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Strong orientations without even directed circuits[1]

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Abstract

We characterize the graphs for which all 2-connected non-bipartite subgraphs have a strongly connected orientation in which each directed circuit has an odd number of edges. We also give a polynomial-time algorithm to find such an orientation in these graphs. Moreover, we give an algorithm that given any orientation of such a graph, determines if it has an even directed circuit.
The proofs of these results are based on a constructive characterization of these graphs.

AMS classification: 05C20; 05C75

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1. Introduction

A directed graph $D=(V(D),A(D))$ is *strongly connected* if between any ordered pair of nodes (u,v) there exists a directed uv-path in D. A strongly connected directed graph without directed circuits with an even number of arcs is called *strong odd.* An *orientation* of an undirected graph $G=(V(G),E(G))$ is a directed graph D obtained from G by replacing each edge in G by a directed edge (arc). In this paper we prove the following result:

Theorem 1. *Let G be a 2-connected non-bipartite graph. If G contains neither an odd-K_4 nor an odd chain as a subgraph, then G has a strong odd orientation.*

Here, an *odd-K_4* is an undirected graph as depicted in Fig. 1(a). A *string* is a graph H for which there exist subgraphs $H_1,\ldots,H_k$, with $k\geqslant 2$, such that $E(H_1),\ldots,E(H_k)$

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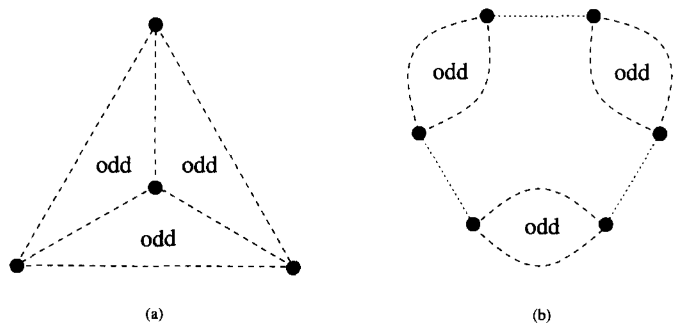

Fig. 1. Dashed and dotted lines denote pairwise openly disjoint paths. Dashed lines correspond to paths with at least one edge, whereas dotted lines may have length 0. The word *odd* in a face indicates that the length of the bounding circuit is odd.

partition $E(H)$ and such that for $i \neq j$,

$$|V(H_i) \cap V(H_j)| = \begin{cases} 2 & \text{if } k=2, \\ 1 & \text{if } k \neq 2 \text{ and } |i-j| = 1 \,(\text{mod}\, k), \\ 0 & \text{else.} \end{cases}$$

$H_1, \ldots, H_k$ are the *beads* of the string. If $k>2$, $h_{i,i+1}$ denotes the unique node in $V(H_i) \cap V(H_{i+1})$ (indices modulo k). If $k=2$, $h_{1,2}$ and $h_{2,1}$ denote the two nodes in $V(H_1) \cap V(H_2)$. The nodes $h_{1,2}, \ldots, h_{k,1}$ are called the *links* of the string. A *chain* is a string in which each bead is a path or an odd circuit. An *m-chain* is a chain in which exactly m beads are odd circuits. A *full* (m-)chain is an (m-)chain in which all beads are odd circuits. An *odd* (*even*) chain is a full m-chain with m odd (even). Fig. 1(b) exhibits a 3-chain. Unless stated otherwise, by *subgraph of G* we do not mean just induced subgraph but any graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

It can be easily checked that odd-K_4's and odd chains have no strong odd orientation; hence, Theorem 1 can also be stated as:

Let G be an undirected graph. Then each 2-connected non-bipartite subgraph of G has a strong odd orientation if and only if G contains neither an odd-K_4 nor an odd chain as a subgraph.

Fig. 2 illustrates that graphs containing an odd-K_4 may have strong odd orientations. In Theorem 1, non-bipartiteness is essential since strongly connected orientations of bipartite graphs always will have even directed circuits. 2-connectedness is almost essential; it can be replaced by: G is connected and each block (= maximal 2-connected subgraph) of G is non-bipartite.

Our proof of Theorem 1 consists of two major phases. First, in Section 3, we derive strong odd orientations for three special types of graphs with no odd-K_4 and no odd chain. In the second phase, in Section 4, we make use of a constructive characterization of graphs with no odd-K_4 and no odd chains (Theorem 7 and Corollary 8 in Section 4),

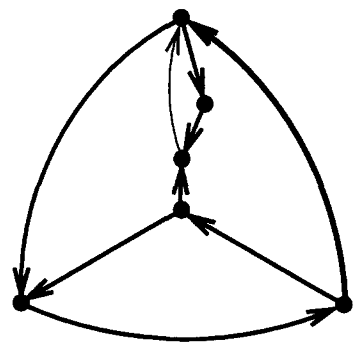

Fig. 2. A strong odd orientation of a graph with an odd-K_4 (depicted in bold).

which says that these graphs can be decomposed into graphs of the three special types. In both phases we make use several times of a small orientation lemma (Lemma 3 in Section 2).

For technical reasons we prove the result in a bit wider context than that of ordinary undirected graphs; namely that of signed graphs (cf. Section 2). Not because this yields a stronger result — essentially it does not — but rather to facilitate stating the arguments.

The results of these paper are motivated by the following computational problem proposed by Bang-Jensen [2]:

(1) *Given a graph, find a strong odd orientation of it.*

We have no clue, as to the complexity of this problem. We do not even know if it is equivalent with the related well-known *even circuit problem*:

(2) *Given an oriented graph, does it contain an even directed circuit?*

The proof of Theorem 1, however, yields the following result:

Theorem 2. *Both* (1) *and* (2) *are solvable in polynomial time for graphs with no odd-K_4 and no odd chain.*

Note that, by Corollary 8, graphs with no odd-K_4 and no odd chain are recognizable in polynomial-time. We conclude this section with a short overview of related results.

1.1. Results on the even circuit problem

The complexity of even circuit problem (2) is a well-known open problem [2] although it has been shown that the problem of determining whether a specified arc is contained in an even directed circuit is NP-hard (Klee, Ladner and Manber [9]; Thomassen [15]).

[2] Recently, this problem has been resolved: McCuaig [12] and, independently, Robertson, Seymour and Thomas [12A] derived a polynomial-time algorithm.

On the other hand, Thomassen [15] has given a polynomial-time algorithm for the even circuit problem in directed planar graphs. Moreover, Galluccio and Loebl [3] have given an algorithm to determine whether all directed circuits in a directed planar graph are of length $p \bmod q$ for arbitrary $0 \leqslant p < q$. In the case of undirected graphs, a polynomial time algorithm has been given to determine whether all circuits in a graph are of length $p \bmod q$ (Arkin, Papadimitriou and Yannakakis [1]). Note that forbidding odd directed circuits instead of even ones, yields a trivial problem: a graph has a strongly connected orientation without odd directed circuits if and only if it is bipartite.

The even cycle problem is polynomially equivalent with any of the following problems: recognizing *even graphs*, i.e. directed graphs for which every subdivision contains an even directed circuit (Seymour and Thomassen [14]); recognizing bipartite graphs with Pfaffian orientations (Vazirani and Yannakakis [18]); recognizing those minimally non-bipartite hypergraphs that have as many edges as vertices (Seymour [13]); and the following problem: given a 0, 1 $n \times n$ matrix A, is there a $-1, 0, 1$ $n \times n$ matrix B such that $perm(A) = det(B)$ (Vazirani and Yannakakis [18]). Seymour and Thomassen [14] gave an NP-characterization of even graphs. Vazirani and Yannakakis [18] show that for any graph the problems of finding a Pfaffian orientation and of checking whether a given orientation is Pfaffian are equivalent. Little [11] fully characterized the class of Pfaffian bipartite graphs in terms of forbidden subconfigurations. The class includes bipartite graphs with no subdivision of K_{33} (Little [10]), so in particular planar bipartite graphs (Kasteleyn [8]). We mention that the problem of determining whether the permanent and determinant of a matrix are equal is NP-hard (Valiant [17]).

1.2. Other orientation results for graphs with no odd-K_4's

There are two other orientation results in which odd-K_4's play a role. The first one is: An undirected graph contains no odd-K_4 and no 3-chain if and only if it has an orientation such that on each circuit the number of forwardly oriented edges differs at most one from the number of backwardly oriented edges (Gerards [6]). The other is: Each undirected graph with no odd-K_4 and no 3-chain can be oriented such that on each circuit the number of forwardly oriented edges minus the number of backwardly oriented edges is a multiple of the length of a shortest odd circuit in the graph (Gerards [4]). (The existence of such an orientation is equivalent with the existence of an adjacency preserving map from the vertices of the graph to the vertices of a shortest odd circuit in that graph.)

2. Preliminaries

2.1. An orientation lemma

In proving Theorem 1 we will use several times the following easy fact. A one node cutset $\{v\}$ *lies between s and t* if the graph $G - v$ (obtained by deleting v) has exactly two components, one containing s and one containing t.

Lemma 3. *Let G be an undirected graph and let $s,t \in V(G)$ such that all one node cutsets lie between s and t. Then G has an acyclic orientation such that each node in G is on a directed st-path in D.*

Proof. First suppose that G has a one node cutset $\{v\}$. Let V_s be the node set of the component of $G - v$ containing s and V_t the node set of the component containing t. Now applying induction to the subgraph of G induced by $V_s \cup \{v\}$ with v instead of t and to $V_t \cup \{v\}$ with v instead of s, we obtain the desired orientation for G, s and t.

So, we may assume G to be 2-connected. Let $\tilde{G}$ be a maximal 2-connected subgraph of G containing s and t, for which such an orientation, $\tilde{D}$ say, exists. This is well defined as G is 2-connected and hence contains a circuit through s and t. If $\tilde{G} = G$ we are done, so suppose this is not the case. Number the nodes of $\tilde{G}$ such the tail of each arc in $\tilde{D}$ has a lower number than the head of that arc. Let R be a uv-path in G with $V(R) \cap V(\tilde{G}) = \{u,v\}$ and $E(R) \cap E(\tilde{G}) = \emptyset$ (R exists as G is 2-connected). Without loss of generality u received the lower number. Orient the edges on R so that R becomes a directed uv-path. Clearly, the directed graph obtained is 2-connected, acyclic and has each node on a directed st-path. But it is larger than $\tilde{G}$ — contradiction! □

2.2. Signed graphs

A *signed graph* is a pair (G,Σ), where $G = (V(G),E(G))$ is an undirected graph and Σ is a subset of $E(G)$. Edges in Σ are called *odd*, the other edges are called *even*. A collection of edges or a subgraph is called *odd* (*even*) if it contains an odd number of odd edges. We call a signed graph (G,Σ) *bipartite* if there exists a set $U \subseteq V(G)$ such that $\Sigma = \delta(U) := \{uv \in E(G) \mid u \in U, v \in V(G)\backslash U\}$. Obviously, a signed graph is bipartite if and only if it has no odd circuits. Note that $(G,E(G))$ is bipartite if and only if G is a bipartite graph in the usual sense. We say that a signed graph (H,Θ) is *contained in* (G,Σ) if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and $\Theta = \Sigma \cap E(H)$.

A *strong odd* orientation of a signed graph (G,Σ) is a strongly connected orientation of G in which no directed circuit is an even circuit in (G,Σ). It is easy to see that Theorem 1 is equivalent to:

(3) *Let (G,Σ) be a non-bipartite signed graph with G 2-connected. If (G,Σ) contains neither an odd-K_4 nor an odd chain, then (G,Σ) has a strong odd orientation.*

Here, *odd-K_4* and *odd chain* are defined similarly as in case of ordinary graphs, with the understanding that in case of signed graphs 'odd' refers not to the cardinality of an edge set but to the number of odd edges contained in it. Similarly, we extend the notions of *string* and of *full*, *m-*, and *even chains* to signed graphs.

Clearly, the strong oddness of orientations does not depend as much on Σ, the collection of odd edges, as on the collection of odd circuits. If (G,Σ) is a signed graph, and $\tilde{\Sigma} \subseteq E(G)$, then (G,Σ) and $(G,\tilde{\Sigma})$ have exactly the same odd circuits if and only if $(G,\Sigma \triangle \tilde{\Sigma})$ is bipartite or, equivalently, if and only if $\tilde{\Sigma} = \Sigma \triangle \delta(U)$ for some $U \subseteq V(G)$. We call the replacement of Σ by $\tilde{\Sigma} = \Sigma \triangle \delta(U)$ a *re-signing on U*.

3. Special cases

We first show the result for three subclasses of signed graphs with no odd-K_4 and no odd chain, namely 'almost bipartite signed graphs', 'planar signed graphs with exactly two odd faces' and chains that are not odd. As we shall see in Section 4, these special classes generate the general case.

3.1. Almost bipartite graphs

A signed graph is called *almost bipartite*, if it contains a node, called a *block node*, that is in each odd circuit. Deleting a block node yields a bipartite signed graph.

Lemma 4. *Let (G,Σ) be an almost bipartite signed graph. If G is 2-connected and (G,Σ) is non-bipartite, then (G,Σ) has a strong odd orientation.*

Proof. Let u be a block node of (G,Σ). Re-sign such that Σ becomes a subset of $\delta(u)$. Construct a new graph G' by splitting u into two new nodes s and t, where odd edges in $\delta(u)$ now become adjacent to s and even edges in $\delta(u)$ to t. As (G,Σ) is non-bipartite neither $\delta(s)$ nor $\delta(t)$ is empty. Moreover, as G is 2-connected, all one node cutsets of G' lie between s and t. Applying Lemma 3 to G', yields an orientation of G' that induces a strong odd orientation of (G,Σ). □

3.2. Planar with two odd faces

Lemma 5. *Let (G,Σ) be a signed graph. If G is 2-connected, planar, and can be embedded in the plane such that exactly two of its faces are bounded by odd circuits in (G,Σ), then (G,Σ) has a strong odd orientation.*

Proof. Let G^* be the planar dual of G and s and t be the nodes of G^* corresponding to the two faces of G bounded by odd circuits. As G is 2-connected so is G^*. Hence, by Lemma 3 there exists an acyclic orientation D^* of G^* such that each node is on a directed st-path in D^*. Take as orientation D of G, the directed dual of D^* by using the right-hand rule.

Because, D^* is acyclic, D has no directed cuts, hence D is strongly connected. If C is a directed circuit in D then it corresponds in D^* to a directed cut. This implies that s and t lie in the plane on different sides of C. Hence, exactly one of the faces inside C is bounded by an odd circuit. As C is the symmetric difference of the boundaries of the faces inside C, circuit C is odd. So, D is a strong odd orientation of G. □

3.3. Chains

Lemma 6. *Even chains have a strong odd orientation, and so do non-bipartite chains that are not full.*

Proof. Let C be an odd circuit with non-empty intersection with all the beads. Orient the edges on C such that C becomes a directed circuit. Orient the other edges in G such that all non-bipartite beads, which are odd circuits, become directed circuits. Clearly, this yields a strongly connected orientation. The only possible directed circuits are C, the odd-circuits forming the non-bipartite beads, and possibly $C' := G \backslash C$ (if it forms a circuit). So, the orientation is odd unless C' is an even circuit in (G, Σ). However, if C' is a circuit, then (G, Σ) is a full chain; if, moreover, $|C' \cap \Sigma|$ is even, then $|\Sigma|$ is odd, so (G, Σ) is an odd chain. □

4. Proof of Theorem 1

As announced we will prove Theorem 1 by proving (3). If (G, Σ) contains (G_1, Σ_1) and (G_2, Σ_2) with $E(G_1) \cup E(G_2) = E(G)$, $E(G_1) \cap E(G_2) = \emptyset$, and $V(G_1) \cup V(G_2) = V(G)$, then we write $(G, \Sigma) = (G_1, \Sigma_1) \oplus_U (G_2, \Sigma_2)$, where $U := V(G_1) \cap V(G_2)$. In proving Theorem 1 we make use of the following decomposition theorem.

Theorem 7 (Gerards et al. [7], cf. Gerards [5, Theorems 3.2.3 and 3.2.5]). *Let (G,Σ) be a signed graph containing no odd-K_4. If G is 2-connected then one of the following holds*:

(4) *(G, Σ) is almost bipartite or can be embedded in the plane such that exactly two of its faces are bounded by odd circuits.*
(5) *(G, Σ) is — up to re-signing — one of the two signed graphs in Fig.* 3.
(6) *$(G, \Sigma) = (G_1, \Sigma_1) \oplus_U (G_2, \Sigma_2)$ such that one of the following holds*:
 (a) *$|U| = 2$, (G_2, Σ_2) is bipartite and $|E(G_2)| \geqslant 2$;*
 (b) *$|U| = 2$ and $|E(G_1)|, |E(G_2)| \geqslant 3$;*
 (c) *$|U| = 3$, (G_2, Σ_2) is bipartite, $|E(G_2)| \geqslant 4$ and (G, Σ) contains no 3-chain.*

Since in this paper we are considering a proper subclass of signed graphs with no odd-K_4, namely those with no odd chain, we need a slight refinement of Theorem 7.

Corollary 8. *Let (G, Σ) be a signed graph containing no odd-K_4. If G is 2-connected then one of the following holds*:

(7) *(G, Σ) is almost bipartite or can be embedded in the plane such that exactly two of its faces are bounded by odd circuits.*
(8) *(G, Σ) is — up to re-signing — the signed graph in Fig.* 3(a).
(9) *$(G, \Sigma) = (G_1, \Sigma_1) \oplus_U (G_2, \Sigma_2)$ such that one of the following holds*:
 (a) *$|U| = 2$, (G_2, Σ_2) is bipartite and $|E(G_2)| \geqslant 2$;*
 (b) *$|U| = 3$, (G_2, Σ_2) is bipartite, $|E(G_2)| \geqslant 4$ and (G, Σ) contains no 3-chain.*
(10) *G is a string, with beads $H_1, \ldots, H_k$ and links $h_{1,2}, \ldots, h_{k,1}$ such that the following hold for each $i = 1, \ldots, k$:*
 (a) *If H_i is non-bipartite in (G, Σ), there exists an odd circuit in H_i containing both $h_{i-1,i}$ and $h_{i,i+1}$.*

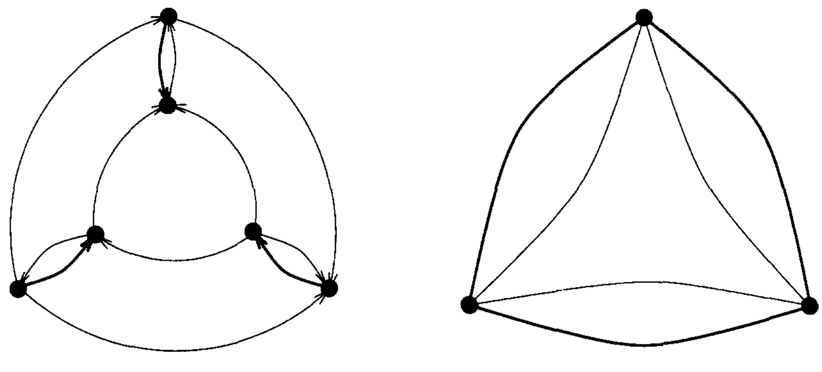

Fig. 3. Bold edges are odd, thin edges are even; in (a) arrows indicate a strong odd orientation.

(b) *If H_i is bipartite in (G,Σ), it consists of a single edge between $h_{i-1,i}$ and $h_{i,i+1}$.*
Moreover, if $k=2$ both H_1 and H_2 have at least 3 edges.

Proof. Let (G,Σ) be a signed graph with no odd-K_4. Assume (7)–(9) do not hold. Then, by Theorem 7, (6b) applies, or (G,Σ) is the graph in Fig. 3(b). Hence, G is a string with at least two non-bipartite beads. Let the beads $H_1,\dots,H_k$ be chosen such that k is as large as possible. Because (9a) does not hold, (10b) follows. So it remains to prove (10a). From maximality of k and 2-connectedness of G we easily get:

(11) *If H_i is non-bipartite, then there exists an odd circuit C in H_i and two (possibly zero-length) node-disjoint paths P_1,P_2 (in H_i) from $\{h_{i-1,i},h_{i,i+1}\}$ to $V(C)$.*

From now, take $i=1$. In H_1, choose C, P_1 and P_2 as in (11) such that $|E(P_1)|+|E(P_2)|$ is as short as possible; assume $|E(P_1)|\geqslant|E(P_2)|$. We prove that P_1 has length 0, which proves (10a). So assume P_1 has positive length. Moreover, assume that P_1 goes from $h_{k,1}$ to $u\in V(C)$. Because of the maximality of k, H_1 is 2-connected. Hence, it contains a vw-path P with $v\in V(P_1)\backslash\{u\}$ and $w\in(V(C)\cup V(P_2))\backslash\{u\}$ that is internally node-disjoint with $V(P_1)\cup V(P_2)\cup V(C)$. As C is odd, the union of P_1, P_2, P and C contains an odd circuit C_1 containing v. By the choice of C, P_1 and P_2 we get that $V(C_1)\cap V(P_2)=\emptyset$. Hence, we are in the situation as depicted by Fig. 4. Since there are at least two non-bipartite beads, at least one of $H_2,\dots,H_k$ is non-bipartite. Hence, by (11), there exists an odd $h_{1,2}h_{k,1}$-path Q_1 and an even $h_{1,2}h_{k,1}$-path Q_2 that are internally node disjoint with H_1 (so lie in $H_2\cup\cdots\cup H_k$). But this implies that either Q_1 or Q_2 closes an odd-K_4 with P_1, P_2, C and C_1 — contradiction! □

Proof of Theorem 1

(12) *Assuming* (3) *wrong, let (G,Σ) be a counterexample with $|E(G)|$ as small as possible.*

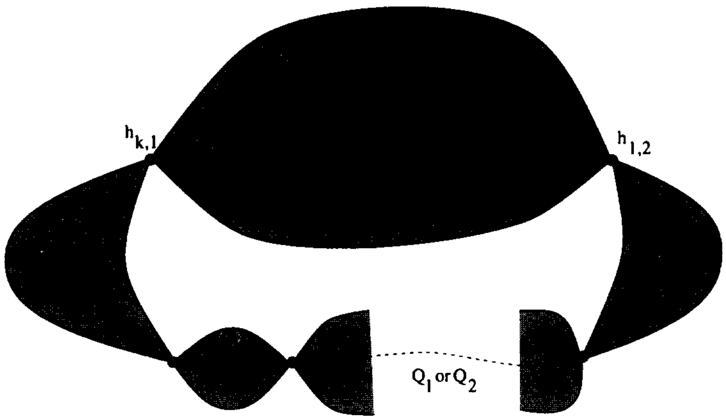

Fig. 4. The shaded areas indicate the beads. Dashed and dotted lines denote pairwise openly disjoint paths. Dashed lines correspond to paths with at least one edge, whereas dotted lines may have length 0. The word *odd* in a face indicates that the length of the bounding circuit is odd.

By Lemmas 4, 5 and 6 and Theorem 7 and because the orientation in Fig. 3(a) is strong odd, (G,Σ) satisfies (9) or (11), but is not a chain. We consider three cases.

Case 1: (G,Σ) *satisfies* (9b) *but not* (9a).

Let $(G,\Sigma)=(G_1,\Sigma_1)\oplus_{\{u_1,u_2,u_3\}}(G_2,\Sigma_2)$ with (G_2,Σ_2) bipartite and $|E(G_2)|\geqslant 4$. Assume that we have chosen (G_1,Σ_1) and (G_2,Σ_2) such that $|E(G_2)|$ is as small as possible. We may assume — by re-signing — that $\Sigma_2=\emptyset$. Let $(\tilde{G}_1,\Sigma_1)$ be obtained by adding to (G_1,Σ_1) three new even edges: $e_1=u_1u_2$, $e_2=u_2u_3$ and $e_3=u_1u_3$. As (9a) does not apply for (G,Σ), (G_2,Σ_2) contains a circuit C (even, of course) with at least three nodes, and three node-disjoint paths from $\{u_1,u_2,u_3\}$ to C. From this it can be proved that if $(\tilde{G}_1,\Sigma_1)$ would contain an odd-K_4, then so would (G,Σ), and if $(\tilde{G}_1,\Sigma_1)$ would contain an odd chain then (G,Σ) would contain a 3-chain or an odd-K_4; we leave the details to the reader. Moreover, $(\tilde{G}_1,\Sigma_1)$ inherits 2-connectedness and non-bipartiteness from (G,Σ). Hence, $(\tilde{G}_1,\Sigma_1)$ has a strong odd orientation $\tilde{D}_1$. In $\tilde{D}_1$, the circuit $\{e_1,e_2,e_3\}$ is not directed (it is even in $(\tilde{G}_1,\Sigma_1)$). So we may assume — by renumbering the indices in $\{u_1,u_2,u_3\}$ — that $\overrightarrow{u_1u_2},\overrightarrow{u_2u_3},\overrightarrow{u_1u_3}\in A(\tilde{D}_1)$. Let D_1 be the orientation of G_1 obtained from $\tilde{D}_1$ by deleting $\overrightarrow{u_1u_2},\overrightarrow{u_2u_3}$ and $\overrightarrow{u_1u_3}$.

Claim 1. *G_2 is the graph in Fig.* 5.

Proof of Claim 1. If G_2 has a one node cutset u that does not lie between u_1 and u_3, then it separates u_2 from u_1 and u_3. So in that case the claim follows because (9a) does not hold and G_2 was chosen such that it has a minimal number of edges.

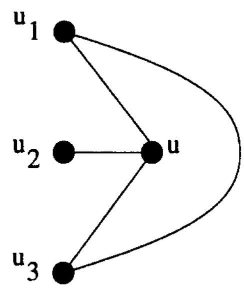

Fig. 5.

Hence we may assume that in G_2 each one node cutset lies between u_1 and u_3. Apply Lemma 3 to G_2 with $s := u_1$ and $t := u_3$; call the resulting orientation D_2. It is not hard to see that the orientation D of G obtained by taking the union of D_1 and D_2 is strongly connected and that none of its directed circuits is even in (G, Σ). This contradicts (12). □

We define two orientations D^{uu_2} and D^{u_2u} in (G, Σ). In both the edges in $E(G_1)$ are oriented as in D_1 and the edges in $E(G_2)\backslash\{uu_2\}$ are oriented as: $\overrightarrow{u_1u}, \overrightarrow{uu_3}$ and $\overrightarrow{u_1u_3}$. In D^{uu_2}, uu_2 is oriented from u to u_2 and in D^{u_2u} from u_2 to u. We will show that either D^{uu_2} or D^{u_2u} is strong odd, contradicting (12).

Claim 2. *Both D^{uu_2} and D^{u_2u} have no even directed circuits.*

Proof of Claim 2. Suppose C is an even directed circuit in D^{uu_2} or D^{u_2u}. As $(\tilde{G}_1, \Sigma_1)$ comes from (G, Σ) by contracting the even edge u_2u and because $\tilde{D}_1$ has no even directed circuits, C is not a circuit in $\tilde{G}_1$. Hence, in G, C contains the nodes u and u_2 but not the edge uu_2. So it contains $\overrightarrow{u_1u}$ and $\overrightarrow{uu_3}$. Replacing in C these two arcs by $\overrightarrow{u_1u_3}$ yields an even directed circuit in $\tilde{D}_1$ — contradiction! □

Claim 3. *Either D^{uu_2} or D^{u_2u} is strongly connected.*

Proof of Claim 3. It is easy to see that if in D_1 there is a directed u_3u_2-path, D^{u_2u} is strongly connected. (Because $\tilde{D}_1$ is strongly connected.) Similarly, if in D_1 there is a directed u_2u_1-path, then D^{uu_2} is strongly connected.

Hence, we may assume that neither u_1 nor u_3 is in the strongly connected component W of D_1 containing u_2. Let vw be an edge in G_1, with $v \in W$ and $w \notin W$. (This edge exists as G_1 is connected.) If $\overrightarrow{vw} \in A(D_1)$, then there exists a directed wu_1-path in D_1, hence also a directed u_2u_1-path (as v is in W). So D^{uu_2} is strongly connected. On the other hand, if $\overrightarrow{wv} \in A(D_1)$, then there exists a directed u_3w-path in D_1, hence also a directed u_3u_1-path. So in that case, D^{u_2u} is strongly connected. □

Hence, Case 1 cannot hold.

Case 2: (G,Σ) *satisfies* (9a).

Let $(G,\Sigma)=(G_1,\Sigma_1)\oplus_{\{u_1,u_2\}}(G_2,\Sigma_2)$ with (G_2,Σ_2) bipartite and $|E(G_2)|\geqslant 2$. From this we have a contradiction against (12). As the proof is just a simplified version of the proof in Case 1 we omit it.

Case 3: (G,Σ) *satisfies* (10).

Let $H_1,\dots,H_k$ be the beads of G, satisfying the conditions in (10). As (G,Σ) contains no odd chain, k is even or one of the beads is bipartite. Assume the numbering of the beads is such that H_k has the maximum number of edges. Define $G_1:=H_1\cup\cdots\cup H_{k-1}$, $\Sigma_1:=\Sigma\cap E(G_1)$, $G_2:=H_k$, $\Sigma_2:=\Sigma\cap E(G_2)$, $u_1:=h_{k-1,k}$ and $u_2:=h_{k,1}$. Then $(G,\Sigma)=(G_1,\Sigma_1)\oplus_{\{u_1,u_2\}}(G_2,\Sigma_2)$; by Lemma 6, (G,Σ) is not a chain, so $|E(G_1)|,|E(G_2)|\geqslant 3$.

For $i=1,2$, we define $(\tilde{G}_i,\tilde{\Sigma}_i)$ by adding to (G_i,Σ_i) two edges e_i^0 and e_i^1 from u_1 to u_2, where e_i^0 is even and e_i^1 is odd (so $\tilde{\Sigma}_i:=\Sigma_i\cup\{e_i^1\}$). For $j=0,1$, $(\tilde{G}_2,\tilde{\Sigma}_2)^j$ is obtained from $(\tilde{G}_2,\tilde{\Sigma}_2)$ by deleting e_2^j. From (10) (and the fact that $|E(G_1)|\geqslant 3$) we deduce:

(13) $(\tilde{G}_1,\tilde{\Sigma}_1),(\tilde{G}_2,\tilde{\Sigma}_2)^0,(\tilde{G}_2,\tilde{\Sigma}_2)^1$ *and* $(\tilde{G}_2,\tilde{\Sigma}_2)$ *are non-bipartite, 2-connected and contain no odd-K_4. Moreover,* $(\tilde{G}_1,\tilde{\Sigma}_1),(\tilde{G}_2,\tilde{\Sigma}_2)^0$ *and* $(\tilde{G}_2,\tilde{\Sigma}_2)^1$ *contain no odd chain. Finally, if all beads are non-bipartite, then also* $(\tilde{G}_2,\tilde{\Sigma}_2)$ *contains no odd chain.*

Claim 4. *For $i=1,2$, the circuit $\{e_i^0,e_i^1\}$ will be a directed circuit in each strong odd orientation $\tilde{D}_i$ of $(\tilde{G}_i,\tilde{\Sigma}_i)$.*

Proof of Claim 4. Let $\tilde{D}_1$ be a counterexample. Assume, both e_1^0 and e_1^1 are directed from u_1 to u_2 in $\tilde{D}_1$. As $\tilde{D}_1$ is strongly connected there exists a directed u_2u_1-path in $\tilde{D}_1$. This path closes a directed even circuit with one of e_1^0 and e_1^1 — contradiction! □

Let $\tilde{D}_1$ be a strong odd orientation of $(\tilde{G}_1,\tilde{\Sigma}_1)$. We may assume that e_1^1 is directed from u_1 to u_2 and e_1^0 from u_2 to u_1 (if not, reverse all orientations). Let D_1 be the restriction of $\tilde{D}_1$ to $E(G_1)$.

Claim 5. *D_1 contains a directed u_1u_2-path or a directed u_2u_1-path.*

Proof of Claim 5. Let W be the set of nodes reachable in D_1 by a directed path from u_1. If $u_2\in W$ we are done, so suppose this is not the case. Let $\vec{uv}\in A(D_1)$, with $u\notin W$ and $v\in W$ ($\vec{uv}$ exists as G_1 is connected). As $\tilde{D}_1$ is strongly connected, there exists in D_1 a directed u_2u-path as well as a directed vu_1-path. Together with $\vec{uv}$ these paths close a directed u_2u_1-path in D_1. □

We consider three cases.

Case 3A: *D_1 contains a directed path from u_1 to u_2 as well as a directed path from u_2 to u_1.*

This case is only possible if all the beads are non-bipartite. So, $(\tilde{G}_2, \tilde{\Sigma}_2)$ contains no odd-K_4 and no odd chain. Let $\tilde{D}_2$ be a strong odd orientation of $(\tilde{G}_2, \tilde{\Sigma}_2)$, where e_2^1 is oriented from u_1 to u_2 and e_2^0 from u_2 to u_1; let D_2 be the restriction of $\tilde{D}_2$ to $E(G_2)$. It is easy to see now that the union D of D_1 and D_2 is a strong odd orientation of (G, Σ) — contradiction!

Case 3B: *D_1 contains a directed path from u_1 to u_2 but none from u_2 to u_1.*

Whereas in Case 3a our main concern was to prevent D to have directed even circuits, now we have to make sure that D becomes strongly connected. Note that the directed u_1u_2-path in D_1 is odd.

Let $\tilde{D}_2$ be a strong odd orientation of $(\tilde{G}_2, \tilde{\Sigma}_2)^0$ such that e_2^1 is oriented from u_1 to u_2. D_2 is the restriction of $\tilde{D}_2$ to $E(G_2)$ and D is the union of D_1 and D_2. Again it is easy to check that D is a strong odd orientation of (G, Σ), yielding again a contradiction.

Case 3C: *D_1 contains a directed path from u_2 to u_1 but none from u_1 to u_2.*

It is not hard to see that this case can be reduced to Case 3b.

We conclude that (12) leads in all cases to a contradiction. Hence, (3) and Theorem 1 are true. □

5. Algorithms — Proof of Theorem 2

In this section we prove Theorem 2. All steps in the proof of Theorem 1 in Section 4 — including the proofs of Lemmas 4–6 in Section 3 — are algorithmic. So if the graph contains no odd-K_4 and no odd chain, then (1) is solvable in polynomial time.

A bit less obvious is the existence of a polynomial algorithm for solving (2) for orientations of graphs with no odd-K_4 and no odd chain. It relies on the fact that the strong odd orientations derived in the previous section are, though not uniquely determined, more or less forced.

Lemma 9. *There exists a polynomial-time algorithm to decide* (2) *in any directed graph that is an orientation of a graph with no odd-K_4's and no odd chain.*

Proof. Let (G, Σ) be a signed graph with no odd-K_4 and no odd chain. Let D be an orientation of (G, Σ). We want to check whether D has a directed circuits that is even with respect to Σ. Clearly, we may restrict ourselves to the blocks of G and the strongly connected components of D. So assume G 2-connected and D strongly connected. We consider several cases.

Case 1: *(G,Σ) is either almost bipartite, planar with two odd faces, the graph of Fig.*3(*a*), *or a chain.*

If (G,Σ) is almost bipartite, let G' be as in the proof of Lemma 4. It is easy to prove that D is strong odd if and only if the corresponding orientation of G' is as in Lemma 3. Similarly, when (G,Σ) is planar with two odd faces, D is strong odd if and only if the dual directed graph D^* is as in Lemma 3 (see the proof of Lemma 5). So in both these cases we can check the existence of even directed circuits in polynomial time. When (G,Σ) is as in Fig. 3(a) we can just check all its circuits. If (G,Σ) is a chain, then all the beads are either paths or odd circuits. If one of these odd circuits is not directed, D cannot be strong odd (compare with Claim 4). If all these odd circuits are directed, there are at most two other directed circuits in D whose eveness can easily be checked (compare with the proof of Lemma 6).

If Case 1 does not hold we know that either (9) or (10) hold. In that case we will proceed recursively, by decomposing (G,Σ) as in the proof of Theorem 1. The only difference is that now the orientation is prescribed.

Case 2: *$(G,\Sigma)=(G_1,\Sigma_1)\oplus_U(G_2,\Sigma_2)$ as in* (9).

Re-sign (G,Σ) such that $\Sigma_2=\emptyset$. For $i=1,2$, D_i denotes the restriction of D to (G_i,Σ_i). If D_2 contains a directed circuit, which is easily checked, D is not strong odd. If that is not the case, add for each pair of nodes $u,v\in U$ such that there exists a directed uv-path in D_2, an even edge uv to (G_1,Σ_1) and an arc $\vec{uv}$ to D_1. Let $(\tilde{G}_1,\tilde{\Sigma}_1)$ be the resulting signed graph and $\tilde{D}_1$ be the resulting orientation. If $|U|=2$, $(\tilde{G}_1,\tilde{\Sigma}_1)$ contains no odd-K_4 and no odd chain; moreover, $\tilde{D}_1$ is strong odd if and only if D is strong odd. The same holds if $|U|=3$, provided that we know that (9a) does not hold.

So, if we decompose according to (9a) until this is no longer possible, and then according to (9b), we can deal with (9) in polynomial time.

Case 3: *Cases* 1 *and* 2 *do not apply.*

So (G,Σ) satisfies (10). Let $H_1,\dots,H_k$ be the beads of G, satisfying the conditions in (10). As (G,Σ) contains no odd chain, k is even or one of the beads is bipartite. In each H_i, search for a directed $h_{i-1,i}h_{i,i+1}$-path R_i and a directed $h_{i,i+1}h_{i-1,i}$-path L_i (indices modulo k). If R_i does not exist we set $R_i:=\emptyset$. We do the same with L_i.

Claim. *If for some $i=1,\dots,k$, R_i and L_i are both non-empty and have the same parity with respect to Σ, D contains an even directed circuit.*

Proof of Claim. Suppose the claim is false with $i=1$. Let $y_0,\dots,y_m$ be the nodes on $V(R_1)\cap V(L_1)$, numbered in the order in which they are visited when traversing R_1 from $y_0=h_{k,1}$ to $y_m=h_{1,2}$.

(14) *Traversing L_1 from $h_{1,2}$ to $h_{k,1}$ the nodes on $V(R_1)\cap V(L_1)$ are visited in the order: $y_m,\dots,y_0$.*

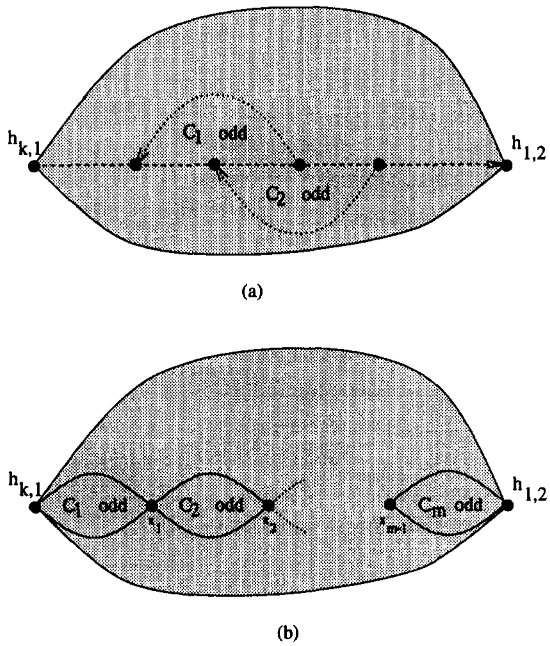

Fig. 6.

If not, R_1 and L_1 would contain a configuration as in Fig. 6(a). The dashed path is R_1, the dotted paths are parts of L_1. The circuits C_1 and C_2 in Fig. 6(a) are directed, hence odd. But this means that (G,Σ) contains a configuration as in Fig. 4. As in the proof of Corollary 8 this would yield the existence of an odd-K_4 in (G,Σ). So (14) holds.

(15) *There exist odd circuits* $C_1,\dots C_m$ *and nodes* $x_0,\dots,x_m$ *in* H_1, *satisfying*: m *is even*; $x_0=h_{k,1}\in V(C_1)$ *and* $x_m=h_{1,2}\in V(C_m)$; $V(C_i)\cap V(C_{i+1})=\{x_i\}$ for $i=1,\dots,m-1$ *and* $V(C_i)\cap V(C_j)=\emptyset$ for $|i-j|>1$ (*see Fig.* 6(b)).

Indeed, the circuits formed by R_1 and L_1 are directed, hence odd; so, by (14), they form such a collection. From now on the orientations do not play a role in the proof of this claim. Assume the odd circuits in (15) are chosen with m as small as possible. As H_1 is non-bipartite it contains a circuit through $h_{k,1}$ and $h_{1,2}$. So x_1 is not a one node cutset in H_1. Hence, there exists a path P in H_1 from some node $y\in V(C_1)\backslash\{x_1\}$ to some node $z\in(V(C_2)\cup\cdots\cup V(C_m))\backslash\{x_1\}$, that is internally node-disjoint from $C_1,\dots,C_m$. If $y\neq h_{k,1}$ or $z\notin\{x_2,\dots,x_m\}$, then using the oddness of the circuits $C_1,\dots,C_m$ we can again derive the existence of a configuration as in Fig. 4. As (G,Σ) has no odd-K_4, this is not possible. So $y=h_{k,1}$ and $z=x_j$ with $j\in\{1,\dots,m\}$. As we have chosen $C_1,\dots,C_m$ with m minimal, j is even. But this implies that (G,Σ) contains an odd chain. Its beads are: $C_1,\dots,C_j$, together with an odd circuit consisting of: P; a path from x_j to x_m in $C_{j+1}\cup\cdots\cup C_m$; and an $x_m h_{k,1}$-path Q of the appropriate parity in $H_2\cup\cdots\cup H_k$. Q exists as at least one of $H_2,\dots,H_k$ is non-bipartite. As (G,Σ) has no odd chain this yields a final contradiction. □

With this claim, the final part of the algorithm is straightforward. Assume the numbering of the beads is such that H_k has the maximum number of edges. Define $G_1 := H_1 \cup \cdots \cup H_{k-1}$, $\Sigma_1 := \Sigma \cap E(G_1)$, $G_2 := H_k$, $\Sigma_2 := \Sigma \cap E(G_2)$, $u_1 := h_{k-1,k}$ and $u_2 := h_{k,1}$. Then $(G,\Sigma) = (G_1,\Sigma_1) \oplus_{\{u_1,u_2\}} (G_2,\Sigma_2)$. As Case 1 does not apply, (G,Σ) is not a chain, so $|E(G_1)|, |E(G_2)| \geqslant 3$.

Define a new oriented signed graph $(\tilde{G}_1, \tilde{\Sigma}_1)$ as follows: Start with (G_1,Σ_1), with the arcs oriented as in D. If L_k is non-empty add to (G_1,Σ_1) a directed arc from u_1 to u_2 with the same parity as L_k. If R_k is non-empty add a directed arc from u_2 to u_1 with the same parity as R_k. Call the resulting directed graph D_1. Similarly we define $(\tilde{G}_2, \tilde{\Sigma}_2)$ and D_2 (where the new arcs are only added if none of $R_1,\ldots,R_{k-1}$, resp. none of $L_1,\ldots,L_{k-1}$ are empty). Obviously D is strong odd, if and only if D_1 and D_2 are strong odd. Moreover, $(\tilde{G}_1, \tilde{\Sigma}_1)$ and $(\tilde{G}_2, \tilde{\Sigma}_2)$ have no odd-K_4 and no odd chain (compare with (13)). □

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