_s = \Theta$ and so it is sufficient to prove that $(\nu^{-1})_a << \mu$.

If the Borel set $A \subset (0,1)$ we derive from $\nu(A) = 0$ and $e = \nu * (\nu^{-1})_a$ that

$$0 = (\nu * (\nu^{-1})_a)(A) = \nu(\{0\})(\nu^{-1})_a(A) .$$

Next observe that $L: \nu \to \nu(\{0\})$ is a homomorphism and so $(\nu^{-1})_a(A) = 0$. Now by induction on the maximal distance of elements of A to zero we easily derive from $e = \nu * (\nu^{-1})_a$ the stated result. □

CHAPTER 2. RENEWAL SEQUENCES

0. Introduction

Let $\underline{X}_1, \underline{X}_2, \dots$ be independent and identically distributed positive integer-valued random variables with distribution F.

Assume F satisfies in this chapter the following properties:

(i) $$F(n) := \sum_{k=0}^{n} f(k) \;;\quad f(0) := 0 \;;\quad \sum_{k=0}^{\infty} f(k) = 1 \,.$$

(ii) $$\text{g.c.d. } \{n: f(n) > 0\} = 1 \,.$$

Set

$$\underline{S}_0 := 0 \,,\quad \underline{S}_n := \sum_{i=1}^{n} \underline{X}_i \qquad (n \in \mathbb{N})$$

and consider the renewal sequence $\{u(n)\}_{n=0}^{\infty}$ defined by

$$u(n) := \mathbb{E}\left(\sum_{k=0}^{\infty} 1_{\{\underline{S}_k = n\}}\right) = P\left\{\bigcup_{k=0}^{\infty} (\underline{S}_k = n)\right\} \,. \tag{1}$$

If $\mathbb{E}(\underline{X}_1)$ is denoted by E the following result is well-known.

<u>THEOREM 2.0.1</u> (cf. [FEL-1]).

$$\lim_{n\to\infty} u(n) = \frac{1}{E} \quad \textit{for } E < \infty \,,$$
$$= 0 \quad \textit{for } E = \infty \,.$$

In the sequel we are interested in the first- and second-order remainder terms in the convergence to this limit. Since the methods of proof are entirely different for $E = \infty$ and $E < \infty$ we will distinguish these two cases. For $E = \infty$ Fourier analysis will be used, while for $E < \infty$ Banach algebra methods can be applied.

An advantage in the latter case is, that, once the Banach algebra machinery is set up, the proofs are short and simple.

1. The behaviour of the renewal sequence in case the expectation is finite

Let us define the sequence $\{f_1(n)\}_{n=0}^{\infty}$, where

$$f_1(n) := \frac{1}{E} \sum_{k=n+1}^{\infty} f(k) = \frac{1}{E}(1-F(n)), \qquad n \in \mathbb{N} .$$

Note that $\sum_{n=0}^{\infty} f_1(n) = 1$ and so $f_1 \in V(\psi_1)$, where $\psi_1(n) := 1$.

<u>LEMMA 2.1.1.</u> *Suppose* f_1 *is invertible in* $V(\psi_1)$. *Then the renewal sequence* $\{u(n)\}_{n=0}^{\infty}$ *has the following representation:*

$$u = f_1^{-1} * \frac{\bar{x}}{E},$$

where $\bar{x}(n) := 1$ *for every* $n \in \mathbb{N}$.

<u>PROOF</u>. For every $z \in \mathbb{C}$ with $|z| \leq 1$ let us define $\varphi_z \in \Delta(V(\psi_1))$ by

$$\varphi_z(x) := \sum_{n=0}^{\infty} x(n)z^n \qquad (x \in V(\psi_1)) .$$

It is easy to see that

$$\varphi_z(f_1) = \frac{1 - \varphi_z(f)}{E(1-z)} \qquad \text{for } |z| \leq 1,\ z \neq 1$$

and

$$\varphi_1(f_1) = \frac{1}{E} \sum_{k=0}^{\infty} (1 - F(k)) = 1 .$$

Since f_1^{-1} exists in $V(\psi_1)$ we obtain

$$\varphi_z(f_1^{-1}) = (\varphi_z(f_1))^{-1} = \frac{E(1-z)}{1 - \varphi_z(f)} . \tag{2}$$

Furthermore, using

$$u(n) = \sum_{k=0}^{\infty} f^{k*}(n) \qquad \text{for every } n \in \mathbb{N},$$

yields

$$\varphi_z(u) = \sum_{k=0}^{\infty} \varphi_z(f^{k*}) = \sum_{k=0}^{\infty} (\varphi_z(f))^k = \frac{1}{1 - \varphi_z(f)} \tag{3}$$

for every $z \in \mathbb{C}$ with $|z| < 1$.

(Fix $|z| < 1$ and choose $\varepsilon > 0$ so that $|z|+\varepsilon < 1$. Observe that the series $\sum_{k=0}^{\infty} f^{k*}$ is norm convergent in $V(\psi_{1-\varepsilon})$, where $\psi_{1-\varepsilon}(n) := (1-\varepsilon)^n$. Now regard φ_z as an element of $\Delta(V(\psi_{1-\varepsilon}))$.)
Observe now that the sequence $\{\bar{x}(n)\}_{n=0}^{\infty}$ belongs to the Banach algebra $V(\psi_{1-\varepsilon})$ for every $\varepsilon > 0$ and that

$$\varphi_z(\bar{x}) = \frac{1}{1-z} \qquad (z \in \mathbb{C},\ |z| \leq 1-\varepsilon) . \tag{4}$$

Substituting (4) and (2) in (3) yields for every $z \in \mathbb{C}$ satisfying $|z| \leq 1-\varepsilon$ (ε fixed) that

$$\varphi_z(u) = \frac{1}{E}\,\varphi_z(\bar{x})\varphi_z(f_1^{-1}) = \varphi_z\Big(\frac{\bar{x}}{E} * f_1^{-1}\Big) .$$

(Regard u, $\bar{x}$ and f_1^{-1} as elements of $V(\psi_{1-\varepsilon})$.) This implies

$$u = \frac{\bar{x}}{E} * f_1^{-1} .$$

□

REMARK 2.1.2. If $f_1 \in V(\psi)$ for some $\psi \in \Psi$ and f_1 is invertible in $V(\psi)$ we obtain the same representation as in Lemma 2.1.1. However, in this case $f_1^{-1} \in V(\psi)$ instead of $f_1^{-1} \in V(\psi_1)$.

Our next result gives a necessary and sufficient condition for $f_1 \in V(\psi)$ to be invertible in $V(\psi)$ if $\lim_{n\to\infty} (\psi(n))^{1/n} = 1$. (Note that this includes in particular the cases $\psi(n) = (1+n)^{\gamma}$ or equivalently $\mathbb{E}(\underline{X}_1^{\gamma+1}) < \infty$, where $\gamma \geq 0$ and $\gamma \in \mathbb{R}$.)

LEMMA 2.1.3. *Let $\psi \in \Psi$ be given and suppose $\lim_{n\to\infty} (\psi(n))^{1/n} = 1$. Then the following conditions are equivalent:*

(a) $f_1 \in V(\psi)$ *and* g.c.d. $\{n: f(n) > 0\} = 1$

(b) f_1 *is invertible in* $V(\psi)$.

Before proving this result we need the following proposition.

PROPOSITION 2.1.4. *Suppose $\underline{Z}$ is a nonnegative integer valued random variable with $p(n) := P\{\underline{Z} = n\}$, $n \geq 0$. Then the next conditions are equivalent:*

(a) g.c.d. $\{n: p(n) > 0\} = 1$

(b) $\mathbb{E}(e^{it\underline{Z}}) = 1$ *if and only if* $t = 2k\pi$ $(k \in \mathbb{Z})$.

PROOF. See p. 94 of [KAW]. □

PROOF OF LEMMA 2.1.3. By Theorem 1.2.3 we have to verify that

$$\varphi_z(f_1) := \sum_{k=0}^{\infty} f_1(k)z^k \neq 0$$

for every $z \in \mathbb{C}$ with $|z| \leqq 1$.
As in Lemma 2.1.1 it follows that

$$\varphi_z(f_1) = \frac{1 - \varphi_z(f)}{E(1-z)}$$

for every $z \in \mathbb{C}$ with $|z| \leqq 1$, $z \neq 1$ and $\varphi_1(f_1) = 1$.
Since $\{f(n)\}_{n=0}^{\infty}$ is a probability distribution, it is clear that $|\varphi_z(f)| < 1$ for every $z \in \mathbb{C}$ with $|z| < 1$. Hence $\varphi_z(f_1) \neq 0$ for every $z \in \mathbb{C}$ with $|z| < 1$. Also by Proposition 2.1.4 and the assumption g.c.d. $\{n: f(n) > 0\} = 1$, we obtain $\varphi_z(f_1) \neq 0$ for every $z \in \mathbb{C}$ with $|z| = 1$, $z \neq 1$. Hence the first part of the lemma is proved.
If f_1^{-1} exists in $V(\psi)$ then in particular $\varphi_z(f_1) \neq 0$ or equivalently $\varphi_z(f) \neq 1$ for every $z \in \mathbb{C}$ with $|z| = 1$, $z \neq 1$. This implies by Proposition 2.1.4 that g.c.d. $\{n: f(n) > 0\} = 1$.
Obviously $f_1 = (f_1^{-1})^{-1}$ exists in $V(\psi)$. □

REMARK 2.1.5.

1. By strengthening the conditions of Lemma 2.1.3 we can prove in a similar way the following stronger result:
   If $f_1 \in V(\psi)$ with $\lim_{n\to\infty} (\psi(n))^{1/n} > 1$, g.c.d. $\{n: f(n) > 0\} = 1$ and $\varphi_z(f) \neq 1$ for all $z \in \mathbb{C}$ with $1 < |z| \leqq \lim_{n\to\infty} (\psi(n))^{1/n}$, then f_1 is invertible in $V(\psi)$, i.e. $f_1^{-1} \in V(\psi)$.
2. By partial summation it follows immediately that $f_1 \in V(\psi)$ is equivalent with $\mathbb{E}(\psi^o(\underline{X}_1)) < \infty$, where $\psi^o(n) := \sum_{k=0}^{n} \psi(k)$.

We now are prepared to prove first-order limit theorems. Although most of the first-order results are already known we will for the sake of completeness mention and prove them.

<u>THEOREM 2.1.6</u> (cf. [GRÜ]). *If the expectation* E *is finite then*

$$\sum_{k=1}^{\infty} |u(k) - u(k-1)|$$

is finite. (Recall we always assume g.c.d. $\{n: f(n) > 0\} = 1$.)

<u>PROOF</u>. Clearly $f_1 \in V(\psi_1)$ and so by Lemma 2.1.3 we obtain $f_1^{-1} \in V(\psi_1)$. This implies, using Lemma 2.1.1, the desired result. □

<u>REMARK 2.1.7</u>. In case $\sum_{k=1}^{\infty} |u(k) - u(k-1)|$ is finite it is easy to see that $\{u(n)\}_{n=0}^{\infty}$ is a Cauchy sequence. Hence $u(n)$ converges to a finite limit as $n \to \infty$ and by elementary computations (use the relation $u = f * u + e$) one can prove that this limit equals $1/E$.

<u>THEOREM 2.1.8</u> (cf. [GRÜ]). *Let* $\gamma > 0$. *Then the following conditions are equivalent.*

(a) $$\mathbb{E}(\underline{X}_1^{1+\gamma}) < \infty$$

(b) $$\sum_{k=1}^{\infty} k^{\gamma} |u(k) - u(k-1)| < \infty .$$

Both imply

$$\sum_{k=1}^{\infty} k^{\gamma-1} |u(k) - \frac{1}{E}| < \infty .$$

<u>PROOF</u>. Since $\mathbb{E}(\underline{X}_1^{1+\gamma}) < \infty$ we obtain $f_1 \in V(\psi)$ with $\psi(n) := (1+n)^{\gamma}$. (Clearly $\psi \in \Psi$.) Then

$$\lim_{n\to\infty} (\psi(n))^{1/n} = \lim_{n\to\infty} (1+n)^{\gamma/n} = 1$$

and so by Lemma 2.1.3 we know that f_1 is invertible in $V(\psi)$, i.e. $f_1^{-1} \in V(\psi)$. This implies (Lemma 2.1.1) $\Delta u \in V(\psi)$, where

$$\Delta u(n) := u(n) - u(n-1) \quad (n \geq 1) \quad \text{and} \quad \Delta u(0) := 1 .$$

Suppose $\Delta u \in V(\psi)$. Then similarly as in Lemma 2.1.3 we can prove that $(\Delta u)^{-1} \in V(\psi)$. This implies (Lemma 2.1.1) $f_1 \in V(\psi)$ or equivalently $\mathbb{E}(\underline{X}_1^{1+\gamma}) < \infty$.
Finally it follows by Fubini's theorem:

$$\sum_{k=1}^{\infty} k^{\gamma-1}|u(k) - \frac{1}{E}| \leq \frac{1}{E} \sum_{k=1}^{\infty} k^{\gamma-1} \sum_{m=k}^{\infty} |f_1^{-1}(m)| =$$

$$= \frac{1}{E} \sum_{m=1}^{\infty} \sum_{k=1}^{m} k^{\gamma-1}|f_1^{-1}(m)|$$

and this series is finite since $f_1^{-1} \in V(\psi)$. □

<u>THEOREM 2.1.9</u> (cf. [GRÜ]). *The following conditions are equivalent.*

(a) $\mathbb{E}(\rho^{X_1}) < \infty$ *for some* $\rho > 1$

(b) $\sum_{k=1}^{\infty} \rho^k|u(k) - u(k-1)| < \infty$ *for some* $\rho > 1$

(not necessarily the <u>same</u> ρ*).*

Both imply

$$\sum_{k=0}^{\infty} \rho^k|u(k) - \frac{1}{E}| < \infty \quad \textit{for some } \rho > 1 .$$

<u>PROOF</u>. Let $\mathbb{E}(\rho^{X_1}) < \infty$ for some $\rho > 1$. Then clearly there exists some η with $1 < \eta \leq \rho$ such that $f_1 \in V(\psi_\eta)$.
Define

$$B(\rho) := \{z \in \mathbb{C}: |z| \leq \rho\}$$

and consider the set

$$W := \{B(\rho): \rho \leq \eta, \varphi_z(f_1) \neq 0 \text{ for all } z \in B(\rho)\} .$$

Suppose there does not exist a $\rho > 1$ with $B(\rho) \in W$. Then we can find a sequence $\{x(n)\}_{n=0}^{\infty}$ with $1 < |x(n)| \leq \eta$, $\lim_{n\to\infty} \downarrow |x(n)| = 1$ and $\varphi_{x(n)}(f_1) = 0$ for every $n \in \mathbb{N}$.
Since the sequence $\{x(n)\}_{n=0}^{\infty}$ is uniformly bounded, there exists some subsequence $\{n_k\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} x(n_k)$ exists, and using $\lim_{n\to\infty} \downarrow |x(n)| = 1$ we can find a $\theta \in (0,2\pi]$ satisfying $\lim_{k\to\infty} x(n_k) = e^{i\theta}$.
Finally by the absolute convergence of the series $\sum_{k=0}^{\infty} f_1(k)z^k$ on $B(\eta)$ we obtain

$$\varphi_{e^{i\theta}}(f_1) = \lim_{k\to\infty} \varphi_{x(n_k)}(f_1) = 0$$

and this contradicts Lemma 2.1.3.

Hence there exists a $\rho > 1$ with $B(\rho) \in W$ and by the same reasoning as in the previous lemma (apply Theorem 1.2.4 and $f_1 \in V(\psi_\eta) \subset V(\psi_\rho)$), we obtain the desired result.
The converse statement as well as the implication can be proved in a similar way as Theorem 2.1.8.

THEOREM 2.1.10. *The following conditions are equivalent.*

(a) $$\lim_{n\to\infty} \frac{f_1^{2*}([n,\infty))}{f_1([n,\infty))} = 2 ;$$

(b) $$\lim_{n\to\infty} \frac{u(n) - \frac{1}{E}}{f_1([n,\infty))} = \frac{1}{E} \quad \textit{and} \quad \lim_{n\to\infty} \frac{f_1([n+1,\infty))}{f_1([n,\infty))} = 1 .$$

PROOF. By assumption

$$\lim_{n\to\infty} \frac{f_1^{2*}([n,\infty))}{f_1([n,\infty))} = 2 .$$

This implies (cf. [ATH])

$$\lim_{n\to\infty} \frac{f_1([n+1,\infty))}{f_1([n,\infty))} = 1 ,$$

and so $f_1 \in ST(1)$ (cf. Definition 1.2.25). On the other hand

$$u(n) - \frac{1}{E} = - \frac{1}{E} \sum_{k=n+1}^{\infty} f_1^{-1}(k)$$

(cf. Lemmas 2.1.1 and 2.1.3) and hence by Theorem 1.2.27 and Remark 1.2.28

$$\lim_{n\to\infty} \frac{u(n) - \frac{1}{E}}{f_1([n,\infty))} = \frac{1}{E} .$$

We now have to prove that

$$\lim_{n\to\infty} \frac{f_1^{2*}([n,\infty))}{f_1([n,\infty))} = 2 .$$

Obviously for every $0 < p \leqq n-1$

$$f_1^{2*}([n,\infty)) = \sum_{k=0}^{p} f_1(k)f_1([n-k,\infty)) +$$

$$+ \sum_{k=0}^{n-p-1} f_1(k)f_1([n-k,\infty)) + f_1([p+1,\infty))f_1([n-p,\infty)) \ . \tag{5}$$

By assumption

$$\lim_{n\to\infty} \frac{f_1([n+1,\infty))}{f_1([n,\infty))} = 1$$

and so

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{p} f_1(k)f_1([n-k,\infty))}{f_1([n,\infty))} = \sum_{k=0}^{p} f_1(k) \quad \text{for every } p > 0 \ . \tag{6}$$

Also by assumption

$$\lim_{n\to\infty} \frac{f_1^{-1}([n,\infty))}{f_1([n,\infty))} = -1$$

(cf. Lemmas 2.1.1 and 2.1.3).

Hence for every $\varepsilon > 0$ we can find some $p_0 = p_0(\varepsilon)$ such that the second part of (5) is bounded from below by

$$-(1-\varepsilon)\left(\sum_{k=0}^{n-p-1} f_1(k)f_1^{-1}([n-k,\infty)) + f_1([p+1,\infty))f_1^{-1}([n-p,\infty))\right)$$

and bounded from above by

$$-(1+\varepsilon)\left(\sum_{k=0}^{n-p-1} f_1(k)f_1^{-1}([n-k,\infty)) + f_1([p+1,\infty))f_1^{-1}([n-p,\infty))\right)$$

for every $p \geq p_0$.

Since $(f_1 * f_1^{-1})([n,\infty)) = 0$ for every $n > 0$ we obtain that (cf. also (5))

$$\sum_{k=0}^{n-p-1} f_1(k)f_1^{-1}([n-k,\infty)) + f_1([p+1,\infty))f_1^{-1}([n-p,\infty))$$

equals

$$-\sum_{k=0}^{p} f_1(k)f_1^{-1}([n-k,\infty)) \ .$$

Using the above observations and the assumptions now yields

$$\lim_{p\to\infty}\overline{\lim_{n\to\infty}}\frac{f_1^{2*}([n,\infty)) - \sum_{k=0}^{p} f_1(k)f_1([n-k,\infty))}{f_1([n,\infty))} = \sum_{k=0}^{\infty} f_1(k) \qquad (7)$$

and combining (7) with (6) we obtain

$$\lim_{n\to\infty}\frac{f_1^{2*}([n,\infty))}{f_1([n,\infty))} = 2 \; . \qquad \square$$

For the class ST(r), r > 1 (= S(r), cf. Remark 1.2.26), a similar result holds.

THEOREM 2.1.11. *The following conditions are equivalent.*

(a) $f_1 \in ST(r)$, $r > 1$ *(cf. Definition 1.2.13) and*

$$\hat{f}_1(z) = \sum_{k=0}^{\infty} f_1(k)z^k \neq 0 \quad \textit{for every } z \in \mathbb{C} \textit{ with } 1 < |z| \leq r \; ;$$

(b) $$\lim_{n\to\infty}\frac{E(u(n-1) - u(n))(\hat{f}_1(r))^2}{f_1(n)} = 1 \quad \textit{and} \quad \lim_{n\to\infty}\frac{f_1(n)}{f_1(n+1)} = r \quad \textit{with } r > 1 \; ;$$

(c) $$\lim_{n\to\infty}\frac{E(u(n) - \frac{1}{E})(\hat{f}_1(r))^2 r}{f_1([n,\infty))} = -1 \quad \textit{and} \quad \lim_{n\to\infty}\frac{f_1([n,\infty))}{f_1([n+1,\infty))} = r \quad \textit{with } r > 1 \; .$$

PROOF. By Theorem 1.2.22, $f_1^{-1} \in \underset{\sim}{V}^a(\psi_r, f_1)$ with $a = -(\hat{f}_1(r))^{-2}$, and so from Remark 2.1.2 result (b) follows.
The proof of (b) → (c) is obvious, so we omit it.
In order to prove (c) → (b) we note that

$$\lim_{n\to\infty}\frac{f_1(n)}{f_1([n,\infty))} = 1 - \lim_{n\to\infty}\frac{f_1([n+1,\infty))}{f_1([n,\infty))} = \frac{r-1}{r} \; .$$

Hence

$$\lim_{n\to\infty}\frac{f_1(n)}{f_1(n+1)} = \lim_{n\to\infty}\frac{f_1(n)}{f_1([n,\infty))}\,\frac{f_1([n,\infty))}{f_1([n+1,\infty))}\,\frac{f_1([n+1,\infty))}{f_1(n+1)} = r \; . \qquad (8)$$

Also (cf. Lemma 2.1.1)

$$\lim_{n\to\infty}\frac{f_1^{-1}([n,\infty))}{f_1([n,\infty))} = -(\hat{f}_1(r))^{-2}$$

and this implies by the same method as in the proof of (8) that

$$\lim_{n\to\infty} \frac{f_1^{-1}(n)}{f_1(n)} = -(\hat{f}_1(r))^{-2}$$

or equivalently

$$\lim_{n\to\infty} \frac{E(u(n) - u(n-1))(\hat{f}_1(r))^2}{f_1(n)} = 1 .$$

Finally for the proof of (b) $\to$ (a) we note that

$$\lim_{n\to\infty} \frac{f_1^{-1}(n)(\hat{f}_1(r))^2}{f_1(n)} = -1 \quad \text{and} \quad \lim_{n\to\infty} \frac{f_1(n)}{f_1(n+1)} = r$$

imply (as in the proof of Theorem 2.1.10)

$$\lim_{n\to\infty} \frac{f_1^{2*}(n)}{f_1(n)} = 2\hat{f}_1(r) .$$

Hence $f_1 \in S(r)$.
Also, by assumption (b) there exists some $n_0 \in \mathbb{N}$ such that

$$\frac{|f_1^{-1}(n)|(\hat{f}_1(r))^2}{f_1(n)} \leq 2 \qquad \text{for all } n \geq n_0$$

and so it is clear that

$$\sum_{k=n_0}^{\infty} r^k |f_1^{-1}(k)| \leq \frac{2\sum_{k=n_0}^{\infty} r^k f_1(k)}{(\hat{f}_1(r))^2} < \infty .$$

Hence

$$f_1^{-1} \in V(\psi_r) . \tag{9}$$

By (9) $L(f_1^{-1}) < \infty$ for every homomorphism $L: V(\psi_r) \to \mathbb{C}$ and since $L(f_1^{-1}) = (L(f_1))^{-1}$ it follows $L(f_1) \neq 0$. Applying Theorem 1.2.3 now yields $\hat{f}_1(z) \neq 0$ for every $z \in \mathbb{C}$ with $1 < |z| \leq r$. □

Without proof we mention the next first-order results, since they follow immediately from Lemmas 2.1.1, 2.1.3, Remark 2.1.5 and the observations after Theorem 1.2.22.

THEOREM 2.1.12. *Let* $\mu \in ST$ *and suppose*

$$\lim_{n\to\infty} (\mu([n,\infty)))^{1/n} = 1 .$$

Then

$$\lim_{n\to\infty} \frac{u(n) - \frac{1}{E}}{\mu([n,\infty))} = 0 \quad \textit{if} \quad \lim_{n\to\infty} \frac{f_1([n,\infty))}{\mu([n,\infty))} = 0 .$$

THEOREM 2.1.13. *Let* $\mu \in ST$ *and suppose*

$$\lim_{n\to\infty} (\mu([n,\infty)))^{1/n} = \beta < 1 \quad \textit{and} \quad \hat{\mu}(\beta^{-1}) < \infty .$$

Then

$$\lim_{n\to\infty} \frac{u(n) - \frac{1}{E}}{\mu([n,\infty))} = 0 \quad \textit{if} \quad \lim_{n\to\infty} \frac{f_1([n,\infty))}{\mu([n,\infty))} = 0 \quad \textit{and} \quad \hat{f}_1(z) \neq 0$$

for every $z \in \mathbb{C}$ *with* $1 < |z| \leq \beta^{-1}$.

THEOREM 2.1.14. *Let* $\mu \in ST$,

$$\lim_{n\to\infty} (\mu(n))^{1/n} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \sup_{p\in\mathbb{N}} \frac{(\mu-\mu_n)^{2*}([p,\infty))}{\mu([p,\infty))} = 0 .$$

Then

$$\sup_{n\geq 0} \frac{|u(n) - \frac{1}{E}|}{\mu([n,\infty))} < \infty \quad \textit{if} \quad \sup_{n\geq 0} \frac{f_1([n,\infty))}{\mu([n,\infty))} < \infty .$$

THEOREM 2.1.15. *Let* $\mu \in ST$,

$$\lim_{n\to\infty} (\mu([n,\infty)))^{1/n} = \beta < 1 , \quad \mu(\beta^{-1}) < \infty$$

and

$$\lim_{n\to\infty} \sup_{p\in\mathbb{N}} \frac{(\mu-\mu_n)^{2*}([p,\infty))}{\mu([p,\infty))} = 0$$

Then

$$\sup_{n\geq 0} \frac{|u(n) - \frac{1}{E}|}{\mu([n,\infty))} < \infty \quad \textit{if} \quad \sup_{n\geq 0} \frac{f_1([n,\infty))}{\mu([n,\infty))} < \infty$$

and $\hat{f}_1(z) \neq 0$ *for every* $z \in \mathbb{C}$ *with* $1 < |z| \leq \beta^{-1}$.

Before discussing second-order limit results we like to give some examples of important subclasses of S, S(r) and

$$SD := \left\{\mu \in S\colon \lim_{n\to\infty} \sup_{p\in\mathbb{N}} \frac{(\mu-\mu_n)^{2*}(p)}{\mu(p)} = 0\right\} .$$

Of course (cf. Lemma 1.2.17)

$$\bigcup_{r\geq 1} S(r) \subseteq SD .$$

DEFINITION 2.1.16.

1. A function $\tau\colon \mathbb{N} \to (0,\infty)$ belongs to $\mathbb{D}$ if

$$D_\tau(a) := \overline{\lim_{n\to\infty}} \sup_{1\leq x\leq a} \frac{\tau(n)}{\tau([nx])} ;$$

   is finite for some $a > 1$.

2. A function $\tau\colon \mathbb{N} \to (0,\infty)$ belongs to $\mathbb{L}$ if

$$\lim_{n\to\infty} \frac{\tau(n+1)}{\tau(n)} = 1 .$$

3. A function $\tau\colon \mathbb{N} \to (0,\infty)$ belongs to $R.V.S._\rho^\infty$ if

$$\lim_{n\to\infty} \frac{\tau([nx])}{\tau(n)} = x^\rho \quad \text{for every } x > 0 .$$

   (This is called a *regularly varying sequence* with index $\rho \in \mathbb{R}$.)

REMARK 2.1.17.

1. A function $\tau\colon \mathbb{N} \to (0,\infty)$ is called a function of *bounded decrease* if τ is nonincreasing and $D_\tau(a) < \infty$ for some $a > 1$.
   (Obviously in this case $D_\tau(a)$ equals $\overline{\lim}_{n\to\infty} \frac{\tau(n)}{\tau([na])}$.)
   A function $\tau\colon \mathbb{N} \to (0,\infty)$ is called a function of *bounded increase* if $1/\tau$ is a function of bounded decrease.

2. In the Appendix we will give a short summary of all the relevant properties of the function classes mentioned in Definition 2.1.16.

Before mentioning the next result we introduce for the sets A, B of positive sequences the notation

$AB := \{\mu\colon \mathbb{N} \to (0,\infty);\ \mu(n) = f(n)h(n)\ (\forall\, n\in\mathbb{N}) \text{ for some } f \in A,\ h \in B\}$.

THEOREM 2.1.18. *Let*

$$\Psi^{-1} := \{\psi\colon \frac{1}{\psi} \in \Psi\} \quad \textit{and} \quad \Psi^{-1}(r) := \{\psi \in \Psi^{-1}\colon \lim_{n\to\infty} \frac{\psi(n)}{\psi(n+1)} = r\} .$$

Then

(a) *The set* $\Psi^{-1}(\mathbb{D}\cap L^1(0,\infty))$ *is a subset of* SD.

(b) *For every* $r \geq 1$ *the set* $\Psi^{-1}(r)(\mathbb{D}\cap \mathbb{L}\cap L^1(0,\infty))$ *is a subset of* $S(r)$.

PROOF. In order to prove (a) we assume $\mu \in \Psi^{-1}(\mathbb{D}\cap L^1(0,\infty))$. Then by definition there exists some $\psi \in \Psi^{-1}$ and $\tau \in \mathbb{D}\cap L^1(0,\infty)$ such that $\mu(n) = \psi(n)\tau(n)$ for every $n \in \mathbb{N}$.
Now for all $n \in \mathbb{N}$ and $p \geq 2n+2$ we obtain by the multiplicative property of ψ that

$$\frac{(\mu-\mu_n)^{2*}(p)}{\mu(p)} = \frac{\sum_{k=n+1}^{p-(n+1)} \mu(k)\mu(p-k)}{\mu(p)} \leq \frac{\sum_{k=n+1}^{p-(n+1)} \tau(k)\tau(p-k)}{\tau(p)} \leq$$

$$\leq 2\,\frac{\sum_{k=n+1}^{\lceil p/2\rceil} \tau(k)\tau(p-k)}{\tau(p)}$$

and so there exists for every $\varepsilon > 0$ some $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\frac{(\mu-\mu_{n_0})^{2*}(p)}{\mu(p)} \leq 2D_\tau(2) \sum_{k=n_0}^{\infty} \tau(k) \leq \varepsilon \qquad \text{for every } p \geq 2n_0+2 .$$

This implies, since $(\mu-\mu_m)^{2*} \leq (\mu-\mu_n)^{2*}$ if $m \geq n$, that

$$\overline{\lim_{n\to\infty}} \sup_{p\in\mathbb{N}} \frac{(\mu-\mu_n)^{2*}(p)}{\mu(p)} = 0 .$$

Hence $\Psi^{-1}(\mathbb{D}\cap L^1(0,\infty))$ is a subset of SD.

In order to prove (b) let $\mu(n) = \psi(n)\tau(n)$ with $\psi \in \Psi^{-1}(r)$ and $\tau \in \mathbb{D} \cap \mathbb{L} \cap L^1(0,\infty)$. Then by the theory of subadditive functions (cf. [HIL]) one easily verifies that

$$r^{-1} = \sup_{n \geq 1} (\psi(n))^{1/n} .$$

This implies $r^k \leq (\psi(k))^{-1}$ for every $k \in \mathbb{N}$ and so

$$\sum_{k=0}^{\infty} r^k \mu(k) \leq \sum_{k=0}^{\infty} \tau(k) < \infty .$$

As in (2) there exists for every $\varepsilon > 0$ a $n_0 = n_0(\varepsilon)$ such that

$$\frac{(\mu-\mu_{n_0})^{2*}(p)}{\mu(p)} \leq \varepsilon \qquad \text{for every } p \geq 2n_0+2 .$$

Hence

$$2\,\frac{\sum_{k=0}^{n_0} \mu(p-k)\mu(k)}{\mu(p)} \leq \frac{\mu^{2*}(p)}{\mu(p)} \leq 2\,\frac{\sum_{k=0}^{n_0} \mu(p-k)\mu(k)}{\mu(p)} + \varepsilon \tag{10}$$

for every $p \geq 2n_0+2$.

Since

$$\lim_{n\to\infty} \frac{\mu(n-1)}{\mu(n)} = r$$

we obtain from (10) that

$$2 \sum_{k=0}^{n_0} r^k \mu(k) \leq \varliminf_{p\to\infty} \frac{\mu^{2*}(p)}{\mu(p)} \leq \varlimsup_{p\to\infty} \frac{\mu^{2*}(p)}{\mu(p)} \leq 2 \sum_{k=0}^{n_0} r^k \mu(k) + \varepsilon .$$

Letting $\varepsilon \downarrow 0$ ($\Rightarrow n_0 \uparrow \infty$) yields using $\sum_{k=0}^{\infty} r^k \mu(k) < \infty$ that

$$\lim_{p\to\infty} \frac{\mu^{2*}(p)}{\mu(p)} = 2\hat{\mu}(r) .$$ □

We will now discuss second-order limit results for renewal sequences having distributions $\{f(n)\}_{n=0}^{\infty}$ for which the tail $1 - F(n)$ decreases sub-exponentially fast. By Theorem 2.1.10 it is then natural to analyze the behaviour of

$$R(n) := u(n) - \frac{1}{E} - \frac{1}{E^2} \sum_{k=n+1}^{\infty} 1 - F(k)$$

if n tends to infinity.

As in the previous case we first give a representation for the sequence $\{R(n)\}_{n=0}^{\infty}$.

THEOREM 2.1.19. *Let*

$$R(n) := u(n) - \frac{1}{E} - \frac{1}{E^2} \sum_{k=n+1}^{\infty} (1 - F(k) \qquad \textit{for every } n \in \mathbb{N} \ .$$

Then the following representation holds:

$$R = (e-f_1)^{2*} * \frac{\bar{x}}{E} * f_1^{-1} \ .$$

(Recall $\bar{x}(n) := 1$ *for every* $n \in \mathbb{N}$.*)*

PROOF. It is well known that the renewal sequence $\{u(n)\}_{n=0}^{\infty}$ satisfies the so-called renewal equation $u = f * u + e$.
Also by definition $f_1 = (e-f) * \frac{\bar{x}}{E}$. This implies

$$f_1 * u = (e-f) * \frac{\bar{x}}{E} * u = (e-f) * u * \frac{\bar{x}}{E} = e * \frac{\bar{x}}{E} = \frac{\bar{x}}{E}$$

and so

$$u - \frac{\bar{x}}{E} = (e-f_1) * u \ .$$

Applying this equality twice yields

$$R := u - \frac{\bar{x}}{E} - (e-f_1) * \frac{\bar{x}}{E} = (e-f_1) * (u-\frac{\bar{x}}{E}) = (e-f_1)^{2*} * u \ .$$

Moreover, since we always assume in this section that g.c.d. $\{n: f(n) > 0\} = 1$, the representation for $\{u(n)\}_{n=0}^{\infty}$ can be applied (Lemmas 2.1.1 and 2.1.3). Hence

$$R = (e-f_1)^{2*} * u = (e-f_1)^{2*} * f_1^{-1} * \frac{\bar{x}}{E} = (e-f_1)^{2*} * \frac{\bar{x}}{E} * f_1^{-1} \ . \qquad \square$$

By imposing moment conditions we obtain the following rate of convergence results for R(n). Since the proofs are slightly different, we distinguish the cases $\mathbb{E}\underline{X}_1^2 < \infty$ and $\mathbb{E}\underline{X}_1^2 = \infty$.

THEOREM 2.1.20. *Suppose*

$$\mathbb{E}(\underline{X}_1^{2+\gamma}) < \infty, \quad \textit{where } \gamma \geq 0 .$$

Then

$$\sum_{k=1}^{\infty} k^{\gamma+1} |u(k) - \frac{1}{E} - \frac{1}{E^2} \sum_{p=k}^{\infty} (1-F(p))| < \infty .$$

PROOF. By the previous theorem we know that $R = (e-f_1)^{2*} * \frac{\bar{x}}{E} * f_1^{-1}$. Since $f_1 \in V(\psi)$, where $\psi(n) = (1+n)^{\gamma+1}$, and f_1 is invertible in $V(\psi)$ (Lemma 2.1.3) it is sufficient to prove that

$$(e-f_1)^{2*} * \frac{\bar{x}}{E} \in V(\psi) .$$

If

$$g(n) := (n+1)f_1(n+1) , \quad f_2(n) := \sum_{k=n+1}^{\infty} f_1(k) \quad \text{and} \quad d(n) := \frac{1}{E} (f_2^{2*} - 2f_2 * g)(n)$$

for every $n \in \mathbb{N}$, then it is easy to verify that the derivative of the generating function of $(e-f_1)^{2*} * (\bar{x}/E)$ equals the generating function of d. Hence

$$d(n) = (n+1)((e-f_1)^{2*} * \frac{\bar{x}}{E})(n+1) \quad \text{for every } n \in \mathbb{N} . \tag{11}$$

Since $\mathbb{E}(\underline{X}_1^{2+\gamma})$ is finite it is also easy to verify that

$$\sum_{n=0}^{\infty} (1+n)^{\gamma} f_2(n) \quad \text{and} \quad \sum_{n=0}^{\infty} (1+n)^{\gamma} g(n)$$

are finite. This implies (Theorem 1.2.1)

$$\sum_{n=0}^{\infty} (1+n)^{\gamma} |d(n)| < \infty$$

and so by (11)

$$(e-f_1)^{2*} * \frac{\bar{x}}{E} \in V(\psi) .$$

□

THEOREM 2.1.21. *Let*

$$f_2^{o}(n) := \sum_{p=0}^{n} f_2(p) := \sum_{p=0}^{n} \sum_{k=p+1}^{\infty} f_1(k) \quad \textit{for every } n \in \mathbb{N}$$

and suppose $\mathbb{E}(\underline{X}_1^{2+\gamma}) < \infty$, *where* $-1 \leq \gamma < 0$. *Then*

$$\sum_{k=1}^{\infty} k^{\gamma+1}(f_2^o(k))^{-1}|u(k) - \frac{1}{E} - \frac{1}{E^2}\sum_{p=k}^{\infty} (1-F(p))| < \infty .$$

<u>PROOF</u>. As in Theorem 2.1.20 we know that $f_1^{-1} \in V(\psi)$, where $\psi(n) = (1+n)^{\gamma+1}$, and

$$d(n) = (n+1)((e-f_1)^{2*} * \frac{\bar{x}}{E})(n+1) \quad \text{for every } n \in \mathbb{N} ,$$

where

$$d := \frac{1}{E}(f_2^{2*} - 2f_2 * g) \quad \text{and} \quad g(n) := (n+1)f_1(n+1) .$$

Hence by Theorem 2.1.19 and the observations in Theorem 2.1.20 we only have to verify that the series

$$\sum_{k=1}^{\infty} k^{\gamma}(f_2^o(k))^{-1} f_2^{2*}(k) \quad \text{and} \quad \sum_{k=1}^{\infty} k^{\gamma}(f_2^o(k))^{-1}(f_2 * g)(k)$$

are finite.
Since $\mathbb{E}(\underline{X}_1^{2+\gamma}) < \infty$ it is easy to see that the series

$$\sum_{k=0}^{\infty} k^{\gamma} f_2(k) \quad \text{and} \quad \sum_{k=0}^{\infty} k^{\gamma+1} f_1(k)$$

are finite. Also by the monotonicity of $f_2(n)$ we obtain

$$f_2^{2*}(n) = \sum_{k=0}^{n} f_2(n-k)f_2(k) \leq 2f_2([\frac{n}{2}])f_2^o(n)$$

and so the finiteness of the first series is established.
On the other hand

$$(f_2 * g)(n) \leq f_2([\frac{n}{2}]) \sum_{k=0}^{n} (k+1)f_1(k+1) + nf_1([\frac{n}{2}])f_2^o(n) =$$

$$= f_2([\frac{n}{2}]) \sum_{k=0}^{n}\sum_{m=0}^{k} f_1(k+1) + nf_1([\frac{n}{2}])f_2^o(n) \leq$$

$$\leq f_2([\frac{n}{2}])f_2^o(n) + nf_1([\frac{n}{2}])f_2^o(n)$$

and this implies by the finiteness of the series $\sum_{k=1}^{\infty} k^{\gamma} f_2(k)$ and $\sum_{k=1}^{\infty} k^{\gamma+1} f_1(k)$ that

$$\sum_{k=1}^{\infty} k^{\gamma}(f_2^o(k))^{-1}(f_2 * g)(k)$$

is finite. It is now clear that

$$\sum_{k=1}^{\infty} (1+k)^{\gamma+1}(f_2^o(k))^{-1}((e-f_1)^{2*} * \frac{\bar{x}}{E})(k)$$

is finite and hence by the submultiplicative property of $(1+n)^{\gamma+1}$ and the monotonicity of $f_2^o(n)$ we obtain by Theorem 2.1.19 that

$$\sum_{k=1}^{\infty} (1+k)^{1+\gamma}(f_2^o(k))^{-1} \left|u(k) - \frac{1}{E} - \frac{1}{E^2}\sum_{p=k}^{\infty} (1-F(p))\right|$$

is finite. □

Before discussing the next theorems we mention the use of the convention that a constant is denoted by the symbol C. This means in particular that two (not necessarily equal) constants are denoted by the same symbol.

THEOREM 2.1.22. *Suppose*

(a) $\mu: \mathbb{N} \to (0,\infty)$ *belongs to* $\mathbb{D}$ *and* $\sup_{n\in\mathbb{N}} \frac{f_1(n)}{\mu(n)} < \infty$.

(b) $\mu_1 \in L^1(0,\infty)$, *where* $\mu_1(n) := \sum_{k=n+1}^{\infty} \mu(k)$ *for every* $n \in \mathbb{N}$.

Then

$$\sup_{n\in\mathbb{N}} \frac{n\left|u(n) - \frac{1}{E} - \frac{1}{E^2}\sum_{k=n}^{\infty}(1-F(k))\right|}{\mu_1(n)} < \infty .$$

PROOF. From the lower bound for functions belonging to $\mathbb{D}$ (cf. Appendix) and $\lim_{n\to\infty} \mu(n) = 0$ it follows that $\lim_{n\to\infty} (\mu(n))^{1/n} = 1$. Moreover by Theorem 2.1.18 $\mu \in S$ and hence by Lemma 2.1.3 and the observations following Theorem 1.2.20 $f_1^{-1} \in V(\psi_1,\mu)$. Since $\mu \in \mathbb{D}$ it can be verified very easily that also $\tilde{\mu} \in \mathbb{D}$, where

$$\tilde{\mu}(n) := \frac{\mu_1(n)}{n} \quad \text{for every } n \geq 1 , \quad \tilde{\mu}(0) := 1 \quad \text{and} \quad \sup_{p\in\mathbb{N}} \frac{\mu(p)}{\tilde{\mu}(p)} < \infty .$$

This last result implies $V(\psi_1,\mu) \subseteq V(\psi_1,\tilde{\mu})$ and so by Theorem 2.1.19 it is sufficient to prove that $(e-f_1)^{2*} * \frac{\bar{x}}{E} \in V(\psi_1,\tilde{\mu})$.
First notice (as in Theorem 2.1.20) that the sequence $\{d(n)\}_{n=0}^{\infty}$, where $d := \frac{1}{E}(f_2 * (f_2-2g))$, satisfies

$$d(n) = (n+1)((e-f_1)^{2*} * \frac{\bar{x}}{E})(n+1) \qquad \text{for every } n \in \mathbb{N} .$$

Since $|(f_2-2g)(n)| \leq C\mu_1(n)$ for some constant $C > 0$ and $\mu_1 \in \mathbb{D} \cap L^1(0,\infty) \subset S$ it is obvious that

$$|d(n)| \leq (f_2 * |f_2-2g|)(n) \leq C\mu_1^{2*}(n) \leq C\mu_1(n) .$$

This implies

$$|((e-f_1)^{2*} * \frac{\bar{x}}{E})(n)| \leq C \frac{\mu_1(n)}{n}$$

and so by the representation for R(n) we obtain the desired result. □

An almost similar proof can be given for the next result.

THEOREM 2.1.23. *Suppose*

(a) $\mu: \mathbb{N} \to (0,\infty)$ *belongs to* $\mathbb{D}$ *and* $\lim_{n\to\infty} \frac{f_1(n)}{\mu(n)} = 0$.

(b) $\mu_1 \in L^1(0,\infty)$.

Then

$$\lim_{n\to\infty} \frac{\left(u(n) - \frac{1}{E} - \frac{1}{E^2} \sum_{k=n}^{\infty} (1-F(k))\right) n}{\mu_1(n)} = 0$$

REMARK 2.1.24.

1. By a slightly different method of proof we can deduce a result which is in special cases stronger than Theorems 2.1.22 and 2.1.23. This method uses the fact that

$$((e-f_1)^{2*} * \bar{x})(n) = f_2^{2*}(n) - f_2^{2*}(n-1) \qquad \text{for every } n \in \mathbb{N}$$

where $f_2 := (e-f_1) * \bar{x}$ and $f_2^{2*}(-1) := 0$.
Now after some elementary calculations we can bound $|((e-f_1)^{2*} * \bar{x})(n)|$ by

$$(f_2(n))^2 + 2 \sum_{p=1}^{\lfloor n/2 \rfloor} f_1(p) \sum_{k=n-p+1}^{n} f_1(k)$$

and so there exists some constant $C > 0$ such that

$$|((e-f_1)^{2*} * \bar{x})(n)| \leq C(\mu_1(n))^2 + C\mu(n) \sum_{k=1}^{n} \mu(k)k \leq$$

$$\leq C(\mu_1(n))^2 + C\mu(n) \qquad \text{for every } n \in \mathbb{N} .$$

This finally yields

$$|R(n)| \leq C((\mu_1(n))^2 + \mu(n))$$

if μ satisfies the conditions of Theorem 2.1.22.
A similar o-result holds if μ satisfies the conditions of Theorem 2.1.23.

2. Note that a special case of Theorem 2.1.22 is given by the sequence $\mu(n) = \frac{1}{E}(1 - F(n)) =: f_1(n)$, if $f_1(n)$ is a function of bounded decrease and the second moment $\mathbb{E}(\underline{X}_1^2)$ is finite.

3. Rogozin obtained by using a more complicated proof the same results for $R(n)$ in case μ is a regularly varying sequence with index < -2.

In order to obtain limit results for $R(n)$ we have to strenghthen the conditions on $f_1(n)$ (see Remark 2.1.24(2)) a little.

<u>THEOREM 2.1.25</u>. *Suppose*

(a) $f_1 \in \mathbb{D} \cap \mathbb{L}$.

(b) $E_2 := \mathbb{E}(\underline{X}_1^2) < \infty$.

Then

$$\lim_{n\to\infty} \frac{u(n) - \frac{1}{E} - \frac{1}{E^2} \sum_{k=n+1}^{\infty} (1-F(k))}{1 - F(n)} = - \frac{2 \sum_{k=1}^{\infty} kf_1(k)}{E^3} .$$

<u>PROOF</u>. Obviously by Theorem 2.1.18: $f_1 \in S(1)$ and so $f_1^{-1} \in \underset{\sim}{V}^{-1}(\psi_1, f_1)$ (Lemma 2.1.3 and Theorem 1.2.22). Now by Theorem 2.1.19 it is sufficient to prove that

$$(e-f_1)^{2*} * \bar{x} \in \underset{\sim}{V}^a(\psi_1, f_1) , \quad \text{where} \quad a = -\frac{2}{E} \sum_{k=1}^{\infty} kf_1(k) .$$

Then the desired result follows by Lemma 1.2.16.
In order to prove $(e-f_1)^{2*} * \bar{x} \in \underset{\sim}{V}^a(\psi_1, f_1)$ we remark that for every $n \geq 0$

$$((e-f_1)^{2*} * \bar{x})(n) = f_2^{2*}(n) - f_2^{2*}(n-1) ,$$

where $f_2 := (e-f_1)^{2*} * \bar{x}$ and $f_2^{2*}(-1) := 0$.

Hence

$$((e-f_1)^{2*} * \bar{x})(n) = \sum_{k=0}^{n-1} (f_2(n-k) - f_2(n-1-k))f_2(k) + f_2(0)f_2(n) =$$

$$= - \sum_{k=0}^{n-1} f_1(n-k)f_2(k) + f_2(0)f_2(n) =$$

$$= - \sum_{k=1}^{n} f_1(k) \sum_{m=0}^{k-1} f_1(n-m) + (f_2(n))^2 .$$

Since the second moment is finite we obtain

$$\lim_{n\to\infty} nf_2(n) = 0$$

and so

$$0 \leq \overline{\lim_{n\to\infty}} \frac{(f_2(n))^2}{f_1(n)} \leq \varepsilon \overline{\lim_{n\to\infty}} \frac{f_2(n)}{nf_1(n)} \leq \varepsilon C$$

for every $\varepsilon > 0$ and C some fixed constant.

Now we only have to verify that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} f_1(k) \sum_{m=0}^{k-1} f_1(n-m)}{f_1(n)} = 2 \sum_{k=1}^{\infty} kf_1(k) . \tag{12}$$

Split the above series into the components

$$I_1(n) := \sum_{k=1}^{\lfloor n/2 \rfloor} f_1(k) \sum_{m=0}^{k-1} f_1(n-m) ,$$

$$I_2(n) := \sum_{k=\lceil n/2 \rceil}^{n} f_1(k) \sum_{m=0}^{k-1} f_1(n-m) . \tag{13}$$

Obviously for every $m_0 \in \mathbb{N}$ and $n > 2m_0$

$$I_1(n) = \sum_{k=1}^{m_0} f_1(k) \sum_{m=0}^{k-1} f_1(n-m) + \sum_{k=m_0+1}^{\lfloor n/2 \rfloor} f_1(k) \sum_{m=0}^{k-1} f_1(n-m) .$$

Since

$$\lim_{n\to\infty} \frac{f_1(n+1)}{f_1(n)} = 1$$

we obtain for every fixed $m_0 \in \mathbb{N}$ that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{m_0} f_1(k) \sum_{m=0}^{k-1} f_1(n-m)}{f_1(n)} = \sum_{k=1}^{m_0} kf_1(k) .$$

Also, since $\sum_{k=1}^{\infty} kf_1(k)$ is finite, we can find for every $\varepsilon > 0$ some $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{k=n_0}^{\infty} kf_1(k) \leq \frac{\varepsilon}{D_{f_1}(2)} .$$

This implies

$$0 \leq \frac{\sum_{k=n_0}^{\lfloor n/2 \rfloor} f_1(k) \sum_{m=0}^{k-1} f_1(n-m)}{f_1(n)} \leq D_{f_1}(2) \sum_{k=n_0}^{\infty} f_1(k)k \leq \varepsilon .$$

Letting $\varepsilon \downarrow 0$ ($\Rightarrow n_0 \uparrow \infty$) finally yields

$$\lim_{n\to\infty} \frac{I_1(n)}{f_1(n)} = \sum_{k=1}^{\infty} kf_1(k) . \tag{14}$$

On the other hand,

$$\begin{aligned} I_2(n) &= \sum_{k=\lceil n/2 \rceil}^{n} f_1(k) \sum_{m=n-k+1}^{n} f_1(m) = \\ &= \sum_{k=\lceil n/2 \rceil}^{n} f_1(k) \sum_{m=n-\lfloor n/2 \rfloor+1}^{n} f_1(m) + \sum_{k=\lceil n/2 \rceil}^{n} f_1(k) \sum_{m=n-k+1}^{n-\lfloor n/2 \rfloor} f_1(m) = \\ &= \sum_{k=\lceil n/2 \rceil}^{n} f_1(k) \sum_{m=n-\lfloor n/2 \rfloor+1}^{n} f_1(m) + \sum_{k=1}^{n-\lfloor n/2 \rfloor} f_1(k) \sum_{m=n-k+1}^{n} f_1(m) \end{aligned} \tag{15}$$

for every $n \in \mathbb{N}$.

One can easily verify that the first term in (15) is $o(f_1(n))$, while the second can be analyzed in a similar way as $I_1(n)$. Hence

$$\lim_{n\to\infty} \frac{I_2(n)}{f_1(n)} = \sum_{k=1}^{\infty} kf_1(k)$$

and this implies together with (14) and (13) the desired result (12). □

In Theorems 2.1.22, 2.1.23 and 2.1.25 we discussed the behaviour of $R(n)$ as $n \uparrow \infty$ under the assumption that $\mathbb{E}\underline{X}_1^2 < \infty$ (and some other conditions). In the next four theorems we will discuss their analogues in case $\mathbb{E}\underline{X}_1^2 = \infty$.

THEOREM 2.1.26. *Suppose*

(a) $\mu: \mathbb{N} \to (0,\infty)$ *belongs to* $\mathbb{D}$ *and* $\sup_{n\in\mathbb{N}} \frac{f_1(n)}{\mu(n)} < \infty$

(b) $\sum_{k=1}^{\infty} \ln(k)\mu(k) < \infty$ & $\sum_{k=1}^{\infty} \mu_1(k) = \infty$.

Then

$$\sup_{n\geq 1} \frac{\left|u(n) - \frac{1}{E} - \frac{1}{E^2}\sum_{k=n+1}^{\infty}(1-F(k))\right|}{\sum_{k=0}^{n}(\mu_1(k))\mu_1(n)} < \infty ,$$

where

$$\mu_1(n) := \sum_{k=n+1}^{\infty} \mu(k) .$$

PROOF. First we will prove that

$$\sup_{p\in\mathbb{N}} \frac{|((e-f_1)^{2*} * \frac{\bar{x}}{E})(p)|p}{\mu_1(p)\sum_{k=0}^{p}(\mu_1(k))} < \infty . \tag{16}$$

As in Theorem 2.1.20 we obtain for every $n \in \mathbb{N}$ that

$$d(n) = (n+1)((e-f_1)^{2*} * \frac{\bar{x}}{E})(n+1) , \tag{17}$$

where

$$d = \frac{1}{E} f_2 * (f_2-2g) .$$

Since μ belongs to $\mathbb{D}$ it follows that there exists some $C > 0$ such that

$$|(f_2-2g)|(n) \leq C\mu_1(n)$$

and so

$$(f_2 * |(f_2-2g)|)(n) \leq C\mu_1^{2*}(n) \qquad \text{for every } n \in \mathbb{N} . \tag{18}$$

Hence by the fact that μ_1 also belongs to $\mathbb{D}$ we obtain from (18)

$$|d(n)| \leqq C\mu_1(n) \sum_{k=0}^{n} \mu_1(k) \qquad \text{for every } n \in \mathbb{N} .$$

This implies by (17) the desired result (16).
As in Theorem 2.1.2 we have $f_1^{-1} \in V(\psi_1,\mu)$ and so there exists a constant $C > 0$ such that

$$|R(n)| \leqq C(\ell * \mu)(n) \qquad \text{for every } n \in \mathbb{N} ,$$

where

$$\ell(n) := \frac{\mu_1(n)}{n} \sum_{k=0}^{n} \mu_1(k) , \quad n \geqq 1 , \quad \text{and} \quad \ell(0) := 1 .$$

(Use representation for R(n) in Theorem 2.1.19.)
This implies by the properties of $\mathbb{D}$ that for some $C > 0$

$$|R(n)| \leqq C \frac{\mu_1(n)}{n} \sum_{k=0}^{n} \mu_1(k) + C\mu(n) \sum_{k=0}^{n} \ell(k) \leqq$$

$$\leqq C \sum_{k=0}^{n} \mu_1(k)\left(\frac{\mu_1(n)}{n} + \mu(n) \sum_{k=1}^{n} \frac{\mu_1(k)}{k}\right) \leqq$$

$$\leqq C \sum_{k=0}^{n} \mu_1(k)\left(\frac{\mu_1(n)}{n} + \mu(n) \sum_{k=1}^{n} \ln(k)\mu(k)\right) \leqq$$

$$\leqq C \sum_{k=0}^{n} \mu_1(k) \frac{\mu_1(n)}{n} .$$

□

<u>REMARK 2.1.27</u>. In case $\mu \in L^1(0,\infty)$ and $\sum_{k=1}^{\infty} \mu_1(k)/k = \infty$ $(\Leftrightarrow \sum_{k=1}^{\infty} \ln(k)\mu(k) = \infty)$ it is easy to see from the proof of Theorem 2.1.26 that

$$R(n) \leqq C \frac{\mu_1(n)}{n} \left(\sum_{k=0}^{n} \mu_1(k) + \sum_{k=1}^{n} \frac{\mu_1(k)}{k} \sum_{p=0}^{k} \mu_1(p)\right)$$

for some constant $C > 0$ and every $n \in \mathbb{N}$.

An almost similar proof can be given for the next result.

THEOREM 2.1.28. *Suppose*

(a) $\mu\colon \mathbb{N} \to (0,\infty)$ *belongs to* $\mathbb{D}$ *and* $\lim_{n\to\infty} \frac{f_1(n)}{\mu(n)} = 0$

(b) $\sum_{k=1}^{\infty} \ln(k)\mu(k) < \infty$ *and* $\sum_{k=1}^{\infty} \mu_1(k) = \infty$.

Then

$$\lim_{n\to\infty} \frac{\left(u(n) - \frac{1}{E} - \frac{1}{E^2}\sum_{k=n}^{\infty} (1 - F(k))\right)n}{\left(\sum_{k=0}^{n} \mu_1(k)\right) \mu_1(n)} = 0 .$$

In order to obtain limit results for the sequence R(n) as n tends to infinity (in case $\mathbb{E}\underline{X}_1^2 = \infty$) we have to strengthen the conditions on f_1.

Since similar results will be proved for the non-lattice case we will only mention in the lattice case one of these results.

THEOREM 2.1.29. *Suppose*

(a) $f_1 \in \text{R.V.S.}_{-2}^{\infty}$

(b) $\mathbb{E}\underline{X}_1^2 = \infty$.

Then

$$\lim_{n\to\infty} \frac{u(n) - \frac{1}{E} - \frac{1}{E^2}\sum_{k=n}^{\infty} (1 - F(k))}{(1 - F(n)) \sum_{k=1}^{n} kf_1(k)} = -\frac{2}{E^3} .$$

Finally we discuss some applications of the previous results in the theory of Markov chains and Markov processes.

Let $X = \{\underline{X}_n : n \in \mathbb{N}\}$ be an irreducible aperiodic time-homogeneous Markov chain on a denumerable set T and suppose all its states are recurrent (cf. [FEL-1]). Define for all $i,j \in T$

$$\begin{aligned} \underline{T}_{ij} &:= \inf\{n \geq 1 : \underline{X}_n = j\} \quad \text{if } \underline{X}_0 = i \\ &:= 0 \qquad\qquad\qquad\qquad \text{otherwise} \end{aligned} \tag{19}$$

and

$$f_{ij}(k) := \mathbb{P}\{\underline{T}_{ij} = k \mid \underline{X}_0 = i\}, \quad k \geq 1$$
$$f_{ij}(0) := 0 . \tag{20}$$

By assumption for all $i \in T$

$$\sum_{k=0}^{\infty} f_{ii}(k) = 1 , \quad \text{g.c.d. } \{n: f_{ii}(n) > 0\} = 1 . \tag{21}$$

If $A \subset T$ is some arbitrary subset then for all $n \geq 1$ and $i \in T$

$$\mathbb{P}\{\underline{X}_n \in A \mid \underline{X}_0 = i\} = \mathbb{P}\{\underline{X}_n \in A; \underline{T}_{ii} > n \mid \underline{X}_0 = i\} +$$
$$\mathbb{P}\{\underline{X}_n \in A; \underline{T}_{ii} \leq n \mid \underline{X}_0 = i\} =$$

$$= \mathbb{P}\{\underline{X}_n \in A; \underline{T}_{ii} > n \mid \underline{X}_0 = i\} + \sum_{k=0}^{n} \mathbb{P}\{\underline{X}_n \in A; \underline{T}_{ii} = k \mid \underline{X}_0 = i\} =$$

$$= \mathbb{P}\{\underline{X}_n \in A; \underline{T}_{ii} > n \mid \underline{X}_0 = i\} + \sum_{k=0}^{n} \mathbb{P}\{\underline{X}_n \in A \mid \underline{T}_{ii} = k, \underline{X}_0 = i\} f_{ii}(k) . \tag{22}$$

By the strong Markov property (cf. [ÇIN]) and the definition of $\underline{T}_{ii}$ we obtain

$$\sum_{k=0}^{n} \mathbb{P}\{\underline{X}_n \in A \mid \underline{T}_{ii} = k, \underline{X}_0 = i\} f_{ii}(k) =$$

$$= \sum_{k=0}^{n} \mathbb{P}\{\underline{X}_n \in A \mid \underline{X}_k = i\} f_{ii}(k) =$$

$$= \sum_{k=0}^{n} \mathbb{P}\{\underline{X}_{n-k} \in A \mid \underline{X}_0 = i\} f_{ii}(k) , \tag{23}$$

and so substituting (23) into (22) yields

$$\mathbb{P}\{\underline{X}_n \in A \mid \underline{X}_0 = i\} = \mathbb{P}\{\underline{X}_n \in A; \underline{T}_{ii} > n \mid \underline{X}_0 = i\} +$$

$$\sum_{k=0}^{n} \mathbb{P}\{\underline{X}_{n-k} \in A \mid \underline{X}_0 = i\} f_{ii}(k) . \tag{24}$$

If we define

$${}_{i}p_{iA}(n) := \mathbb{P}\{\underline{X}_n \in A; \underline{T}_{ii} > n \mid \underline{X}_0 = i\}$$

and

$$p_{iA}(n) := \mathbb{P}\{\underline{X}_n \in A \mid \underline{X}_0 = i\}$$

it is easy to see that (24) means

$$p_{iA}(n) := {}_ip_{iA}(n) + (p_{iA} * f_{ii})(n) \qquad \text{for all } n \geqq 0 .$$

This implies for all $|z| < 1$

$$\hat{P}_{iA}(z) = {}_i\hat{P}_{iA}(z) + \hat{P}_{iA}(z)\hat{F}_{ii}(z)$$

or

$$\hat{P}_{iA}(z) = {}_i\hat{P}_{iA}(z) \cdot \frac{1}{1 - \hat{F}_{ii}(z)} , \tag{25}$$

where

$$\hat{P}_{iA}(z) := \sum_{n=0}^{\infty} p_{iA}(n) z^n ,$$

$${}_i\hat{P}_{iA}(z) := \sum_{n=0}^{\infty} {}_ip_{iA}(n) z^n ,$$

$$\hat{F}_{ii}(z) := \sum_{n=0}^{\infty} f_{ii}(n) z^n .$$

(Remember $|\hat{F}_{ii}(z)| < 1$ if $|z| < 1$ since $\{f_{ii}(n)\}_{n=0}^{\infty}$ is a probability distribution!)

It is well known that the sequence of n-steps transition probabilities $\{p_{ii}(n)\}_{n=0}^{\infty}$ of the Markov chain satisfies (cf. [FEL-1])

$$p_{ii}(n) = (f_{ii} * p_{ii})(n) , \qquad n \geqq 1$$
$$p_{ii}(0) = 1$$

and hence for all $|z| < 1$

$$\hat{P}_{ii}(z) = \frac{1}{1 - \hat{F}_{ii}(z)} , \tag{26}$$

where $\hat{P}_{ii}(z) := \hat{P}_{iA}(z)$ with $A = \{i\}$.

Combining (25) and (26) yields

$$\hat{P}_{iA}(z) = {}_i\hat{P}_{iA}(z) \cdot \hat{P}_{ii}(z) = (\widehat{{}_ip_{iA} * p_{ii}})(z)$$

and this implies by the uniqueness of the generating function that

$$p_{iA}(n) = ({}_ip_{iA} * p_{ii})(n) , \qquad \text{for all } n \geq 0 . \tag{27}$$

If the Markov chain is also ergodic (cf. [FEL-1]) we can proceed as follows. Let $j \in T$ and take $A = \{j\}$. By (27)

$$p_{ij}(n) = ({}_ip_{ij} * p_{ii})(n) , \qquad \text{for all } n \geq 0 . \tag{28}$$

Define now

$$m_{ii} := \mathbb{E}(\underline{T}_{ii} \mid \underline{X}_0 = i) \quad (< \infty) ,$$

$$f_{ii,1}(n) := \frac{1}{m_{ii}} \sum_{k=n+1}^{\infty} f_{ii}(k) := \frac{1}{m_{ii}} (1-F_{ii}(n)) . \tag{29}$$

From (26) and (21) we learn that $p_{ii}(n)$ $(f_{ii}(n))$ plays the role of $u(n)$ $(f(n))$ for all $n \in \mathbb{N}$. Hence by Lemma 2.1.3, $f_{ii,1}^{-1}$ exists and $f_{ii,1}^{-1} \in V(\psi_1)$. Moreover, by Lemma 2.1.1

$$p_{ii} = f_{ii,1}^{-1} * \frac{\bar{x}}{m_{ii}} . \tag{30}$$

Substituting (30) into (28) yields

$$p_{ij}(n) = \left({}_ip_{ij} * \frac{\bar{x}}{m_{ii}} * f_{ii,1}^{-1}\right)(n) , \qquad \text{for all } n \geq 0 . \tag{31}$$

We know from Chapter 1, § 9 of [CHU] that

$$\alpha_{ij} := \sum_{n=0}^{\infty} {}_ip_{ij}(n)$$

is finite and hence by (31) $(\alpha_{ij}^* := \alpha_{ij}/m_{ii})$

$$|p_{ij}(n) - \alpha_{ij}^*| \leq \left({}_ip_{ij} * \left|\frac{\bar{x}}{m_{ii}} * (f_{ii,1}^{-1} - e)\right|\right)(n) + \frac{1}{m_{ii}} \sum_{k=n+1}^{\infty} {}_ip_{ij}(k)$$

or equivalently (use $\sum_{k=0}^{\infty} f_{ii,1}^{-1}(k) = 1$)

$$|p_{ij}(n) - \alpha_{ij}^*| \leq \frac{1}{m_{ii}} \sum_{k=0}^{n} {}_ip_{ij}(k) |f_{ii,1}^{-1}|([n-k+1,\infty)) + \frac{1}{m_{ii}} \sum_{k=n+1}^{\infty} {}_ip_{ij}(k) . \tag{32}$$

Using

$$\sum_{j \in T} {}_ip_{ij}(k) =: {}_ip_{iT}(k) = 1 - F_{ii}(k) = m_{ii}\, f_{ii,1}(k)$$

we obtain from (32)

$$\sum_{j\in T} |p_{ij}(n) - \alpha^*_{ij}| \leq \sum_{k=0}^{n} f_{ii,1}(k)|f^{-1}_{ii,1}|([n-k+1,\infty)) + \sum_{k=n+1}^{\infty} f_{ii,1}(k)$$

$$\leq (f_{ii,1} * |f^{-1}_{ii,1}|)([n,\infty)) \ . \qquad (33)$$

(Observe that $|f^{-1}_{ii,1}|([0,\infty)) \geq 1$.)

Also $f_{ii,1} * |f^{-1}_{ii,1}| \in V(\psi_1)$ by Theorem 1.2.1 and $f_{ii,1}, |f^{-1}_{ii,1}| \in V(\psi_1)$. Hence

$$\lim_{n\to\infty} \sum_{j\in T} |p_{ij}(n) - \alpha^*_{ij}| = 0$$

and knowing that the limit *exists* we can easily deduce that

$$\alpha^*_{ij} = \frac{1}{m_{jj}} \qquad \text{(cf. [FEL-1]).}$$

By (33) it is also possible to prove stronger results under stronger conditions. However, before mentioning those results, we introduce the following notations.

If $\psi: \mathbb{N} \to (0,\infty)$ is some arbitrary sequence then

$$\psi^o(n) := \sum_{k=0}^{n} \psi(k) \qquad \text{for all } n \geq 0 ,$$

$$(\Delta\psi)(n) := \psi(n) - \psi(n-1) , \quad n \geq 1 ; \qquad (\Delta\psi)(0) := 0 .$$

Moreover, for $X = \{\underline{X}_n : n \in \mathbb{N}\}$ some *ergodic Markov chain* on a denumerable set T we introduce

$$\|P_i^{\underline{X}_n} - \pi\| := \sum_{j\in T} |p_{ij}(n) - \frac{1}{m_{jj}}| \qquad (i \in T) .$$

THEOREM 2.1.30.

(i) *If*

$$\sum_{n=0}^{\infty} \psi^o(n) P\{\underline{T}_{ii} = n \mid \underline{X}_0 = i\}$$

is finite for some $i \in T$, *where* ψ *nondecreasing and* $\psi \in \Psi$ *with* $\lim_{n\to\infty} (\psi(n))^{1/n} = 1$, *then*

$$\sum_{n=0}^{\infty} (\Delta\psi)(n) \| P_i^{\underline{X}_n} - \pi \| < \infty .$$

(ii) *If*

$$\sum_{n=0}^{\infty} r^n \, P\{\underline{T}_{ii} = n \mid \underline{X}_0 = i\}$$

is finite for some $r > 1$ *and* $i \in T$, *then there exists some* α *with* $1 < \alpha \leq r$ *such that*

$$\sum_{n=0}^{\infty} \alpha^n \| P_i^{\underline{X}_n} - \pi \| < \infty .$$

(iii) *If* $\mathbb{E}(\underline{T}_{ii} \mid \underline{X}_0 = i)$ *is finite and*

$$\lim_{n\to\infty} \frac{\sum_{k=n}^{\infty} P\{\underline{T}_{ii} > k \mid \underline{X}_0 = i\}}{\mu([n,\infty))} = 0$$

for some $i \in T$ *and* $\mu \in ST$ *with* $\lim_{n\to\infty} (\mu([n,\infty)))^{1/n} = 1$, *then*

$$\lim_{n\to\infty} \frac{\| P_i^{\underline{X}_n} - \pi \|}{\mu([n,\infty))} = 0 .$$

(iv) *If* $\mathbb{E}(\underline{T}_{ii} \mid \underline{X}_0 = i)$ *is finite and*

$$\sup_{n\geq 0} \frac{\sum_{k=n}^{\infty} P\{\underline{T}_{ii} > k \mid \underline{X}_0 = i\}}{\mu([n,\infty))} < \infty$$

for some $i \in T$ *and* $\mu \in ST$ *with*

$$\lim_{n\to\infty} (\mu([n,\infty)))^{1/n} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \sup_{p\in\mathbb{N}} \frac{(\mu-\mu_n)^{2*}([p,\infty))}{\mu([p,\infty))} = 0$$

then

$$\sup_{n\in\mathbb{N}} \frac{\| P_i^{\underline{X}_n} - \pi \|}{\mu([n,\infty))} < \infty .$$

<u>PROOF</u>. We only prove (i) since the other results can be proved similarly using different Banach algebras (see also proof different results for

renewal sequences). Since $f_{ii} \in V(\psi^o)$ for some i we obtain easily $f_{ii,1} \in V(\psi)$. Hence by Lemma 2.1.3 and Theorem 1.2.1

$$f_{ii,1} * |f_{ii,1}^{-1}| \in V(\psi) \ . \tag{34}$$

By (33) and ψ nondecreasing

$$\sum_{n=1}^{\infty} (\Delta\psi)(n) \| P_i^{X_n} - \pi \| \leq \sum_{n=1}^{\infty} (\Delta\psi)(n)(f_{ii,1} * |f_{ii,1}^{-1}|)([n,\infty))$$

and so using Fubini's theorem (observe that the double serie consist of positive terms):

$$\begin{aligned}\sum_{n=1}^{\infty} (\Delta\psi)(n) \| P_i^{X_n} - \pi \| &\leq \sum_{n=1}^{\infty} (\Delta\psi)(n) \sum_{k=n}^{\infty} (f_{ii,1} * |f_{ii,1}^{-1}|)(k) = \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^{k} (\Delta\psi)(n)(f_{ii,1} * |f_{ii,1}^{-1}|)(k) = \\ &= \sum_{k=1}^{\infty} (\psi(k) - \psi(1))(f_{ii,1} * |f_{ii,1}^{-1}|)(k) \ . \end{aligned} \tag{35}$$

Applying (34) yields the desired result. □

REMARK 2.1.31. A part of the above results are also given by [GRÜ] using a more complicated proof.
Moreover, these results slightly generalize Theorem 1.1 of [NUM].

This concludes our application to Markov chains. In order to apply our results to Markov processes we first summarize some well-known properties of a time-homogeneous Markov process $X = \{\underline{X}_t : t \geq 0\}$ on a denumerable set T with standard Markov semi-group $P = \{P(t) : t \geq 0\}$.

In this case (cf. [FREE], [CHU]):

(i) $\lim_{h \downarrow 0} \frac{p_{ii}(h) - 1}{h} =: Q_{ii}$ exists and $-\infty \leq Q_{ii} \leq 0$ for all $i \in T$.

(Remember $p_{ij}(t) := \mathbb{P}\{\underline{X}(t) = j \mid \underline{X}(0) = i\}$.)

(ii) $\lim_{h \downarrow 0} \frac{p_{ij}(h)}{h} =: Q_{ij}$ exists and $Q_{ij} < \infty$ for every $i,j \in T$ with $i \neq j$.

iii) $\inf_{i\in T} Q_{ii} > -\infty \Leftrightarrow P = \{P(t): t \geq 0\}$ is a uniform Markov semi-group.

(iv) $\inf_{i\in T} Q_{ii} > -\infty \Rightarrow \lim_{h\downarrow 0} \|\frac{P(h) - I}{h} - Q\| = 0$ with $\|A\| := \sup_{i\in T} \sum_{j\in T} |a_{ij}|$ and

$P(t) = e^{Qt}$, where $e^{Qt} := \sum_{k=0}^{\infty} \frac{Q^k t^k}{k!}$.

By the equality

$$\sum_{j\in T} Q_{ij} = \sum_{j\in T} \left(Q_{ij} - \frac{p_{ij}(h) - \delta_{ij}}{h}\right)$$

and (iv) it follows immediately letting $h \downarrow 0$ that

$$\sum_{j\in T} Q_{ij} = 0 \quad \forall i\in T \tag{36}$$

if $-q_{inf} := \inf_{i\in T} Q_{ii} > -\infty$.

Define in this case for every $q \geq q_{inf}$

$$\hat{Q}_q = I + \frac{Q}{q} . \tag{37}$$

Then $\hat{Q}_q$ is a stochastic matrix by (i), (ii) and (36). Also

$$\begin{aligned} P(t) = e^{Qt} &= e^{-qIt+q\hat{Q}_q t} \\ &= e^{-qIt}\, e^{q\hat{Q}_q t} \quad (qIt \text{ and } q\hat{Q}_q t \text{ commute}) \\ &= e^{-qt} \sum_{k=0}^{\infty} \frac{(qt)^k}{k!} \hat{Q}_q^k . \end{aligned}$$

This implies for all $i,j \in T$

$$p_{ij}(t) = e^{-qt} \sum_{k=0}^{\infty} \frac{(qt)^k}{k!} \hat{Q}_{q,ij}(k) , \tag{38}$$

where $\hat{Q}_{q,ij}(k)$ is the k-step transition probability from i to j belonging to the Markov chain $X(q) := \{\underline{X}_n(q): n \in \mathbb{N}\}$ having the stochastic matrix $\hat{Q}_q$ as its transition probability matrix.

Since the realizations of a Poisson process are right-continuous step functions (cf. [ÇIN]) it follows immedately from (38) that there exists a version of the Markov process $X = \{\underline{X}(t): t \geq 0\}$ having right-continuous realizations with a finite number of jumps in every finite interval. (From now on we will always consider this version.) Hence the following random variables are well-defined:

$$\underline{\rho}_i := \inf\{t > 0 : \underline{X}_t \neq i\} \quad \text{if } \underline{X}_0 = i$$
$$:= 0 \qquad \text{otherwise.} \tag{39}$$

and

$$\underline{\alpha}_{ii} := \inf\{t > 0,\ t > \underline{\rho}_i \ \&\ \underline{X}_t = i\} \quad \text{if } \underline{X}_0 = i$$
$$:= 0 \qquad \text{otherwise.} \tag{40}$$

DEFINITION 2.1.32. Let $X = \{\underline{X}(t) : t \geqq 0\}$ be a time-homogeneous Markov process on a denumerable set S with uniform Markov semi-group $P = \{P(t) : t \geqq 0\}$. (This is called a *uniform Markov process*!)

(i) If $\underline{X}_0 = i$ we call i *recurrent* if $P\{\underline{\alpha}_{ii} < \infty \mid \underline{X}_0 = i\} = 1$, otherwise i is called *transient*.

(ii) A recurrent state i is called *positive recurrent* if $\mathbb{E}(\underline{\alpha}_{ii} \mid \underline{X}_0 = i) < \infty$, otherwise it is called *null recurrent*.

(iii) The Markov process $X = \{\underline{X}(t) : t \geqq 0\}$ is called *irreducible* if for every $i,j \in S$ there exists some $t_1 = t(i,j) > 0$ such that $p_{ij}(t_1) > 0$. (Since $P = \{P(t) : t \geqq 0\}$ is a standard Markov semi-group this immediately implies $p_{ij}(t) > 0$ for every $t \geqq t_1$.)

Finally we prove the following rate of convergence result for uniform Markov processes.

THEOREM 2.1.33. *Let* $X = \{\underline{X}(t) : t \geqq 0\}$ *be an irreducible uniform Markov process and suppose there exists a recurrent state* i *with* $\mathbb{E}(\underline{T}_{ii}^k \mid \underline{X}_0 = i)$ *is finite for some* $k > 1$ *(*$k \in \mathbb{N}$*). Then*

$$\lim_{t\to\infty} t^{k-1} \sum_{j\in T} |p_{ij}(t) - \pi_j| = 0 \tag{41}$$

where $\{\pi_j\}_{j\in T}$ *is the unique solution of the set of equations*

$$v_j > 0\ ,\quad \sum_{j\in T} v_j = 1\ ,\quad \sum_{i\in T} v_i\, Q(i,j) = v_j\ . \tag{42}$$

PROOF. By (38) (fix $q > q_{inf}$)

$$p_{ij}(t) = e^{-qt} \sum_{k=0}^{\infty} \frac{(qt)^k}{k!} \hat{Q}_{q,ij}(k) \qquad (i,j \in T) \tag{43}$$

and this implies, since $q > q_{inf}$, i recurrent and $X = \{\underline{X}(t): t \geq 0\}$ irreducible, that the underlying Markov chain $X(q) = \{\underline{X}_n(q): n \in \mathbb{N}\}$ is also irreducible and consists of aperiodic recurrent states.
If as always

$$f_{ii}(k) := \mathbb{P}\{\underline{T}_{ii}(q) = k \mid \underline{X}_0(q) = i\} ,$$

where $\underline{T}_{ii}(q)$ is the first return time from i to i of the Markov chain $X(q) = \{\underline{X}_n(q): n \in \mathbb{N}\}$ and

$$\hat{F}_{ii}(z) := \sum_{n=0}^{\infty} z^n f_{ii}(k) , \qquad |z| < 1 ,$$

we obtain from (43)

$$\int_0^\infty \exp(-st)p_{ii}(t)dt = \frac{1}{s+q} \sum_{k=0}^{\infty} \left(\frac{q}{s+q}\right)^k \hat{Q}_{q,ii}(k) =$$

$$= \frac{1}{s+q} \cdot \frac{1}{1 - \hat{F}_{ii}(\frac{q}{s+q})} . \tag{44}$$

Also (cf. [COH])

$$\int_0^\infty \exp(-st)p_{ii}(t)dt = \frac{1}{s+q_i} \cdot \frac{1}{1 - \varphi_{ii}(s)} , \tag{45}$$

where

$$q_i := -Q_{ii} , \quad \varphi_{ii}(s) := \int_0^\infty e^{-st} F_{ii}(dt) \quad \text{and} \quad F_{ii}(t) := P\{\underline{\alpha}_{ii} \leq t \mid \underline{X}_0 = i\} .$$

Combining (44) and (45) yields

$$(s+q_i) \frac{1 - \varphi_{ii}(s)}{s} = \frac{1 - \hat{F}_{ii}(\frac{q}{s+q})}{1 - \frac{q}{s+q}}$$

and this implies

$$\mathbb{E}\left((\underline{T}_{ii}(q))^k \mid \underline{X}_0(q) = i\right) < \infty \Leftrightarrow \mathbb{E}\left(\underline{\alpha}_{ii}^k \mid \underline{X}_0 = i\right) < \infty . \tag{46}$$

Hence it follows from (46) that the recurrent states of the underlying Markov chain are ergodic and applying Theorem 2.1.30 (iii) we have

$$\lim_{n\to\infty} n^{k-1} \sum_{j\in S} |\hat{Q}_{q,ij}(n) - \pi_j| = 0 . \tag{47}$$

Moreover, by considering (43), we obtain

$$t^{k-1} \sum_{j\in T} |P_{ij}(t) - \pi_j| \leq t^{k-1} e^{-qt} \sum_{k=0}^{\infty} \frac{(qt)^k}{k!} \sum_{j\in T} |\hat{Q}_{q,ij}(k) - \pi_j| \leq$$

$$\leq 2t^{k-1} e^{-qt} \sum_{k=0}^{\lfloor qt/2 \rfloor} \frac{(qt)^k}{k!} + t^{k-1} e^{-qt} \sum_{k=\lceil qt/2 \rceil}^{\infty} \frac{(qt)^k}{k!} \sum_{j\in T} |\hat{Q}_{q,ij}(k) - \pi_j| . \tag{48}$$

The second term in (48) converges to zero by (47) while the first term equals $2t^{k-1} P\{\underline{N}(t) \leq \lfloor qt/2 \rfloor\}$ where $\{\underline{N}(t): t \geq 0\}$ is a Poisson process with rate q (cf. [ÇIN]).
This implies that the first term also converges to zero as $t \uparrow \infty$ and so by the above observations and (48) we have proved (41).
Also by a well-known result for Markov chains (cf. [ÇIN]) we obtain that $\{\pi_j\}_{j\in T}$ is the unique solution of the set of equations

$$v_j > 0 , \quad \sum_{j\in T} v_j = 1 , \quad v_j = \sum_{i\in T} \hat{Q}(i,j) v_i .$$

This implies (42) and so the proof of Theorem 2.1.33 is finished. □

<u>REMARK 2.1.34.</u>

(i) In case the state space T is finite it is clear that a Markov process on T is a uniform Markov process. Moreover, it is not difficult to prove that every finite irreducible Markov chain converges exponentially fast to its invariant distribution (cf. [LAM]).
Combining the above observations yields that every finite irreducible Markov process also converges exponentially fast to its limit $\{\pi_j\}_{j\in T}$ (cf. [DOO]).

(ii) In applications one often knows only the matrix Q and so by imposing conditions on $\hat{Q}_q$ for some $q > q_{inf}$ we can derive similar results as in Theorem 2.1.30.

(iii) A similar type of result as stated in Theorem 2.1.33 can be proved for even a bigger class of standard Markov processes by using general results for regenerative processes (see Chapter 4). However, since uniform Markov processes have a very nice representation for their transition probabilities, we thought it useful to give an alternative proof of this class.

This concludes our section on the behaviour of the renewal sequence $\{u(n)\}_{n=0}^{\infty}$ in case the expectation E is finite. In the next section we consider the rate of convergence in case $E = \infty$.

2. The behaviour of the renewal sequence in case the expectation is infinite

If $E = \infty$ we have to use another method of proof for analyzing the rate of convergence of u(n) to zero. This is due to the fact that the total variation norm of the sequence $\{1 - F(n)\}_{n=0}^{\infty}$ is infinite.

In this case we will use Fourier analysis to obtain a representation of u(n). Although this method can also be used in the lattice case if $E < \infty$ (cf. [STO-3]), it is not as powerful as the Banach algebra method. Further, we will only discuss in this section the case of regularly varying tails (cf. Definition 2.1.16).

LEMMA 2.2.1. *Suppose* $\{c(n)\}_{n=-\infty}^{\infty}$ *is a sequence satisfying*

$$\sum_{n=-\infty}^{+\infty} |c(n)| < \infty .$$

If

$$\varphi(\theta) := \sum_{n=-\infty}^{+\infty} c(n) \exp(in\theta)$$

then

$$c(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-in\theta)\varphi(\theta)d\theta .$$

PROOF. By Lebesgue's dominated convergence theorem we obtain that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(-in\theta)\varphi(\theta)d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(-in\theta)\sum_{k=-\infty}^{+\infty} c(k)\exp(ik\theta)d\theta =$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} c(k)\int_{-\pi}^{\pi}\exp(-i(n-k)\theta)d\theta = c(n)\ .$$

A very important result is stated in the following theorem.

THEOREM 2.2.2. (cf. [ERI].) *Suppose the sequence* $\{f(n)\}_{n=-\infty}^{+\infty}$ *defines a lattice probability distribution function with g.c.d.* $\{n \in \mathbb{N}: f(n) > 0\} = 1$ *and let* $\varphi(\theta)$ *be its characteristic function (i.e.* $\varphi(\theta) := \sum_{n=-\infty}^{\infty}\exp(in\theta)f(n)$*.) If the absolute first moment* m_1 *is finite and the expectation* E *positive then for all continuous functions* h *on* $[-\pi,\pi]$

$$\lim_{r\uparrow 1}\int_{-\pi}^{\pi} h(\theta)\ \mathrm{Re}\left(\frac{1}{1-r\varphi(\theta)}\right)d\theta = \frac{\pi h(0)}{E} + \int_{-\pi}^{\pi} h(\theta)\ \mathrm{Re}\left(\frac{1}{1-\varphi(\theta)}\right)d\theta\ .$$

If $\lim_{\theta\to 0}\frac{\mathrm{Im}\ \varphi(\theta)}{\theta} = \infty$ *the above relation is also true with the interpretation that in this case* $\pi h(0)/E$ *equals zero.*

PROOF. We omit the proof for the lattice case as it is similar to the proof for the nonlattice case (cf. Section 3, Chapter 3). □

From now on we will only consider in this section (unless stated otherwise) the case where the lattice distribution f(n) is concentrated on the nonnegative integers and g.c.d. $\{n > 0: f(n) > 0\} = 1$.

THEOREM 2.2.3. (cf. [ERI], [GAR].) *The renewal sequence* $\{u(n)\}_{n=0}^{\infty}$ *possesses the following representation for all* $n \geq 1$

$$u(n) = \begin{cases} \frac{1}{E} + \frac{2}{\pi}\int_0^{\pi}\cos(n\theta)\ \mathrm{Re}\left(\frac{1}{1-\varphi(\theta)}\right)d\theta & \text{if } 0 < E < \infty \\ \frac{2}{\pi}\int_0^{\infty}\cos(n\theta)\ \mathrm{Re}\left(\frac{1}{1-\varphi(\theta)}\right)d\theta & \text{if } E = \infty\ . \end{cases}$$

PROOF. From the probabilistic interpretation for u(n) we easily obtain

$$u(n) = \sum_{p=0}^{\infty} f^{p*}(n)$$

with f^{p*} the p-fold convolution of the sequence $\{f(n)\}_{n=0}^{\infty}$ and f^{0*} the probability distribution concentrated at 0. Hence

$$u(n) = \lim_{r\uparrow 1} \sum_{p=0}^{\infty} r^p f^{p*}(n) =$$

$$= \lim_{r\uparrow 1} \frac{1}{2\pi} \sum_{p=0}^{\infty} r^p \int_{-\pi}^{\pi} \exp(-in\theta)(\varphi(\theta))^p \, d\theta =$$

$$= \lim_{r\uparrow 1} \frac{1}{2\pi} \sum_{p=0}^{\infty} r^p \int_{-\pi}^{\pi} \mathrm{Re}(\exp(-in\theta)(\varphi(\theta))^p) d\theta \quad \forall\, n\geq 0 \tag{1}$$

Consider now the integral $\int_{-\pi}^{\pi} \mathrm{Re}(\exp(-in\theta)(\varphi(\theta))^p)d\theta$ for every $n \geq 1$ and $p \geq 0$.
It follows

$$\int_{-\pi}^{\pi} \mathrm{Re}(\exp(-in\theta)(\varphi(\theta))^p)d\theta = \int_{-\pi}^{\pi} \cos(n\theta)\, \mathrm{Re}((\varphi(\theta))^p)d\theta + $$

$$+ \int_{-\pi}^{\pi} \sin(n\theta)\, \mathrm{Im}((\varphi(\theta))^p)d\theta .$$

Moreover by the dominated convergence theorem and the definition of the Fourier transform we obtain for every $n \geq 1$

$$\int_{-\pi}^{\pi} \cos(n\theta)\, \mathrm{Re}((\varphi(\theta))^p)d\theta - \int_{-\pi}^{\pi} \sin(n\theta)\, \mathrm{Im}((\varphi(\theta))^p)d\theta =$$

$$= \sum_{k=0}^{\infty} f^{p*}(k) \int_{-\pi}^{\pi} (\cos(n\theta)\cos(k\theta) - \sin(n\theta)\sin(k\theta))d\theta =$$

$$= \sum_{k=0}^{\infty} f^{p*}(k) \int_{-\pi}^{\pi} \cos((n+k)\theta)d\theta = 0 .$$

Hence

$$\int_{-\pi}^{\pi} \mathrm{Re}(\exp(-in\theta)(\varphi(\theta))^p)\,d\theta = 2\int_{-\pi}^{\pi} \cos(n\theta)\ \mathrm{Re}((\varphi(\theta))^p)\,d\theta$$

for every $n \geqq 1$ and this implies by (1) and Theorem 2.2.2

$$u(n) = \lim_{r\uparrow 1} \frac{1}{\pi} \sum_{p=0}^{\infty} r^p \int_{-\pi}^{\pi} \cos(n\theta)\ \mathrm{Re}((\varphi(\theta))^p)\,d\theta =$$

$$= \lim_{r\uparrow 1} \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(n\theta)\ \mathrm{Re}\Big(\sum_{p=0}^{\infty} r^p(\varphi(\theta))^p\Big)\,d\theta =$$

$$= \lim_{r\uparrow 1} \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(n\theta)\ \mathrm{Re}\Big(\frac{1}{1 - r\varphi(\theta)}\Big)\,d\theta =$$

$$= \begin{cases} \dfrac{1}{E} + \dfrac{2}{\pi}\displaystyle\int_{0}^{\pi} \cos(n\theta)\ \mathrm{Re}\Big(\frac{1}{1-\varphi(\theta)}\Big)\,d\theta & \text{if } 0 < E < \infty \\[2ex] \dfrac{2}{\pi}\displaystyle\int_{0}^{\pi} \cos(n\theta)\ \mathrm{Re}\Big(\frac{1}{1-\varphi(\theta)}\Big)\,d\theta & \text{if } \displaystyle\lim_{\theta\to 0} \frac{\mathrm{Im}\,\varphi(\theta)}{\theta} = \infty \ (\Leftrightarrow E = \infty) \end{cases}$$

□

The main results in case $E = \infty$ are stated in the next two theorems.

<u>THEOREM 2.2.4</u>. (cf. [ERI], [GAR].)

(a) *For* $\frac{1}{2} < \alpha < 1$

$$1 - F(n) \in \mathrm{R.V.S.}_{-\alpha}^{\infty} \Leftrightarrow u(n) \in \mathrm{R.V.S.}_{\alpha-1}^{\infty}\ .$$

Either relation implies

$$\lim_{n\to\infty} m(n)u(n) = \frac{\sin \pi\alpha}{\pi(1-\alpha)} \quad \textit{with} \quad m(n) := \sum_{k=0}^{n} (1 - F(k))\ .$$

(b) *For* $0 < \alpha \leqq \frac{1}{2}$

$$1 - F(n) \in \mathrm{R.V.S.}_{-\alpha}^{\infty} \Rightarrow \liminf_{n\to\infty} m(n)u(n) = \frac{\sin \pi\alpha}{\pi(1-\alpha)}\ .$$

PROOF. It is easy to deduce (using Karamata's Abel-Tauber theorem) that $u(n) \in R.V.S.^{\infty}_{\alpha-1}$ implies $1 - F(n) \in R.V.S.^{\infty}_{-\alpha}$ for $\frac{1}{2} < \alpha < 1$. (A proof of this result is only given for the nonlattice case, cf. Chapter 3.)
The proof of the other results (using the Fourier representation given in Theorem 2.2.3) can be found in [GAR] or [ERI]. □

REMARK 2.2.5. It is possible to construct a probability distribution $\{f(n)\}_{n=0}^{\infty}$ with g.c.d. $\{n > 0: f(n) > 0\} = 1$ such that

$$1 - F(n) := \sum_{k=n+1}^{\infty} f(k) \in R.V.S.^{\infty}_{-\frac{1}{2}}$$

and

$$\limsup_{n \to \infty} m(n)u(n) > 0 . \qquad \text{(cf. [GOE].)}$$

This means that there is no analog of Theorem 2.2.4, part (a), for $0 < \alpha \leq \frac{1}{2}$. Similar results can be obtained only by stronger conditions on the tail of the lattice probability distribution.

The following theorem discusses a key result in case $1 - F(n) \in R.V.^{\infty}_{-1}$.

THEOREM 2.2.6. (cf. [FRE].)

$$1 - F(n) \in R.V.S.^{\infty}_{-1} \Rightarrow \lim_{n\to\infty} \frac{u(n) - u([np])}{n(1-F(n))(m(n))^{-2}} = \ln(p) \quad \forall\, p>0 .$$

PROOF. Since $1 - F(n) \in R.V.S.^{\infty}_{-1}$ we may apply the representation for the sequence $\{u(n)\}$ mentioned in Theorem 2.2.3. This means that for all $p > 1$ with $W(\theta) := \mathrm{Re}(\frac{1}{1 - \varphi(\theta)})$

$$\frac{\pi}{2}\,(u(n) - u([np])) = \left(\int_0^{B/n} \cos(n\theta)W(\theta)d\theta - \int_0^{B/[np]} \cos([np]\theta)W(\theta)d\theta\right) +$$

$$+ \int_{B/n}^{\varepsilon} \cos(n\theta)W(\theta)d\theta - \int_{B/[np]}^{\varepsilon} \cos([np]\theta)W(\theta)d\theta +$$

$$+ \int_{\varepsilon}^{\pi} (\cos(n\theta) - \cos([np]\theta))W(\theta)d\theta . \qquad (2)$$

We will consider these three parts separately and prove

(a) $$\lim_{n\to\infty} \frac{2}{\pi} \frac{\int_0^{B/n} \cos(n\theta)W(\theta)d\theta - \int_0^{B/[np]} \cos([np]\theta)W(\theta)d\theta}{n(1-F(n))m^{-2}(n)} = \ln(p) .$$

(b) $$\overline{\lim_{n\to\infty}} \frac{|\int_\varepsilon^\pi (\cos(n\theta) - \cos([np]\theta))W(\theta)d\theta|}{n(1-F(n))m^{-2}(n)} = 0 .$$

(c) $$\overline{\lim_{n\to\infty}} \frac{|\int_{B/n}^\varepsilon \cos(n\theta)W(\theta)d\theta - \int_{B/[np]}^\varepsilon \cos([np]\theta)W(\theta)d\theta|}{n(1-F(n))m^{-2}(n)} = O(B^{-\delta})$$

for some $\delta > 0$.

In order to prove (b) and (c) it is sufficient to prove

(b') $$\overline{\lim_{n\to\infty}} \frac{|\int_\varepsilon^\pi \cos(n\theta)W(\theta)d\theta|}{n(1-F(n))m^{-2}(n)} = 0 .$$

(c') $$\overline{\lim_{n\to\infty}} \frac{|\int_{B/n}^\varepsilon \cos(n\theta)W(\theta)d\theta|}{n(1-F(n))m^{-2}(n)} = O(B^{-\delta}) \quad \text{for some } \delta > 0 .$$

We will first provide the proof of (a) and (b') since the proof of (c') is lengthy and rather technical.

<u>PROOF of (a)</u>. Using integration by parts we obtain for every $p \geqq 1$ and $B > 0$

$$\int_0^{B/[np]} \cos([np]\theta)W(\theta)d\theta = \cos(B) \int_0^{B/[np]} W(\theta)d\theta +$$

$$+ [np] \int_0^{B/[np]} \sin([np]\theta) \int_0^\theta W(z)dz\, d\theta . \tag{3}$$

Hence by (3)

$$\int_0^{B/n} \cos(n\theta)W(\theta)d\theta - \int_0^{B/[np]} \cos([np]\theta)W(\theta)d\theta =$$

$$= \frac{\cos B}{n} \int_{nB/[np]}^{B} W(\tfrac{\theta}{n})d\theta + \frac{1}{n}\int_0^B \sin(\theta) \int_{n\theta/[np]}^{\theta} W(\tfrac{z}{n})dz\,d\theta \ . \qquad (4)$$

Since $1-F(n) \in R.V.S._{-1}^{\infty}$ we have (cf. Appendix or [ERI])

$$W(\tfrac{1}{n}) \in R.V._{-1}^{\infty} \quad \text{and} \quad W(\tfrac{1}{n}) \sim \frac{\pi}{2}\cdot\frac{n^2(1-F(n))}{m^2(n)} \qquad (n\to\infty) \ . \qquad (5)$$

Combining now (4) and (5) it is easy to deduce

$$\lim_{n\to\infty} \frac{2}{\pi}\, \frac{\int_0^{B/n} \cos(n\theta)W(\theta)d\theta - \int_0^{B/[np]} \cos([np]\theta)W(\theta)d\theta}{n(1-F(n))m^{-2}(n)} = \ln(p) \ .$$

<u>PROOF of (b')</u>. Since $\cos(n\theta) = -\cos(n\theta+\pi)$ we obtain

$$2\int_\varepsilon^\pi \cos(n\theta)W(\theta)d\theta = \int_\varepsilon^\pi \cos(n\theta)(W(\theta) - W(\theta-\tfrac{\pi}{n}))d\theta \ +$$

$$+ \int_\varepsilon^{\varepsilon+\pi/n} \cos(n\theta)\ W(\theta-\tfrac{\pi}{n})d\theta - \int_\pi^{\pi+\pi/n} \cos(n\theta)\ W(\theta-\tfrac{\pi}{n})d\theta \ . \qquad (6)$$

Because, by Proposition 2.1.4, $W(\theta)$ is bounded on $[a,b]$ with $0 < a < b \le \pi$ and $W(\frac{1}{n}) \to \infty$ $(n\to\infty)$ we get

$$\overline{\lim_{n\to\infty}} \frac{\left|\int_\varepsilon^{\varepsilon+\pi/n} \cos(n\theta)\ W(\theta-\tfrac{\pi}{n})d\theta\right|}{\frac{1}{n}\ W(\frac{1}{n})} \le M \overline{\lim_{n\to\infty}} \ (W(1/n))^{-1} = 0 \qquad (7)$$

and

$$\overline{\lim_{n\to\infty}} \frac{\left|\int_\pi^{\pi+\pi/n} \cos(n\theta)\ W(\theta-\tfrac{\pi}{n})d\theta\right|}{\frac{1}{n}\ W(\frac{1}{n})} \le M \overline{\lim_{n\to\infty}} \ (W(1/n))^{-1} = 0 \ . \qquad (8)$$

By [ERI, Lemma 5]

$$|\varphi(\theta_1) - \varphi(\theta_2)| \le 2|\theta_1-\theta_2|\, m\left(\frac{1}{|\theta_1-\theta_2|}\right) \ \forall\, \theta_1 \neq \theta_2$$

and thus, using the definition of $W(\theta)$ and Proposition 2.1.4 we obtain for all $\theta \in [\varepsilon,\pi]$

$$|W(\theta) - W(\theta - \tfrac{\pi}{n})| \leq \frac{\frac{\pi}{n}\, m(\frac{n}{\pi})}{|1-\varphi(\theta)|\,|1-\varphi(\theta-\frac{\pi}{n})|} \leq C(\varepsilon,\pi)\,\frac{\pi}{n}\,m(\tfrac{n}{\pi}) \qquad (9)$$

with

$$C(\varepsilon,\pi) := \max_{\theta\in[\varepsilon,\pi]} \left(\frac{1}{|1-\varphi(\theta)|\,|1-\varphi(\theta-\frac{\pi}{n})|}\right) < \infty .$$

Hence by (5) and (9)

$$\overline{\lim_{n\to\infty}} \frac{\left|\int_{\varepsilon}^{\pi} \cos(n\theta)(W(\theta) - W(\theta-\frac{\pi}{n}))d\theta\right|}{\frac{1}{n}\,W(\frac{1}{n})} = 0 . \qquad (10)$$

Combining (6) up to (10) yields the desired result.

PROOF of (c'). We write

$$2\int_{B/n}^{\varepsilon} \cos(n\theta)W(\theta)d\theta = \frac{1}{n}\int_{B}^{B+\pi} \cos(\theta)\; W(\tfrac{\theta-\pi}{n})d\theta +$$

$$-\frac{1}{n}\int_{\varepsilon n}^{\varepsilon n+\pi} \cos(\theta)\; W(\tfrac{\theta-\pi}{n})d\theta + \frac{1}{n}\int_{B}^{\varepsilon n} \cos(\theta)(W(\tfrac{\theta}{n}) - W(\tfrac{\theta-\pi}{n}))d\theta . \qquad (11)$$

Obviously

$$\lim_{n\to\infty} \frac{\int_{B}^{B+\pi} \cos(\theta)\; W(\frac{\theta-\pi}{n})d\theta}{W(\frac{1}{n})} = \int_{B}^{B+\pi} \frac{\cos(\theta)}{\theta-\pi}\, d\theta$$

and hence

$$\overline{\lim_{n\to\infty}} \frac{\frac{1}{n}\int_{B}^{B+\pi} \cos(\theta)\; W(\frac{\theta-\pi}{n})d\theta}{\frac{1}{n}\,W(\frac{1}{n})} = O(B^{-1}) . \qquad (12)$$

Also, since $W(\frac{1}{n}) \to \infty$ $(n \to \infty)$ and $W(\frac{\theta-\pi}{n})$ is bounded on $\theta \in [\varepsilon n, \varepsilon n+\pi]$ we find

$$\overline{\lim_{n\to\infty}} \frac{\frac{1}{n}\left|\int_{\varepsilon n}^{\varepsilon n+\pi} \cos(\theta)\; W(\frac{\theta-\pi}{n})d\theta\right|}{\frac{1}{n}\,W(\frac{1}{n})} = 0 . \qquad (13)$$

Finally we have to consider

$$\frac{1}{n}\int_B^{\varepsilon n} \cos(\theta)(W(\tfrac{\theta}{n}) - W(\tfrac{\theta-\pi}{n}))d\theta .$$

By the definition of $W(\theta)$ we get

$$W(\tfrac{\theta}{n}) - W(\tfrac{\theta-\pi}{n}) = \frac{\mathrm{Re}\,\varphi(\frac{\theta-\pi}{n}) - \mathrm{Re}\,\varphi(\frac{\theta}{n})}{|1-\varphi(\frac{\theta}{n})|^2} +$$

$$+ (1 - \mathrm{Re}\,\varphi(\tfrac{\theta-\pi}{n}))\left(\frac{1}{|1-\varphi(\frac{\theta}{n})|^2} - \frac{1}{|1-\varphi(\frac{\theta-\pi}{n})|^2}\right) . \tag{14}$$

Denote the first term in (14) by $\text{Integrand}_1(\theta,n)$ and the second by $\text{Integrand}_2(\theta,n)$. Consider then

$$\frac{1}{n}\int_B^{\varepsilon n} \cos(\theta)\ \text{Integrand}_1(\theta,n)d\theta .$$

It is proved in [ERI, Lemma 5] that

$$\theta m(\tfrac{1}{\theta}) \leq k|1-\varphi(\theta))| \qquad \text{for all } \theta \in (0,\pi) \text{ and } k \text{ some constant.}$$

By this inequality and the results in the Appendix we find for $h = \frac{1-\delta}{4} > 0$ and n sufficiently large

$$\frac{|\frac{1}{n}\int_B^{\varepsilon n} \cos(\theta)\ \text{Integrand}_1(\theta,n)d\theta}{n(1-F(n))m^{-2}(n)} \leq Ck^2 \int_B^{\varepsilon n} \frac{\theta^\delta\ m^2(n)}{\theta^2\ m^2(\frac{n}{\theta})}\, d\theta =$$

$$= O\left(\int_B^{\varepsilon n} \theta^{\delta+2h-2}\ d\theta\right) = O(B^{\frac{1}{2}(\delta-1)}) . \tag{15}$$

Consider next

$$\frac{1}{n}\int_B^{\varepsilon n} \cos(\theta)\ \text{Integrand}_2(\theta,n)d\theta .$$

Since

$$|1-\varphi(\theta)|^2 = (1 - \mathrm{Re}\,\varphi(\theta))^2 + (\mathrm{Im}\,\varphi(\theta))^2$$

and

$$\frac{1}{|1-\varphi(\frac{\theta}{n})|^2} - \frac{1}{|1-\varphi(\frac{\theta-\pi}{n})|^2} = \frac{|1-\varphi(\frac{\theta-\pi}{n})|^2 - |1-\varphi(\frac{\theta}{n})|^2}{|1-\varphi(\frac{\theta}{n})|^2|1-\varphi(\frac{\theta-\pi}{n})|^2}$$

we get the following relation:

$\text{Integrand}_2(\theta,n) =$

$$= \frac{(1-\text{Re}\,\varphi(\frac{\theta-\pi}{n}))(\text{Re}\,\varphi(\frac{\theta}{n}) - \text{Re}\,\varphi(\frac{\theta-\pi}{n}))(1-\text{Re}\,\varphi(\frac{\theta-\pi}{n}) + 1 - \text{Re}\,\varphi(\frac{\theta}{n}))}{|1-\varphi(\frac{\theta}{n})|^2|1-\varphi(\frac{\theta-\pi}{n})|^2} +$$

$$+ \frac{(1-\text{Re}\,\varphi(\frac{\theta-\pi}{n}))(\text{Im}\,\varphi(\frac{\theta-\pi}{n}) - \text{Im}\,\varphi(\frac{\theta}{n}))(\text{Im}\,\varphi(\frac{\theta-\pi}{n}) + \text{Im}\,\varphi(\frac{\theta}{n}))}{|1-\varphi(\frac{\theta}{n})|^2|1-\varphi(\frac{\theta-\pi}{n})|^2} \,. \qquad (16)$$

Notice that one can prove in a similar way as done by [ERI, Lemma 5]

$$|\text{Im}\,\varphi(\frac{\theta-\pi}{n}) - \text{Im}\,\varphi(\frac{\theta}{n})| \leq \frac{2\pi}{n}\, m(\frac{n}{\pi})$$

and (17)

$$|\text{Re}\,\varphi(\frac{\theta-\pi}{n}) - \text{Re}\,\varphi(\frac{\theta}{n})| \leq \frac{2\pi}{n}\, m(\frac{n}{\pi}) \,.$$

By (16), (17) and the mentioned inequality for $|1-\varphi(\theta)|$ we find

$|\text{Integrand}_2(\theta,n)| \leq$

$$\leq \frac{2\pi k^4 n^3 m(\frac{n}{\pi})(1-\text{Re}\,\varphi(\frac{\theta-\pi}{n}))(2-\text{Re}\,\varphi(\frac{\theta-\pi}{n}) - \text{Re}\,\varphi(\frac{\theta}{n}))}{m^2(\frac{n}{\theta-\pi})m^2(\frac{n}{\theta})\theta^2(\theta-\pi)^2} +$$

$$+ \frac{2\pi k^4 n^3 m(\frac{n}{\pi})(1-\text{Re}\,\varphi(\frac{\theta-\pi}{n}))(|\text{Im}\,\varphi(\frac{\theta-\pi}{n}) + \text{Im}\,\varphi(\frac{\theta}{n})|)}{m^2(\frac{n}{\theta-\pi})m^2(\frac{n}{\theta})\theta^2(\theta-\pi)^2} \,. \qquad (18)$$

Denote the first term in (18) by $\text{Integrand}_3(\theta,n)$ and the second by $\text{Integrand}_4(\theta,n)$. Consider now $\text{Integrand}_3(\theta,n)$.
Since $1 - \text{Re}\,\varphi(\frac{1}{n}) \sim \frac{\pi}{2}(1-F(n))$ $(n \to \infty)$ and $w^{1+h}(1-\text{Re}\,\varphi(\frac{1}{w}))$ is bounded in the neighbourhood of zero we can apply the results from the Appendix to the following integral and find for $0 < \eta < \frac{1}{6}$ and n sufficiently large

$$\frac{\left|\frac{1}{n}\int_B^{\varepsilon n} \cos(\theta)\ \text{Integrand}_3(\theta,n)d\theta\right|}{n^2(1-F(n))^2\ m^{-3}(n)}$$

$$= O\left(\int_B^{\varepsilon n} \frac{(\theta-\pi)^{1+\eta}(\theta^{1+\eta} + (\theta-\pi)^{1+\eta})\theta^{2\eta}(\theta-\pi)^{2\eta}\ d\theta}{\theta^2(\theta-\pi)^2}\right) =$$

$$= O(B^{-(1-6\eta)}) \ . \tag{19}$$

Hence

$$\overline{\lim_{n\to\infty}} \frac{\left|\frac{1}{n}\int_B^{\varepsilon n} \cos(\theta)\ \text{Integrand}_3(\theta,n)d\theta\right|}{n(1-F(n))m^{-2}(n)} =$$

$$= O(B^{-(1-6\eta)})\ \overline{\lim_{n\to\infty}} \frac{n(1-F(n))}{m(n)} = 0 \ . \tag{20}$$

Since $\operatorname{Im}\varphi(\frac{1}{n}) \sim \frac{m(n)}{n}$ $(n \to \infty)$ and η sufficiently small we find analogously

$$\overline{\lim_{n\to\infty}} \frac{\left|\frac{1}{n}\int_B^{\varepsilon n} \cos(\theta)\ \text{Integrand}_4(\theta,n)d\theta\right|}{n(1-F(n))m^{-2}(n)} = O(B^{-(1-6\eta)}) \ . \tag{21}$$

By (18), (20) and (21) we obtain for $1-6\eta > 0$

$$\overline{\lim_{n\to\infty}} \frac{\left|\frac{1}{n}\int_B^{\varepsilon n} \cos(\theta)\ \text{Integrand}_2(\theta,n)d\theta\right|}{n(1-F(n))m^{-2}(n)} = O(B^{-(1-6\eta)}) \tag{22}$$

and this implies by (14) and (15) (take $1-6\eta > \frac{1}{2}(1-\delta)$)

$$\overline{\lim_{n\to\infty}} \frac{\left|\frac{1}{n}\int_B^{\varepsilon n} \cos(\theta)(W(\frac{\theta}{n}) - W(\frac{\theta-\pi}{n}))d\theta\right|}{n(1-F(n))m^{-2}(n)} = O(B^{\frac{1}{2}(\delta-1)}) \tag{23}$$

with $\delta-1 < 0$.

Finally, combining (11), (12), (13) and (23), we get with $\delta-1 < 0$

$$\overline{\lim_{n\to\infty}} \frac{\left|\int_{B/n}^{\varepsilon} \cos(n\theta)\ W(\theta)d\theta\right|}{n(1-F(n))m^{-2}(n)} = O(B^{\frac{1}{2}(\delta-1)}) \ . \tag{24}$$

The proof of (c') is now completed and by combining (a), (b), (c) and (2) we find as B tends to infinity

$$\lim_{n\to\infty} \frac{u(n) - u([np])}{n(1-F(n))m^{-2}(n)} = \ln(p) \ . \qquad \square$$

In order to prove the main result we need the following definition.

DEFINITION 2.2.7. (cf. [GEL].) A sequence of positive numbers $\{c(n)\}_{n=0}^{\infty}$ belongs to the class $\Pi.S.^{\infty}$ if there exists a sequence $\{L(n)\}_{n=0}^{\infty}$ such that

(i) $L(n) \in R.V.S._0^{\infty}$

(ii) $$\lim_{n\uparrow\infty} \frac{c([nx]) - c(n)}{L(n)} = \ln(x) \quad \forall x>0 \ .$$

The next theorem yields the extension of Theorem 2.2.4 in case $\alpha = 1$.

THEOREM 2.2.8. (cf. [FRE].) *The following result holds*

$$1-F(n) \in R.V.S._{-1}^{\infty} \Leftrightarrow \frac{1}{u(n)} \in \Pi.S.^{\infty}$$

Either relation implies

$$\lim_{n\uparrow\infty} \frac{m(n)u(n) - 1}{n(1-F(n))(m(n))^{-1}} = 0 \ .$$

PROOF. It follows immediately by Theorem 2.2.6 that $1-F(n) \in R.V.S._{-1}^{\infty}$ yields $\frac{1}{u(n)} \in \Pi.S.^{\infty}$. Also one can derive using Tauberian arguments (cf. [GEL]) and the monotonicity of $1-F$ that $\frac{1}{u(n)} \in \Pi.S.^{\infty}$ implies $1-F(n) \in R.V.S._{-1}^{\infty}$. (We only give a proof of this result in the nonlattice case, cf. Chapter 3.) Hence we only gave to prove that

$$\lim_{n\uparrow\infty} \frac{m(n)u(n) - 1}{n(1-F(n))(m(n))^{-1}} = 0 \ .$$

This can be obtained in the following way. Let

$$f(t) := u([t]) \quad \text{and} \quad \hat{f}(\tfrac{1}{t}) := \frac{1}{t}\int_0^{\infty} \exp(-\tfrac{x}{t}) f(x)\,dx \ .$$

Then by Theorem 2.2.6

$$\lim_{t\uparrow\infty} \frac{f(t) - f(tp)}{L(t)} = \ln(p) \quad \forall p>0$$

where

$$L(t) = [t](1-F([t]))(m([t]))^{-2} .$$

Now by Corollary 1.15 of [GEL] (see also the Appendix)

$$\lim_{t\uparrow\infty} \frac{f(t) - \hat{f}(1/t)}{L(t)} = \lim_{t\uparrow\infty} \frac{\int_0^\infty \exp(-x)(f(t) - f(tx))dx}{L(t)} =$$

$$= \int_0^\infty \exp(-x) \ln(x)dx = -\gamma$$

(γ denotes Euler's constant, cf. [ABR, p. 230]).
Notice that $\hat{f}(1/t)$ equals $(1-\exp(-1/t))(1-\hat{F}(\exp(-1/t))^{-1}$ where $\hat{F}(z) := \sum_{k=0}^{\infty} f(k)z^k$ is the so-called moment generating function of the probability distribution $\{f(k)\}_{k=0}^{\infty}$. Hence

$$\lim_{t\uparrow\infty} \frac{u([t]) - (1-\exp(-1/t))(1-\hat{F}(\exp(-1/t)))^{-1}}{L(t)} = -\gamma . \tag{25}$$

On the other hand one can prove in a similar way using $m(n) \in \Pi.S.^{\infty}$ that

$$\lim_{t\uparrow\infty} \frac{m([t]) - \hat{m}(1/t)}{[t](1 - F([t]))} =$$

$$= \lim_{t\uparrow\infty} \frac{m([t]) - (1-\exp(-1/t))^{-1}(1-\hat{F}(\exp(-1/t)))}{t\ (1-F([t]))} = \gamma$$

and this implies

$$\lim_{t\to\infty} \frac{\frac{1}{m([t])} - (1-\hat{F}(\exp(-1/t)))^{-1}(1-\exp(-1/t))}{L(t)} = -\gamma . \tag{26}$$

Combining (25) and (26) now yields the desired result. □

CHAPTER 3. RENEWAL MEASURES

0. Introduction

Let $\underline{X}_1, \underline{X}_2, \ldots$ be positive independent and identically distributed random variables with nonlattice probability distribution function F, and corresponding probability measure ν_F. We assume that

$$F(0^+) = 0 \ , \quad F(\infty) = 1 \quad \text{and} \quad E := \int_0^\infty x\, \nu_F(dx) \ .$$

Set

$$\underline{S}_0 := 0 \ , \quad \underline{S}_n := \sum_{i=1}^{n} \underline{X}_i \qquad (n \geqq 1)$$

and consider the renewal measure U defined by

$$U(B) := E\left(\sum_{n=0}^{\infty} 1_{\{\underline{S}_n \in B\}} \right) \qquad (B \in \mathcal{B}([0,\infty))) \ . \tag{1}$$

(= expected number of renewals in B.)

With e denoting, as always, the probability measure of a random variable degenerate in zero, (1) can clearly be written equivalently as

$$U(B) = e(B) + \sum_{n=1}^{\infty} \nu_F^n(B) \ . \tag{2}$$

We mention the following well-known result.

THEOREM 3.0.1 (cf. [FEL-2]). *For every nonlattice probability distribution function* F *with* $F(0^+) = 0$ *and* $F(\infty) = 1$ *it follows that* $U([t,t+h]) \to h/E$ $(t \to \infty)$ *for any* $h > 0$. *(If* $E = \infty$, h/E *equals* 0.*)*

As in the lattice case (Chapter 2) we are interested in the first and second-order remainder terms in the convergence to this limit. Moreover, we will distinguish the cases $E = \infty$ and $E < \infty$.

1. The behaviour of the renewal measure in case the expectation is finite

Our results on the asymptotic behaviour of the renewal measure U (in case $E < \infty$) will be based on a certain representation of U (Lemma 3.1.2). Before stating this result we recall the following well-known fact about Laplace transforms.

LEMMA 3.1.1. *Let* ν *be a (complex) measure on* $[0,\infty)$ *and assume that*

$$\int_0^\infty \exp(ax)|\nu|(dx) < \infty$$

for some $a \in \mathbb{R}$. *Then if*

$$\int_0^\infty \exp(\lambda x)\nu(dx) = 0$$

for all real $\lambda < a$, *it follows that* $\nu \equiv 0$.

PROOF. It clearly suffices to prove this for real ν. Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν. Then

$$\int_0^\infty \exp(ax)\nu^+(dx) \quad \text{and} \quad \int_0^\infty \exp(ax)\nu^-(dx)$$

are both finite and

$$\int_0^\infty \exp(\lambda x)\nu^+(dx) = \int_0^\infty \exp(\lambda x)\nu^-(dx) \qquad \text{for all } \lambda < a \text{ .}$$

Since a measure is uniquely determined by the restriction of its Laplace transform to any interval of the form $(-\infty,b)$ we conclude that $\nu^+ = \nu^-$, hence $\nu = 0$. □

Now let us define the following two measures on $[0,\infty)$:

$$\nu_m(B) := \int_B (1-F(x))dx = ((e-\nu_F)*\ell)(B) \tag{1}$$

$$\nu_E(B) := e(B) - \nu_F(B) + \frac{1}{E}\nu_m(B) \text{ .} \tag{2}$$

for all $B \in \mathcal{B}(0,\infty))$. Note that $\nu_m \in S(\psi_o)$, where $\psi_o \equiv 1$, and hence $\nu_E \in S(\psi_o)$.

LEMMA 3.1.2. *Suppose that* ν_E *is invertible in* $S(\psi_o)$. *Then we have the following representation for the renewal measure* U *(ℓ denotes the Lebesgue measure)*

$$U(B) = \frac{\ell(B)}{E} + \nu_E^{-1}(B) - \frac{1}{E}\int_B \nu_E^{-1}([x,\infty))dx \qquad (B \in \mathcal{B}([0,\infty))) \,. \quad (3)$$

PROOF. For every $\lambda \in \mathbb{C}$ with Re $\lambda \leq 0$ let us define $\varphi_\lambda \in \Delta(S(\psi_o))$ by

$$\varphi_\lambda(\nu) := \int_0^\infty \exp(\lambda x)\nu(dx) \qquad (\nu \in S(\psi_o)) \,. \tag{4}$$

One checks without difficulty that

$$\varphi_\lambda(\nu_m) = \varphi_\lambda((e-\nu_F)*\ell) = -\frac{1-\varphi_\lambda(\nu_F)}{\lambda}$$

whenever $\lambda \neq 0$, hence

$$\varphi_\lambda(\nu_E) = 1 - \varphi_\lambda(\nu_F) - \frac{1}{E}\cdot\frac{1-\varphi_\lambda(\nu_F)}{\lambda} = \frac{1-\varphi_\lambda(\nu_F)}{E\lambda}(E\lambda-1) \,. \tag{5}$$

In the remainder of this proof λ will always be real and < 0.
Since $\varphi_\lambda(\nu_E^{-1}) = (\varphi_\lambda(\nu_E))^{-1}$ it easily follows from (5) that

$$\frac{1}{1-\varphi_\lambda(\nu_F)} = -\frac{1}{E\lambda} + \varphi_\lambda(\nu_E^{-1}) + \frac{1-\varphi_\lambda(\nu_E^{-1})}{E\lambda} \,. \tag{6}$$

Furthermore, by Fubini's theorem,

$$\int_0^\infty (1-\exp(\lambda x))\nu_E^{-1}(dx) = -\lambda\int_0^\infty \nu_E^{-1}([x,\infty))\exp(\lambda x)dx \,. \tag{7}$$

Observe next that $E = \nu_m([0,\infty))$, so $\nu_E([0,\infty)) = 1$. Since

$$1 = e([0,\infty)) = (\nu_E * \nu_E^{-1})([0,\infty)) = \nu_E([0,\infty))\nu_E^{-1}([0,\infty)) \,,$$

also $\nu_E^{-1}([0,\infty)) = 1$ and therefore (7) can be rewritten as

$$\frac{1-\varphi_\lambda(\nu_E^{-1})}{-\lambda} = \int_0^\infty \nu_E^{-1}([x,\infty))\exp(\lambda x)dx \ . \tag{8}$$

Finally, let us note that the definition of U in the introduction (see (2)) implies

$$\varphi_\lambda(U) = \sum_{n=0}^\infty (\varphi_\lambda(\nu_F))^n = \frac{1}{1-\varphi_\lambda(\nu_F)} \ . \tag{9}$$

(Fix $\lambda < 0$ and choose $\varepsilon > 0$ so that $\lambda+\varepsilon < 0$. Observe that the series $U = e + \sum_{n=1}^\infty \nu_F^n$ is norm convergent in the Banach algebra $S(\psi_{-\varepsilon})$, where $\psi_{-\varepsilon}(x) := \exp(-\varepsilon x)$. Now regard φ_λ as an element of $\Delta(S(\psi_{-\varepsilon}))$.) Substitution of (8) and (9) into (6) yields

$$\int_0^\infty \exp(\lambda x)U(dx) = \frac{1}{E}\int_0^\infty \exp(\lambda x)dx + \int_0^\infty \exp(\lambda x)\nu_E^{-1}(dx) +$$

$$- \frac{1}{E}\int_0^\infty \exp(\lambda x)\nu_E^{-1}([x,\infty))dx \ .$$

Lemma 3.1.1 now implies (3). □

<u>REMARK 3.1.3</u>. If $\nu_E \in S(\psi)$ for some $\psi \in \Psi$ and ν_E is invertible in $S(\psi)$, we obtain the same representation as in Lemma 3.1.2. However, in this case $\nu_E^{-1} \in V(\psi)$ instead of $\nu_E^{-1} \in V(\psi_o)$.

Our next result gives a sufficient condition for ν_E to be invertible in $S(\psi)$ if $\lim_{x\to\infty} \frac{\ln \psi(x)}{x} = 0$.

<u>LEMMA 3.1.4</u>. *Let $\psi \in \Psi$ be given and suppose $\lim_{x\to\infty} \frac{\ln \psi(x)}{x} = 0$. If $\nu_F^{n_0}$ is nonsingular for some $n_0 \in \mathbb{N}$ then ν_E is invertible in $S(\psi)$ whenever ν_E belongs to $S(\psi)$.*

<u>PROOF</u>. We will first prove the result for $\psi = \psi_o$. By Theorems 1.1.15 and 1.1.21 ν_E is invertible iff $\varphi(\nu_E) \neq 0$ for all $\varphi \in \Delta(S(\psi_o))$.
We first show that $\varphi_\lambda(\nu_E) \neq 0$ for all $\lambda \in \mathbb{C}$ with Re $\lambda \leq 0$. This is clear for $\lambda = 0$. Suppose $\lambda \neq 0$. If Re $\lambda = 0$, then $|\varphi_\lambda(\nu_F)| < 1$ because ν_F is non-lattice (cf. [KAW]) and if Re $\lambda < 0$ then $|\varphi_\lambda(\nu_F)| < 1$ holds trivially. Thus in both cases (5) implies that $\varphi_\lambda(\nu_E) \neq 0$.

By Proposition 1.3.3 or Theorem 4.18.2 of [HIL] every φ belonging to $\Delta(S(\psi_o)) \setminus \{\varphi_\lambda : \operatorname{Re} \lambda \leq 0\}$ is identically zero on the set of ℓ-absolutely continuous measures in $S(\psi_o)$. Hence

$$|\varphi(\nu_F)| = |\varphi(\nu_F^{n_0})|^{1/n_0} \leq \|(\nu_F^{n_0})_s\|_{\psi_o}^{1/n_0} .$$

($(\nu_F^{n_0})_s$ denotes the singular part of $\nu_F^{n_0}$.)
By assumption

$$\|(\nu_F^{n_0})_s\|_{\psi_o} < \|\nu_F^{n_0}\|_{\psi_o} = 1 ,$$

so $|\varphi(\nu_F)| < 1$. Since $\varphi(\nu_E) = 1 - \varphi(\nu_F)$ we again have $\varphi(\nu_E) \neq 0$ and the proof for $\psi = \psi_o$ is complete.
Suppose now $\nu_E \in S(\psi)$ for some $\psi \in \Psi$ with $\lim\limits_{x\to\infty} \frac{\ln \psi(x)}{x} = 0$. Then by [HIL] this limit equals $\inf\limits_{x>0} \frac{\ln \psi(x)}{x}$ and so $\psi(x) \geq 1$ for every $x \in [0,\infty)$.
By Theorem 1.3.4 we know that any $L_1 \in \Delta(S(\psi))$ is the restriction to $S(\psi)$ of a unique $L_2 \in \Delta(S(\psi_o))$. Hence, if we assume that $\varphi(\nu_E) = 0$ for some $\varphi \in \Delta(S(\psi))$, there exists some $\bar{\varphi} \in \Delta(S(\psi_o))$ with $\bar{\varphi}(\nu_E) = 0$. This contradicts ν_E is invertible in $S(\psi_o)$. □

<u>REMARK 3.1.5.</u> It is not known to the author whether the condition in Lemma 3.1.4 is also necessary.

We are now prepared for the main results of this section.

<u>THEOREM 3.1.6.</u> *Let* $\psi \in \Psi$, ψ *nondecreasing,* $\lim\limits_{x\to\infty} \frac{\ln \psi(x)}{x} = 0$, *and define* $(\Delta\psi)(x) := \psi(x+1) - \psi(x)$, $x \geq 0$.
If $\nu_m \in S(\psi)$ *and* $\nu_F^{n_0}$ *is nonsingular for some* $n_0 \in \mathbb{N}$ *then*

$$\int_0^\infty (\Delta\psi)(x)\left|U - \frac{\ell}{E}\right| (dx) < \infty .$$

<u>PROOF.</u> Since $\psi \in \Psi$ and ψ nondecreasing we obtain

$$\int_0^\infty \psi(x)\nu_F(dx) \leq \psi(3) + \int_3^\infty \psi(x)\nu_F(dx) \leq$$

$$\leq \psi(3) + \psi(1) \int_3^\infty \int_{x-1}^x \psi(z)dz\ \nu_F(dx) \leq$$

$$\leq \psi(3) + \psi(1) \int_0^\infty \psi^o(z)\nu_F(dz) \qquad (10)$$

with

$$\psi^o(z) := \int_0^z \psi(x)dx .$$

By Fubini's theorem the integral ${}_0\int^\infty \psi(x)\nu_F([x,\infty))dx$ equals ${}_0\int^\infty \psi^o(x)\nu_F(dx)$ and this implies, since $\nu_m \in S(\psi)$, that $\nu_E \in S(\psi)$ (use (10)). Applying Lemma 3.1.4 yields $\nu_E^{-1} \in S(\psi)$. Hence

$$0 \leq \int_0^\infty (\Delta\psi)(x)|\nu_E^{-1}|(dx) \leq \int_0^\infty \psi(x+1)|\nu_E^{-1}|(dx) \leq$$

$$\leq \psi(1) \int_0^\infty \psi(x)|\nu_E^{-1}|(dx) < \infty \qquad (11)$$

and

$$0 \leq \int_0^\infty (\Delta\psi)(x)|\nu_E^{-1}([x,\infty))|(dx) \leq \int_0^\infty (\Delta\psi)(x) \int_x^\infty |\nu_E^{-1}|(dz)(dx) =$$

$$= \int_0^\infty \int_0^z (\Delta\psi)(x)(dx)|\nu_E^{-1}|(dz) \leq \int_0^\infty \int_z^{z+1} \psi(x)dx|\nu_E^{-1}|(dz) \leq$$

$$\leq \psi(1) \int_0^\infty \psi(z)|\nu_E^{-1}|(dz) < \infty . \qquad (12)$$

Finally using the representation of U (Lemma 3.1.2) we obtain from (11) and (12) the desired result. □

REMARK 3.1.7.

(i) Theorem 3.1.6 slightly improves results of [NUM] and [STO-4]. The above authors derive their results by completely different methods, [NUM] by coupling methods and [STO-4] by Fourier analysis.

(ii) A special case of Theorem 3.1.6 is given by the following result. If $\mathbb{E}(\underline{X}_1^{1+\gamma}) < \infty$ for some $\gamma > 0$ and $\nu_F^{n_0}$ is nonsingular for some $n_0 \in \mathbb{N}$ then

$$\int_1^\infty x^{\gamma-1} |U - \frac{\ell}{E}| \ (dx) < \infty \ .$$

THEOREM 3.1.8. *Let* $\psi_\alpha(x) := \exp(\alpha x)$, $x \geq 0$. *If* $\nu_F^{n_0}$ *is nonsingular for some* $n_0 \in \mathbb{N}$ *and* $\nu_m \in S(\psi_\alpha)$ *for some* $\alpha > 0$ *then there exists some* $0 < \alpha' \leq \alpha$ *such that*

$$\int_0^\infty \exp(\alpha' x) |U - \frac{\ell}{E}| \ (dx) < \infty \ .$$

PROOF. Set

$$h(\beta) := \|(\nu_F^{n_0})_s\|_{\psi_\beta} \ , \qquad \text{where } \beta \in (-\infty,\alpha] \ .$$

Since $\nu_F^{n_0}$ is nonsingular we have $h(0) < 1$. Clearly h is continuous and non-decreasing and so there exists some $0 < \bar{\alpha} \leq \alpha$ such that $h(\beta) < 1$ for every $\beta \in (-\infty,\bar{\alpha}]$.
Suppose now ν_E is not invertible in any $S(\psi_\beta)$ where $0 < \beta \leq \bar{\alpha}$. This means that for every $0 < \beta \leq \bar{\alpha}$ there exists a homomorphism $L \in \Delta(S(\psi_\beta))$ such that $L(\nu_E) = 0$. From $\|(\nu_F^{n_0})_s\|_{\psi_\beta} < 1$ for every $0 \leq \beta \leq \bar{\alpha}$ it follows (see also the proof of Lemma 3.1.4) that this implies the existence of a sequence $\{\lambda_n\} \subseteq \mathbb{C}$ with

$$0 < \text{Re}\ \lambda_n \leq \min(\bar{\alpha},\frac{1}{2E}) \ , \quad \lim_{n\to\infty} \text{Re}\ \lambda_n = 0 \ , \quad |\lambda_n| \not\to 0 \quad \text{and} \quad \varphi_{\lambda_n}(\nu_F) = 1 \ . \qquad (13)$$

Since for every $\lambda \in \mathbb{C}$ with $0 < \text{Re}\ \lambda < \bar{\alpha}$ the inequality

$$|\varphi_\lambda(\nu_F) - \varphi_{i\text{Im}\lambda}(\nu_F)| \leq |\text{Re}\ \lambda| \int_0^\infty \exp((\text{Re}\ \lambda)x)\ x\ \nu_F(dx)$$

holds we obtain from (13) that

$$\lim_{n\to\infty} \varphi_{i\mathrm{Im}\lambda_n}(\nu_F) = 1 \ . \tag{14}$$

In case the sequence $\{|\lambda_n|\}$ (remember $|\lambda_n| \not\to 0$) is uniformly bounded, we obtain a contradiction since φ_λ is continuous in λ and ν_F nonlattice. Hence $\lim_{n\to\infty} |\lambda_n| = \infty$. However in that case we have, since $\nu_F^{n_0}$ is nonsingular, that

$$\overline{\lim_{n\to\infty}} |\varphi_{i\mathrm{Im}\lambda_n}(\nu_F)| < 1 \qquad \text{(Riemann-Lebesgue lemma).}$$

This contradicts (14) again and so there exists some $0 < \alpha' \le \bar{\alpha}$ such that ν_E is invertible in $S(\psi_{\alpha'})$. The rest of the proof follows the same lines as Theorem 3.1.6. □

In the next theorems we discuss the behaviour of $|U - \frac{\ell}{E}|\,((t,t+h])$ as t tends to infinity in case the tail of the distribution function F tends to zero at a subexponential rate.

THEOREM 3.1.9. *Suppose that*

(i) $\nu_F^{n_0}$ *is nonsingular for some* $n_0 \in \mathbb{N}$ *and*

(ii) $$\lim_{t\to\infty} \frac{\nu_m([t,\infty))}{\mu([t,\infty))} = 0 \ ,$$

where μ *is a probability measure on* $[0,\infty)$ *satisfying*

$$\sup_{t\ge 0} \frac{(\mu*\mu)([t,\infty))}{\mu([t,\infty))} < \infty \ \textit{(i.e. } \mu \in \mathrm{SMT}\textit{)} \quad \textit{and} \quad \lim_{t\to\infty} \frac{\ln \mu([t,\infty))}{t} = 0 \ .$$

Then

$$\lim_{t\to\infty} \frac{|U - \frac{\ell}{E}|\,([t,t+h])}{\mu([t,\infty))} = 0 \qquad \textit{for every } h > 0 \ .$$

PROOF. (i) and Lemma 3.1.4 (take $\psi = \psi_0$) imply that the representation (3) for U is valid. For each $a \ge 0$ the trivial inequality

$$\mu([t-a,\infty))\mu([a,\infty)) \le (\mu*\mu)([t,\infty)) \qquad (t \ge a)$$

implies that

$$\sup_{t\ge a} \frac{\mu([t-a,\infty))}{\mu([t,\infty))} \le \frac{1}{\mu([a,\infty))} \sup_{t\ge a} \frac{(\mu*\mu)([t,\infty))}{\mu([t,\infty))} < \infty \ .$$

Hence, by (ii)

$$0 \leq \overline{\lim_{t\to\infty}} \frac{1-F(t)}{\mu([t,\infty))} \leq \overline{\lim_{t\to\infty}} \frac{\int_{t-1}^{\infty}(1-F(x))dx}{\mu([t-1,\infty))} \cdot \overline{\lim_{t\to\infty}} \frac{\mu([t-1,\infty))}{\mu([t,\infty))} = 0 \ .$$

Thus $\nu_F \in ST^0(\psi_o,\mu)$. Since (ii) states that $\nu_m \in ST^0(\psi_o,\mu)$ we also have $\nu_E \in ST^0(\psi_o,\mu)$.
Remark 1.3.19 (A) now says that the inverse ν_E^{-1} (which exists in $S(\psi_o)$ by (i) and Lemma 3.1.4) actually belongs to $ST^0(\psi_o,\mu)$. The conclusion then follows from (3). □

THEOREM 3.1.10. *Suppose that*

(i) $\nu_F^{n_0}$ *is nonsingular for some* $n_0 \in \mathbb{N}$ *and*

(ii) $$\overline{\lim_{t\to\infty}} \frac{\nu_m([t,\infty))}{\mu([t,\infty))} < \infty \ ,$$

where μ *is a probability measure on* $[0,\infty)$ *satisfying*

$$\sup_{t\geq 0} \frac{(\mu*\mu)([t,\infty))}{\mu([t,\infty))} < \infty \ ,$$

$$\lim_{n\to\infty} \sup_{t\geq 0} \frac{(\mu-\mu_n)^2([t,\infty))}{\mu([t,\infty))} = 0$$

and

$$\lim_{t\to\infty} \frac{\ln \mu([t,\infty))}{t} = 0 \ .$$

Then

$$\sup_{t\geq 0} \frac{|U - \frac{\ell}{E}|([t,t+h))}{\mu([t,\infty))} < \infty \qquad \textit{for every } h > 0 \ .$$

PROOF. Argue as in the preceding proof with $ST(\psi_o,\mu)$ instead of $ST^0(\psi_o,\mu)$ and use Remark 1.3.19 (B). □

THEOREM 3.1.11. *Suppose that*

(i) $\nu_F^{n_0}$ *is nonsingular for some* $n_0 \in \mathbb{N}$ *and*

(ii) $$\lim_{t\to\infty} \frac{(\nu_m*\nu_m)([t,\infty))}{\nu_m([t,\infty))} = 2E \ .$$

Then

(a) $$\sup_{t\geq 0} \frac{|U - \frac{\ell}{E}|\ ([t,t+h])}{\nu_m([t,\infty))} < \infty ,$$

(b) $$\lim_{t\to\infty} \frac{(U - \frac{\ell}{E})([t,t+h])}{\nu_m([t,\infty))} = \frac{h}{E^2} \quad \textit{for every } h > 0 .$$

PROOF. Note that $E = \hat{\nu}_m(0)$ and that (ii) implies

$$\lim_{t\to\infty} \frac{\nu_m([t+1,\infty))}{\nu_m([t,\infty))} = 1 \qquad \text{(cf. [ATH])} . \tag{15}$$

It follows that $\frac{1}{E}\nu_m \in SMT(0)$.
Using also Lemmas 1.3.15 and 1.3.17 the assumptions of Theorem 3.1.10 are satisfied with $\mu = \nu_m/E$. Hence (a) follows.
For the proof of (b) we observe that (15) easily implies that

$$\lim_{t\to\infty} \frac{\nu_F([t,\infty))}{\nu_m([t,\infty))} = 0$$

and consequently

$$\lim_{t\to\infty} \frac{\nu_E([t,\infty))}{\frac{1}{E}\nu_m([t,\infty))} = 1 , \quad \text{i.e. } \nu_E \in \underset{\sim}{ST}^1(\psi_o,\frac{\nu_m}{E}) .$$

Since ν_E^{-1} exists in $S(\psi_o)$ by (i) we now obtain from Theorem 1.3.21 with $\widetilde{\Lambda}(z) = z^{-1}$ (see also Remark 1.1.31) that

$$\lim_{t\to\infty} \frac{\nu_E^{-1}([t,\infty))}{\frac{1}{E}\nu_m([t,\infty))} = -1 .$$

Now (b) follows from (3) with the aid of (15). □

REMARK 3.1.12. The so-called coupling method for proving first order renewal theorems only yields O- en o-results ([NEY], [NUM]), comparable with Theorems 3.1.9 and 3.1.10. (For weaker results proved by Fourier analytic methods, see [STO-1], [STO-2], [STO-3], [STO-4].)
The coupling method only gives an upper bound on the error due to replacing the original process by its corresponding stationary version. It is therefore

unlikely that this method will ever produce limit results such as Theorem 3.1.11.

We will now give an example of a limit result for $|U - \frac{\ell}{E}|$ ([t,t+h]) as t tends to infinity in case the tail of the distribution function F decreases exponentially fast to zero. However, before mentioning this result, we have to make the following observation.

REMARK 3.1.13. Let

$$\nu_c(B) := e(B) - \nu_F(B) + \frac{1}{c}\,\nu_m(B) \qquad (B \in \mathcal{B}([0,\infty)))$$

for any fixed $c > 0$.
As in Lemma 3.1.2 we now obtain in case ν_c is invertible in $S(\psi)$ ($\psi \in \Psi$) that the following representation of U holds:

$$U(B) = \frac{\ell(B)}{E} + \nu_c^{-1}(B) - \frac{1}{c}\int_B \nu_c^{-1}([x,\infty))dx \qquad (B \in \mathcal{B}([0,\infty))). \quad (16)$$

THEOREM 3.1.14. *Suppose that*

(i) $\nu_F \in SMT(\alpha)$ *for some* $\alpha > 0$ *(cf. Definition 1.3.12),*

(ii) $\varphi_\lambda(\nu_F) \neq 1$ *for every* $\lambda \in \mathbb{C}$ *with* $0 < \operatorname{Re}\lambda \leq \alpha$,

(iii) $\|(\nu_F^{n_0})_s\|_{\psi_\alpha} < 1$, *where* $\psi_\alpha(x) := \exp(\alpha x)$, $x \geq 0$.

Then

(a) $$\sup_{t\geq 0} \frac{|U - \frac{\ell}{E}|\ ([t,t+h])}{\nu_F([t,\infty))} < \infty \qquad \textit{for every } h > 0 ,$$

(b) $$\lim_{t\to\infty} \frac{(U - \frac{\ell}{E})([t,t+h])}{\nu_F([t,\infty))} = (1 - \exp(-\alpha h))(1 - \varphi_\alpha(\nu_F))^{-2} \qquad \textit{for every } h > 0 .$$

PROOF. Since $\nu_F \in SMT(\alpha)$ it is easy to verify that

$$\overline{\lim_{t\to\infty}} \frac{\nu_m([t,\infty))}{\nu_F([t,\infty))} \leq \overline{\lim_{t\to\infty}} \frac{\sum_{k=0}^{\infty} \nu_F([t+kh,\infty))h}{\nu_F([t,\infty))} \leq h \sum_{k=0}^{\infty} \exp((-\alpha+\varepsilon)kh) \qquad (17)$$

for any $\varepsilon,h > 0$ (cf. Definition 1.3.12(iii)).
Also for every fixed $B > 0$

$$\underline{\lim}_{t\to\infty} \frac{\nu_m([t,\infty))}{\nu_F([t,\infty))} \geq \underline{\lim}_{t\to\infty} \frac{{}_0\int^B \nu_F([t+x,\infty))dx}{\nu_F([t,\infty))} = \int_0^B \exp(-\alpha x)dx \ . \tag{18}$$

(cf. Remark 1.3.13.(c).) Hence by (17) and (18) it follows that

$$\lim_{t\to\infty} \frac{\nu_m([t,\infty))}{\nu_F([t,\infty))} = \frac{1}{\alpha}$$

and this implies

$$\nu_c \in \underset{\sim}{ST}^b(\psi_\alpha,\nu_F) \ , \quad \text{where } b = \frac{1 - c\alpha}{c\alpha} \ .$$

Take $c := 1/2\alpha$. Then by (ii) and (iii) one can prove in a similar way as in Lemma 3.1.4 that ν_c is invertible in $S(\psi_\alpha)$. Hence

$$\nu_c^{-1} \in \underset{\sim}{ST}^d(\psi_\alpha,\nu_F) \ , \ \text{where } d = -\frac{1}{(\varphi_\alpha(\nu_c))^2} \ ,$$

(cf. Theorem 1.3.21) and by the representation of U ((16)) the results (a) and (b) follow easily. □

REMARK 3.1.15. It is also possible in this case to prove O- and o-results. We leave the details to the reader.

Before discussing second-order limit results we need some definitions.

DEFINITION 3.1.16.

1. A measurable function τ: $[0,\infty) \to (0,\infty)$ belongs to $\mathbb{D}$ if

$$D_\tau(a) := \overline{\lim_{t\to\infty}} \sup_{1\leq x\leq a} \frac{\tau(t)}{\tau(tx)}$$

is finite for some $a > 1$.

2. A measurable function τ: $[0,\infty) \to (0,\infty)$ belongs to $\mathbb{L}$ if

$$\lim_{t\to\infty} \frac{\tau(t+x)}{\tau(t)} = 1 \quad \text{for every } x \geq 0 \ .$$

3. A measurable function τ: $[0,\infty) \to (0,\infty)$ belongs to $R.V.^{\infty}_{\rho}$ if

$$\lim_{t\to\infty} \frac{\tau(tx)}{\tau(t)} = x^{\rho} \qquad \text{for every } x > 0 .$$

(This is called a *regularly varying function* with index ρ.)

REMARK 3.1.17.

(i) A measurable function τ: $[0,\infty) \to (0,\infty)$ is called a function of *bounded decrease* if τ is nonincreasing and $D_{\tau}(a) < \infty$ for some $a > 1$. (Clearly in this case $D_{\tau}(a)$ equals $\overline{\lim}_{t\to\infty} \frac{\tau(t)}{\tau(ta)}$.) A measurable function τ: $[0,\infty) \to (0,\infty)$ is called a function of *bounded increase* (notation $\tau \in$ B.I.) if $1/\tau$ is a function of *bounded decrease*.

(ii) In the Appendix we will give a short summary of all the relevant properties of the function classes mentioned in Definition 3.1.16.

THEOREM 3.1.18. *Suppose that*

(i) $\nu_F^{n_0}$ *is nonsingular for some* $n_0 \in \mathbb{N}$,

(ii) $\frac{1}{1-F} \in$ B.I.,

(iii) $\int_0^{\infty} x^2 \, \nu_F(dx) < \infty$.

Then

$$\overline{\lim_{t\to\infty}} \frac{t\left(\left|U - \frac{\ell}{E} - \frac{1}{E^2}\, \nu_m([x,\infty))dx\right|([t,t+h])\right)}{\nu_m(\,t,\infty))} < \infty \qquad \textit{for every } h > 0 .$$

PROOF. Fix $h > 0$. Using (ii) one easily verifies that $1-F \in SM$ (see Definition 1.3.22). Since also

$$|\nu_E|(\,0,\infty)) < \infty \quad \text{and} \quad \sup_{t\geq 0} \frac{|\nu_E|([t,t+h])}{1 - F(t)} < \infty ,$$

we have $\nu_E \in S(\psi_0, 1-F)$, with $\psi_0 \equiv 1$.
It is also not difficult to check, using (ii) and the inequality for functions of bounded increase (see Appendix), that the assumptions of Theorem

1.3.29 are satisfied with $m = 1-F$ and $\psi = \psi_o$. Since by (i) and Lemma 3.1.4 ν_E is invertible in $S(\psi_o)$, Theorem 1.3.29 therefore tells us that

$$\nu_E^{-1} \in S(\psi_o, 1-F) \ . \tag{19}$$

The definition of ν_E (cf. (2)) and (3) yield

$$|U - \frac{\ell}{E} - \frac{1}{E^2}\nu_m([x,\infty))dx|([t,t+h]) \leq$$

$$\leq |\nu_E^{-1}|([t,t+h]) + \frac{1}{E}\int_t^{t+h}(1-F(x))dx +$$

$$+ \frac{1}{E}\int_t^{t+h}|\nu_E^{-1}([x,\infty)) + \nu_E([x,\infty))|dx \ . \tag{20}$$

Also it follows from (19), (ii) and the inequality

$$\nu_m([t,\infty)) \geq (a-1)t(1-F(ta)) \qquad (t \geq 0,\ a > 1) \tag{21}$$

that

$$\overline{\lim_{t\to\infty}} \frac{t(|\nu_E^{-1}|([t,t+h]) + \frac{1}{E}\int_t^{t+h}(1-F(x))dx)}{\nu_m([t,\infty))} < \infty \ . \tag{22}$$

Hence the desired result will follow from (20) and (22) once we have shown that

$$\overline{\lim_{t\to\infty}} \frac{t\int_t^{t+h}|\nu_E^{-1}([x,\infty)) + \nu_E([x,\infty))|dx}{\nu_m([t,\infty))} < \infty \ . \tag{23}$$

For the proof of (23) we first observe that, by a simple computation,

$$\varphi_\lambda(\nu_E^{-1}([x,\infty))dx + \nu_E([x,\infty))dx) = \frac{1-\varphi_\lambda(\nu_E^{-1})}{-\lambda} + \frac{1-\varphi_\lambda(\nu_E)}{-\lambda} =$$

$$= \frac{2-\varphi_\lambda(\nu_E) - \frac{1}{\varphi_\lambda(\nu_E)}}{-\lambda} = \frac{(\varphi_\lambda(\nu_E) - 1)^2}{\lambda\varphi_\lambda(\nu_E)} =$$

$$= \frac{(\varphi_\lambda(\nu_E) - 1)^2}{\lambda}\varphi_\lambda(\nu_E^{-1}) \ . \tag{24}$$

(Here and in the rest of the proof, λ will be an arbitrary complex number with Re $\lambda < 0$.)
Let us define $\rho := -(\nu_E - e)^2 * \ell$. Using formula (30) of section 3 in Chapter 1 with $\Lambda(z) = (z-1)^2$, and the fact that $\varphi_\lambda(-\ell) = 1/\lambda$, we find that

$$\varphi_\lambda(\rho) = \frac{(\varphi_\lambda(\nu_E) - 1)^2}{\lambda} , \tag{25}$$

hence by (24),

$$\varphi_\lambda(\rho * \nu_E^{-1}) = \varphi_\lambda(\nu_E^{-1}([x,\infty))dx + \nu_E([x,\infty))dx) ,$$

so, by Lemma 3.1.1,

$$\rho * \nu_E^{-1} = \nu_E^{-1}([x,\infty))dx + \nu_E([x,\infty))dx . \tag{26}$$

(Later we shall see that $\rho \in S(\psi_o)$. For the moment is suffices to observe that $\rho \in S(\psi_{-\varepsilon})$ for every $\varepsilon > 0$ so that $\varphi_\lambda(\rho)$ is well-defined.)
A consequence of (25) is that

$$\int_0^\infty \exp(\lambda x)\, x\, \rho(dx) = \frac{d}{d\lambda}(\varphi_\lambda(\rho)) =$$

$$= \frac{2(\varphi_\lambda(\nu_E) - 1)}{\lambda} \frac{d}{d\lambda}(\varphi_\lambda(\nu_E)) - \left(\frac{\varphi_\lambda(\nu_E) - 1}{\lambda}\right)^2 .$$

Hence by Lemma 3.1.1 and the already used fact that

$$\varphi_\lambda(\nu_E([x,\infty))dx) = \frac{\varphi_\lambda(\nu_E) - 1}{\lambda}$$

we get that

$$x\rho(dx) = 2(\nu_E([x,\infty))dx) * (x\nu_E(dx)) - (\nu_E([x,\infty))dx) * (\nu_E([x,\infty))dx) . \tag{27}$$

We shall use (27) to obtain an estimate for ρ that will finish the proof in combination with (26).
First let us observe that, by (iii),

$$\int_0^\infty |\nu_E|([x,\infty))dx < \infty \quad \text{and} \quad \int_0^\infty x|\nu_E|(dx) < \infty . \tag{28}$$

We also have by (ii) that

$$\sup_{t\geq 0} \frac{\int_t^{t+h} |\nu_E|([x,\infty))dx}{\nu_m([t,\infty))} \leq h \sup_{t\geq 0} \frac{|\nu_E|([t,\infty))}{\nu_m([t,\infty))} < \infty \tag{29}$$

and, using (21), that

$$\sup_{t\geq 0} \frac{\int_t^{t+h} x|\nu_E|(dx)}{\nu_m([t,\infty))} \leq \sup_{t\geq 0} \frac{h(t+h)|\nu_E|([t,t+h])}{\nu_m([t,\infty))} < \infty \ . \tag{30}$$

For simplicity let us write $\tau(t) := \nu_m([t,\infty))$, $t \geq 0$. Since $1/(1-F) \in$ B.I. it is easily seen that also $1/\tau \in$ B.I. (see Appendix) which implies (see the first line of this proof) that $\tau \in$ SM.
We can now reformulate (28), (29) and (30) as

$$\nu_E([x,\infty))dx \ , \quad x\nu_E(dx) \in S(\psi_o,\tau) \ . \tag{31}$$

Since $S(\psi_o,\tau)$ is an algebra by Proposition 1.3.25, (27) now yields that

$$x\rho(dx) \in S(\psi_o,\tau) \ . \tag{32}$$

Let us put

$$\tau_1(t) := \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ \frac{\tau(t)}{t} & \text{if } t > 1 \ . \end{cases}$$

Clearly $\tau_1 \in$ SM and (32) is equivalent to

$$\rho \in S(\psi_o,\tau_1) \ . \tag{33}$$

Moreover, (21) shows that $S(\psi_o,1-F) \subseteq S(\psi_o,\tau_1)$. Thus, by (19) and (33),

$$\rho * \nu_E^{-1} \in S(\psi_o,\tau_1)$$

since $S(\psi_o,\tau_1)$ is an algebra (Proposition 1.3.25).
This last fact combined with (26) proves (23). □

<u>REMARK 3.1.19.</u>

(i) If the second moment of ν_F is finite and if $1/(1-F) \in$ B.I. (as assumed in Theorem 3.1.18), then it is easy to prove by contradiction that the upper index of $1/(1-F)$ is ≥ 2 (cf. Appendix).

(ii) [ROG-5] has proved a somewhat weaker result than Theorem 3.1.18 (under the implicit assumption that Theorem 1.3.4 holds). The present proof is a considerably simplified version of his.

Before proving the next result we need the following lemma.

LEMMA 3.1.20. *Suppose that* $1/(1-F) \in$ B.I. *with upper index* $\alpha < 2$. *Then*

(i) $$\lim_{t\to\infty} t\nu_m([t,\infty)) = \infty ;$$

(ii) $$\overline{\lim_{t\to\infty}} \frac{\int_0^t \nu_m([x,\infty))dx}{t\nu_m([t,\infty))} < \infty .$$

PROOF. Fix c such that $\alpha < c < 2$.

(i) By the inequality for functions of bounded increase (cf. Appendix) we have for some constant M and for all $t \geq 1$ that

$$\frac{1-F(2)}{1-F(2t)} \leq Mt^c ,$$

so

$$1 - F(2t) \geq \frac{t^{-c}}{M} (1-F(2))$$

and therefore

$$t\nu_m([t,\infty)) \geq t^2(1-F(2t)) \geq \frac{t^{2-c}}{M} (1-F(2)) . \tag{34}$$

This implies $\lim_{t\to\infty} t\nu_m([t,\infty)) = \infty$ since $c < 2$.

(ii) By the inequality for functions of bounded increase (cf. Appendix) we have for all $y \geq x \geq 1$ that

$$\frac{\nu_m([x,\infty))}{\nu_m([y,\infty))} = \frac{\int_x^\infty (1-F(z))dz}{\int_y^\infty (1-F(z))dz} = \frac{\frac{x}{y}\int_y^\infty (1-F(\frac{x}{y}z))dz}{\int_y^\infty (1-F(z))dz} \leq$$

$$\leq \frac{x}{y} M(\frac{y}{x})^c = M(\frac{y}{x})^{c-1} .$$

Hence for $t \geq 1$

$$\frac{\int_1^t \nu_m([x,\infty))dx}{t\nu_m([t,\infty))} \leq \frac{1}{t} M \int_1^t (\frac{t}{x})^{c-1} dx \leq \frac{M}{2-c} .$$

This implies (ii), by (i). □

THEOREM 3.1.21. *Suppose that*

(i) $\nu_F^{n_0}$ *is nonsingular for some* $n_0 \in \mathbb{N}$.

(ii) $\frac{1}{1-F} \in$ B.I. *with upper index* $\alpha < 2$.

Then

$$\overline{\lim_{t\to\infty}} \frac{|U - \frac{\ell}{E} - \frac{1}{E^2}\, \nu_m([x,\infty))dx|([t,t+h])}{(\nu_m([t,\infty)))^2} < \infty \quad \textit{for every } h > 0 \ .$$

PROOF. We argue as in the proof of Theorem 3.1.18 up to formula (27). For simplicity we introduce the notations

$$\rho_1 := |\nu_E|([x,\infty))dx \ , \quad \rho_2 := x|\nu_E|(dx) \ .$$

and, as before, $\tau(t) := \nu_m([t,\infty))$ $(t \geq 0)$.
Recall that, by (ii), $1/\tau \in$ B.I. and that (29) and (30) state that $\bar{P}_\tau(\rho_i) < \infty$ $(i = 1,2)$.
Furthermore, by the proof of Lemma 1.3.24 there existst a constant $M > 0$ such that for all $t \geq 0$ we have

$$(\rho_1 * \rho_2)([t,t+h]) \leq M\bar{P}_\tau(\rho_1)\bar{P}_\tau(\rho_2) \int_0^t \nu_m([t-x,\infty))\nu_m([x,\infty))dx \ +$$

$$+ \ M\bar{P}_\tau(\rho_1)\bar{P}_\tau(\rho_2)\nu_m([t,\infty)) \ . \qquad (35)$$

By Lemma 3.1.20 (i) and (ii) and by the fact that $1/\tau \in$ B.I. we infer from (35) that

$$\sup_{t\geq 1} \frac{(\rho_1 * \rho_2)([t,t+h])}{t(\nu_m([t,\infty)))^2} < \infty \ .$$

The same argument proves that

$$\sup_{t\geq 1} \frac{(\rho_1 * \rho_1)([t,t+h])}{t(\nu_m([t,\infty)))^2} < \infty$$

and thus, by (27),

$$\sup_{t\geq 1} \frac{\int_t^{t+h} x|\rho|(dx)}{t(\nu_m([t,\infty)))^2} < \infty \ ,$$

or equivalently,

$$\sup_{t\geq 1} \frac{|\rho|([t,t+h))}{(\nu_m([t,\infty)))^2} < \infty \ . \tag{36}$$

Using (19) and (36) one easily shows by an obvious modification of Lemma 1.3.24 that for all $t \geq 0$

$$(|\rho|*|\nu_E^{-1}|)([t,t+h]) \leq M \int_0^t (\nu_m([t-x,\infty)))^2(1-F(x))dx + M(1-F(t)) \tag{37}$$

for some constant M.

We now estimate both terms in the right member of (37).

It follows from the first half of (34) and from (ii) that

$$\lim_{t\to\infty} \frac{1 - F(t)}{\nu_m([t,\infty))^2} = 0 \ . \tag{38}$$

It is also not difficult to verify, using $1/\tau \in$ B.I., that there exist constants M, $t_1 > 0$ such that for all $t \geq t_1$

$$\frac{\int_{t/2}^{t} (\nu_m([t-x,\infty)))^2(1-F(x))dx}{(\nu_m([t,\infty)))^2} \leq M \frac{\int_0^{t/2} \nu_m([x,\infty))dx}{t\nu_m([t,\infty))} < \infty \tag{39}$$

where the last inequality also uses Lemma 3.1.20 (ii).

Also, again using $1/\tau \in$ B.I., there exists a $t_2 > 0$ such that

$$\sup_{t\geq t_2} \frac{\int_0^{t/2} (\nu_m([t-x,\infty)))^2(1-F(x))dx}{(\nu_m([t,\infty)))^2} < \infty \ . \tag{40}$$

Substitution of (38), (39) and (40) into (37) yields

$$\overline{\lim_{t\to\infty}} \frac{(|\rho|*|\nu_E^{-1}|)([t,t+h])}{(\nu_m([t,\infty)))^2} < \infty \ . \tag{41}$$

Therefore, by (26),

$$\overline{\lim_{t\to\infty}} \frac{\int_t^{t+h} |\nu_E^{-1}([x,\infty)) + \nu_E([x,\infty))|dx}{(\nu_m([t,\infty)))^2} < \infty \ . \tag{42}$$

Finally, it is clear from (22) and Lemma 3.1.20 (i) that

$$\overline{\lim_{t\to\infty}} \frac{|\nu_E^{-1}|([t,t+h]) + \frac{1}{E}\,{}_t\!\int^{t+h} (1-F(x))dx}{(\nu_m([t,\infty)))^2} = 0 \; . \tag{43}$$

The desired result now follows from (42), (43) and (20). □

REMARK 3.1.22.

(i) By the same proof as in Theorem 3.1.18 (with some obvious modifications) it is possible to show the following more general result. Suppose (i), (iii) of Theorem 3.1.18 hold and

$$\sup_{t\geq 0} \frac{1 - F(t)}{\tau(t)} < \infty \; , \quad \text{where } \frac{1}{\tau} \in \text{B.I. and } {}_0\!\int^{\infty} \tau(z)dz < \infty \; ,$$

then

$$\overline{\lim_{t\to\infty}} \frac{t|U - \frac{\ell}{E} - \frac{1}{E^2}\,\nu_m([x,\infty))dx|([t,t+h])}{{}_t\!\int^{\infty} \tau(z)dz} < \infty \; .$$

If, in addition, $\lim_{t\to\infty} \frac{1 - F(t)}{\tau(t)} = 0$, it follows that (44)

$$\lim_{t\to\infty} \frac{t|U - \frac{\ell}{E} - \frac{1}{E^2}\,\nu_m([x,\infty))dx|([t,t+h])}{{}_t\!\int^{\infty} \tau(z)dz} = 0 \; .$$

(ii) A similar remark applies to Theorem 3.1.21. We leave the details to the reader.

Finally, we will discuss *second-order limit results.*

THEOREM 3.1.23. *Suppose that*

(i) $\nu_F^{n_0}$ *is nonsingular for some* $n_0 \in \mathbb{N}$.

(ii) $1-F \in R.V._{-\alpha}^{\infty}$, $\alpha \geq 2$ (cf. Definition 3.1.16 (3)).

(iii) $E_2 := \int_0^{\infty} x^2 \, \nu_F(dx) < \infty$.

Then

a. $$\varlimsup_{t\to\infty} \frac{\left|U - \frac{\ell}{E} - \frac{1}{E^2}\,\nu_m([x,\infty))dx\right|([t,t+h])}{1 - F(t)} < \infty \quad \textit{for every } h > 0 \ .$$

b. $$\lim_{t\to\infty} \frac{U([t,t+h]) - \frac{h}{E} - \frac{1}{E^2}\int_t^{t+h} \nu_m([x,\infty))dx}{1 - F(t)} = -\frac{hE_2}{E^3}$$

for every $h > 0$.

PROOF. Since (ii) implies $1/(1-F) \in$ B.I. we can apply Theorem 3.1.18, and so by this result and

$$\lim_{t\to\infty} \frac{t(1-F(t))}{\nu_m([t,\infty))} = \alpha - 1$$

(cf. Appendix) (a) follows immediately.
In order to prove (b) we first fix $h > 0$. Obviously $1-F \in \text{SMT}(0)$ by (ii). Since

$$\lim_{t\to\infty} \frac{\nu_E([t,t+h])}{1 - F(t)} = \frac{h}{E}$$

it is easy to verify that $\nu_E \in \underset{\sim}{S}^a(\psi_o, 1-F)$, where $a := h/E$.
By (i) ν_E^{-1} exists in $S(\psi_o)$ and so (cf. Remark 1.3.34)

$$\nu_E^{-1} \in \underset{\sim}{S}^{-a}(\psi_o, 1-F) \ . \tag{45}$$

The definition of ν_E and formula (3) yield

$$\begin{aligned} U([t,t+h]) - \frac{h}{E} - \frac{1}{E^2}\int_t^{t+h} \nu_m([x,\infty))dx &= \\ = \nu_E^{-1}([t,t+h]) - \frac{1}{E}\int_t^{t+h} (1-F(x))dx &+ \\ - \frac{1}{E}\int_t^{t+h} (\nu_E^{-1}([x,\infty)) + \nu_E([x,\infty)))dx &\ . \end{aligned} \tag{46}$$

From (45) and (46) it is now clear that we only have to verify that

$$\lim_{t\to\infty} \frac{\int_t^{t+h} (\nu_E^{-1}([x,\infty)) + \nu_E([x,\infty)))dx}{1 - F(t)} = h\left(\frac{E_2 - 2E^2}{E^2}\right) \ . \tag{47}$$

in order to obtain the desired result. Als in Theorem 3.1.18 we have

$$\rho * \nu_E^{-1} = \nu_E^{-1}([x,\infty))dx + \nu_E([x,\infty))dx \ , \tag{48}$$

where $\rho := -(\nu_E - e)^2 * \ell$.

By (iii) and (ii) it is easy to show that

$$\nu_m \in SMT(0) \ , \quad \lim_{t\to\infty} \frac{\int_t^{t+h} \nu_E([x,\infty))dx}{\nu_m([t,\infty))} = \frac{h}{E}$$

and

$$\lim_{t\to\infty} \frac{\int_t^{t+h} x\nu_E(dx)}{\nu_m([t,\infty))} = \frac{h(\alpha-1)}{E} \ . \tag{49}$$

(For the last result use Karamata's theorem, cf. Appendix.)

Combining (28), (29), (30) and (49) yields

$$\nu_E([x,\infty))dx \in \underset{\sim}{S}^a(\psi_o,\tau) \ , \qquad x\nu_E(dx) \in \underset{\sim}{S}^b(\psi_o,\tau) \ , \tag{50}$$

where

$$b := \frac{h(\alpha-1)}{E} \ , \quad a = \frac{h}{E} \quad \text{and} \quad \tau(t) := \nu_m([t,\infty)) \ ; \qquad t \geqq 0 \ .$$

Hence by (27) and formula (47) of Proposition 1.3.32 it follows that

$$\bar{\rho} \in \underset{\sim}{S}^c(\psi_o,\nu_m) \ , \tag{51}$$

where

$$c := \frac{2h(\alpha-1)}{E} \int_0^\infty \nu_E([x,\infty))dx \quad \text{and} \quad \bar{\rho} := x\rho(dx) \ .$$

Since

$$\rho(B) = \int_B \frac{1}{z}\,\bar{\rho}(dz) \qquad (B \in \mathcal{B}([0,\infty)))$$

it is obvious from (51) and Karamata's theorem that

$$\lim_{t\to\infty} \frac{\rho([t,t+h])}{1-F(t)} = \frac{c}{\alpha-1}$$

and so

$$\rho \in \underset{\sim}{S}^d(\psi_o,1-F) \ , \quad \text{where } d := \frac{c}{\alpha-1} \ . \tag{52}$$

Hence by (45), (48), (52) and formula (47) of Proposition 1.3.32 we obtain

$$\lim_{t\to\infty} \frac{{}_t\int^{t+h} (\nu_E^{-1}([x,\infty)) + \nu_E([x,\infty)))dx}{1 - F(t)} =$$

$$= \frac{2h}{E} \int_0^\infty \nu_E([x,\infty))dx - \frac{h}{E}\, \rho([0,\infty)) = \frac{2h}{E} \int_0^\infty \nu_E([x,\infty))dx =$$

$$= h\left(\frac{E_2 - 2E^2}{E^2}\right) . \qquad \square$$

THEOREM 3.1.24. *Suppose that*

(i) $\nu_F^{n_0}$ *is nonsingular for some* $n_0 \in \mathbb{N}$.

(ii) $1-F \in R.V._{-2}^{\infty}$.

(iii) $E_2 := \int_0^\infty x^2\, \nu_F(dx) = \infty$.

Then

a. $$\overline{\lim_{t\to\infty}} \frac{|U - \frac{\ell}{E} - \frac{1}{E^2}\nu_m([x,\infty))dx|([t,t+h])}{(1-F(t))\; {}_0\int^t \nu_m([x,\infty))dx} < \infty$$

for every $h > 0$.

b. $$\lim_{t\to\infty} \frac{U([t,t+h]) - \frac{h}{E} - \frac{1}{E^2}\; {}_t\int^{t+h} \nu_m([x,\infty))dx}{(1-F(t))\; {}_0\int^t \nu_m([x,\infty))dx} = -\frac{2h}{E^3} .$$

PROOF. We only prove the second part since the proof of the first part can be done along similar lines. Let $h > 0$ be given. As in Theorem 3.1.23 $\nu_E^{-1} \in \underset{\sim}{S}^{-a}(\psi_o, 1-F)$ with $a := h/E$ (cf. (45)). Also by the representation for the renewal measure

$$U([t,t+h]) - \frac{h}{E} - \frac{1}{E^2} \int_t^{t+h} \nu_m([x,\infty))dx =$$

$$= \nu_E^{-1}([t,t+h]) - \frac{1}{E} \int_t^{t+h} (1-F(z))dz - \frac{1}{E} \int_t^{t+h} (\nu_E^{-1}([x,\infty)) + \nu_E([x,\infty)))dx \qquad (53)$$

Clearly

$$\lim_{t\to\infty} \frac{\nu_E^{-1}([t,t+h]) - \frac{1}{E}\,{}_t\int^{t+h} (1-F(z))dz}{(1-F(t))\,{}_0\int^t \nu_m([x,\infty))dx} = 0$$

and so we only have to prove that

$$\lim_{t\to\infty} \frac{{}_t\int^{t+h} \nu_E^{-1}([x,\infty)) + \nu_E([x,\infty))dx}{(1-F(t))\,{}_0\int^t \nu_m([x,\infty))dx} = \frac{2h}{E} .$$

Since we already know that $\nu_E^{-1} \in \underset{\sim}{S}^{-a}(\psi_o, 1-F)$ and $(\nu_E^{-1}([x,\infty)) + \nu_E([x,\infty)))dx = = \rho * \nu_E^{-1}$ (cf. (26)) with $\rho := -(\nu_E - e)^{2*} * \ell$ it is obvious that we first have to analyse the behaviour of ρ in case $1-F \in R.V.{}_{-2}^{\infty}$ and $E_2 = \infty$.
By (27)

$$\rho([t,t+h]) = -\int_0^\infty\int_0^\infty \frac{1}{x+y}\, 1_{\{t\le x+y\le t+h\}}\, \nu_E([y,\infty))dy\, \nu_E([x,\infty))dx +$$

$$+ 2\int_0^\infty\int_0^\infty \frac{y}{x+y}\, 1_{\{t\le x+y\le t+h\}}\, \nu_E(dy)\nu_E([x,\infty))dx . \qquad (54)$$

The first integral in (54) equals

$$-\int_0^t \int_{t-x}^{t+h-x} \frac{1}{x+y}\, \nu_E([y,\infty))dy\, \nu_E([x,\infty))dx +$$

$$-\int_t^{t+h} \int_0^{t+h-x} \frac{1}{x+y}\, \nu_E([y,\infty))dy\, \nu_E([x,\infty))dx . \qquad (55)$$

Clearly, using $\lim_{t\to\infty} {}_0\int^t \nu_m([x,\infty))dx = \infty$, we obtain

$$\lim_{t\to\infty} \frac{{}_t\int^{t+h}\,{}_0\int^{t+h-x} \frac{1}{x+y}\, \nu_E([y,\infty))dy\, \nu_E([x,\infty))dx}{1 - F(t)\,{}_0\int^t \nu_m([x,\infty))dx} = 0 . \qquad (56)$$

Also by the definition of ν_E it is easy to check that the first part of (55) is bounded from above by

$$\frac{1}{E^2}\int_0^t \int_{t-x}^{t+h-x} \frac{1}{x+y}\, \nu_m([y,\infty))dy\, \nu_m([x,\infty))dx . \qquad (57)$$

Applying

$$\left|\int_0^t \int_{t-x}^{t+h-x} (1-F(y))\, \frac{1}{x+y}\, dy\ \nu_E([x,\infty))dx\right| \leq \frac{C}{t}\,(1-F(\tfrac{t}{2})) \int_0^{t/2} \nu_m([x,\infty))dx\ ,$$

$$\left|\int_0^t \int_{t-x}^{t+h-x} \nu_E([y,\infty))\, \frac{1}{x+y}\, dy(1-F(x))dx\right| \leq \frac{C}{t}\,(1-F(\tfrac{t}{2})) \int_0^{t/2} \nu_m([x,\infty))dx\ ,$$

$$\left|\int_0^t \int_{t-x}^{t+h-x} (1-F(y))\, \frac{1}{x+y}\, (1-F(x))\, dy\, dx\right| \leq \frac{C}{t}\,(1-F(\tfrac{t}{2}))$$

and

$$\int_0^t \int_{t-x}^{t+h-x} \nu_m([y,\infty))\, \frac{1}{x+y}\, \nu_m([x,\infty))\, dy\, dx \geq \frac{C}{t}\, \nu_m([\tfrac{1}{2}t,\infty)) \int_0^{t/2} \nu_m([x,\infty))dx$$

for sufficiently large t, yields in combination with (57) that the ratio of

$$\int_0^t \int_{t-x}^{t+h-x} \frac{1}{x+y}\, \nu_E([y,\infty))dy\ \nu_E([x,\infty))dx$$

and

$$\frac{1}{E^2} \int_0^t \int_{t-x}^{t+h-x} \frac{1}{x+y}\, \nu_m([y,\infty))dy\ \nu_m([x,\infty))dx$$

tends to one as $t \uparrow \infty$.

Hence by the monotonicity of $\nu_m([t,\infty))$ and

$$\lim_{t\to\infty} \frac{\nu_m([t+z,\infty))}{\nu_m([t,\infty))} = 1 \qquad \text{for every } z > 0$$

it follows that

$$\lim_{t\to\infty} \frac{t \int_0^t \int_{t-x}^{t+h-x} \frac{1}{x+y}\, \nu_E([y,\infty))dy\ \nu_E([x,\infty))dx}{\frac{1}{E^2} \int_0^t \nu_m([t-x,\infty))\nu_m([x,\infty))dx} = h\ . \tag{58}$$

Since by assumption

$$\lim_{t\to\infty} \frac{\nu_m([t,\infty))}{t(1-F(t))} = 1 \quad \text{and} \quad \int_0^t \nu_m([x,\infty))dx \in R.V._0^\infty$$

(58) now implies

$$\lim_{t\to\infty} \frac{{}_0\int^t {}_{t-x}\int^{t+h-x} \frac{1}{x+y} \nu_E([y,\infty))dy \; \nu_E([x,\infty))dx}{1 - F(t) \; {}_0\int^t \nu_m([x,\infty))dx} =$$

$$= \lim_{t\to\infty} \frac{2h \; {}_0\int^{t/2} \nu_m([t-x,\infty))\nu_m([x,\infty))dx}{E^2 \; t(1-F(t)) \; {}_0\int^t \nu_m([x,\infty))dx} = \frac{2h}{E^2} \tag{59}$$

and so by (55) and (56)

$$\lim_{t\to\infty} - \frac{{}_0\int^\infty {}_0\int^\infty \frac{1}{x+y} 1_{\{t\le x+y\le t+h\}} \nu_E([y,\infty))dy \; \nu_E([x,\infty))dx}{(1-F(t)) \; {}_0\int^t \nu_m([x,\infty))dx} = - \frac{2h}{E^2} . \tag{60}$$

Consider now the second part of (54). As for the first part one can easily check that the ratio of

$$\int_0^\infty \int_0^\infty \frac{y}{x+y} 1_{\{t\le x+y\le t+h\}} \nu_E(dy)\nu_E([x,\infty))dx$$

and (61)

$$\frac{1}{E^2} \int_0^\infty \int_0^\infty \frac{y}{x+y} 1_{\{t\le x+y\le t+h\}} \nu_m(dy)\nu_m([x,\infty))dx$$

tends to one as $t \uparrow \infty$.
Also it is easy to see that

$$\lim_{t\to\infty} \frac{{}_0\int^\infty {}_0\int^\infty \frac{y}{x+y} 1_{\{t\le x+y\le t+h\}} \nu_m(dy)\nu_m([x,\infty))dx}{{}_0\int^t {}_{t-x}\int^{t+h-x} \frac{y}{x+y} (1-F(y))dy \; \nu_m([x,\infty))dx} = 1 . \tag{62}$$

Since

$$\lim_{t\to\infty} \frac{1 - F(t+y)}{1 - F(t)} = 1 \quad \text{for all } y \ge 0$$

and

$$\lim_{t\to\infty} \frac{{}_{t-A}\int^t (t-x)(1-F(t-x))\nu_m([x,\infty))dx}{{}_0\int^{t-A} (t-x)(1-F(t-x))\nu_m([x,\infty))dx} = 0 \quad \text{for every } A > 0$$

we obtain

$$\lim_{t\to\infty} \frac{t \; {}_0\int^t \; {}_{t-x}\int^{t+h-x} \frac{y}{x+y} \, (1-F(y))dy \;\; \nu_m([x,\infty))dx}{h \; {}_0\int^t \; (1-F(t-x))(t-x)\nu_m([x,\infty))dx} = 1 \tag{63}$$

and hence by assumption (ii) (decompose nominator ${}_0\int^t \ldots$ into the parts ${}_0\int^{\delta t} \ldots$, ${}_{(1-\delta)t}\int^t \ldots$ and ${}_{\delta t}\int^{(1-\delta)t} \ldots$):

$$\lim_{t\to\infty} \frac{{}_0\int^t \; {}_{t-x}\int^{t+h-x} \frac{y}{x+y} \, (1-F(y))dy \;\; \nu_m([x,\infty))dx}{2h(1-F(t)) \; {}_0\int^t \; \nu_m([x,\infty))dx} = 1 \; . \tag{64}$$

By (64), (62) and (61)

$$\lim_{t\to\infty} \frac{{}_0\int^\infty \; {}_0\int^\infty \frac{y}{x+y} \, 1_{\{t\leq x+y\leq t+h\}} \; \nu_E(dy)\nu_E([x,\infty))dx}{(1-F(t)) \; {}_0\int^t \; \nu_m([x,\infty))dx} = \frac{2h}{E^2}$$

and this yields (cf. (54), (60))

$$\lim_{t\to\infty} \frac{\rho([t,t+h])}{(1-F(t)) \; {}_0\int^t \; \nu_m([x,\infty))dx} = \frac{2h}{E^2} \; . \tag{65}$$

Finally, using $\nu_E^{-1} \in \underset{\sim}{S}^{-a}(\psi_o, 1-F)$ and (65) we obtain

$$\lim_{t\to\infty} \frac{(\rho*\nu_E^{-1})([t,t+h])}{(1-F(t)) \; {}_0\int^t \; \nu_m([x,\infty))dx} = \frac{2h}{E^2} \; .$$

Hence by (26)

$$\lim_{t\to\infty} \frac{{}_t\int^{t+h} \; \nu_E^{-1}([x,\infty)) \; + \; \nu_E([x,\infty))dx}{(1-F(t)) \; {}_0\int^t \; \nu_m([x,\infty))dx} = \frac{2h}{E^2}$$

and this implies the desired result. □

REMARK 3.1.25. If we combine Theorems 3.1.23 and 3.1.24 the result reads as follows:

(i) $\nu_F^{n_0}$ is nonsingular for some $n_0 \in \mathbb{N}$.

(ii) $1-F \in R.V._{-\alpha}^{\infty}$; $\alpha \geq 2$.

Then for every h > 0

a. $$\overline{\lim_{t\to\infty}} \frac{|U - \frac{\ell}{E} - \frac{1}{E^2} \nu_m([x,\infty))dx|([t,t+h])}{(1-F(t)) \int_0^t \nu_m([x,\infty))dx} < \infty \; ;$$

b. $$\lim_{t\to\infty} \frac{U([t,t+h]) - \frac{h}{E} - \frac{1}{E^2} \int_t^{t+h} \nu_m([x,\infty))dx}{(1-F(t)) \int_0^t \nu_m([x,\infty))dx} = -\frac{2h}{E^3} \, .$$

THEOREM 3.1.26. *Suppose that*

(i) $\nu_F^{n_0}$ *is nonsingular for some* $n_0 \in \mathbb{N}$.

(ii) $1-F \in R.V.^{\infty}_{-\alpha}$, $1 < \alpha < 2$.

Then for every h > 0

a. $$\overline{\lim_{t\to\infty}} \frac{|U - \frac{\ell}{E} - \frac{1}{E^2} \nu_m([x,\infty))dx|([t,t+h])}{(1-F(t)) \int_0^t \nu_m([x,\infty))dx} < \infty \; ;$$

b. $$\lim_{t\to\infty} \frac{U([t,t+h]) - \frac{h}{E} - \frac{1}{E^2} \int_t^{t+h} \nu_m([x,\infty))dx}{(1-F(t)) \int_0^t \nu_m([x,\infty))dx} =$$

$$= -\frac{(2\alpha-3)(2-\alpha)h}{E^3(\alpha-1)} \int_0^1 (1-x)^{1-\alpha} x^{1-\alpha} \, dx \, .$$

PROOF. Since the method of proof is similar as in the previous theorem we will only discuss the differences.
As in Theorem 3.1.24

$$\lim_{t\to\infty} \frac{t \int_0^\infty \int_0^\infty \frac{1}{x+y} 1_{\{t\le x+y\le t+h\}} \nu_E([y,\infty))dy \; \nu_E([x,\infty))dx}{\int_0^t \nu_m([t-x,\infty))\nu_m([x,\infty))dx} = \frac{h}{E^2} \qquad (66)$$

and hence by the properties of regularly varying functions (cf. Appendix)

$$\lim_{t\to\infty} - \frac{\int_0^\infty \int_0^\infty \frac{1}{x+y} 1_{\{t\le x+y\le t+h\}} \nu_E([y,\infty))dy \; \nu_E([x,\infty))dx}{(\nu_m([t,\infty)))^2} =$$

$$= -\frac{h}{E^2} \int_0^1 (1-x)^{1-\alpha} x^{1-\alpha} dx . \tag{67}$$

Moreover (cf. Theorem 3.1.24),

$$\lim_{t\to\infty} \frac{t \int_0^\infty \int_0^\infty \frac{y}{x+y} 1_{\{t\leq x+y\leq t+h\}} \nu_E(dy)\nu_E([x,\infty))dx}{\int_0^t (1-F(t-x))(t-x)\nu_m([x,\infty))dx} = \frac{h}{E^2}$$

and again by the properties of regularly varying functions

$$\lim_{t\to\infty} \frac{\int_0^\infty \int_0^\infty \frac{y}{x+y} 1_{\{t\leq x+y\leq t+h\}} \nu_E(dy)\nu_E([x,\infty))dx}{t(1-F(t))\nu_m([t,\infty))} =$$

$$= \frac{h}{E^2} \int_0^1 (1-x)^{1-\alpha} x^{1-\alpha} dx . \tag{68}$$

Formulas (54), (67) and (68) now imply (use also $\lim_{t\to\infty} \frac{\nu_m([t,\infty))}{t(1-F(t))} = \frac{1}{\alpha-1}$) that

$$\lim_{t\to\infty} \frac{\rho([t,t+h])}{(\nu_m([t,\infty)))^2} = (2\alpha-3) \frac{h}{E^2} \int_0^1 (1-x)^{1-\alpha} x^{1-\alpha} dx . \tag{69}$$

Hence by (68) and $\nu_E^{-1} \in \underset{\sim}{S}^{-a}(\psi_o, 1-F)$ with $a = h/E$ we obtain

$$\lim_{t\to\infty} \frac{\int_t^{t+h} \nu_E^{-1}([x,\infty)) + \nu_E([x,\infty))dx}{(\nu_m([t,\infty)))^2} =$$

$$= \lim_{t\to\infty} \frac{(\rho*\nu_E^{-1})([t,t+h])}{(\nu_m([t,\infty)))^2} = (2\alpha-3) \frac{h}{E^2} \int_0^1 (1-x)^{1-\alpha} x^{1-\alpha} dx$$

and this implies (using the representation for the renewal measure) the desired result. □

<u>REMARK 3.1.27.</u> If we denote the integral $\int_0^\infty x^{\beta-1} e^{-x} dx$ for $\beta > 0$ by $\Gamma(\beta)$ (*the so-called gamma function*) then one can prove (cf. [ABR])

$$\int_0^1 x^{1-\alpha}(1-x)^{1-\alpha} dx = \frac{(\Gamma(2-\alpha))^2}{\Gamma(4-2\alpha)} \quad \text{and} \quad \lim_{\beta\downarrow 0} \beta\Gamma(\beta) = 1 .$$

This implies

$$\lim_{\alpha \uparrow 2} - \frac{(2\alpha-3)(2-\alpha)h}{E^3(\alpha-1)} \int_0^1 (1-x)^{1-\alpha}\, x^{1-\alpha}\, dx = - \frac{2h}{E^3}$$

and this limit, as one expects, is the same constant as mentioned in Theorem 3.1.24.

2. On the behaviour of the renewal measure for a special class of distributions

In this section we introduce a class of distributions on $[0,\infty)$ for which analytical expressions for the corresponding renewal measure can be derived. This class includes many well-known distributions and is therefore an important subset of the set of all distributions on $[0,\infty)$. Moreover, the expression for the renewal measure can be computed by an elementary method and is well suited for numerical evaluations. Therefore the elementary approach is preferable above the Banach algebra approach used in the previous section. However, as always, one has to pay a price for using this method, since it can only be applied to a limited number of distributions. In order to start we first introduce these distributions.

DEFINITION 3.2.1. Let ν_F be a probability measure on $[0,\infty)$. We say $\nu_F \in K_m$ if

$$\varphi_\lambda(\nu_F) := \int_0^\infty \exp(\lambda x)\nu_F(dx) \ , \qquad \text{Re}\ \lambda \leq 0 \ ,$$

equals $P_1(\lambda)/P_2(\lambda)$, where $P_i(\lambda)$ $(i = 1,2)$ are polynomials without common factors and $m = \text{degree}(P_2) > \text{degree}(P_1)$.

Suppose now in the remainder of this section that $\nu_F \in K_m$ for some $m \in \mathbb{N}$. As always

$$\varphi_\lambda(U) = \frac{1}{1 - \varphi_\lambda(\nu_F)} \ , \qquad \text{Re}\ \lambda < 0$$

and this reduces for $\nu_F \in K_m$ to

$$\varphi_\lambda(U) = \frac{P_2(\lambda)}{P_2(\lambda) - P_1(\lambda)} , \qquad \mathrm{Re}\ \lambda < 0 . \tag{1}$$

If we define $P(\lambda) := P_2(\lambda) - P_1(\lambda)$ then it follows immediately by (1)

$$\varphi_\lambda(U) = 1 + \frac{P_1(\lambda)}{P(\lambda)} . \tag{2}$$

Moreover, by the definition of K_m we obtain

$$\text{degree}(P_1) < \text{degree}(P) .$$

In order to expand the rational function $P_1(\lambda)/P(\lambda)$ by partial fractions ([CON]) we have to determine the factors of $P(\lambda)$. Note that these factors equal the roots of the equation $P_1(\lambda)/P_2(\lambda) = 1$, since P_1 and P_2 have no common zeroes. Now the following observations about these factors can be made.

In case c_i is a complex-valued root of $P(\lambda) = 0$, then also $\bar{c}_i$ is a root of $P(\lambda) = 0$ where $\bar{c}_i$ denotes the complex conjugate of c_i. This follows from the fact that the coefficients of the polynomial $P(\lambda)$ are real-valued and hence $P(\bar{\lambda}) = \overline{P(\lambda)}$.

Also using $\varphi_o(\nu_F) = 1$ it is obvious that $P(0) = 0$ and so $\lambda = 0$ is a real-valued root. This root has multiplicity 1 since by the factorization theorem for polynomials

$$\lim_{\lambda\to 0} \frac{1 - \varphi_\lambda(\nu_F)}{\lambda} = \lim_{\lambda\to 0} \frac{P(\lambda)}{P_2(\lambda)\lambda} = \lim_{\lambda\to 0} \frac{\lambda^{k-1}\, G(\lambda)}{P_2(\lambda)} = 0$$

in case $k \geqq 2$, where k denotes the multiplicity of $\lambda = 0$. This contradicts

$$\lim_{\lambda\to 0} \frac{1 - \varphi_\lambda(\nu_F)}{\lambda} = E > 0$$

and hence k equals one.

So in the general case the roots of the equation $P(\lambda) = 0$ are either real-valued nonnegative numbers $r_1,\ldots,r_p$, each with multiplicity k_i $(i = 1,\ldots,p)$ and/or conjugate pairs $(c_1,\bar{c}_1),\ldots,(c_\ell,\bar{c}_\ell)$, each with multiplicity n_i and $\mathrm{Re}\ c_i > 0$, $i = 1,\ldots,\ell$. (Note ν_F is nonlattice.)

However, we will not discuss this general case but focus our attention on two important subcases.

CASE A. No complex-valued roots (i.e. $\ell = 0$).

CASE B. There exist complex-valued roots and the multiplicity of all distinct roots equals one.

THEOREM 3.2.2. *Let $\nu_F \in K_m$ for some $m \in \mathbb{N}$ and suppose the different roots r_i $(i = 1,\dots,p)$ of the equation $P(\lambda) = 0$ are real-valued. Then*

$$U([0,t]) = \frac{t}{E} + \frac{E_2}{2E^2} - \sum_{i=1}^{p} \sum_{j=1}^{k_i} (-r_i)^{-j} A_{ij} \sum_{k=0}^{j-1} \frac{(r_i t)^k}{k!} \exp(-r_i t) \qquad (3)$$

where k_i denotes the multiplicity of the nonzero root r_i and

$$A_{ik_i-j} = (D^j H_i)(r_i)/j! \,, \qquad j = 0,\dots,k_i-1$$

with

$$H_i(\lambda) := \frac{(\lambda-r_i)^{k_i}}{1-\varphi_\lambda(\nu_F)} \qquad (4)$$

and $D^j H_i$ the j^{th}-derivative of H_i $(H_i^{(0)} := H_i)$.

PROOF. By partial fraction expansion (cf. [CON]) we obtain for every λ with $\text{Re}\,\lambda < 0$

$$\frac{P_1(\lambda)}{P(\lambda)} = \sum_{i=1}^{p} \sum_{j=1}^{k_i} A_{ij}(\lambda-r_i)^{-j} + A_{01}\,\lambda^{-1} \qquad (5)$$

where p denotes the number of different nonzero roots $r_i > 0$ and k_i the multiplicity of r_i.
Hence by (3)

$$\varphi_\lambda(U) = 1 + A_{01}\,\lambda^{-1} + \sum_{i=1}^{p} \sum_{j=1}^{k_i} A_{ij}(\lambda-r_i)^{-j} \,. \qquad (6)$$

Since the Laplace-Stieltjes transform of the negative exponential distribution with parameter r_i equals $-r_i/(\lambda-r_i)$ it follows from (6) and Lemma 3.1.1 that the renewal measure equals

$$U([0,t]) = 1 - A_{01}\,t + \sum_{i=1}^{p}\sum_{j=1}^{k_i} (-1)^j A_{ij}\, r_i^{-j}\, F_{r_i}^{j*}(t) \tag{7}$$

where $F_{r_i}(t) := 1 - \exp(-r_i t)$.
It is well known that

$$1 - F_{r_i}^{j*}(t) = \exp(-r_i t) \sum_{k=0}^{j-1} \frac{(r_i t)^k}{k!}$$

and so from (7)

$$U([0,t]) = 1 + \sum_{i=1}^{p}\sum_{j=1}^{k_i} (-1)^j A_{ij}\, r_i^{-j}\left(1 - \exp(-r_i t) \sum_{k=0}^{j-1} \frac{(r_i t)^k}{k!}\right) - A_{01}\, t\ . \tag{8}$$

In order to determine the constants we observe by (6) that

$$A_{01} = \lim_{\lambda \downarrow 0} \lambda \varphi_\lambda(U) = -\frac{1}{E} \tag{9}$$

and

$$1 + \sum_{i=1}^{p}\sum_{j=1}^{k_i} (-1)^j A_{ij}\, r_i^{-j} = \lim_{\lambda \downarrow 0} \varphi_\lambda(U) + \frac{1}{E\lambda} =$$

$$= \lim_{\lambda \downarrow 0} \frac{1 - \frac{1}{E}\varphi_\lambda(\nu_m)}{1 - \varphi_\lambda(\nu_F)} = \frac{E_2}{2E^2}\ . \tag{10}$$

Finally noting that the constants A_{ik_i-j} $(j = 0,\ldots,k_i-1)$ equal $(D^j H_i)(r_i)/j!$ where $H_i(\lambda) = (\lambda - r_i)^{k_i}/(1-\varphi_\lambda(\nu_F))$ and $D^j H_i$ denotes the j^{th} derivative of H_i $(D^0 H_i := H_i)$ (cf. (6)) and combining (7), (8) and (9) yield the desired result. □

<u>REMARK 3.2.3.</u>

(i) In case $k_i = 1$ $(i = 1,\ldots,p)$ $(\Rightarrow p = m-1)$ it is easy to see that formula (4) reduces to

$$U([0,t]) = \frac{t}{E} + \frac{E_2}{2E^2} + \sum_{i=1}^{m-1} A_{i1}\, r_i^{-1} \exp(-r_i t) \tag{11}$$

where

$$A_{i1} = -(D\varphi_{r_i}(\nu_F))^{-1} = -\left(\int_0^\infty x \exp(r_i x)\nu_F(dx)\right)^{-1}$$

(ii) Let $\nu_F \in K_2$. Then by definition $\varphi_\lambda(\nu_F)$ equals $(1+a_0\lambda)/(1+a_1\lambda+a_2\lambda^2)$, where $a_2 \neq 0$. Also by definition it is forbidden that both polynomials have common factors and so we have to assume in case $a_0 \neq 0$ that

$$a_0^{-1} \neq (2a_2)^{-1}(a_1 + (a_1^2-4a_2)^{\frac{1}{2}}) \quad \text{and} \quad a_0^{-1} \neq (2a_2)^{-1}(a_1 - (a_1^2-4a_2)^{\frac{1}{2}}) \ .$$

Now the roots of the equation $(1+a_0\lambda)/(1+a_1\lambda+a_2\lambda^2) = 1$ are given by $\lambda = 0$ and $\lambda = (a_0-a_1)/a_2$, $\lambda > 0$, and after some calculations and applying (i) we obtain

$$U([0,t]) = \frac{t}{E} + \frac{E_2}{2E^2} + \frac{-a_2 - a_0^2 + a_0a_1}{(a_1-a_0)^2} \exp\left(-\frac{(a_0-a_1)}{a_2}\, t\right) .$$

(Note that the class of K_2-distributions contains hyperexponential distributions and mixtures of Erlang-1 and Erlang-2 distributions with the same scale parameters (cf. [KOK]).)

Finally we discuss case B.

THEOREM 3.2.4. *Let* $\nu_F \in K_m$ *for some* $m \in \mathbb{N}$ *and suppose the nonzero roots of the equation* $P(\lambda) = 0$ *are either real-valued or complex-valued and have multiplicity* 1. *Then*

$$U([0,t]) = \frac{t}{E} + \frac{E_2}{2E^2} + \sum_{i=1}^{p} A_{i1}\, r_i^{-1} \exp(-r_it) +$$

$$+ 2 \sum_{i=1}^{\ell} \mathrm{Re}(B_{i1}c_i^{-1} \exp(-i\, \mathrm{Im}\, c_it)) \exp(-\mathrm{Re}\, c_it) \qquad (11)$$

(*)

where p *denotes the number of nonzero real-valued roots* $r_1,\dots,r_p$ *and* ℓ *the number of pairs of conjugate roots* $(c_1,\bar{c}_1),\dots,(c_\ell,\bar{c}_\ell)$. *Also*

$$A_{i1} = -(D\varphi_{r_i}(\nu_F))^{-1}, \qquad i = 1,\dots,p \ ;$$

$$B_{i1} = -(D\varphi_{c_i}(\nu_F))^{-1}, \qquad i = 1,\dots,\ell \ .$$

PROOF. By partial fraction expansion

$$\frac{P_1(\lambda)}{P(\lambda)} = \sum_{i=1}^{p} A_{i1}(\lambda-r_i)^{-1} + \sum_{i=1}^{\ell} B_{i1}(\lambda-c_i)^{-1} + \sum_{i=1}^{\ell} B_{i1}^*(\lambda-\bar{c}_i)^{-1} + A_{01}\lambda^{-1} \ . \qquad (12)$$

(*) If p = 0 then the third term of (11) vanishes.

Hence (as in Theorem 3.3.2) we obtain

$$U([0,t]) = 1 - \sum_{i=1}^{p} A_{i1}\, r_i^{-1}(1 - \exp(-r_i t)) - A_{01}t +$$

$$- \sum_{i=1}^{\ell} B_{i1}\, c_i^{-1}(1 - \exp(-c_i t)) - \sum_{i=1}^{\ell} B_{i1}^{*}\, \bar{c}_i^{-1}(1 - \exp(-\bar{c}_i t)) \quad . \quad (13)$$

Now it is easy to check that for all $i = 1,\ldots,\ell$

$$B_{i1} = -\,(D\varphi_{c_i}(\nu_F))^{-1} \quad ; \quad B_{i1}^{*} = -\,(D\varphi_{\bar{c}_i}(\nu_F))^{-1}$$

and this implies

$$B_{i1} = \overline{B_{i1}^{*}} \quad . \qquad (14)$$

By (14) formula (13) reduces to

$$U([0,t]) = 1 - 2\sum_{i=1}^{\ell} \mathrm{Re}(B_{i1}c_i^{-1}) + \sum_{i=1}^{p} A_{i1}\, r_i^{-1} \exp(-r_i t) - A_{01}t +$$

$$- \sum_{i=1}^{p} A_{i1}\, r_i^{-1} + 2\sum_{i=1}^{\ell} \mathrm{Re}(B_{i1}\, c_i^{-1} \exp(-\, i\, \mathrm{Im}\, c_i t)) \exp(-\,\mathrm{Re}\, c_i t) \qquad (15)$$

and following the same method as in Theorem 3.2.2 to determine the constants in (15) we can easily establish from (15) the desired result. □

We conclude this section with an example of a distribution satisfying the conditions of Theorem 3.2.4.

Let ν_F be the Erlang-n distribution, where n is an integer. For this distribution $\varphi_\lambda(\nu_F) = (1-\lambda)^{-n}$ and now the roots of the equation $(1-\lambda)^{-n} = 1$ are given by

$$-\exp(\frac{2k\pi i}{n}) + 1 \ , \qquad k = 0,\ldots,n-1 \ .$$

This implies (see Theorem 3.2.4)

$$U([0,t]) = \frac{t}{n} + \frac{n+1}{2n} + O(\exp(-(1-\cos(2\pi/n))t)) \quad . \qquad (16)$$

3. The behaviour of the renewal measure in case the expectation is infinite

In this section we will only consider (unless stated otherwise) nonlattice probability distribution functions F on $(0,\infty)$ with $F(0^+) = 0$, $E = \infty$ and regularly varying tails (cf. Definition 3.1.16).

First we will discuss the asymptotic behaviour of the *renewal function* and after that the behaviour of the same function on $[t,t+h]$ as $t \uparrow \infty$. The results for the case $1-F \in R.V._{-1}^{\infty}$ are new. Before mentioning these results let us introduce the following notations:

$$U(t) := \mathbb{E}\left(\sum_{n=0}^{\infty} 1_{\{\underline{S}_n \le t\}}\right), \qquad t \geq 0$$

$$m(t) := \int_0^t (1-F(z))dz, \qquad t \geq 0. \tag{1}$$

The corresponding Laplace-Stieltjes transforms (cf. [WID]) are given by

$$\hat{U}(\lambda) := \int_0^\infty \exp(-\lambda x)U(dx), \qquad \lambda > 0$$

$$\hat{m}(\lambda) := \int_0^\infty \exp(-\lambda x)m(dx), \qquad \lambda > 0. \tag{2}$$

These notations are slightly different from the ones used in the previous sections. This is due to the fact that we deal in this section with functions instead of measures.

<u>THEOREM 3.3.1</u>. (cf. [FEL-2].) *For* $0 \leq \alpha \leq 1$ *the next result holds:*

$$m(t) \in R.V._{1-\alpha}^{\infty} \Leftrightarrow U(t) \in R.V._{\alpha}^{\infty}.$$

Either relation implies

$$\lim_{t\uparrow\infty} \frac{m(t)U(t)}{t} = \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}.$$

<u>PROOF</u>. By Karamata's Abel-Tauber theorem (cf. [GEL] or the Appendix) and the relation $\hat{U}(\lambda)\hat{m}(\lambda) = \lambda$ for all $\lambda > 0$ we obtain the following equivalent statements:

$$m(t) \in R.V.^{\infty}_{1-\alpha} \Leftrightarrow \hat{m}(\lambda) \in R.V.^{0(*)}_{\alpha-1} \Leftrightarrow \hat{U}(\lambda) \in R.V.^{0}_{-\alpha} \Leftrightarrow U(t) \in R.V.^{\infty}_{\alpha} .$$

Moreover, by the same Abel-Tauber theorem,

$$\lim_{t\uparrow\infty} \frac{m(t)U(t)}{t} = \lim_{t\uparrow\infty} \frac{\hat{m}(1/t)\hat{U}(1/t)}{t\Gamma(2-\alpha)\Gamma(1+\alpha)} = \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)} . \qquad \square$$

REMARK 3.3.2.

(i) Note that for every (nonlattice) probability distribution F we have $1 \leq \frac{m(t)U(t)}{t} \leq 2$ for all $t > 0$. This inequality is easily verified, since

$$t = \int_{0^-}^{t} m(t-x)U(dx)$$

and hence

$$t \leq m(t)U(t) \quad \text{and} \quad t \geq \int_{0^-}^{t/2} m(t-x)U(dx) \geq m(t/2)U(t/2) .$$

(ii) If the first moment is finite then $\lim_{t\to\infty} m(t) = \mu < \infty$, so $m(t) \in R.V.^{\infty}_{0}$. This implies by Theorem 3.3.1 (take $\alpha = 1$) that

$$\lim_{t\to\infty} \frac{m(t)U(t)}{t} = 1 \quad \text{or equivalently} \quad \lim_{t\to\infty} \frac{U(t)}{t} = \frac{1}{\mu} .$$

(iii) Since for $0 < \alpha < 1$, $m(t) \in R.V.^{\infty}_{1-\alpha}$ if and only if $1 - F(t) \in R.V.^{\infty}_{-\alpha}$ (cf. Appendix), an equivalent formulation of Theorem 3.3.1 is given by

$$1 - F(t) \in R.V.^{\infty}_{-\alpha} \Leftrightarrow U(t) \in R.V.^{\infty}_{\alpha} \qquad (0 < \alpha < 1) .$$

Either relation implies

$$\lim_{t\uparrow\infty} (1 - F(t))U(t) = \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} .$$

If the tail of the distribution function F belongs to $R.V.^{\infty}_{-1}$ we can prove a stronger result than the one mentioned in Theorem 3.3.1.

(*) $\tau \in R.V.^{0}_{\rho}$ if $g \in R.V.^{\infty}_{-\rho}$ where $g(t) := \tau(1/t)$ (cf. Definition 3.1.16).

However, before proving this, we need the following.

THEOREM 3.3.3. *For $\frac{1}{2} < \alpha \leq 1$ the next result holds*

$$m(t) \in R.V.^{\infty}_{1-\alpha} \Leftrightarrow U(t+1) - U(t) \in R.V.^{\infty}_{\alpha-1} .$$

Either relation implies

$$\lim_{t\uparrow\infty} m(t)(U(t+h) - U(t)) = \frac{h}{\Gamma(\alpha)\Gamma(2-\alpha)} \quad \forall h>0 .$$

PROOF. It is proved by [ERI] using Fourier analysis that for $\frac{1}{2} < \alpha \leq 1$ $m(t) \in R.V.^{\infty}_{1-\alpha}$ implies $\lim_{t\uparrow\infty} m(t)(U(t+h) - U(t)) = \frac{h}{\Gamma(\alpha)\Gamma(2-\alpha)}$. So we only need to verify that $U(t+1) - U(t) \in R.V.^{\infty}_{\alpha-1}$ implies $m(t) \in R.V.^{\infty}_{1-\alpha}$.
We have

$$\int_t^{t+1} U(x)dx = \int_0^t (U(x+1) - U(x))dx + \int_0^1 U(x)dx$$

and this yields since $\alpha > 0$ and $U(t+1) - U(t) \in R.V.^{\infty}_{\alpha-1}$ that

$$\int_t^{t+1} U(x)dx \in R.V.^{\infty}_{\alpha} .$$

Now by the monotonicity of U we obtain

$$U(t) \leq \int_t^{t+1} U(x)dx \leq U(t+1)$$

and since $U(t+1) - U(t)$ remains bounded (cf. [FEL-2]) we get

$$\lim_{t\to\infty} \frac{\int_t^{t+1} U(x)dx}{U(t)} = 1 .$$

Hence $U(t) \in R.V.^{\infty}_{\alpha}$ and by Theorem 3.3.1 the desired result follows. □

REMARK 3.3.4. For $0 < \alpha \leq \frac{1}{2}$ the next result holds (cf. [ERI]):

$$m(t) \in R.V.^{\infty}_{-\alpha} \Rightarrow \underline{\lim}_{t\to\infty} m(t)(U(t+h) - U(t)) = \frac{h}{\Gamma(\alpha)\Gamma(2-\alpha)} . \quad (3)$$

This result cannot be improved without imposing stronger conditions on F. (See also the lattice case.)

In order to prove the next theorem we need the following definition.

DEFINITION 3.3.5. (cf. [GEL].)

(i) A measurable function $\tau: [0,\infty) \to \mathbb{R}$ belongs to the class Π^∞ if there exists a positive measurable function L such that

$$\lim_{t\uparrow\infty} \frac{\tau(tx) - \tau(t)}{L(t)} = \ln(x)$$

for all $x > 0$.

(ii) A measurable function $\tau: [0,\infty) \to \mathbb{R}$ belongs to the class Π^0 if g belongs to the class Π^∞ where $g(t) = \tau(1/t)$.

REMARK 3.3.6. The auxiliary function L belongs automatically to $R.V._0^\infty$. This is easily verified from the equality

$$\frac{L(ty)}{L(t)} \frac{\tau(txy) - \tau(ty)}{L(ty)} = \frac{\tau(txy) - \tau(t)}{L(t)} - \frac{\tau(ty) - \tau(t)}{L(t)}$$

which is valid for all $x,y > 0$.
Moreover, one can prove that Π^∞ is a proper subclass of $R.V._0^\infty$ (cf. [GEL] or Appendix).

THEOREM 3.3.7. *Let the function* $\tau \in \Pi^\infty$ *with auxiliary function* L *be positive and nondecreasing. Then for every nondecreasing function* $R(t) \in R.V._\alpha^\infty$ *with* $\alpha \geq 0$ *and* $\limsup_{t\uparrow\infty} \frac{t(R(t+1) - R(t))}{R(t)} < \infty$ *we obtain*

$$\lim_{t\uparrow\infty} \int_{0^-}^{t} \frac{(\tau(t) - \tau(t-x))R(dx)}{L(t)R(t)} = -\alpha \int_0^1 \ln(1-x)x^{\alpha-1}\, dx \ .$$

PROOF. Without loss of generality we assume that L is locally bounded on $[0,\infty)$. Obviously for every $0 < \eta < 1$

$$\int_{0^-}^{t} (\tau(t) - \tau(t-x))R(dx) = \int_{0^-}^{\eta t} (\tau(t) - \tau(t-x))R(dx) + $$

$$+ \int_{\eta t}^{t} (\tau(t) - \tau(t-x))R(dx) \ . \qquad (4)$$

We first prove that the latter term on the right in (4) is small for η close to 1, if t tends to infinity. The proof of this result is carried out as follows.

Since τ and R are nondecreasing we obtain for this term the upperbound

$$\int_{\lfloor\eta t\rfloor}^{\lfloor t\rfloor} (\tau(t)-\tau(t-x))R(dx) + \tau(t)(R(t)-R(t-1)) . \tag{5}$$

For the last term in (5) we have

$$0 \leqq \overline{\lim_{t\uparrow\infty}} \frac{\tau(t)(R(t)-R(t-1))}{L(t)R(t)} \leqq$$

$$\leqq \overline{\lim_{t\uparrow\infty}} \frac{t(R(t)-R(t-1))}{R(t)} \overline{\lim_{t\uparrow\infty}} \frac{\tau(t)}{tL(t)} \leqq M \overline{\lim_{t\uparrow\infty}} \frac{\tau(t)}{tL(t)} = 0 . \tag{6}$$

Consider next

$$\int_{\lfloor\eta t\rfloor}^{\lfloor t\rfloor} (\tau(t)-\tau(t-x))R(dx) .$$

From the monotonicity of R and τ we obtain

$$0 \leqq \int_{\lfloor\eta t\rfloor}^{\lfloor t\rfloor} (\tau(t)-\tau(t-x))R(dx) = \sum_{k=\lfloor\eta t\rfloor}^{\lfloor t\rfloor-1} \int_{k}^{k+1} (\tau(t)-\tau(t-x))R(dx) \leqq$$

$$\leqq \sum_{k=\lfloor\eta t\rfloor}^{\lfloor t\rfloor-1} (\tau(t)-\tau(t-k-1))(R(k+1)-R(k)) \leqq$$

$$\leqq M \sum_{k=\lfloor\eta t\rfloor}^{\lfloor t\rfloor-1} (\tau(t)-\tau(t-k-1)) \frac{R(k)}{k} \quad \text{for some } M > 0$$

and hence, since R is nondecreasing and $\eta > 0$

$$0 \leqq \int_{\lfloor\eta t\rfloor}^{\lfloor t\rfloor-1} (\tau(t)-\tau(t-x))R(dx) \leqq$$

$$\leqq \frac{MR(t)}{\eta t} \sum_{k=\lfloor\eta t\rfloor}^{\lfloor t\rfloor-1} (\tau(t)-\tau(t-k-1)) . \tag{7}$$

Consider the latter term on the right in (7). Since τ is nondecreasing and positive we get

$$0 \leq \sum_{k=\lfloor \eta t \rfloor}^{\lfloor t \rfloor - 1} (\tau(t) - \tau(t-k-1)) =$$

$$= \tau(t - \lfloor \eta t \rfloor) - \tau(t - \lfloor t \rfloor) + \sum_{k=\lfloor \eta t \rfloor}^{\lfloor t \rfloor - 1} (\tau(t) - \tau(t-k))$$

$$\leq \tau(t - \lfloor \eta t \rfloor) - \tau(t - \lfloor t \rfloor) + \int_{\lfloor \eta t \rfloor}^{\lfloor t \rfloor} (\tau(t) - \tau(t-x))dx .$$

So, in order to prove

$$\lim_{\eta \uparrow 1} \overline{\lim_{t \uparrow \infty}} \frac{\int_{\eta t}^{t} (\tau(t) - \tau(t-x))R(dx)}{L(t)R(t)} = 0$$

it is sufficient to show that

$$\lim_{\eta \uparrow 1} \overline{\lim_{t \uparrow \infty}} \frac{\int_{\lfloor \eta t \rfloor}^{\lfloor t \rfloor} (\tau(t) - \tau(t-x))dx}{tL(t)} = 0 .$$

In order to do so we need the following representation for $\tau \in \Pi^{\infty}$ given in the Appendix:

$$\tau(t) = c + L(t) + \int_1^t \frac{L(p)}{p} dp$$

with c some constant and L the auxiliary function. Using this representation we obtain for t sufficiently large

$$0 \leq \int_{\lfloor \eta t \rfloor}^{\lfloor t \rfloor} (\tau(t) - \tau(t-x))dx \leq \int_{\eta^2 t}^{t} (\tau(t) - \tau(t-x))dx =$$

$$= t \int_{\eta^2}^{1} \int_{1-x}^{1} \frac{L(ty)}{y} dy dx + t \int_{\eta^2}^{1} (L(t) - L(t(1-x)))dx . \tag{8}$$

Note that by the uniform convergence theorem (cf. Appendix)

$$\lim_{t\uparrow\infty}\int_{\eta^2}^{1}\int_{1-x}^{1}\frac{L(ty)}{L(t)}\frac{dy}{y}\,dx =$$

$$= \lim_{t\uparrow\infty}\int_{\eta^2}^{1}\int_{1-x}^{1}\frac{(ty)^{\delta}\,L(ty)}{t^{\delta}\,L(t)}\frac{dy}{y^{1+\delta}}\,dx = -\int_{\eta^2}^{1}\ln(1-x)dx \tag{9}$$

and

$$\lim_{t\uparrow\infty}\int_{\eta^2}^{1}\frac{L(t(1-x))}{L(t)}\,dx =$$

$$= \lim_{t\uparrow\infty}\int_{\eta^2}^{1}\frac{(t(1-x))^{\delta}\,L(t(1-x))}{t^{\delta}\,L(t)}\frac{dx}{(1-x)^{\delta}} = 1-\eta^2 \qquad (\text{take } \delta\in(0,1))\ .$$

Hence by (8) and (9)

$$0 \leqq \lim_{\eta\uparrow 1}\overline{\lim_{t\uparrow\infty}}\int_{\lfloor\eta t\rfloor}^{\lfloor t\rfloor}\frac{\tau(t)-\tau(t-x)}{tL(t)}\,dx = 0$$

and so

$$\lim_{\eta\uparrow 1}\overline{\lim_{t\uparrow\infty}}\frac{\int_{\eta t}^{t}(\tau(t)-\tau(t-x))R(dx)}{L(t)R(t)} = 0\ . \tag{10}$$

This takes care of the second term on the right of (4).
We proceed with the first term of (4). By the representation for functions $\tau\in\Pi^{\infty}$ and Fubini's theorem we obtain for the first term on the right of (4) that

$$\lim_{t\uparrow\infty}\frac{\int_{0^-}^{\eta t}(\tau(t)-\tau(t-x))R(dx)}{L(t)R(t)} = \lim_{t\uparrow\infty}\frac{\int_{0^-}^{\eta t}\left(1-\frac{L(t-x)}{L(t)}\right)R(dx)}{R(t)} +$$

$$+ \lim_{t\uparrow\infty}\frac{\int_{0^-}^{\eta t}\int_{t-x}^{t}\frac{L(y)}{y}\,dy\,R(dx)}{L(t)R(t)} =$$

$$= \lim_{t\uparrow\infty} \frac{\int_{1-\eta}^{1} (R(t\eta) - R(t(1-x)))L(xt)\,\frac{dx}{x}}{R(t)L(t)} =$$

$$= \int_{1-\eta}^{1} \frac{\eta^{\alpha} - (1-x)^{\alpha}}{x}\,dx = -\alpha \int_{0}^{\eta} \ln(1-x)x^{\alpha-1}\,dx \;.$$

This takes care of the first term on the right of (4) and so the proof is now complete. □

REMARK 3.3.8. Note that

$$\alpha \int_0^1 \ln(1-x)x^{\alpha-1}\,dx = \int_0^1 \frac{1-x^{\alpha}}{1-x}\,dx = (D(\ln\circ\Gamma))(\alpha+1) + \gamma$$

where γ is Euler's constant, $(\ln\circ\Gamma)(x) := \ln(\Gamma(x))$ and $(Df)(x)$ denotes the derivative of a function f in x (cf. [ABR, p. 259]).

THEOREM 3.3.9. *The next result holds:*

$$1 - F(t) \in R.V._{-1}^{\infty} \Leftrightarrow -\frac{U(t)}{t} \in \Pi^{\infty} \;.$$

Either relation implies

$$\lim_{t\uparrow\infty} \frac{\frac{U(t)m(t)}{t} - 1}{t(1-F(t))(m(t))^{-1}} = 1$$

PROOF. Suppose $-\frac{U(t)}{t} \in \Pi^{\infty}$. Then by [HAA-2] we obtain

$$-\int_0^t x^2\, d\!\left(\frac{U(x)}{x}\right) = 2\int_0^t U(x)dx - tU(t) \in R.V._2^{\infty}$$

and hence by Karamata's Abel-Tauber theorem its Laplace-Stieltjes transform is also regularly varying, i.e.

$$\frac{\hat{U}(\lambda)}{\lambda} + (D(\hat{U}))(\lambda) \in R.V._{-2}^{0} \;.$$

This implies $(D(\lambda\hat{U}))(\lambda) \in R.V._{-1}^{0}$ and so $-\lambda\hat{U}(\lambda) \in \Pi^{0}$. From $\hat{m}(\lambda)\hat{U}(\lambda) = \lambda^{-1}$ we get $\hat{m}(\lambda) \in \Pi^{0}$ and hence the Abel-Tauber theorem for π-varying functions (cf.

[GEL] or the Appendix) yields $m(t) \in \Pi^{\infty}$ or equivalently $1-F(t) \in R.V._{-1}^{\infty}$ (cf. Appendix).

We now prove the other part of this theorem. Let $1-F(t) \in R.V._{-1}^{\infty}$. Since

(i) m nondecreasing, $m(t) \in \Pi^{\infty}$ with auxiliary function $L(t) := t(1-F(t))$,

(ii) U nondecreasing, $U(t) \in R.V._{1}^{\infty}$ and $\overline{\lim\limits_{t\to\infty}} \frac{t(U(t)-U(t-1))}{U(t)} < \infty$ (use Theorems 3.3.3 and 3.3.1)

we may apply Theorem 3.3.7 and so we obtain

$$\lim_{t\to\infty} \frac{m(t)U(t) - t}{t(1-F(t))U(t)} = \lim_{t\to\infty} \frac{\int_{0^-}^{t} (m(t) - m(t-y))U(dy)}{t(1-F(t))U(t)} = 1 .$$

This implies by Theorem 3.3.1 the desired result. □

REMARK. For $1-F(t) \in R.V._{-1}^{\infty}$ we obtain $\lim\limits_{t\to\infty} \frac{m(t)}{t(1-F(t))} = \infty$ and hence the result, stated in Theorem 3.3.9 implies $\lim\limits_{t\to\infty} \frac{m(t)U(t)}{t} = 1$.

Since the first moment is not necessarily finite if $1-F(t) \in R.V._{-1}^{\infty}$ we cannot use the methods from Banach algebra theory for the analysis of the function $U(t+h)-U(t)$. Therefore (cf. [STO-3]) Fourier analysis will be used to investigate this behaviour. Before starting we introduce the characteristic function of a probability distribution F, i.e.

$$\varphi(\theta) := \int_{-\infty}^{+\infty} \exp(i\theta x)F(dx) .$$

LEMMA 3.3.10. *For any probability distribution function* F *we have*

$$\lim_{\theta\to 0} \frac{\operatorname{Im} \varphi(\theta)}{\theta} = \infty \Rightarrow \int_{-\infty}^{+\infty} |x|\, dF(x) = \infty .$$

PROOF. Clearly

$$\left|\frac{\operatorname{Im} \varphi(\theta)}{\theta}\right| \leq \int_{-\infty}^{+\infty} |x| \left|\frac{\sin x\theta}{x\theta}\right| dF(x) \leq \int_{-\infty}^{+\infty} |x|\, dF(x) .$$ □

THEOREM 3.3.11. *Suppose* F *is an arbitrary nonlattice distribution function (not necessarily on the positive halfline) and* $\lim_{\theta\to 0} \frac{\text{Im}\,\varphi(\theta)}{\theta} = \infty$. *Then for all continuous functions* h *with compact support*

$$\lim_{s\uparrow 1} \int_{-\infty}^{+\infty} h(\theta)\ \text{Re}\left(\frac{1}{1-s\varphi(\theta)}\right)d\theta = \int_{-\infty}^{+\infty} h(\theta)\ \text{Re}\left(\frac{1}{1-\varphi(\theta)}\right)d\theta .$$

PROOF. For the greater part this proof is similar to the proof for the case where the absolute first moment m_1 is finite and $0 < E < \infty$, given by [BRE, Lemma 10.11]. Checking Breiman's proof we see that we have to verify the following:

(i) $\text{Re}\left(\frac{1}{1-\varphi(\theta)}\right)$ is absolutely integrable on $(-b,b)$ for some positive constant b.

(ii) $$\lim_{s\uparrow 1} \int_{-\varepsilon}^{\varepsilon} \frac{(1-s)h(\theta)}{|1-s\varphi(\theta)|^2}\ \text{Re}\left(\frac{\varphi(\theta)(1-\bar{\varphi}(\theta))}{1-\varphi(\theta)}\right)d\theta = 0 ,$$

with $\bar{\varphi}(\theta)$ the complex conjugate of $\varphi(\theta)$ and ε some positive constant.

We first verify (i).
Since $\lim_{\theta\to 0} \frac{\text{Im}\,\varphi(\theta)}{\theta} = \infty$, we obtain by Lemma 3.3.10 that $\int_{-\infty}^{+\infty} |x| F(dx) = \infty$.
This implies by [BRE, p. 52] that $|S_n|/n$ diverges almost surely to infinity as $n \to \infty$, where S_n is the partial sum of independent random variables, each distributed according to F.
Hence $P\{S_n \in I \text{ i.o.}\} = 0$ for all bounded intervals I and so by [FEL-2, p. 202] $U(I) < \infty$ for all bounded intervals I. This implies by [FEL-2, p. 616] that for all $b > 0$

$$\int_0^b \text{Re}\left(\frac{1}{1-\varphi(\theta)}\right)d\theta < \infty$$

and since $\text{Re}\left(\frac{1}{1-\varphi(\theta)}\right)$ is even and positive the desired result follows.
We now verify (ii).
Since $h(\theta)$ is bounded (say $|h(\theta)| \leq 1$), $(1 - s\ \text{Re}\ \varphi(\theta))^2 \geq (1-s)^2$ and

$$\left|\text{Re}\left(\frac{\varphi(\theta)(1-\bar{\varphi}(\theta))}{1-\varphi(\theta)}\right)\right| \leq |\varphi(\theta)| \leq 1 ,$$

we obtain for every $\varepsilon > 0$ that

$$\left| \int_{-\varepsilon}^{\varepsilon} \frac{(1-s)h(\theta)}{|1-s\varphi(\theta)|^2} \operatorname{Re}\left(\frac{\varphi(\theta)(1-\bar{\varphi}(\theta))}{1-\varphi(\theta)}\right) d\theta \right| \leq$$

$$\leq \int_{-\varepsilon}^{\varepsilon} \frac{(1-s)}{(1-s)^2 + s^2(\operatorname{Im}\varphi(\theta))^2} d\theta \ . \tag{11}$$

Also, since $\frac{\operatorname{Im}\varphi(\theta)}{\theta} \to \infty$ $(\theta \to 0)$, there exists for every $M > 0$ a positive constant ε such that for all θ with $0 < |\theta| \leq \varepsilon$, $\frac{\operatorname{Im}\varphi(\theta)}{\theta} \geq M$ and this implies

$$\int_{-\varepsilon}^{\varepsilon} \frac{(1-s)}{(1-s)^2 + s^2(\operatorname{Im}\varphi(\theta))^2} d\theta \leq \int_{-\varepsilon}^{\varepsilon} \frac{(1-s)}{(1-s)^2 + s^2 M^2 \theta^2} d\theta \ . \tag{12}$$

Using

$$0 \leq \overline{\lim_{s\uparrow 1}} \int_{-\varepsilon}^{\varepsilon} \frac{(1-s)}{(1-s)^2 + s^2 M^2 \theta^2} d\theta \leq \overline{\lim_{s\uparrow 1}} \int_{-\varepsilon}^{\varepsilon} \frac{(1-s)}{(1-s)^2 + \frac{M^2}{2}\theta^2} d\theta =$$

$$= \overline{\lim_{s\uparrow 1}} \int_{-\varepsilon/(1-s)}^{\varepsilon/(1-s)} \frac{1}{1 + \frac{M^2\theta^2}{2}} d\theta = \frac{2\pi}{M^2}$$

we finally obtain by combining (11) and (12) the desired result. □

<u>REMARK 3.3.12</u>. As noted in the proof of this theorem Breiman proved for nonlattice probability distribution functions with $0 < E < \infty$ and finite absolute first moment m_1 that for all continuous functions h with compact support

$$\lim_{s\uparrow 1} \int_{-\infty}^{+\infty} h(\theta) \operatorname{Re}\left(\frac{1}{1-s\varphi(\theta)}\right) d\theta = \frac{\pi h(0)}{E} + \int_{-\infty}^{+\infty} h(\theta) \operatorname{Re}\left(\frac{1}{1-\varphi(\theta)}\right) d\theta \ .$$

An easy consequence of Theorem 3.3.11 is the next result.

<u>THEOREM 3.3.13</u>. *Suppose* F *is an arbitrary nonlattice probability distribution function (not necessarily on the positive halfline) for which* $\frac{\operatorname{Im}\varphi(\theta)}{\theta} \to \infty$ *(*$\theta \to 0$*). Then for all continuous functions* h *with compact support and* $x \in \mathbb{R}$ *we have*

$$\lim_{s\uparrow 1} \int_{-\infty}^{+\infty} h(\theta)\ \mathrm{Re}\left(\frac{\exp(-ix\theta)}{1 - s\varphi(\theta)}\right) d\theta = \int_{-\infty}^{+\infty} h(\theta)\ \mathrm{Re}\left(\frac{\exp(-ix\theta)}{1-\varphi(\theta)}\right) d\theta .$$

PROOF. It is easy to verify that

$$\int_{-\infty}^{+\infty} h(\theta)\ \mathrm{Re}\left(\frac{\exp(-ix\theta)}{1 - s\varphi(\theta)}\right) d\theta = \int_{-\infty}^{+\infty} h(\theta) \cos(x\theta)\ \mathrm{Re}\left(\frac{1}{1 - s\varphi(\theta)}\right) d\theta +$$

$$+ x \int_{-\infty}^{+\infty} h(\theta) \frac{\sin(x\theta)}{x\theta}\ \mathrm{Im}\left(\frac{\theta}{1 - s\varphi(\theta)}\right) d\theta . \tag{13}$$

By Theorem 3.3.11 and (13) it is now sufficient to show that

$$\lim_{s\uparrow 1} \int_{-\infty}^{+\infty} h(\theta) \frac{\sin(x\theta)}{x\theta}\ \mathrm{Im}\left(\frac{\theta}{1 - s\varphi(\theta)}\right) d\theta =$$

$$= \int_{-\infty}^{+\infty} h(\theta) \frac{\sin(x\theta)}{x\theta}\ \mathrm{Im}\left(\frac{\theta}{1-\varphi(\theta)}\right) d\theta .$$

This proof can be carried out as follows. Since $h(\theta)$ is a continuous function with compact support and for $|\theta|$ sufficiently small,

$$\left|\mathrm{Im}\left(\frac{\theta}{1 - s\varphi(\theta)}\right)\right| = \frac{|\theta s\ \mathrm{Im}\ \varphi(\theta)|}{|1 - s\varphi(\theta)|^2} \leq \frac{|\theta s\ \mathrm{Im}\ \varphi(\theta)|}{s^2(\mathrm{Im}\ \varphi(\theta))^2} =$$

$$= \frac{1}{s} \left|\frac{\theta}{\mathrm{Im}\ \varphi(\theta)}\right| \leq 2 \left|\frac{\theta}{\mathrm{Im}\ \varphi(\theta)}\right| \leq M$$

for all $\frac{1}{2} \leq s \leq 1$ with M some constant, we may apply Lebesgue's dominated convergence theorem (use also in case $|\theta|$ away from zero the nonlattice property of F and the continuity in both variables of $p(\theta,s) := \mathrm{Im}\left(\frac{\theta}{1-s\varphi(\theta)}\right)$) and so we get

$$\lim_{s\uparrow 1} \int_{-\infty}^{+\infty} h(\theta) \frac{\sin(x\theta)}{x\theta}\ \mathrm{Im}\left(\frac{\theta}{1 - s\varphi(\theta)}\right) d\theta =$$

$$= \int_{-\infty}^{+\infty} h(\theta) \frac{\sin(x\theta)}{x\theta}\ \mathrm{Im}\left(\frac{\theta}{1-\varphi(\theta)}\right) d\theta .$$

This finishes the proof of the theorem. □

Suppose now F is a nonlattice probability distribution function on $(-\infty,+\infty)$ with finite moment $E > 0$ or $\frac{\operatorname{Im} \varphi(\theta)}{\theta} \to \infty$ $(\theta \to 0)$ and denote the average number of renewals on $[x-\frac{h}{2}, x+\frac{h}{2}]$ by $U(x,h)$, i.e.

$$U(x,h) := \sum_{n=0}^{\infty} \left(F^{n*}(x+\frac{h}{2}) - F^{n*}(x-\frac{h}{2})\right) .$$

By [BRE, Prop. 3.39] we get $S(\bar{h}) < \infty$ with

$$S(\bar{h}) := \sup\{U(x,h) : -\infty < x < \infty,\ 0 \leq h \leq \bar{h}\} \quad \text{for every } \bar{h} > 0 .$$

Choose subsequently a probability density function k on $(-\infty,+\infty)$, satisfying the conditions (cf. Appendix)

(i) its characteristic function $\hat{k}(\theta) := \int_{-\infty}^{+\infty} \exp(i\theta x)k(x)dx$ is zero for $|\theta| \geq 1$,

(ii) $k(x)$ is even and continuous,

(iii) $k(x)$ has finite moments of all order;

and set

$$V(x,h,a) := \int_{-\infty}^{+\infty} a^{-1} k(a^{-1} y)U(x-y,h)dy \qquad (a > 0,\ x \in \mathbb{R}) . \tag{14}$$

Since $U(\cdot,h)$ is uniformly bounded on $(-\infty,+\infty)$ and k is a probability density function we obtain that $V(\cdot,h,a)$ is uniformly bounded and so this function is well defined.

We now derive a Fourier representation for $V(x,h,a)$. Using integration by parts we have for every $n \in \mathbb{N}$

$$\int_{-\infty}^{+\infty} \exp(ix\theta)\left(F^{n*}(x+\frac{h}{2}) - F^{n*}(x-\frac{h}{2})\right)dx = \frac{2 \sin(h\theta/2)}{\theta} \varphi^n(\theta)$$

and this implies for every $0 < s < 1$ by Fubini's theorem, the inversion formula for Fourier transforms and the monotone convergence theorem that

$$\int_{-\infty}^{+\infty} \left(\sum_{n=0}^{\infty} s^n\ F^{n*}(x-y+\frac{h}{2}) - F^{n*}(x-y-\frac{h}{2})\right) a^{-1} k(a^{-1} y)dy =$$

$$= \frac{1}{2\pi} \sum_{n=0}^{\infty} s^n \int_{-\infty}^{+\infty} F^{n*}(x-y+\frac{h}{2}) - F^{n*}(x-y-\frac{h}{2}) \int_{-1/a}^{1/a} \exp(-iy\theta)\hat{k}(a\theta)d\theta\, dy =$$

$$= \frac{1}{2\pi} \sum_{n=0}^{\infty} s^n \int_{-1/a}^{1/a} \exp(-ix\theta)\hat{k}(a\theta) \int_{-\infty}^{+\infty} \exp(i(x-y)\theta)\left(F^{n*}(x-y+\tfrac{h}{2}) - F^{n*}(x-y-\tfrac{h}{2})\right) dy\, d\theta =$$

$$= \frac{h}{2\pi} \sum_{n=0}^{\infty} s^n \int_{-1/a}^{1/a} \exp(-ix\theta)\hat{k}(a\theta) \frac{\sin(h\theta/2)}{h\theta/2} \varphi^n(\theta)\, d\theta =$$

$$= \frac{h}{2\pi} \int_{-1/a}^{1/a} \exp(-ix\theta)\hat{k}(a\theta) \frac{\sin(h\theta/2)}{h\theta/2} \frac{1}{1 - s\varphi(\theta)}\, d\theta =$$

$$= \frac{h}{2\pi} \int_{-1/a}^{1/a} \hat{k}(a\theta) \frac{\sin(h\theta/2)}{h\theta/2} \operatorname{Re}\left(\frac{\exp(-ix\theta)}{1 - s\varphi(\theta)}\right) d\theta\ . \tag{15}$$

Hence by the relations (14), (15), the monotone convergence theorem and Theorem 3.3.13 we finally obtain that

$$V(x,h,a) = \lim_{s\uparrow 1} \int_{-\infty}^{+\infty} a^{-1} k(a^{-1}y) \sum_{n=0}^{\infty} s^n\left(F^{n*}(x-y+\tfrac{h}{2}) - F^{n*}(x-y-\tfrac{h}{2})\right) dy =$$

$$= \frac{h}{2\pi} \lim_{s\uparrow 1} \int_{-1/a}^{1/a} \hat{k}(a\theta) \frac{\sin(h\theta/2)}{h\theta/2} \operatorname{Re}\left(\frac{\exp(-ix\theta)}{1 - s\varphi(\theta)}\right) d\theta =$$

$$= \begin{cases} \dfrac{h}{2E} + \dfrac{h}{2\pi} \displaystyle\int_{-1/a}^{1/a} \hat{k}(a\theta) \frac{\sin(h\theta/2)}{h\theta/2} \operatorname{Re}\left(\frac{\exp(-ix\theta)}{1-\varphi(\theta)}\right) d\theta & \text{if } 0 < E < \infty \text{ and } m_1 \text{ finite,} \\ \dfrac{h}{2\pi} \displaystyle\int_{-1/a}^{1/a} \hat{k}(a\theta) \frac{\sin(h\theta/2)}{h\theta/2} \operatorname{Re}\left(\frac{\exp(-ix\theta)}{1-\varphi(\theta)}\right) d\theta & \text{if } \lim_{\theta\to 0} \dfrac{\operatorname{Im}\varphi(\theta)}{\theta} = \infty\ . \end{cases} \tag{16}$$

Since the integrand in (16) is even we obtain the following result.

<u>THEOREM 3.3.14</u>. *For every nonlattice probability distribution function* F *(not necessarily concentrated on the positive halfline) we have*

$$V(x,h,a) = \begin{cases} \dfrac{h}{2E} + \dfrac{h}{\pi}\displaystyle\int_0^{1/a} \hat{k}(a\theta)\, \frac{\sin(h\theta/2)}{h\theta/2}\, \mathrm{Re}\left(\frac{\exp(-ix\theta)}{1-\varphi(\theta)}\right) d\theta \\ \qquad\qquad \text{if } m_1 \text{ is finite and } 0 < E < \infty , \\ \dfrac{h}{\pi}\displaystyle\int_0^{1/a} \hat{k}(a\theta)\, \frac{\sin(h\theta/2)}{h\theta/2}\, \mathrm{Re}\left(\frac{\exp(-ix\theta)}{1-\varphi(\theta)}\right) d\theta \\ \qquad\qquad \text{if } \lim\limits_{\theta\to 0} \dfrac{\mathrm{Im}\, \varphi(\theta)}{\theta} = +\infty . \end{cases}$$

REMARK 3.3.15. Stone (cf. [STO-3]) uses the above representation for $V(x,h,a)$ to analyze the behaviour of $U(t+h) - U(t) - \frac{h}{E}$ in case the m-th absolute moment is finite for $m \geq 2$.

The following theorem links the function $V(x,h,a)$ to the function $U(x,h)$.

THEOREM 3.3.16. *Let* $M > 0$ *be given. Then for every* $\varepsilon > 0$ *there exists some* $x_0 = x_0(\varepsilon)$ *such that for all* $|x| \geq x_0$ *and* $2|x|^{-3} \leq h \leq M$ *the following inequality holds*

$$V(x,h-2|x|^{-3},|x|^{-4}) - \frac{S(M)\varepsilon}{|x|^2} \leq U(x,h) \leq$$

$$\leq (1-\varepsilon|x|^{-2})^{-1}\, V(x,h+2|x|^{-3},|x|^{-4})$$

where $S(M) := \sup\{U(x,h) : 0 \leq h \leq M, -\infty < x < \infty\} < \infty$.

PROOF. See [STO-3, p. 340]. □

Since $U(x,h)$ is estimated in terms of $V(x,h,a)$ it is sufficient for the analysis of $U(x,h)$ to analyze the asymptotic behaviour of $V(x,h,a)$. This is done in the next theorem.

DEFINITION 3.3.17. A distribution function F on $(-\infty,+\infty)$ is called *strongly nonlattice* if

(a) F is nonlattice,

(b) $\limsup\limits_{\theta \uparrow \infty} |\varphi(\theta)| < 1$.

THEOREM 3.3.18. *If* F *is strongly nonlattice and* $1-F \in R.V._{-1}^{\infty}$ *then for* h *and* p *strictly positive:*

$$\lim_{x\to\infty} \frac{(V(x,h,x^{-4}) - V(px,h,(px)^{-4}))}{L(x)} = h\ \ln(p)$$

where L(x) *is some slowly varying function (i.e.* $L \in R.V._0^{\infty}$*).*

PROOF. By Theorem 3.3.14 we obtain

$$V(x,h,x^{-4}) - V(px,h,(px)^{-4}) =$$

$$= \frac{h}{\pi} \int_0^{x^4} \hat{k}(x^{-4}\,\theta)\ g(h\theta)\ \mathrm{Re}\left(\frac{\exp(-ix\theta)}{1-\varphi(\theta)}\right)d\theta\ +$$

$$- \int_0^{(xp)^4} \hat{k}((xp)^{-4}\,\theta)\ g(h\theta)\ \mathrm{Re}\left(\frac{\exp(-ix\theta)}{1-\varphi(\theta)}\right)d\theta$$

with

$$g(\theta) := \frac{\sin(\theta/2)}{\theta/2}\ .$$

Now the major contribution to the integral V comes from a small neighbourhood of zero. Since L(x) is slowly varying we may neglect terms of the order 1/x.

Consider the difference

$$D = \int_0^{B/x} \hat{k}(x^{-4}\,\theta)\ g(h\theta)\ \mathrm{Re}\left(\frac{\exp(-ix\theta)}{1-\varphi(\theta)}\right)d\theta\ +$$

$$- \int_0^{B/xp} \hat{k}((xp)^{-4}\,\theta)\, g(h\theta)\, \mathrm{Re}\left(\frac{\exp(-ix\theta)}{1-\varphi(\theta)}\right)d\theta$$

for some fixed $B > 0$.

We may delete the first two factors $\hat{k}\cdot g$ in both integrals. This only alters D by $O(1/x)$ $(x \to \infty)$ since the functions $\hat{k}$, g are differentiable in zero, $x \geqq 1$ and $\theta/|1-\varphi(\theta)|$ is bounded on $(0,1]$.

A change of variables, $\tau = \theta x$ in the first integral and $\tau = \theta px$ in the second, yields

$$D_0 = \int_0^B \mathrm{Re}(\exp(-i\tau)\left(\frac{\tau/x}{1-\varphi(\tau/x)} - \frac{\tau/px}{1-\varphi(\tau/px)}\right)\frac{d\tau}{\tau} = D + o(L(x)) \ .$$

Write

$$(1-\varphi(\theta))^{-1} = \rho(\theta) + i\sigma(\theta) \ .$$

Then

$$\rho(1/x) \sim \frac{\pi}{2}\, x^2(1-F(x))/(m(x))^2$$

since $1-F \in R.V._{-1}^{\infty}$ (cf. Appendix) and this implies $\rho(1/x)\cdot 1/x \sim L(x)$ $(x \to \infty)$. For any slowly varying function $L(x)$ we have

$$\lim_{x\uparrow\infty} \frac{1}{L(x)} \int_0^B \psi(\tau)\left(L(\tfrac{x}{\tau}) - L(\tfrac{px}{\tau})\right)\frac{d\tau}{\tau} =$$

$$= \lim_{x\uparrow\infty} \frac{\psi(0)}{L(x)} \int_0^B \left(L(\tfrac{x}{\tau}) - L(\tfrac{px}{\tau})\right)\frac{d\tau}{\tau} +$$

$$- \lim_{x\to\infty} \frac{1}{L(x)} \int_0^B \left(\frac{\psi(0)-\psi(\tau)}{\tau}\right)\left(L(\tfrac{x}{\tau}) - L(\tfrac{px}{\tau})\right) d\tau =$$

$$= \psi(0)\ \ln(p) \qquad \text{if} \quad \sup_{\tau\in[0,B]} \left|\frac{\psi(0)-\psi(\tau)}{\tau}\right| < \infty \ .$$

Thus we obtain

$$\lim_{x\to\infty} \frac{1}{L(x)} \int_0^B \cos(\tau)\left(\frac{\tau}{x}\,\rho(\tfrac{\tau}{x}) - \frac{\tau}{px}\,\rho(\tfrac{\tau}{px})\right)\frac{d\tau}{\tau} = \ln(p) \ .$$

The second part of D_0 yields almost the same limit but the proof is different.

Note that (cf. Appendix)

$$\lim_{x\uparrow\infty} \frac{\frac{1}{x}\,\sigma(1/x) - \frac{1}{x\ \mathrm{Im}(\varphi(1/x))}}{-\,x^2(1-F(x))^2(m(x))^{-3}} =$$

$$= \lim_{x\uparrow\infty} \frac{\frac{x\ \mathrm{Im}(\varphi(1/x))}{x^2/|1-\varphi(1/x)|^2} - \frac{1}{x\ \mathrm{Im}(\varphi(1/x))}}{-\,x^2(1-F(x))^2(m(x))^{-3}} =$$

$$= \lim_{x \uparrow \infty} \frac{(x^2|1-\varphi(1/x)|^2)^{-1} - (x \text{ Im } \varphi(1/x))^{-1}}{-x^2(1-F(x))^2(m(x))^{-4}} = 1$$

and this implies $\frac{1}{x}\sigma(1/x)$ is a Π-varying function with auxiliary function $x(1-F(x))(m(x))^{-2}$. Hence, since $\frac{\sin(\tau)}{\tau}$ is bounded, we obtain for $p \neq 1$

$$\lim_{x \uparrow \infty} \frac{1}{L(x)} \int_0^B \left(\frac{\sin(\tau)}{\tau}\right) \left(\frac{\tau}{x}\sigma(\frac{\tau}{x}) - \frac{\tau}{px}\sigma(\frac{\tau}{px})\right) d\tau =$$

$$= \frac{2}{\pi} \int_0^B \frac{\sin(\tau)}{\tau} d\tau \cdot \ln(p) .$$

(Remember: $L(x) := \frac{\pi}{2} x(1-F(x))(m(x))^{-2}$.)

For $B \uparrow \infty$ the right-hand side converges to $\ln(p)$ which also was the case for the first part of D_0.

In order to complete the proof it suffices to show that for some $M > 0$

$$\left| \int_{B/x}^{x^4} \hat{k}(x^{-4}\theta)\, g(h\theta) \text{ Re}\left(\frac{e^{-ix\theta}}{1-\varphi(\theta)}\right) d\theta \right| \leq M \cdot L(x)/B$$

for $x \geq x_B$ and B fixed.

This can be done in a similar way (by splitting the integrals over the intervals $[B/x,\varepsilon]$ and $[\varepsilon,x^4]$) as in the lattice case and so we will omit the proof of it. □

Finally, the main result follows, which is an extension of Theorem 3.3.3 ($\frac{1}{2} < \alpha < 1$) to the case $\alpha = 1$.

<u>THEOREM 3.3.19</u>. *Suppose* F *is a strong nonlattice probability distribution on* $(0,\infty)$. *Then the following result holds*

$$1-F(t) \in R.V._{-1}^{\infty} \Leftrightarrow -(U(x+h)-U(x)) \in \Pi^{\infty} \quad \forall h>0 .$$

Either relation implies

$$\lim_{t \uparrow \infty} \frac{U(t+h) - U(t) - h/m(t)}{t(1-F(t))(m(t))^{-2}} = 0 .$$

PROOF. Let F be strongly nonlattice with -1-varying tail. Then as in Theorem 3.3.18 one can prove

$$\lim_{x\uparrow\infty} \frac{V(x,h+O(x^{-3}),x^{-4}) - V(x,h,x^{-4})}{L(x)} = 0$$

and this implies by Theorems 3.3.16 and 3.3.18 and $x^{-\alpha} = o(L(x))$ for every $\alpha > 0$ (cf. Appendix) that

$$\lim_{x\uparrow\infty} \frac{U(x,h) - U(xp,h)}{L(x)} = h\ \ln(p) \quad \forall h>0\ \forall p>0$$

where $L(x)$ equals $x(1-F(x))(m(x))^{-2}$.
Thus

$$-\ (U(x+h) - U(x)) \in \Pi^{\infty}\ .$$

In order to prove the other implication we note that $-\ (U(x+h) - U(x)) \in \Pi^{\infty}$ yields $-\frac{U(x)}{x} \in \Pi^{\infty}$ (cf. proof of Theorem 3.3.3) and hence by Theorem 3.3.9 $1-F(t) \in R.V._{-1}^{\infty}$.
For the proof of the limit result we can easily adapt the proof of the same result in the lattice case (cf. Theorem 2.2.8) and so we will omit it. □

CHAPTER 4. REGENERATIVE PROCESSES

0. Introduction

In this chapter we introduce the notion of a regenerative process and prove for these processes some first order limit results for the convergence of the probability distribution of the process at time t to its stationary distribution.

1. The behaviour of a regenerative process in case its distribution of the time between regeneration points has finite mean

Let (Ω, F, P) be a probability space and consider a set of random variables $\{X_t: t \in T\}$ where the index set T denotes either $\mathbb{N}$ or $\mathbb{R}^+$ and $X_t: \Omega \to E$ for every $t \in T$. Assume E is a topological space equipped with some topology E and $B(E)$ is the corresponding Borel σ-algebra.

DEFINITION 4.1.1. (cf. [ÇIN]) The stochastic process $X = \{\underline{X}_t: t \in T\}$ is called *regenerative* if there exists a sequence of stopping times $\underline{S}_0, \underline{S}_1, \ldots$ such that

(i) $S = \{\underline{S}_n: n \in \mathbb{N}\}$ is a renewal process.

(ii) For any $m,n \in \mathbb{N}$, $t_1,\ldots,t_n \in T$

$$\mathbb{P}\{\underline{X}_{\underline{S}_m+t_1},\ldots,\underline{X}_{\underline{S}_m+t_n} \in A \mid \underline{X}_u: u \leq \underline{S}_m\} =$$

$$= \mathbb{P}\{\underline{X}_{t_1},\ldots,\underline{X}_{t_n} \in A\} \tag{1}$$

where A belongs to the product σ-algebra $\Pi_1^n B(E)$.

In order to derive limit theorems for regenerative processes we need the following well-known result.

THEOREM 4.1.2. *Let* g *be a Borel-measurable function* $(-\infty,+\infty) \to \mathbb{R}$, *vanishing for* $x < 0$ *and bounded on finite intervals, and* F *some probability distribution on* $[0,\infty)$ *with* $F(0) < 1$. *Then* $Z = g * U$, *where*

$$(g * U)(t) := \int_{-\infty}^{+\infty} g(t-y)U(dy)$$

and U *denotes the renewal measure* $\sum_0^\infty \nu_F^{n*}$, *satisfies the renewal equation* $H = g + H * F$ *and* Z *is the only solution of the renewal equation vanishing for* $t < 0$ *and bounded on finite intervals.*

PROOF. Since it is very easy to check that Z satisfies the desired properties we will only prove the uniqueness.
Suppose f_1, f_2 are two solutions of the renewal equation with the above properties. Then $h := f_1 - f_2$ vanishes for $t < 0$, is bounded on finite intervals and satisfies

$$h = h * F \; . \tag{2}$$

This implies $h = h * F^{n*}$ for every $n \in \mathbb{N}$ (iterate (2) n times) and hence

$$|h(t)| \leq \sup_{0 \leq x \leq t} |h(x)| F^{n*}(t) \qquad \text{for every } t > 0 \; .$$

We are now finished once we have proved that $F^{n*}(t) \to 0$ $(n \to \infty)$ for every $t > 0$. This can be seen as follows.
Since the distribution function F is by definition right-continuous and $F(0) < 1$, there exists some $b > 0$ such that $F(b) < 1$. Then

$$\begin{aligned} F^{k*}(t) &\leq F^{k*}(kb) = 1 - \mathbb{P}\{\underline{S}_k > kb\} \leq \\ &\leq 1 - \mathbb{P}\{\underline{X}_1 > b, \ldots, \underline{X}_k > b\} = 1 - (1 - F(b))^k < 1 \end{aligned} \tag{3}$$

where $k \in \mathbb{N}$ is chosen in such a way that $kb \geq t$.
Finally, using $F^{(nk)*}(t) \leq ((F^{k*})(t))^n$ and applying (3) the desired result follows. □

Before stating the next result we introduce for the remainder of this chapter the following notations.

The stochastic process $X = \{\underline{X}_t : t \in T\}$ denotes a regenerative process and F the probability distribution associated with the underlying renewal process $S = \{\underline{S}_n : n \in \mathbb{N}\}$. For this distribution

$$F(0^+) = 0 \quad \text{and} \quad E := \int_0^\infty x F(dx) < \infty .$$

Sometimes we will also use the corresponding probability measure ν_F instead of F. Moreover,

$$K_A(t) := \mathbb{P}\{\underline{X}_t \in A, \underline{S}_1 > t\} \qquad \text{for every } t \geq 0 ,$$

$$P_t(A) := \mathbb{P}\{\underline{X}_t \in A\} \quad \text{and} \quad P_\infty(A) := \int_0^\infty K_A(z)dz .$$

Observe

$$\int_0^\infty K_A(t)dt = \int_0^\infty \mathbb{E}(\underline{I}_A(t))dt = \mathbb{E}\left(\int_0^\infty \underline{I}_A(t)dt\right) = \mathbb{E}\left(\int_0^{\underline{S}_1} 1_{\{\underline{X}_t \in A\}} dt\right)$$

where

$$\underline{I}_A(t) := \begin{cases} 1 & \text{if } \underline{X}_t \in A \text{ and } \underline{S}_1 > t \\ 0 & \text{otherwise.} \end{cases}$$

Hence $P_\infty(A)$ equals the expected amount of time the process $X = \{\underline{X}_t : t \in T\}$ stays in A during the first cycle.

Finally, U denotes the renewal measure $\sum_{n=0}^\infty \nu_F^n$ and $f: E \to \mathbb{R}$ some Borel-measurable function.

THEOREM 4.1.3. *Suppose* $\mathbb{E}(|f(\underline{X}_t)|) < \infty$ *for every* $t \in T$. *Then*

$$\mathbb{E}(f(\underline{X}_t)) = \int_0^t \mathbb{E}(f(X_{t-y})1_{\{\underline{S}_1 > t-y\}})U(dy) . \tag{4}$$

PROOF. By standard arguments from measure theory (cf. [BIL], [LAH]) we obtain

$$\mathbb{E}(f(\underline{X}_{t+\underline{S}_1}) \mid \underline{S}_1) = \mathbb{E}(f(\underline{X}_{t+\underline{S}_1}) \mid \underline{X}_u, u \leq \underline{S}_1) = \mathbb{E}(f(\underline{X}_t)) . \tag{5}$$

Hence for every $t \geq 0$

$$\mathbb{E}(f(\underline{X}_t)) = \mathbb{E}(f(\underline{X}_t)1_{\{\underline{S}_1 > t\}}) + \mathbb{E}(f(\underline{X}_t)1_{\{\underline{S}_1 \leq t\}}) =$$

$$= \mathbb{E}(f(\underline{X}_t)1_{\{\underline{S}_1 > t\}}) + \int_0^t \mathbb{E}(f(\underline{X}_t) \mid \underline{S}_1 = y)F(dy) =$$

$$= \mathbb{E}(f(\underline{X}_t)1_{\{\underline{S}_1>t\}}) + \int_0^t \mathbb{E}(f(\underline{X}_{t-y}))F(dy) \quad . \tag{6}$$

Since

$$p(t) := \mathbb{E}(f(\underline{X}_t)1_{\{\underline{S}_1>t\}})$$

is a Borel-measurable function, bounded on finite intervals, the desired result follows from (6) and Theorem 4.1.2. □

LEMMA 4.1.4.

$$P_t(A) = \int_0^t K_A(t-y)U(dy) \qquad \textit{for every } A \in \mathcal{B}(E) \textit{ and } t \in T \; .$$

PROOF. Apply Theorem 4.1.3 with $f = 1_A$. □

By Theorem 4.1.3 and Lemma 4.1.4 it is easy to see that we can obtain limit results for $X = \{\underline{X}_t : t \in T\}$ by decomposing the renewal measure U (see Chapters 2 and 3).

Since this decomposition is most difficult for $T = \mathbb{R}^+$ we will only prove results for this case. As the reader can easily verify, similar results hold for $T = \mathbb{N}$.

THEOREM 4.1.5. (cf. [MIL]) *Let* $E = \mathbb{R}$ *and* $D[0,\infty)$ *the set of real-valued functions on* $[0,\infty)$ *which are right-continuous and for which left-hand limits exist.*

If either

(i) *the sample paths of the process* $X = \{\underline{X}_t : t \geq 0\}$ *belong to* $D[0,\infty)$

or

(ii) $\nu_F^{n_0}$ *is nonsingular for some* $n_0 \in \mathbb{N}$

then $P_t \overset{w}{\Rightarrow} P_\infty$, *where* $\overset{w}{\Rightarrow}$ *denotes weak convergence (cf.* [LAH]*).*

THEOREM 4.1.6. *Suppose*

(i) $\nu_F^{n_0}$ *is nonsingular for some* $n_0 \in \mathbb{N}$.

(ii) $\int_0^\infty \psi(x)\nu_m(dx) < \infty$, *where* $\psi \in \Psi$, ψ *nondecreasing and* $\lim_{x\to\infty} \frac{\ln \psi(x)}{x} = 0$.

Then

$$\psi(t)\|P_t - P_\infty\| \to 0 \qquad (t \to \infty) ,$$

where $\| \ \|$ *denotes the total variation norm.*

<u>PROOF</u>. By a similar reasoning as in Theorem 3.1.6 we obtain $\nu_F \in S(\psi)$ and hence $\nu_E \in S(\psi)$ (for definition ν_E, see p. 116). Furthermore, by Lemma 3.1.4, ν_E^{-1} exists and $\nu_E^{-1} \in S(\psi)$. Hence we can decompose the renewal measure (Lemma 3.1.2) and so by Lemma 4.1.4

$$P_t(A) - P_\infty(A) = -\frac{1}{E}\int_t^\infty K_A(z)dz + \int_0^t K_A(t-y)\nu_E^{-1}(dy) + - \frac{1}{E}\int_0^t K_A(y)\nu_E^{-1}([t-y,\infty))dy \tag{7}$$

for every $A \in \mathcal{B}(E)$ and $t \geq 0$.
Observe $K_A(t) \leq \nu_F([t,\infty))$ for every $A \in \mathcal{B}(E)$ and $t \geq 0$ and this implies using (7)

$$\|P_t - P_\infty\| \leq \frac{\nu_m([t,\infty))}{E} + \int_0^t \nu_F([t-y,\infty))|\nu_E^{-1}|(dy) + + \frac{1}{E}\int_0^t \nu_F([y,\infty))|\nu_E^{-1}|([t-y,\infty))dy . \tag{8}$$

Rewriting this upper bound yields

$$\|P_t - P_\infty\| \leq \frac{\nu_m([t,\infty))}{E} + (\nu_F * |\nu_E^{-1}|)([t,\infty)) + + \frac{1}{E}(|\nu_E^{-1}| * \nu_m)([t,\infty)) . \tag{9}$$

Since $S(\psi)$ is closed under convolutions (cf. [HIL]) it follows that the upper bound in (9) belongs to $S(\psi)$ and hence

$$\lim_{t\to\infty} \psi(t) \left[\frac{\nu_m([t,\infty))}{E} + (\nu_F * |\nu_E^{-1}|)([t,\infty)) + \right.$$

$$\left. + \frac{1}{E} (|\nu_E^{-1}| * \nu_m)([t,\infty))\right] = 0 \ .$$

Substituting this in (9) finally yields the desired result. □

REMARK 4.1.7. By taking $\psi(x) \equiv 1$ in Theorem 4.1.6 we can easily deduce case (ii) of Theorem 4.1.5. (Note that the result in Theorem 4.1.6 is actually stronger than the one stated in Theorem 4.1.5.)

A closely related result is given in the next theorem.

THEOREM 4.1.8. *Suppose*

i) $\nu_F^{n_0}$ *is nonsingular for some* $n_0 \in \mathbb{N}$.

ii) $\nu_m \in ST(\psi_0,\mu)$, *where* $\mu \in SMT$ *and* $\lim_{t\to\infty} \frac{\ln \mu([t,\infty))}{t} = 0$.

Then

$$\sup_{t\geq 0} \frac{\|P_t - P_\infty\|}{\mu([t,\infty))} < \infty \ .$$

REMARK 4.1.9. Since any regular standard irreducible Markov process (cf. [CHU]) and any regular irreducible semi-Markov process (cf. [ÇIN]) is a regenerative process with sample paths in $D([0,\infty))$, the results of theorems 4.1.6 and 4.1.8 immediately carry over to these processes. Hence we have extended the results proved for uniform Markov processes in Chapter 2.

APPENDIX

A.1. Functions of bounded increase and related concepts

In this section every function is defined on $[0,\infty)$, Lebesgue measurable and positive (unless stated otherwise). Also we relate to every function τ the function D_τ with

$$D_\tau(a) = \begin{cases} \overline{\lim\limits_{t\to\infty}} \sup\limits_{a\leq x\leq 1} \dfrac{\tau(t)}{\tau(tx)}, & 0 < a \leq 1, \\[2ex] \overline{\lim\limits_{t\to\infty}} \sup\limits_{1\leq x\leq a} \dfrac{\tau(t)}{\tau(tx)}, & a > 1. \end{cases}$$

THEOREM A.1.1. *The following statements are equivalent*

a) *For some* $a > 1$ $D_\tau(a) < \infty$.

b) *There exist positive constants* M, c, t_0 *such that for all* $y \geq x \geq t_0$

$$\frac{\tau(x)}{\tau(y)} \leq M\left(\frac{y}{x}\right)^c .$$

c) *For every* $a > 1$ $D_\tau(a) < \infty$.

If $D_\tau(a) < \infty$ *for some* $a > 1$, *then* b) *holds for any* $c > \dfrac{\ln(D_\tau(a))}{\ln(a)}$.

PROOF. We only give a proof of a) $\to$ b) since b) $\to$ c) and c) $\to$ a) are obvious.

Suppose for some $a > 1$ $D_\tau(a) < \infty$. By the definition of $D_\tau(a)$ there exists for every $\varepsilon > 0$ a constant $t_0 = t_0(\varepsilon)$ such that

$$\sup_{1\leq x\leq a} \frac{\tau(t)}{\tau(tx)} \leq D_\tau(a) + \varepsilon \qquad \text{for all } t \geq t_0 .$$

Let $y \geq x \geq t_0$ and choose p in such a way that $xa^p \leq y \leq xa^{p+1}$. Then for $n = 0,1,\ldots,p-1$

$$\frac{\tau(xa^n)}{\tau(xa^{n+1})} \leq D_\tau(a) + \varepsilon$$

and

$$\frac{\tau(xa^p)}{\tau(y)} \leqq D_\tau(a) + \varepsilon .$$

Hence

$$\frac{\tau(x)}{\tau(y)} \leqq (D_\tau(a) + \varepsilon)^{p+1} . \qquad (1)$$

Notice that

$$p \leqq \frac{\ln(x/y)}{\ln(a)}$$

and so

$$\ln((D_\tau(a) + \varepsilon)^p) \leqq c \ln(x/y)$$

where c equals $\dfrac{\ln(D_\tau(a) + \varepsilon)}{\ln(a)} > 0$. Now by (1) the desired result follows. □

LEMMA A.1.2. *Suppose* $D_\tau(a) < \infty$ *for some* $a > 1$. *Then there exist some positive constants* c, M, t_0 *such that for all* $y \geqq t_0$ $\tau(y) \geqq My^{-c}$.

PROOF. Take in Theorem A.1.1: $t = t_0$. □

THEOREM A.1.3. *The following statements are equivalent:*

a) *There exists some constant* $0 < a < 1$ *for which* $D_\tau(a) < \infty$.

b) *There exist positive constants* M, c, t_0 *such that for all* $y \geqq x \geqq t_0$

$$\frac{\tau(x)}{\tau(y)} \leqq M\left(\frac{y}{x}\right)^{-c} .$$

c) *For every* $0 < a < 1$ $D_\tau(a) < \infty$.

If $D_\tau(a) < \infty$ *for some* $0 < a < 1$ *then* b) *holds for any* $c > \dfrac{\ln(D_\tau(a))}{\ln(a)}$.

PROOF. Apply Theorem A.1.1 with τ replaced by $1/\tau$. □

REMARK A.1.4.

a) If $\{\tau(n)\}_{n=0}^{\infty}$ is a sequence of positive numbers satisfying

$$\overline{\lim_{n\to\infty}} \sup_{1\leqq x\leqq a} \frac{\tau(n)}{\tau([nx])} < \infty \qquad \text{for some } a > 1 ,$$

then it is easily verified that there exists some $1 < b \leqq a$ such that

$D_f(b) < \infty$ where $f(t) = \tau([t])$. The same applies to $0 < a < 1$ and so every result for functions also holds for sequences.

b) If τ is a nondecreasing function satisfying $D_\tau(a) < \infty$ for some $0 < a < 1$ then τ is called a function of *bounded increase* ($\tau \in$ B.I.). For such a function it is easy to prove that the conditions in Theorem A.1.3 are equivalent with

$$\lim_{x\to\infty} \frac{\ln \overline{\lim\limits_{t\to\infty}} \frac{\tau(xt)}{\tau(t)}}{\ln(x)} < \infty .$$

c) If τ is a function such that $1/\tau$ is a function of *bounded increase* then τ is called a function of *bounded decrease*.

DEFINITION A.1.5. For every function $\tau \in$ B.I. we call

$$\lim_{x\to\infty} \frac{\ln \overline{\lim\limits_{t\to\infty}} \frac{\tau(xt)}{\tau(t)}}{\ln(x)}$$

the *upper index* of τ. (Notation: $\overline{\text{index}}\ \tau$.)

An important subclass of the set of functions for which $D_\tau(a) < \infty$ for all $a > 0$ is the set of functions of *regular variation*.

DEFINITION A.1.6.

a) A function $\tau: \mathbb{R}^+ \to \mathbb{R}$ which is eventually positive is called *regularly varying* (at infinity) if

$$\lim_{t\to\infty} \frac{\tau(tx)}{\tau(t)} = x^\rho$$

for some $\rho \in \mathbb{R}$ and all $x > 0$.
Notation: $\tau \in \text{R.V.}_\rho^\infty$.

b) A sequence $\tau: \mathbb{N} \to \mathbb{R}^+$ is called *regularly varying* (at infinity) if

$$\lim_{n\to\infty} \frac{\tau([nx])}{\tau(n)} = x^\rho$$

for some $\rho \in \mathbb{R}$ and all $x > 0$.
Notation: $\tau \in \text{R.V.S.}_\rho^\infty$.

Without proof we mention the following results. For a proof of these results the reader is referred to [GEL] or [HAA-1].

THEOREM A.1.7. (*Uniform convergence theorem*) *If* $\tau \in R.V._{\rho}^{\infty}$ *then the limit relation* $\lim_{t\to\infty} \frac{\tau(tx)}{\tau(t)} = x^{\rho}$ *holds uniformly for* $x \in [a,b]$ *with* $0 < a < b < \infty$.

THEOREM A.1.8. (*Karamata's theorem*) *Suppose* $\tau \in R.V._{\rho}^{\infty}$. *Then the following results hold:*

a) *There exists some* $t_0 > 0$ *such that* $\tau(t)$ *is positive and locally bounded for* $t \geqq t_0$.

b) *If* $\rho \geqq -1$ *then*

$$\lim_{t\to\infty} \frac{t\tau(t)}{\int_{t_0}^{t} \tau(s)dx} = \rho + 1 . \tag{2}$$

c) *If* $\alpha < -1$, *or* $\alpha = -1$ *and* ${}_0\int^{\infty} \tau(s)ds < \infty$ *then*

$$\lim_{t\to\infty} \frac{t\tau(t)}{\int_{t}^{\infty} \tau(s)ds} = -\rho - 1 . \tag{3}$$

d) *Conversely: if* (2) *holds with* $-1 < \rho < \infty$, *then* $\tau \in R.V._{\rho}^{\infty}$; *if* (3) *holds with* $-\infty < \rho < -1$, *then* $\tau \in R.V._{\rho}^{\infty}$.

From these last two theorems the following results can easily be deduced.

THEOREM A.1.9. (cf. [GEL])

a) *If* $\tau \in R.V._{\rho}^{\infty}$, *there exist some measurable functions* $a: [0,\infty) \to \mathbb{R}$ *and* $c: [0,\infty) \to \mathbb{R}$ *with*

$$\lim_{t\to\infty} c(t) = c_0 \quad (0 < c_0 < \infty) \quad \textit{and} \quad \lim_{t\to\infty} a(t) = \rho \tag{4}$$

and $t_0 > 0$ *such that for all* $t > t_0$

$$\tau(t) = c(t) \exp\left(\int_{t_0}^{t} \frac{a(s)}{s} ds \right) . \tag{5}$$

Conversely, if (5) *holds with* a *and* c *satisfying* (4), *then* $\tau \in R.V._\rho^\infty$.

b) *If* $\tau \in R.V._\rho^\infty$ *then*

$$\lim_{t\to\infty} \frac{\ln(\tau(t))}{\ln(t)} = \rho \ .$$

This implies

$$\lim_{t\to\infty} \tau(t) = \begin{cases} 0 & \text{if } \rho < 0 \\ \infty & \text{if } \rho > 0 \ . \end{cases}$$

c) *Suppose* $\tau \in R.V._\rho^\infty$. *Then for every* $\varepsilon, \delta > 0$ *there exists a constant* $t_0 = t_0(\varepsilon,\delta)$ *such that for* $t \geqq t_0$ *and* $x \geqq 1$

$$(1-\varepsilon)x^{\rho-\delta} \leqq \frac{\tau(tx)}{\tau(t)} \leqq (1+\varepsilon)x^{\rho+\delta} \ .$$

d) *If* $\tau \in R.V._\rho^\infty$, $\rho \leqq 0$, *is bounded on finite intervals of* $[0,\infty)$ *and* $\delta,\lambda > 0$ *arbitrary, then there exist some constants* $c > 0$ *and* $t_0 = t_0(\delta,\lambda)$ *such that*

$$\frac{\tau(tx)}{\tau(t)} \leqq cx^{\rho-\delta}$$

for all $t \geqq t_0$ *and* $0 < x \leqq \lambda$.

e) *If* $\tau \in R.V._\rho^\infty$ *(*$\rho > 0$*) and* τ *is bounded on finite intervals, then the limit relation* $\lim_{t\to\infty} \frac{\tau(tx)}{\tau(t)} = x^\rho$ *holds uniformly on* $(0,b]$ *with* $b < \infty$. *For* $\tau \in R.V._\rho^\infty$ *(*$\rho < 0$*) the uniformity of the limit relation holds on* $[a,\infty)$ *with* $a > 0$. (*)

f) *If* $\tau \in R.V._\rho^\infty$ *(*$\rho \geqq 0$*) and* $\tau(t) = \tau(t_0) + \int_{t_0}^{t} g(s)ds$ *for* $t \geqq t_0$ *and* $g(s)$ *is monotone, then*

$$\lim_{t\to\infty} \frac{tg(t)}{\tau(t)} = \rho \ .$$

g) *If* $\tau \in R.V._\rho^\infty$ *(*$\rho \leqq 0$*) and* $\tau(t) = \int_t^\infty g(s)ds < \infty$ *with* $g(s)$ *nonincreasing, then*

$$\lim_{t\to\infty} \frac{tg(t)}{\tau(t)} = -\rho \ .$$

(*) Note that e) is an extension of the uniform convergence theorem.

h) *If* $\{\tau(n)\}_{n=0}^{\infty}$ *is a regularly varying sequence (with index* ρ*), then* $f \in R.V._{\rho}^{\infty}$ *where* $f(t) := \tau([t])$.

An important subclass of the slowly varying functions (i.e. $R.V._0^{\infty}$) is given in the next definition.

DEFINITION A.1.10. (cf. [GEL])

(i) A Lebesgue measurable function τ: $[0,\infty) \to \mathbb{R}$ is said to belong to the class Π^{∞} if there exists a positive measurable function L such that

$$\lim_{t\to\infty} \frac{\tau(tx) - \tau(t)}{L(t)} = \ln(x) \qquad \text{for all } x > 0 \, .$$

(ii) A sequence of positive numbers $\{\tau(n)\}_{n=0}^{\infty}$ is said to belong to the class $\Pi.S.^{\infty}$ if there exists a sequence $\{L(n)\}_{n=0}^{\infty}$ such that

a) $L(n) \in R.V.S._0^{\infty}$,

b) $$\lim_{n\to\infty} \frac{\tau([nx]) - \tau(n)}{L(n)} = \ln(x) \qquad \text{for all } x > 0 \, .$$

Without proof we mention the following results for the class Π^{∞}. For a proof the reader is referred to [GEL] or [HAA-1].

THEOREM A.1.11. (*Uniform convergence theorem*) *If* $\tau \in \Pi^{\infty}$ *then the limit relation* $\lim_{t\to\infty} \frac{\tau(tx) - \tau(t)}{L(t)} = \ln(x)$ *holds uniformly for* $x \in [a,b]$ *with* $0 < a < b < \infty$.

THEOREM A.1.12.

a) *If* $\tau \in \Pi^{\infty}$, *then for every* $\varepsilon > 0$ *there exist some constants* $t_0 = t_0(\varepsilon)$ *and* $c > 0$ *such that*

$$\left|\frac{\tau(tx) - \tau(t)}{L(t)}\right| \leq cx^{\varepsilon}$$

for every $t \geq t_0$ *and* $x \geq 1$.

b) $\tau \in \Pi^{\infty}$ *if and only if there exists a slowly varying function* ρ *and a constant* c *such that*

$$\tau(t) = c + \rho(t) + \int_1^t \rho(x)dx \ .$$

c) *If the sequence* $\{\tau(n)\}_{n=0}^{\infty}$ *belongs to* $\Pi.S.^{\infty}$ *then the function* $\tau(t) := \tau([t])$ *belongs to* Π^{∞}.

Originally, the class of regularly varying functions was introduced by Karamata for use as a suitable condition for Abelian and Tauberian theorems for Laplace transforms.

A slight extension of his famous Abel-Tauber result is given in the next theorem.

THEOREM A.1.13. (cf. [GEL]) *Suppose* $f\colon [0,\infty) \to \mathbb{R}$ *is a Lebesgue measurable function and*

$$\int_0^{\infty} \exp(-\lambda x)|f(x)|dx < \infty \qquad \textit{for all } \lambda > 0 \ .$$

Then the following holds for all $\rho \geq 0$*:*

a) $f(t) \in R.V._{\rho}^{\infty}$ *implies* $1/t\ \hat{f}(1/t) \in R.V._{\rho}^{\infty}$ *with*

$$\hat{f}(s) := \int_0^{\infty} \exp(-sx)f(x)dx \ .$$

Moreover

$$\lim_{t\to\infty} \frac{1/t\ \hat{f}(1/t)}{f(t)} = \Gamma(1+\rho) \ .$$

b) *Conversely, if* $x^{\beta} f(x)$ *is nondecreasing for some* $0 \leq \beta < 1$*, then* $1/t\ \hat{f}(1/t) \in R.V._{\rho}^{\infty}$ *implies* $f(t) \in R.V._{\rho}^{\infty}$.

For the class Π^{∞} a stronger result can be proved.

THEOREM A.1.14. (cf. [GEL]) *Suppose* f *satisfies the same conditions as in Theorem A.1.13. Then the following holds:*

a) $f(t) \in \Pi^{\infty}$ *implies* $1/t\ \hat{f}(1/t) \in \Pi^{\infty}$.

Moreover,

$$\lim_{t\to\infty} \frac{f(t) - 1/t\ \hat{f}(1/t)}{L(t)} = \gamma$$

with γ *Euler's constant and* L(t) *the auxiliary function of* f(t).

b) *Conversely, if* f *is nondecreasing, then* $1/t\ \hat{f}(1/t) \in \Pi^{\infty}$ *implies* $f(t) \in \Pi^{\infty}$.

Finally, we discuss in this section the behaviour of the characteristic function

$$\varphi(\theta) := \int_0^{\infty} \exp(i\theta) dF(x)$$

in the neighbourhood of the origin in case the tail of the distribution function is regularly varying.

THEOREM A.1.15. *The following results hold:*

a) $$1-F \in R.V.^{\infty}_{-\alpha}\ (0<\alpha<1) \Rightarrow \begin{cases} \lim\limits_{\theta\to 0} \dfrac{1 - \operatorname{Re}\varphi(\theta)}{1 - F(1/\theta)} = \Gamma(1-\alpha)\cos(\frac{\pi\alpha}{2}) \\[2ex] \lim\limits_{\theta\to 0} \dfrac{\operatorname{Im}\varphi(\theta)}{1 - F(1/\theta)} = \Gamma(1-\alpha)\sin(\frac{\pi\alpha}{2}) \end{cases}$$

b) $$1-F \in R.V.^{\infty}_{-1} \Rightarrow \begin{cases} \lim\limits_{\theta\to 0} \dfrac{1 - \operatorname{Re}\varphi(\theta)}{1 - F(1/\theta)} = \frac{\pi}{2} \\[2ex] \lim\limits_{\theta\to 0} \dfrac{\operatorname{Im}\varphi(\theta) - \theta m(1/\theta)}{1 - F(1/\theta)} = -\gamma \quad \text{(Euler's constant)} \end{cases}$$

PROOF. Clearly for every $k \in \mathbb{N}$ and $\theta > 0$

$$1-\varphi(\theta) = \int_0^{2k\pi/\theta} (1-\exp(i\theta x))dF(x) + \int_{2k\pi/\theta}^{\infty} (1-\exp(i\theta x))dF(x) \tag{6}$$

and

$$\overline{\lim_{\theta\to 0}} \left| \int_{2k\pi/\theta}^{\infty} \frac{(1-\exp(i\theta x))dF(x)}{1 - F(1/\theta)} \right| \le \lim_{\theta\to 0} \frac{2(1 - F(2k\pi/\theta))}{1 - F(1/\theta)} =$$

$$= 2\cdot(2k\pi)^{-\alpha}\ . \tag{7}$$

If we consider the first integral in (6) then we obtain by Fubini's theorem

$$\int_0^{2k\pi/\theta} (1-\exp(i\theta x))\,dF(x) = -i\int_0^{2k\pi} (1-F(z/\theta))\exp(iz)\,dz$$

and hence by the dominated convergence theorem and Theorem A.1.9, part d),

$$\lim_{\theta\to 0} \frac{\int_0^{2k\pi/\theta} (1-\exp(i\theta x))\,dF(x)}{1-F(1/\theta)} = -i\int_0^{2k\pi} \exp(iz)\,z^{-\alpha}\,dz\ . \tag{8}$$

Letting $k\uparrow\infty$ in (7) and (8) and using [GRA, 3.761] yields for every $0<\alpha<1$

$$\lim_{\theta\to 0}\frac{1-\varphi(\theta)}{1-F(1/\theta)} = -i\int_0^{\infty}\exp(iz)\,z^{-\alpha}\,dz =$$

$$= \Gamma(1-\alpha)\exp\left(-\frac{i\pi\alpha}{2}\right)\ . \tag{9}$$

This implies the result stated in a).
If $1-F\in R.V.^{\infty}_{-1}$ we can prove in a similar way as in a) that

$$\lim_{\theta\to 0}\frac{1-\mathrm{Re}\,\varphi(\theta)}{1-F(1/\theta)} = \int_0^{\infty}\frac{\sin z}{z}\,dz = \frac{\pi}{2}\ .$$

In order to prove the second part of b) we notice that $\mathrm{Im}\,\varphi(\theta) - \theta m(1/\theta)$ equals

$$\int_0^1 (1-F(z/\theta))(\cos(z)-1)\,dz + \int_1^{2k\pi}(1-F(z/\theta))\cos(z)\,dz +$$

$$+\int_{2k\pi/\theta}^{\infty}\sin(z)\,dF(z) \tag{10}$$

Now by Theorem A.1.9, part d), and the dominated convergence theorem

$$\lim_{\theta\to 0}\frac{\int_0^1 (1-F(z/\theta))(\cos(z)-1)\,dz}{1-F(1/\theta)} = \int_0^1 \frac{\cos(z)-1}{z}\,dz \tag{11}$$

and

$$\lim_{\theta\to 0} \frac{\int_1^{2k\pi} (1-F(z/\theta))\cos(z)dz}{1-F(1/\theta)} = \int_1^{2k\pi} \frac{\cos(z)}{z}\, dz \; . \qquad (12)$$

Hence by (10), (11) and (12) (Let $k \uparrow \infty$)

$$\lim_{\theta\to 0} \frac{\operatorname{Im}\varphi(\theta) - \theta m(1/\theta)}{1-F(1/\theta)} = \int_0^1 \frac{\cos(z)-1}{z}\, dz + \int_1^\infty \frac{\cos(z)}{z}\, dz = -\gamma$$

(cf. [GRA]), where γ denotes Euler's constant. □

THEOREM A.1.16. *If* F *is a nonlattice probability distribution function, concentrated on* $[0,\infty)$, *and* $1-F \in R.V.^{\infty}_{-1}$ *then there exist for every* $0 < \varepsilon < 1$ *some constants* $c > 0$, $0 < \delta < 1$, *and* $t_0 = t_0(\varepsilon)$ *such that for all* $t \geq t_0$ *and* $\theta \in [2\pi, \varepsilon t]$

$$\frac{|\operatorname{Re}\varphi(\frac{\theta-\pi}{t}) - \operatorname{Re}\varphi(\frac{\theta}{t})|}{1-F(t)} \leqq M\theta^{\delta} \; .$$

(In case F *is a lattice probability distribution function we have to replace* $t \in \mathbb{R}$ *by* $n \in \mathbb{N}$*).*

PROOF. We only give the proof for the nonlattice case, since the proof of the lattice case is completely similar. Also we denote in this proof every constant (not necessarily equal) by c.
The definition of φ yields for every finite constant p and m (use Fubini's theorem)

$$1 - \operatorname{Re}\varphi(m) = \int_0^p (1-\cos(mx))dF(x) + \int_p^\infty (1-\cos(mx))dF(x) =$$

$$= -(1-F(p))(1-\cos(mp)) + \int_0^{mp} \sin(x)(1-F(\tfrac{x}{m}))dx +$$

$$+ \int_p^\infty (1-\cos(mx))dF(x) \; .$$

Hence (let $p \uparrow \infty$)

$$1 - \operatorname{Re}\varphi(m) = \int_0^\infty \sin(x)(1-F(\tfrac{x}{m}))dx$$

and this implies

$$\mathrm{Re}\,\varphi(\tfrac{\theta-\pi}{t}) - \mathrm{Re}\,\varphi(\tfrac{\theta}{t}) = \int_0^\infty \sin(x)(F(\tfrac{xt}{\theta}) - F(\tfrac{xt}{\theta-\pi}))dx$$

or equivalently

$$\frac{1}{\theta}\,(\mathrm{Re}\,\varphi(\tfrac{\theta-\pi}{t}) - \mathrm{Re}\,\varphi(\tfrac{\theta}{t})) = \int_0^\infty \sin(\theta x)(F(xt) - F(\tfrac{xt\theta}{\theta-\pi}))dx\ .$$

Split the last integral into two parts, the first part

$$I_1(\theta,t,\eta) = \int_0^{\theta^{-\eta}} \sin(\theta x)(F(xt) - F(\tfrac{xt\theta}{\theta-\pi}))dx$$

and the second

$$I_2(\theta,t,\eta) = \int_{\theta^{-\eta}}^\infty \sin(\theta x)(F(xt) - F(\tfrac{xt\theta}{\theta-\pi}))dx$$

where η is an arbitrary chosen number between 0 and $\frac{1}{2}(\sqrt{5}-1)$.
Since

$$I_2(\theta,t,\eta) = \int_{\theta^{-\eta}}^\infty \sin(\theta x)(1-F(\tfrac{xt\theta}{\theta-\pi}))dx - \int_{\theta^{-\eta}}^\infty \sin(\theta x)(1-F(xt))dx$$

and $1-F$ is a positive bounded and nonincreasing function, we can apply Bonnet's form of the second mean value theorem (cf. [KAW, p. 24]) and this yields

$$|I_2(\theta,t,\eta)| \leqq \frac{4}{\theta}\,(1-F(t\theta^{-\eta}))\ . \tag{13}$$

By Theorem A.1.9, part d) we obtain from (13) the existence of some $t_1 = t_1(\eta)$ and some constant c such that for all $t \geqq t_1$ and $\theta \geqq 2\pi$

$$\left|\frac{I_2(\theta,t,\eta)}{1-F(t)}\right| \leqq c\theta^{\eta^2+\eta-1}\ . \tag{14}$$

Consider now $I_1(\theta,t,\eta)$. By applying Fubini's theorem again we get

$$I_1(\theta,t,\eta) = \theta \int_0^{\theta^{-\eta}} \int_p^{\theta^{-\eta}} \left(F(tz(1+\frac{\pi}{\theta-\pi})) - F(tz)\right) dz \cos(\theta p)dp \ . \qquad (15)$$

Since

$$\int_p^{\theta^{-\eta}} \left(1 - F(tz(1+\frac{\pi}{\theta-\pi}))\right) dz = \frac{\theta-\pi}{\theta} \int_{p(1+\frac{\pi}{\theta-\pi})}^{\theta^{-\eta}(1+\frac{\pi}{\theta-\pi})} (1-F(tz))dz$$

we obtain that

$$\int_p^{\theta^{-\eta}} \left(F(tz(1+\frac{\pi}{\theta-\pi})) - F(tz)\right) dz = \frac{\pi}{\theta} \int_{p(1+\frac{\pi}{\theta-\pi})}^{\theta^{-\eta}(1+\frac{\pi}{\theta-\pi})} (1-F(tz))dz +$$

$$+ \int_p^{p(1+\frac{\pi}{\theta-\pi})} (1-F(tz))dz - \int_{\theta^{-\eta}}^{\theta^{-\eta}(1+\frac{\pi}{\theta-\pi})} (1-F(tz))dz \ . \qquad (16)$$

Combining (15) and (16) yields

$$I_1(\theta,t,\eta) = \theta I_{11}(\theta,t,\eta) + \theta I_{12}(\theta,t,\eta) - \theta I_{13}(\theta,t,\eta) \qquad (17)$$

with

$$I_{11}(\theta,t,\eta) := \frac{\pi}{\theta} \int_0^{\theta^{-\eta}} \int_{p(1+\frac{\pi}{\theta-\pi})}^{\theta^{-\eta}(1+\frac{\pi}{\theta-\pi})} (1-F(tz))dz \cos(\theta p)dp \ ,$$

$$I_{12}(\theta,t,\eta) := \int_0^{\theta^{-\eta}} \int_p^{p(1+\frac{\pi}{\theta-\pi})} (1-F(tz))dz \cos(\theta p)dp \ ,$$

$$I_{13}(\theta,t,\eta) := \int_0^{\theta^{-\eta}} \int_{\theta^{-\eta}}^{\theta^{-\eta}(1+\frac{\pi}{\theta-\pi})} (1-F(tz))dz \cos(\theta p)dp \ .$$

Since $t^{1+\eta}(1-F(t))$ is locally bounded on $[0,\infty)$ we can apply Theorem A.1.9, part d) and so there exist some constants c and $t_2 = t_2(\eta)$ such that

$$\frac{1-F(t\lambda)}{1-F(t)} \leq c\lambda^{-(1+\eta)} \qquad \text{for every } t \geq t_2 \text{ and } \lambda \leq 2 \ . \qquad (18)$$

Hence by (18) for every $t \geq t_2$ and $\theta \geq 2\pi$

$$\frac{|I_{11}(\theta,t,\eta)|}{1 - F(t)} \leq \frac{c}{\theta} \int_0^{\theta^{-\eta}} \int_{p(1+\frac{\pi}{\theta-\pi})}^{\theta^{-\eta}(1+\frac{\pi}{\theta-\pi})} z^{-1-\eta}\, dz\, dp \leq c\theta^{\eta^2-\eta-1} \tag{19}$$

and

$$\frac{|I_{12}(\theta,t,\eta)|}{1 - F(t)} \leq c \int_0^{\theta^{-\eta}} \int_{p}^{p(1+\frac{\pi}{\theta-\pi})} z^{-1-\eta}\, dz\, dp \leq c\theta^{\eta^2-\eta-1} \tag{20}$$

and

$$\frac{|I_{13}(\theta,t,\eta)|}{1 - F(t)} \leq c \int_{\theta^{-\eta}}^{\theta^{-\eta}(1+\frac{\pi}{\theta-\pi})} z^{-1-\eta}\, dz \, \left| \int_0^{\theta^{-\eta}} \cos(\theta p)\, dp \right| \leq$$

$$\leq c\theta^{\eta^2}\left((1+\frac{\pi}{\theta-\pi})^{-\eta} - 1\right)\theta^{-1} = c\theta^{\eta^2-2} \ . \tag{21}$$

Combining (15), (17), (19), (20) and (21) finally yields

$$\frac{|I_1(\theta,t,\eta)|}{1-F(t)} \leq c\theta^{\eta^2-\eta} \qquad \text{for every } t \geq t_2 \text{ and } \theta \geq 2\pi \ . \tag{22}$$

Taking $t_0 = \max(t_1,t_2)$ and $\delta = \max(\eta^2-\eta+1, \eta^2+\eta)$ and using (22) and (14) implies

$$\frac{|\mathrm{Re}\, \varphi(\frac{\theta-\pi}{t}) - \mathrm{Re}\, \varphi(\frac{\theta}{t})|}{1 - F(t)} \leq \frac{\theta|I_1(\theta,t,\eta)|}{1 - F(t)} + \frac{\theta|I_2(\theta,t,\eta)|}{1 - F(t)} \leq$$

$$\leq c\theta^{\eta^2-\eta+1} + c\theta^{\eta^2+\eta} = c\theta^{\delta}$$

for every $t \geq t_0$ and $\theta \geq 2\pi$.

Since $0 < \eta < \frac{1}{2}(\sqrt{5}-1)$, we have $0 < \delta < 1$ and this completes the proof of Theorem A.1.16. □

A.2. On the Fourier transform

In this section of the Appendix we discuss some important properties of the Fourier transform

$$\hat{f}(\theta) := \int_{-\infty}^{+\infty} \exp(ix\theta) f(x) dx$$

of a complex measurable function f belonging to $L^1(\mathbb{R})$ (cf. [RUD-2]).

Note that the term 'Fourier transform' also applies to the mapping $\hat{}$ which takes f to $\hat{f}$.

It is well-known (cf. [RUD-2]) that

(i) If $f \in L^1(\mathbb{R})$ then $\hat{f} \in C_0$ where C_0 denotes the class of continuous functions h on $\mathbb{R}$ for which $\lim_{|x|\uparrow\infty} h(x) = 0$.

(ii) If $f \in L^1(\mathbb{R})$, $\hat{f} \in L^1(\mathbb{R})$ and

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(t) \exp(-ixt) dt$$

then $g \in C_0$ and $g(x) = f(x)$ a.e. (with respect to the Lebesgue measure).

Since we are interested in the behaviour of $\hat{}$ on a subset of $L^1(\mathbb{R})$ we introduce the following class of functions.

DEFINITION A.2.1. (cf. [DYM]) The function $f: \mathbb{R} \to \mathbb{C}$ belongs to the class $C_{\downarrow}^{\infty}(\mathbb{R})$ if

(i) f is an infinitely differentiable function on $\mathbb{R}$, i.e. $f \in C^{\infty}(\mathbb{R})$.

(ii) For every nonnegative integral p and q

$$\lim_{|x|\uparrow\infty} x^p (D^q f)(x) = 0$$

where $D^q f$ denotes the q-th derivative of f.

It is clear that for every $f \in C_{\downarrow}^{\infty}(\mathbb{R})$ also $D^p(x^q f)$ belongs to $C_{\downarrow}^{\infty}(\mathbb{R})$ for every nonnegative integral p and q. Moreover, by partial integration

$$(\widehat{Df})(\theta) = \int_{-\infty}^{+\infty} (Df)(x)\exp(ix\theta)dx = \tag{1}$$

$$= i\theta \int_{-\infty}^{+\infty} f(x)\exp(ix\theta)dx = i\theta\,\hat{f}(\theta) \quad \forall f \in C_{\downarrow}^{\infty}(\mathbb{R}) .$$

On the other hand, by the dominated convergence theorem and the inequality $|\exp(ix) - 1| \leqq |x| \quad \forall x \in \mathbb{R}$

$$(\widehat{xf})(\theta) = \int_{-\infty}^{+\infty} xf(x)\exp(ix\theta)dx =$$

$$= -i \int_{-\infty}^{+\infty} \exp(ix\theta)f(x) \lim_{h\to 0} \left(\frac{\exp(ixh) - 1}{h}\right) dx =$$

$$= -i \lim_{h\to 0} \int_{-\infty}^{+\infty} \exp(ix\theta)f(x)\left(\frac{\exp(ixh) - 1}{h}\right) dx =$$

$$= -i \lim_{h\to 0} \frac{\hat{f}(\theta+h) - \hat{f}(\theta)}{h} =$$

$$= -i\,(D\hat{f})(\theta) \quad \forall f \in C_{\downarrow}^{\infty}(\mathbb{R}) . \tag{2}$$

We are now able to prove the following result.

<u>THEOREM A.2.2.</u>

a) $\hat{}$ *maps* $C_{\downarrow}^{\infty}(\mathbb{R})$ *into* $C_{\downarrow}^{\infty}(\mathbb{R})$.

b) $$\forall f \in C_{\downarrow}^{\infty}(\mathbb{R}) \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\theta)\exp(-ix\theta)d\theta .$$

c) *The mapping* $\hat{}$ *is one-to-one and onto.*

d) *The inverse mapping* $\check{}$: $f \to f$ *of* $\hat{}$ *is given by*

$$\check{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(y)\exp(-iyx)dy .$$

e) *If* $f \in C_{\downarrow}^{\infty}(\mathbb{R})$ *is even then* $2\pi(f * f)$ *is the Fourier transform of* $(\hat{f})^2$.

PROOF. We only give a proof of a), b) and e) since c) and d) follow immediately from a) and b).
In order to prove a) we consider some $f \in C_{\downarrow}^{\infty}(\mathbb{R})$. Now for every nonnegative integral p and q we obtain by (1)

$$\widehat{(D^p(x^q f))}(\theta) = \widehat{(D(D^{p-1}(x^q f)))}(\theta) = i\theta\widehat{(D^{p-1}(x^q f))}(\theta)$$

and so by iteration

$$\widehat{(D^p(x^q f))}(\theta) = (i\theta)^p \widehat{(x^q f)}(\theta) \ .$$

Similarly by (2)

$$\widehat{(x^q f)}(\theta) = \widehat{(x(x^{q-1} f))}(\theta) = - i(D\widehat{(x^{q-1} f)})(\theta)$$

and again by iteration

$$\widehat{(x^q f)}(\theta) = (-i)^q (D^q \hat{f})(\theta) \ .$$

Hence

$$\widehat{(D^p(x^q f))}(\theta) = (-1)^q \, i^{p+q} \, \theta^p (D^q \hat{f})(\theta)$$

and this implies since $D^p(x^q f) \in C_{\downarrow}^{\infty}(\mathbb{R}) \subset L^1(\mathbb{R})$ that

$$\lim_{|\theta|\to\infty} \theta^p (D^q \hat{f})(\theta) = 0 \ . \tag{3}$$

This completes the proof of a).
Since $f, \hat{f} \in C_{\downarrow}^{\infty}(\mathbb{R}) \subset L^1(\mathbb{R})$ we obtain (cf. [RUD-2]) that $g(x) = f(x)$ a.e. where $g \in C_0$ and

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-ix\theta)\, \hat{f}(\theta) d\theta \ .$$

Hence the continuity of f yields $f(x) = g(x)$.
For the proof of e) we notice that the mapping ^ maps $f * f$ to $(\hat{f})^2$ and so by by d)

$$2\pi (f * f)(x) = \int_{-\infty}^{+\infty} (\hat{f}(y))^2 \exp(-ixy) dy \ .$$

Since f is even it follows easily that $\hat{f}$ is also even and hence

$$\int_{-\infty}^{+\infty} (\hat{f}(y))^2 \exp(-ixy)dy = \int_{-\infty}^{+\infty} (\hat{f}(y))^2 \exp(ixy)dy .$$

This implies the desired result. □

Finally we can prove the main result.

THEOREM A.2.3. *There exists a probability density function* k(x) *which satisfies the following conditions:*

a) *Its characteristic function*

$$\hat{k}(\theta) := \int_{-\infty}^{+\infty} \exp(ix\theta)k(x)dx$$

belongs to $C^\infty(\mathbb{R})$ *and is zero for* $|\theta| \geqq 1$.

b) k(x) *is even and belongs to* $C_\downarrow^\infty(\mathbb{R})$.

PROOF. Define

$$f(x) := \begin{cases} 0 & |x| \geqq \frac{1}{2} \\ \exp(-\frac{1}{\frac{1}{4} - x^2}) & |x| < \frac{1}{2} . \end{cases}$$

Then $f \in C^\infty$ with compact support (cf. [SCH]) and f is even. Also $\hat{f}$ is real-valued and even.
Now by Theorem A.2.2, part e), the desired result follows by taking $k(x) = c(\hat{f}(x))^2$, where c is a normalization constant. □

LIST OF SYMBOLS

LIST OF KEYWORDS

J

K

L

M

O

P

Q

R

S

T

U

V

W

Z

REFERENCES

[ABR] ABRAMOWITZ, M., STEGUN, I.A., Handbook of Mathematical Functions, Nat. Bureau of Standards, Washington, 1970.

[ATH] ATHREYA, K.B., NEY, P.E., Branching Processes, Springer Verlag, New York, 1972.

[BRE] BREIMAN, L., Probability, Addison-Wesley, London, 1968.

[CHO] CHOVER, J., NEY, P.E., WAIGNER, S., Functions of Probability Measures, Journal d'Analyse Math. 26 (1973), 255-302.

[CHU] CHUNG, K.L., Markov Chains with Stationary Transition Probabilities, 2nd ed., Springer Verlag, Berlin, 1967.

[ÇIN] ÇINLAR, E., Introduction to Stochastic Processes, Prentice-Hall, New Jersey, 1975.

[COH] COHEN, J.W., The Single Server Queue, North-Holland, Amsterdam, 1969.

[CON] CONWAY, J.B., Functions of one complex variable, Springer Verlag, New York, 1973.

[DOO] DOOB, J.L., Stochastic Processes, Wiley, New York, 1963.

[DUL] DULST, D. van, FRENK, J.B.G., On Banach algebras, subexponential distributions and renewal theory, Report 84-20, Department of Mathematics, University of Amsterdam, 1984.

[DUN] DUNFORD, N., SCHWARTZ, J.T., Linear Operators, Part I: General Theory, Interscience Publishers Inc., New York, 1957.

[DYM] DYM, H., McKEAN, H.P., Fourier Series and Integrals, Academic Press, 1972.

[EMB-1] EMBRECHTS, P., OMEY, E., Functions of Power Series, Yokohama Math. J., 32 (1984), 77-88.

[EMB-2] EMBRECHTS, P., GOLDIE, C.M., On Closure and Factorization Properties of Subexponential and Related Distributions, J. Austral. Math. Soc. (Ser A) 29 (1980), 243-256.

[EMB-3] EMBRECHTS, P., GOLDIE, C.M., On Convolution Tails, Stochastic Processes Appl. 13 (1982), 263-278.

[ERI] ERICKSON, K.B., Strong Renewal Theorems with Infinite Mean, Trans. Amer. Math. Soc. 151 (1970), 263-291.

[FEL-1] FELLER, W., An Introduction to Probability Theory and its Applications, Vol. I, Wiley, New York, 1970.

[FEL-2] FELLER, W., An Introduction to Probability Theory and is Applications, Vol. II, Wiley, New York, 1971.

[FEL-3] FELLER, W., OREY, S., A Renewal Theorem, J. Math. and Mech. 10 (1961), 619-624.

[FREE] FREEDMAN, D., Markov Chains, Holden-Day, San Francisco, 1971.

[FRE] FRENK, J.B.G., The behaviour of the renewal sequence in case the tail of the waiting-time distribution is regularly varying with index -1, Adv. Appl. Prob. 14 (1982), 870-884.

[GAR] GARSIA, A., LAMPERTI. J., A discrete renewal theorem with infinite mean, Comment. Math. Helv. 37 (1962/63), 221-234.

[GEL] GELFAND, T.M., RAIKOV, D.A., SCHILOV, G.E., Kommutative Normierte Algebren, VEB Deutscher Verlag der Wissenschaften, Berlin, 1964.

[GEL] GELUK, J.L., de HAAN, L., Regular Variation, Extensions and Tauberian Theorems, forthcoming monograph.

[GOE] GOEMAERE, E., Reguliere Variatie en Vernieuwingstheorie, unpublished manuscript (in Dutch).

[GRA] GRADSHTEYN, I.S., RYZNIK, I.M., Tables of Integrals, Series and Products, Academic Press, 1980.

[GRÜ] GRÜBEL, R., Über die Geschwindigkeit der Konvergenz beim Erneuerungssatz und dem Hauptgrenzwertsatz für Markoffketten, Ph.D. thesis, Universität Essen, 1979.

[HAA-1] de HAAN, L.F.M., On Regular Variation and its Applications to the Weak Convergence of Sample Extremes, Math. Centre Tract 32, Amsterdam, 1970.

[HAA-2] de HAAN, L.F.M., An Abel-Tauber Theorem for Laplace Transforms, J. London Math. Soc. (2) 13 (1976), 537-542.

[HIL] HILLE, E., PHILLIPS, R.S., Functional Analysis and Semi-Groups, Americam Mathematical Society, Colloquium Publications, Vol. 31, 1981.

[KAN] KANTOROVICH, L.V., AKILOV, G.P., Functional Analysis (second edition), Pergamon Press, Oxford, 1982.

[KAW] KAWATA, T., Fourier Analysis in Probability Theory, Academic Press, New York, 1972.

[KOD] KODAIRA, K., Introduction to Complex Analysis, Cambridge University Press, Cambridge, 1984.

[KOK] de KOK, A.G., Production-inventory control models; approximations and algorithms, Ph.D. thesis, University of Amsterdam, 1985.

[LAH] LAHA, R.G., ROHATGI, V.K., Probability Theory, Wiley, New York, 1979.

[LAM] LAMPERTI, J., Stochastic Processes, Springer Verlag, New York, 1977.

[LUE] LUENBERGER, D.G., Optimization by Vector Space Methods, Wiley, New York, 1969.

[NAI] NAIMARK, M.A., Normed Algebras, Wolters-Noordhoff, Groningen, 1972.

[NEY] NEY, P.E., A refinement of the coupling method in renewal theory, Stochastic Processes and their Applications 11 (1981), 11-26.

[NUM] NUMMELIN, E., TUOMINEN, P., The rate of convergence in Orey's theorem for Harris recurrent Markov chains with applications to renewal theory, Stochastic Processes and their Applications 15 (1983), 295-311.

[MIL] MILLER, P.R., Existence of limits in regenerative processes, Ann. Math. Stat. 43 (1972), 1275-1282.

[RIC] RICKART, C.E., General Theory of Banach Algebras, Van Nostrand, New York, 1960.

[ROG-1] ROGOZIN, B.A., Asymptotics of the coefficients in the Wiener-Lévy theorem on absolutely convergent trigonometric series, Sibirsk. Matem. Zh. 14 (1973), 1304-1312.

[ROG-2] ROGOZIN, B.A., Asymptotic behavior of the coefficients of power series and Fourier series, Sibirsk. Matem. Zh. 17 (1976), 897-906.

[ROG-3] ROGOZIN, B.A., Banach algebras of measures on a straight line, connected with asymptotic behavior of the measure at infinity, Sibirsk. Matem. Zh. 17 (1976), 640-647.

[ROG-4] ROGOZIN, B.A., An estimate of the remainder term in limit theorems of renewal theory, Theory Prob. Appl. 18 (1973), 662-677.

[ROG-5] ROGOZIN, B.A., Asymptotics of renewal functions, Theory Prob. Appl. 21 (1976), 669-686.

[RUD-1] RUDIN, W., Functional Analysis, Tata-McGraw-Hill Publishing Company, New Dehli, 1981.

[RUD-2] RUDIN, W., Real and Complex Analysis, Tata-McGraw-Hill Publishing Company, New Dehli, 1982.

[SCH] SCHWARTZ, L., Mathematics for the Physical Sciences, Addison Wesley, 1966.

[SEN] SENETA, E., Regularly Varying Functions, Springer Verlag, New York, 1976.

[ŠRE] ŠREIDER, YU.A., The structure of maximal ideals in rings of measures with convolutions, Amer. Math. Soc. Trans., serie 1, 8 (1962).

[STO-1] STONE, C., On moment generating functions and renewal theory, Ann. of Math. Stat. 36 (1965), 1298-1301.

[STO-2] STONE, C., On absolutely continuous components and renewal theory, Ann. of Math. Stat. 37 (1966), 271-275.

[STO-3] STONE, C., On characteristic functions and renewal theory, Trans. Amer. Math. Soc. 120 (1965), 327-342.

[STO-4] STONE, C., WAINGER, S., One-sided error estimates in renewal theory, Journal d'Analyse Math. 20 (1967), 325-352.

[TEU] TEUGELS, J.L., The Class of Subexponential Distributions, Ann. Prob. 3 (1975), 1000-1011.

[TIT] TITCHMARSH, E.C., Theory of functions, Oxford University Press, London, 1952.

[WID] WIDDER, P.V., The Laplace Transform, Princeton University Press, Princeton, 1972.

MATHEMATICAL CENTRE TRACTS

1 T. van der Walt. *Fixed and almost fixed points.* 1963.

2 A.R. Bloemena. *Sampling from a graph.* 1964.

3 G. de Leve. *Generalized Markovian decision processes, part I: model and method.* 1964.

4 G. de Leve. *Generalized Markovian decision processes, part II: probabilistic background.* 1964.

5 G. de Leve, H.C. Tijms, P.J. Weeda. *Generalized Markovian decision processes, applications.* 1970.

6 M.A. Maurice. *Compact ordered spaces.* 1964.

7 W.R. van Zwet. *Convex transformations of random variables.* 1964.

8 J.A. Zonneveld. *Automatic numerical integration.* 1964.

9 P.C. Baayen. *Universal morphisms.* 1964.

10 E.M. de Jager. *Applications of distributions in mathematical physics.* 1964.

11 A.B. Paalman-de Miranda. *Topological semigroups.* 1964.

12 J.A.Th.M. van Berckel, H. Brandt Corstius, R.J. Mokken, A. van Wijngaarden. *Formal properties of newspaper Dutch.* 1965.

13 H.A. Lauwerier. *Asymptotic expansions.* 1966, out of print; replaced by MCT 54.

14 H.A. Lauwerier. *Calculus of variations in mathematical physics.* 1966.

15 R. Doornbos. *Slippage tests.* 1966.

16 J.W. de Bakker. *Formal definition of programming languages with an application to the definition of ALGOL 60.* 1967.

17 R.P. van de Riet. *Formula manipulation in ALGOL 60, part 1.* 1968.

18 R.P. van de Riet. *Formula manipulation in ALGOL 60, part 2.* 1968.

19 J. van der Slot. *Some properties related to compactness.* 1968.

20 P.J. van der Houwen. *Finite difference methods for solving partial differential equations.* 1968.

21 E. Wattel. *The compactness operator in set theory and topology.* 1968.

22 T.J. Dekker. *ALGOL 60 procedures in numerical algebra, part 1.* 1968.

23 T.J. Dekker, W. Hoffmann. *ALGOL 60 procedures in numerical algebra, part 2.* 1968.

24 J.W. de Bakker. *Recursive procedures.* 1971.

25 E.R. Paërl. *Representations of the Lorentz group and projective geometry.* 1969.

26 European Meeting 1968. *Selected statistical papers, part I.* 1968.

27 European Meeting 1968. *Selected statistical papers, part II.* 1968.

28 J. Oosterhoff. *Combination of one-sided statistical tests.* 1969.

29 J. Verhoeff. *Error detecting decimal codes.* 1969.

30 H. Brandt Corstius. *Exercises in computational linguistics.* 1970.

31 W. Molenaar. *Approximations to the Poisson, binomial and hypergeometric distribution functions.* 1970.

32 L. de Haan. *On regular variation and its application to the weak convergence of sample extremes.* 1970.

33 F.W. Steutel. *Preservation of infinite divisibility under mixing and related topics.* 1970.

34 I. Juhász, A. Verbeek, N.S. Kroonenberg. *Cardinal functions in topology.* 1971.

35 M.H. van Emden. *An analysis of complexity.* 1971.

36 J. Grasman. *On the birth of boundary layers.* 1971.

37 J.W. de Bakker, G.A. Blaauw, A.J.W. Duijvestijn, E.W. Dijkstra, P.J. van der Houwen, G.A.M. Kamsteeg-Kemper, F.E.J. Kruseman Aretz, W.L. van der Poel, J.P. Schaap-Kruseman, M.V. Wilkes, G. Zoutendijk. *MC-25 Informatica Symposium.* 1971.

38 W.A. Verloren van Themaat. *Automatic analysis of Dutch compound words.* 1972.

39 H. Bavinck. *Jacobi series and approximation.* 1972.

40 H.C. Tijms. *Analysis of (s,S) inventory models.* 1972.

41 A. Verbeek. *Superextensions of topological spaces.* 1972.

42 W. Vervaat. *Success epochs in Bernoulli trials (with applications in number theory).* 1972.

43 F.H. Ruymgaart. *Asymptotic theory of rank tests for independence.* 1973.

44 H. Bart. *Meromorphic operator valued functions.* 1973.

45 A.A. Balkema. *Monotone transformations and limit laws.* 1973.

46 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 1: the language.* 1973.

47 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 2: the compiler.* 1973.

48 F.E.J. Kruseman Aretz, P.J.W. ten Hagen, H.L. Oudshoorn. *An ALGOL 60 compiler in ALGOL 60, text of the MC-compiler for the EL-X8.* 1973.

49 H. Kok. *Connected orderable spaces.* 1974.

50 A. van Wijngaarden, B.J. Mailloux, J.E.L. Peck, C.H.A. Koster, M. Sintzoff, C.H. Lindsey, L.G.L.T. Meertens, R.G. Fisker (eds.). *Revised report on the algorithmic language ALGOL 68.* 1976.

51 A. Hordijk. *Dynamic programming and Markov potential theory.* 1974.

52 P.C. Baayen (ed.). *Topological structures.* 1974.

53 M.J. Faber. *Metrizability in generalized ordered spaces.* 1974.

54 H.A. Lauwerier. *Asymptotic analysis, part 1.* 1974.

55 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 1: theory of designs, finite geometry and coding theory.* 1974.

56 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 2: graph theory, foundations, partitions and combinatorial geometry.* 1974.

57 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 3: combinatorial group theory.* 1974.

58 W. Albers. *Asymptotic expansions and the deficiency concept in statistics.* 1975.

59 J.L. Mijnheer. *Sample path properties of stable processes.* 1975.

60 F. Göbel. *Queueing models involving buffers.* 1975.

63 J.W. de Bakker (ed.). *Foundations of computer science.* 1975.

64 W.J. de Schipper. *Symmetric closed categories.* 1975.

65 J. de Vries. *Topological transformation groups, 1: a categorical approach.* 1975.

66 H.G.J. Pijls. *Logically convex algebras in spectral theory and eigenfunction expansions.* 1976.

68 P.P.N. de Groen. *Singularly perturbed differential operators of second order.* 1976.

69 J.K. Lenstra. *Sequencing by enumerative methods.* 1977.

70 W.P. de Roever, Jr. *Recursive program schemes: semantics and proof theory.* 1976.

71 J.A.E.E. van Nunen. *Contracting Markov decision processes.* 1976.

72 J.K.M. Jansen. *Simple periodic and non-periodic Lamé functions and their applications in the theory of conical waveguides.* 1977.

73 D.M.R. Leivant. *Absoluteness of intuitionistic logic.* 1979.

74 H.J.J. te Riele. *A theoretical and computational study of generalized aliquot sequences.* 1976.

75 A.E. Brouwer. *Treelike spaces and related connected topological spaces.* 1977.

76 M. Rem. *Associons and the closure statement.* 1976.

77 W.C.M. Kallenberg. *Asymptotic optimality of likelihood ratio tests in exponential families.* 1978.

78 E. de Jonge, A.C.M. van Rooij. *Introduction to Riesz spaces.* 1977.

79 M.C.A. van Zuijlen. *Emperical distributions and rank statistics.* 1977.

80 P.W. Hemker. *A numerical study of stiff two-point boundary problems.* 1977.

81 K.R. Apt, J.W. de Bakker (eds.). *Foundations of computer science II, part 1.* 1976.

82 K.R. Apt, J.W. de Bakker (eds.). *Foundations of computer science II, part 2.* 1976.

83 L.S. van Benthem Jutting. *Checking Landau's "Grundlagen" in the AUTOMATH system.* 1979.

84 H.L.L. Busard. *The translation of the elements of Euclid from the Arabic into Latin by Hermann of Carinthia (?), books vii-xii.* 1977.

85 J. van Mill. *Supercompactness and Wallman spaces.* 1977.

86 S.G. van der Meulen, M. Veldhorst. *Torrix I, a programming system for operations on vectors and matrices over arbitrary fields and of variable size.* 1978.

88 A. Schrijver. *Matroids and linking systems.* 1977.

89 J.W. de Roever. *Complex Fourier transformation and analytic functionals with unbounded carriers.* 1978.

90 L.P.J. Groenewegen. *Characterization of optimal strategies in dynamic games.* 1981.

91 J.M. Geysel. *Transcendence in fields of positive characteristic.* 1979.

92 P.J. Weeda. *Finite generalized Markov programming.* 1979.

93 H.C. Tijms, J. Wessels (eds.). *Markov decision theory.* 1977.

94 A. Bijlsma. *Simultaneous approximations in transcendental number theory.* 1978.

95 K.M. van Hee. *Bayesian control of Markov chains.* 1978.

96 P.M.B. Vitányi. *Lindenmayer systems: structure, languages, and growth functions.* 1980.

97 A. Federgruen. *Markovian control problems; functional equations and algorithms.* 1984.

98 R. Geel. *Singular perturbations of hyperbolic type.* 1978.

99 J.K. Lenstra, A.H.G. Rinnooy Kan, P. van Emde Boas (eds.). *Interfaces between computer science and operations research.* 1978.

100 P.C. Baayen, D. van Dulst, J. Oosterhoff (eds.). *Proceedings bicentennial congress of the Wiskundig Genootschap, part 1.* 1979.

101 P.C. Baayen, D. van Dulst, J. Oosterhoff (eds.). *Proceedings bicentennial congress of the Wiskundig Genootschap, part 2.* 1979.

102 D. van Dulst. *Reflexive and superreflexive Banach spaces.* 1978.

103 K. van Harn. *Classifying infinitely divisible distributions by functional equations.* 1978.

104 J.M. van Wouwe. *Go-spaces and generalizations of metrizability.* 1979.

105 R. Helmers. *Edgeworth expansions for linear combinations of order statistics.* 1982.

106 A. Schrijver (ed.). *Packing and covering in combinatorics.* 1979.

107 C. den Heijer. *The numerical solution of nonlinear operator equations by imbedding methods.* 1979.

108 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science III, part 1.* 1979.

109 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science III, part 2.* 1979.

110 J.C. van Vliet. *ALGOL 68 transput, part I: historical review and discussion of the implementation model.* 1979.

111 J.C. van Vliet. *ALGOL 68 transput, part II: an implementation model.* 1979.

112 H.C.P. Berbee. *Random walks with stationary increments and renewal theory.* 1979.

113 T.A.B. Snijders. *Asymptotic optimality theory for testing problems with restricted alternatives.* 1979.

114 A.J.E.M. Janssen. *Application of the Wigner distribution to harmonic analysis of generalized stochastic processes.* 1979.

115 P.C. Baayen, J. van Mill (eds.). *Topological structures II, part 1.* 1979.

116 P.C. Baayen, J. van Mill (eds.). *Topological structures II, part 2.* 1979.

117 P.J.M. Kallenberg. *Branching processes with continuous state space.* 1979.

118 P. Groeneboom. *Large deviations and asymptotic efficiencies.* 1980.

119 F.J. Peters. *Sparse matrices and substructures, with a novel implementation of finite element algorithms.* 1980.

120 W.P.M. de Ruyter. *On the asymptotic analysis of large-scale ocean circulation.* 1980.

121 W.H. Haemers. *Eigenvalue techniques in design and graph theory.* 1980.

122 J.C.P. Bus. *Numerical solution of systems of nonlinear equations.* 1980.

123 I. Yuhász. *Cardinal functions in topology - ten years later.* 1980.

124 R.D. Gill. *Censoring and stochastic integrals.* 1980.

125 R. Eising. *2-D systems, an algebraic approach.* 1980.

126 G. van der Hoek. *Reduction methods in nonlinear programming.* 1980.

127 J.W. Klop. *Combinatory reduction systems.* 1980.

128 A.J.J. Talman. *Variable dimension fixed point algorithms and triangulations.* 1980.

129 G. van der Laan. *Simplicial fixed point algorithms.* 1980.

130 P.J.W. ten Hagen, T. Hagen, P. Klint, H. Noot, H.J. Sint, A.H. Veen. *ILP: intermediate language for pictures.* 1980.

131 R.J.R. Back. *Correctness preserving program refinements: proof theory and applications.* 1980.

132 H.M. Mulder. *The interval function of a graph.* 1980.

133 C.A.J. Klaassen. *Statistical performance of location estimators.* 1981.

134 J.C. van Vliet, H. Wupper (eds.). *Proceedings international conference on ALGOL 68.* 1981.

135 J.A.G. Groenendijk, T.M.V. Janssen, M.J.B. Stokhof (eds.). *Formal methods in the study of language, part I.* 1981.

136 J.A.G. Groenendijk, T.M.V. Janssen, M.J.B. Stokhof (eds.). *Formal methods in the study of language, part II.* 1981.

137 J. Telgen. *Redundancy and linear programs.* 1981.

138 H.A. Lauwerier. *Mathematical models of epidemics.* 1981.

139 J. van der Wal. *Stochastic dynamic programming, successive approximations and nearly optimal strategies for Markov decision processes and Markov games.* 1981.

140 J.H. van Geldrop. *A mathematical theory of pure exchange economies without the no-critical-point hypothesis.* 1981.

141 G.E. Welters. *Abel-Jacobi isogenies for certain types of Fano threefolds.* 1981.

142 H.R. Bennett, D.J. Lutzer (eds.). *Topology and order structures, part 1.* 1981.

143 J.M. Schumacher. *Dynamic feedback in finite- and infinite-dimensional linear systems.* 1981.

144 P. Eijgenraam. *The solution of initial value problems using interval arithmetic; formulation and analysis of an algorithm.* 1981.

145 A.J. Brentjes. *Multi-dimensional continued fraction algorithms.* 1981.

146 C.V.M. van der Mee. *Semigroup and factorization methods in transport theory.* 1981.

147 H.H. Tigelaar. *Identification and informative sample size.* 1982.

148 L.C.M. Kallenberg. *Linear programming and finite Markovian control problems.* 1983.

149 C.B. Huijsmans, M.A. Kaashoek, W.A.J. Luxemburg, W.K. Vietsch (eds.). *From A to Z, proceedings of a symposium in honour of A.C. Zaanen.* 1982.

150 M. Veldhorst. *An analysis of sparse matrix storage schemes.* 1982.

151 R.J.M.M. Does. *Higher order asymptotics for simple linear rank statistics.* 1982.

152 G.F. van der Hoeven. *Projections of lawless sequences.* 1982.

153 J.P.C. Blanc. *Application of the theory of boundary value problems in the analysis of a queueing model with paired services.* 1982.

154 H.W. Lenstra, Jr., R. Tijdeman (eds.). *Computational methods in number theory, part I.* 1982.

155 H.W. Lenstra, Jr., R. Tijdeman (eds.). *Computational methods in number theory, part II.* 1982.

156 P.M.G. Apers. *Query processing and data allocation in distributed database systems.* 1983.

157 H.A.W.M. Kneppers. *The covariant classification of two-dimensional smooth commutative formal groups over an algebraically closed field of positive characteristic.* 1983.

158 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science IV, distributed systems, part 1.* 1983.

159 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science IV, distributed systems, part 2.* 1983.

160 A. Rezus. *Abstract AUTOMATH.* 1983.

161 G.F. Helminck. *Eisenstein series on the metaplectic group, an algebraic approach.* 1983.

162 J.J. Dik. *Tests for preference.* 1983.

163 H. Schippers. *Multiple grid methods for equations of the second kind with applications in fluid mechanics.* 1983.

164 F.A. van der Duyn Schouten. *Markov decision processes with continuous time parameter.* 1983.

165 P.C.T. van der Hoeven. *On point processes.* 1983.

166 H.B.M. Jonkers. *Abstraction, specification and implementation techniques, with an application to garbage collection.* 1983.

167 W.H.M. Zijm. *Nonnegative matrices in dynamic programming.* 1983.

168 J.H. Evertse. *Upper bounds for the numbers of solutions of diophantine equations.* 1983.

169 H.R. Bennett, D.J. Lutzer (eds.). *Topology and order structures, part 2.* 1983.

CWI TRACTS

1 D.H.J. Epema. *Surfaces with canonical hyperplane sections.* 1984.

2 J.J. Dijkstra. *Fake topological Hilbert spaces and characterizations of dimension in terms of negligibility.* 1984.

3 A.J. van der Schaft. *System theoretic descriptions of physical systems.* 1984.

4 J. Koene. *Minimal cost flow in processing networks, a primal approach.* 1984.

5 B. Hoogenboom. *Intertwining functions on compact Lie groups.* 1984.

6 A.P.W. Böhm. *Dataflow computation.* 1984.

7 A. Blokhuis. *Few-distance sets.* 1984.

8 M.H. van Hoorn. *Algorithms and approximations for queueing systems.* 1984.

9 C.P.J. Koymans. *Models of the lambda calculus.* 1984.

10 C.G. van der Laan, N.M. Temme. *Calculation of special functions: the gamma function, the exponential integrals and error-like functions.* 1984.

11 N.M. van Dijk. *Controlled Markov processes; time-discretization.* 1984.

12 W.H. Hundsdorfer. *The numerical solution of nonlinear stiff initial value problems: an analysis of one step methods.* 1985.

13 D. Grune. *On the design of ALEPH.* 1985.

14 J.G.F. Thiemann. *Analytic spaces and dynamic programming: a measure theoretic approach.* 1985.

15 F.J. van der Linden. *Euclidean rings with two infinite primes.* 1985.

16 R.J.P. Groothuizen. *Mixed elliptic-hyperbolic partial differential operators: a case-study in Fourier integral operators.* 1985.

17 H.M.M. ten Eikelder. *Symmetries for dynamical and Hamiltonian systems.* 1985.

18 A.D.M. Kester. *Some large deviation results in statistics.* 1985.

19 T.M.V. Janssen. *Foundations and applications of Montague grammar, part 1: Philosophy, framework, computer science.* 1986.

20 B.F. Schriever. *Order dependence.* 1986.

21 D.P. van der Vecht. *Inequalities for stopped Brownian motion.* 1986.

22 J.C.S.P. van der Woude. *Topological dynamix.* 1986.

23 A.F. Monna. *Methods, concepts and ideas in mathematics: aspects of an evolution.* 1986.

24 J.C.M. Baeten. *Filters and ultrafilters over definable subsets of admissible ordinals.* 1986.

25 A.W.J. Kolen. *Tree network and planar rectilinear location theory.* 1986.

26 A.H. Veen. *The misconstrued semicolon: Reconciling imperative languages and dataflow machines.* 1986.

27 A.J.M. van Engelen. *Homogeneous zero-dimensional absolute Borel sets.* 1986.

28 T.M.V. Janssen. *Foundations and applications of Montague grammar, part 2: Applications to natural language.* 1986.

29 H.L. Trentelman. *Almost invariant subspaces and high gain feedback.* 1986.

30 A.G. de Kok. *Production-inventory control models: approximations and algorithms.* 1987.

31 E.E.M. van Berkum. *Optimal paired comparison designs for factorial experiments.* 1987.

32 J.H.J. Einmahl. *Multivariate empirical processes.* 1987.

33 O.J. Vrieze. *Stochastic games with finite state and action spaces.* 1987.

34 P.H.M. Kersten. *Infinitesimal symmetries: a computational approach.* 1987.

35 M.L. Eaton. *Lectures on topics in probability inequalities.* 1987.

36 A.H.P. van der Burgh, R.M.M. Mattheij (eds.). *Proceedings of the first international conference on industrial and applied mathematics (ICIAM 87).* 1987.

37 L. Stougie. *Design and analysis of algorithms for stochastic integer programming.* 1987.

38 J.B.G. Frenk. *On Banach algebras, renewal measures and regenerative processes.* 1987.