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**APPLICATION OF THE WIGNER  
DISTRIBUTION  
TO HARMONIC ANALYSIS OF  
GENERALIZED STOCHASTIC  
PROCESSES**

A.J.E.M. JANSSEN

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## PREFACE

The present tract deals with applications of the Wigner distribution to harmonic analysis of (generalized) stochastic processes.

The Wigner distribution was introduced in the thirties by E.P. Wigner in his paper [Wi]: On the quantum correction for thermodynamic equilibrium, Phys. Rev. 40 (1932), 749-759, as a new concept in quantum mechanics (we refer to [GSu] for an exposition of the role of the Wigner distribution in quantum mechanics; cf. also [B1], section 26 and 27.26. 2,3,4 and 5). In the past 30 years the Wigner distribution has received further attention as a useful tool in several branches of applied mathematics and engineering, such as radar analysis, Fourier optics and geometrical optics (cf. [Ba], [Br], [P], [PH], [Re], [St], [Su]). In some of these branches the Wigner distribution appears in a somewhat different form, and is called the ambiguity function.

In 1948 J. Ville ([V]: Théorie et application de la notion de signal analytique, Câbles et Transmission 2 (1948), 61-74) proposed the Wigner distribution as a tool for harmonic analysis of signals. The theory of harmonic analysis (as created in the thirties by Wiener and others) is satisfactory for signals with certain stationarity properties. This excludes signals like those which are limited in time, such as pieces of music. Signals of the latter kind need something like a local spectrum which depends on observation time.

Let us go into some more detail. Let  $f: \mathbb{R} \rightarrow \mathbb{C}$  be measurable, and assume that  $f$  belongs to the Wiener class (cf. section 4.1 of this thesis), i.e.

$$(1) \quad \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(\xi + x) \overline{f(x)} dx =: \varphi(\xi)$$

exists for every  $\xi \in \mathbb{R}$ . The spectral density function  $s$  can be defined roughly as the Fourier transform of  $\varphi$ . It may be shown that

$$(2) \quad s(\lambda) = \lim_{T \rightarrow \infty} \frac{1}{2T} \left| \int_{-T}^T f(x) e^{-2\pi i \lambda x} dx \right|^2$$

(the right hand side is to be interpreted as a limit in distributional sense). E.g., if  $f(x) = \sum_{n=1}^N A_n \exp(2\pi i \lambda_n x)$  ( $x \in \mathbb{R}$ ) is a trigonometric polynomial, then  $s$  turns to be the measure on  $\mathbb{R}$  concentrated in the points  $\lambda_1, \dots, \lambda_N$  with masses  $|A_1|^2, \dots, |A_N|^2$  (we have assumed  $\lambda_n \neq \lambda_m$  for  $n \neq m$ ).

The above analysis does not apply to functions  $f$  for which the limits in (1) do not exist. Even in case they do exist (e.g., if  $f$  is limited in time, whence  $\varphi$  is identically equal to zero) the formulas (1) and (2) may fail to give a useful description. A further objection concerns the fact that the spectral density function does not contain a time variable, and this certainly does not agree with the idea one has of the spectral density function when the signal  $f$  represents a piece of music.

In 1970 W.D. Mark published a paper ([Ma]: Spectral analysis of the convolution and filtering of non-stationary processes, J. Sound Vib. (1970) 11 (1), 19-63) in which a modification of the theory of harmonic analysis was proposed so as to be able to handle more general signals as well. In this paper expressions like

$$(3) \quad \left| \int_{-\infty}^{\infty} w(\eta - \xi) f(\xi) e^{-2\pi i \xi \lambda} d\xi \right|^2 =: S_w(\eta, \lambda)$$

(with  $\eta \in \mathbb{R}$ ,  $\lambda \in \mathbb{R}$ ) occur. Here  $f$  is the signal to be analyzed, and  $w$  is a weight function with  $\int_{-\infty}^{\infty} |w(\xi)|^2 d\xi = 1$ . The function  $S_w$  was called by Mark the physical spectrum of  $f$ . Note that this physical spectrum contains both a frequency variable  $\lambda$  and a time variable  $\eta$ .

It was pointed out by M.B. Priestley (cf. [Pri]: Some notes on the physical interpretation of spectra of non-stationary stochastic processes, J. Sound Vib. (1971) 17 (1), 51-54) that the term "physical spectrum" for  $S_w$  is not quite correct, as  $S_w$  heavily depends on the choice of the weight function  $w$ . It is more proper to say that  $S_w$  is a "candidate" for the physical spectrum of  $f$ : E.g., if  $f$  belongs to the Wiener class, it seems to be adequate to take a  $w$  that averages over a long range of the real line (cf. (2)).

We sketch an alternative way leading to expressions like (3). Let  $g$  be a weight function of two variables, and put



$$(4) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, \omega) e^{-2\pi i \omega \xi} f(x + \eta + \xi) \overline{f(x + \eta - \xi)} dx d\omega := \varphi_g(\eta, \xi).$$

If  $f$  belongs to the Wiener class, and  $g$  is the characteristic function of the set  $[-T, T] \times [-T^{-1}, T^{-1}]$ , then we get something like (1) if  $T$  tends to infinity. Hence (1) can be regarded as a limit case of (4).

There are two differences between (1) and (4). Firstly, (4) involves a time-frequency average, whereas in (1) only time averages occur. Secondly, the time variable  $\eta$  occurs explicitly in (4).

Performing Fourier transformation in (4) with respect to  $\xi$  we obtain (by a formal appeal to Fubini's theorem)

$$(5) \quad \begin{aligned} S'_g(\eta, \lambda) &:= \int_{-\infty}^{\infty} e^{-2\pi i \lambda \xi} \varphi_g(\eta, \xi) d\xi = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, \omega) \left\{ \int_{-\infty}^{\infty} e^{-2\pi i \xi (\lambda + \omega)} f(x + \eta + \xi) \overline{f(x + \eta - \xi)} d\xi \right\} dx d\omega. \end{aligned}$$

The expression between { } is called (apart from a simple transformation of variables) the Wigner distribution of  $f$  at the point  $(x + \eta, \lambda + \omega)$  in time-frequency plane. Although the Wigner distribution of  $f$  may assume negative values, the function  $S'_g$  is non-negative for a fairly large class of weight functions  $g$ . It is, however, in general not possible to concentrate  $g$  in arbitrarily small areas without destroying non-negativity of  $S'_g$ . We refer to [B1], [B2] and [PH] where this fact is related to Heisenberg's uncertainty principle.

The  $S'_g$ 's of (5) are closely related to the  $S'_w$ 's of (3). If  $w$  is a weight function as in (3), then it may be shown that  $S'_w = S'_g$ , where  $g$  equals (apart from a transformation of variables) the Wigner distribution of  $w$  (cf. 2.5 of appendix 2 of this tract and [Ma], (82)). In particular,  $S'_g$  is non-negative.

The above discussion about (local) spectra applies in a more general setting, viz. in the case of signals with a random character (noise process). This motivates us to study (generalized) stochastic processes with or

without stationarity properties (examples of non-stationary processes in electrical engineering are the Barkhausen effect and  $\frac{1}{f}$  - noise). Now we have to consider averaged Wigner distributions (averaged over the collection of random signals), and this involves integrals like

$$(6) \quad \int_{-\infty}^{\infty} e^{-2\pi i \xi (\lambda + \omega)} R(x + \eta + \xi, x + \eta - \xi) d\xi.$$

Here  $R$  is a positive definite function (autocorrelation function) of two variables (compare (6) and the expression in (5) between  $\{ \}$ ). The integral in (6) is the averaged Wigner distribution of the process at the point  $(x + \eta, \lambda + \omega)$  in time-frequency plane (also cf. [Ma], (82)).

A further application of averaged Wigner distributions concerns (second order) simulation of noise processes. To explain this, let there be given some stochastic process with finite second order moments. The problem is to construct an "elementary" process that agrees as much as possible with the given process as far as second order moments are concerned (the first order moments are usually assumed to be zero). For the elementary processes we take shot noise processes, i.e. processes of the form  $\sum_n p_n g(a_n - x)$ , and "random Fourier series" processes, i.e. processes of the form  $\sum_n p_n e^{-2\pi i b_n x} g(x)$ , or a generalized version of both types, viz. processes of the form  $\sum_n p_n e^{-2\pi i b_n x} g(a_n - x)$  which we call "noise shower" processes.

Here  $p_n$ ,  $a_n$  and  $b_n$  are random variables, and  $g$  is a fixed function (generalized or not). The first two kinds of processes are suited for simulation of processes with a stationary character, but the third one allows to handle non-stationary processes as well. The Wigner distribution of the process to be simulated indicates how to distribute the parameters  $a_n$  and  $b_n$  of the noise quanta over the time-frequency plane.

By now it will be clear that Fourier theory is essential for our investigations. As we want to study not just stochastic processes, but generalized stochastic processes (like white noise and the processes mentioned in the previous paragraph) we have to start from a theory of generalized functions adapted to the needs of Fourier analysis. Such a theory can be built on the test function space  $S$  to which appendix 1 is devoted; we note that the Fourier transform is a continuous linear bijection of  $S$ . Although there is a large amount of literature on generalized

stochastic processes (we refer in particular to the recent book of Schwartz [S]), it seems that the test function space  $S$  with its facilities for harmonic analysis has hardly been studied in this respect. It is to be noted that important theorems about cylindrical measures on the dual space  $S^*$  (space of generalized functions), such as Minlos' theorem, Bochner's theorem and theorems on regularity, still hold (certain cylindrical measures on  $S^*$  can be identified with our generalized stochastic processes). This is due to the fact that  $S$  can be endowed with a nuclear topology.

The fact that our space of generalized functions is suited for Fourier analysis has a further consequence. It is possible to develop a satisfactory theory of convolution operators of  $S$  and  $S^*$  (cf. [J2] and appendix 2). This convolution theory turns out to be convenient, particularly for stationary and ergodic processes.

The space  $S$  is a good starting point for a theory of generalized stochastic processes, but not in every respect. For our present aims it is certainly quite satisfactory, but difficulties arise with local behaviour of generalized functions and processes. This is connected with the fact that  $S$  does not contain functions of compact support.

We shall now give a survey of this tract. In chapter 1 we give a number of (more or less) equivalent definitions of the notion of generalized stochastic process, and we prove a version of Minlos' theorem (on the  $\sigma$ -additivity of cylindrical measures). Furthermore we consider strict sense stationary and ergodic generalized stochastic processes, and we also pay attention to Gaussian processes. Finally, chapter 1 contains a section about the embedding of "ordinary" processes defined on  $\mathbb{R}$ .

Chapter 2 is devoted to the first and second order moments of generalized stochastic processes, and the notions of expectation function, auto-correlation function and Wigner distribution of these processes are introduced. We pay attention to (second order) time stationarity and frequency stationarity, and we indicate a relation between processes with independent values and frequency stationary processes. Moreover, we define spectral density function and measure of time stationary processes, and we consider random measures in their connection with time stationary processes.

In chapter 3 convolution theory (cf. [J2]) is applied to both stationary and non-stationary processes. We prove a theorem on the representation of time stationary processes as filtered white noise processes, and we also

prove an ergodic theorem. We further consider shot noise processes, "random Fourier series" processes and "noise shower" processes. This gives rise to simulation theorems.

In chapter 4 the Wiener theory of generalized harmonic analysis (spectral analysis) is generalized in two respects. Firstly generalized functions are admitted, and secondly a generalization is obtained by considering Wigner distributions of functions instead of their spectral density functions (the Wigner distribution of a function always makes sense, but the spectral density function may not exist). Some applications to (generalized) stochastic processes are given.

This thesis contains 4 appendices. The first one lists all we need concerning the spaces  $S$  and  $S^*$ . E.g., it contains a survey of the theory of generalized functions as presented in [B1], information about the topological structure of  $S$  and  $S^*$ , as well as theorems on continuous linear transformations in these spaces. Furthermore we provide  $S^*$  with a  $\sigma$ -algebra (the  $\sigma$ -algebra generated by all open sets in  $S^*$ ) that has among its members (the embeddings) of the  $L_p(\mathbb{R})$ -spaces, (the embeddings) of the classes of embeddable continuous and measurable functions and the (generalized) Wiener class.

The second appendix gives a survey of the main notions and theorems of [J2] on convolution theory in  $S$  and  $S^*$ .

The third appendix contains information about the Wigner distribution for smooth and generalized functions, and about time-frequency convolution operators.

The fourth appendix contains a theorem on translation invariance of generalized functions and a theorem on generalized functions of positive type.

Notation. We use Church's lambda calculus notation, but instead of his  $\lambda$  we have  $\Upsilon$ , as suggested by Freudenthal: if  $S$  is a set, then putting  $\Upsilon_{x \in S}$  in front of an expression (usually containing  $x$ ) means to indicate the function with domain  $S$  and with the function values given by the expression. We write  $\Upsilon_x$  instead of  $\Upsilon_{x \in S}$  if it is clear from the context which set  $S$  is meant.

We further have the usual notations for the set theoretical operations (we have the symbol  $\div$  for the symmetric difference, and, if  $f$  is a mapping from a set  $A$  into a set  $B$ , then  $f^{-1}(C)$  denotes the set  $\{a \in A \mid f(a) \in C\}$

for any subset  $C$  of  $B$ ).

The sets of all real numbers and complex numbers are denoted by  $\mathbb{R}$  and  $\mathbb{C}$  respectively. The set of all integers, all positive integers, all non-negative integers, all rationals are denoted by  $\mathbb{Z}$ ,  $\mathbb{N}$ ,  $\mathbb{N}_0$  and  $\mathbb{Q}$  respectively.

Let  $(\Omega, \Lambda, P)$  be a  $\sigma$ -finite measure space ( $\Omega$  is a non-empty set,  $\Lambda$  is a  $\sigma$ -algebra of subsets of  $\Omega$ ,  $P$  is a  $\sigma$ -finite positive measure on  $\Lambda$ ). We shall in general not assume  $\Lambda$  to be completed with respect to  $P$ . Let  $1 \leq p \leq \infty$ . We denote by  $L_p(\Omega)$  the collection of all mappings  $f: \Omega \rightarrow \mathbb{C}$  such that  $f$  is measurable over  $\Omega$  and  $(\int_{\Omega} |f|^p dP)^{1/p} < \infty$  (if  $p = \infty$  the left hand side of this inequality is interpreted as the essential supremum of  $|f|$ ). The collection of all classes of equivalent functions in  $L_p(\Omega)$  is denoted by  $L_p(\Omega)$ . The  $p$ -norm in  $L_p(\Omega)$  or  $L_p(\Omega)$  is denoted by  $\| \cdot \|_p$ . In cases where it is not necessary to discriminate between functions or classes of functions, we shall use the notation  $L_p(\Omega)$  for both the set of functions and the set of classes of functions. We shall use  $L_p(\Omega)$  only in cases where we want to emphasize that functions and not function classes are meant.

If  $1 \leq p \leq \infty$ ,  $f \in L_p(\Omega)$ ,  $g \in L_q(\Omega)$  ( $q$  denotes the conjugate exponent of  $p$ ), then we write

$$\langle f, g \rangle = \int_{\Omega} f \cdot \bar{g} dP.$$

If  $n \in \mathbb{N}$ ,  $\Omega = \mathbb{R}^n$ ,  $\Lambda$  the class of all Borel sets of  $\mathbb{R}^n$ ,  $P$  Lebesgue measure on  $\mathbb{R}^n$ , then we write

$$(f, g) = \int_{\mathbb{R}^n} f(x) \overline{g(x)} dx, \quad \|f\| = ((f, f))^{1/2}$$

for  $f \in L_2(\mathbb{R}^n)$ ,  $g \in L_2(\mathbb{R}^n)$ .

The classes of all Borel sets in  $\mathbb{R}^n$  and  $\mathbb{C}^n$  (with  $n \in \mathbb{N}$ ) are denoted by  $\mathcal{B}(\mathbb{R}^n)$  and  $\mathcal{B}(\mathbb{C}^n)$  respectively.

We give a further notational convention. Ordinary functions and stochastic processes are denoted by lower case characters, whereas generalized functions and stochastic processes are denoted by capitals (an exception is made for the elements of  $C$  and  $M$ ; cf. 3 and 7 of appendix 2). We also refer to the index of symbols on page 160.

## CHAPTER 1

## GENERALIZED STOCHASTIC PROCESSES

In section 1 of this chapter we give several definitions of the notion of generalized stochastic process; all these definitions are to some extent equivalent. Our definitions may seem to be rather complex and abstract, and in fact they have an indirect character in the sense that things are defined by the effect they have on other things. Therefore we shall try here to say in more common words what it is all about. We have to be a bit vague at this stage, of course.

We may think of a stochastic process as a complex-valued function of two variables  $t$  and  $\omega$ ;  $t$  runs through the reals ( $t$  may represent the time), and  $\omega$  runs through some probability space  $\Omega$ . If  $t$  is fixed we have a function defined on  $\Omega$ , i.e. a stochastic variable. If  $\omega$  is fixed we have a complex-valued function of  $t$ . Hence choosing an  $\omega$  means choosing an element from a collection of complex-valued functions of  $t$ , according to some probability measure on the set of all those functions.

Our generalization of this concept of stochastic process amounts to replacing, in some form or other, the functions of  $t$  by generalized functions. This transition is modelled after the one leading from the set  $S$  of smooth functions to the set  $S^*$  of generalized functions (cf. appendix 1, 1.9 and theorem 3.3). The elements of  $S^*$  are no longer proper functions. Two ways have been used for the introduction of  $S^*$ . The first one depends on the smoothing operators  $N_\alpha$  (cf. appendix 1, 1.4). For each  $\alpha > 0$ ,  $N_\alpha$  maps  $S$  into  $S$ . Many "bad" functions (which are outside  $S$  but still functions) are mapped by the  $N_\alpha$ 's into  $S$  too (cf. appendix 1, 1.5). A bad function  $f$  leaves a trace  $N_\alpha f$  ( $\alpha > 0$ ) in  $S$ . Noting the basic property of these traces, viz.  $N_\alpha(N_\beta f) = N_{\alpha+\beta}f$ , we get a definition of  $S^*$ . That is, elements of  $S^*$  are described by means of their traces. The second way amounts to defining  $S^*$  as a kind of dual of  $S$ , i.e. the elements of  $S^*$  are defined as certain linear functionals on  $S$ . This  $S^*$  contains an embedding of  $S$ , since  $\int_{g \in S} (f, g)$  (where  $(f, g)$  is the ordinary inner product  $\int_{-\infty}^{\infty} f(t) \overline{g(t)} dt$ ) is such a linear functional. It turns out that we get the same  $S^*$  as by the previous generalization (cf. theorem 3.3).

We can now discuss the set  $S_{\Omega, P}^*$  of all generalized stochastic processes

of order  $p$  (with  $1 \leq p \leq \infty$ ). We are inclined to conceive these processes as functions of  $t$  and  $\omega$ , but they are not. Nevertheless we can get to a set  $S_{\Omega, p}^*$  in which we have some operations that are extensions of known operations on functions. We mention five possibilities.

(i) If  $g$  is a smooth function of time, and if  $\underline{X} \in S_{\Omega, p}^*$  we can form an inner product  $(\underline{X}, g)$  over  $\mathbb{R}$ ; its value is a function of  $\omega$  only (in fact it lies in  $L_p(\Omega)$ ). This idea gives rise to the first definition (cf. 1.1.1.).

(ii) If  $\alpha$  is a positive number, the "generalized" dependence on  $t$  can be smoothed: if  $\underline{X} \in S_{\Omega, p}^*$  then  $N_{\alpha} \underline{X}$  is a function of  $t$  and  $\omega$  depending smoothly on  $t$ . That is,  $N_{\alpha} \underline{X}$  is a smooth process, and  $\underline{X}$  can be described by its trace of smooth processes. This idea leads to the definition in 1.1.5.

(iii) If  $h \in L_q(\Omega)$  then we can take the inner product over  $\Omega$  with every  $f \in L_p(\Omega)$ . Let us denote it by  $\langle f, h \rangle$ . The same operation applied to  $\underline{X}$  instead of  $f$  is expected to lead to a function  $\langle \underline{X}, h \rangle$  of  $t$  only, but it is still a generalized function. So  $\underline{X}$  can be described as a mapping of  $L_q(\Omega)$  into  $S^*$ . This gives rise to the definition in 1.1.23.

The transition from (i) to (iii) is easy to grasp in the following terms. In (i) a stochastic process is described by a mapping of  $S$  into  $L_p(\Omega)$ , and such a thing gives rise to a mapping of the dual space of  $L_p(\Omega)$  into the dual space of  $S$ , that is of  $L_q(\Omega)$  into  $S^*$  (if  $p \neq \infty$ ).

(iv) The idea of describing a stochastic process as a set of functions of  $t$  (for each  $\omega \in \Omega$  we consider  $\bigcup_{t \in \mathbb{R}} \underline{X}(t, \omega)$ ) with a probability measure on the set of these functions, can be generalized, simply by taking generalized functions of  $t$  instead of ordinary functions. This leads to the definition of 1.1.15.

(v) In order to describe a function of two variables it is often convenient to represent it by separation of variables, i.e. by a sum  $\sum_k \psi_k(t) \varphi_k(\omega)$ . If the sum is "decently" convergent, this represents a function of  $t$  and  $\omega$ , but with a weaker notion of convergence we can get generalized functions. In the theory of  $S^*$  this is achieved by series  $\sum_{k=0}^{\infty} c_k \cdot \psi_k$  where the  $\psi_k$ 's are the Hermite functions, and the  $c_k$ 's are complex numbers with  $c_k = O(e^{-k\epsilon})$  ( $k \in \mathbb{N}_0$ ) for every  $\epsilon > 0$  (cf. appendix 1, 1.10). This idea can be used with stochastic processes too (cf. 1.1.17).

Conceptually this method (v) seems to be the simplest of all and the least indirect one: it describes a generalized stochastic process by means of a sequence of elements of  $L_p(\Omega)$ . But (v) is not always the most convenient

one. In particular, the behaviour of these expansions under time shifts  $t \rightarrow t + a$  (with  $a \in \mathbb{R}$ ) is definitely unpleasant.

We conclude the discussion on the notion of generalized stochastic process by the following remark. It depends on the kind of application which one of the above methods (i), ..., (v) is to be preferred. It certainly pays to show their equivalence, so that we are always able to apply the most convenient one.

Section 2 of this chapter is devoted to the concepts of strict sense stationarity and ergodicity for generalized stochastic processes. These concepts can be formulated conveniently in terms of the probability measure in  $S^*$  arising from a generalized stochastic process (cf. (iv) above). We speak of strict sense time stationarity, e.g., if this probability measure is invariant with respect to the time shifts  $T_a$  ( $a \in \mathbb{R}$ ). If there are no non-trivial sets in  $S^*$  which are invariant with respect to the time shift we speak of time ergodicity (the trivial sets are the sets with measure 0 or 1). We further pay attention to Gaussian generalized stochastic processes and Gaussian white noise.

In section 3 we introduce a class of embeddable "ordinary" stochastic processes, and prove a theorem on the embedding of these processes in our system of generalized stochastic processes (this embedding theorem is conveniently formulated in terms of the first method of the above). We further prove a theorem, stating that a large class of strict sense time stationary ordinary processes (in the classical sense) have strict sense time stationary embeddings. A theorem of the same kind is proved for ordinary processes that are strict sense time stationary and ergodic.

### 1.1. DEFINITION OF GENERALIZED STOCHASTIC PROCESSES

1.1.1. Let  $\Omega$  be a non-empty set,  $\Lambda$  a  $\sigma$ -algebra of subsets of  $\Omega$  and  $P$  a probability measure on  $\Lambda$  (whence  $(\Omega, \Lambda, P)$  is a probability space).

Let  $p$  be an element of the extended real number system with  $1 \leq p \leq \infty$ .

DEFINITION. A *generalized stochastic process* of order  $p$  is an anti-linear mapping  $\underline{X} = \bigvee_{f \in S} (\underline{X}, f)$  of  $S$  into  $L_p(\Omega)$  such that  $\|(\underline{X}, f_n)\|_p \rightarrow 0$

( $n \rightarrow \infty$ ) for every sequence  $(f_n)_{n \in \mathbb{N}}$  in  $S$  with  $f_n \rightarrow 0$ . (cf. appendix 1, 1,12)

The class of all generalized stochastic processes of order  $p$  is denoted by  $S_{\Omega, p}^*$ . If  $\underline{X} \in S_{\Omega, p}^*$ , and  $\hat{\underline{X}} = \bigvee_{(f, \omega) \in S \times \Omega} \hat{\underline{X}}(f, \omega)$  is a mapping of  $S \times \Omega$  into  $\mathbb{C}$ ,



then we call  $\hat{\underline{X}}$  a *representative* of  $\underline{X}$  if  $\Psi_{\omega \in \Omega} \hat{\underline{X}}(f, \omega)$  is a representative of  $(\underline{X}, f)$  for every  $f \in S$ .

1.1.2. REMARKS. 1. Although it will not be used in this thesis, the following fact is of interest. Let  $\underline{X} = \bigcup_{f \in S} (\underline{X}, f)$  be an anti-linear mapping of  $S$  into  $L_p(\Omega)$ . Since  $S$  is a bornological space with the topology  $\tau$ ,  $\underline{X} \in S_{\Omega, p}^*$  if and only if  $\underline{X}$  is continuous (cf. the proof of appendix 1, theorem 4.5 (iii), 2.6 and [FW], §11.32). Compare also [GW], Kap. III, §1.2. If  $\underline{X} \in S_{\Omega, p}^*$ , and  $S$  is endowed with the weak topology of appendix 1, 2.2, then  $\underline{X}$  need not be continuous.

2.  $S_{\Omega, p}^*$  is a linear space if addition and scalar multiplication are defined in the obvious way. We have  $S_{\Omega, p_2}^* \subset S_{\Omega, p_1}^*$  if  $1 \leq p_1 \leq p_2 \leq \infty$ .

3. It is often not necessary to discriminate between elements of  $S_{\Omega, p}^*$  and their representatives.

1.1.3. We consider linear mappings of  $S_{\Omega, p}^*$ .

THEOREM. Let  $T$  be a linear operator of  $S$  with an adjoint (cf. appendix 1, 4.3), and let  $\underline{X} \in S_{\Omega, p}^*$ . If  $\underline{Y}$  is defined by

$$(\underline{Y}, f) := (\underline{X}, T^* f) \quad (f \in S),$$

then  $\underline{Y} \in S_{\Omega, p}^*$ .

PROOF. It follows from appendix 1, theorem 4.7 (ii) that  $T^*$  is a continuous linear operator of  $S$ , hence  $\underline{Y}$  is continuous as a mapping of  $S$  into  $L_p(\Omega)$ . Anti-linearity of  $\underline{Y}$  is obvious, so  $\underline{Y} \in S_{\Omega, p}^*$ .  $\square$

1.1.4. Theorem 1.1.3 motivates the following definition.

DEFINITION. Let  $T$  be a linear operator of  $S$  with an adjoint, and let  $\underline{X} \in S_{\Omega, p}^*$ . Then  $T\underline{X}$  is defined by

$$(T\underline{X}, f) = (\underline{X}, T^* f) \quad (f \in S).$$

Note that  $T$  is a linear mapping of  $S_{\Omega, p}^*$  into itself.

1.1.5. In [J1], 1.3 generalized stochastic processes are introduced in a somewhat different way. First of all smooth stochastic processes of order  $p$  are defined as mappings  $\underline{x}$  of  $\mathbb{T}$  into  $L_p(\Omega)$  satisfying

$$(*) \int_{\mathbb{T}} \int_{\Omega} \underline{x}(t) \bar{f} dP \in S \quad (f \in L_q(\Omega))$$

(cf. [J1], 1.1.2;  $q$  denotes the conjugate exponent of  $p$ ). The class of all smooth stochastic processes of order  $p$  is denoted by  $S_{\Omega,p}$  (this is a linear space). Denote for  $f \in L_q(\Omega)$  and  $\underline{x} \in S_{\Omega,p}$  by  $\langle \underline{x}, f \rangle$  the function given in (\*).

It has been proved (cf. [J1], 1.2.5) that every continuous linear operator  $T$  of  $S$  (cf. appendix 1, 4.2) can be extended to a linear operator of  $S_{\Omega,p}$  (again denoted by  $T$ ) such that

$$T(\langle \underline{x}, f \rangle) = \langle T\underline{x}, f \rangle \quad (\underline{x} \in S_{\Omega,p}, f \in L_q(\Omega)).$$

The word "extended" is motivated as follows: we can regard  $S$  as a subspace of  $S_{\Omega,p}$  by identifying  $h \in S$  and  $\hat{h} := \int_{(t,\omega) \in \mathbb{T} \times \Omega} h(t) \in S_{\Omega,p}$ ; now  $T\hat{h} = \int_{(t,\omega) \in \mathbb{T} \times \Omega} (Th)(t)$ .

Next generalized stochastic processes of order  $p$  are defined (cf. [J1], 1.3.1) as mappings  $\underline{X} = \int_{\alpha > 0} X_{\alpha}$  of the set of positive real numbers into  $S_{\Omega,p}$  satisfying

$$\underline{X}_{\alpha+\beta} = N_{\alpha} X_{\beta} \quad (\alpha > 0, \beta > 0);$$

here the  $N_{\alpha}$ 's are the smoothing operators of appendix 1,1.4 ( $N_{\alpha} X_{\beta}$  is well-defined for  $\alpha > 0, \beta > 0$  according to the foregoing: take  $T = N_{\alpha}, \underline{x} = X_{\beta}$ ). Denote the class of all these  $\underline{X}$  by  $\tilde{S}_{\Omega,p}^*$ .

If  $f \in L_q(\Omega), \underline{X} \in \tilde{S}_{\Omega,p}^*$ , then  $\int_{\alpha > 0} \langle X_{\alpha}, f \rangle$  is a generalized function:  $N_{\alpha} \langle X_{\beta}, f \rangle = \langle N_{\alpha} X_{\beta}, f \rangle = \langle X_{\alpha+\beta}, f \rangle$  according to the foregoing (cf. appendix 1, 1.9 and [J1], 1.3.1).

The following theorems on linear transformations of the space  $\tilde{S}_{\Omega,p}^*$  have been proved (cf. [J1], 1.4.3 and 1.4.4).

(i) If  $L$  is a continuous linear functional of  $S^*$  (cf. appendix 1, 3.2), then  $L$  can be extended to a linear mapping of  $\tilde{S}_{\Omega,p}^*$  into  $L_p(\Omega)$  such that

$$L(\langle \underline{X}, f \rangle) = \int_{\Omega} L\underline{X} \bar{f} dP \quad (\underline{X} \in \tilde{S}_{\Omega,p}^*, f \in L_q(\Omega)).$$

(ii) If  $T$  is a linear operator of  $S$  with an adjoint, then  $T$  can be extended to a linear operator  $\tilde{S}_{\Omega, p}^*$  (again denoted by  $T$ ) such that

$$T(\langle \underline{X}, f \rangle) = \langle T\underline{X}, f \rangle \quad (\underline{X} \in \tilde{S}_{\Omega, p}^*, f \in L_q(\Omega)).$$

1.1.6. The following result (stated as a theorem without proof) provides a link between the definitions given in 1.1.1 and [J1], 1.3. Let  $L_f$  denote for  $f \in S$  the continuous linear functional  $\bigcup_{F \in S^*} (F, f)$  of  $S^*$ .

**THEOREM.** (i) If  $\underline{X} \in S_{\Omega, p}^*$  (cf. 1.1.1), then there exists exactly one  $\underline{Y} \in \tilde{S}_{\Omega, p}^*$  (cf. 1.1.5) such that  $L_f \underline{Y} = (\underline{X}, f)$  for  $f \in S$ .

(ii) If  $\underline{Y} \in \tilde{S}_{\Omega, p}^*$ , then there exists exactly one  $\underline{X} \in S_{\Omega, p}^*$  such that  $(\underline{X}, f) = L_f \underline{Y}$  for  $f \in S$ .

1.1.7. **REMARK.** Let  $T$  be a linear operator of  $S$  with an adjoint. If  $\underline{X} \in S_{\Omega, p}^*$  and  $\underline{Y} \in \tilde{S}_{\Omega, p}^*$  are related to each other as in theorem 1.1.6, then the same holds for  $T\underline{X}$  and  $T\underline{Y}$  (cf. 1.1.4 and 1.1.5). Hence  $L_{T^*g} \underline{Y} = (\underline{X}, T^*g) = (T\underline{X}, g) = L_g T\underline{Y}$  for  $g \in S$ .

1.1.8. We define a notion of convergence for sequences of generalized stochastic processes.

**DEFINITION.** Let  $\underline{X}_n \in S_{\Omega, p}^*$  ( $n \in \mathbb{N}$ ), and assume that  $\|(\underline{X}_n, f)\|_p \rightarrow 0$  ( $n \rightarrow \infty$ ) for every  $f \in S$ . We then say that  $\underline{X}_n$  converges to zero in  $S_{\Omega, p}^*$ -sense, and write  $\underline{X}_n \rightarrow 0$  ( $S_{\Omega, p}^*$ ). If  $\underline{X} \in S_{\Omega, p}^*$ ,  $\underline{X}_n \in S_{\Omega, p}^*$  ( $n \in \mathbb{N}$ ), we say that  $\underline{X}_n$  converges to  $\underline{X}$  in  $S_{\Omega, p}^*$ -sense and write  $\underline{X}_n \rightarrow \underline{X}$  ( $S_{\Omega, p}^*$ ) if  $\underline{X}_n - \underline{X} \rightarrow 0$  ( $S_{\Omega, p}^*$ ).

1.1.9. **THEOREM.** Let  $T$  be a linear operator of  $S$  with an adjoint, and let  $(\underline{X}_n)_{n \in \mathbb{N}}$  be a sequence in  $S_{\Omega, p}^*$  with  $\underline{X}_n \rightarrow 0$  ( $S_{\Omega, p}^*$ ). Then  $T\underline{X}_n \rightarrow 0$  ( $S_{\Omega, p}^*$ ).

**PROOF.** Follows from definition 1.1.8 and 1.1.4.  $\square$

1.1.10. We can define convergence in the spaces  $S_{\Omega, p}$  and  $\tilde{S}_{\Omega, p}^*$  as well (cf. 1.1.5). If  $\underline{x}_n \in S_{\Omega, p}$  ( $n \in \mathbb{N}$ ), and if there is an  $A > 0$ ,  $B > 0$  such that

$$\|\underline{x}_n(t)\|_p \exp(\pi A(\operatorname{Re} t)^2 - \pi B(\operatorname{Im} t)^2) \rightarrow 0 \quad (n \rightarrow \infty)$$

uniformly in  $t \in \mathbb{C}$ , then we say that  $\underline{x}_n$  converges to zero in  $S_{\Omega, p}$ -sense and write  $\underline{x}_n \rightarrow 0 (S_{\Omega, p})$ . If  $\underline{x}_n \in \tilde{S}_{\Omega, p}^*$  and  $(\underline{x}_n)_\alpha \rightarrow 0 (S_{\Omega, p})$  for every  $\alpha > 0$ , then we say that  $\underline{x}_n$  converges to zero in  $\tilde{S}_{\Omega, p}^*$ -sense and write  $\underline{x}_n \rightarrow 0 (\tilde{S}_{\Omega, p}^*)$ .

The following theorem holds (its proof is omitted).

**THEOREM.** Let  $\underline{x}_n \in S_{\Omega, p}^*$ ,  $\underline{y}_n \in \tilde{S}_{\Omega, p}^*$ , and assume that  $\underline{x}_n$  and  $\underline{y}_n$  are related to each other as  $\underline{x}$  and  $\underline{y}$  in theorem 1.1.6 ( $n \in \mathbb{N}$ ). Then  $\underline{x}_n \rightarrow 0 (S_{\Omega, p}^*)$  if and only if  $\underline{y}_n \rightarrow 0 (\tilde{S}_{\Omega, p}^*)$ .

1.1.11. We give a useful criterion for  $S_{\Omega, p}^*$ -convergence.

**THEOREM.** Let  $\underline{x}_n \in S_{\Omega, p}^*$  ( $n \in \mathbb{N}$ ), and assume that  $((\underline{x}_n, f))_{n \in \mathbb{N}}$  converges in  $L_p(\Omega)$ -sense for every  $f \in S$ . There is exactly one  $\underline{x} \in S_{\Omega, p}^*$  such that  $\underline{x}_n \rightarrow \underline{x} (S_{\Omega, p}^*)$ .

**PROOF.** Define  $(\underline{x}, f) := \lim_{n \rightarrow \infty} (\underline{x}_n, f)$  for  $f \in S$ . We are going to show continuity of  $\underline{x}$ . Let  $\alpha > 0$ . Then  $\bigcup_{f \in L_2(\mathbb{R})} (\underline{x}_n, N_\alpha f)$  is a continuous anti-linear mapping of  $L_2(\mathbb{R})$  into  $L_p(\Omega)$  ( $n \in \mathbb{N}$ ), and we have  $\|(\underline{x}_n, N_\alpha f) - (\underline{x}, N_\alpha f)\|_p \rightarrow 0$  for every  $f \in L_2(\mathbb{R})$ . Hence  $(\|(\underline{x}_n, N_\alpha f)\|_p)_{n \in \mathbb{N}}$  is bounded for every  $f \in L_2(\mathbb{R})$ . It follows easily from the Banach-Steinhaus theorem that there is an  $M > 0$  such that

$$\|(\underline{x}_n, N_\alpha f)\|_p \leq M \|f\| \quad (n \in \mathbb{N}, f \in L_2(\mathbb{R})).$$

This implies that

$$\|(\underline{x}, N_\alpha f)\|_p \leq M \|f\| \quad (f \in L_2(\mathbb{R})).$$

It follows easily from appendix 1, 1.12 that  $\underline{x} \in S_{\Omega, p}^*$ . Also  $\underline{x}_n \rightarrow \underline{x} (S_{\Omega, p}^*)$ .

It is easy to see that there is at most one  $\underline{x} \in S_{\Omega, p}^*$  with  $\underline{x}_n \rightarrow \underline{x} (S_{\Omega, p}^*)$ .  $\square$

1.1.12. We shall meet in what follows (cf. chapter 3 and 4) conditional expectations of generalized stochastic processes. We give the following definition (compare also [Ur], II).

**DEFINITION.** Let  $\Lambda_0$  be a  $\sigma$ -algebra of subsets of  $\Omega$ , and assume that  $\Lambda_0 \subset \Lambda$ . Let  $\underline{x} \in S_{\Omega, p}^*$ . The *conditional expectation* of  $\underline{x}$  with respect to  $\Lambda_0$ , denoted by  $E(\underline{x} \mid \Lambda_0)$ , is defined by

$$E(\underline{x} \mid \Lambda_0) := \bigcup_{f \in S} E((\underline{x}, f) \mid \Lambda_0),$$

where  $E(\cdot | \Lambda_0)$  denotes ordinary conditional expectation with respect to  $\Lambda_0$  in  $L_1(\Omega, \Lambda, P)$  (cf. [Lo], Ch. VII, §24.2).

1.1.13. THEOREM. Let  $\Lambda_0$  and  $\underline{X}$  be as in definition 1.1.12.

Then  $E(\underline{X} | \Lambda_0) \in S_{\Omega, P}^*$ .

PROOF. If  $f \in S$ , then  $E((\underline{X}, f) | \Lambda_0) \in L_P(\Omega, \Lambda, P)$ , for  $E((\underline{X}, f) | \Lambda_0)$  is measurable with respect to  $\Lambda_0$  (whence with respect to  $\Lambda$ ), and

$$(*) \|E((\underline{X}, f) | \Lambda_0)\|_P \leq \|(\underline{X}, f)\|_P$$

(cf. [Lo], Ch. VIII, section 25.1,2). It follows from elementary properties of conditional expectations that  $E(\underline{X} | \Lambda_0)$  is anti-linear as a mapping of  $S$  into  $L_P(\Omega, \Lambda, P)$ , and also (from  $(*)$ ) that  $E(\underline{X} | \Lambda_0)$  is continuous. Hence  $E(\underline{X} | \Lambda_0) \in S_{\Omega, P}^*$ .  $\square$

1.1.14. We shall now indicate a relation (cf. theorem 1.1.15) between our generalized stochastic processes and probability measures on  $(S^*, \Lambda^*)$ .

Here  $\Lambda^*$  is the  $\sigma$ -algebra of  $S^*$  generated by all Borel cylinders in  $S^*$  (= the  $\sigma$ -algebra generated by all weakly open sets in  $S^*$  = the  $\sigma$ -algebra generated by all strongly open sets in  $S^*$  = the  $\sigma$ -algebra generated by all sets of the form  $\{F \in S^* | F_\alpha(t) \in O\}$ , where  $\alpha > 0$ ,  $t \in \mathbb{T}$ ,  $O$  open in  $\mathbb{T}$ ; cf. appendix 1, 5.2 and 5.2, remark).

1.1.15. Let  $P^*$  be a probability measure on  $(S^*, \Lambda^*)$  satisfying  $\|\bigcup_{F \in S^*} (F, f_n)\|_P \rightarrow 0$  ( $n \rightarrow \infty$ ) for every sequence  $(f_n)_{n \in \mathbb{N}}$  with  $f_n \xrightarrow{S} 0$  (the  $p$ -norm is taken relative to  $P^*$ ). Define  $\hat{\underline{X}}(f) := \bigcup_{F \in S^*} (F, f)$  for every  $f \in S$ . Now  $\hat{\underline{X}}$  maps  $S$  into  $L_P(S^*, \Lambda^*, P^*)$  in an anti-linear and continuous way. Hence  $\hat{\underline{X}}$  is a representative of an element of  $S_{S^*, P}^*$  (cf. 1.1.1).  $S_{S^*, P}^*$

We shall show the following converse.

THEOREM. If  $\underline{X} \in S_{\Omega, P}^*$ , then there exists exactly one probability measure  $P^*$  on  $(S^*, \Lambda^*)$  such that the simultaneous distributions of  $((\underline{X}, f_1), \dots, (\underline{X}, f_n))$  with respect to  $P$  and of  $\bigcup_{F \in S^*} ((F, f_1), \dots, (F, f_n))$  with respect to  $P^*$  are the same for every  $n \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_n \in S$ .

A generalized stochastic process  $\underline{X}$  gives rise to a cylindrical measure on the class of all Borel cylinders of  $S^*$  (cf. appendix 1, 5.2 remark) in the following way. Associate with  $f_1 \in S, \dots, f_n \in S$  (where  $n \in \mathbb{N}$ ) the probability measure on  $\mathbb{T}^n$  generated by the distribution function

of  $((\underline{X}, f_1), \dots, (\underline{X}, f_n))$ . We refer to the work of Schwartz ([S]) where general theorems on  $\sigma$ -additivity of cylindrical measures on arbitrary topological spaces are proved. These theorems also apply to our case, and can be used to prove theorem 1.1.15. Our setting, however, allows a fairly simple proof (1.1.16-18) of theorem 1.1.15. This proof is based on the properties of the Hermite functions (compare also appendix 1, 5.6).

1.1.16. Let  $\underline{X} \in S_{\Omega, p}^*$ . We need some lemmas for the proof of theorem 1.1.15.

LEMMA. Let  $q_k$  be a representative of  $(\underline{X}, \psi_k)$  for  $k \in \mathbb{N}_0$  ( $\psi_k$  is the  $k^{\text{th}}$  Hermite function; cf. appendix 1, 1.7 (iv)). Then the set  $\Omega_0$  of all  $\omega \in \Omega$  with  $\forall_{\varepsilon > 0} [q_k(\omega) = O(e^{k\varepsilon}) \ (k \in \mathbb{N}_0)]$  is measurable and has probability one.

PROOF. It is easily seen that  $\Omega_0$  is a measurable subset of  $\Omega$ .

Now let  $\varepsilon > 0$ . Since  $N_{\varepsilon/2} \xrightarrow{S} 0$  (cf. appendix 1, 1.7 (ii) and (iv)), we have

$$\|q_k\|_p e^{-(k+\frac{1}{2})\varepsilon/2} = \|(\underline{X}, N_{\varepsilon/2} \psi_k)\|_p \rightarrow 0 \quad (k \rightarrow \infty),$$

hence

$$\|q_k\|_p = O(e^{k\varepsilon/2}) \quad (k \in \mathbb{N}_0).$$

It follows that

$$\int_{\Omega} \sum_{k=0}^{\infty} |q_k|^p e^{-(k+\frac{1}{2})\varepsilon p} dP = \sum_{k=0}^{\infty} e^{-(k+\frac{1}{2})\varepsilon p} \|q_k\|_p^p < \infty.$$

This implies that

$$\sum_{k=0}^{\infty} |q_k(\omega)|^p e^{-(k+\frac{1}{2})\varepsilon p} < \infty,$$

and hence that  $q_k(\omega) = O(e^{k\varepsilon})$  ( $k \in \mathbb{N}_0$ ) for almost every  $\omega \in \Omega$ .

Now  $\Omega_0 = \bigcap_{n=1}^{\infty} \{\omega \in \Omega \mid q_k(\omega) = O(e^{k/n}) \ (k \in \mathbb{N}_0)\}$  is the countable

intersection of sets in  $\Omega$  whose complements are null sets.

Hence  $P(\Omega_0) = 1$ . □

1.1.17. We define a mapping  $U$  of  $\Omega$  into  $S^*$  by putting

$$U(\omega) = \begin{cases} \bigvee_{\alpha > 0} \sum_{k=0}^{\infty} q_k(\omega) N_{\alpha} \psi_k & (\omega \in \Omega_0), \\ 0 & (\omega \notin \Omega_0). \end{cases}$$

Here  $q_k$  and  $\Omega_0$  are as in lemma 1.1.16. Now the following lemma holds.

LEMMA.  $U^{-1}(\Lambda^*) \subset \Lambda$ .

PROOF. Let  $D$  be the set of all elements  $A$  of  $\Lambda^*$  satisfying  $U^{-1}(A) \in \Lambda$ . If  $f \in S$  and if  $B$  is a Borel set in  $\mathfrak{E}$  then  $U^{-1}(\{F \in S^* \mid (F, f) \in B\})$  is an element of  $D$  since it differs from  $\{\omega \in \Omega \mid \sum_{k=0}^{\infty} q_k(\omega) (\psi_k, f) \in B\}$  by an element of  $\Lambda$  of measure zero. It easily follows that  $D$  contains all Borel cylinders in  $S^*$  (cf. appendix 1, 5.2, remark). Also,  $D$  is a  $\sigma$ -algebra of subsets of  $S^*$  contained in  $\Lambda^*$ . Hence, by appendix 1, 5.2, remark,  $D = \Lambda^*$ . Hence  $U^{-1}(\Lambda^*) \subset \Lambda$ .  $\square$

1.1.18. Define the set function  $P^*$  by

$$P^*(A) = P(U^{-1}(A)) \quad (A \in \Lambda^*).$$

LEMMA. (i)  $P^*$  is a probability measure on  $\Lambda^*$ .

(ii) If  $h: S^* \rightarrow \mathfrak{E}$  is measurable over  $S^*$ , then  $h \circ U$  is measurable over  $\Omega$ , and

$$\int_{S^*} h \, dP^* = \int_{\Omega} h \circ U \, dP$$

in the sense that if either integral exists, then so does the other and the two are equal.

PROOF. This follows from lemma 1.1.17 and [Ha], Ch. VIII, Section 39, theorem B and C.  $\square$

It is now easy to prove the theorem in 1.1.15. For if  $f \in S$ , then  $k := \bigvee_{F \in S^*} (F, f)$  is measurable over  $S^*$ . It follows easily from (ii) of the above lemma that the distributions of  $k \circ U$  (with respect to  $P$ ) and  $k$  (with respect to  $P^*$ ) are the same. Since  $k \circ U$  is a representative of  $(\underline{X}, f)$  we conclude that the distributions of  $(\underline{X}, f)$  and  $\bigvee_{F \in S^*} (F, f)$  are the same. With a similar proof the same thing can be shown for the simultaneous

distributions occurring in the assertion of the theorem in 1.1.15. Hence  $P^*$  satisfies our requirements. It is not hard to show (from Caratheodory's extension theorem) that there is at most one probability measure on  $\Lambda^*$  with the assigned properties.

1.1.19. DEFINITION. Let  $\underline{X} \in S_{\Omega, P}^*$ . Then we denote by  $P_{\underline{X}}^*$  the unique probability measure of theorem 1.1.15. We call  $P_{\underline{X}}^*$  the *probability measure associated with  $\underline{X}$* .

If we regard  $S^*$  as a measure space with  $P_{\underline{X}}^*$  as measure, then  $\bigcup_{[f, F]} (F, f)$  can be viewed as a representative of  $\underline{X}$ ; we call this representative the *canonical representative of  $\underline{X}$* .

1.1.20. Let  $T$  be a linear operator of  $S$  with an adjoint, and let  $\underline{X} \in S_{\Omega, P}^*$ . We want to derive a relation between  $P_{\underline{X}}^*$  and  $P_{T\underline{X}}^*$  (cf. definition 1.1.4), and we claim that  $P_{T\underline{X}}^* = \bigcup_{A \in \Lambda^*} P_{\underline{X}}^*(T^{-1}(A))$  (in view of appendix 1, 5.4 the right hand side of this equality makes sense). Let therefore  $n \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_n \in S$ . The distribution of  $\bigcup_F ((F, f_1), \dots, (F, f_n))$  with respect to  $P_{T\underline{X}}^*$  equals the one of  $((T\underline{X}, f_1), \dots, (T\underline{X}, f_n)) = ((\underline{X}, T^*f_1), \dots, (\underline{X}, T^*f_n))$  with respect to  $P$ . Also, the distribution of  $((\underline{X}, T^*f_1), \dots, (\underline{X}, T^*f_n))$  with respect to  $P$  equals the one of  $\bigcup_F ((F, T^*f_1), \dots, (F, T^*f_n)) = \bigcup_F ((TF, f_1), \dots, (TF, f_n))$  with respect to  $P_{\underline{X}}^*$ . We easily infer that  $P_{T\underline{X}}^*(A) = P_{\underline{X}}^*(T^{-1}(A))$  for all Borel cylinders  $A$  in  $S^*$ . Since both  $P_{T\underline{X}}^*$  and  $\bigcup_{A \in \Lambda^*} P_{\underline{X}}^*(T^{-1}(A))$  are probability measures on  $(S^*, \Lambda^*)$ , we conclude that  $P_{T\underline{X}}^*(A) = P_{\underline{X}}^*(T^{-1}(A))$  for all  $A \in \Lambda^*$ .

1.1.21. The following lemma will be convenient sometimes.

LEMMA. Let  $\underline{X} \in S_{\Omega, P}^*$ , and let  $U$  and  $P_{\underline{X}}^*$  be as in 1.1.17, 1.1.18 and 1.1.19. Let  $\Lambda_0^*$  be a  $\sigma$ -algebra contained in  $\Lambda_{*X}^*$ . Then  $U^{-1}(\Lambda_0^*)$  is a  $\sigma$ -algebra contained in  $\Lambda$ , and if  $h: S^* \rightarrow \mathbb{C}$  is integrable over  $S^*$ , then  $E(h \mid \Lambda_0^*) \circ U = E(h \circ U \mid U^{-1}(\Lambda_0^*))$ .

PROOF. It is easy to see that  $U^{-1}(\Lambda_0^*)$  is a  $\sigma$ -algebra contained in  $\Lambda$ . Let  $h: S^* \rightarrow \mathbb{C}$  be integrable over  $S^*$ . For  $B \in \Lambda_0^*$  we have

$$\begin{aligned} \int_{U^{-1}(B)} E(h \mid \Lambda_0^*) \circ U \, dP &= \int_{\Omega} (\chi_B \cdot E(h \mid \Lambda_0^*)) \circ U \, dP = \\ &= \int_{S^*} \chi_B \cdot E(h \mid \Lambda_0^*) \, dP_{\underline{X}}^* = \int_B E(h \mid \Lambda_0^*) \, dP_{\underline{X}}^* \end{aligned}$$



by lemma 1.1.18 (ii). Similarly,

$$\begin{aligned} \int_B E(h \mid \Lambda_0^*) dP_{\underline{X}}^* &= \int_B h dP_{\underline{X}}^* = \\ &= \int_{U^+(B)} h \circ U dP = \int_{U^+(B)} E(h \circ U \mid U^+(\Lambda_0^*)) dP \end{aligned}$$

by the definition of conditional expectation. Hence  $E(h \circ U \mid U^+(\Lambda_0^*)) = E(h \mid \Lambda_0^*) \circ U$ .  $\square$

1.1.22. We consider yet another way to introduce generalized stochastic processes, and therefore we give the following lemma.

LEMMA. Let  $\underline{X} \in S_{\Omega, p}^*$  and let  $f \in L_q(\Omega)$ . There exists exactly one  $F \in S^*$  such that  $\int_{\Omega} (\underline{X}, g) \cdot \bar{f} dP = (F, g)$  for every  $g \in S$ .

PROOF. Note that  $\int_{\Omega} (\underline{X}, g) \cdot \bar{f} dP$  is a continuous anti-linear functional of  $S$ . The lemma follows from appendix 1, theorem 3.3 (i).  $\square$

1.1.23. DEFINITION. If  $\underline{X} \in S_{\Omega, p}^*$  and  $f \in L_q(\Omega)$ , then we denote the  $F$  of lemma 1.1.22 by  $\langle \underline{X}, f \rangle$ .

Note that the mapping  $f \in L_q(\Omega) \rightarrow \langle \underline{X}, f \rangle \in S^*$  is continuous (and anti-linear): if  $(f_n)_{n \in \mathbb{N}}$  is a sequence in  $L_q(\Omega)$  with  $\|f_n\|_q \rightarrow 0$  ( $n \rightarrow \infty$ ), then  $(\langle \underline{X}, f_n \rangle, g) \rightarrow 0$  ( $n \rightarrow \infty$ ) for every  $g \in S$ , hence  $\langle \underline{X}, f_n \rangle \xrightarrow{S^*} 0$  by appendix 1, 1.15. Furthermore we have  $\langle \underline{X}, f \rangle = 0$  ( $f \in L_q(\Omega)$ ) if and only if  $\underline{X} = 0$ . Hence, if  $\underline{X} \in S_{\Omega, p}^*$ , there is exactly one continuous anti-linear mapping  $\underline{Y}$  of  $L_q(\Omega)$  into  $S^*$  such that  $\langle \underline{X}, f \rangle = \underline{Y}(f)$  ( $f \in L_q(\Omega)$ ). For  $p > 1$  the converse may also be proved (compare the proof of theorem 1.3.3), i.e. for every continuous anti-linear mapping  $\underline{Y}$  of  $L_q(\Omega)$  into  $S^*$  there exists exactly one  $\underline{X} \in S_{\Omega, p}^*$  such that  $\langle \underline{X}, f \rangle = \underline{Y}(f)$  for every  $f \in L_q(\Omega)$ . We thus see that (in case  $p > 1$ ) we could have defined generalized stochastic processes of order  $p$  as continuous anti-linear mappings of  $L_q(\Omega)$  into  $S^*$ . We mention that things are more complicated if  $p = 1$ .

REMARK. Let  $\underline{X} \in S_{\Omega, p}^*$ ,  $f \in L_q(\Omega)$ . We have  $\langle T\underline{X}, f \rangle = T\langle \underline{X}, f \rangle$ . To see this note that for  $g \in S$

$$\langle \langle T\underline{X}, f \rangle, g \rangle = \int_{\Omega} (T\underline{X}, g) \cdot \bar{f} dP = \int_{\Omega} (\underline{X}, T^*g) \cdot \bar{f} dP = (\langle \underline{X}, f \rangle, T^*g) = (T\langle \underline{X}, f \rangle, g).$$

See also 1.1.5.

1.1.24. We considered in this section four more or less equivalent definitions of the notion of generalized stochastic process (cf. 1.1.1, 1.1.5, 1.1.15 and 1.1.23). We shall elaborate in this thesis mainly those given in 1.1.1 and 1.1.15. There are cases that definition 1.1.1 is easier to handle than definition 1.1.15 (e.g. when considering the shot noise processes of 3.3), but often definition 1.1.15 is more convenient (e.g. when dealing with ergodic processes).

1.1.25. It should be noted that our generalized stochastic processes are complex-valued, i.e. the random variables  $(\underline{X}, f)$  with  $\underline{X} \in S_{\Omega, p}^*$  and  $f \in S$  take complex values. It is of course also possible to consider real processes: we may call an  $\underline{X} \in S_{\Omega, p}^*$  real if  $(\underline{X}, f)$  is a real random variable for every  $f \in S$  with  $f(x) \in \mathbb{R}$  ( $x \in \mathbb{R}$ ). The reason to consider complex-valued processes is the fact that important operators as the Fourier transform  $\bar{F}$  and the frequency shifts map real-valued elements of  $S$  to not necessarily real-valued elements of  $S$  (cf. appendix 1, 1.8). In most parts of literature only real-valued processes are considered (cf., however, [GW], Ch. III, §2.2 and [D], Ch. II, §3), but in our theory (where Fourier analysis plays a dominant role) we have to consider complex-valued processes as well.

1.1.26. We shall sometimes consider generalized stochastic processes depending on several time variables (especially the case with two time variables). A generalized stochastic process of order  $p$  depending on two variables is a continuous anti-linear mapping of  $S^2$  into  $L_p(\Omega)$ . The class of all these processes is denoted by  $S_{\Omega, p}^{2*}$ .

Let  $\underline{X} \in S_{\Omega, p}^*$ ,  $\underline{Y} \in S_{\Omega, q}^*$  ( $q$  is the conjugate exponent of  $p$ ). An important example of an element of  $S_{\Omega, 1}^{2*}$  is the tensor product  $\underline{X} \otimes \underline{Y}$  of  $\underline{X}$  and  $\underline{Y}$ :

$$\underline{X} \otimes \underline{Y} := \left\|_{f \in S^2} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} (\underline{X}, \psi_k) \cdot (\underline{Y}, \psi_\ell) (\psi_k \otimes \psi_\ell, f) \right\|$$

$(\psi_k)$  ( $k \in \mathbb{N}_0$ ) denotes as usual the  $k^{\text{th}}$  Hermite function; cf. appendix 1, 1.7 (iv)). It can be proved that  $\underline{X} \otimes \underline{Y}$  is an element of  $S_{\Omega, 1}^{2*}$  by noting that  $\|(\underline{X}, \psi_k) \cdot (\underline{Y}, \psi_\ell)\|_1 = O(e^{(k+\ell)\varepsilon})$  ( $k \in \mathbb{N}_0$ ,  $\ell \in \mathbb{N}_0$ ) for every  $\varepsilon > 0$ .

We sketch another way (which gives the same result as the one above) to define the tensor product of  $\underline{X}$  and  $\underline{Y}$ . Let  $\underline{B}$  be a mapping of  $S \times S$  into  $L_r(\Omega)$  (where  $1 \leq r \leq \infty$ ), and assume that  $\underline{B}$  is continuous and anti-linear

in each variable separately. It can be proved (cf. also appendix 1, 3.6) that there exists exactly one  $\underline{Z} \in S_{\Omega, r}^{2*}$  such that  $(\underline{Z}, f \otimes g) = \underline{B}(f, g)$  for  $f \in S, g \in S$ . If we take  $\underline{B} := \bigvee_{[f, g] \in S \times S} (\underline{X}, f) \cdot (\underline{Y}, g)$ , then it is easy to see that  $\underline{B}$  is continuous ( $r = 1$  in this case) and anti-linear in each variable separately. Now we put  $\underline{X} \otimes \underline{Y} := \underline{Z}$ , where  $\underline{Z}$  is the unique element of  $S_{\Omega, 1}^{2*}$  satisfying  $(\underline{Z}, f \otimes g) = \underline{B}(f, g)$  for  $f \in S, g \in S$ . (We refer to [J1], 1.3.5(ii) for a third way to introduce tensor products of generalized stochastic processes.)

As to linear transformations in the spaces of generalized stochastic processes depending on several time variables, we remark that theorem 1.1.3 still holds (with proper modifications) for the present case. We further mention the following theorem. Let  $\underline{X} \in S_{\Omega, p}^*$ ,  $\underline{Y} \in S_{\Omega, q}^*$ , and let  $T_1$  and  $T_2$  be two linear operators of  $S$  with an adjoint. Then we have  $(T_1 \otimes T_2)(\underline{X} \otimes \underline{Y}) = T_1 \underline{X} \otimes T_2 \underline{Y}$  (cf. appendix 1, 4.16 and theorem 1.1.3 of this chapter). This may be proved by noting that

$$\begin{aligned} ((T_1 \otimes T_2)(\underline{X} \otimes \underline{Y}), f \otimes g) &= (\underline{X} \otimes \underline{Y}, (T_1^* \otimes T_2^*)(f \otimes g)) = \\ &= (\underline{X} \otimes \underline{Y}, T_1^* f \otimes T_2^* g) = (X, T_1^* f) \cdot (\underline{Y}, T_2^* g) = (T_1 \underline{X}, f) \cdot (T_2 \underline{Y}, g) \end{aligned}$$

for  $f \in S, g \in S$  (cf. also [J1], appendix 1, 3.12).

## 1.2. STRICT SENSE STATIONARITY AND ERGODICITY ; GAUSSIAN PROCESSES

1.2.1. We introduce in this section the notions of strict sense stationarity and ergodicity. We further consider briefly Gaussian processes and we give some references to literature on these processes. As usual  $(\Omega, \Lambda, P)$  is a fixed probability space, and  $p$  is an element of the extended real number system with  $1 \leq p \leq \infty$ .

1.2.2. DEFINITION. Let  $V$  be a group of linear operators of  $S$  with an adjoint, and let  $\underline{X} \in S_{\Omega, p}^*$ . If for every  $n \in \mathbb{N}, f_1 \in S, \dots, f_n \in S$  the distribution of  $((V\underline{X}, f_1), \dots, (V\underline{X}, f_n))$  is independent of  $V \in V$ , then we say that  $\underline{X}$  is *strict sense  $V$ -stationary*. In the special case that  $V = (T_a)_{a \in \mathbb{R}}$  (cf. appendix 1, 1.8 (ii)) we speak of *strict sense time stationarity*, and in case that  $V = (R_b)_{b \in \mathbb{R}}$  we speak of *strict sense frequency stationarity*.

1.2.3. EXAMPLE. Let  $\underline{X} \in S_{\Omega, P}^*$ . We have  $FT_a = R_{-a}F$  by [B1], (11.1) (cf. appendix 1, 1.8). Hence, if  $n \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_n \in S$ , then  $((R_{-a}F\underline{X}, f_1), \dots, (R_{-a}F\underline{X}, f_n)) = ((T_a\underline{X}, F^*f_1), \dots, (T_a\underline{X}, F^*f_n))$  for  $a \in \mathbb{R}$ . Since  $F^*$  is a bijection of  $S$  we conclude that  $\underline{X}$  is strict sense time stationary if and only if  $F\underline{X}$  is strict sense frequency stationary. The same holds with  $F^*\underline{X}$  instead of  $F\underline{X}$ .

1.2.4. Stationarity properties of a generalized stochastic process has consequences for the associated probability measure on  $(S^*, \Lambda^*)$  (cf. 1.1.19).

THEOREM. Let  $V$  be a group of linear operators of  $S$  with an adjoint, and let  $\underline{X} \in S_{\Omega, P}^*$ . Then  $\underline{X}$  is strict sense  $V$ -stationary if and only if every  $V \in V$  induces a measure preserving transformation of  $(S^*, \Lambda^*)$  with respect to  $P_{\underline{X}}^*$ .

PROOF. Assume  $\underline{X}$  is strict sense  $V$ -stationary. If  $\Lambda_0^*$  is the set of all elements  $A$  of  $\Lambda^*$  with  $P_{\underline{X}}^*(V(A)) = P_{\underline{X}}^*(A)$  for every  $V \in V$ , then  $\Lambda_0^*$  is a  $\sigma$ -algebra containing all Borel cylinders in  $S^*$ . This is easily seen from theorem 1.1.15 and definition 1.1.19. Hence  $\Lambda_0^* = \Lambda^*$  by appendix 1, 5.2, remark.

Conversely, assume that every  $V \in V$  induces a measure preserving transformation in  $(S^*, \Lambda^*)$  with respect to  $P_{\underline{X}}^*$ . If  $V \in V$ ,  $n \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_n \in S$ , the distributions of  $((V\underline{X}, f_1), \dots, (V\underline{X}, f_n))$  and the one of  $\bigcup_{F \in S^*} ((VF, f_1), \dots, (VF, f_n))$  are the same. The latter one is known to be independent of  $V \in V$ .  $\square$

1.2.5. We next define ergodicity. For notational convenience we formulate the definition in terms of the associated measure on  $S^*$  (cf. 1.1.19).

DEFINITION. Let  $V$  be a group of linear operators of  $S$  with an adjoint, and let  $\underline{X} \in S_{\Omega, P}^*$ . We say that  $\underline{X}$  is *ergodic* if for every  $A \in \Lambda^*$  ( $\div$  stands for symmetric difference)

$$P_{\underline{X}}^*(V(A) \div A) = 0 \quad (V \in V) \Rightarrow P_{\underline{X}}^*(A) = 0 \text{ or } 1.$$

In the special case that  $V = (T_a)_{a \in \mathbb{R}}$  we speak of *time ergodicity*, and in the (rare) case that  $V = (R_b)_{b \in \mathbb{R}}$  we speak of *frequency ergodicity*. If we say " $\underline{X}$  is ergodic" without any further specification, then we mean that  $\underline{X}$  is time ergodic.

1.2.6. It is sometimes necessary to have definitions of stationarity and ergodicity which are formulated exclusively in terms of the measure space  $(\Omega, \Lambda, P)$ . We restrict ourselves to time stationarity and ergodicity.

Let  $\underline{X} \in S_{\Omega, P}^*$ , and let  $\hat{X}$  be a representative of  $\underline{X}$  (cf. 1.1.1). Let  $\Gamma_0$  be the collection of all sets  $\{\omega \mid (\hat{X}(f_1, \omega), \dots, \hat{X}(f_n, \omega)) \in B\}$  where  $n \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_n \in S$ ,  $B \in \mathcal{B}(\mathbb{C}^n)$ , and let  $\Lambda_0$  be the  $\sigma$ -algebra generated by  $\Gamma_0$ . Finally, let  $[\Lambda_0]$  be the system of equivalence classes of sets in  $\Lambda_0$  (equivalence with respect to  $P$ ).

Let  $a \in \mathbb{R}$ , and define the mapping  $\hat{T}_a$  of  $\Gamma_0$  into itself by

$$\hat{T}_a(\{\omega \mid (\hat{X}(f_1, \omega), \dots, \hat{X}(f_n, \omega)) \in B\}) = \{\omega \mid (\hat{X}(T_{-a}f_1, \omega), \dots, \hat{X}(T_{-a}f_n, \omega)) \in B\}$$

for  $n \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_n \in S$ ,  $B \in \mathcal{B}(\mathbb{C}^n)$ . If  $\hat{T}_a$  is measure preserving on  $\Gamma_0$ , then  $\hat{T}_a$  can be extended in exactly one way to a measure preserving mapping of  $[\Lambda_0]$  into itself. It is easy to see that  $\underline{X}$  is strict sense time stationary if and only if  $(\hat{T}_a)_{a \in \mathbb{R}}$  is a group of measure preserving transformations of  $[\Lambda_0]$ .

Assume that  $\underline{X}$  is strict sense time stationary, and let  $I$  be the class of invariant elements of  $[\Lambda_0]$  (i.e.  $E \in I \iff E \in [\Lambda_0], \hat{T}_a(E) = E (a \in \mathbb{R})$ ). We are going to show that  $\underline{X}$  is time ergodic if and only if  $I$  is trivial (i.e.  $I$  consists of the class containing  $\phi$  and the class containing  $\Omega$ ). Since ergodicity was defined in terms of the probability space  $(S^*, \Lambda^*, P_{\underline{X}}^*)$ , we construct a mapping  $T$  of  $[\Lambda_0]$  into  $[\Lambda^*]$ , the system of classes of equivalent sets in  $\Lambda^*$  (equivalence with respect to  $P_{\underline{X}}^*$ ).

Define

$$T(\{\omega \mid (\hat{X}(f_1, \omega), \dots, \hat{X}(f_n, \omega)) \in B\}) = \{F \mid ((F, f_1), \dots, (F, f_n)) \in B\}$$

for  $n \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_n \in S$ ,  $B \in \mathcal{B}(\mathbb{C}^n)$ . It is easy to see that  $T$  can be extended in exactly one way to a bijective, measure preserving mapping of  $[\Lambda_0]$  into  $[\Lambda^*]$  (cf. 1.1.15).

It follows from stationarity of  $\underline{X}$  (cf. 1.2.4) that the mapping  $T_a = \bigcup_{A \in \Lambda^*} T_a(A)$  can be extended in exactly one way to a measure preserving mapping of  $[\Lambda^*]$  into itself ( $a \in \mathbb{R}$ ). It is easy to see that  $T_a \hat{T}_a = T_a T, \hat{T}_a T^{-1} = T^{-1} T_a (a \in \mathbb{R})$ .

Let  $I^*$  be the class of invariant elements of  $[\Lambda^*]$ . It is not hard to prove that  $I^* = T(I)$ . Since  $T$  is bijective, this means that  $I^*$  is trivial if and only if  $I$  is trivial. But  $I^*$  is trivial if and only if  $\underline{X}$  is ergodic.

1.2.7. An important role in any theory of generalized stochastic processes is played by Gaussian processes. These are processes  $\underline{X}$  for which the distribution function of  $((\underline{X}, f_1), \dots, (\underline{X}, f_n))$  is Gaussian for every  $n \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_n \in S$  (note that the  $(\underline{X}, f_k)$ 's are complex-valued Gaussian random variables; cf. [GW], Ch. III, §2.2 or [D], Ch. II, §3 for the definition of complex-valued Gaussian variables).

There is a large amount of literature on Gaussian processes. Especially the case where the setting is a triple  $(E, H, E')$  (with  $E$  a nuclear space of test functions,  $H$  a Hilbert space (usually  $L_2(\mathbb{R})$ ),  $E'$  the dual of  $E$ ) has received much attention. Note that  $(S, L_2(\mathbb{R}), S^*)$  is such a triple if  $S$  is endowed with the inductive limit topology  $\tau$  of appendix 1, 2.6.

An important example of a Gaussian process is Gaussian white noise, where  $\int_{\Omega} (\underline{X}, f) dP = 0$ ,  $\int_{\Omega} (\underline{X}, f) \overline{(\underline{X}, g)} dP = (f, g)$  for  $f \in E$ ,  $g \in E$ . A detailed analysis of Gaussian white noise can be found in [Hi], Part III. In the reference just given the relation between Gaussian white noise on one hand and multiple Wiener integrals and Brownian motion on the other one is discussed. (In [Hi] the test function space  $E$  is assumed to contain non-trivial elements of compact support; this assumption plays no significant role in the analysis given in [Hi], Part III of white noise, Brownian motion etc.)

More general Gaussian processes are studied in [Um]. E.g., theorems about quasi-invariance of the measures arising from these processes are derived there.

### 1.3. EMBEDDING OF ORDINARY STOCHASTIC PROCESSES

1.3.1. We are going to embed a certain class of ordinary stochastic processes (with real time parameter) in our system of generalized stochastic processes. The class of embeddable processes can be compared to some extent with the space  $S^+$  of appendix 1, 1.5. As usual,  $(\Omega, \Lambda, P)$  is a probability space, and  $p$  is an element of the extended real number system with  $1 \leq p \leq \infty$ .

1.3.2. DEFINITION. The class  $S_{\Omega, p}^+$  consists of all mappings  $\underline{x}: \mathbb{R} \times \Omega \rightarrow \mathbb{C}$  satisfying

- (1)  $\underline{x}(t) := \bigvee_{\omega \in \Omega} \underline{x}(t, \omega) \in L_p(\Omega)$  for almost every  $t \in \mathbb{R}$ ,
- (2) there is an  $h \in S^+$  with  $\|\underline{x}(t)\|_p \leq h(t)$  for almost every  $t \in \mathbb{R}$ ,
- (3)  $\langle \underline{x}, f \rangle := \bigvee_{t \in \mathbb{R}} \langle \underline{x}(t), f \rangle := \bigvee_{t \in \mathbb{R}} \int_{\Omega} \underline{x}(t) \bar{f} dP \in S^+$  for every  $f \in L_q(\Omega)$ .

Here  $q$  denotes the conjugate exponent of  $p$ . The class  $\bar{S}_{\Omega, p}^+$  consists of all equivalence classes of elements of  $S_{\Omega, p}^+$  (two elements  $\underline{x}_1$  and  $\underline{x}_2$  of  $S_{\Omega, p}^+$  are considered equivalent if  $\underline{x}_1(t) = \underline{x}_2(t)$  (a.e.) for every  $t \in \mathbb{R}$ ).

REMARKS. 1. Especially condition (3) looks a bit awkward, but we shall give in 1.3.5 a number of examples where conditions (1)-(3) are readily verified.

2. The definition of  $S_{\Omega, p}^+$  deals with mappings  $\underline{x}: \mathbb{R} \times \Omega \rightarrow \mathbb{C}$ . At first sight it looks more appropriate to consider the elements of  $\bar{S}_{\Omega, p}^+$ . These can be identified with mappings  $\underline{x}$  of  $\mathbb{R}$  into  $L_p(\Omega)$  such that (2) and (3) hold. In view of 1.3.6-8, however, it is convenient to have mappings  $\underline{x}$  defined on the product space  $\mathbb{R} \times \Omega$  and taking values in  $\mathbb{C}$ .

1.3.3. In the proof of the embedding theorem below there is no harm in identifying functions and function classes; we thus write  $L_p(\Omega)$  and  $L_q(\Omega)$  instead of  $L_p(\Omega)$  and  $L_q(\Omega)$ .

THEOREM. Let  $\underline{x} \in S_{\Omega, p}^+$ . There exists exactly one  $\underline{X} \in S_{\Omega, p}^*$  such that  $\langle \underline{X}, f \rangle = \text{emb}(\langle \underline{x}, f \rangle)$  for every  $f \in L_q(\Omega)$  (cf. 1.1.23).

PROOF. The uniqueness of an  $\underline{X} \in S_{\Omega, p}^*$  with the assigned properties is seen from 1.1.23, so we only have to show existence.

Let  $g \in S$  be fixed, and consider the linear functional

$$L_g := \bigvee_{f \in L_q(\Omega)} (g, \text{emb}(\langle \underline{x}, f \rangle)).$$

We show that there is exactly one  $h_g \in L_p(\Omega)$  such that

$$(*) \quad L_g(f) = \int_{\Omega} f \cdot h_g \, dP \quad (f \in L_q(\Omega)).$$

We have by appendix 1, 1.11 (i) for  $f \in L_q(\Omega)$

$$(**) \quad |L_g(f)| = \left| \int_{-\infty}^{\infty} g(t) \langle \underline{x}(t), f \rangle dt \right| \leq \|f\|_q \int_{-\infty}^{\infty} |g(t)| h(t) dt.$$

Since  $\int_{-\infty}^{\infty} |g(t)| h(t) dt < \infty$ , the Riesz representation theorem (cf. [Za], Ch. 12, §50, theorem 2) applies for the case  $1 < p \leq \infty$ : there is exactly one  $h_g \in L_p(\Omega)$  satisfying (\*). For the case  $p = 1$  we note that the set function  $\bigcup_{A \in \Lambda} L_g(X_A)$  is completely additive, and we can complete the proof of the existence of an  $h_g \in L_1(\Omega)$  satisfying (\*) in the same way as the proof of [J1], theorem 1.2.4. There is just one  $h_g$  satisfying (\*).

We next show that the mapping  $g \rightarrow h_g$  is linear and continuous. Linearity is easily seen, so we only have to show continuity. Let  $(g_n)_{n \in \mathbb{N}}$  be a sequence in  $S$ , and assume that  $g_n \xrightarrow{S} 0$ . By (\*) and (\*\*) we get

$$\left| \int_{\Omega} f \cdot h_{g_n} dP \right| = |L_{g_n}(f)| \leq \|f\|_q \int_{-\infty}^{\infty} |g_n(t)| h(t) dt$$

for every  $f \in L_q(\Omega)$ . Since  $\int_{-\infty}^{\infty} |g_n(t)| h(t) dt \rightarrow 0$ , we easily conclude that  $\|h_{g_n}\|_p \rightarrow 0$ . This establishes continuity.

Now we put  $\underline{x} := \bigcup_{g \in S} \bar{h}_g$ . Then  $\underline{x} \in S_{\Omega, p}^*$ . Also,  $\langle \underline{x}, f \rangle = \text{emb}(\langle \underline{x}, f \rangle)$  if  $f \in L_q(\Omega)$ , for (cf. 1.1.23)

$$\langle \underline{x}, f \rangle, g = \int_{\Omega} (\underline{x}, g) \cdot \bar{f} dP = \int_{\Omega} \bar{h}_g \cdot \bar{f} dP = (\text{emb}(\langle \underline{x}, f \rangle), g)$$

for every  $g \in S$ . Hence  $\underline{x}$  satisfies the requirements.  $\square$

REMARK. We note that we have a similar theorem if we take  $\underline{x} \in \bar{S}_{\Omega, p}^+$  instead of  $S_{\Omega, p}^+$ .

1.3.4. DEFINITION. If  $\underline{x} \in S_{\Omega, p}^+$  (or  $\bar{S}_{\Omega, p}^+$ ), we put  $\text{emb}(\underline{x}) := \underline{x}$ , where  $\underline{x}$  is the unique element of  $S_{\Omega, p}^*$  satisfying  $\langle \underline{x}, f \rangle = \text{emb}(\langle \underline{x}, f \rangle)$  for every  $f \in L_q(\Omega)$ .

Note that the mapping  $\text{emb}$  is in general not injective as a mapping of  $S_{\Omega, p}^+$  (or  $\bar{S}_{\Omega, p}^+$ ) into  $S_{\Omega, p}^*$ .

1.3.5. In the examples below  $\underline{x}: \mathbb{R} \times \Omega$  satisfies (1) of 1.3.2.

(i) If  $\underline{x}$  is continuous in  $p^{\text{th}}$  order (i.e.  $\|\underline{x}(t) - \underline{x}(t_0)\|_p \rightarrow 0$  ( $t \rightarrow t_0$ ) for  $t_0 \in \mathbb{R}$ ) and if  $\bigcup_t \|\underline{x}(t)\|_p \in S^+$ , then  $\underline{x} \in S_{\Omega, p}^+$ . Now  $\langle \underline{x}, f \rangle$  is continuous for every  $f \in L_q(\Omega)$ .

(ii) If  $\langle \underline{x}, f \rangle$  is measurable for every  $f \in L_q(\Omega)$  and if  $\bigcup_t \|\underline{x}(t)\|_p \in S^+$ , then  $\underline{x} \in S_{\Omega, p}^+$ .



(iii) If  $p = 2$ , if  $\bigvee_{(t,s)} \langle \underline{x}(t), \underline{x}(s) \rangle \in S^{2+}$  (cf. appendix 1, 1.17) and if  $\bigvee_t \|\underline{x}(t)\|_2 \in S^+$ , then  $\underline{x} \in S_{\Omega,2}^+$ . In view of (ii) the only thing we have to check is measurability of  $\langle \underline{x}, f \rangle$  for  $f \in L_2(\Omega)$ . To prove this, define  $U$  as the closure in  $L_2(\Omega)$  of the set  $\{\sum_{i=1}^n \alpha_i \underline{x}(t_i) \mid n \in \mathbb{N}, \alpha_i \in \mathbb{C}, t_i \in \mathbb{R} (i = 1, \dots, n)\}$ . Every  $f \in L_2(\Omega)$  can be written as  $f_1 + f_2$  with  $f_1 \in U, f_2 \in U^\perp$ . Now  $\langle \underline{x}, f \rangle = \langle \underline{x}, f_1 \rangle$  is measurable.

(iv) Let  $\underline{x}$  be a real Brownian motion process:  $\underline{x}(0) = 0$  and the distribution of  $(\underline{x}(t_2) - \underline{x}(t_1), \dots, \underline{x}(t_n) - \underline{x}(t_{n-1}))$  is Gaussian with zero expectation function and var-covar-matrix  $\text{diag}(t_2 - t_1, \dots, t_n - t_{n-1})$  for  $n \in \mathbb{N}, t_1 < t_2 < \dots < t_n$ . This  $\underline{x}$  is embeddable on account of (i). It can be shown that the derivative of  $\text{emb}(\underline{x})$  in  $S_{\Omega,2}^*$  is a real Gaussian white noise process (cf. 1.1.25 and 1.2.6).

1.3.6. If  $\underline{x} \in S_{\Omega,p}^+$  and  $\underline{x}$  is measurable over the product space  $\mathbb{R} \times \Omega$ , embedding of  $\underline{x}$  is performed by simply taking integrals over the real axis.

THEOREM. Let  $\underline{x} \in S_{\Omega,p}^+$  be measurable over  $\mathbb{R} \times \Omega$ , and let  $g \in S$ . Then  $\bigvee_t \underline{x}(t, \omega) \in S^+$  for almost every  $\omega \in \Omega$ , and

$$(\text{emb}(\underline{x}), g) = \bigvee_{\omega \in \Omega} \int_{-\infty}^{\infty} \underline{x}(t, \omega) \overline{g(t)} dt$$

almost everywhere in  $\Omega$ .

PROOF. Let  $\varepsilon > 0$ . The function  $\bigvee_{(t,\omega)} \underline{x}(t, \omega) \exp(-\pi \varepsilon t^2)$  is measurable over  $\mathbb{R} \times \Omega$ , and we have by Fubini's theorem and Hölder's inequality

$$\begin{aligned} & \int_{\Omega} \left( \int_{-\infty}^{\infty} |\underline{x}(t, \omega)| \exp(-\pi \varepsilon t^2) dt \right) dP(\omega) = \\ & = \int_{-\infty}^{\infty} \left( \int_{\Omega} |\underline{x}(t, \omega)| dP(\omega) \right) \exp(-\pi \varepsilon t^2) dt \leq \int_{-\infty}^{\infty} \|\underline{x}(t)\|_p \exp(-\pi \varepsilon t^2) dt < \infty \end{aligned}$$

(cf. 1.3.2 (2) and appendix 1, 1.5). Hence  $\bigvee_{t \in \mathbb{R}} \underline{x}(t, \omega) \exp(-\pi \varepsilon t^2) \in L_1(\mathbb{R})$  for almost every  $\omega \in \Omega$ . We now have for almost every  $\omega \in \Omega$

$$\forall \varepsilon > 0 \left[ \bigvee_t \underline{x}(t, \omega) \in S^+ \right],$$

since  $\forall_{n \in \mathbb{N}} [\int_t^{\cup} \underline{x}(t, \omega) \exp(-\pi n^{-1} t^2) \in L_1(\mathbb{R})]$  holds for almost every  $\omega \in \Omega$  (cf. appendix 1, 1.5).

Now let  $A \in \Lambda$ . We have by definition 1.3.4

$$\int_A (\text{emb}(\underline{x}), g) dP = \int_{-\infty}^{\infty} \left( \int_A \underline{x}(t, \omega) dP(\omega) \right) \overline{g(t)} dt.$$

Since  $\int_{(t, \omega)} \underline{x}(t, \omega) \overline{g(t)}$  is absolutely integrable over  $\mathbb{R} \times \Omega$ , we obtain from Fubini's theorem

$$\int_A (\text{emb}(\underline{x}), g) dP = \int_A \left( \int_{-\infty}^{\infty} \underline{x}(t, \omega) \overline{g(t)} dt \right) dP(\omega).$$

Hence (since  $A \in \Lambda$  is arbitrarily chosen)

$$(\text{emb}(\underline{x}), g) = \int_{\omega \in \Omega} \int_{-\infty}^{\infty} \underline{x}(t, \omega) \overline{g(t)} dt$$

almost everywhere in  $\Omega$ . □

1.3.7. We are going to show that certain elements of  $S_{\Omega, P}^+$  that are strict sense time stationary (and ergodic) in the sense of [D], Ch. XI, §1 have embeddings that are strict sense time stationary (and ergodic) in the sense of definition 1.2.2 (1.2.5).

**THEOREM.** Let  $\underline{x} \in S_{\Omega, P}^+$ , and assume that  $\underline{x}$  is strict sense time stationary (in the sense of [D], Ch. XI, §9) and measurable with respect to the product  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R}) \otimes \Lambda_{\underline{x}}$  on  $\mathbb{R} \times \Omega$ . Here  $\Lambda_{\underline{x}}$  denotes the completed  $\sigma$ -algebra generated by all sets of the form  $\{\omega \mid \underline{x}(t_1, \omega), \dots, \underline{x}(t_n, \omega) \in B\}$  with  $n \in \mathbb{N}$ ,  $t_1 \in \mathbb{R}, \dots, t_n \in \mathbb{R}$ ,  $B \in \mathcal{B}(\mathbb{C}^n)$ . Then  $\text{emb}(\underline{x})$  is strict sense time stationary (in the sense of 1.2.2), and if  $\underline{x}$  is ergodic (in the sense of [D], Ch. XI, §1), then so is  $\text{emb}(\underline{x})$  (in the sense of 1.2.5).

**PROOF.** Let  $f \in S$ . The set  $E_f$  of all  $\omega$ 's for which  $\int_{-\infty}^{\infty} \underline{x}(t, \omega) \overline{f(t)} dt$  makes sense is an element of  $\Lambda_{\underline{x}}$  and has measure one (cf. the proof of theorem 1.3.6).

Now put for every  $f \in S$

$$\hat{X}(f, \omega) = \begin{cases} \int_{-\infty}^{\infty} \underline{x}(t, \omega) \overline{f(t)} dt & (\omega \in E_f) \\ 0 & (\omega \notin E_f). \end{cases}$$

Then  $\hat{\underline{x}}$  is a representative of  $\underline{x} := \text{emb}(\underline{x})$  according to theorem 1.3.6 and definition 1.1.1.

If we adopt the notation of 1.2.6, then  $\Lambda_0 \subset \Lambda_{\underline{x}}$ , since  $\bigvee_{\omega} \hat{\underline{x}}(f, \omega)$  is measurable with respect to  $\Lambda_{\underline{x}}$  for every  $f \in S$ . According to the definition of stationarity given in [D], Ch. XI, §1,  $\hat{T}_a$  is measure preserving on  $[\Lambda_0]$  for  $a \in \mathbb{R}$ . Hence  $\underline{x}$  is strict sense time stationary by 1.2.6.

If  $\underline{x}$  is ergodic, then  $\Lambda_{\underline{x}}$  has no trivial invariant elements according to the definition of ergodicity given in [D], Ch. XI, §1. Hence  $[\Lambda_0]$  has no trivial invariant elements. By 1.2.6 we conclude that  $\underline{x}$  is ergodic.  $\square$

REMARK. It may occur that an  $\underline{x} \in S_{\Omega, P}^+$  is measurable with respect to  $\mathcal{B}(\mathbb{R}) \otimes \Lambda$  ( $\Lambda$  is the  $\sigma$ -algebra we started with) but not measurable with respect to  $\mathcal{B}(\mathbb{R}) \otimes \Lambda_{\underline{x}}$  (cf. [D], Ch. II, p.p. 68 and 69 where things like this are discussed in connection with processes of function space type). In that case the proof of the above theorem does not work. However, many interesting cases are covered by the above theorem and the following one.

1.3.8. THEOREM. Let  $\underline{y}$  be an ordinary process with zero expectation function, finite second order moments and stationary and independent increments (cf. [D], Ch. II, §9). Let  $h \in L_2(\mathbb{R})$  (whence  $h$  is a function and not a function class). Define the process

$$\int_{-\infty}^{\infty} h(t-s) d\underline{y}(s, \omega) =: \underline{x}(t, \omega) \quad (t \in \mathbb{R}, \omega \in \Omega)$$

as in [D], Ch. IX, §2. Then  $\underline{x} \in S_{\Omega, 2}^+$ , and  $\text{emb}(\underline{x})$  is strict sense time stationary and ergodic.

PROOF. The mapping  $\underline{x}: \mathbb{R} \times \Omega \rightarrow \mathbb{C}$  is not known to be measurable with respect to  $\mathcal{B}(\mathbb{R}) \otimes \Lambda_{\underline{x}}$  (notation as in theorem 1.3.7); it may, however, be altered to become so. To prove this we invoke [D], Ch. II, §2, theorem 6, and show that  $\|\underline{x}(t_1) - \underline{x}(t)\|_2 \rightarrow 0$  ( $t_1 \rightarrow t$ ) for every  $t \in \mathbb{R}$ . We have by [D], Ch. IX, §2 for  $t_1 \in \mathbb{R}$ ,  $t \in \mathbb{R}$ .

$$\begin{aligned} \|\underline{x}(t_1) - \underline{x}(t)\|_2^2 &= \int_{\Omega} \left| \int_{-\infty}^{\infty} (h(t_1 - s) - h(t - s)) d\underline{y}(s) \right|^2 dP \\ &= \int_{-\infty}^{\infty} |h(t_1 - s) - h(t - s)|^2 ds \rightarrow 0 \end{aligned}$$

if  $t_1 \rightarrow t$  since  $h \in L_2(\mathbb{R})$ .

By [D], Ch. II, §2, theorem 2.6 we can find a process  $\underline{x}_1$ , measurable with respect to  $\mathcal{B}(\mathbb{R}) \otimes \Lambda_{\underline{x}}$ , such that  $\underline{x}_1(t) = \underline{x}(t)$  (a.e.) for every  $t \in \mathbb{R}$ . We have  $\underline{x} \in S_{\Omega,2}^+$ ,  $\underline{x}_1 \in S_{\Omega,2}^+$  by the above and 1.3.5 (i), and  $\text{emb}(\underline{x}) = \text{emb}(\underline{x}_1)$ . As in the proof of theorem 1.3.7 we can find a representative  $\hat{\underline{X}}$  of  $\underline{X} := \text{emb}(\underline{x}_1)$  such that  $\bigcup_{\omega} \hat{\underline{X}}(f, \omega)$  is measurable with respect to  $\Lambda_{\underline{x}}$  for every  $f \in S$ .

It is known from [D], Ch. XI, §1, example 3 that  $\underline{x}$  is strict sense time stationary and ergodic. Proceeding as in the proof of theorem 1.3.7 we conclude that  $\underline{X}$  is strict sense time stationary and ergodic.  $\square$

1.3.9. We note that the definition of embeddable process of  $n$  variables (with  $n \in \mathbb{N}$ ) can be given, and that the embedding theorem 1.3.3 can be given and proved for such a process. The classes of embeddable processes are denoted by  $S_{\Omega,p}^{n+}$  and  $S_{\Omega,p}^{-n+}$  respectively (cf. definition 1.3.2).

## CHAPTER 2

EXPECTATION FUNCTION, AUTOCORRELATION FUNCTION AND WIGNER DISTRIBUTION  
OF GENERALIZED STOCHASTIC PROCESSES

This chapter is devoted to the functions constituting an important part of the statistics of generalized stochastic processes, viz. the expectation function and the autocorrelation function. Furthermore we shall introduce the Wigner distribution (or, actually, the expected Wigner distribution) of generalized stochastic processes.

If  $\underline{x}$  is an ordinary stochastic process (defined on some probability space  $\Omega$ ) with finite second order moments, then the expectation function of  $\underline{x}$  is defined as  $\int_{\Omega} \underline{x}(t) dP$ , and the autocorrelation function of  $\underline{x}$  is defined as  $\int_{\Omega} \underline{x}(t) \overline{\underline{x}(s)} dP$ . If we are dealing with generalized stochastic processes, expectation function, autocorrelation function and (expected) Wigner distribution are not just ordinary functions, but generalized functions. The expectation function of a generalized stochastic process  $\underline{X}$  (defined on some probability space  $\Omega$ ) is defined with the aid of the values of  $\int_{\Omega} (\underline{X}, f) dP$  ( $f \in S$ ). The definition of the autocorrelation function of  $\underline{X}$  involves the values of  $\int_{\Omega} (\underline{X}, f) \overline{(\underline{X}, g)} dP$  ( $f \in S, g \in S$ ). Our definitions are such that if  $\underline{x}$  is the embedding of a reasonably well-behaved, ordinary process  $\underline{x}$ , then expectation function and autocorrelation function of  $\underline{X}$  are obtained by taking the embeddings of  $\int_{\Omega} \underline{x}(t) dP$  and  $\int_{\Omega} \underline{x}(t) \overline{\underline{x}(s)} dP$  respectively (cf. 2.1.9).

The expected Wigner distribution of a generalized stochastic process is defined roughly as the image of the autocorrelation function under the mapping  $f \rightarrow \int_{-\infty}^{\infty} e^{-2\pi i y t} f\left(\frac{x+t}{\sqrt{2}}, \frac{x-t}{\sqrt{2}}\right) dt$  (this mapping can be extended to a linear mapping of  $S^{2*}$  into  $S^{2*}$ ; cf. appendix 1, 4.16). The word "expected" may be motivated as follows. If we assume for a while that  $\underline{X}$  is a sufficiently well-behaved ordinary process, we can write down for each  $\omega \in \Omega$  the function  $\int_{-\infty}^{\infty} e^{-2\pi i y t} \underline{X}\left(\frac{x+t}{\sqrt{2}}, \omega\right) \overline{\underline{X}\left(\frac{x-t}{\sqrt{2}}, \omega\right)} dt$ , i.e. the Wigner distribution of  $\int_{\Omega} \underline{X}(t, \omega)$ . On integrating over  $\omega$  and inverting the order of integration we get  $\int_{-\infty}^{\infty} e^{-2\pi i y t} R_{\underline{X}}\left(\frac{x+t}{\sqrt{2}}, \frac{x-t}{\sqrt{2}}\right) dt$ , where  $R_{\underline{X}}$  is the autocorrelation function of  $\underline{X}$ .

In section 1 of this chapter definition and main properties of expectation function, autocorrelation function and Wigner distribution are given.

We show positive definiteness of the autocorrelation function, and prove a theorem on the existence of Gaussian processes with prescribed expectation function and autocorrelation function.

In section 2 we pay attention to various kinds of second order stationarity; we are particularly interested in the general form assumed by expectation function, autocorrelation function and Wigner distribution of stationary processes. We shall also relate Wigner distribution to spectral density function of second order time stationary processes. We further consider second order frequency stationary processes in their relation with processes with independent values at every moment, and we comment on the representation of second order time stationary processes as Fourier transforms of random measures.

The Wigner distribution of a generalized stochastic process is easier than the autocorrelation function as far as physical interpretations are concerned. The Wigner distribution provides a picture of the energy distribution of the process over time and frequency. This appears clearly from the general form of the Wigner distribution of a second order time stationary process (cf. 2.2.5): it is the tensor product of a constant function and the spectral density function. A similar thing holds for the Wigner distribution of a second order frequency stationary process: it is the tensor product of a generalized function of positive type and a constant function. In particular, the Wigner distribution of a second order white noise process (time and frequency stationary; cf. theorem 2.2.13) is constant in both time and frequency. If, however, non-stationary processes are considered, certain difficulties with physical interpretations arise: the Wigner distribution is not necessarily of positive type (whence it can no longer be interpreted as an energy distribution). Then certain smoothing operations have to be performed; we shall go into more detail in chapter 3, section 4 and 5.

## 2.1. DEFINITIONS AND MAIN PROPERTIES

2.1.1. In this section  $(\Omega, \Lambda, P)$  is a fixed probability space. We consider generalized stochastic processes of order 1 and 2.

2.1.2. DEFINITION. Let  $\underline{X} \in S_{\Omega, 1}^*$ . The *expectation function*  $E\underline{X}$  of  $\underline{X}$  is defined as the generalized function  $F$  satisfying

$$(F, f) = \int_{\Omega} (\underline{X}, f) dP$$

for  $f \in S$  (by definition 1.1.1 and appendix 1, theorem 3.3 this definition makes sense).

Of course we have a similar definition for processes depending on several variables.

2.1.3. Now let  $\underline{X} \in S_{\Omega, 2}^*$ . To introduce the autocorrelation function of  $\underline{X}$  we consider the functional

$$\Psi_{[f, g] \in S \times S} \int_{\Omega} (\underline{X}, f) \cdot \overline{(\underline{X}, g)} dP.$$

This functional depends anti-linearly and continuously on each variable separately. According to appendix 1.3.6 there exists exactly one  $F \in S^{2*}$  such that

$$(*) \quad (F, f \otimes g) = \int_{\Omega} (\underline{X}, f) \cdot \overline{(\underline{X}, g)} dP \quad (f \in S, g \in S).$$

DEFINITION. The *autocorrelation function*  $R_{\underline{X}}$  (or, shortly,  $R$  if it is clear which process  $\underline{X}$  is meant) is defined as the unique  $F$  satisfying (\*).

We give an alternative way to introduce the autocorrelation function. If we define  $\bar{\underline{X}} := \Psi_{f \in S} (\underline{X}, \bar{f})$ , then  $\bar{\underline{X}} \in S_{\Omega, 2}^*$ . Now the  $R_{\underline{X}}$  of the above definition equals the expectation function of  $\underline{X} \otimes \bar{\underline{X}}$  (cf. 1.1.26), for we have

$$\begin{aligned} (E(\underline{X} \otimes \bar{\underline{X}}), f \otimes g) &= \int_{\Omega} (\underline{X} \otimes \bar{\underline{X}}, f \otimes g) dP = \\ &= \int_{\Omega} (\underline{X}, f) \cdot \overline{(\underline{X}, g)} dP = (R_{\underline{X}}, f \otimes g) \quad (f \in S, g \in S). \end{aligned}$$

We can also consider the *autocovariance function*  $C_{\underline{X}}$  (or, shortly,  $C$ ) of  $\underline{X}$ , defined as the autocorrelation function of the centralized process  $\underline{X} - E\underline{X}$ . It is easily checked that  $C_{\underline{X}} = R_{\underline{X}} - E\underline{X} \otimes E\bar{\underline{X}} = R_{\underline{X}} - E\underline{X} \otimes \overline{E\underline{X}}$ . We shall often assume  $E\underline{X} = 0$  (so that  $R_{\underline{X}} = C_{\underline{X}}$ ).

We refer to [J1], definition 3.2 where a third way is given to define the autocorrelation functions for elements of  $S_{\Omega, 2}^*$ .

2.1.4. Let  $\underline{X} \in S_{\Omega, 2}^*$ . We introduce the Wigner distribution of  $\underline{X}$ .

DEFINITION. The (*expected*) *Wigner distribution*  $V_{\underline{X}}$  (or, shortly,  $V$ ) of  $\underline{X}$  is defined by  $V_{\underline{X}} := F^{(2)} \underset{U}{Z} R_{\underline{X}}$  (cf. appendix 1, 4.16 and appendix 3, 1.4 and 2.1).

The word "expected" may be motivated as follows. We have  $E(F^{(2)}_{Z_U}(\underline{X} \otimes \bar{\underline{X}})) = F^{(2)}_{Z_U}(E(\underline{X} \otimes \bar{\underline{X}})) = V_{\underline{X}}$  (cf. 2.1.3 and 2.1.7), and  $F^{(2)}_{Z_U}(\underline{X} \otimes \bar{\underline{X}})$  may be interpreted as the Wigner distribution of  $\underline{X}$  (this is a generalized stochastic process depending on two variables).

2.1.5. Let  $\underline{X} \in S_{\Omega, 2}^*$ ,  $f \in S$ . We have  $(R_{\underline{X}}, f \otimes \bar{f}) \geq 0$ .

PROOF. We have by definition 2.1.3

$$(R_{\underline{X}}, f \otimes \bar{f}) = \int_{\Omega} |(X, f)|^2 dP \geq 0. \quad \square$$

If  $F \in S^{2*}$  satisfies  $(F, f \otimes \bar{f}) \geq 0$  for every  $f \in S$ , then we call  $F$  *positive definite*. Hence, if  $\underline{X} \in S_{\Omega, 2}^*$ , then  $R_{\underline{X}}$  is a positive definite function, and so is  $C_{\underline{X}} = R_{\underline{X}} - E\underline{X} \otimes E\underline{X}$ . The following theorem gives an interesting converse.

2.1.6. THEOREM. Let  $F \in S^*$ ,  $R \in S^{2*}$ , and assume that  $R - F \otimes \bar{F}$  is a positive definite function. There exists a Gaussian generalized stochastic process  $\underline{X}$  such that  $E\underline{X} = F$ ,  $R_{\underline{X}} = R$  (cf. 1.2.7).

PROOF. The proof uses [D], Ch. II, §3, theorem 3.1 with a suitable choice for the time space  $T$ , the function  $\mu$  and the positive definite function  $r$ .

We take  $T = \mathbb{N}_0$ , and put

$$\mu(k) = (F, \psi_k), \quad r(k, l) = (R - F \otimes \bar{F}, \psi_k \otimes \psi_l) \quad (k \in T, l \in T).$$

Since  $\psi_k$  is real on the reals (hence  $\bar{\psi}_k = \psi_k$ ) for  $k \in \mathbb{N}_0$ , the function  $r$  satisfies

$$\sum_{i=1}^n \sum_{j=1}^n \alpha_i \bar{\alpha}_j r(k_i, k_j) = (R - F \otimes \bar{F}, \sum_{i=1}^n \bar{\alpha}_i \psi_{k_i} \otimes \overline{\sum_{j=1}^n \alpha_j \psi_{k_j}}) \geq 0$$

for every  $n \in \mathbb{N}$ ,  $\alpha_1 \in \mathbb{C}, \dots, \alpha_n \in \mathbb{C}$ ,  $k_1 \in T, \dots, k_n \in T$ . Furthermore

$$r(k, l) = \overline{r(l, k)} \quad (k \in T, l \in T).$$

The conditions of the theorem mentioned above are satisfied. Hence we can find a Gaussian stochastic process  $\{q_k(\omega)\}_{(k, \omega) \in T \times \Omega'}$  defined on some probability space  $(\Omega', \Lambda', P')$  such that

$$\int_{\Omega'} q_k dP' = \mu_k, \quad \int_{\Omega'} q_k \bar{q}_l dP' = r(k, l) + \mu(k) \overline{\mu(l)} \quad (k \in T, l \in T).$$



We define the generalized stochastic process  $\underline{X}$  by putting

$$(\underline{X}, f) = \sum_{k=0}^{\infty} q_k(\psi_k, f) \quad (f \in S).$$

This  $\underline{X}$  is well-defined:  $\|q_k\|_2^2 = r(k, k) + |\mu(k)|^2 = O(e^{k\varepsilon})$  ( $k \in \mathbb{N}_0$ ) for every  $\varepsilon > 0$  since  $F \in S^*$ ,  $R \in S^{2*}$  (cf. appendix 1, 1.10 and 1.17). If  $f_n \in S$  ( $n \in \mathbb{N}$ ) and  $f_n \xrightarrow{S} 0$ , then  $\|(\underline{X}, f_n)\|_2 \rightarrow 0$  ( $n \rightarrow \infty$ ) by appendix 1, 1.12 and 1.7 (i) and (iv). Since  $\underline{X}$  is anti-linear we conclude  $\underline{X} \in S_{\Omega', 2}^*$ .

The expectation function of  $\underline{X}$  equals  $F$ , for (cf. appendix 1, 1.10)

$$(\underline{EX}, \psi_k) = \int_{\Omega'} (\underline{X}, \psi_k) dP = \mu(k) = (F, \psi_k) \quad (k \in \mathbb{N}_0).$$

The autocorrelation function of  $\underline{X}$  equals  $R$ , for

$$\begin{aligned} (R_{\underline{X}}, \psi_k \otimes \psi_l) &= \int_{\Omega'} (X, \psi_k) \overline{(\underline{X}, \psi_l)} dP = \int_{\Omega'} q_k \bar{q}_l dP = \\ &= r(k, l) + \mu(k) \overline{\mu(l)} = (R, \psi_k \otimes \psi_l) \quad (k \in \mathbb{N}_0, l \in \mathbb{N}_0) \end{aligned}$$

(cf. appendix 1, 1.17).

It remains to show that  $\underline{X}$  is a Gaussian process, i.e. that the distribution of  $((X, f_1), \dots, (X, f_n))$  is Gaussian for every  $n \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_n \in S$ . For the case  $n = 1$  this follows from the fact that for  $f \in S$

$$(\underline{X}, f) = \lim_{K \rightarrow \infty} \sum_{k=0}^K q_k(\psi_k, f)$$

(the limit is in  $L_2(\Omega')$ -sense), so that  $(\underline{X}, f)$  has a Gaussian distribution with mean  $(F, f)$  and variance  $(R - F \otimes \bar{F}, f \otimes \bar{f})$ . The cases with  $n > 1$  are treated analogously.  $\square$

We refer to [GW], Ch. III, §2.3 where a theorem as the above one is proved for the case that the space  $K$  is point of departure of the theory.

2.1.7. The following theorem determines the effect on expectation function and autocorrelation function of linear transformations of the process.

**THEOREM.** Let  $T$  be a linear operator of  $S$  with an adjoint.

- (i) If  $\underline{X} \in S_{\Omega,1}^*$ , then  $E\underline{TX} = T E\underline{X}$ .  
(ii) If  $\underline{X} \in S_{\Omega,2}^*$ , then  $R_{\underline{TX}} = T \otimes \bar{T} R_{\underline{X}}$  (cf. appendix 1, 4.6, 4.7 and 4.16).

**PROOF.** (i) Let  $\underline{X} \in S_{\Omega,1}^*$ . We have for every  $f \in S$

$$(E\underline{TX}, f) = \int_{\Omega} (\underline{TX}, f) dP = \int_{\Omega} (\underline{X}, T^* f) dP = (E\underline{X}, T^* f) = (T E\underline{X}, f).$$

Hence  $E\underline{TX} = T E\underline{X}$ .

- (ii) Let  $\underline{X} \in S_{\Omega,2}^*$ . We have

$$\begin{aligned} (R_{\underline{TX}}, f \otimes g) &= \int_{\Omega} (\underline{TX}, f) \cdot \overline{(\underline{TX}, g)} dP = \int_{\Omega} (\underline{X}, T^* f) \cdot \overline{(\underline{X}, T^* g)} dP = \\ &= (R_{\underline{X}}, T^* f \otimes T^* g) = (T \otimes \bar{T} R_{\underline{X}}, f \otimes g) \end{aligned}$$

by appendix 1, 4.6 and 4.7. Hence  $R_{\underline{TX}} = T \otimes \bar{T} R_{\underline{X}}$ .  $\square$

2.1.8. We have the following theorem about convergence of expectation functions and autocorrelation functions.

**THEOREM.** (i) Let  $\underline{X} \in S_{\Omega,1}^*$ ,  $\underline{X}_n \in S_{\Omega,1}^*$  ( $n \in \mathbb{N}$ ), and let  $\underline{X}_n \rightarrow \underline{X}$  ( $S_{\Omega,1}^*$ ). Then  $E\underline{X}_n \xrightarrow{S^*} E\underline{X}$ .

(ii) Let  $\underline{X} \in S_{\Omega,2}^*$ ,  $\underline{X}_n \in S_{\Omega,2}^*$  ( $n \in \mathbb{N}$ ), and let  $\underline{X}_n \rightarrow \underline{X}$  ( $S_{\Omega,2}^*$ ). Then  $R_{\underline{X}_n} \xrightarrow{S^{2*}} R_{\underline{X}}$ .

**PROOF.** (i) We have for  $f \in S$

$$(E\underline{X}_n, f) = \int_{\Omega} (\underline{X}_n, f) dP \rightarrow \int_{\Omega} (\underline{X}, f) dP = (E\underline{X}, f) \quad (n \rightarrow \infty).$$

Hence by appendix 1, 1.15,  $E\underline{X}_n \xrightarrow{S^*} E\underline{X}$ .

- (ii) We have for  $f \in S$ ,  $g \in S$

$$(R_{\underline{X}_n}, f \otimes g) = \int_{\Omega} (\underline{X}_n, f) \cdot \overline{(\underline{X}_n, g)} dP \rightarrow \int_{\Omega} (\underline{X}, f) \cdot \overline{(\underline{X}, g)} dP = (R_{\underline{X}}, f \otimes g) \quad (n \rightarrow \infty).$$

Hence, by appendix 1, 3.7,  $R_{\underline{x}_n} \xrightarrow{S^{2*}} R_{\underline{x}}$ . □

REMARK. If  $\underline{x} \in S_{\Omega,2}^*$ ,  $\underline{x}_n \in S_{\Omega,2}^*$  ( $n \in \mathbb{N}$ ), and  $R_{\underline{x}_n - \underline{x}} \xrightarrow{S^{2*}} 0$ , then  $\underline{x}_n \rightarrow \underline{x}$  ( $S_{\Omega,2}^*$ ). In general  $E \underline{x}_n \xrightarrow{S^{2*}} E \underline{x}$ ,  $R_{\underline{x}_n} \xrightarrow{S^{2*}} R_{\underline{x}}$  does not imply  $\underline{x}_n \rightarrow \underline{x}$  ( $S_{\Omega,2}^*$ ).

2.1.9. Let  $\underline{x} \in S_{\Omega,2}^+$  (cf. 1.3.2), and assume that  $r := \bigvee_{(t,s)} \int_{\Omega} \underline{x}(t) \overline{\underline{x}(s)} dP$  is measurable over  $\mathbb{R}^2$ . Now both  $r$  and  $f := \bigvee_t \int_{\Omega} \underline{x}(t) dP$  are embeddable functions (of two and one variable respectively). It is not hard to see that  $E(\text{emb}(\underline{x})) = \text{emb}(f)$ . It is somewhat harder to show that  $R_{\text{emb}(\underline{x})} = \text{emb}(r)$ ; we omit the proof.

## 2.2. SECOND ORDER STATIONARITY

2.2.1. Let  $(\Omega, \Lambda, P)$  be a fixed probability space. We are going to define second order stationarity for elements of  $S_{\Omega,2}^*$ .

2.2.2. DEFINITION. Let  $V$  be a group of linear operators of  $S$  with an adjoint, and let  $\underline{x} \in S_{\Omega,2}^*$ . If the numbers  $\int_{\Omega} (V\underline{x}, f) dP$ ,  $\int_{\Omega} (V\underline{x}, f) \overline{(V\underline{x}, g)} dP$  are independent of  $V \in V$  for every  $f \in S$ ,  $g \in S$ , then we say that  $\underline{x}$  is *second order V-stationary*. In the special case that  $V = (T_a)_{a \in \mathbb{R}}$ , we speak of *second order time stationarity*, and in case  $V = (R_b)_{b \in \mathbb{R}}$ , we speak of *second order frequency stationarity*.

In terms of expectation functions and autocorrelation functions  $V$ -stationarity of an  $\underline{x} \in S_{\Omega,2}^*$  means that  $V E \underline{x} = E \underline{x}$ ,  $V \otimes \bar{V} R_{\underline{x}} = R_{\underline{x}}$  ( $V \in V$ ). Cf. 2.1.7.

Note that if  $V$  is a group of linear operators of  $S$  with an adjoint, and if  $\underline{x} \in S_{\Omega,2}^*$  is strict sense  $V$ -stationary, then  $\underline{x}$  is second order  $V$ -stationary (cf. 1.2.2).

2.2.3. EXAMPLE. We find as in 1.2.3 that  $\underline{x} \in S_{\Omega,2}^*$  is second order time stationary if and only if  $F \underline{x}$  is second order frequency stationary. Such a thing also holds with  $F^* \underline{x}$  instead of  $F \underline{x}$ .

2.2.4. We want to characterize expectation function of a second order stationary process. The following theorem deals with the time stationary case.

THEOREM. Let  $\underline{x} \in S_{\Omega,2}^*$ . Then  $\underline{x}$  is second order time stationary if and only if there exists  $c \in \mathbb{C}$  and  $K \in S^*$  with  $F K \geq 0$  (cf. appendix 4, definition 2.1) such that  $E \underline{x} = c H$ ,  $R_{\underline{x}} = Z_U (H \otimes K)$ . Here  $H$  denotes  $\text{emb}(\bigvee_t 1)$ .

PROOF. Assume that  $\underline{X}$  is second order time stationary. We have (by 2.2.2)  $T_a \underline{E\underline{X}} = \underline{E\underline{X}}$  ( $a \in \mathbb{R}$ ). Hence  $\underline{E\underline{X}} = cH$  for some  $c \in \mathbb{C}$  by appendix 4, theorem 1.2 (applied with  $n = 1$ ,  $m = 0$ ).

Secondly, if  $F = Z_U R_{\underline{X}}$ , then  $T_a^{(1)} F = F$  ( $a \in \mathbb{R}$ ). This is proved as follows. By the formula  $T_a^{(1)} Z_U = Z_U T_{a/\sqrt{2}} \otimes T_{a/\sqrt{2}}$  ( $a \in \mathbb{R}$ ) we get

$$T_a^{(1)} F = T_a^{(1)} Z_U R_{\underline{X}} = Z_U T_{a/\sqrt{2}} \otimes T_{a/\sqrt{2}} R_{\underline{X}} = Z_U R_{\underline{X}} = F$$

for  $a \in \mathbb{R}$  (cf. 2.2.2; note that  $\bar{T}_{a/\sqrt{2}} = T_{a/\sqrt{2}}$  for  $a \in \mathbb{R}$ ). Application of appendix 4, theorem 1.2 shows that  $R_{\underline{X}} = Z_U (H \otimes K)$  with some  $K \in S^*$ . We next show that  $\bar{F}K \geq 0$ . Let therefore  $f \in S$ ,  $f(x) \geq 0$  ( $x \in \mathbb{R}$ ). It is known from elementary complex function theory that we can write  $f$  as  $h \cdot \bar{h}$  with some analytic  $h$ . Since  $f \in S$  we have  $h \in S$ . If we put  $g = F^* h$ , we get

$$(H \otimes \bar{F}K, F^{(2)} Z_U (g \otimes \bar{g})) = (R, g \otimes \bar{g}) \geq 0.$$

The left hand side of this equality can be written (by [B1], (21.4)) as

$$(FK, \int_{-\infty}^{\infty} (F^{(2)} Z_U (g \otimes \bar{g})) (x, y) dx).$$

Since for every  $y \in \mathbb{C}$

$$\int_{-\infty}^{\infty} (F^{(2)} Z_U (g \otimes \bar{g})) (x, y) dx = h(y/\sqrt{2}) \overline{h(\bar{y}/\sqrt{2})},$$

we find  $(FK, \int_{-\infty}^{\infty} f(y/\sqrt{2}) dy) \geq 0$ . It easily follows that  $(FK, k) \geq 0$  for every  $k \in S$  with  $k(x) \geq 0$  ( $x \in \mathbb{R}$ ). Hence  $\bar{F}K \geq 0$ .

Now assume that there is a  $c \in \mathbb{C}$ ,  $K \in S^*$  with  $\underline{E\underline{X}} = cH$ ,  $R_{\underline{X}} = Z_U (H \otimes K)$ . It follows at once that  $T_a \underline{E\underline{X}} = \underline{E\underline{X}}$ ,  $T_a \otimes \bar{T}_a R_{\underline{X}} = R_{\underline{X}}$  for  $a \in \mathbb{R}$ , hence  $\underline{X}$  is second order time stationary.  $\square$

REMARK. If  $\underline{X} \in S_{\Omega, 2}^*$ , and if  $R_{\underline{X}} \neq 0$  has the form  $Z_U (G \otimes K)$  with some  $G \in S^*$ ,  $K \in S^*$ , then it can be proved that there is an  $a \in \mathbb{C}$  with  $|a| = 1$  such that both  $aG$  and  $\bar{a}\bar{F}K$  are of positive type.

2.2.5. DEFINITION. Let  $\underline{X}$  be a second order time stationary process with autocorrelation function  $Z_U (H \otimes K)$ . We call the generalized function  $L$ , given by  $(L, f) = (FK, \int_Y f(y/\sqrt{2}))$  for  $f \in S$  (cf. appendix 1, theorem 3.3) the *spectral density function* of the process, and the measure associated with  $L$  (cf. appendix 4, theorem 2.2) is called the *spectral measure* of the process.

If  $\underline{X}$  is a second order time stationary process with autocorrelation function  $R_{\underline{X}} = Z_U (H \otimes K)$ , then the Wigner distribution  $F^{(2)} Z_U R_{\underline{X}}$  of  $\underline{X}$  equals the tensor product of the functions  $H$  and  $FK$ . We thus see that the Wigner distribution of a second order time stationary process is constant in the first (= time) variable. We also have a converse: if the Wigner distribution of a process (with constant expectation function) is constant in its first variable, then the process is second order time stationary.

For later references we mention the following result: if  $\underline{X} \in S_{\Omega, 2}^*$  is second order time stationary, and  $E\underline{X} = cH$ ,  $R_{\underline{X}} = Z_U (H \otimes K)$  with some  $c \in \mathbb{C}$ ,  $K \in S^*$ , then

$$(*) \quad E\underline{F\underline{X}} = c \delta_0, \quad R_{\underline{F\underline{X}}} = Z_U (FK \otimes \delta_0).$$

To show this we first note that  $\overline{FH} = \delta_0$  (for  $(\overline{FH}, f) = (H, F^*f) = \int_{-\infty}^{\infty} (F^*f)(x) dx = \overline{f(0)} = (\delta_0, f)$  if  $f \in S$ ), hence  $E\underline{F\underline{X}} = \overline{E\underline{X}} = c \delta_0$  by 2.1.7 (i). We further have by 2.1.7 (ii)

$$R_{\underline{F\underline{X}}} = F \otimes \overline{F} Z_U (H \otimes K) = F \otimes F^* Z_U (H \otimes K).$$

It is not hard to show that  $F \otimes F^* Z_U (F_1 \otimes F_2) = Z_U (\overline{F} F_2 \otimes F F_1)$  for  $F_1 \in S^*$ ,  $F_2 \in S^*$ . Therefore  $R_{\underline{F\underline{X}}} = Z_U (FK \otimes FH) = Z_U (FK \otimes \delta_0)$ .

Let  $\underline{X} \in S_{\Omega, 2}^*$  be second order frequency stationary. Then  $\underline{X} = FF^* \underline{X}$ , and  $\overline{F^* \underline{X}}$  is second order time stationary (cf. 2.2.3). We conclude from (\*) that  $E\underline{X} = c \delta_0$ ,  $R_{\underline{X}} = Z_U (L \otimes \delta_0)$  for some  $c \in \mathbb{C}$  and  $L \in S^*$  with  $L \geq 0$ . We further see that the Wigner distribution of  $\underline{X}$  is constant in the second variable:  $F^{(2)} Z_U R_{\underline{X}} = F^{(2)} Z_U (Z_U (L \otimes \delta_0)) = L \otimes F \delta_0 = L \otimes H$ .

2.2.6. In literature more general notions of time stationarity than that of 2.2.2 are studied. E.g., if  $n \in \mathbb{N}_0$ ,  $\underline{X} \in S_{\Omega, 2}^*$ , then  $\underline{X}$  is said to have *stationary increments of order n* if the distribution of  $((T_a (\frac{d}{dz})^n \underline{X}, f_1), \dots, (T_a (\frac{d}{dz})^n \underline{X}, f_m))$  is independent of  $a \in \mathbb{R}$  for every

$m \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_m \in S$  ( $(\frac{d}{dz})^n$  denotes  $n$ -fold differentiation; cf. appendix 1, 1.8 (ii)). In [GW], Ch. II, § 5 the general form of expectation function and autocorrelation function of processes of this type was determined (the result proved there also holds under the weaker assumption that the first and second order moments of  $T_a(\frac{d}{dz})^n \underline{X}$  are independent of  $a \in \mathbb{R}$ ). Although in [GW] the test function space  $K$  was taken as point of departure, it is not hard (but pretty laborious) to reproduce the proof of [GW], Ch. II, § 5, Satz 5.

**THEOREM.** Let  $\underline{X}$  be as above. Then  $E\underline{X}$  is the embedding of a polynomial of degree  $\leq n$ . For the autocorrelation function  $R$  the following holds:

Let  $\theta_0, \theta_1, \dots, \theta_{n-1}$  be elements of  $S$  satisfying  $\int_{-\infty}^{\infty} x^i \theta_j(x) dx = \delta_{ij}$  ( $i = 0, 1, \dots, n-1, j=0, 1, \dots, n-1$ ). Denote for  $\varphi \in S$  by  $t_\varphi$  the function  $\varphi - \sum_{j=0}^{n-1} b_j(\varphi) \theta_j$ , where  $b_j(\varphi) := \int_{-\infty}^{\infty} x^j \varphi(x) dx$  for  $j \in \mathbb{N}_0$ . There exists a measure  $\nu$  defined on  $\mathcal{B}(\mathbb{R})$ , satisfying  $\int_{-\infty}^{\infty} \exp(-\pi \epsilon x^2) d\nu(x) < \infty$  for every  $\epsilon > 0$ , such that for every  $\varphi \in S, \psi \in S$

$$\begin{aligned} (R, \varphi \otimes \bar{\psi}) &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R} \setminus \{0\}} (F \bar{t}_\varphi)(x) (F t_\psi)(x) x^{-2n} d\nu(x) + \\ &+ \nu(\{0\}) \frac{\overline{b_n(\varphi) b_n(\psi)}}{(2n)!} + \sum_{j=0}^{n-1} b_j(\psi) (F_j, \varphi) + \sum_{j=0}^{n-1} \overline{b_j(\varphi) (F_j, \psi)} + \\ &+ \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} a_{ij} \overline{b_i(\varphi)} b_j(\psi). \end{aligned}$$

Here  $F_j$  is the element of  $S^*$  determined by  $(F_j, f) = (R, f \otimes \bar{\theta}_j)$  ( $f \in S$ ), and  $a_{ij} = (R, \theta_i \otimes \bar{\theta}_j)$  for  $i=0, 1, \dots, n-1, j=0, 1, \dots, n-1$ . The measure  $\nu$  is uniquely determined by  $R$ .

We note that for the Wigner distribution  $V_{\underline{X}}$  of a process with stationary increments of order  $n$  the following holds:

$$((Q^{(2)})^2 - (P^{(1)})^2)^n V_{\underline{X}} = H \otimes L$$

with some  $L \in S^*$  of positive type.

2.2.7. We next consider second order frequency stationary processes in more detail. Let  $\underline{X}$  be such a process. According to 2.2.5 the autocorrelation function of  $\underline{X}$  has the form  $Z_U(L \otimes \delta_0)$  where  $L$  is a generalized function of positive type. This autocorrelation function resembles the autocorrelation function of a process with independent values at every moment. Our setting is not ideal for studying this kind of processes since the space  $S$  does not contain functions of compact support (except the zero function).

If the space  $K$  (of real-valued functions defined on  $\mathbb{R}$  of compact support with derivatives up to any order) is point of departure, processes with independent values at every moment can be defined as linear mappings  $\Phi$  of  $K$  into the set of all random variables such that  $\Phi(\varphi)$  and  $\Phi(\psi)$  are independent whenever  $\varphi \in K$  and  $\psi \in K$  have pairwise disjoint supports (cf. [GW], Ch. III, §4.1 and [Hi], Part II, §4.5). It is proved in [GW], Ch. III, §4.7 that the autocorrelation function (when defined, of course) of such a process is "concentrated on the diagonal". This amounts to say (in our language) that the autocorrelation function assumes the form  $Z_U(L \otimes \delta_0)$  with some generalized function  $L$ .

2.2.8. In spite of lack of elements of  $S$  of compact support, it is yet possible to study generalized stochastic processes with independent values at every moment. We restrict ourselves to generalized stochastic processes that are second order frequency stationary (this is not a very severe restriction in view of 2.2.7: only processes with order less than 2 cannot be handled). The following theorem is fundamental.

THEOREM. Let  $\underline{X} \in S_{\Omega, 2}^*$  be second order frequency stationary, and let  $\mu$  be the spectral measure of  $F^* \underline{X}$  (cf. 2.2.3 and 2.2.5). Let  $(,)_\mu$  be the inner product on  $S$  given by

$$(f, g)_\mu := \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, d\mu(x) \quad (f \in S, g \in S),$$

and let  $S_\mu$  be the completion of  $S$  with respect to  $\mu$ . Then  $\underline{X}$  can be extended to a continuous, anti-linear, norm preserving mapping of  $S_\mu$  into  $L_2(\Omega)$ .

PROOF. As we know from 2.2.5,  $R_{\underline{X}}$  has the form  $Z_U(L \otimes \delta_0)$  with  $L \geq 0$ . We have for  $f \in S, g \in S$  (by [B1], (21.4))

$$\int_{\Omega} (\underline{X}, f) \cdot \overline{(\underline{X}, g)} \, dP = (R_{\underline{X}}, f \otimes \bar{g}) = (Z_U(L \otimes \delta_0), f \otimes \bar{g}) = (L_1, f \cdot \bar{g}),$$

where  $L_1 \in S^*$  is determined by  $(L_1, h) = (L, \int_{-\infty}^{\infty} h(x/\sqrt{2}) dx)$  for  $h \in S$ . We show that  $L_1$  is the spectral density function of  $F^* \underline{X}$ . We have  $R_{\underline{X}} = R_{FF^* \underline{X}} = Z_U(L \otimes \delta_0)$ . Hence, if  $R_{F^* \underline{X}} = Z_U(H \otimes K)$  (cf. 2.2.3), then (by 2.2.5(\*))  $L = FK$ . It follows from definition 2.2.5 that  $L_1$  is the spectral density function of  $F^* \underline{X}$ .

We thus have (cf. appendix 4, theorem 2.2) for  $f \in S, g \in S$

$$\int_{\Omega} (\underline{X}, f) \overline{(\underline{X}, g)} dP = \int_{-\infty}^{\infty} \overline{f(x)} g(x) d\mu(x).$$

The remaining part of the proof can be given by using elementary Hilbert space techniques; it is omitted.  $\square$

REMARK. The  $S_{\mu}$  of the above theorem equals the collection of all (equivalence classes of) functions that are integrable with respect to  $\mu$ . To prove this it suffices to show that  $S_{\mu}$  contains all continuous functions of compact support. If  $f$  is such a function, then  $N_{\alpha} f \rightarrow f(\alpha \downarrow 0)$  uniformly in  $\mathbb{R}$  and dominatedly by a function of the form  $\int_{-\infty}^{\infty} M \exp(-\pi A z^2) dz$ . Hence, by appendix 4, theorem 4, 2.2,  $f \in S_{\mu}$ .

2.2.9. DEFINITION. Let  $\underline{X} \in S_{\Omega, 2}^*$  be second order frequency stationary, and let  $\mu$  and  $S_{\mu}$  be as in theorem 2.2.8. Let  $C$  be a subset of  $S_{\mu}$ . We say that  $\underline{X}$  has *independent values with respect to the class C* if  $(\underline{X}, \varphi)$  and  $(\underline{X}, \psi)$  are independent whenever  $\varphi \in C$  and  $\psi \in C$  have pairwise disjoint supports. In case  $C = K$  (or, more precisely,  $C$  is the collection of all classes in  $S_{\mu}$  containing an element of  $K$ ), and  $\underline{X}$  has independent values with respect to  $C$ , then we say that  $\underline{X}$  has *independent values at every moment*.

2.2.10. EXAMPLE. If  $\underline{X}$  is a second order frequency stationary Gaussian process (defined on  $S$ ) with zero expectation, then  $\underline{X}$  has independent values at every moment. This is seen as follows. It is not hard to show that the distribution of  $((\underline{X}, \varphi), (\underline{X}, \psi))$  is Gaussian if  $\varphi \in K, \psi \in K$  ( $L_2(\Omega)$ -limits of Gaussian random variables are Gaussian random variables). If now  $\varphi \in K$  and  $\psi \in K$  have pairwise disjoint supports then it follows from

$$\int_{\Omega} (\underline{X}, \varphi) \overline{(\underline{X}, \psi)} dP = (\psi, \varphi)_{\mu} = \int_{-\infty}^{\infty} \psi(x) \overline{\varphi(x)} d\mu(x) = 0$$

that  $(\underline{X}, \varphi)$  and  $(\underline{X}, \psi)$  are independent random variables.



2.2.11. REMARK. Let  $\Phi$  be a (real) generalized stochastic process (defined on  $K$ ) with independent values at every moment. We want to embed  $\Phi$  in our system  $S_{\Omega,2}^*$ . We therefore assume that  $\Phi$  is a continuous linear mapping of  $K$  into  $L_2(\Omega)$  (continuous with respect to the topology of  $K$  and the  $\|\cdot\|_2$ -topology of  $L_2(\Omega)$ ). The autocorrelation function of  $\Phi$  exists in that case, and is concentrated on the diagonal  $x = y$  ([GW], Ch. III, §4.7): there exists an  $L \in K'$  (dual of  $K$ ) such that

$$\int_{\Omega} \Phi(\varphi) \cdot \Phi(\psi) dP = (L, \varphi \cdot \psi) \quad (\varphi \in K, \psi \in K).$$

This  $L$  is of positive type: it follows from  $(L, \varphi^2) \geq 0$  for  $\varphi \in K$  that  $L$  is of multiplicative positive type, and in the space  $K'$  the notions of multiplicative positivity and positivity are equivalent (cf. [GW], Ch. II, §2.4). We conclude from [GW], Ch. II, §2.1, Satz 1 that there is a (unique) measure  $\mu$  defined on the Borel sets of  $\mathbb{R}$  such that

$$(L, \varphi) = \int_{-\infty}^{\infty} \varphi(x) d\mu(x) \quad (\varphi \in K).$$

Proceeding as in 2.2.8 (with  $\Phi$  and  $K$  instead of  $X$  and  $S$ ), we let  $K_{\mu}$  be the completion of  $K$  with respect to the inner product  $(\cdot, \cdot)_{\mu}$  and denote the unique extension of  $\Phi$  to a continuous linear mapping of  $K_{\mu}$  into  $L_2(\Omega)$  again by  $\Phi$ .

Now assume that  $\mu$  satisfies  $\int_{-\infty}^{\infty} \exp(-\pi \epsilon x^2) d\mu(x) < \infty$  for every  $\epsilon > 0$ . Let  $f \in S$ , and denote  $f_1 := \bigcup_{x \in \mathbb{R}} \operatorname{Re} f(x)$ ,  $f_2 := \bigcup_{x \in \mathbb{R}} \operatorname{Im} f(x)$ . It is easy to see that  $f_1 \in K_{\mu}$ ,  $f_2 \in K_{\mu}$ , hence  $\Phi(f_1)$  and  $\Phi(f_2)$  are well-defined. Let us put now

$$\Phi(f) := \Phi(f_1) - i \Phi(f_2).$$

It is not hard to show that  $\Phi$  depends continuously and anti-linearly on  $f \in S$ . Hence  $\bigcup_{f \in S} \Phi(f) \in S_{\Omega,2}^*$ .

2.2.12. The proof of theorem 2.2.8 uses a (well-known) technique of extending the domain of certain mappings to a complete space. The same technique can be used to prove the following theorem (cf. [GW], Ch. III, §3.4 for more details).

THEOREM. If  $\underline{X}$  is a second order time stationary process, and if  $\mu$  is the spectral measure of  $\underline{X}$  (cf. 2.2.5), then there exists a random measure  $Z$  such that

$$(\underline{X}, f) = \int_{-\infty}^{\infty} (F\bar{f})(\lambda) dZ(\lambda) \quad (f \in S).$$

This  $Z$  further satisfies  $\int_{\Omega} Z(A_1) \overline{Z(A_2)} dP = \mu(A_1 \cap A_2)$  for every pair of Borel sets  $A_1$  and  $A_2$  in  $\mathbb{R}$  with  $\mu(A_1) < \infty$ ,  $\mu(A_2) < \infty$ .

PROOF. Exactly the same as the proof of [GW], Ch. III, §3.4, Satz 3.  $\square$

REMARK. We sketch an alternative way to obtain a representation as in the above theorem for a second order time stationary process. It is possible to show that the equation  $\frac{d}{dz} \underline{y} = F\underline{x}$  has a solution  $\underline{y} \in S_{\Omega, 2}^*$ . (This is readily seen if one rewrites this equation as an infinite system of equations involving the Hermite coefficients  $(\underline{y}, \psi_k)$  and  $(F\underline{x}, \psi_k)$ , and makes use of the relations  $\frac{d}{dz} \psi_0 = -\pi^{1/2} \psi_1$ ,  $\frac{d}{dz} \psi_k = -((k+1)\pi)^{1/2} \psi_{k+1} + (k\pi)^{1/2} \psi_{k-1}$  for  $k \in \mathbb{N}$ .)

Since  $F\underline{x}$  is a second order frequency stationary process (cf. 2.2.3), we can extend it (as in 2.2.8) to an anti-linear, continuous mapping of  $S_{\mu}$  into  $L_2(\Omega)$ . Here  $\mu$  is the spectral measure of  $\underline{X}$ . We find

$$\begin{aligned} & \int_{\Omega} \left( \frac{d}{dz} \underline{y}, \chi_{A_1} \right) \cdot \overline{\left( \frac{d}{dz} \underline{y}, \chi_{A_2} \right)} dP = \\ & = \int_{\Omega} (F\underline{x}, \chi_{A_1}) \cdot \overline{(F\underline{x}, \chi_{A_2})} dP = \int_{A_1 \cap A_2} d\mu(x) \end{aligned}$$

for every pair of Borel sets  $A_1$  and  $A_2$  in  $\mathbb{R}$  with  $\mu(A_1) < \infty$ ,  $\mu(A_2) < \infty$  (if  $A$  is a Borel set in  $\mathbb{R}$  with  $\mu(A) < \infty$ , then  $\chi_A \in S_{\mu}$ ).

2.2.13. We are going to consider now what we call second order white noise.

DEFINITION. A generalized stochastic process  $\underline{w} \in S_{\Omega, 2}^*$  is called a *second order white noise process* if  $E\underline{w} = 0$ ,  $R_{\underline{w}} = c Z_U(H \otimes \delta_0)$  for some  $c \geq 0$ .

The following theorem lists some characterizations of second order white noise.

THEOREM. Let  $\underline{X} \in S_{\Omega, 2}^*$ , and assume  $E\underline{X} = 0$ . The following statements are equivalent.

- (i)  $\underline{X}$  is a second order white noise process.
- (ii) There is a  $d \geq 0$  such that  $\int_{\Omega} (\underline{X}, f) \cdot \overline{(\underline{X}, g)} dP = d(g, f)$  for every  $f \in S, g \in S$ .
- (iii)  $\underline{X}$  is a second order time and frequency stationary process.
- (iv) The Wigner distribution of  $\underline{X}$  is the embedding of a constant function of 2 variables.
- (v)  $f \in S, g \in S, (f, g) = 0 \Rightarrow \int_{\Omega} (\underline{X}, f) \cdot \overline{(\underline{X}, g)} dP = 0$ .

PROOF. The equivalence of (i) and (iv) is obvious:  $F^{(2)} Z_U(Z_U(H \otimes \delta_0)) = H \otimes H$ .

Assume that (i) is satisfied:  $R_{\underline{X}} = cZ_U(H \otimes \delta_0)$ . We have for  $f \in S, g \in S$

$$\begin{aligned} \int_{\Omega} (\underline{X}, f) \cdot \overline{(\underline{X}, g)} dP &= c(Z_U(H \otimes \delta_0), f \otimes \bar{g}) = \\ &= c \int_{-\infty}^{\infty} g\left(\frac{x}{\sqrt{2}}\right) \overline{f\left(\frac{x}{\sqrt{2}}\right)} dx = c\sqrt{2} (g, f). \end{aligned}$$

Hence (i)  $\Rightarrow$  (ii). If, on the other hand, there is a  $d \geq 0$  such that

$$(R_{\underline{X}}, f \otimes \bar{g}) = \int_{\Omega} (\underline{X}, f) \cdot \overline{(\underline{X}, g)} dP = d(g, f) \quad (f \in S, g \in S),$$

then  $(R_{\underline{X}}, f \otimes \bar{g}) = \frac{d}{\sqrt{2}} (Z_U(H \otimes \delta_0), f \otimes \bar{g})$ . Hence  $R_{\underline{X}} = \frac{d}{\sqrt{2}} Z_U(H \otimes \delta_0)$ .

It follows that (ii)  $\Rightarrow$  (i).

The equivalence of (iii) and (iv) follows from 2.2.5.

It is obvious that (ii)  $\Rightarrow$  (v). Assume that (v) is satisfied. To show that (ii) holds, we note that for  $f \in S, g \in S$

$$\int_{\Omega} (\underline{X}, f) \cdot \overline{(\underline{X}, g)} dP = \sum_{n=0}^{\infty} (\psi_n, f) (g, \psi_n) c_n,$$

where  $c_n = \int_{\Omega} |(\underline{X}, \psi_n)|^2 dP$  and  $\psi_n$  denotes (as usual) the  $n^{\text{th}}$  Hermite function ( $n \in \mathbb{N}_0$ ). We only have to show that  $c_n$  does not depend on  $n \in \mathbb{N}_0$ . Let

$n \in \mathbb{N}_0, m \in \mathbb{N}_0$ . Since  $(\psi_n + \psi_m, \psi_n - \psi_m) = 0$  we have

$$0 = \int_{\Omega} (\underline{X}, \psi_n + \psi_m) \cdot \overline{(\underline{X}, \psi_n - \psi_m)} dP = c_n - c_m, \text{ hence } c_n = c_m. \quad \square$$

A consequence of the above theorem is: if  $T$  is a unitary linear operator of  $S$ , and if  $\underline{X}$  is a second order white noise process, then so is  $\underline{TX}$ .

## CHAPTER 3

CONVOLUTION THEORY AND GENERALIZED STOCHASTIC PROCESSES;  
WIGNER DISTRIBUTION AND SECOND ORDER SIMULATION

In this chapter we give applications of convolution theory (as presented in [J2]; cf. also appendix 2) to the theory of generalized stochastic processes. We further describe second order simulation of generalized stochastic processes with the aid of "noise showers" and shot noise processes.

Convolution theory is for several reasons useful for the study of generalized stochastic processes. In section 3.1 we define the convolution product  $T_g \underline{X}$  with  $g \in C$  (cf. appendix 2,3) and a generalized stochastic process  $\underline{X}$ . If  $g \in S$ , this convolution product can be identified with the ordinary process  $\bigcup_t (T_t \underline{X}, g_-)$  (more precisely,  $T_g \underline{X}$  is the embedding of  $\bigcup_t (T_t \underline{X}, g_-)$ ). The set of processes  $T_g \underline{X}$ , where  $g$  runs through  $S$ , gives in some sense a complete description of the process  $\underline{X}$ . It thus appears that questions about  $\underline{X}$  on stationarity (or ergodicity) can be reduced to the same questions about  $T_g \underline{X}$  with  $g \in S$  (cf. 3.2.6 and 3.2.9 (i)).

In section 3.2 we deal with representation of second order time stationary processes as filtered white noise (cf. theorem 3.2.4). In [BT], page 6 the theorem is attributed to S.O. Rice, but we were not able to locate its proof in literature. The theorem says roughly that every second order time stationary process can be regarded as the result of passing a second order white noise process through a linear filter. We further prove in section 3.2 some ergodic theorems.

In section 3.3 we consider shot noise processes and "random Fourier series" processes as typical examples of generalized stochastic processes. These processes can be used for describing simulation of time and/or frequency stationary processes.

The above applications mainly deal with stationary processes. An important application (cf. section 3.4) of convolution theory for processes of general type is obtained by considering what we call time-frequency convolutions (instead of the ordinary (= time) convolutions). We presently sketch what this is all about.

Let  $g \in S$ , and let  $\underline{X}$  be a sufficiently well-behaved stochastic process. The time-frequency convolution  $S_g \underline{X}$  of  $g$  and  $\underline{X}$  is defined as

$\int_{-\infty}^{\infty} e^{-\pi i a b - 2\pi i b t} \underline{X}(t+a, \omega) \overline{g(t)} dt$ . We can define  $S_g \underline{X}$  as well if  $\underline{X}$  is a generalized stochastic process, simply by taking  $S_g \underline{X} = \text{emb}(S'_g \underline{X}) = \text{emb} \left( \int_{(a,b)} (e^{\pi i a b} T_{a,b} R_{\underline{X},g}) \right)$ . Now  $S_g$  maps generalized processes depending on one (time) variable onto generalized processes depending on two (time and frequency) variables. If  $\underline{X}$  is a generalized stochastic process, then the expected value of  $|(S'_g \underline{X})(a,b)|^2$  equals the (2-dimensional) convolution of  $V(g,g)$  (Wigner distribution of  $g$ ) and  $V_{\underline{X}}$  (Wigner distribution of  $\underline{X}$ ) evaluated at the point  $(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$ :

$$(T_{V(g,g)} V_{\underline{X}}) \left( \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right) = \int_{\Omega} |(S'_g \underline{X})(a,b)|^2 dP$$

(cf. 3.4.4 and 3.4.5; for convenience we have assumed  $g$  to be even).

This implies in particular that  $T_{V(g,g)} V_{\underline{X}}$  is a non-negative function of two variables, and thus allows an interpretation as an energy distribution in time and frequency. In this connection it seems to be adequate to take a  $g \in S$  such that the operation  $T_{V(g,g)}$  affects  $V_{\underline{X}}$  as little as possible. According to this rule, a time stationary  $\underline{X}$  (whose Wigner distribution is constant in the first variable) demands a  $g$  for which the operation  $T_{V(g,g)}$  has the flavour of averaging over a long horizontal ellipse. We note that there is no  $g \in S$  such that  $T_{V(g,g)}$  is close to the identity operator of  $S^2$  (or  $S^{2*}$ ). This fact is related to the non-existence of functions  $g$  whose Wigner distributions are concentrated in a very small area in the time-frequency plane.

We shall present details in section 3.4, where we also comment on related, fairly recent literature on physical spectra and evolutionary spectral density functions (cf. [Ma] and [Pr1], [Pr2]).

Let  $\underline{X}$  be a generalized stochastic process and let  $g \in S$ . The function  $T_{V(g,g)} V_{\underline{X}}$  occurring in the previous paragraphs is of importance when describing second order simulation of  $\underline{X}$  with the aid of what we call noise showers. This is the subject of section 3.5. A noise shower is a generalized stochastic process; it can be thought of as the superposition of a countable number of mutually independent noise quanta (also to be called random notes). The notes of the noise shower can be random in time, in frequency, and in pureness. Such a note can be symbolized as  $e^{-\pi i a b} R_{-b} T_{-a} Z_{\gamma} g$  (cf. appendix 1, 1.8 (ii)), where  $a$  stands for (random) time,  $b$  for (random) frequency, and where  $\gamma$  controls the (random) bandwidth of the note (the product of

bandwidth and pureness is constant). If we assume  $V_{\underline{X}}$  to be of positive type, we can find a noise shower, with non-random  $\gamma$ 's, whose expected Wigner distribution equals  $T_{V(g,g)} V_{\underline{X}}$ ; the function  $V_{\underline{X}}$  indicates how to distribute the random notes over the time-frequency plane (cf. 3.5.14). If, e.g.,  $\underline{X}$  is a second order white noise process (whence  $V_{\underline{X}}$  is a constant function), then the notes have to be distributed uniformly over the time-frequency plane. The general case (with  $V_{\underline{X}}$  possibly not of positive type) can be handled too (cf. 3.5.16), but then a smoothing operation on  $V_{\underline{X}}$  has to be performed so as to get a function of positive type. We can also describe simulation with showers of notes whose pureness is random as well. This does not give better results: it turns out that every note of the shower can be simulated by a shower of notes with non-random pureness (cf. 3.5.10-3.5.12).

### 3.1. PREPARATION

3.1.1. We give in this section some general applications of convolution theory to the theory of generalized stochastic processes. As usual  $(\Omega, \Lambda, P)$  is a fixed probability space, and  $p$  is an element of the extended real number system with  $1 \leq p \leq \infty$ .

3.1.2. Let  $g \in C$  (cf. appendix 2,3). It follows from appendix 2, theorem 5 (iii) and 1.1.3 that we can extend  $T_g$  to a linear operator of  $S_{\Omega, p}^*$  such that  $(T_g \underline{X}, f) = (\underline{X}, T_{\bar{g}} f)$  for every  $\underline{X} \in S_{\Omega, p}^*$ ,  $f \in S$ .

If  $\underline{X} \in S_{\Omega, 2}^*$ , then we have for the expectation function and the autocorrelation function of  $T_g \underline{X}$

$$E(T_g \underline{X}) = T_g(E\underline{X}), R_{T_g \underline{X}} = T_g \otimes T_{\bar{g}} R_{\underline{X}}$$

(cf. 2.1.7 and appendix 2, theorem 5 (ii)).

In a similar way we can handle with multiplication operators: if  $h \in \text{emb}^{-1}(M)$  (cf. appendix 2,7), and if  $\underline{X} \in S_{\Omega, p}^*$ , then  $h \cdot \underline{X}$  is well-defined. If  $\underline{X} \in S_{\Omega, 2}^*$  then we have for the expectation function and the autocorrelation function of  $h \cdot \underline{X}$

$$E(h \cdot \underline{X}) = h \cdot E\underline{X}, R_{h \cdot \underline{X}} = (h \otimes \bar{h}) \cdot R_{\underline{X}}$$

A number of theorems of appendix 2 have a pendant for generalized stochastic processes. We mention in particular the stochastic version of

appendix 2, theorem 8: if  $\underline{X} \in S_{\Omega, P}^*$ ,  $g \in C$ , then  $F(T_{\underline{g}} \underline{X}) = \text{emb}^{-1}(Fg) \cdot F\underline{X}$ .

If  $\underline{X}$  is a second order time stationary process, then we have for  $g \in C$ ,  $a \in \mathbb{R}$

$$T_a(E(T_{\underline{g}} \underline{X})) = E(T_{\underline{g}} \underline{X}), T_a \otimes \bar{T}_a R_{T_{\underline{g}} \underline{X}} = R_{T_{\underline{g}} \underline{X}}$$

(this follows from the above and the fact that the convolution operators commute with the time shifts; cf. also appendix 2, 4(ii) and 5(v)). We conclude from 2.2.2 that convolution operators preserve second order time stationarity. Similarly, multiplication operators preserve second order frequency stationarity.

3.1.3. THEOREM. Let  $g \in S$ ,  $\underline{X} \in S_{\Omega, P}^*$ . Then  $T_{\underline{g}} \underline{X} := \bigcup_{x \in \mathbb{R}} (T_{\underline{x}} \underline{X}, g_{\underline{x}}) \in \bar{S}_{\Omega, P}^+$  (cf. 1.3.2), and  $T_{\underline{g}} \underline{X} = \text{emb}(T_{\underline{g}} \underline{X})$ .

PROOF. We combine the proof of the two assertions. It is obvious that  $(T_{\underline{x}} \underline{X}, g_{\underline{x}}) \in L_P(\Omega)$ . Now let  $f \in L_Q(\Omega)$ . We shall show that  $\langle T_{\underline{g}} \underline{X}, f \rangle = \text{emb}^{-1}(\langle T_{\underline{g}} \underline{X}, f \rangle)$ . We have by 1.1.23, 1.3.2 and appendix 2,9

$$\begin{aligned} \langle T_{\underline{g}} \underline{X}, f \rangle &= \bigcup_x \int_{\Omega} (T_{\underline{x}} \underline{X}, g_{\underline{x}}) \cdot \bar{f} dP = \\ &= \bigcup_x \int_{\Omega} (\underline{X}, T_{-x} g_{\underline{x}}) \cdot \bar{f} dP = \bigcup_x (\langle \underline{X}, f \rangle, T_{-x} g_{\underline{x}}) = \\ &= \text{emb}^{-1}(T_{\underline{g}}(\langle \underline{X}, f \rangle)) = \text{emb}^{-1}(\langle T_{\underline{g}} \underline{X}, f \rangle). \end{aligned}$$

By the uniqueness part of theorem 1.3.3 it suffices to show that condition (2) of 1.3.2 is satisfied. Let  $\varepsilon > 0$ . We conclude from the above that  $\bigcup_x \int_{\Omega} (T_{\underline{g}} \underline{X})(x) \cdot \bar{f} dP \in \text{emb}^{-1}(M)$  for every  $f \in L_Q(\Omega)$ . Hence, for every  $f \in L_Q(\Omega)$  there exists an  $M > 0$  such that

$$\left| \int_{\Omega} (T_{\underline{g}} \underline{X})(x) \cdot \bar{f} dP \right| \leq M \exp(\pi \varepsilon x^2) \quad (x \in \mathbb{R}).$$

It follows from the Banach-Steinhaus theorem that  $\| (T_{\underline{g}} \underline{X})(x) \|_P = 0(\exp(\pi \varepsilon x^2))$  ( $x \in \mathbb{R}$ ). This holds for every  $\varepsilon > 0$ . It easily follows that condition (2) of 1.3.2 is satisfied.  $\square$



3.1.4. We have the following result on convergence.

THEOREM. Let  $\underline{X} \in S_{\Omega, P}^*$ , and assume that  $g \in C$ ,  $g_n \in C$  ( $n \in \mathbb{N}$ ),  $g_n \xrightarrow{C} g$  (cf. appendix 2, 10). Then  $T_{g_n} \underline{X} \rightarrow T_{g^-} \underline{X} (S_{\Omega, P}^*)$ .

PROOF. We have for  $f \in S$  by appendix 2, 5 (iii) and 10 (note that  $(\bar{g}_n)_- \xrightarrow{C} \bar{g}_-$ )

$$(T_{g_n} \underline{X}, f) = (\underline{X}, T_{(\bar{g}_n)_-} f) \rightarrow (\underline{X}, T_{\bar{g}_-} f) = (T_{g^-} \underline{X}, f)$$

if  $n \rightarrow \infty$ . Hence  $T_{g_n} \underline{X} \rightarrow T_{g^-} \underline{X} (S_{\Omega, P}^*)$  by 1.1.8.  $\square$

## 3.2. CONVOLUTION THEORY AND TIME STATIONARITY

3.2.1. We are going to apply convolution theory to time stationary and ergodic processes. We shall prove a version of a classical theorem of Rice (theorem 3.2.4) stating that a pretty large class of second order time stationary processes can be represented as the convolution product of a second order white noise process (cf. 2.2.13) and a function depending on the spectral measure of the process (cf. 2.2.5). We shall also prove some ergodic theorems for strict sense time stationary and ergodic processes.

Once again  $(\Omega, \Lambda, P)$  is a probability space.

3.2.2. Before giving the proof of theorem 3.2.4 we need some preparations. Let  $\underline{W}$  be a second order white noise process. We shall define  $T_{g^-} \underline{W}$  for a considerably larger class of  $g$ 's than the class  $C$ ; our main tool for doing so is theorem 2.2.8. This larger class  $V$  consists of all generalized functions  $g$  with the property that  $T_{g^-} f = \int_{\mathbb{R}} (T_{x^-} f, g_-) \in L_2(\mathbb{R})$  for every  $f \in S$ . The set  $V$  can be described in an alternative way: we have  $g \in V$  if and only if  $Fg$  is the embedding of an element of  $S^+$  whose square also belongs to  $S^+$ . To show this we note that for  $f \in S$ ,  $g \in S^*$

$$(1) \quad F(\text{emb}(T_{g^-} f)) = F(T_{f^-} g) = Ff \cdot Fg$$

by appendix 2, theorem 9 and 8. Let  $g \in V$ . Then also  $\bar{g} \in V$  (and  $g_- \in V$ ,  $\bar{g}_- \in V$ ). We conclude that  $Ff \cdot Fg \in \text{emb}(L_2(\mathbb{R}))$  for every  $f \in S$ . It is not hard (but somewhat laborious) to show that  $Fg = \text{emb}(h)$  for some  $h \in S^+$  with  $h^2 \in S^+$ . Conversely, if  $Fg = \text{emb}(h)$  for some  $h \in S^+$  with  $h^2 \in S^+$ , then the right hand side in (1) belongs to  $\text{emb}(L_2(\mathbb{R}))$  for every  $f \in S$ , and therefore

$T_g f \in L_2(\mathbb{R})$  (and also  $T_g f \in L_2(\mathbb{R})$ ) for every  $f \in S$ .

Now let  $g \in V$ . Note that  $R_{\underline{W}} = Z_U(H \otimes \delta_0)$  (cf. 2.2.13). It follows from theorem 2.2.8 that  $(\underline{W}, f)$  can be defined for every  $f \in L_2(\mathbb{R})$  (cf. also 2.2.8, remark). We have for  $f \in L_2(\mathbb{R})$

$$\int_{\Omega} |(\underline{W}, f)|^2 dP = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

3.2.3. DEFINITION. Let  $\underline{W}$  be a second order white noise process, and let  $g \in V$ . We define  $T_g \underline{W}$  by

$$(T_g \underline{W}, f) = (\underline{W}, T_{g^-} f) \quad (f \in S).$$

It is not hard to see that  $T_g \underline{W} \in S_{\Omega, 2}^*$  if  $\underline{W}$  and  $g$  are as in the definition (as noted earlier  $g \in V \Rightarrow \bar{g} \in V$ ). And if  $g \in C$ , then the  $T_g \underline{W}$  of the above definition equals the one of definition 3.1.3 (it is obvious that  $C \subset V$ ).

REMARK. As to multiplication operators we have a similar situation: if  $\underline{Z}$  is a second order white noise process, and if  $k \in S^+$  is such that  $k^2 \in S^+$ , then we can define the process  $k \cdot \underline{Z}$  by putting

$$(k \cdot \underline{Z}, f) = (\underline{Z}, \bar{k} \cdot f) \quad (f \in S).$$

Note that the relation  $F(T_g \underline{W}) = k \cdot \underline{F}\underline{W}$  still holds if  $\underline{W}$  and  $g$  are as in the above definition ( $\underline{F}\underline{W}$  is a second order white noise process). Here  $k \in S^+$  is such that  $Fg = \text{emb}(k)$  (whence  $k^2 \in S^+$ ). To prove this let  $f \in S$ . We have  $(F T_g \underline{W}, f) = (\underline{W}, T_{g^-} F^* f)$ . It follows from 3.2.2 (1) that  $\text{emb}(T_{g^-} F^* f) = F^* \text{emb}(\bar{k} \cdot f)$ . Now let  $u \in L_2(\mathbb{R})$ ,  $v \in L_2(\mathbb{R})$  be such that  $\text{emb}(u) = F^* \text{emb}(v)$ . Then  $(\underline{W}, u) = (\underline{F}\underline{W}, v)$ . For  $(\underline{W}, u) = (\underline{F}\underline{W}, v)$  holds if  $u \in S$ ,  $v \in S$ , and the general case easily follows from 2.2.8 (especially 2.2.8, remark) and the fact that the Fourier transform is a unitary linear operator of  $L_2(\mathbb{R})$ .

3.2.4. THEOREM. Let  $\underline{X} \in S_{\Omega, 2}^*$  be a second order time stationary process with zero expectation function. Write  $R_{\underline{X}} = Z_U(H \otimes K)$ , and assume that  $FK = \text{emb}(k_1)$  with some  $k_1 \in S^+$ ,  $k_1(x) > 0$  ( $x \in \mathbb{R}$ ). Then there is a  $g \in V$  and a second order white noise process  $\underline{W}$  such that  $\underline{X} = T_g \underline{W}$ .

PROOF. Define  $\underline{Y} := \underline{FX}$  (whence  $R_{\underline{Y}} = Z_U(FK \otimes \delta_0)$ , cf. 2.2.5), and let  $k := k_1^{\frac{1}{2}}$ . We are going to define a second order white noise process  $\underline{Z}$  whose inverse Fourier transform will turn out to be the second order white noise process with the desired properties. This  $\underline{Z}$  is defined formally as  $\underline{Y}/k$ . In order to give the division a meaning, we apply theorem 2.2.8.

It is not hard to show that the completion of  $S$  with respect to the inner product norm  $\|h\|_k$  given by

$$\|h\|_k = \left( \int_{-\infty}^{\infty} |h(x)k(x)|^2 dx \right)^{\frac{1}{2}} \quad (h \in S)$$

equals the collection of (classes of) functions  $\varphi$  defined on  $\mathbb{R}$  with  $\varphi \cdot k \in L_2(\mathbb{R})$  (cf. also 2.2.8, remark). Hence  $(\underline{Y}, f/k)$  is well-defined for  $f \in L_2(\mathbb{R})$  according to 2.2.8, and we have

$$\int_{\Omega} |(\underline{Y}, f/k)|^2 dP = \|f/k\|_k^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

Now put  $\underline{Z} := \bigvee_{f \in L_2(\mathbb{R})} (\underline{Y}, f/k)$ . Then (the restriction to  $S$  of)  $\underline{Z}$  is a second order white noise process (this follows easily from 2.2.13), and we have for  $f \in S$

$$(\underline{Z}, k \cdot f) = (\underline{Y}, f).$$

Finally define  $W := F^* \underline{Z}$ ,  $g := F(\text{emb}(k))$ . Then  $W$  is a second order white noise process for which  $T_g W$  makes sense (cf. definition 3.2.3), and  $Fg = \text{emb}(k)$ . We have for  $f \in S$

$$\begin{aligned} (T_g W, f) &= (F(T_g W), Ff) = (k \cdot \underline{Z}, Ff) = \\ &= (\underline{Z}, k \cdot Ff) = (\underline{Y}, Ff) = (FX, Ff) = (\underline{X}, f) \end{aligned}$$

by 3.2.3, remark and the fact that  $F$  is a unitary operator of  $S$ .

Hence  $T_g W = \underline{X}$ . □

3.2.5. REMARKS. 1. The restriction " $k_1(x) > 0$  ( $x \in \mathbb{R}$ )" in theorem 3.2.4 can be removed if we pass to product spaces. Let  $k$  and  $k_1$  be as in the proof of theorem 3.2.4, but drop the assumption  $k_1(x) > 0$  ( $x \in \mathbb{R}$ ). Since  $\tilde{F}K$  is of positive type (cf. 2.2.4) we may assume  $k_1(x) \geq 0$  ( $x \in \mathbb{R}$ ). Let  $A := \{x \in \mathbb{R} \mid k_1(x) > 0\}$ ,  $A_1 := \mathbb{R} \setminus A$ . We can find (as in the proof of theorem 3.2.4) a mapping  $\underline{Z}$  of  $L_2(\mathbb{R})$  into  $L_2(\Omega)$  such that for  $f \in L_2(\mathbb{R})$ ,  $h \in S$

$$(1) \quad \int_{\Omega} |(\underline{Z}, f)|^2 dP = \int_A |f(x)|^2 dx, \quad (\underline{Z}, k \cdot h) = (F\underline{X}, Fh).$$

(This is done by putting  $(f/k)(x) = (f(x)/k(x)) \cdot \chi_A(x)$  ( $x \in \mathbb{R}$ ) for  $f \in L_2(\mathbb{R})$ .)

According to 2.1.6 we can find a probability space  $(\Omega_1, \mathcal{A}_1, P_1)$  and a  $\underline{Z}_1 \in S_{\Omega_1, 2}^*$  such that  $E\underline{Z}_1 = 0$ ,  $R_{\underline{Z}_1} = Z_U(L \otimes \delta_0)$ , where  $L = \text{emb}(\bigcup_U \chi_{A_1}(\frac{U}{\sqrt{2}}))$ . Let  $\Omega_0 := \Omega \times \Omega_1$ , and take product measure on  $\Omega_0$ . Let  $\hat{\underline{X}}$ ,  $\hat{\underline{Z}}$  and  $\hat{\underline{Z}}_1$  be representatives (cf. 1.1.1) of  $\underline{X}$ ,  $\bigcup_{f \in S} (\underline{Z}, f)$  and  $\underline{Z}_1$  respectively. Now  $\bigcup_{(f, (\omega, \omega_1))} \hat{\underline{X}}(f, \omega)$ ,  $\bigcup_{(f, (\omega, \omega_1))} \hat{\underline{Z}}(f, \omega)$  and  $\bigcup_{(f, (\omega, \omega_1))} \hat{\underline{Z}}_1(f, \omega_1)$  are representatives of elements  $\underline{X}_0$ ,  $\underline{Z}_0$  and  $\underline{Z}_1$  of  $S_{\Omega_0, 2}^*$ . Let  $\underline{Z}_2 := \underline{Z}_0 + \underline{Z}_1$ . Since  $E\underline{Z}_0 = E\underline{Z}_1 = 0$ ,  $\int_{\Omega_0} (\underline{Z}_0, f) \cdot \overline{(\underline{Z}_1, f)} dP_0 = 0$  ( $f \in S$ ), we have  $E\underline{Z}_2 = 0$  and

$$\begin{aligned} \int_{\Omega_0} |(\underline{Z}_2, f)|^2 dP_0 &= \int_{\Omega_0} |(\underline{Z}_0, f)|^2 dP_0 + \int_{\Omega_0} |(\underline{Z}_1, f)|^2 dP_0 = \\ &= \int_A |f(x)|^2 dx + \int_{A_1} |f(x)|^2 dx = \int_{-\infty}^{\infty} |f(x)|^2 dx \end{aligned}$$

for  $f \in S$ . By 2.2.13 we conclude that  $\underline{Z}_2$  is a second order white noise process. We further note that  $\underline{Z}_1$  is second order frequency stationary, and that  $(\underline{Z}_1, f)$  makes sense for every  $f \in L_2(\mathbb{R})$  ( $i = 0, 1, 2$ ). Now if  $f \in S$ , then

$$(\underline{Z}_2, k \cdot f) = (\underline{Z}_0, k \cdot f) + (\underline{Z}_1, k \cdot f) = (\underline{Z}_0, k \cdot f),$$

and  $(\underline{Z}_0, k \cdot f) = (F\underline{X}_0, Ff)$  according to (1). Proceeding as in the proof of theorem 3.2.4 we find  $\underline{X}_0 = T_{g=0} \underline{W}$  with  $\underline{W}_0 = F^* \underline{Z}_2$ ,  $g = F(\text{emb}(k))$ .

2. The condition " $FK \in \text{emb}(S^+)$ " is equivalent to the condition that the spectral measure of  $\underline{X}$  (cf. 2.2.5) is absolutely continuous with respect to Lebesgue measure. This follows easily from the uniqueness part of appendix 4, theorem 2.2.

3.2.6. We next apply convolution theory to strict sense time stationary processes.

**THEOREM.** Let  $\underline{X} \in S_{\Omega, P}^*$ . If  $\underline{X}$  is strict sense time stationary, then  $T_{\underline{g}} \underline{X}$  is strict sense time stationary for every  $g \in \mathcal{C}$ . If  $T_{\underline{g}} \underline{X}$  is strict sense time stationary for every  $g \in S$ , then  $\underline{X}$  is strict sense time stationary.

**PROOF.** Assume that  $\underline{X}$  is strict sense time stationary, and let  $g \in \mathcal{C}$ . If  $n \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_n \in S$ , then the distribution of  $((T_{\underline{a}} \underline{X}, T_{\underline{g}} f_1), \dots, (T_{\underline{a}} \underline{X}, T_{\underline{g}} f_n))$  is independent of  $a \in \mathbb{R}$ . Since  $T_{\underline{g}}$  and  $T_{\underline{g}^-}$  are adjoint operators and  $T_{\underline{a}} T_{\underline{g}} = T_{\underline{g}^-} T_{\underline{a}}$  ( $a \in \mathbb{R}$ ) we conclude that the distribution of  $((T_{\underline{a}} T_{\underline{g}} \underline{X}, f_1), \dots, (T_{\underline{a}} T_{\underline{g}} \underline{X}, f_n))$  is independent of  $a \in \mathbb{R}$ . Hence  $T_{\underline{g}} \underline{X}$  is strict sense time stationary.

Now assume that  $T_{\underline{g}} \underline{X}$  is strict sense time stationary for every  $g \in S$ . Let  $n \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_n \in S$ . We can find  $g \in S$  and  $h_1 \in S$  with  $T_{\underline{g}} h_i = f_i$  ( $i = 1, \dots, n$ ). Now we see that the distribution of  $((T_{\underline{a}} T_{\underline{g}} \underline{X}, h_1), \dots, (T_{\underline{a}} T_{\underline{g}} \underline{X}, h_n))$  (and hence that of  $((T_{\underline{a}} \underline{X}, f_1), \dots, (T_{\underline{a}} \underline{X}, f_n))$ ) is independent of  $a \in \mathbb{R}$ .  $\square$

3.2.7. We next prove an ergodic theorem for strict sense time stationary processes.

**THEOREM.** Let  $\underline{X} \in S_{\Omega, P}^*$  with  $1 \leq p < \infty$ , and assume that  $\underline{X}$  is strict sense time stationary. Then we have  $T_{h_\tau} \underline{X} \rightarrow E(\underline{X} \mid \Lambda_1)$  ( $S_{\Omega, P}^*$ ) if  $\tau \rightarrow \infty$  (cf. 1.1.13 and 1.1.8). Here  $h_\tau$  denotes  $\text{emb}(\frac{1}{2\tau} \chi_{[-\tau, \tau]}) \in \mathcal{C}$  (cf. appendix 2,11 (iii)), and  $\Lambda_1$  is a  $\sigma$ -algebra of invariant sets (cf. 1.2.6).

**PROOF.** We shall first consider the canonical representative of  $\underline{X}$  (cf. 1.1.19) instead of  $\underline{X}$  itself, and then use 1.2.6 and 1.1.21 to carry over the results. We start by showing that for all  $A \in \Lambda^*$

$$(*) \quad \bigcup_{(t, F) \in \mathbb{R} \times S^*} \chi_A(T_t F)$$

is measurable over the product space  $\mathbb{R} \times S^*$ . If  $f \in S$ , then  $\bigcup_{(t, F)} (T_t F, f)$  is measurable over  $\mathbb{R} \times S^*$ :  $(T_t F, f) = \sum_{k=0}^{\infty} (F, \psi_k) (\psi_k, T_{-t} f)$  for  $t \in \mathbb{R}$ ,  $f \in S$ . Hence the function in (\*) is measurable for all sets  $A$  of the form

$\{F \mid (F, f) \in B\}$  with  $f \in S$ ,  $B \in \mathcal{B}(\mathbb{C})$ . Since the collection of all elements  $A$  of  $\Lambda^*$  for which the function in (\*) is measurable forms a  $\sigma$ -algebra, we easily conclude from appendix 1, 5.2, remark that this collection equals  $\Lambda^*$ .

We conclude from the above and 1.2.4 that (in the terminology of [D], Ch. XI, §1)  $(T_t)_{t \in \mathbb{R}}$  is a measurable translation group of measure preserving 1 - 1 point transformations of the probability space  $(S^*, \Lambda^*, P_{\underline{X}}^*)$ . According to [D], Ch. XI, 2 (in particular theorem 2.1) this implies that for every integrable  $k: S^* \rightarrow \mathbb{C}$

$$(1) \quad \bigvee_F \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} k(T_t F) dt = E(k \mid \Lambda_1^*)$$

almost everywhere in  $S^*$ , and that, if  $\int_{S^*} |k|^p dP_{\underline{X}} < \infty$ ,

$$(2) \quad \lim_{\tau \rightarrow \infty} \left\| \bigvee_F \frac{1}{2\tau} \int_{-\tau}^{\tau} k(T_t F) dt - E(k \mid \Lambda_1^*) \right\|_p = 0.$$

Here  $\Lambda_1^*$  is the  $\sigma$ -algebra of invariant sets.

Now let  $f \in S$ . If  $F \in S^*$ , then

$$\frac{1}{2\tau} \int_{-\tau}^{\tau} (T_t F, f) dt = (T_{h_\tau} F, f)$$

(this identity holds if  $F \in \text{emb}(S)$ ; the general case can be handled by noting that  $(T_t N_\alpha F, f) \rightarrow (T_t F, f)$  ( $\alpha \downarrow 0$ ) locally uniformly in  $t \in \mathbb{R}$ ). Hence, if we take  $k := \bigvee_F (F, f)$  in (2), we get

$$(3) \quad \lim_{\tau \rightarrow \infty} \left\| \bigvee_F (T_{h_\tau} F, f) - E\left(\bigvee_F (F, f) \mid \Lambda_1^*\right) \right\|_p = 0.$$

We now carry over the results proved. Define  $\hat{\underline{X}}$  by

$$\hat{\underline{X}}(f, \omega) := \begin{cases} \sum_{k=0}^{\infty} q_k(\omega) (\psi_k, f) & (\omega \in \Omega_0) \\ 0 & (\omega \notin \Omega_0) \end{cases}$$

for  $f \in S$ . Here  $q_k$  ( $k \in \mathbb{N}_0$ ) and  $\Omega_0$  are as in 1.1.16. Now  $\hat{\underline{X}}$  is a representative of  $\underline{X}$ . Further, let  $U$  be as in 1.1.17 and put  $\Lambda_1 := U^{-1}(\Lambda_1^*)$ . It is clear from 1.2.6 that the elements of  $\Lambda_1$  are invariant, and it follows from 1.1.21

that  $\Lambda_1$  is a  $\sigma$ -algebra. Finally, by (3) and 1.1.21,

$$\lim_{\tau \rightarrow \infty} \| (T_{h_\tau} \underline{X}, f) - E((\underline{X}, f) \mid \Lambda_1) \|_P = 0$$

for every  $f \in S$ . Hence  $T_{h_\tau} \underline{X} \rightarrow E(\underline{X} \mid \Lambda_1)$  ( $S_{\Omega, P}^*$ ) if  $\tau \rightarrow \infty$ .  $\square$

COROLLARY. If  $\underline{X}$  is strict sense time stationary and ergodic, then  $\Lambda_1$  consists of sets of probability zero or one. Now the conditional expectation occurring in the above theorem can be replaced by  $E\underline{X}$ . We refer for related results to [Ur].

3.2.8. THEOREM. Let  $\underline{X} \in S_{\Omega, P}^*$ . Then  $\underline{X}$  is ergodic if and only if  $T_g \underline{X}$  is ergodic for every  $g \in C$ .

PROOF. Assume that  $\underline{X}$  is ergodic, and let  $g \in C$ . Let  $A \in \Lambda^*$  be an invariant set with respect to  $P_{T_g \underline{X}}^*$ , i.e.  $P_{T_g \underline{X}}^*(T_a A \div A) = 0$  ( $a \in \mathbb{R}$ ). We have by

$$1.1.20 \text{ and } T_a T_g = T_g T_a$$

$$\begin{aligned} P_{T_g \underline{X}}^*(T_a A \div A) &= P_{\underline{X}}^*(T_g^{-1}(T_a A \div A)) = \\ &= P_{\underline{X}}^*(T_a(T_g^{-1}(A)) \div T_g^{-1}(A)) = 0 \end{aligned}$$

for every  $a \in \mathbb{R}$ . Hence  $P_{\underline{X}}^*(T_g^{-1}(A)) = P_{T_g \underline{X}}^*(A) = 0$  or 1. This means that  $T_g \underline{X}$  is ergodic.

If  $T_g \underline{X}$  is ergodic for every  $g \in C$ , then  $\underline{X}$  is ergodic, for  $T_{\delta_0} \underline{X} = \underline{X}$  and  $\delta_0 \in C$ .  $\square$

3.2.9. Without proofs we mention some further results.

(i) Let  $\underline{X}$  be strict sense time stationary. If  $T_g \underline{X}$  is ergodic for every  $g \in S$ , then  $\underline{X}$  is ergodic. This theorem is very useful as a tool in checking ergodicity of a given generalized stochastic process. For, if  $\underline{X} \in S_{\Omega, P}^*$ ,  $g \in S$  then  $T_g \underline{X} = \text{emb}(\bigcup_t (T_t \underline{X}, g_-))$  by 3.1.3. By 1.3.8,  $T_g \underline{X}$  is ergodic if  $\bigcup_t (T_t \underline{X}, g_-)$  is ergodic in the sense of [D], Ch. XI, §1 (we can take a representative of  $\bigcup_t (T_t \underline{X}, g_-)$  satisfying the measurability conditions of theorem 1.3.8; also cf. the proof of 1.3.9).

(ii) Let  $\underline{X} \in S_{\Omega, 2}^*$  be a strict sense time stationary process with independent values at every moment (cf. 2.2.9) and  $E\underline{X} = 0$ . Then  $\underline{X}$  is ergodic. The proof of this statement uses (i). Compare [Ur] for related results.

(iii) Let  $\underline{X}$  be a time stationary Gaussian process with absolute continuous spectral measure (cf. 2.2.5). Then  $\underline{X}$  is ergodic. The proof uses (i) and a theorem of Wiener and Akutowicz (cf. [WA], section 5, theorem 3).

### 3.3. SHOT NOISE PROCESSES

3.3.1. By shot noise processes we mean processes of the form  $\sum_n p_n \delta(a_n)$  (or, more generally, of the form  $\sum_n p_n T_{-a_n} g$ , where  $g \in S^*$  is fixed). Here  $p_n$  is a complex-valued random variable (to be referred to as random phase factor), and  $a_n$  is a real-valued random variable (to be referred to as random excitation time) for every  $n \in \mathbb{N}$ . These shot noise processes are of great practical importance; they occur in all situations where charged particles have to overcome some barrier (cf. [Zi], [BS]). As we shall see in this section shot noise processes play a vital role in the second order simulation of stationary processes.

Very similar to shot noise processes are (what we call) "random Fourier series" processes. These processes have the form  $\sum_n p_n e(a_n)$  (with  $e(a_n) = \text{emb}(\int_t e^{-2\pi i a_n t}$ )), where  $p_n$  and  $a_n$  are as above for  $n \in \mathbb{N}$ .

We shall only be concerned with second order properties of the above processes.

3.3.2. In the remainder of this section  $(\Omega, \Lambda, P)$  is a fixed probability space.

DEFINITION. If  $a$  is a real random variable defined on  $\Omega$ , then  $\delta(a)$  is the generalized stochastic process of which a representative is given by  $\int_{(f, \omega)} \overline{f(a(\omega))}$ , and  $e(a)$  is the generalized stochastic process of which a representative is given by  $\int_{(f, \omega)} (F\overline{f})(a(\omega))$ .

It is not hard to see that  $\delta(a)$  and  $e(a)$  are indeed generalized stochastic processes, viz. of order  $p = \infty$ , if  $a$  is as in the definition.

3.3.3. It is possible to define, more generally, processes  $T_{-a} g$  where  $g \in S^*$  and  $a$  is a real random variable. Then certain assumptions about  $g$  and the distribution function of  $a$  must be made, and in general the order of the process is finite. As we are mainly interested in second order properties, we give the following definition.

DEFINITION. Let  $g \in S^*$ , and let  $a$  be a real random variable with distribution function  $F$ . Assume that  $\int_{-\infty}^{\infty} |(T_g f)(x)|^2 dF(x) < \infty$  for every  $f \in S$  (cf. appendix 2,2). Then  $T_{-a} g$  is the generalized stochastic process of which a



representative is given by  $\Psi_{(f,\omega)} (T_{-a(\omega)} g, f)$ .

3.3.4. THEOREM. Let  $a, g$  and  $F$  be as in definition 3.3.3. Then  $T_{-a} g \in S_{\Omega, 2}^*$ .

PROOF. We first note that  $(T_{-a} g, f) \in L_2(\Omega)$ , and that

$$\int_{\Omega} |(T_{-a} g, f)|^2 dP = \int_{-\infty}^{\infty} |(T_{-x} g, f)|^2 dF(x)$$

for  $f \in S$ .

Now let (for  $n \in \mathbb{N}$ )  $X_n$  be the element of  $S_{\Omega, 2}^*$  of which a representative is given by  $\Psi_{(f,\omega) \in S \times \Omega} (T_{-a(\omega)} g, f) \cdot \chi_{A_n}(\omega)$ , where  $A_n := \{\omega \in \Omega \mid |a(\omega)| \leq n\}$ . It is easy to see that  $X_n \in S_{\Omega, 2}^*$  indeed: if  $(f_k)_{k \in \mathbb{N}}$  is a sequence in  $S$  with  $f_k \xrightarrow{S} 0$ , then  $(T_{-x} g, f_k) \rightarrow 0$  ( $k \rightarrow \infty$ ) uniformly in  $[-n, n]$  (cf. [J2], 5.2), hence

$$\int_{\Omega} |(X_n, f_k)|^2 dP = \int_{[-n, n]} |(T_{-x} g, f_k)|^2 dF(x) \rightarrow 0 \quad (k \rightarrow \infty).$$

We further have  $(X_n, f) \rightarrow (T_{-a} g, f)$  ( $n \rightarrow \infty$ ) in  $L_2(\Omega)$ -sense for every  $f \in S$ . Hence, by 1.1.11,  $X_n \rightarrow T_{-a} g$  ( $S_{\Omega, 2}^*$ ), so  $T_{-a} g \in S_{\Omega, 2}^*$ .  $\square$

EXAMPLE. Let  $g \in V$  (cf. 3.2.2), and let  $a$  be a real random variable with distribution function  $F$ . Let  $f \in S$ . Then  $T_g f$  is bounded over  $\mathbb{R}$ . To see this we note that both  $T_g f$  and its derivative  $(T_g f)' = T_g f'$  belong to  $L_2(\mathbb{R})$ , hence

$$\begin{aligned} |(T_g f)^2(x) - (T_g f)^2(0)|^2 &= 4 \left| \int_0^x (T_g f)(t) (T_g f')(t) dt \right|^2 \leq \\ &\leq 4 \int_{-\infty}^{\infty} |(T_g f)(t)|^2 dt \int_{-\infty}^{\infty} |(T_g f')(t)|^2 dt < \infty \quad (x \in \mathbb{R}). \end{aligned}$$

This implies that  $\int_{-\infty}^{\infty} |(T_g f)(x)|^2 dF(x) < \infty$  for  $f \in S$ . Hence  $T_{-a} g$  makes sense according to theorem 3.3.4.

3.3.5. We are going to prove a theorem on the convergence of the series  $\sum_{n=1}^{\infty} p_n \delta(a_n)$ . This theorem involves series  $\sum_{n,m} f_{nm}$  where  $f_{nm} \in S^{2*}$  is of positive type (cf. appendix 4, 2.1) for  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}$ . If

$\lim_{N \rightarrow \infty, M \rightarrow \infty} \sum_{n=1}^N \sum_{m=1}^M f_{nm}$  exists in  $S^{2*}$ -sense, then  $\sum_{n,m} f_{nm}$  is unconditionally convergent in  $S^{2*}$ -sense, i.e. the order of summation is immaterial. The proof of this fact is not hard.

**THEOREM.** Assume that the phase factors  $p_n$  are uniformly bounded. Let  $F_{nm}$  be the joint distribution function of  $(a_n, a_m)$ , and let  $f_{nm} := \frac{\partial^2}{\partial x \partial y} \text{emb}(F_{nm})$  for  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}$  (cf. appendix 1, 4.16). Assume that  $\sum_{n,m} f_{nm}$  converges in  $S^{2*}$ -sense. Then  $\lim_{N \rightarrow \infty} \sum_{n=1}^N p_n \delta(a_n)$  exists in  $S_{\Omega, 2}^*$ -sense.

**PROOF.** By the assumptions on the  $p_n$ 's there is a  $C > 0$  such that for  $f \in S$ ,  $N \in \mathbb{N}$ ,  $M \in \mathbb{N}$ ,  $M \geq N$

$$\begin{aligned} \int_{\Omega} \left| \left( \sum_{n=N}^M p_n \delta(a_n), f \right) \right|^2 dP &= \int_{\Omega} \sum_{n=N}^M \sum_{m=N}^M p_n \overline{p_m} f(a_n) \overline{f(a_m)} dP \leq \\ &\leq C \sum_{n=N}^M \sum_{m=N}^M \int_{\mathbb{R}^2} |f(a) \overline{f(b)}| dF_{nm}(a, b). \end{aligned}$$

Let  $\gamma > 0$ ,  $\kappa > 0$  be such that

$$|f(a) \overline{f(b)}| \leq h(a, b) := \kappa \exp(-\pi\gamma(a^2 + b^2)) \quad (a \in \mathbb{R}, b \in \mathbb{R}).$$

Then

$$\begin{aligned} \int_{\Omega} \left| \sum_{n=N}^M (p_n \delta(a_n), f) \right|^2 dP &\leq C \sum_{n=N}^M \sum_{m=N}^M \int_{\mathbb{R}^2} h(a, b) dF_{nm}(a, b) = \\ &= C(h, \sum_{n=N}^M \sum_{m=N}^M f_{nm}) \end{aligned}$$

by appendix 4, 2.4. Since  $h \in S^2$  we conclude from  $S^{2*}$ -convergence of  $\sum_{n,m} f_{nm}$  that the right hand side of the above inequality tends to zero if  $N \rightarrow \infty$ ,  $M \rightarrow \infty$ . Hence  $\lim_{N \rightarrow \infty} \sum_{n=1}^N p_n \delta(a_n), f$  exists in  $L_2(\Omega)$ -sense. By 1.1.11 we see that  $\lim_{N \rightarrow \infty} \sum_{n=1}^N p_n \delta(a_n)$  exists in  $S_{\Omega, 2}^*$ -sense.  $\square$

3.3.6. DEFINITION. If  $p_n$  and  $a_n$  are as in theorem 3.3.5, then we define the generalized stochastic process  $\sum_{n=1}^{\infty} p_n \delta(a_n)$  as the limit  $\lim_{N \rightarrow \infty} \sum_{n=1}^N p_n \delta(a_n)$ .

The series  $\sum_{n=1}^{\infty} p_n \delta(a_n)$  is unconditionally convergent: if  $\pi$  is any permutation of  $\mathbb{N}$ , then  $\sum_{n=1}^{\infty} p_{\pi(n)} \delta(a_{\pi(n)})$  is well-defined (in the sense of the above definition) and equals  $\sum_{n=1}^{\infty} p_n \delta(a_n)$ . We also write  $\sum_n p_n \delta(a_n)$  instead of  $\sum_{n=1}^{\infty} p_n \delta(a_n)$ .

We have a similar definition as above for the process  $\sum_{n=1}^{\infty} p_n e(a_n)$ ; the condition for convergence of the series is again:  $\sum_{n,m} f_{nm}$  is an  $S^{2^*}$ -convergent series.

It is easy to see that  $F(\sum_{n=1}^{\infty} p_n \delta(a_n)) = \sum_{n=1}^{\infty} p_n e(a_n)$ .

3.3.7. EXAMPLE. Let  $p_n \equiv 1$  ( $n \in \mathbb{N}$ ), and assume that the  $a_n$ 's are distributed in accordance with a Poisson process at a rate 1 (cf. [D], Ch. VIII, §4). In this case  $\sum_{n,m} \text{emb}(F_{nm}) = \text{emb}(\bigvee_{(x,y)} (xy + \min(x,y)))$ . And if  $\underline{X} := \sum_n \delta(a_n)$ , then we have for  $f \in S$

$$\int_{\Omega} |(\underline{X}, f)|^2 dP = \int_{-\infty}^{\infty} |f(x)|^2 dx + \left| \int_{-\infty}^{\infty} f(x) dx \right|^2$$

(Campbell's theorem).

3.3.8. We consider a case in which we can settle convergence of the series  $\sum_{n=1}^{\infty} p_n \delta(a_n)$  with the aid of the Riesz-Fischer theorem.

THEOREM. Assume that  $\int_{\Omega} p_n \bar{p}_m dP = c_n \delta_{nm}$  for some  $c_n \geq 0$ , and that the pairs  $(p_n, p_m)$  and  $(a_n, a_m)$  are mutually independent for every  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}$ . Let  $F_n$  be the distribution function of  $a_n$ , and let  $f_n := \frac{d}{dx} F_n$  (cf. appendix 1, 1.8(ii)) for  $n \in \mathbb{N}$ . Assume that  $\sum_n c_n f_n$  converges in  $S^*$ -sense (the order of summation is immaterial; cf. the beginning of 3.3.5). Then  $\lim_{N \rightarrow \infty} \sum_{n=1}^N p_n \delta(a_n)$  exists in  $S_{\Omega, 2}^*$ -sense.

PROOF. Let  $f \in S$ . The terms  $p_n(\delta(a_n), f)$  ( $n \in \mathbb{N}$ ) are mutually orthogonal.

Hence  $\lim_{N \rightarrow \infty} \sum_{n=1}^N p_n(\delta(a_n), f)$  exists in  $L_2(\Omega)$ -sense if  $\sum_{n=1}^{\infty} \int_{\Omega} |p_n(\delta(a_n), f)|^2 dP < \infty$ . We have for  $N \in \mathbb{N}$

$$\begin{aligned} \sum_{n=1}^N \int_{\Omega} |p_n(\delta(a_n), f)|^2 dP &= \sum_{n=1}^N c_n \int_{-\infty}^{\infty} |f(x)|^2 dF_n(x) = \\ &= \left( \sum_{n=1}^N c_n f_n, f \cdot \bar{f} \right) \end{aligned}$$

by appendix 4, theorem 2.4. It easily follows from  $S^*$ -convergence of the series  $\sum_n c_n f_n$  that  $\sum_{n=1}^{\infty} \int_{\Omega} |p_n(\delta(a_n), f)|^2 dP < \infty$ , hence  $\lim_{N \rightarrow \infty} \sum_{n=1}^N p_n(\delta(a_n), f)$  exists in  $L_2(\Omega)$ -sense. By theorem 1.1.11 we conclude that  $\lim_{N \rightarrow \infty} \sum_{n=1}^N p_n \delta(a_n)$  exists in  $S_{\Omega, 2}^*$ -sense.  $\square$

DEFINITION. If  $p_n, a_n$  ( $n \in \mathbb{N}$ ) are as in the above theorem, then we define

$$\sum_{n=1}^{\infty} p_n \delta(a_n) := \lim_{N \rightarrow \infty} \sum_{n=1}^N p_n \delta(a_n).$$

We note that the series  $\sum_{n=1}^{\infty} p_n \delta(a_n)$  is unconditionally convergent: if  $\pi$  is a permutation of  $\mathbb{N}$ , then  $\lim_{N \rightarrow \infty} \sum_{n=1}^N p_{\pi(n)} \delta(a_{\pi(n)})$  exists and equals  $\sum_{n=1}^{\infty} p_n \delta(a_n)$ . We shall also write  $\sum_n p_n \delta(a_n)$  instead of  $\sum_{n=1}^{\infty} p_n \delta(a_n)$ .

We have a similar definition (and a similar condition of convergence) for the process  $\sum_n p_n e(a_n)$ .

3.3.9. The processes of 3.3.8 have some interesting properties.

THEOREM. Let  $p_n$  and  $a_n$  be as in 3.3.8, and assume that  $\int_{\Omega} p_n dP = 0$  ( $n \in \mathbb{N}$ )

(i) The process  $\underline{X} := \sum_n p_n \delta(a_n)$  is second order frequency stationary with zero expectation function and autocorrelation function  $Z_U(f_0 \otimes \delta_0)$ .

Here  $f_0$  denotes  $Z_{1/\sqrt{2}}(\sum_n c_n f_n)$  (cf. appendix 1, 1.8 (ii)).

(ii) The process  $\underline{Y} := \sum_n p_n e(a_n)$  is second order time stationary with zero expectation function and autocorrelation function  $Z_U(H \otimes Ff_0)$  ( $f_0$  as in (i)).

PROOF. (i) It is trivial that  $E\underline{X} = 0$ . We have by definition 2.1.3 and 3.3.8

$$\begin{aligned} (R_{\underline{X}}, f \otimes \bar{g}) &= \int_{\Omega} (\underline{X}, f) \cdot \overline{(\underline{X}, g)} dP = \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N c_n \int_{-\infty}^{\infty} \overline{f(a)} g(a) dF_n(a) = \lim_{N \rightarrow \infty} \sum_{n=1}^N c_n (f_n, f \cdot \bar{g}). \end{aligned}$$

It is not hard to check (with the aid of [B1], (21.4)) that  $(f_n, f \cdot \bar{g}) = (Z_U(Z_{1/\sqrt{2}} f_n \otimes \delta_0), f \otimes \bar{g})$  for  $n \in \mathbb{N}$  (notice that  $Z_U$  maps  $S^{2*}$  into  $S^{2*}$  whereas  $Z_{1/\sqrt{2}}$  maps  $S^*$  into  $S^*$ ; cf. appendix 1, 1.8 (ii), 4.9, 4.15 and 4.16). Hence  $R_{\underline{X}} = Z_U(f_0 \otimes \delta_0)$ . We see from 2.2.5 that  $\underline{X}$  is second order frequency stationary.

(ii) This follows from 2.2.5 (note that  $\underline{Y} = F\underline{X}$ ).  $\square$

3.3.10. We conclude this section with some examples.

EXAMPLE 1. If the function  $f_0$  of the previous theorem equals  $\text{emb}(\bigvee_x 1)$ , then both  $\sum_{n=1}^{\infty} p_n \delta(a_n)$  and  $\sum_{n=1}^{\infty} p_n e(a_n)$  are second order white noise processes.

EXAMPLE 2. Let  $\underline{X} \in S_{\Omega, 2}^*$  be a second order time stationary process with zero expectation function and autocorrelation function  $R_{\underline{X}} = Z_U(H \otimes K)$ , and assume that the spectral measure of  $\underline{X}$  is absolutely continuous with respect to Lebesgue measure. Write  $FK = \text{emb}(g)$  with  $g \in S^+$ ,  $h := g^{\frac{1}{2}}$ ,  $k := F(\text{emb}(h))$  (cf. also the proof of theorem 3.2.4 and the remark at the end of 3.2.5). We are going to define a shot noise process  $\underline{Y}$  of the type defined in 3.3.3 with  $E\underline{X} = 0$ ,  $R_{\underline{Y}} = R_{\underline{X}}$ .

Take random variables  $p_n, a_n$  as in theorem 3.3.9, and assume  $c_n = 1$  ( $n \in \mathbb{N}$ ),  $f_0 = Z_{1/\sqrt{2}}(\sum_n c_n f_n) = \text{emb}(\bigvee_x 1)$ . (As to the existence of  $p_n$ 's and  $a_n$ 's with the assigned properties, we refer to the proof of theorem 3.5.14 where a similar problem is handled.) Now  $\underline{W} := \sum_n p_n \delta(a_n)$  is a second order white noise process. As  $k \in V$  (cf. 3.2.2) we conclude that  $T_{\underline{k}} \underline{W}$  makes sense according to 3.2.3. We further have

$$T_{\underline{k}} \underline{W} = \lim_{N \rightarrow \infty} \sum_{n=1}^N p_n T_{-a_n} \bar{k}$$

(note that  $T_{-a_n} \bar{k}$  ( $n \in \mathbb{N}$ ) makes sense by example 3.3.4; the existence in  $S_{\Omega, 2}^*$ -sense of the right hand side limit may be proved in the same way as theorem 3.3.8). The autocorrelation function of  $\underline{Y} = T_{\underline{k}} \underline{W}$  equals  $R_{\underline{X}}$  (compare the proof of theorem 3.2.4).

EXAMPLE 3. Let  $\underline{X} \in S_{\Omega, 2}^*$  be a second order frequency stationary process with  $E\underline{X} = 0$  and with  $R_{\underline{X}} = Z_U(L \otimes \delta_0)$  (cf. 2.2.5). As we know from 2.2.5 this  $L$  is of positive type, and it can be shown with the aid of appendix 4, theorem 2.2 that there are non-negative numbers  $c_n$  and distribution functions  $F_n$  such that  $L = Z_{1/\sqrt{2}}(\sum_n c_n \frac{d}{dx} (\text{emb}(F_n)))$ . It is possible to construct a probability space  $(\Omega', \Lambda', P')$  (a suitable countable product; compare the proof of theorem 3.5.14) and random variables  $p_n$  and  $a_n$  (satisfying the conditions of theorem 3.3.9) defined on it such that  $\int_{\Omega'} |p_n|^2 dP' = c_n$  and  $F_n$  is the distribution function of  $a_n$  for  $n \in \mathbb{N}$ . This means (cf. theorem 3.3.9) that we can simulate  $\underline{X}$  up to second order by means of a shot noise process.

We have of course a corresponding result for second order time stationary processes (this involves the random Fourier series processes of theorem

3.3.9 (ii)).

### 3.4. TIME-FREQUENCY CONVOLUTIONS AND THE WIGNER DISTRIBUTION FOR GENERALIZED STOCHASTIC PROCESSES.

3.4.1. This section deals with applications of the time-frequency convolution theory of appendix 3. We comment (after introductory definitions and theorems) on uncertainty principles in the measurement of autocorrelation function and Wigner distribution of a non-stationary generalized stochastic process. Also some references to recent literature on the subject are given.

As usual,  $(\Omega, \Lambda, P)$  is a fixed probability space, and  $p$  is an element of the extended real number system with  $1 \leq p \leq \infty$ .

3.4.2. Let  $g \in S$ . We know from a slight generalization of theorem 1.1.3 that the mapping  $S_g$  (cf. appendix 3, 1.5 and 2.2) can be defined on  $S_{\Omega, p}^*$  such that  $S_g \underline{X} \in S_{\Omega, p}^{2*}$  and  $(S_g \underline{X}, h) = (\underline{X}, S_g^* h)$  ( $\underline{X} \in S_{\Omega, p}^*$ ,  $h \in S^2$ ). We further have (compare 1.1.22, 1.1.23 and 1.1.27)  $\langle S_g \underline{X}, f \rangle = S_g \langle \underline{X}, f \rangle$  for  $\underline{X} \in S_{\Omega, p}^*$ ,  $f \in L_q(\Omega)$ .

On the other hand we can define  $S'_g \underline{X}$  for  $\underline{X} \in S_{\Omega, p}^*$  by

$$S'_g \underline{X} := \bigvee_{(a,b) \in \mathbb{R}^2} (e^{\pi i a b} T_{a \ b} R_{\underline{X}, g}).$$

(cf. appendix 3, 2.3). The following theorem relates  $S_g$  and  $S'_g$ .

3.4.3. THEOREM. If  $\underline{X} \in S_{\Omega, p}^*$ ,  $g \in S$ , then  $S'_g \underline{X} \in \bar{S}_{\Omega, p}^{2+}$  (cf. 1.3.2 and 1.3.9), and  $S_g \underline{X} = \text{emb}(S'_g \underline{X})$ .

PROOF. The proof is almost the same as the one of theorem 3.1.3 (use appendix 3, theorem 2.3); we therefore omit it.  $\square$

3.4.4. In fact we are not interested in  $S_g \underline{X}$  (or  $S'_g \underline{X}$ ), but in  $\int_{\Omega} |(S'_g \underline{X})(a, b)|^2 dP$  ( $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$ ) if  $\underline{X} \in S_{\Omega, 2}^*$ . The following theorem is the stochastic version of appendix 3, theorem 2.4.

THEOREM. Let  $g \in S$ ,  $\underline{X} \in S_{\Omega, 2}^*$ , and define  $K := F^{(2)} Z_U(g \otimes \bar{g})$ ,  $L := F^{(2)} Z_U R_{\underline{X}}$  ( $K$  and  $L$  are, apart from a transformation of variables, the Wigner distributions of  $g$  and  $\underline{X}$  respectively; cf. appendix 3, 1.4 and 2.1.4). We have

$$\int_{\Omega} |(S'_g \underline{X})(a, b)|^2 dP = (L, T_{-a/2}^{(1)} T_{-b/2}^{(2)} K)$$

for  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$ .

PROOF. Let  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$ . We have by definition 2.1.3 and 3.4.2

$$\int_{\Omega} \left| \left( \frac{S'X}{g} \right) (a,b) \right|^2 dP = (R_{\underline{x}}, R_{-b} T_{-a} g \otimes \overline{R_{-b} T_{-a} g})$$

(the factors  $e^{\pi i a b}$  and  $e^{-\pi i a b}$  drop out since  $a$  and  $b$  are real). From the fact that  $F^{(2)}_{Z_U}$  is a unitary linear operator of  $S^2$  we infer

$$\int_{\Omega} \left| \left( \frac{S'X}{g} \right) (a,b) \right|^2 dP = (L, F^{(2)}_{Z_U} (R_{-b} T_{-a} g \otimes \overline{R_{-b} T_{-a} g})),$$

and an easy calculation (compare appendix 3, 1.2 (ii)) gives the required result.  $\square$

3.4.5. According to the above theorem and the 2-dimensional version of appendix 2, theorem 9 we find (note that  $\bar{K}_- = K_-$  since  $K_-$  is real on  $\mathbb{R}^2$ )

$$T_{K_-} L = 2 \text{ emb}(Z_{1/\sqrt{2}}^{(1)} Z_{1/\sqrt{2}}^{(2)} E |S'X/g|^2).$$

Here  $E |S'X/g|^2$  denotes  $\int_{(a,b) \in \mathbb{R}^2} \int_{\Omega} \left| \left( \frac{S'X}{g} \right) (a,b) \right|^2 dP$ . Note that  $T_{K_-} L$  is the embedding of a non-negative analytic function of two variables. Hence, certain averages of the expected Wigner distribution of  $\underline{x}$  are non-negative (cf. appendix 3, 2.4 and [B1], 27.12.1, 27.15 and [B2], theorem 4.2).

We investigate the case  $g := g_{\gamma} = \int_{z \in \mathbb{C}} \left( \frac{2}{\gamma} \right)^{\frac{1}{2}} \exp(-\pi \gamma^{-1} |z|^2)$  with  $\gamma > 0$  in some more detail. We then have  $K = K_- = g_{\gamma} \otimes g_{\gamma^{-1}}$ , hence

$$(*) \quad T_{g_{\gamma}}^{(1)} T_{g_{\gamma^{-1}}}^{(2)} L = 2 \text{ emb}(Z_{1/\sqrt{2}}^{(1)} Z_{1/\sqrt{2}}^{(2)} E |S'X/g|^2) =: G_{\gamma}.$$

If we take  $F^{(2)*}$  at both sides, and use appendix 2, theorem 8 then we get (since  $F^{(2)*} g_{\gamma^{-1}} = g_{\gamma}$ )

$$(**) \quad T_{g_{\gamma}}^{(1)} M_{g_{\gamma}}^{(2)} Z_U R = F^{(2)*} G_{\gamma}.$$

If  $R$  has the form  $Z_U (H \otimes R_0)$  with some  $R_0 \in S^*$  (second order time stationarity), then (\*\*) becomes

$$(2\gamma)^{\frac{1}{2}} H \otimes (g_{\gamma} \cdot R_0) = F^{(2)*} G_{\gamma}.$$

Note that  $(2\gamma)^{\frac{1}{2}} g_\gamma \cdot R_0 \xrightarrow{S^*} 2^{\frac{1}{2}} R_0$  if  $\gamma \rightarrow \infty$ , and that  $G_\gamma \geq 0$  for every  $\gamma > 0$ . This shows again that  $R_0$  is the Fourier transform of a function of positive type (cf. 2.2.4).

A similar thing holds of course for the case that  $R$  has the form  $Z_U(L_0 \otimes \delta_0)$  with some  $L_0 \in S^*$  (second order frequency stationarity). Then (\*) takes the form

$$\left(\frac{2}{\gamma}\right)^{\frac{1}{2}} T_{g_\gamma} L_0 \otimes \left(\frac{2}{\gamma}\right)^{\frac{1}{2}} g_{\gamma-1} = \left(\frac{2}{\gamma}\right)^{\frac{1}{2}} G_\gamma .$$

Now  $\left(\frac{2}{\gamma}\right)^{\frac{1}{2}} T_{g_\gamma} L_0 \xrightarrow{S^*} L_0$  by appendix 2, 11(iii), hence  $\left(\frac{2}{\gamma}\right)^{\frac{1}{2}} T_{g_\gamma} L_0 \xrightarrow{S^*} L_0$ , and  $\text{emb}\left(\left(\frac{2}{\gamma}\right)^{\frac{1}{2}} g_{\gamma-1}\right) \xrightarrow{S^*} \text{emb}\left(\bigcup_t 1\right)$  if  $\gamma \downarrow 0$ . This shows again that  $L_0$  is of positive type (cf. 2.2.5).

In the general, non-stationary, case it will be of interest to choose an "optimal" value for  $\gamma$  (optimal in the sense that  $T_{g_\gamma}^{(1)} M_{g_\gamma}^{(2)} Z_U R$  is as close to  $Z_U R$  as possible). The choice of  $\gamma$  depends of course on  $R$ . In case that  $R$  is approximately of the form  $Z_U(H \otimes R_0)$  with some  $R_0 \in S^*$ , we can take  $\gamma$  pretty large, but in case  $R$  looks like  $Z_U(L_0 \otimes \delta_0)$  with some  $L_0 \in S^*$ , we better take  $\gamma$  near to zero. Note that there are no values  $\gamma > 0$  such that  $T_{g_\gamma}^{(1)} M_{g_\gamma}^{(2)}$  (or  $T_{g_\gamma}^{(1)} T_{g_{\gamma-1}}^{(2)}$ ) is near to the identity operator. This means that (if we measure the Wigner distribution of  $\underline{x}$  according to (\*)) a precise measurement in the first (time) variable ( $\gamma$  small) will cause an imprecise measurement in the second (frequency) variable, and vice versa ( $\gamma$  large).

To get a nice picture of the state of affairs we have taken in (\*) Gauss functions  $g_\gamma$  ( $\gamma > 0$ ), but the above discussion also applies when more general functions  $g$  are taken: it is not possible to concentrate the Wigner distribution of a function in an area in phase plane of arbitrarily small measure. For inequalities expressing this impossibility we refer to [B1], theorem 15.2 and [B2], theorem 4.4 and 4.5.

3.4.6. The results of this section are closely related to work of Mark and Priestley (cf. [Ma], [Pr1], [Pr2]). Mark develops in his paper a theory of spectral analysis for non-stationary stochastic processes. His definition of "physical spectrum" of a stochastic process involves the expression  $E|S_{\underline{g}}' \underline{x}|^2$  of the beginning of 3.4.5. (Mark does not use the word "Wigner distribution": he did not seem to be aware of the existence of it at the time he wrote his paper.) Mark also derives a version of theorem 3.4.4 for his physical spectrum. Furthermore he studies convolutions of certain



mutually independent processes, and derives formulas for the resulting autocorrelation functions and physical spectra. At this point some criticism seems to be justified. The formalism developed in [Ma], section 4.2 makes sense for certain well-behaved, mutually independent stochastic processes. But if, e.g.,  $\underline{X}_1$  and  $\underline{X}_2$  are mutually independent white noise processes (so both  $F^{(2)}_{Z_U R_{\underline{X}_1}}$  and  $F^{(2)}_{Z_U R_{\underline{X}_2}}$  equal the constant function of 2 variables), then [Ma], formula (24a), written in our notation as

$$(F^{(2)}_{Z_U R_{\underline{X}_1} * \underline{X}_2})(t, \lambda) = \int_{-\infty}^{\infty} (F^{(2)}_{Z_U R_{\underline{X}_1}})(u, \lambda) (F^{(2)}_{Z_U R_{\underline{X}_2}})(t-u, \lambda) du,$$

is not valid, not even when interpreted in distributional sense.

Priestley defines a notion of evolutionary spectral density function for what he calls oscillatory processes. These are processes  $\underline{X}$  that admit a representation of the form

$$(*) \quad \underline{X}(t) = \int_{-\infty}^{\infty} e^{2\pi i f t} A(t, f) d\underline{Z}(f) \quad (t \in \mathbb{R}),$$

where  $\underline{Z}$  is a stochastic process with orthogonal increments and  $A$  is a mapping of  $\mathbb{R}^2$  into  $\mathbb{C}$  such that for every  $f \in \mathbb{R}$  the function  $\psi_g \left| \int_{-\infty}^{\infty} e^{-2\pi i g s} A(s, f) ds \right|$  is maximal at  $g = 0$ . The evolutionary spectral density function  $h$  of  $\underline{X}$  with respect to  $A$  is then defined by

$$h(t, f) := |A(t, f)|^2 h_0(f) \quad (t \in \mathbb{R}, f \in \mathbb{R}).$$

Here  $h_0$  is the density function (if it exists) associated with  $\underline{Z}$  (roughly:  $h_0(f) df = E |d\underline{Z}(f)|^2$ ).

It is noted in [Pr1] that the spectral density function thus obtained depends on the choice of  $A$  (in general, if  $\underline{X}$  has a representation as in (\*), then  $\underline{X}$  has many such representations). Priestley further indicates a relation between his evolutionary spectral density function (for oscillatory processes) and the physical spectrum of Mark: if  $g$  is a weight function then the expression  $E |S'_g \underline{X}|^2$  is a smoothed form of  $h$  (smoothed over frequency). The smoothing operation to be performed depends on  $A$ .

It is pretty clear that Priestley's approach gives an evolutionary spectral density function admitting nice physical interpretations. It covers the time stationary case (where we can then take a constant  $A$  in the repre-

sentation of  $\underline{X}$  in (\*)), but it is not likely to cover all possible cases: it depends on the existence of a representation as in (\*). Priestley's approach leads to a function  $h$  of two variables in which the time variable is treated as a parameter (Priestley himself writes  $h_t(f)$  instead of  $h(t,f)$ ). In the Wigner distribution time and frequency variable are treated completely symmetrically. It is only in special cases (such as stationarity) that one of the variables has a preference over the other. This is seen e.g. in 3.5.5 where  $\gamma > 0$  in the expression  $E|S_{g\sqrt{\gamma}}^{\dagger} \underline{X}|^2$  is taken large or small accordingly as  $\underline{X}$  is (approximately) time or frequency stationary.

We further remark that Priestley's approach can be used to give a precise meaning to notions as local stationarity and degree of stationarity of a stochastic process (cf. [Pr2]).

### 3.5. SECOND ORDER SIMULATION BY MEANS OF NOISE SHOWERS

3.5.1. In this section we consider generalized stochastic processes of type

$$(*) \quad \sum_{n=1}^{\infty} p_n e^{-\pi i a_n b_n} R_{-b_n} T_{-a_n} g.$$

Here  $p_n$  is a complex random variable (random phase factor), and  $a_n$  and  $b_n$  are real random variables (random time and random frequency variable respectively) for  $n \in \mathbb{N}$ , and  $g$  is a fixed element of  $S^*$ . We shall often take  $g \in S$ : we shall have to deal with expressions like  $T_K V$  where  $K$  is the Wigner distribution of  $g$  and  $V \in S^{2*}$  (by appendix 3, 2.4, remark,  $K \in C^2$  if and only if  $g \in S$ ).

We shall also consider processes of type

$$(**) \quad \sum_{n=1}^{\infty} p_n G_{\gamma_n}(a_n, b_n).$$

Here  $p_n$ ,  $a_n$  and  $b_n$  are as above, and  $\gamma_n$  is a positive random variable for every  $n \in \mathbb{N}$ . If  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$ ,  $\gamma > 0$ , then  $G_{\gamma}(a,b)$  is the Gabor function localized in the time-frequency plane at the point  $(a,b)$ . The "shape" of this function is controlled by  $\gamma$ :

$$\begin{aligned} G_{\gamma}(a,b) &= \int_{\mathbb{C}} \left(\frac{2}{\gamma}\right)^{\frac{1}{4}} \exp(-\pi\gamma^{-1}(t-a)^2 + 2\pi i b t - \pi i a b) = \\ &= e^{-\pi i a b} R_{-b} T_{-a} g_{\gamma}, \end{aligned}$$

where  $g_\gamma = \int_t \left(\frac{2}{\gamma}\right)^{\frac{1}{2}} \exp(-\pi\gamma^{-1}t^2) \quad (\text{cf. [B1], 27.7}).$

The process in (\*\*) is a special case of the process  $\sum_{n=1}^{\infty} p_n e^{-\pi i a_n b_n} R_{-b_n} T_{-a_n} Z_{\gamma_n}^{\frac{1}{2}} g$ , where  $p_n$ ,  $a_n$ ,  $b_n$  and  $\gamma_n$  are as above and  $g \in S$  (cf. appendix 1, 1.8 (ii) for the definition of  $Z_\gamma$  for  $\gamma > 0$ ).

The function  $G_\gamma(a,b)$  may be thought of as a note at time  $a$  with frequency  $b$  and with degree of "pureness"  $\gamma$ : the larger  $\gamma$ , the longer this note, the more  $G_\gamma(a,b)$  looks like  $\int_t \left(\frac{2}{\gamma}\right)^{\frac{1}{2}} e^{-\pi i a b + 2\pi i b t}$ . These notes all have the same energy:  $\int_{-\infty}^{\infty} |G_\gamma(a,b)(t)|^2 dt = 1$  for all  $a,b$  and  $\gamma$ . We may thus regard (\*\*) as a shower of noise quanta in which each quantum is a note, random in time, frequency and pureness.

The processes in (\*) and (\*\*) resemble those of section 3.3. We shall see that the latter processes can be obtained as limits of processes of type (\*\*) (by making  $\gamma \downarrow 0$  or  $\gamma \rightarrow \infty$  accordingly as we consider shot noise processes or "random Fourier series" processes).

We use the processes in (\*) and (\*\*) for second order simulation of generalized stochastic processes (with zero expectation function). We shall see that every generalized stochastic process can be simulated approximately by means of processes of type (\*) and (\*\*) with mutually independent random notes. Here (a smoothed form of) the expected Wigner distribution of the process to be simulated plays an important role: it says how to distribute the random variables  $a_n$  and  $b_n$  over the time-frequency plane. We shall further see that each noise quantum of a process of type (\*\*) can be simulated exactly (as far as first and second order moments are concerned) by means of a series  $\sum_{n=1}^{\infty} p_n e^{-\pi i a_n b_n} R_{-b_n} T_{-a_n} g_n$ .

3.5.2. In the remainder of this section  $(\Omega, \Lambda, P)$  is a probability space.

DEFINITION. Let  $g \in S$ , and let  $a$  and  $b$  be two real random variables defined on  $\Omega$ . We denote by  $e^{-\pi i a b} R_{-b} T_{-a} g$  the generalized stochastic process of which a representative is given by

$$\int_{(f, \omega)} (e^{-\pi i a(\omega) b(\omega)} R_{-b(\omega)} T_{-a(\omega)} g, f).$$

It is easy to see that  $e^{-\pi i a b} R_{-b} T_{-a} g \in S_{\Omega, \infty}^*$  if  $g$ ,  $a$  and  $b$  are as in the above definition. For if  $f \in S$ , then  $\int_{\omega} \frac{1}{2} W(\frac{1}{2} a(\omega), \frac{1}{2} b(\omega); f, g)$  is a representative of  $(e^{-\pi i a b} R_{-b} T_{-a} g, f)$  (cf. appendix 3, 11).

We shall use for  $e^{-\pi i a b} R_{-b} T_{-a} g$  the shorter notation  $V_a g$ , where

$a = (a, b)$ . We also use this notation in case  $a$  and  $b$  are non-stochastic.

3.5.3. THEOREM. Assume that the phase factors  $p_n$  are uniformly bounded. Let  $F_{nm}$  be the joint distribution function of the pair  $(a_n, a_m)$  for  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}$ , and assume that  $\sum_{n,m} \frac{\partial^4 \text{emb}(F_{nm})}{\partial a_1 \partial b_1 \partial a_2 \partial b_2}$  converges in  $S^{4*}$ -sense. Then  $\lim_{N \rightarrow \infty} \sum_{n=1}^N p_n V_a g$  exists in  $S_{\Omega, 2}^*$ -sense.

PROOF. The proof can be given along the same lines as that of theorem 3.3.5; we therefore omit it.  $\square$

REMARK. We have a similar theorem for the series  $\sum_{n=1}^{\infty} p_n G_{\gamma_n}(a_n, b_n)$  or, more generally, for the series  $\sum_{n=1}^{\infty} p_n e^{-\pi i a_n b_n} R_{-b_n} T_{-a_n} Z_{\gamma_n} g$  if we add the requirement that the  $\gamma_n$ 's are stochastic variables with  $m \leq \gamma_n(\omega) \leq M$  ( $\omega \in \Omega$ ,  $n \in \mathbb{N}$ ) for some  $m > 0$ ,  $M > 0$ . The proof uses the following fact. For every  $K > 0$ ,  $A > 0$ ,  $B > 0$  there are numbers  $K' > 0$ ,  $A' > 0$ ,  $B' > 0$  such that

$$|W(a, b; f, g)| \leq K' \exp(-\pi A'((\text{Re}a)^2 + (\text{Re}b)^2) + \pi B'((\text{Im}a)^2 + (\text{Im}b)^2))$$

for  $a \in \mathbb{C}$ ,  $b \in \mathbb{C}$  whenever  $f \in S$ ,  $g \in S$  satisfy

$$\max(|f(a)|, |g(a)|) \leq K \exp(-\pi A(\text{Re}a)^2 + \pi B(\text{Im}a)^2)$$

for  $a \in \mathbb{C}$ .

3.5.4. DEFINITION. Let  $g$ ,  $p_n$  and  $a_n$  ( $n \in \mathbb{N}$ ) be as in 3.5.3. We define  $\sum_{n=1}^{\infty} p_n V_a g := \lim_{N \rightarrow \infty} \sum_{n=1}^N p_n V_a g$ . If furthermore the  $\gamma_n$ 's are as in 3.5.3, remark, then we define  $\sum_{n=1}^{\infty} p_n e^{-\pi i a_n b_n} R_{-b_n} T_{-a_n} Z_{\gamma_n} g := \lim_{N \rightarrow \infty} \sum_{n=1}^N p_n e^{-\pi i a_n b_n} R_{-b_n} T_{-a_n} Z_{\gamma_n} g$ .

We note that the convergence of  $\sum_{n=1}^{\infty} p_n V_a g$  (and  $\sum_{n=1}^{\infty} p_n e^{-\pi i a_n b_n} R_{-b_n} T_{-a_n} Z_{\gamma_n} g$ ) is unconditional in the sense that the order of the terms is immaterial. We shall also write  $\sum_n p_n V_a g$  (and  $\sum_n p_n e^{-\pi i a_n b_n} R_{-b_n} T_{-a_n} Z_{\gamma_n} g$ ).

3.5.5. We next show that the shot noise processes of section 3.3 appear as  $S_{\Omega, 2}^*$ -limits of processes of the form  $\sum_{n=1}^{\infty} p_n G_{\gamma}(a_n, b_n)$  ( $\gamma$  independent of  $n \in \mathbb{N}$  and non-stochastic). Let  $p_n$  and  $a_n$  be as in 3.3.5. Now  $p_n$  and  $a_n := (a_n, 0)$  satisfy the conditions of theorem 3.5.3. Hence both

$\sum_n p_n \delta(a_n)$  and  $\sum_n p_n V_{a_n} g_\gamma = \sum_n p_n G_{\gamma^2}(a_n, 0)$  are well defined for  $\gamma > 0$  ( $g_\gamma$  denotes as usual  $\int_t^{\infty} (\frac{z}{\gamma})^{\frac{1}{2}} \exp(-\pi \gamma^{-1} t^2)$ ). Since  $V_{a_n} g_\gamma = T_{g_\gamma} \delta(a_n)$  for  $n \in \mathbb{N}$ , we have by theorem 1.1.9.

$$\left(\frac{1}{2\gamma}\right)^{\frac{1}{2}} \sum_n p_n V_{a_n} g_\gamma = \left(\frac{1}{2\gamma}\right)^{\frac{1}{2}} T_{g_\gamma} \left(\sum_n p_n \delta(a_n)\right).$$

And since  $\left(\frac{1}{2\gamma}\right)^{\frac{1}{2}} g_\gamma \xrightarrow{C} \delta_0$  if  $\gamma \uparrow 0$  (cf. appendix 2, 11(iii)), we have by theorem 3.1.4

$$\left(\frac{1}{2\gamma}\right)^{\frac{1}{2}} \sum_n p_n V_{a_n} g_\gamma \rightarrow \sum_n p_n \delta(a_n)$$

in the sense of  $S_{\Omega, 2}^*$  if  $\gamma \downarrow 0$ .

We have of course a similar theorem for the "random Fourier series" processes of section 3.3: if  $p_n$  and  $a_n$  are as in 3.3.5 (cf. also 3.3.6), then we have  $\left(\frac{\gamma}{2}\right)^{\frac{1}{2}} \sum_n p_n V_{a_n} g_\gamma \rightarrow \sum_n p_n \delta(a_n)$  in the sense of  $S_{\Omega, 2}^*$  if  $\gamma \rightarrow \infty$ . Here we have taken  $a_n := (0, -a_n)$  ( $n \in \mathbb{N}$ ).

3.5.6. We consider a case in which we can settle the convergence of the series  $\sum_n p_n V_{a_n} g$  with the aid of the Riesz-Fischer theorem (compare the corresponding case for shot noise processes in 3.3.8). Assume that  $\int_{\Omega} p_n \cdot \bar{p}_m dP = c_n \delta_{nm}$  for some  $c_n \geq 0$ , and that the pairs  $(p_n, p_m)$  and  $(a_n, a_m)$  are mutually independent for every  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}$ . We find as in 3.3.8 that  $\lim_{N \rightarrow \infty} \sum_{n=1}^N p_n V_{a_n} g$  exists in  $S_{\Omega, 2}^*$ -sense if  $\sum_n c_n f_n$  is  $S^{2*}$ -convergent. Here  $f_n = \frac{\partial^2}{\partial x \partial y} \text{emb}(F_n)$ , where  $F_n$  denotes the distribution function of  $a_n$  ( $n \in \mathbb{N}$ ). We define  $\sum_{n=1}^{\infty} p_n V_{a_n} g := \lim_{N \rightarrow \infty} \sum_{n=1}^N p_n V_{a_n} g$ . Again the order of the terms in the series  $\sum_{n=1}^{\infty} p_n V_{a_n} g$  is immaterial; we shall also write  $\sum_n p_n V_{a_n} g$ .

3.5.7. The following theorem gives a nice picture of the energy distribution of the proces  $\sum_n p_n V_{a_n} g$  over the time-frequency plane.

**THEOREM.** Let  $p_n$ ,  $a_n$  and  $F_n$  be as in 3.5.6, and let  $f_0 := \sum_n c_n f_n$ . Let  $R$  be the autocorrelation function of  $\underline{X} := \sum_{n=1}^{\infty} p_n V_{a_n} g$ , and let  $L := F^{(2)} Z_{UR}$  (whence  $L$  is the expected Wigner distribution of  $\underline{X}$ ). Then we have (cf. appendix 1, 1.8 (ii))

$$L = \frac{1}{2} T_k Z \frac{1}{\sqrt{2}} Z \frac{1}{\sqrt{2}} f_0,$$

where  $K$  denotes  $F^{(2)}_{Z_U}(g \otimes \bar{g})$  (Wigner distribution of  $g$ ).

PROOF. Let  $h_1 \in S$ ,  $h_2 \in S$ . We have by the assumptions on  $p_n$ ,  $a_n$  and  $f_n$  ( $n \in \mathbb{N}$ ).

$$\begin{aligned} (L, F^{(2)}_{Z_U}(h_1 \otimes \bar{h}_2)) &= (R, h_1 \otimes \bar{h}_2) = \\ &= \int_{\Omega} (\underline{X}, h_1) \cdot \overline{(\underline{X}, h_2)} dP = \sum_{n=1}^{\infty} c_n \int_{\mathbb{R}^2} (V_a g, h_1) \overline{(V_a g, h_2)} dF_n(a). \end{aligned}$$

By appendix 4, lemma 2.4 and  $S^{2*}$ -convergence of the series  $\sum_n c_n f_n$ , the right hand side can be written as

$$\begin{aligned} &(\sum_n c_n f_n, \bigvee_a (h_1, V_a g) \overline{(h_2, V_a g)}) = \\ &= (f_0, \bigvee_a (h_1, V_a g) \overline{(h_2, V_a g)}). \end{aligned}$$

If  $a = (a, b) \in \mathbb{R}^2$ , then

$$\begin{aligned} (h_1, V_a g) \overline{(h_2, V_a g)} &= (h_1 \otimes \bar{h}_2, V_a g \otimes \overline{V_a g}) = \\ &= (F^{(2)}_{Z_U}(h_1 \otimes \bar{h}_2), F^{(2)}_{Z_U}(V_a g \otimes \overline{V_a g})), \end{aligned}$$

and we also have (cf. appendix 3, 1.2 (ii))

$$F^{(2)}_{Z_U}(V_a g \otimes \overline{V_a g}) = T_{-a/2}^{(1)} T_{-b/2}^{(2)} K.$$

Hence (cf. also 3.4.4)

$$(h_1, V_a g) \overline{(h_2, V_a g)} = (T_{K_-} F^{(2)}_{Z_U}(h_1 \otimes \bar{h}_2)) (a/2, b/2)$$

for  $a = (a, b) \in \mathbb{R}^2$ . We thus find

$$\begin{aligned} (L, F^{(2)}_{Z_U}(h_1 \otimes \bar{h}_2)) &= (f_0, \frac{1}{2} Z_{\sqrt{2}}^{(1)} Z_{\sqrt{2}}^{(2)} T_{K_-} F^{(2)}_{Z_U}(h_1 \otimes \bar{h}_2)) = \\ &= (\frac{1}{2} T_{K_-}^{(1)} Z_{1/\sqrt{2}}^{(2)} f_0, F^{(2)}_{Z_U}(h_1 \otimes \bar{h}_2)). \end{aligned}$$

Here we used the fact that  $Z_{\sqrt{2}}$  and  $Z_{1/\sqrt{2}}$  are adjoint linear operators of  $S$ , and the 2-dimensional version of appendix 2, theorem 5 (iii) (cf. also [J2], 5.11). By noting that  $K = \bar{K}$  (as  $K$  is real-valued on  $\mathbb{R}^2$ ), the theorem easily follows.  $\square$

3.5.8. We next study processes of type  $p V_a g$  with  $g \in L_2(\mathbb{R})$ . (The condition " $g \in S$ " in 3.5.6 and 3.5.7 was made to ensure convergence of the series  $\sum_n p_n V_{a_n} g$ ; such a condition is superfluous now.)

Let  $g \in L_2(\mathbb{R})$ . Let  $p$  be a complex random variable with  $\int_{\Omega} p \, dP = 0$ ,  $\int_{\Omega} |p|^2 \, dP =: c < \infty$ , and let  $a = (a, b)$  be an  $\mathbb{R}^2$ -valued random vector with distribution function  $F$ . Assume that  $p$  and  $a$  are independent. We denote by  $\underline{X} = p V_a g$  the generalized stochastic process of which a representative is given by  $\Psi_{(f, \omega)} (p(\omega) e^{-\pi i a(\omega) b(\omega)} R_{-b(\omega)} T_{-a(\omega)} g_0, f)$ . Here  $g_0 := \text{emb}(g)$ . Then  $X \in S_{\Omega, 2}^*$ , for we have for  $f \in S$

$$\begin{aligned} \int_{\Omega} |(\underline{X}, f)|^2 \, dP &= \int_{\Omega} |(p e^{-\pi i a b} R_{-b} T_{-a} g_0, f)|^2 \, dP = \\ &= c \int_{\mathbb{R}^2} |(e^{-\pi i a b} R_{-b} T_{-a} g_0, f)|^2 \, dF(a), \end{aligned}$$

and

$$|(e^{-\pi i a b} R_{-b} T_{-a} g_0, f)|^2 \leq \|g\|^2 \|f\|^2 \quad (a \in \mathbb{R}, b \in \mathbb{R}).$$

Let

$$K := F^{(2)} Z_U(g \otimes \bar{g}) = \Psi_{(x, y) \in \mathbb{R}^2} \int_{-\infty}^{\infty} e^{-2\pi i y t} g\left(\frac{x+t}{\sqrt{2}}\right) \overline{g\left(\frac{x-t}{\sqrt{2}}\right)} \, dt.$$

Then  $K$  is continuous and bounded over  $\mathbb{R}^2$ , and  $K \in L_2(\mathbb{R}^2)$ . The expected Wigner distribution  $L$  of  $\underline{X}$  is given by

$$L = \text{emb} \left( \Psi_{(u, v)} \int_{\mathbb{R}^2} c K(u - a/2, v - b/2) \, dF(a) \right).$$

This can be proved by using the result of theorem 3.5.7 for  $N_{\alpha} g$  (instead of  $g$ ) with  $\alpha > 0$  and  $c_1 = c$ ,  $c_n = 0$  ( $n \geq 2$ ), and then taking  $\alpha \downarrow 0$  (we have,

by boundedness and continuity of  $K$ ,  $F^{(2)} Z_U (N_\alpha g \otimes \overline{N_\alpha g}) =$   
 $= N_\alpha^{(1)} N_\alpha^{(2)} F^{(2)} Z_U (g \otimes \bar{g}) \rightarrow K(\alpha + 0)$  uniformly and boundedly in  $\mathbb{R}^2$ .

3.5.9. We now consider processes  $\underline{X}$  of the form  $pe^{-\pi iab} R_{-b} T_{-a} Z_\gamma g$ . Here  $p$  and  $a = (a, b)$  are as in 3.5.8, and  $\gamma$  is a positive random variable (for notational convenience, we take  $\gamma$  rather than  $\gamma^{\frac{1}{2}}$ ; cf. 3.5.1). We assume that  $g \in L_2(\mathbb{R})$ , and that the joint distribution function of  $(p, a, \gamma)$  has the form  $\int_{(u, (v, w), x) \in \mathbb{T} \times \mathbb{R}^2 \times (0, \infty)} F^{(1)}(u) F^{(2)}(v, w) F^{(3)}(x)$ , where  $F^{(1)}$ ,  $F^{(2)}$  and  $F^{(3)}$  are the distribution functions of  $p$ ,  $a$  and  $\gamma$  respectively (whence  $p$ ,  $a$  and  $\gamma$  are mutually independent).

Under these conditions the process  $\underline{X}$  (of which a representative is given by  $\int_{(f, \omega)} (p(\omega) e^{-\pi i a(\omega) b(\omega)} R_{-b(\omega)} T_{-a(\omega)} Z_{\gamma(\omega)} g_0, f)$  with  $g_0 := \text{emb}(g)$ ) is an element of  $S_{\Omega, 2}^*$ . For, if  $f \in S$ , then we find

$$\int_{\Omega} |(\underline{X}, f)|^2 dP = \int_{\Omega} |(pe^{-\pi iab} R_{-b} T_{-a} Z_\gamma g_0, f)|^2 dP =$$

$$(*) = c \int_{\mathbb{R}^2} \int_0^\infty |(e^{-\pi iab} R_{-b} T_{-a} Z_\gamma g_0, f)|^2 dF^{(3)}(\gamma) dF^{(2)}(a),$$

and ( $\|\cdot\|$  denotes ordinary  $L_2(\mathbb{R})$ -norm)

$$|(e^{-\pi iab} R_{-b} T_{-a} Z_\gamma g_0, f)|^2 \leq \|Z_\gamma g\|^2 \|f\|^2 = \|g\|^2 \|f\|^2$$

for  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$ ,  $\gamma > 0$ .

3.5.10. Let  $\underline{X}$  be as in 3.5.9. We shall show that there is a sequence  $(\underline{X}_n)_{n \in \mathbb{N}}$  of processes (defined on some probability space  $(\Omega', \Lambda', P')$ ) of the type studied in 3.5.8 such that  $\underline{Y} := \sum_{n=1}^\infty \underline{X}_n$  converges in  $S_{\Omega, 2}^*$ -sense and  $E\underline{Y} = E\underline{X} = 0$ ,  $R_{\underline{Y}} = R_{\underline{X}}$ . Moreover, the  $\underline{X}_n$ 's satisfy

$$\int_{\Omega} (\underline{X}_n, f) \cdot \overline{(\underline{X}_m, h)} dP = 0 \quad (f \in S, h \in S)$$

if  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}$ ,  $n \neq m$ . Hence, for proving simulation theorems, we can restrict ourselves to series of processes of the type occurring in 3.5.8.

We shall give a proof of the above statement in the following few sections. We first observe that  $R_{\underline{X}}$  is completely determined by the values



of  $\int_{\Omega} |\underline{X}, f|^2 dP$  for  $f \in S$ . This involves according to 3.5.9 (\*) the expression

$$\int_0^{\infty} |e^{-\pi i a b} R_{-b} T_{-a} Z_{\gamma} g_0, f|^2 dF^{(3)}(\gamma).$$

We can write this expression as

$$\frac{1}{4} \int_0^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\pi i b(t-s)} \overline{f(\frac{1}{2}(a+t))} f(\frac{1}{2}(a+s)) \times \right. \\ \left. \times \gamma g(\frac{\gamma}{2}(-a+t)) \overline{g(\frac{\gamma}{2}(-a+s))} dt ds \right) dF^{(3)}(\gamma).$$

By interchanging the order of integration this becomes

$$\frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\pi i b(t-s)} \overline{f(\frac{a}{2} + \frac{t}{2})} f(\frac{a}{2} + \frac{s}{2}) \times \\ \times \left\{ \int_0^{\infty} \gamma g(\frac{\gamma}{2}(-a+t)) \overline{g(\frac{\gamma}{2}(-a+s))} dF^{(3)}(\gamma) \right\} dt ds.$$

We are going to study the expression between { } in more detail. In view of the proof of the following theorem it is convenient to assume that  $g$  is Borel measurable (if  $g$  is not Borel measurable, then we can take a Borel measurable  $g_1$  (with  $g = g_1$  almost everywhere) that gives rise to the same process  $\underline{X}$ ).

3.5.11. THEOREM. For almost every  $(u, v) \in \mathbb{R}^2$

$$P(u, v) := \int_0^{\infty} \gamma g(\gamma u) \overline{g(\gamma v)} dF^{(3)}(\gamma)$$

is defined, and  $P \in L_2(\mathbb{R}^2)$ . Furthermore,  $P$  is positive definite, and there exists a sequence  $(c_n)_{n \in \mathbb{N}}$  of non-negative real numbers and a complete orthonormal sequence  $(g_n)_{n \in \mathbb{N}}$  in  $L_2(\mathbb{R})$  such that  $\sum_{n=1}^{\infty} c_n < \infty$  and

$$P = \sum_{n=1}^{\infty} c_n g_n \otimes \bar{g}_n$$

with convergence in  $L_2(\mathbb{R}^2)$ -sense.

PROOF. It is easy to see that  $\int_{(u,v,\gamma)} \gamma g(\gamma u) \overline{g(\gamma v)}$  is Borel measurable over  $\mathbb{R} \times \mathbb{R} \times (0, \infty)$ . We have by the Cauchy-Schwarz inequality

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_0^{\infty} |\gamma g(\gamma u) \overline{g(\gamma v)}| dF^{(3)}(\gamma) \right)^2 dudv \leq \\ & \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_0^{\infty} \gamma_1 |g(\gamma_1 u)|^2 dF^{(3)}(\gamma_1) \right) \left( \int_0^{\infty} \gamma_2 |g(\gamma_2 v)|^2 dF^{(3)}(\gamma_2) \right) dudv. \end{aligned}$$

Hence, by Fubini's theorem,

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_0^{\infty} |\gamma g(\gamma u) \overline{g(\gamma v)}| dF^{(3)}(\gamma) \right)^2 dudv \leq \\ & \leq \left( \int_0^{\infty} \left( \int_{-\infty}^{\infty} \gamma |g(\gamma u)|^2 du \right) dF^{(3)}(\gamma) \right)^2 = \left( \int_{-\infty}^{\infty} |g(u)|^2 du \right)^2. \end{aligned}$$

This implies that  $\int_0^{\infty} |\gamma g(\gamma u) \overline{g(\gamma v)}| dF^{(3)}(\gamma) < \infty$  for almost every  $(u,v) \in \mathbb{R}^2$ . Therefore  $P$  is defined almost everywhere in  $\mathbb{R}^2$ . Furthermore, it follows from the above that  $P \in L_2(\mathbb{R}^2)$ .

In order to show that  $P$  is positive definite, we take an  $f \in L_2(\mathbb{R})$ . We have by Fubini's theorem

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(u,v) f(u) \overline{f(v)} dudv = \int_0^{\infty} \left| \int_{-\infty}^{\infty} \gamma^{\frac{1}{2}} g(\gamma u) f(u) du \right|^2 dF^{(3)}(\gamma) \geq 0.$$

It follows from well-known Hilbert space theory that there exists a sequence  $(c_n)_{n \in \mathbb{N}}$  of non-negative numbers and a complete orthonormal sequence  $(g_n)_{n \in \mathbb{N}}$  in  $L_2(\mathbb{R})$  such that  $\sum_{n=1}^{\infty} c_n^2 < \infty$  and

$$P = \sum_{n=1}^{\infty} c_n g_n \otimes \bar{g}_n$$

with convergence in  $L_2(\mathbb{R}^2)$ -sense.

We shall show that  $\sum_{n=1}^{\infty} c_n < \infty$ . We have for  $n \in \mathbb{N}$

$$\begin{aligned} c_n &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(u,v) \overline{g_n(u)} g_n(v) \, dudv = \\ &= \int_0^{\infty} \left| \int_{-\infty}^{\infty} \gamma^{\frac{1}{2}} g(\gamma u) \overline{g_n(u)} \, du \right|^2 dF^{(3)}(\gamma). \end{aligned}$$

Hence, by monotone convergence and Parseval's identity,

$$\begin{aligned} \sum_{n=1}^{\infty} c_n &= \int_0^{\infty} \sum_{n=1}^{\infty} \left| \int_{-\infty}^{\infty} \gamma^{\frac{1}{2}} g(\gamma u) \overline{g_n(u)} \, du \right|^2 dF^{(3)}(\gamma) = \\ &= \int_0^{\infty} \left( \int_{-\infty}^{\infty} |\gamma^{\frac{1}{2}} g(\gamma u)|^2 \, du \right) dF^{(3)}(\gamma) = \int_{-\infty}^{\infty} |g(u)|^2 \, du. \quad \square \end{aligned}$$

3.5.12. We conclude from the above theorem and 3.5.10 that for  $f \in S$

$$\begin{aligned} &\int_0^{\infty} \left| (e^{-\pi i a b} R_{-b} T_{-a} Z_{\gamma} g_0, f) \right|^2 dF^{(3)}(\gamma) = \\ &= \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\pi i b(t-s)} \overline{f\left(\frac{a}{2} + \frac{t}{2}\right)} f\left(\frac{a}{2} + \frac{s}{2}\right) P\left(-\frac{a}{2} + \frac{t}{2}, -\frac{a}{2} + \frac{s}{2}\right) dt ds = \\ &= \frac{1}{4} \sum_{n=1}^{\infty} c_n \left| \int_{-\infty}^{\infty} e^{\pi i b t} \overline{f\left(\frac{a}{2} + \frac{t}{2}\right)} g_n\left(-\frac{a}{2} + \frac{t}{2}\right) dt \right|^2 = \\ &= \frac{1}{4} \sum_{n=1}^{\infty} c_n \left| (e^{-\pi i a b} R_{-b} T_{-a} g_n, f) \right|^2. \end{aligned}$$

Note that the right hand side series converges uniformly and boundedly:

$$\sum_{n=1}^{\infty} c_n < \infty, \quad \left| (e^{-\pi i a b} R_{-b} T_{-a} g_n, f) \right|^2 \leq \|g_n\|^2 \|f\|^2 = \|f\|^2$$

$((a,b) \in \mathbb{R}^2, n \in \mathbb{N})$ . Hence we have (cf. 3.5.9)

$$\int_{\Omega} |\underline{X}, f|^2 dP = \frac{1}{4} c \sum_{n=1}^{\infty} c_n \int_{\mathbb{R}^2} |(e^{-\pi i a b} R_{-b} T_{-a} g_n, f)|^2 dF^{(2)}(a).$$

There exists a probability space  $(\Omega', \Lambda', P')$  with random variables  $p_n, a_n = (a_n, b_n)$  ( $n \in \mathbb{N}$ ) as in 3.5.6 defined on it such that  $\int_{\Omega'} p_n dP' = 0, \int_{\Omega'} |p_n|^2 dP' = c c_n, F_n = F^{(2)}$  for  $n \in \mathbb{N}$  (cf. also the proof of theorem 3.5.14). If we define  $\underline{X}_n := p_n e^{-\pi i a_n b_n} R_{-b_n} T_{-a_n} g_n$  (cf. 3.5.8), then we have for  $f \in S, n \in \mathbb{N}, m \in \mathbb{N}$

$$\int_{\Omega'} (\underline{X}_n, f) \overline{(\underline{X}_m, f)} dP' = \begin{cases} 0 & (n \neq m) \\ c c_n \int_{\mathbb{R}^2} |(e^{-\pi i a b} R_{-b} T_{-a} g_n, f)|^2 dF^{(2)}(a) & (n = m). \end{cases}$$

The series  $\sum_{n=1}^{\infty} \underline{X}_n$  is convergent in  $S_{\Omega, 2}^*$ -sense, and the autocorrelation function of the sum equals the one of  $\underline{X}$ . For the Wigner distribution of  $\underline{X}_n$  we have

$$F^{(2)}_{Z_U R_{\underline{X}_n}} = \text{emb} \left( \int_{(u,v)} c c_n \int_{\mathbb{R}^2} K_n(u - a/2, v - b/2) dF^{(2)}(a) \right),$$

where  $K_n$  is the Wigner distribution of  $g_n$  for  $n \in \mathbb{N}$  (cf. 3.5.8). Note that  $F^{(2)}_{Z_U R_{\underline{X}}} = \sum_{n=1}^{\infty} F^{(2)}_{Z_U R_{\underline{X}_n}}$ .

It can be shown that the function  $K := \int_{(x,y)} \int_{-\infty}^{\infty} e^{-2\pi i y t} p\left(\frac{x+t}{\sqrt{2}}, \frac{x-t}{\sqrt{2}}\right) dt$  is defined everywhere in  $\mathbb{R}^2$  and continuous and bounded, and that

$$F^{(2)}_{Z_U R_{\underline{X}}} = \text{emb} \left( \int_{\mathbb{R}^2} K(u - a/2, v - b/2) dF^{(2)}(a) \right).$$

It can be shown furthermore that the series  $\sum_{n=1}^{\infty} c_n K_n$  converges boundedly and uniformly to  $K$ , and that

$$\sum_{n=1}^{\infty} c_n \int_{\mathbb{R}^2} K_n(u - a/2, v - b/2) dF^{(2)}(a) = \int_{\mathbb{R}^2} K(u - a/2, v - b/2) dF^{(2)}(a)$$

for every  $(u, v) \in \mathbb{R}^2$  (uniform convergence).

3.5.13. We use this opportunity to state some further results about  $P$  and about smooth positive definite functions in general.

(i) If the  $g$  of 3.5.9 is continuous and bounded, and if  $\int_0^{\infty} \gamma dF^{(3)}(\gamma) < \infty$ , then  $P$  is continuous and bounded, and the series  $\sum_{n=1}^{\infty} c_n g_n \otimes \bar{g}_n$  converges locally uniformly to  $P$  (cf. theorem 3.5.11). Moreover,  $g_n$  is continuous whenever  $n \in \mathbb{N}$ ,  $c_n \neq 0$ .

(ii) If  $g \in S$ , and  $F$  is constant outside an interval of the form  $[a, b]$  with  $0 < a \leq b < \infty$ , then  $P \in S^2$ .

(iii) If  $Q \in S^2$  is positive definite, then there exists a sequence  $(c_n)_{n \in \mathbb{N}}$  of non-negative numbers (and a complete orthonormal sequence  $(g_n)_{n \in \mathbb{N}}$  in  $L_2(\mathbb{R})$  such that  $g_n \in S$  for  $n \in \mathbb{N}$ ,  $c_n \neq 0$ , and  $Q = \sum_{n=1}^{\infty} c_n g_n \otimes \bar{g}_n$  where the convergence is in  $S^2$ -sense.

3.5.14. We now come to theorems about second order simulation of generalized stochastic processes with zero expectation function by means of noise showers. Theorems of this kind were also proved in section 3.3, but there we used shot noise processes  $\sum_n p_n \delta(a_n)$  and random Fourier series processes  $\sum_n p_n e(a_n)$ , and the processes to be simulated were stationary.

**THEOREM.** Let  $f \in S^{2*}$  be of positive type (cf. appendix 3, 2.1), let  $g \in S$ , and let  $K := F^{(2)}_{Z_U}(g \otimes \bar{g})$ . There is a generalized stochastic process  $\underline{X} = \sum_{n=1}^{\infty} p_n v_{a_n} g$  of the type discussed in 3.5.6 and 3.5.7 such that  $E \underline{X} = 0$  and

$$F^{(2)}_{Z_U \underline{X}} = T_K f.$$

**PROOF.** It is not hard to prove from appendix 4, theorem 2.2 that there are non-negative numbers  $c_n$  and 2-dimensional distribution functions  $F_n$  ( $n \in \mathbb{N}$ ) such that

$$f = \frac{1}{2} Z_{1/\sqrt{2}}^{(1)} Z_{1/\sqrt{2}}^{(2)} \left( \sum_{n=1}^{\infty} c_n \frac{\partial^2}{\partial x \partial y} \text{emb}(F_n) \right).$$

For each  $n \in \mathbb{N}$  we can find a probability space  $(\Omega_{n1}, \Lambda_{n1}, P_{n1})$  and a random variable  $p'_n := \Omega_{n1} \rightarrow \mathbb{T}$  such that  $\int_{\Omega_{n1}} p'_n dP_{n1} = 0$ ,  $\int_{\Omega_{n1}} |p'_n|^2 dP_{n1} = c_n$ . Also, for each  $n \in \mathbb{N}$  we can find a probability space  $(\Omega_{n2}, \Lambda_{n2}, P_{n2})$  and a random vector  $a'_n = (a'_n, b'_n): \Omega_{n2} \rightarrow \mathbb{R}^2$  such that  $F_n$  is the distribution function of  $a'_n$ . Now let  $\Omega' := \prod_{n=1}^{\infty} \Omega_{n1} \times \Omega_{n2}$ , and take product measure  $P'$  on  $\Omega'$ . Put furthermore for  $n \in \mathbb{N}$

$$p_n := \bigcup_{\omega = ((\omega_{m1}, \omega_{m2}))_{m \in \mathbb{N}} \in \Omega'} p'_n(\omega_{n1}),$$

$$a_n := \bigcup_{\omega = ((\omega_{m1}, \omega_{m2}))_{m \in \mathbb{N}} \in \Omega'} a'_n(\omega_{n2}).$$

Then  $\int_{\Omega'} p_n dP = 0$ ,  $\int_{\Omega'} p_n \bar{p}_m dP = c_n \delta_{nm}$ ,  $F_n$  is the distribution function of  $a_n$ , and the pairs  $(p_n, p_m)$  and  $(a_n, a_m)$  are mutually independent for  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}$ . Hence, by  $S^{2*}$ -convergence of the series  $\sum_n c_n \text{emb}(F_n)$ ,  $\underline{X} := \sum_n p_n \nabla_{a_n} g$  is a process of the type discussed in 3.5.6 and 3.5.7 with  $E\underline{X} = 0$ , and we have for the Wigner distribution

$$F^{(2)}_{\underline{U}, \underline{R}, \underline{X}} = \frac{1}{2} T_K Z_{1/\sqrt{2}}^{(1)} Z_{1/\sqrt{2}}^{(2)} \left( \sum_n c_n \frac{\partial^2}{\partial x \partial y} \text{emb}(F_n) \right) = T_K f$$

by theorem 3.5.7. □

REMARK. We have a similar theorem as the one above if we take  $F^{(2)}_{\underline{U}, P}$  instead of  $K$ , where  $P = \bigcup_{(u,v)} \int_0^{\infty} (Z_{\gamma} g)(u) (Z_{\gamma} g)(\bar{v}) dF(\gamma)$ . Here  $F$  is a distribution function defined on  $(0, \infty)$  and constant outside an interval  $[a, b]$  with  $0 < a \leq b < \infty$  (cf. 3.5.13, remark (ii)). We can take a process  $\sum_n p_n e^{-\pi i a_n b_n} R_{-b_n} T_{-a_n} Z_{\gamma_n} g$ , where the distribution function of each  $\gamma_n$  equals  $F$  ( $n \in \mathbb{N}$ ). We note that not every positive definite element of  $S^2$  can be represented as  $\bigcup_{(u,v)} \int_0^{\infty} (Z_{\gamma} g)(u) (Z_{\gamma} g)(\bar{v}) dF(\gamma)$  with  $F$  and  $g$  as above.

3.5.15. Theorem 3.5.14 is in particular useful for describing second order simulation of generalized stochastic processes with a zero expectation function and an expected Wigner distribution of positive type. Second order white noise processes, e.g., have a zero expectation function and a constant, non-negative Wigner distribution (cf. 2.2.13). For such processes we can take in theorem 3.5.14 any  $g \in S$  to obtain exact second order simulation (i.e. the autocorrelation function of the noise shower equals the one of

the process to be simulated); the noise quanta are to be distributed uniformly over the time-frequency plane, i.e. the function

$$\frac{1}{2}(Z_{1/\sqrt{2}}^{(1)} Z_{1/\sqrt{2}}^{(2)} (\sum_n c_n \frac{\partial^2}{\partial x \partial y} \text{emb}(F_n))) \text{ has to equal } \text{emb}(\psi(x,y) | 1).$$

We may not expect to obtain higher order simulation than second order simulation, unless further assumptions are made on the distribution functions of the  $p_n$ 's and  $a_n$ 's. This is due to the fact that the Wigner distribution only involves second order moments of the process to be simulated.

The restriction "zero expectation function" in theorem 3.5.14 is a natural one in case noise processes are to be simulated (these processes have in general zero expectation function). It is not very likely that the processes  $\sum_n p_n V_{a_n} g$  of 3.5.6 can be used for a simple description of the simulation (first and second order) of processes with non-zero expectation function.

We noted above that for white noise processes we can take any  $g \in S$  we wish. In general we shall have to adapt our  $g$  to the process to be simulated. If the process is close to being second order time stationary (so that its Wigner distribution is almost constant in the first variable), it seems to be adequate to take a  $g \in S$  for which the operator  $T_K$  has the flavour of averaging over an ellipse with a long horizontal axis and a small vertical one (cf. 3.4.5). Similar things hold for processes which are close to being second order frequency stationary.

It is, of course, also possible to describe second order simulation with the aid of processes of type  $\sum_n p_n e^{-\pi i a_n b_n} R_{-b_n} T_{-a_n} Z_{\gamma_n} g$ , where the tuples  $(p_n, (a_n, b_n), \gamma_n)$  are as in 3.5.14, remark. Processes of this kind are of interest for simulation of processes that look second order time stationary in some areas of the time-frequency plane, and second order frequency stationary in other areas. In areas where the process to be simulated is more or less time stationary, we have to distribute quanta having a small value of  $\gamma$ , and in areas where the process is more or less frequency stationary, we have to take quanta having a large value of  $\gamma$ .

3.5.16. As said in the beginning of 3.5.15, theorem 3.5.14 is useful for the case we want to simulate a process with a Wigner distribution of positive type. In order to handle the general case we give the following theorem.

**THEOREM.** Let  $V$  be the Wigner distribution of a generalized stochastic process, and let  $g_1 \in S$ ,  $g_2 \in S$ . Let  $K_1 := F^{(2)} Z_U(g_1 \otimes \bar{g}_1)$ ,  $K_2 := F^{(2)} Z_U(g_2 \otimes \bar{g}_2)$ . There exists a generalized stochastic process  $\underline{X} = \sum_{n=1}^{\infty} p_n V_{a_n} g_1$  of the type discussed in 3.5.6 and 3.5.7 such that

$$F^{(2)}_{Z_U} R_{\underline{X}} = T_{K_1} T_{K_2} V .$$

PROOF. This follows from theorem 3.5.13 by noting that  $T_{K_2} V$  is of positive type (cf. 3.4.4).  $\square$

Note that the sole purpose of the operation  $T_{K_2}$  is to obtain a function of positive type. The above theorem can be generalized somewhat by taking a  $K_2$  of the form  $F^{(2)}_{Z_U} P$  where  $P \in S^2$  is positive definite (cf. theorem 3.5.13, remark (iii)).



## CHAPTER 4

## THE WIGNER DISTRIBUTION AND GENERALIZED HARMONIC ANALYSIS

This chapter establishes the relation between the Wigner distribution and the Wiener theory of generalized harmonic analysis (cf. [W1] for a treatment of generalized harmonic analysis). We sketch below what we are aiming at.

Let  $\underline{x}$  be an ordinary, strict sense time stationary and ergodic process (defined on some probability space  $\Omega$ ) with finite second order moments (whence the autocorrelation function of  $\underline{x}$  exists). It is common practice in engineering to estimate the value at  $\lambda$  of the spectral density function of  $\underline{x}$  by the number

$$(1) \quad \frac{1}{2\tau} \left| \int_{-\tau}^{\tau} e^{-2\pi i \lambda t} f(t) dt \right|^2$$

with some large positive value of  $\tau$ . Here  $f$  is a realization of the process, i.e. a function  $\bigcup_t \underline{x}(t, \omega)$  with  $\omega \in \Omega$  ( $\underline{x}$  must satisfy certain integrability conditions, of course). This can be motivated by applying ergodic theorems and some results of the theory of generalized harmonic analysis.

In order to go into some more detail, we define the Wiener class  $W$  as the set of all measurable functions  $f: \mathbb{R} \rightarrow \mathbb{C}$  such that

$$(2) \quad \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} f(x + \xi) \overline{f(\xi)} d\xi$$

exists for every  $x \in \mathbb{R}$ . Denote the limit in (2) by  $\varphi_f(x)$  for  $x \in \mathbb{R}$ . The spectral density function of  $f \in W$  is defined roughly as the Fourier transform of  $\varphi_f$ ; we shall denote it here by  $s_f$ . It is a well-known result in engineering that the function whose values are given by the expression in (1) tends in some sense to  $s_f$  if  $\tau \rightarrow \infty$  (cf. [BT], Part II, (B-2.9)). Now, if  $\underline{x}$  is as above, almost every realization belongs to  $W$ . If, in addition,  $\underline{x}$  is ergodic, the spectral density functions of almost all realizations equal the spectral density function of  $\underline{x}$ .

As we want to study spectral density functions of generalized stochastic

processes (and not just of ordinary processes), we have to consider limits like

$$(3) \quad \lim_{\tau \rightarrow \infty} T_{h_\tau \otimes \delta_0} Z_U(F \otimes \bar{F}),$$

or, as will appear more proper for our aims,

$$(4) \quad \lim_{\epsilon \rightarrow 0} \epsilon^{-\frac{1}{2}} T_{k_\epsilon}^{(1)} M_{k_\epsilon}^{(2)} Z_U(F \otimes \bar{F})$$

for  $F \in S^*$  (cf. 3 and 6 of appendix 2, and 1.17 of appendix 1). Here  $h_\tau := \frac{1}{2\tau} \chi_{[-\tau, \tau]}$  ( $\tau > 0$ ) and  $k_\epsilon := \prod_x \epsilon^{\frac{1}{2}} \exp(-\pi \epsilon x^2)$ . We have for a reasonably behaved, ordinary function  $F$

$$T_{h_\tau \otimes \delta_0} Z_U(F \otimes \bar{F}) = \prod_{(x, y)} \frac{1}{2\tau} \int_{-\tau}^{\tau} F\left(\frac{x+t+y}{\sqrt{2}}\right) \overline{F\left(\frac{x+t-y}{\sqrt{2}}\right)} dt$$

for  $\tau > 0$ , and

$$\epsilon^{-\frac{1}{2}} T_{k_\epsilon}^{(1)} M_{k_\epsilon}^{(2)} Z_U(F \otimes \bar{F}) = \prod_{(x, y)} \epsilon^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp(-\pi \epsilon t^2 - \pi \epsilon y^2) F\left(\frac{x+t+y}{\sqrt{2}}\right) \overline{F\left(\frac{x+t-y}{\sqrt{2}}\right)} dt$$

for  $\epsilon > 0$  (compare 2). The reason for considering (3) (or (4)) is the fact that the ordinary product of two generalized functions may be undefined, whereas the tensor product always makes sense.

The class of all  $F \in S^*$  for which the limit in (3) exists in  $S^{2*}$ -sense is called the generalized Wiener class and denoted by  $W^*$ . It is possible to present a direct link between the classes  $W$  and  $W^*$ : we have  $F \in W^*$  if and only if  $T_F f = \prod_t (T_t f, F_-) \in W$  for every  $f \in S$  (cf. 4.3.4). The limit in (3) is a generalized function (of 2 variables) of the form  $H \otimes \phi_F$ . Here  $H$  is the constant function, and  $\phi_F$  is a generalized function whose Fourier transform  $F\phi_F$ , called the spectral density function of  $F$ , is of positive type (cf. 4.3.5 and 4.3.7).

It can be proved (cf. 4.3.4, remark and 4.4.3) that  $F \in W^*$  if and only if the limit in (4) exists in  $S^{2*}$ . Also, if  $F \in W^*$ , the limit in (4) equals  $H \otimes \phi_F$  where  $\phi_F$  is as in the previous paragraph.

Using the limit in (4) is to be preferred to using the one in (3): we can use (4) to present a link between the spectral density function of  $F$  and the Wigner distribution of  $F$ . If  $\varepsilon > 0$ ,  $F \in S^*$ , then Fourier transformation of  $\varepsilon^{-\frac{1}{2}} T_{k_\varepsilon}^{(1)} M_{k_\varepsilon}^{(2)} Z_U(F \otimes \bar{F})$  with respect to the second variable gives  $T_{k_\varepsilon \otimes k_\varepsilon - 1} V(F, F)$ . Here  $V(F, F)$  is the Wigner distribution of  $F$ . Thus we see that  $F \in W^*$  if and only if  $\lim_{\varepsilon \downarrow 0} T_{k_\varepsilon \otimes k_\varepsilon - 1} V(F, F)$  exists. Also, if  $F \in W^*$ ,  $\lim_{\varepsilon \downarrow 0} T_{k_\varepsilon \otimes k_\varepsilon - 1} V(F, F) = H \otimes F_{\Phi_F}^{\varepsilon \downarrow 0}$  (cf. 4.4.4). If  $\bar{F} \in S^*$  does not belong to  $W^*$ , then  $F_{\Phi_F}$  is not defined. But  $T_{k_\varepsilon \otimes k_\varepsilon - 1} V(F, F)$  still makes sense, and is of positive type for  $\varepsilon > 0$  (cf. 4.4.4). For such an  $F$ ,  $T_{k_\varepsilon \otimes k_\varepsilon - 1} V(F, F)$  is not constant in the first (= time) variable. We may therefore regard  $T_{k_\varepsilon \otimes k_\varepsilon - 1} V(F, F)$  as a time dependent spectral density function (it depends, of course, on  $\varepsilon$  as well). This fact can be illustrated further if we take a reasonably behaved, ordinary function  $f: \mathbb{R} \rightarrow \mathbb{C}$  in the role of  $F$ . We have (cf. 4.4.5)

$$(5) \quad T_{k_\varepsilon \otimes k_\varepsilon - 1} V(f, f) = \bigvee_{(t, \lambda)} \varepsilon^{\frac{1}{2}} \left| \int_{-\infty}^{\infty} \exp(-\pi\varepsilon(u-t/\sqrt{2})^2 - 2\pi i u \lambda / \sqrt{2}) f(u) du \right|^2.$$

This resembles the expression in (1), but we note that in (5) the time variable  $t$  occurs explicitly.

The sketch given in the above paragraphs is detailed and elaborated in the four sections of this chapter. Section 4.1 is intended to make the reader familiar with the main notions in generalized harmonic analysis. Section 4.2 gives a Tauberian theorem (cf. 4.2.2) of the type occurring in [W1], Ch. II, §10 and Ch. III, §20. This theorem is needed in the proof of the statement that if one of the limits in (3) and (4) exists in  $S^{2*}$ -sense, then the other one exists too, with the same value. Section 4.3 is devoted to the generalized Wiener class  $W^*$ ; it is proved there (among other things) that  $W^*$  is a measurable subset of  $S^*$ . Section 4.4 points out the relation between the Wigner distribution and generalized harmonic analysis. Furthermore, it gives some applications of ergodic theorems to strict sense stationary and ergodic generalized stochastic processes and their spectral density functions. It is finally indicated how to handle certain non-stationary processes occurring in practice (e.g.  $\frac{1}{f}$ -noise and the Barkhausen effect).

## 4.1. SOME IMPORTANT NOTIONS IN GENERALIZED HARMONIC ANALYSIS

4.1.1. We give here some main notions in the Wiener theory of generalized harmonic analysis. For the proofs of the theorems mentioned and further details we refer to [W1], Ch. IV, §21 and §22.

4.1.2. DEFINITION. Let  $f$  be a complex-valued measurable function defined on  $\mathbb{R}$ . Then  $f$  is said to belong to the Wiener class  $W$  if

$$\lim_{A \rightarrow \infty} \frac{1}{2A} \int_{-A}^A f(x + \xi) \overline{f(\xi)} d\xi$$

exists for every  $x \in \mathbb{R}$ . (Wiener uses the letter  $S$  instead of  $W$ , but we have reserved  $S$  for the space of smooth functions.) If  $f \in W$ , then we define  $\varphi_f$  (or, shortly  $\varphi$ ) as the function given by

$$\varphi_f(x) := \lim_{A \rightarrow \infty} \frac{1}{2A} \int_{-A}^A f(x + \xi) \overline{f(\xi)} d\xi \quad (x \in \mathbb{R}).$$

4.1.3. Let  $f \in W$ . The following properties hold

- (i) If  $a \in \mathbb{R}$ , then  $T_a f \in W$ , and  $\varphi_{T_a f} = \varphi_f$ .
- (ii)  $|\varphi(x)| \leq \varphi(0)$  for  $x \in \mathbb{R}$ .
- (iii) If  $\varphi$  is continuous at  $x = 0$ , then  $\varphi$  is continuous everywhere.
- (iv)  $\varphi(x) = \overline{\varphi(-x)}$  for  $x \in \mathbb{R}$ .
- (v)  $\varphi$  is positive definite, i.e.  $\sum_{i=1}^n \sum_{j=1}^n a_i \overline{a_j} \varphi(x_i - x_j) \geq 0$  for every  $n \in \mathbb{N}$ ,  $a_1 \in \mathbb{C}, \dots, a_n \in \mathbb{C}$ ,  $x_1 \in \mathbb{R}, \dots, x_n \in \mathbb{R}$ .
- (vi) (Bochner). If  $\varphi$  is continuous, there is a non-decreasing bounded function  $F$ , continuous from the right, such that

$$\varphi(x) = \int_{-\infty}^{\infty} e^{i\lambda x} dF(\lambda) \quad (x \in \mathbb{R}).$$

It should be noted that  $W$  is not a linear space: it is not closed under addition. We can give an example of a real-valued  $f \in W$  such that

$\lim_{A \rightarrow \infty} \frac{1}{2A} \int_{-A}^A f(\xi) d\xi$  does not exist, hence  $\int_x (f(x) + 1) \notin W$  (note that  $\int_x 1 \in W$ ).

4.1.4. Let  $f \in W$ . Wiener defines the *spectrum*  $\sigma_f$  (or, shortly,  $\sigma$ ) of  $f$  by putting

$$\sigma_f := \lim_{A \rightarrow \infty} \frac{1}{2\pi} \int_{-A}^A \left[ \int_{-1}^1 \frac{\varphi(x) e^{-iux}}{-ix} dx + \int_{-1}^1 \varphi(x) \frac{e^{-iux} - 1}{-ix} dx \right] du$$

(cf. [W1], Ch. IV (21.21)). The limit is in  $L_2(\mathbb{R})$ -sense. It is shown in [W1], Ch. IV, §22, lemma 30<sub>a</sub> that  $\sigma_f$  may be assumed to be bounded, real-valued, non-decreasing and continuous from the right, and that

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iux} d\sigma_f(u)$$

for almost every  $x \in \mathbb{R}$  (cf. also [W1], Ch. IV, §23, theorem 34). This generalizes 4.1.3 (vi) where  $\varphi$  was assumed to be continuous. If  $\sigma_f$  is absolutely continuous with respect to Lebesgue measure, then  $\sigma_f'$  is called the *spectral density function* of  $f$ .

4.1.5. A further result of Wiener's theory deals with the spectrum of certain linear transforms (convolutions) of elements of  $W$ . Let  $f \in W$ , and let  $K$  be a complex-valued function defined on  $\mathbb{R}$  satisfying some integrability and boundedness conditions. Define the function  $g$  by

$$g(x) := \int_{-\infty}^{\infty} K(x - \xi) f(\xi) d\xi \quad (x \in \mathbb{R}).$$

It is shown in [W1], Ch. IV, §22, theorem 30 that  $g \in W$ , and that

$$\sigma_g(u) = C + \int_0^u \left| \int_{-\infty}^{\infty} K(\xi) e^{-i v \xi} d\xi \right|^2 d\sigma_f(v) \quad (u \in \mathbb{R})$$

for some constant  $C$ .

4.1.6. Hopf (cf. [Ho], Kap. II, §5, Def. 1) uses a slightly different definition of the class  $W$ . A measurable complex-valued function  $f$  is said to have a spectrum if

$$(*) \quad \lim_{A \rightarrow -\infty, B \rightarrow \infty} \frac{1}{B-A} \int_A^B f(x+\xi) \overline{f(\xi)} d\xi =: \varphi(x)$$

exists for almost every  $x \in \mathbb{R}$  including  $x = 0$ , and if

$$(**) \quad \limsup_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \varphi(x) dx = \varphi(0).$$

The spectrum of such an  $f$  is defined as in 4.1.4.

This definition gives a Wiener class  $W_1$  that neither contains nor is contained in the class  $W$ .

In section 4.3 we give the  $S^*$ -generalizations of both  $W$  and  $W_1$ , and it will appear that the generalized  $W_1$  is a proper subset of the generalized  $W$ . Generalized harmonic analysis in the class  $W_1$  runs somewhat smoother than in the class  $W$ , but this advantage disappears almost entirely when we pass to the  $S^*$ -generalizations of  $W$  and  $W_1$ .

## 4.2. A TAUBERIAN THEOREM

4.2.1. In this section we deal with the following question. What conditions imposed on a measurable function  $h$  (defined on  $[0, \infty)$ ) ensure validity of the assertion "if one of the limits

$$\lim_{A \rightarrow \infty} \frac{1}{A} \int_0^A h(\xi) d\xi$$

and

$$\lim_{\varepsilon \downarrow 0} 2\varepsilon^{\frac{1}{2}} \int_0^{\infty} \exp(-\pi\varepsilon\xi^2) h(\xi) d\xi$$

exists, then the other one exists and assumes the same value"?

It can be proved with the aid of the well-known Wienerian Tauberian theorem (cf. [La], theorem 8.2.1) that the above assertion is true if  $h$  is essentially bounded. Unfortunately, we shall have to consider cases with unbounded  $h$ 's as well. There seems to be no recent literature in which problems like this are attacked. In [W1], Ch. III, §20, however, a special Tauberian theorem on a similar problem is proved; our next theorem uses several arguments of the proof of [W1], Ch. III, §20, theorem 21.

4.2.2. Denote by  $D$  the collection of all measurable functions  $h$  for which the assertion of the first paragraph of 4.2.1 holds.

THEOREM. Let  $h$  be measurable over  $[0, \infty)$ , and assume that  $h(x) \geq 0$  ( $x \in \mathbb{R}$ ) or that  $\bigvee_{\eta \in \mathbb{R}} \int_0^\eta |h(e^y)| dy$  is uniformly continuous over  $\mathbb{R}$ . Then  $h \in D$ .

PROOF. We proceed as in the proof of [W1], Ch. III, §20, theorem 21, and put  $A = e^x$ ,  $\varepsilon = \pi^{-1} e^{-2x}$ . On substituting  $\xi = e^y$  ( $y \in \mathbb{R}$ ), we get for the limits to be compared

$$\lim_{A \rightarrow \infty} \frac{1}{A} \int_0^A h(\xi) d\xi = \lim_{x \rightarrow \infty} \int_{-\infty}^x e^{y-x} \ell(y) dy$$

and

$$\lim_{\varepsilon \rightarrow 0} 2\varepsilon^{\frac{1}{2}} \int_0^\infty \exp(-\pi\varepsilon\xi^2) h(\xi) d\xi = \lim_{x \rightarrow \infty} \pi^{-\frac{1}{2}} \int_{-\infty}^\infty e^{y-x} \exp(-e^{2(y-x)}) \ell(y) dy$$

respectively. Here  $\ell$  denotes  $\bigvee_{y \in \mathbb{R}} h(e^y)$ .

We further put  $K_1 := \bigvee_{x \in \mathbb{R}} e^{-x} \chi_{[0, \infty)}(x)$  and  $K_2 := \bigvee_{x \in \mathbb{R}} 2\pi^{-\frac{1}{2}} e^{-x} \exp(-e^{-2x})$ . With this notation the assertion to be proved takes the following form

"for  $C \in \mathbb{C}$  the propositions

$$(1) \lim_{x \rightarrow \infty} \int_{-\infty}^\infty K_1(x-y) \ell(y) dy \text{ exists and equals } C$$

and

$$(2) \lim_{x \rightarrow \infty} \int_{-\infty}^\infty K_2(x-y) \ell(y) dy \text{ exists and equals } C$$

are equivalent".

Let  $C \in \mathbb{C}$ . In the remainder of the proof we assume that (1) or (2) holds, and that  $\ell(y) = 0$  if  $y \leq 0$  (i.e.  $h = 0$  in  $[0,1]$ ; this is clearly no restriction). We are going to apply some theorems of [W1], Ch. II, §10, and therefore we first show that  $\int_n^{n+1} |\ell(y)| dy$  is bounded in  $n \in \mathbb{N}$ . If  $\bigcup_{n \in \mathbb{R}} \int_0^n |\ell(y)| dy$  is uniformly continuous over  $\mathbb{R}$ , there is nothing to prove. If  $\ell(y) \geq 0$  ( $y \in \mathbb{R}$ ), then we have for  $i = 1, 2$

$$\limsup_{x \rightarrow \infty} \int_{x-1}^x K_i(x-y) \ell(y) dy \geq \min_{0 \leq u \leq 1} K_i(u) \limsup_{x \rightarrow \infty} \int_{x-1}^x \ell(y) dy.$$

Since  $K_i(x) \geq 0$  ( $x \in \mathbb{R}$ ),  $\min_{0 \leq u \leq 1} K_i(u) > 0$  ( $i=1,2$ ), we have by our assumptions

$$\limsup_{x \rightarrow \infty} \int_{x-1}^x \ell(y) dy < \infty.$$

We next note that  $K_2 \in M_1$  (cf. [W1], Ch. II, (10.01)), and that the Fourier transform of  $K_2$  has no real zeros. For, if  $\lambda \in \mathbb{R}$ , then

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-2\pi i \lambda x} K_2(x) dx &= 2\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{(1-2\pi i \lambda)x} \exp(-e^{-2x}) e^{-2x} dx = \\ &= \pi^{-\frac{1}{2}} \int_0^{\infty} t^{-\frac{1}{2}(1-2\pi i \lambda)} e^{-t} dt = \pi^{-\frac{1}{2}} \Gamma(\frac{1}{2} + \pi i \lambda) \neq 0. \end{aligned}$$

Also,  $\int_{-\infty}^{\infty} K_2(x) dx = 1$ .

As to  $K_1$  we note that  $K_1 \notin M_1$  by discontinuity. As in the proof of [W1], Ch. III, §20, theorem 21 we define for  $\varepsilon > 0$

$$K_{1,\varepsilon}(x) := \frac{1}{\varepsilon} \int_x^{x+\varepsilon} K_1(y) dy \quad (x \in \mathbb{R}).$$

Then  $K_{1,\varepsilon} \in M_1$ , and



$$\int_{-\infty}^{\infty} e^{-2\pi i \lambda x} K_{1,\varepsilon}(x) dx = \begin{cases} \varepsilon^{-1} (e^{2\pi i \lambda \varepsilon} - 1) / 2\pi i \lambda (1 + 2\pi i \lambda) & (\lambda \neq 0) \\ 1 & (\lambda = 0) \end{cases}$$

for every  $\varepsilon > 0$ . Hence, there is no real value  $\lambda$  such that the Fourier transform of  $K_{1,\varepsilon}$  vanishes at  $\lambda$  for all  $\varepsilon > 0$ .

It is not hard to see from Fubini's theorem and Lebesgue's theorem on dominated convergence that (1) implies

$$(3) \quad \lim_{x \rightarrow \infty} \int_{-\infty}^{\infty} K_{1,\varepsilon}(x-y) \ell(y) dy = c$$

for every  $\varepsilon > 0$ . Hence, by [W1], Ch. II, §10, theorem 7 applied with  $\Sigma = \{K_{1,\varepsilon} \mid \varepsilon > 0\}$  and  $g := \int_0^y \ell(z) dz$ , (1) implies (2). Also, by [W1], Ch. II, §10, theorem 5, (2) implies (3) for every  $\varepsilon > 0$ .

We complete the proof by showing that the validity of (3) for every  $\varepsilon > 0$  implies that of (1). For the case that  $\ell(z) \geq 0$  ( $z \in \mathbb{R}$ ) we refer to the end of the proof of [W1], Ch. III, §20, theorem 21. For the case that  $\int_0^\eta |\ell(y)| dy$  is uniformly continuous, we note that for  $\varepsilon > 0$

$$K_1(x) = \frac{\varepsilon}{1 - e^{-\varepsilon}} K_{1,\varepsilon}(x) - \frac{\varepsilon}{1 - e^{-\varepsilon}} p_\varepsilon(x) \quad (x \in \mathbb{R}),$$

where  $p$  is defined by

$$p_\varepsilon(x) := \begin{cases} \varepsilon^{-1} (1 - e^{-x-\varepsilon}) & (-\varepsilon \leq x \leq 0) \\ 0 & (x > 0 \text{ or } x < -\varepsilon). \end{cases}$$

Now it easily follows that

$$\lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{1 - e^{-\varepsilon}} \int_{-\infty}^{\infty} p_\varepsilon(x-y) \ell(y) dy = 0$$

uniformly in  $x \in \mathbb{R}$ . From this it is not hard to see that validity of (3) for every  $\epsilon > 0$  implies that of (1).  $\square$

4.2.3. THEOREM. Let  $h$  be measurable over  $[0, \infty)$ , and let  $p$ ,  $1 < p < \infty$  be such that at least one of the numbers  $\limsup_{A \rightarrow \infty} \frac{1}{A} \int_0^A |h(\xi)|^p d\xi$  and  $\limsup_{\epsilon \rightarrow 0} 2\epsilon^{1/2} \int_0^\infty \exp(-\pi\epsilon\xi^2) |h(\xi)|^p d\xi$  is finite. Then  $h \in D$ .

PROOF. By theorem 4.2.2 it suffices to show that  $\Psi_{\eta \in \mathbb{R}} \int_0^\eta |h(e^y)| dy$  is uniformly continuous over  $\mathbb{R}$ . It is clearly no restriction to assume  $h(x) = 0$  ( $0 \leq x \leq 1$ ).

Assume that  $\limsup_{A \rightarrow \infty} \frac{1}{A} \int_0^A |h(\xi)|^p d\xi$  is finite. Then (cf. the proof of theorem 4.2.2)

$$\limsup_{\eta \rightarrow \infty} \int_{-\infty}^{\eta} e^{y-\eta} |h(e^y)|^p dy$$

is finite. Hence

$$\int_{\eta-1}^{\eta} e^{y-\eta} |h(e^y)|^p dy$$

is bounded in  $\eta \in \mathbb{R}$ . It follows that  $\int_{\eta-1}^{\eta} |h(e^y)|^p dy$  is bounded in  $\eta \in \mathbb{R}$ . If now  $\eta_2 > \eta_1$ , then we conclude from

$$\int_{\eta_1}^{\eta_2} |h(e^y)| dy \leq \left( \int_{\eta_1}^{\eta_2} |h(e^y)|^p dy \right)^{1/p} (\eta_2 - \eta_1)^{1/q}$$

(where  $q$  is the conjugate exponent of  $p$ ) that  $\Psi_{\eta} \int_0^\eta |h(e^y)| dy$  is uniformly continuous.

If  $\limsup_{\epsilon \rightarrow 0} 2\epsilon^{1/2} \int_0^\infty \exp(-\pi\epsilon\xi^2) |h(\xi)|^p d\xi$  is assumed to be finite, then  $\Psi_{\eta} \int_0^\eta |h(e^y)| dy$  is uniformly continuous. The proof is similar to the one above.  $\square$

REMARK. The above theorem covers the case that  $h$  is measurable and essentially bounded over  $[0, \infty)$ : we can take then any  $p > 1$ .

4.2.4. It is easy to see that the previous theorem can be stated and proved with  $\mathbb{R}$  instead of  $[0, \infty)$  as integration interval.

The following theorem is an important application of theorem 4.2.2.

**THEOREM.** Let  $f$  be measurable over  $\mathbb{R}$ . If one of the limits

$$\lim_{A \rightarrow \infty} \frac{1}{2A} \int_{-A}^A f(x + \xi) \overline{f(\xi)} d\xi$$

and

$$\lim_{\epsilon \rightarrow 0} \epsilon^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp(-\pi \epsilon \xi^2) f(x + \xi) \overline{f(\xi)} d\xi$$

exists for every  $x \in \mathbb{R}$ , then the other one exists and assumes the same value for every  $x \in \mathbb{R}$ .

**PROOF.** Write for  $x \in \mathbb{R}$ ,  $\xi \in \mathbb{R}$

$$\begin{aligned} f(x + \xi) \overline{f(\xi)} &= \\ &= \frac{1}{4} \{ |f(x + \xi) + f(\xi)|^2 - |f(x + \xi) - f(\xi)|^2 + \\ &+ i |f(x + \xi) + i f(\xi)|^2 - i |f(x + \xi) - i f(\xi)|^2 \}, \end{aligned}$$

and apply theorem 4.2.2 to each of the four terms occurring between the curly brackets.  $\square$

4.2.5. An immediate consequence of theorem 4.2.4 is the following theorem.

**THEOREM.** Let  $f$  be measurable over  $\mathbb{R}$ . Then  $f \in W$  if and only if

$$\lim_{\epsilon \rightarrow 0} \epsilon^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp(-\pi \epsilon \xi^2) f(x + \xi) \overline{f(\xi)} d\xi =: \psi(x)$$

exists for every  $x \in \mathbb{R}$ . If  $f \in W$ , then  $\varphi_f(x)$  (cf. 4.1.2) equals  $\psi(x)$  for every  $x \in \mathbb{R}$ .

**PROOF.** Obvious.  $\square$

## 4.3. GENERALIZED WIENER CLASSES

4.3.1. In this section we introduce the  $S^*$ -generalizations of the Wiener classes of section 4.1. We generalize the theorems of section 4.1 correspondingly, and give some examples. We further show measurability (as subsets of  $S^*$ ) of the generalized Wiener classes.

4.3.2. DEFINITION. Let  $F \in S^*$ . Then  $F$  is said to belong to the generalized Wiener class  $W^*$  if there is a  $G \in S^{2*}$  such that

$$T_{k_\varepsilon \otimes \delta_0} Z_U(F \otimes \bar{F}) \xrightarrow{S^{2*}} G \quad (\varepsilon \downarrow 0).$$

Here  $k_\varepsilon$  denotes  $\int_{\mathbb{Z}} \varepsilon^{\frac{1}{2}} \exp(-\pi \varepsilon z^2)$  for every  $\varepsilon > 0$ .

4.3.3. It is not yet obvious that this class  $W^*$  is an  $S^*$ -version of the Wiener class  $W$ , but this will be clear after the theorems 4.3.4, 4.3.5 and 4.3.8. The following lemma will be useful in the proofs of the theorems 4.3.4 and 4.3.5.

LEMMA. If  $h \in S$ ,  $f \in S$ ,  $g \in S$ ,  $F \in S^*$ ,  $G \in S^*$ , then

$$(T_{h \otimes \delta_0} Z_U(F \otimes \bar{G}), Z_U(f \otimes \bar{g})) = \int_{-\infty}^{\infty} \overline{(T_F f_-)(\frac{x}{\sqrt{2}})} (T_G g_-)(\frac{x}{\sqrt{2}}) \overline{h(-x)} dx$$

(cf. appendix 1, 4.6 and appendix 2,2).

PROOF. First assume that  $F \in \text{emb}(S)$ ,  $G \in \text{emb}(S)$ . The above formula then follows from a simple calculation. The general case is handled as follows. Take sequences  $(F_n)_{n \in \mathbb{N}}$  and  $(G_n)_{n \in \mathbb{N}}$  in  $\text{emb}(S)$  with  $F_n \xrightarrow{S^*} F$ ,  $G_n \xrightarrow{S^*} G$ , and apply [J2], lemma 5.2.  $\square$

4.3.4. The following theorem characterizes the class  $W^*$  in terms of the class  $W$  of 4.1.2.

THEOREM. Let  $F \in S^*$ . Then  $F \in W^*$  if and only if  $T_F f \in W$  for every  $f \in S$ .

PROOF. Assume that  $F \in W^*$ , and let  $f \in S$ . If  $a \in \mathbb{R}$ , then it follows from lemma 4.3.3. (with  $k = k_\varepsilon$  ( $\varepsilon > 0$ ),  $g = T_a f$ ,  $G = F$ ) that

$$\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} \varepsilon^{\frac{1}{2}} \exp(-\pi \varepsilon x^2) \overline{(T_F f)(\frac{x}{\sqrt{2}})} (T_F f)(\frac{x+a}{\sqrt{2}}) dx =$$

$$= \lim_{\varepsilon \downarrow 0} (T_{k_\varepsilon \otimes \delta_0} Z_U(F \otimes \bar{F}), Z_U(f_- \otimes \overline{T_a f_-})).$$

This means by 4.2.5 that  $T_F f \in W$ .

Now assume that  $T_F f \in W$  for every  $f \in S$ . It follows from lemma 4.3.3 (cf. also 4.2.5) that  $\lim_{\varepsilon \downarrow 0} (T_{k_\varepsilon \otimes \delta_0} Z_U(F \otimes \bar{F}), Z_U(f \otimes \bar{f}))$  exists for every  $f \in S$ . It is further seen from the formula

$$\begin{aligned} Z_U(f \otimes \bar{g}) &= \\ &= \frac{1}{2} Z_U \{ (f+g) \otimes \overline{(f+g)} - (f-g) \otimes \overline{(f-g)} + i(f+ig) \otimes \overline{(f+ig)} - i(f-ig) \otimes \overline{(f-ig)} \} \end{aligned}$$

that  $\lim_{\varepsilon \downarrow 0} (T_{k_\varepsilon \otimes \delta_0} Z_U(F \otimes \bar{F}), Z_U(f \otimes \bar{g}))$  exists for every  $f \in S, g \in S$ . It is not hard to prove (by using a continuous version of appendix 1, theorem 3.7) that  $\lim_{\varepsilon \downarrow 0} T_{k_\varepsilon \otimes \delta_0} Z_U(F \otimes \bar{F})$  exists in  $S^{2*}$ -sense.  $\square$

REMARK. We also could have introduced the class  $W^*$  with the aid of the functions  $h_A := \text{emb}(\frac{1}{2A} \chi_{[-A, A]})$  ( $A > 0$ ) instead of  $k_\varepsilon$  ( $\varepsilon > 0$ ). The resulting generalized Wiener class coincides with the one of 4.3.2. To see this, we note that lemma 4.3.3 can also be proved with  $h = h_A$  ( $A > 0$ ). Now the above theorem, with  $h_A$  ( $A > 0$ ) instead of  $k_\varepsilon$  ( $\varepsilon > 0$ ), can be proved, and then 4.2.5 shows that we obtain the same generalized Wiener class. We also have for  $F \in W^*$

$$\lim_{\varepsilon \downarrow 0} T_{k_\varepsilon \otimes \delta_0} Z_U(F \otimes \bar{F}) = \lim_{A \rightarrow \infty} T_{h_A \otimes \delta_0} Z_U(F \otimes \bar{F}).$$

4.3.5. The following theorem shows that the  $G$  of definition 4.3.2 is a function of the second variable only.

THEOREM. If  $F \in W^*$ , then there is exactly one  $\phi \in S^*$  such that

$$\lim_{\varepsilon \downarrow 0} T_{k_\varepsilon \otimes \delta_0} Z_U(F \otimes \bar{F}) = H \otimes \phi.$$

Here  $H$  denotes as usual  $\text{emb}(\bigcup_t 1)$ .

PROOF. Denote  $G := \lim_{\epsilon \rightarrow 0} T_{k_\epsilon \otimes \delta_0} Z_U(F \otimes \bar{F})$ . Using the relation  $T_a Z_U = Z_U T_{a/\sqrt{2}}^{(1)} T_{a/\sqrt{2}}^{(2)}$  ( $a \in \mathbb{R}$ ), we conclude by lemma 4.3.3, theorem 4.3.4, theorem 4.2.5 and 4.1.3(i) that

$$\begin{aligned} (T_{-a}^{(1)} G, Z_U(f \otimes \bar{f})) &= \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} |(T_{F^-} f_-)(\frac{x+a}{\sqrt{2}})|^2 k_\epsilon(x) dx = \\ &= \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} |(T_{F^-} f_-)(\frac{x}{\sqrt{2}})|^2 k_\epsilon(x) dx = (G, Z_U(f \otimes \bar{f})) \end{aligned}$$

for every  $a \in \mathbb{R}$ ,  $f \in S$ . It follows (cf. the second part of the proof of theorem 4.3.4) that  $(T_{-a}^{(1)} G, Z_U(f \otimes \bar{g})) = (G, Z_U(f \otimes \bar{g}))$  for every  $a \in \mathbb{R}$ ,  $f \in S$ ,  $g \in S$ . It is easy to see now that  $T_{-a}^{(1)} G = G$  for every  $a \in \mathbb{R}$ . By appendix 4, theorem 1.2 we conclude that  $G$  has the form  $H \otimes \phi$  with some  $\phi \in S^*$ .

It is almost trivial that  $\phi$  is uniquely determined by  $F$ .  $\square$

DEFINITION. If  $F \in W^*$ , then  $\phi_F$  (or, shortly,  $\phi$ ) denotes the unique element  $\psi$  of  $S^*$  with  $H \otimes \psi = \lim_{\epsilon \rightarrow 0} T_{k_\epsilon \otimes \delta_0} Z_U(F \otimes \bar{F})$ .

4.3.6. We next show that the convolution operators of appendix 2,3 map  $W^*$  into  $W^*$ .

THEOREM. Let  $F \in W^*$ ,  $g \in \mathcal{C}$ . Then  $T_g F \in W^*$ , and  $\phi_{T_g F} = T_h \phi_F$ , where  $h = 1/\sqrt{2} Z_{\sqrt{2}} T_{g^-}$  (cf. appendix 1, 1.8 (ii)).

PROOF. We have for  $\epsilon > 0$

$$T_{k_\epsilon \otimes \delta_0} Z_U(T_g F \otimes \overline{T_g F}) = T_{k_\epsilon \otimes \delta_0} Z_U T_g \otimes T_{\bar{g}} (F \otimes \bar{F}).$$

Using the relation  $Z_U T_g \otimes T_{\bar{g}} = T_{Z_U(g \otimes \bar{g})} Z_U$  and the 2-dimensional version of appendix 2, 5 (iv) (cf. also [J2], 5.11) we obtain

$$T_{k_\epsilon \otimes \delta_0} Z_U(T_g F \otimes \overline{T_g F}) = T_{Z_U(g \otimes \bar{g})} T_{k_\epsilon \otimes \delta_0} Z_U(F \otimes \bar{F}).$$

It easily follows that  $T_g F \in W^*$  and that

$$\lim_{\epsilon \rightarrow 0} T_{k_\epsilon \otimes \delta_0} Z_U(T_g F \otimes \overline{T_g F}) = T_{Z_U(g \otimes \bar{g})} (H \otimes \phi_F).$$

We complete the proof by showing that

$$T_{Z_U(g \otimes \bar{g})} (H \otimes \Psi) = H \otimes T_h \Psi$$

for  $\Psi \in S^*$ . Let  $\Psi \in S^*$ , and first assume that  $g \in S$ . Then (according to the 2-dimensional version of appendix 2, 9)

$$T_{Z_U(g \otimes \bar{g})} (H \otimes \Psi) = \text{emb}(\bigcup_{(a,b)} (H \otimes T_b \Psi, Z_U(g_- \otimes \bar{g}_-)).$$

We have for  $b \in \mathbb{R}$  by an application of [B1], (21.4)

$$(H \otimes T_b \Psi, Z_U(g_- \otimes \bar{g}_-)) = (T_b \Psi, \bigcup_Y \int_{-\infty}^{\infty} g_- \left(\frac{x+Y}{\sqrt{2}}\right) \bar{g}_- \left(\frac{x-Y}{\sqrt{2}}\right) dx).$$

It is easy to verify that

$$\bigcup_Y \int_{-\infty}^{\infty} g_- \left(\frac{x+Y}{\sqrt{2}}\right) \bar{g}_- \left(\frac{x-Y}{\sqrt{2}}\right) dx = \frac{1}{\sqrt{2}} Z_{\sqrt{2}} T_g g_-.$$

Hence the formula holds for smooth functions  $g$ , since

$$T_f \Psi = \text{emb}(\bigcup_b (T_b \Psi, f_-))$$

for  $f \in S$ .

The general case (with  $g \in C$ ) can be handled as follows. Denote  $h^{(\alpha)} := \frac{1}{\sqrt{2}} Z_{\sqrt{2}} T(g_\alpha)_- g_\alpha$  for  $\alpha > 0$  ( $g = N_\alpha g$ ). Then

$$T_{Z_U(g_\alpha \otimes \bar{g}_\alpha)} (H \otimes \Psi) = H \otimes T_h(\alpha) \Psi \quad (\alpha > 0).$$

Also,  $g_\alpha \xrightarrow{C} g$  ( $\alpha \downarrow 0$ ),  $h^{(\alpha)} \xrightarrow{C} h$  ( $\alpha \downarrow 0$ ) as is easily seen from appendix 2, 11(iii) and [J2], 5.9. Hence  $H \otimes T_h(\alpha) \Psi \xrightarrow{S^{2*}} H \otimes T_h \Psi$  ( $\alpha \downarrow 0$ ). Moreover,

$T_{Z_U(g_\alpha \otimes \bar{g}_\alpha)} (H \otimes \Psi) \xrightarrow{S^{2*}} T_{Z_U(g \otimes \bar{g})} (H \otimes \Psi)$  ( $\alpha \downarrow 0$ ). Consequently,

$$T_{Z_U(g \otimes \bar{g})} (H \otimes \Psi) = H \otimes T_h \Psi. \quad \square$$

REMARK. The following fact is an easy consequence of the above theorem.

If  $F \in W^*$ ,  $a \in \mathbb{R}$ , then  $T_a F \in W^*$  and  $\phi_{T_a F} = \phi_F$ .

4.3.7. THEOREM. If  $F \in W^*$ , then  $F\phi_F$  is of positive type (cf. appendix 4,2.1).

PROOF. We have for every  $f \in S$  (cf. the proof of theorem 4.3.6)

$$\begin{aligned} (\phi_F, \frac{1}{\sqrt{2}} Z_{\sqrt{2}} T_{f_-} f) &= (H \otimes \phi_F, Z_U(f \otimes \bar{f})) = \\ &= \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} |(T_{F_-} f_-)(\frac{x}{\sqrt{2}})|^2 k_{\epsilon}(x) dx \geq 0 \end{aligned}$$

by lemma 4.3.3. Hence  $(\phi_F, T_{h_-} h) \geq 0$  for every  $h \in S$ .

If  $g \in S$  and  $g(x) \geq 0$  ( $x \in \mathbb{R}$ ), then we can write  $g = g_1 \cdot \bar{g}_1$  with some  $g_1 \in S$ . Noting that  $F^*(g_1 \cdot \bar{g}_1) = T_{k_-} k$  (with  $k = F^* g_1$ ), we obtain

$$(F\phi_F, g) = (\phi_F, F^* g) = (\phi_F, T_{k_-} k) \geq 0.$$

Hence  $F\phi_F$  is of positive type. □

DEFINITION. If  $F \in W^*$ , then we call  $F\phi_F$  the *spectral density function* of  $F$ .

4.3.8. It is natural to ask whether the class  $W$  of 4.1.2 can be regarded as a subset of  $W^*$ . An affirmative answer will be given by means of the following lemma.

LEMMA. Let  $f \in W$ . There exists a  $C > 0$  such that

$$\int_{-\infty}^{\infty} \epsilon^{\frac{1}{2}} \exp(-\pi \epsilon t^2) |f(t+a)|^2 dt \leq C(1-K^{-1})^{-\frac{1}{2}} \exp(\pi(K-1)\epsilon a^2)$$

for every  $K > 1$ ,  $a \in \mathbb{R}$  and every  $\epsilon$  with  $0 < \epsilon \leq 1$ .

PROOF. Define  $\psi := \bigvee_{x \in \mathbb{R}} |f(x)|^2$ , and let  $K > 1$ ,  $a \in \mathbb{R}$ ,  $0 < \epsilon \leq 1$ . We find (by using the inequality  $(u-a)^2 \geq u^2(1-K^{-1}) + a^2(1-K)$  where  $u \in \mathbb{R}$ ,  $a \in \mathbb{R}$ )



$$\int_{-\infty}^{\infty} \varepsilon^{\frac{1}{2}} \exp(-\pi \varepsilon t^2) |f(t+a)|^2 dt = \int_{-\infty}^{\infty} \varepsilon^{\frac{1}{2}} \exp(-\pi \varepsilon (u-a)^2) \psi(u) du \leq$$

$$\leq (1-K^{-1})^{-\frac{1}{2}} \exp(\pi(K-1)\varepsilon a^2) \sup_{0 < \delta \leq 1} (\delta^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp(-\pi \delta u^2) \psi(u) du).$$

This proves the inequality with  $C = \sup_{0 < \delta \leq 1} (\delta^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp(-\pi \delta u^2) \psi(u) du)$ .  $\square$

4.3.9. THEOREM. Let  $f \in W$ . Then  $f \in S^+$ ,  $\text{emb}(f) \in W^*$ ,  $\varphi_f \in S^+$  (cf. 4.1.2),  $\Phi \text{emb}(f) = 2^{-\frac{1}{2}} Z_{\sqrt{2}} \text{emb}(\varphi_f)$ ,  $\sigma_f \in S^+$  (cf. 4.1.4), and  $F\Phi \text{emb}(f) = 2^{-\frac{1}{2}} Z_{\pi\sqrt{2}} \left( \frac{d}{dx} (\text{emb}(\sigma_f)) \right)$ .

PROOF. It is easily seen from lemma 4.3.8 and appendix 1, 1.5 that  $f \in S^+$ . Put  $F := \text{emb}(f)$ . We want to show that

$$(*) \quad T_{k_\varepsilon \otimes \delta_0} Z_U (F \otimes \bar{F}) = \text{emb} \left( \int_{(x,y)} \int_{-\infty}^{\infty} k_\varepsilon(t) f\left(\frac{t+x+y}{\sqrt{2}}\right) \overline{f\left(\frac{t+x-y}{\sqrt{2}}\right)} dt \right)$$

for every  $\varepsilon > 0$  (note that, by lemma 4.3.8, the right hand side is in  $S^{2*}$  indeed). Therefore let  $l \in S^{2+}$ ,  $f \in S$ ,  $g \in S$ ,  $\varepsilon > 0$ . We have (with  $L := \text{emb}(l)$ ) by appendix 1, 4.11 and 4.16 and 1.11(i)

$$\begin{aligned} (T_{k_\varepsilon \otimes \delta_0} L, f \otimes g) &= (L, (T_{k_\varepsilon} f) \otimes g) = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l(t,y) \left( \int_{-\infty}^{\infty} k_\varepsilon(x-t) \overline{f(x)} dx \right) \overline{g(y)} dt dy. \end{aligned}$$

Hence, by Fubini's theorem,

$$(T_{k_\varepsilon \otimes \delta_0} L, f \otimes g) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} k_\varepsilon(x-t) l(t,y) dt \right) \overline{f(x)} \overline{g(y)} dx dy$$

$$= (\text{emb}(\bigcup_{(x,y)})) \int_{-\infty}^{\infty} k_{\varepsilon}(t-x) l(t,y) dt, f \otimes g),$$

as  $k_{\varepsilon}$  is even (note that  $\bigcup_{(x,y)} \int_{-\infty}^{\infty} k_{\varepsilon}(t-x) l(t,y) dt \in S^{2+}$ ). If we take  $l := \bigcup_{(x,y)} f(\frac{x+y}{\sqrt{2}}) f(\frac{x-y}{\sqrt{2}})$  in the above, the required result easily follows.

Now denote  $\tilde{\varphi} = \bigcup_{a \in \mathbb{R}} \varphi_f(a/\sqrt{2})$ . It is easily seen from lemma 4.3.8 that  $\varphi_f$ , and hence  $\tilde{\varphi}$ , is an element of  $S^+$ . It further follows from lemma 4.3.8, 4.2.5 and (\*) that

$$T_{k_{\varepsilon} \otimes \delta_0} Z_U(F \otimes \bar{F}) \xrightarrow{S^{2+}} H \otimes \text{emb}(\tilde{\varphi}).$$

Hence  $F \in W^*$ , and  $\Phi_F = \text{emb}(\tilde{\varphi}) = 2^{-1/4} Z_{\sqrt{2}} \text{emb}(\varphi_f)$  by the uniqueness part of theorem 4.3.5.

We easily infer from 4.1.4 that  $\sigma_f \in S^+$ . It follows from the above, 4.1.4, appendix 4, lemma 2.4 and some calculations that  $F\Phi_{\text{emb}(f)} = 2^{-1/2} Z_{\pi/\sqrt{2}} \frac{d}{dx}(\text{emb}(\sigma_f))$ .  $\square$

4.3.10. EXAMPLE. Let  $F \in S^*$ , and assume that  $F$  is periodic with period 1 (i.e.  $T_1 F = F$ ). According to [B1], 27.6.3 we can develop  $F$  in a generalized Fourier series:  $F = \sum_{n=-\infty}^{\infty} d_n \zeta_n$ , where  $\zeta_{\lambda} := \text{emb}(\bigcup_x e^{-2\pi i \lambda x})$  for  $\lambda \in \mathbb{R}$ , and  $d_n = O(e^{\varepsilon n^2})$  ( $n \in \mathbb{Z}$ ) for every  $\varepsilon > 0$ .

If we calculate  $T_{k_{\varepsilon} \otimes \delta_0} Z_U(F \otimes \bar{F})$ , then we find

$$T_{k_{\varepsilon} \otimes \delta_0} Z_U(F \otimes \bar{F}) = \sum_{n,m=-\infty}^{\infty} d_n \bar{d}_m \exp(-\pi \varepsilon^{-1} (n-m)^2) Z_U(\zeta_n \otimes \bar{\zeta}_m),$$

and it is not hard to show that this tends to

$$H \otimes \sum_{n=-\infty}^{\infty} |d_n|^2 2^{-1/4} Z_{\sqrt{2}} \zeta_n$$

in  $S^{2+}$ -sense if  $\varepsilon \downarrow 0$ . This shows that  $F \in W^*$ , and that

$\Phi_F = \sum_{n=-\infty}^{\infty} |d_n|^2 2^{-1/4} Z_{\sqrt{2}} \zeta_n$ . We find for the spectral density function  $F\Phi_F$  of  $F$

$$F_{\Phi} = \sum_{n=-\infty}^{\infty} |d_n|^2 2^{-\frac{1}{2}n} z_{1/\sqrt{2}} \delta_n.$$

This is indeed a generalized function of positive type.

4.3.11. We next introduce the class  $W_1^*$  ( $S^*$ -generalization of the class  $W_1$  of 4.1.6).

DEFINITION. Let  $F \in S^*$ . Then  $F$  is said to belong to the class  $W_1^*$  if there is a  $G \in S^{2*}$  such that

$$T_{h_{A,B} \otimes \delta_0} Z_U(F \otimes \bar{F}) \xrightarrow{S^{2*}} G \quad (A \rightarrow -\infty, B \rightarrow \infty).$$

Here  $h_{A,B} := \text{emb}(\frac{1}{B-A} \chi_{[A,B]})$  for  $A < 0, B > 0$ .

It is less convenient here to work with limits that involve the functions  $k_\varepsilon$  ( $\varepsilon > 0$ ) as was done in 4.3.2.

It is obvious from theorem 4.3.5 that the  $G$  of the above theorem has the form  $H \otimes \phi$  with  $\phi \in S^*$ .

4.3.12. We are going to show that  $W_1^* \subset W^*$  and that  $F \in W_1^*$  if and only if  $T_F f \in W_1$  for every  $f \in S$ . We first prove a lemma.

LEMMA. Let  $h: \mathbb{R} \rightarrow \mathbb{C}$  be measurable, and assume that

$$\lim_{A \rightarrow -\infty, B \rightarrow \infty} \frac{1}{B-A} \int_A^B h(\xi+x) \overline{h(x)} dx =: \phi(\xi)$$

exists for almost every  $\xi \in \mathbb{R}$  including  $\xi = 0$ . Let  $f \in S$ . Then

$$\frac{1}{B-A} \int_A^B (f * h)(\xi + x) \overline{(f * h)(x)} dx \rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(t-s) f(t-\xi) \overline{f(s)} dt ds$$

( $A \rightarrow -\infty, B \rightarrow \infty$ ) locally uniformly in  $\xi \in \mathbb{R}$  (here  $*$  denotes ordinary convolution:  $f * h = \int_x^\psi f(t-x) h(t) dt$ ).

PROOF. Let  $A < 0, B > 0, \xi \in \mathbb{R}$ . We have

$$\begin{aligned}
& \frac{1}{B-A} \int_A^B (f * h)(\xi + x) \overline{(f * h)(x)} dx = \\
& = \frac{1}{B-A} \int_A^B \int_{-\infty}^{\infty} f(t - (\xi + x)) h(t) dt \int_{-\infty}^{\infty} \overline{f(s - x) h(s)} ds dx = \\
& = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{1}{B-A} \int_A^B h(x+t) \overline{h(x+s)} dx \right) f(t - \xi) \overline{f(s)} dt ds.
\end{aligned}$$

Here application of Fubini's theorem can be justified by noting that

$$(1) \quad \frac{1}{B-A} \int_A^B |h(x+t) h(x+s)| dx \leq \left( \frac{1}{B-A} \int_A^B |h(x+t)|^2 dx \right)^{\frac{1}{2}} \left( \frac{1}{B-A} \int_{-A}^B |h(x+s)|^2 dx \right)^{\frac{1}{2}}$$

and

$$\begin{aligned}
(2) \quad \frac{1}{B-A} \int_A^B |h(x+t)|^2 dx & \leq \left(1 + \frac{2|t|}{B-A}\right) \frac{1}{B-A+2|t|} \int_{A-|t|}^{B+|t|} |h(x)|^2 dx \leq \\
& \leq \left(1 + \frac{2|t|}{B-A}\right) M
\end{aligned}$$

for some  $M > 0$  independent of  $t \in \mathbb{R}$ ,  $A < 0$ ,  $B > 0$ .

It is easy to prove that

$$\lim_{A \rightarrow -\infty, B \rightarrow \infty} \frac{1}{B-A} \int_A^B h(x+t) \overline{h(x+s)} dx = \varphi(t-s)$$

for all  $t \in \mathbb{R}$ ,  $s \in \mathbb{R}$  for which  $\varphi(t-s)$  is defined. Note therefore that

$$\frac{1}{B-A} \int_A^B h(x+t) \overline{h(x+s)} dx = \frac{1}{B-A} \int_{A+s}^{B+s} h(x+t-s) \overline{h(x)} dx \rightarrow \varphi(t-s)$$

if  $\varphi(t-s)$  is defined and  $A \rightarrow -\infty$ ,  $B \rightarrow \infty$ . We infer from the estimations in (1) and (2) and Lebesgue's theorem on dominated convergence that

$$\frac{1}{B-A} \int_A^B (f * h)(\xi + x) \overline{(f * h)(x)} dx \rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(t-s) f(t-\xi) \overline{f(s)} dt ds$$

( $A \rightarrow -\infty, B \rightarrow \infty$ ) locally uniformly in  $\xi \in \mathbb{R}$ . □

COROLLARY. In a similar way we can prove that for  $h \in W, f \in S$

$$\frac{1}{2T} \int_{-T}^T (f * h)(\xi + x) \overline{(f * h)(x)} dx \rightarrow \varphi_{f * h}(\xi)$$

( $T \rightarrow \infty$ ) locally uniformly in  $\xi \in \mathbb{R}$ .

4.3.13. THEOREM. Let  $F \in S^*$ . If  $F \in W_1^*$ , then  $T_F f \in W \cap W_1$  for  $f \in S$ . If  $T_F f \in W_1$  for every  $f \in S$ , then  $F \in W_1^*$ .

PROOF. Let  $F \in W_1^*$ ,  $f \in S$ . As in the proof of theorem 4.3.4 we see that

$$\lim_{A \rightarrow -\infty, B \rightarrow \infty} \frac{1}{B-A} \int_A^B (T_F f)(x+a) \overline{(T_F f)(x)} dx$$

exists for every  $a \in \mathbb{R}$ . Hence  $T_F f \in W$ . We show that the above limit depends continuously on  $a \in \mathbb{R}$ . Therefore write  $f = f_1 * f_2$  with some  $f_1 \in S, f_2 \in S$ . Then  $T_F f = f_2 * T_F f_1$  (this is easy to prove if  $F \in \text{emb}(S)$ ; the general case can be handled by using [J2], lemma 5.2). Now apply lemma 4.3.12 with  $T_F f_1$  in the role of  $h$  and  $f_2$  in the role of  $f$ . It follows from definition 4.1.6 that  $T_F f \in W_1$ .

Next assume that  $T_F f \in W_1$  for every  $f \in S$ . If  $f_1 \in S, f_2 \in S$ , then

$$\lim_{A \rightarrow -\infty, B \rightarrow \infty} \frac{1}{B-A} \int_A^B (f_2 * T_F f_1)(\xi + x) \overline{(f_2 * T_F f_1)(x)} dx$$

exists locally uniformly in  $\xi \in \mathbb{R}$  according to lemma 4.3.12. Hence, if  $f \in S$ ,

$$\lim_{A \rightarrow -\infty, B \rightarrow \infty} \frac{1}{B-A} \int_A^B (T_F f)(\xi + x) \overline{(T_F f)(x)} dx$$

exists for every  $\xi \in \mathbb{R}$  (cf. the first part of the proof). We can prove now (compare the second part of the proof of theorem 4.3.4) that

$$\lim_{A \rightarrow -\infty, B \rightarrow \infty} T_{h_{A,B} \otimes \delta_0} Z_U(F \otimes \bar{F}) \text{ exists in } S^{2*}\text{-sense.} \quad \square$$

REMARK. We conclude from the above theorem that  $W_1^* \subset W^*$ ;  $W_1^*$  is a proper subset of  $W^*$  since  $\text{emb}(\chi_{[0,\infty)}) \in W^* \setminus W_1^*$ . Furthermore we note that  $h \in S^+$  and  $\text{emb}(h) \in W_1^*$  if  $h \in W_1$ .

4.3.14. We next show that the class  $W^*$  is a measurable subset of  $S^*$  (cf. appendix 1, 5.1). We need the following lemma.

LEMMA. If  $h \in W$ ,  $f \in S$ , then the set of functions

$$(*) \quad \bigcup_{\xi \in \mathbb{R}} \frac{1}{2T} \int_{-T}^T (f * h)(\xi + x) \overline{(f * h)(x)} dx$$

with  $T \geq 1$  is equicontinuous in every interval  $[a,b]$  with  $-\infty < a < b < \infty$ .

PROOF. Denote for  $T \geq 1$  the function in (\*) by  $g_T$ , and let  $-\infty < a < b < \infty$ . We know from 4.3.12, corollary that  $g(\xi) := \lim_{T \rightarrow \infty} g_T(\xi)$  exists uniformly in  $\xi \in [a,b]$ . Also  $g$  is a continuous function. Let  $\varepsilon > 0$ , and let  $\delta_1 > 0$  be such that  $|g(\xi) - g(\eta)| < \frac{\varepsilon}{3}$  if  $\xi \in [a,b]$ ,  $\eta \in [a,b]$ ,  $|\xi - \eta| < \delta_1$ . Also, let  $T_0 \geq 1$  be such that  $|g_T(\xi) - g(\xi)| < \varepsilon/3$  if  $T > T_0$ ,  $\xi \in [a,b]$ . It is not hard to see that the set of functions  $g_T$  with  $1 \leq T \leq T_0$  is equicontinuous in  $[a,b]$ . We can therefore find a  $\delta_2 > 0$  such that  $|g_T(\xi) - g_T(\eta)| < \varepsilon$  if  $1 \leq T \leq T_0$ ,  $\xi \in [a,b]$ ,  $\eta \in [a,b]$ ,  $|\xi - \eta| < \delta_2$ . If we put  $\delta := \min(\delta_1, \delta_2)$ , then we have  $|g_T(\xi) - g_T(\eta)| < \varepsilon$  if  $T \geq 1$ ,  $\xi \in [a,b]$ ,  $\eta \in [a,b]$ ,  $|\xi - \eta| < \delta$ . □

4.3.15. THEOREM.  $W^*$  is a measurable subset of  $S^*$ .

PROOF. We have  $F \in W^*$  if and only if  $T_F g_n \in W$  for every  $n \in \mathbb{N}$ . Here  $g_n := \bigcup_x n^{\frac{1}{2}} \exp(-\pi n x^2)$  ( $n \in \mathbb{N}$ ). For, if  $f \in S$ , we can write  $f = g_n * f_1$  with some  $f_1 \in S$ ,  $n \in \mathbb{N}$ , and  $T_F f = f_1 * T_F g_n \in W$  according to 4.3.12, corollary (cf. also the proof of theorem 4.3.13).

It is easily seen from lemma 4.3.14 (as  $T_F g_n = T_F(g_{2n} * g_{2n}) = g_{2n} * T_F g_{2n}$ ) that for  $F \in W^*$  the following conditions are satisfied:

- (1) the set of functions

$$\forall_{\xi \in \mathbb{R}} \frac{1}{2T} \int_{-T}^T (\mathbb{T}_F g_n)(\xi + x) \overline{(\mathbb{T}_F g_n)(x)} dx$$

with  $T \geq 1$ ,  $T \in \mathbb{Q}$  is locally equicontinuous for every  $n \in \mathbb{N}$ ,

(2) the limit

$$\lim_{T \rightarrow \infty, T \in \mathbb{Q}} \frac{1}{2T} \int_{-T}^T (\mathbb{T}_F g_n)(\xi + x) \overline{(\mathbb{T}_F g_n)(x)} dx$$

exists for every  $\xi \in \mathbb{Q}$  and every  $n \in \mathbb{N}$ .

On the other hand, if the conditions (1) and (2) are satisfied, then  $\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T (\mathbb{T}_F g_n)(\xi + x) \overline{(\mathbb{T}_F g_n)(x)} dx$  exists for every  $\xi \in \mathbb{R}$ ,  $n \in \mathbb{N}$ , i.e.  $\mathbb{T}_F g_n \in W$  for every  $n \in \mathbb{N}$ . As in the examples of appendix 1, 5.5, we can obtain the class of all  $F \in S^*$  satisfying the conditions (1) and (2) as a finite chain of unions and intersections involving a countable number of measurable sets. Hence  $W^*$  is a measurable subset of  $S^*$ .  $\square$

REMARK. In a similar way as in the proof of the above theorem we can show that  $W_1^*$  (cf. 4.3.11) is a measurable subset of  $S^*$ .

#### 4.4. GENERALIZED HARMONIC ANALYSIS AND THE WIGNER DISTRIBUTION; APPLICATIONS TO GENERALIZED STOCHASTIC PROCESSES

4.4.1. This section relates the Wigner distribution to generalized harmonic analysis. We shall see that the spectral density functions of the elements of the Wiener class can be obtained as limits of certain averages of their Wigner distributions. We further elaborate the relation between averaged Wigner distributions and the spectra occurring in engineering. Finally, the theorems of the previous sections are applied to strict sense time stationary and ergodic processes.

4.4.2. Let  $F \in S^*$ . In section 4.3 we considered the expressions  $T_{k_\varepsilon \otimes \delta_0} Z_U(F \otimes \bar{F})$  with  $\varepsilon > 0$ , and  $F$  was said to belong to the class  $W^*$  whenever  $\lim_{\varepsilon \downarrow 0} T_{k_\varepsilon \otimes \delta_0} Z_U(F \otimes \bar{F})$  exists in  $S^{2*}$ -sense. We modify this limit procedure slightly so as to involve non-negative averages of the Wigner distribution of  $F$ . The following lemma deals with this modification.

LEMMA. Let  $F \in S^*$ . Then  $\lim_{\varepsilon \downarrow 0, \delta \downarrow 0} \delta^{-\frac{1}{2}} T_{k\varepsilon}^{(1)} M_{k\delta}^{(2)} Z_U(F \otimes \bar{F})$  exists in  $S^{2*}$ -sense if and only if  $F \in W^*$  ( $M_{k\delta}$  stands for the multiplication operator  $\bigcup_{G \in S^*} k_\delta \cdot G$ ; cf. appendix 2, 6). If  $F \in W^*$ , then

$$\lim_{\varepsilon \downarrow 0, \delta \downarrow 0} \delta^{-\frac{1}{2}} T_{k\varepsilon}^{(1)} M_{k\delta}^{(2)} Z_U(F \otimes \bar{F}) = H \otimes \phi_F \quad (\text{cf. 4.3.5}).$$

PROOF. Assume that  $F \in W^*$ , and let  $f \in S^2$ . We have for  $\varepsilon > 0, \delta > 0$

$$(\delta^{-\frac{1}{2}} T_{k\varepsilon}^{(1)} M_{k\delta}^{(2)} Z_U(F \otimes \bar{F}), f) = (T_{k\varepsilon \otimes \delta 0} Z_U(F \otimes \bar{F}), \delta^{-\frac{1}{2}} M_{k\delta}^{(2)} f).$$

Since  $T_{k\varepsilon \otimes \delta 0} Z_U(F \otimes \bar{F}) \xrightarrow{S^{2*}} H \otimes \phi_F$  ( $\varepsilon \downarrow 0$ ),  $\delta^{-\frac{1}{2}} M_{k\delta}^{(2)} f \xrightarrow{S^2} f$  ( $\delta \downarrow 0$ ), we conclude from a 2-dimensional, continuous version of appendix 1, 1.16 that

$$\lim_{\varepsilon \downarrow 0, \delta \downarrow 0} (\delta^{-\frac{1}{2}} T_{k\varepsilon}^{(1)} M_{k\delta}^{(2)} Z_U(F \otimes \bar{F}), f) = (H \otimes \phi_F, f).$$

Hence  $\lim_{\varepsilon \downarrow 0, \delta \downarrow 0} \delta^{-\frac{1}{2}} T_{k\varepsilon}^{(1)} M_{k\delta}^{(2)} Z_U(F \otimes \bar{F})$  exists in  $S^{2*}$ -sense, and equals  $H \otimes \phi_F$  since  $f \in S^2$  is arbitrarily chosen.

Now assume that  $\lim_{\varepsilon \downarrow 0, \delta \downarrow 0} \delta^{-\frac{1}{2}} T_{k\varepsilon}^{(1)} M_{k\delta}^{(2)} Z_U(F \otimes \bar{F})$  exists in  $S^{2*}$ -sense, and let  $f \in S^2$ . It is easy to see that for small values of  $\delta > 0$  there is exactly one  $f_\delta \in S^2$  such that  $f = \delta^{-\frac{1}{2}} M_{k\delta}^{(2)} f_\delta$ ; we have  $f_\delta \xrightarrow{S^2} f$  if  $\delta \downarrow 0$ .

Now

$$(T_{k\varepsilon \otimes \delta 0} Z_U(F \otimes \bar{F}), f) = (\delta^{-\frac{1}{2}} T_{k\varepsilon}^{(1)} M_{k\delta}^{(2)} Z_U(F \otimes \bar{F}), f_\delta),$$

and  $\lim_{\varepsilon \downarrow 0, \delta \downarrow 0} (\delta^{-\frac{1}{2}} T_{k\varepsilon}^{(1)} M_{k\delta}^{(2)} Z_U(F \otimes \bar{F}), f_\delta)$  exists (again by a 2-dimensional, continuous version of appendix 1, 1.16). Hence  $\lim_{\varepsilon \downarrow 0} T_{k\varepsilon \otimes \delta 0} Z_U(F \otimes \bar{F})$  exists in  $S^{2*}$ -sense since  $f \in S^2$  is arbitrarily chosen.  $\square$

REMARK. Let  $F \in S^{2*}$ . It is easily seen from the proof of the above lemma that  $F \in W^*$  if and only if the following condition is satisfied. There exists a function  $p: (0, \infty) \rightarrow (0, \infty)$  with  $p(\varepsilon) \rightarrow 0$  ( $\varepsilon \downarrow 0$ ) such that  $\lim_{\varepsilon \downarrow 0} (p(\varepsilon))^{-\frac{1}{2}} T_{k\varepsilon}^{(1)} M_{k p(\varepsilon)}^{(2)} Z_U(F \otimes \bar{F})$  exists in  $S^{2*}$ -sense.

4.4.3. In the remainder of this section  $V(F)$  denotes for  $F \in S^*$  the Wigner distribution of  $F$ .

THEOREM. Let  $F \in S^*$ . The following conditions are equivalent



- (i)  $F \in W^*$ ,
- (ii)  $\lim_{\epsilon \downarrow 0, \delta \downarrow 0} T_{k_\epsilon \otimes k_{\delta-1}} V(F)$  exists in  $S^{2*}$ -sense,
- (iii)  $\lim_{\epsilon \downarrow 0} \text{emb} \left( \left| \left( S_{k_\epsilon}^1 F \right) \left( \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right) \right|^2 \right)$  exists in  $S^{2*}$ -sense.

Furthermore, if  $F \in W^*$ , the limits in (ii) and (iii) equal  $H \otimes L$ , where  $L$  is the spectral density function of  $F$  (cf. 4.3.7).

PROOF. Note that  $T_{k_\epsilon \otimes k_{\delta-1}} V(F) = \delta^{-1/2} F^{(2)} T_{k_\epsilon}^{(1)} M_{k_\delta}^{(2)} Z_U(F \otimes \bar{F})$  for  $\epsilon > 0$ ,  $\delta > 0$ . We further have  $k_\epsilon \otimes k_{\delta-1} = \epsilon^{-1/2} F^{(2)} Z_U(k_\epsilon \otimes k_\delta)$ , therefore it follows from appendix 3, 2.4 that  $T_{k_\epsilon \otimes k_{\delta-1}} V(F) = \text{emb} \left( \left| \left( S_{k_\epsilon}^1 F \right) \left( \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right) \right|^2 \right)$ . The theorem easily follows from 4.4.2 and 4.4.2, remark.  $\square$

4.4.4. The reason for studying  $\lim_{\epsilon \downarrow 0, \delta \downarrow 0} T_{k_\epsilon \otimes k_{\delta-1}} V(F)$  for  $F \in W^*$  (rather than  $\lim_{\epsilon \downarrow 0} T_{k_\epsilon \otimes \delta} V(F)$ ) is the fact that  $T_{k_\epsilon \otimes k_{\delta-1}} V(F)$  is of positive type if  $\epsilon \leq \delta$  (cf. also [B2], theorem 4.2).

THEOREM. Let  $F \in S^*$ , and let  $\epsilon > 0$ ,  $\delta > 0$ . If  $\epsilon \leq \delta$ , then  $T_{k_\epsilon \otimes k_{\delta-1}} V(F)$  is the embedding of an analytic function  $g$  with  $g(x,y) \geq 0$  ( $x \in \mathbb{R}, y \in \mathbb{R}$ ).

PROOF. It follows from the 2-dimensional version of appendix 2, theorem 9 that

$$T_{k_\epsilon \otimes k_{\delta-1}} V(F) = \text{emb} \left( \left| \left( T_a^{(1)} T_b^{(2)} V(F), k_\epsilon \otimes k_{\delta-1} \right) \right| \right),$$

and the embedded function at the right hand side is analytic (cf. [J2], 3.3 and 5.11).

For  $\epsilon = \delta$ , positivity is easily seen from the proof of theorem 4.4.3. The general case is handled as follows. We have  $k_\delta = k_\gamma * k_{\epsilon-1}$  with  $\gamma = \frac{1}{\delta-\epsilon}$ . Now

$$T_{k_\epsilon \otimes k_{\delta-1}} V(F) = T_{k_\gamma}^{(2)} \left( T_{k_\epsilon \otimes k_{\epsilon-1}} V(F) \right).$$

Since  $k_\gamma$  is positive everywhere on  $\mathbb{R}$ , the theorem follows.  $\square$

4.4.5. In order to indicate relations between weighted Wigner distributions and power spectra (as occurring in applied signal analysis), let  $F = \text{emb}(f)$  with  $f \in W$  (cf. 4.3.9). We have for  $\epsilon > 0$ ,  $\delta > 0$

$$\delta^{-\frac{1}{2}} \mathbb{T}_{k_\epsilon}^{(1)} \mathbb{M}_{k_\delta}^{(2)} Z_U(F \otimes \bar{F}) = \text{emb}(G_{\epsilon, \delta}),$$

where

$$G_{\epsilon, \delta} := \mathbb{Y}_{(t, y)} \epsilon^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp(-\pi \delta y^2 - \pi \epsilon (x-t)^2) f\left(\frac{x+y}{\sqrt{2}}\right) \overline{f\left(\frac{x-y}{\sqrt{2}}\right)} dx$$

(cf. also the proof of 4.3.9).

If  $\delta > 0$  is fixed, then we have for  $t \in \mathbb{R}, y \in \mathbb{R}$

$$(1) \quad \lim_{\epsilon \downarrow 0} G_{\epsilon, \delta}(t, y) = \exp(-\pi \delta y^2) \varphi_f(y/\sqrt{2})$$

(cf. 4.1.2 and 4.2.5, and also 4.3.9). Fourier transformation with respect to  $y$  in (1) gives (lemma 4.3.8 legitimates interchanging of  $\lim$  and the Fourier integral)

$$(2) \quad \lim_{\epsilon \downarrow 0} (F^{(2)} G_{\epsilon, \delta})(t, \lambda) = \int_{-\infty}^{\infty} \exp(-\pi \delta y^2 - 2\pi i \lambda y) \varphi_f(y/\sqrt{2}) dy.$$

By theorem 4.4.3 and analyticity of  $F^{(2)} G_{\epsilon, \delta}$  we have

$$(3) \quad (F^{(2)} G_{\epsilon, \delta})(t, \lambda) = \langle \mathbb{T}_t^{(1)} \mathbb{T}_\lambda^{(2)} \mathbb{V}(\text{emb}(f)), k_\epsilon \otimes k_{\delta-1} \rangle \geq 0$$

if  $\epsilon \leq \delta$ . We can regard the right hand side of (2) as an estimation of the spectral density function of  $f$  at the point  $\lambda$ : we have

$$\text{emb} \left( \mathbb{Y}_\lambda \int_{-\infty}^{\infty} \exp(-\pi \delta y^2 - 2\pi i \lambda y) \varphi_f(y/\sqrt{2}) dy \right) \xrightarrow{S^*} F \Phi \text{emb}(f)$$

if  $\delta \downarrow 0$  (cf. 4.3.7).

For  $\epsilon = \delta$  we have in (3)

$$(4) \quad \begin{aligned} (F^{(2)} G_{\epsilon, \epsilon})(t, \lambda) &= \left| \epsilon^{-\frac{1}{4}} (S_{k_\epsilon}^1(\text{emb}(f))) \left( \frac{t}{\sqrt{2}}, \frac{\lambda}{\sqrt{2}} \right) \right|^2 = \\ &= \left| \int_{-\infty}^{\infty} \epsilon^{\frac{1}{4}} \exp(-\pi \epsilon (u - t/\sqrt{2})^2) e^{-2\pi i u \lambda / \sqrt{2}} f(u) du \right|^2 \end{aligned}$$

if  $\lambda \in \mathbb{R}$ ,  $t \in \mathbb{R}$  (cf. the proof of theorem 4.4.3 and appendix 3, 2.3). Compare (4) with the formula (54) for  $S_x(f,t;w)$  in [Ma] (cf. also 3.4.6).

In engineering literature (cf. [BS], [BT], [RD], [Zi]) the expression

$$(5) \quad \frac{1}{2T} \left| \int_{-T}^T e^{-2\pi i \lambda u} f(u) du \right|^2$$

is quite often used for estimating, what is usually called, the power spectrum of  $f$  at  $\lambda$ . (In fact the word "spectrum" is not correct: if (5) estimates something, then it estimates the spectral density function of  $f$  at  $\lambda$ ; cf. 4.1.4 and 4.3.7.) The expressions in (4) and (5) show great resemblance, but there is a difference. The first expression contains the time variable  $t$ , whereas the latter one does not. This makes (4) more appropriate in case we drop the restriction  $f \in W$  (so that signals  $f$  representing e.g. a piece of music can be treated as well).

4.4.6. We now apply the results of the previous sections to generalized stochastic processes. Let  $(\Omega, \Lambda, P)$  be a probability space.

THEOREM. Let  $\underline{X} \in S_{\Omega, 2}^*$  be strict sense time stationary, and let  $P_{\underline{X}}^*$  be the probability measure on  $(S^*, \Lambda^*)$  associated with  $\underline{X}$  (cf. 1.1.19). Then  $P_{\underline{X}}^*(W_1^*) = 1$  (cf. 4.3.11). If, in addition,  $\underline{X}$  is ergodic, then we have for almost every  $F \in S^*$

$$\frac{1}{2\tau} \int_{-\tau}^{\tau} \overline{(T_F f)(t)} (T_F g)(t) dt \rightarrow (R_{\underline{X}}, f_- \otimes \bar{g}_-) \quad (\tau \rightarrow \infty)$$

for every  $f \in S$ ,  $g \in S$ . The above limit is assumed in  $L_1(S^*, \Lambda^*, P_{\underline{X}}^*)$ -sense for every  $f \in S$ ,  $g \in S$ .

PROOF. It follows from 4.3.15, remark that  $W_1^* \in \Lambda^*$ . According to 4.3.13 we have to show that  $T_F g \in W_1$  ( $g \in S$ ) for almost every  $F \in S^*$ .

Let  $n \in \mathbb{N}$ , and let  $g_n := \bigvee_x \exp(-\pi n x^2)$ . The function  $\bigvee_{F \in S^*} (g_n, F)$  is measurable and square integrable over  $S^*$ . Since  $(T_{-t})_{t \in \mathbb{R}}$  is a measure preserving group of transformations of  $(S^*, \Lambda^*)$  with respect to  $P_{\underline{X}}^*$ , we conclude by [Ho], Ch. IV, §14, Satz 14.2 that the function  $\bigvee_t (g_n, T_{-t} F) = T_F g_n \in W_1$  for almost every  $F \in S^*$ .

Since  $F \in W_1^*$  if and only if  $T_F g_n \in W_1$  for every  $n \in \mathbb{N}$  (cf. the first part of the proof of theorem 4.3.15 and 4.3.15, remark) we easily conclude

that  $P_{\underline{X}}^*(W_1^*) = 1$ .

Now assume, in addition, that  $\underline{X}$  is ergodic. We conclude from the above that for almost every  $F \in S^*$

$$(1) \quad \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |(T_F f)(t)|^2 dt$$

exists for every  $f \in S$ . If  $f \in S$ , then we get, by applying [D], Ch. XI, section 2, theorem 2.1 to  $\bigcup_F (F, f_-)$ , for almost every  $F \in S^*$

$$(2) \quad \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |(T_{\underline{F}} f, f_-)|^2 dt = E(\bigcup_F |(F, f_-)|^2) = (R_{\underline{X}}, f_- \otimes \bar{f}_-).$$

By combining (1) and (2) we get for almost every  $F \in S^*$  (note that  $(T_F f)(t) = \overline{(T_{\underline{F}} f, f_-)}$  for  $f \in S, t \in \mathbb{R}$ )

$$\lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |(T_F f)(t)|^2 dt = (R_{\underline{X}}, f_- \otimes \bar{f}_-)$$

for every  $f \in S$ . For fixed  $f \in S$  this limit is achieved in  $L_1(S^*, \Lambda^*, P_{\underline{X}}^*)$ -sense (cf. the remark at the end of the proof of [D], Ch. XI, section 2, theorem 2.1). It is not hard to complete the proof (cf. 4.2.5 and 4.3.4).  $\square$

REMARK 1. Since  $W_1^* \subset W^*$  (cf. 4.3.13, remark), we have  $P_{\underline{X}}^*(W^*) = 1$  if  $\underline{X} \in S_{\Omega, 2}^*$  is strict sense time stationary.

REMARK 2. Let  $\underline{X} \in S_{\Omega, 2}^*$  be strict sense time stationary and ergodic. Then it can be proved that for almost every  $F \in S^*$

$$\int_{-\infty}^{\infty} k_{\epsilon}(t) \overline{(T_F f)(t)} (T_F g)(t) dt \rightarrow (R_{\underline{X}}, f_- \otimes \bar{g}_-) \quad (\epsilon \downarrow 0)$$

for every  $f \in S, g \in S$ , and the above limit is achieved in  $L_1(S^*, \Lambda^*, P_{\underline{X}}^*)$ -sense for every  $f \in S, g \in S$ .

4.4.7. THEOREM. Assume that  $\underline{X} \in S_{\Omega, 2}^*$  is strict sense time stationary and ergodic. Then  $T_{h_{\tau} \otimes \delta_0} Z_U(\underline{X} \otimes \bar{\underline{X}}) \rightarrow Z_{U R_{\underline{X}}}(S_{\Omega, 1}^{2*})$  if  $\tau \rightarrow \infty$ .

PROOF. It follows from an application of appendix 1, 3.5, remark (with  $R = L_1(\Omega)$ ) that it suffices to show that for  $f \in S$ ,  $g \in S$

$$(T_{h_\tau \otimes \delta_0} Z_U(\underline{X} \otimes \bar{X}), Z_U(f \otimes \bar{g})) \rightarrow (R_{\underline{X}}, f \otimes \bar{g})$$

in  $L_1(\Omega)$ -sense.

Let  $h \in S^2$ . If  $N \in \mathbb{N}$ , then  $\sum_{k=0}^N \sum_{\ell=0}^N (\underline{X} \otimes \bar{X}, \psi_k \otimes \psi_\ell)(\psi_k \otimes \psi_\ell, h)$  and  $\prod_F \sum_{k=0}^N \sum_{\ell=0}^N (F \otimes \bar{F}, \psi_k \otimes \psi_\ell)(\psi_k \otimes \psi_\ell, h)$  have the same distribution by 1.1.15 (with respect to  $P$  and  $P_{\underline{X}}^*$  respectively). It easily follows that  $(\underline{X} \otimes \bar{X}, h) = \lim_{N \rightarrow \infty} \sum_{k=0}^N \sum_{\ell=0}^N (\underline{X} \otimes \bar{X}, \psi_k \otimes \psi_\ell)(\psi_k \otimes \psi_\ell, h)$  and  $\prod_F (F \otimes \bar{F}, h) = \lim_{N \rightarrow \infty} \prod_F \sum_{k=0}^N \sum_{\ell=0}^N (F \otimes \bar{F}, \psi_k \otimes \psi_\ell)(\psi_k \otimes \psi_\ell, h)$  have the same distribution.

Now let  $f \in S$ ,  $g \in S$ . It follows from the above that

$(T_{h_\tau \otimes \delta_0} Z_U(\underline{X} \otimes \bar{X}), Z_U(f \otimes \bar{g}))$  and  $\prod_F (T_{h_\tau \otimes \delta_0} Z_U(F \otimes \bar{F}), Z_U(f \otimes \bar{g}))$  have the same distribution for every  $\tau > 0$ . Hence, it suffices to show that  $\prod_F (T_{h_\tau \otimes \delta_0} Z_U(F \otimes \bar{F}), Z_U(f \otimes \bar{g})) \rightarrow (R_{\underline{X}}, f \otimes \bar{g})$  ( $\tau \rightarrow \infty$ ) in  $L_1(S^*, \Lambda^*, P_{\underline{X}}^*)$ -sense. We have for  $\tau > 0$ ,  $F \in S^*$

$$\begin{aligned} & (T_{h_\tau \otimes \delta_0} Z_U(F \otimes \bar{F}), Z_U(f \otimes \bar{g})) = \\ & = \frac{1}{2\tau} \int_{-\tau}^{\tau} (T_{t/\sqrt{2}}^{F, f}) \overline{(T_{t/\sqrt{2}}^{F, g})} dt \end{aligned}$$

(cf. also lemma 4.3.3 and 4.3.4, remark). Since  $(T_{t/\sqrt{2}}^{F, h}) = \overline{(T_F^{h_-})(t/\sqrt{2})}$  for every  $F \in S^*$ ,  $h \in S$ ,  $t \in \mathbb{R}$ , we easily conclude from theorem 4.4.6 that

$$\prod_F (T_{h_\tau \otimes \delta_0} Z_U(F \otimes \bar{F}), Z_U(f \otimes \bar{g})) \rightarrow E(\prod_F (F, f) \overline{(F, g)}) = (R_{\underline{X}}, f \otimes \bar{g}) \quad (\tau \rightarrow \infty)$$

in  $L_1(S^*, \Lambda^*, P_{\underline{X}}^*)$ -sense.  $\square$

REMARK. Let  $\underline{X}$  as in the above theorem. It can be shown that  $T_{k_\epsilon \otimes \delta_0} Z_U(\underline{X} \otimes \bar{X}) \rightarrow Z_{UR_{\underline{X}}}(S_{\Omega, 1}^{2*})$  if  $\epsilon > 0$ . Also (cf. 4.4.3 and 3.4.2),  $T_{k_\epsilon \otimes k_{\epsilon-1}} F(2) Z_U(\underline{X} \otimes \bar{X}) \rightarrow F(2) Z_{UR_{\underline{X}}} = H \otimes L(S_{\Omega, 1}^{2*})$  and  $\text{emb}_{(a, b)}^\psi \left| (S_{k_\epsilon \underline{X}}^1(a/\sqrt{2}, b/\sqrt{2})) \right|^2 \rightarrow H \otimes L(S_{\Omega, 1}^{2*})$ . Here  $L$  is the spectral density function of  $\underline{X}$  (cf. 2.2.5).

4.4.8. The theorems 4.4.6 and 4.4.7 deal with strict sense time stationary and ergodic processes. For general processes it is often hard to estimate spectral density functions (or, more properly spoken, weighted Wigner distributions) from the observation of a single realization of the process. There exist, however, certain non-stationary processes for which spectral density functions have been calculated. We mention in this respect  $\frac{1}{f}$ -noise and the Barkhausen effect ([BS], 3.2.3 and 3.6). The first process has the property that its statistics vary only very slowly with time ("locally stationary process"). For such a process the spectral density function is estimated by employing one realization of the process (cf. 4.4.5 (4) and (5); in (4) a small value of  $\epsilon$  can be taken). We further note that the Barkhausen effect is a "periodic" noise process, so that each period of a realization can be used. Now one considers averages of the expressions in 4.4.5 (4) and 4.4.5 (5) (averaged over the various periods; this is, in fact, an ensemble average, i.e. an average over a collection  $\Omega$  of signals).

## APPENDIX 1

THE SPACES  $S$  AND  $S^*$ 

This appendix is devoted to the spaces  $S$  and  $S^*$ ; it contains all available information about these spaces as far as relevant for the main text.

Section 1 is introductory. It gives a survey (with some supplements) of De Bruijn's theory of generalized functions.

Section 2 is devoted to convergence and topology in the spaces  $S$  and  $S^*$ . Various topologies, compatible with the respective notions of sequential convergence in  $S$  and  $S^*$ , are introduced there. Furthermore it is shown that both  $S$  and  $S^*$  can be regarded as nuclear spaces (when provided with the proper topology).

Section 3 studies continuous linear functionals of  $S$  and  $S^*$ . It is shown that the set of continuous linear functionals of  $S$  ( $S^*$ ) can be identified with  $S^*$  ( $S$ ), no matter which one of the topologies of section 2 is taken for defining continuity. Continuous bi-linear functionals of  $S \times S$  are studied, and a kernel theorem as well as a theorem about convergence in  $S^{2*}$  is proved.

Section 4 presents the main results of [J1], appendix 1, section 2 and 3 about continuous linear operators of  $S$  and  $S^*$ . It is shown that all topologies of section 2 for  $S$  ( $S^*$ ) determine the same class of continuous linear operators of  $S$  ( $S^*$ ).

In section 5 the space  $S^*$  is regarded as a measure space. It is shown that the various topologies of section 2 all generate the same  $\sigma$ -algebra on  $S^*$ , and that  $S^*$  is a Radon space with respect to each one of these topologies. A number of examples of measurable subsets of  $S^*$  are given.

## 1. INTRODUCTION

1.1. We give a survey of the fundamental notions and theorems of De Bruijn's theory of generalized functions as far as they are relevant for this tract. A detailed treatment of this theory can be found in [B1]. Also, some supplements are given.

1.2. If  $A$  and  $B$  are positive numbers, we denote by  $S_{A,B}$  the class of analytic functions  $f$  of a single complex variable for which there is a positive number  $M$  such that

$$|f(t)| \leq M \exp(-\pi A(\operatorname{Re}t)^2 + \pi B(\operatorname{Im}t)^2) \quad (t \in \mathbb{C}).$$

The set  $S$  of *smooth functions* of a single complex variable is defined as  $\bigcup_{A>0} \bigcup_{B>0} S_{A,B}$  (cf. [B1], 2.1).

1.3. In  $S$  we take the usual inner product and norm:

$$(f, g) := \int_{-\infty}^{\infty} f(t) \overline{g(t)} dt \quad (f \in S, g \in S),$$

$$\|f\| := ((f, f))^{\frac{1}{2}} \quad (f \in S).$$

1.4. We consider a semigroup  $(N_{\alpha})_{\alpha>0}$  of linear operators of  $S$  (the *smoothing operators*). The  $N_{\alpha}$ 's satisfy  $N_{\alpha} N_{\beta} = N_{\alpha+\beta}$  ( $\alpha > 0, \beta > 0$ ), where the product is the usual composition of mappings. These  $N_{\alpha}$ 's are integral operators:

$$N_{\alpha} f := \int_{z \in \mathbb{C}} K_{\alpha}(z, t) f(t) dt \quad (f \in S, \alpha > 0),$$

where the kernel  $K_{\alpha}$  ( $\alpha > 0$ ) is given by

$$K_{\alpha} := \int_{(z, t) \in \mathbb{C}^2} 2(\sinh \alpha)^{-\frac{1}{2}} \exp\left(\frac{-\pi}{\sinh \alpha} ((z^2 + t^2) \cosh \alpha - 2zt)\right)$$

(cf. [B1], section 3, 4, 5 and 6).

1.5. In fact the smoothing operators can be defined on the space  $S^+$ , consisting of all complex-valued measurable functions defined on the reals with the property that for every  $\epsilon > 0$

$$\int_{-x}^x |f(t)| dt = O(\exp(\epsilon x^2)) \quad (x \geq 0)$$

(cf. [B1], section 20). For  $f \in S^+, \alpha > 0$



$$N_\alpha f = \int_{-\infty}^{\infty} K_\alpha(z, t) f(t) dt$$

is well-defined, and it is a smooth function of one variable.  $N_\alpha$  thus maps  $S^+$  into  $S$  for  $\alpha > 0$ .

Let  $f$  be measurable over  $\mathbb{R}$ . It is to be noted that  $f \in S^+$  if and only if

$$\int_{-\infty}^{\infty} \exp(-\pi \epsilon t^2) |f(t)| dt < \infty$$

for every  $\epsilon > 0$ . Note that  $L_p(\mathbb{R}) \subset S^+$  for every  $p$  with  $1 \leq p \leq \infty$ .

1.6. The following theorem is useful.

**THEOREM.** Let  $1 \leq p < \infty$ , and let  $f \in L_p(\mathbb{R})$ . Then  $\|N_\alpha f - f\|_p \rightarrow 0$  ( $\alpha \downarrow 0$ ).

**PROOF.** Let  $\epsilon > 0$ , and let  $b$  a function of the form  $\sum_{n=1}^N \alpha_n \chi_{(a_n, b_n]}$  where  $N \in \mathbb{N}$ ,  $\alpha_n \in \mathbb{C}$ ,  $a_n \in \mathbb{R}$ ,  $b_n \in \mathbb{R}$ ,  $a_n < b_n$  ( $n = 1, \dots, N$ ), such that  $\|f - b\|_p < \epsilon/3$ . We then have

$$\begin{aligned} \|N_\alpha f - f\|_p &\leq \|N_\alpha(f - b)\|_p + \|f - b\|_p + \|N_\alpha b - b\|_p \leq \\ &\leq \|N_\alpha(f - b)\|_p + \|N_\alpha b - b\|_p + \frac{\epsilon}{3}. \end{aligned}$$

Let  $h \in L_p(\mathbb{R})$ . We are going to show that  $\|N_\alpha h\|_p \leq \|h\|_p$  for  $\alpha > 0$ . We have for  $z \in \mathbb{R}$ ,  $p > 1$  (the case  $p = 1$  is almost trivial)

$$\begin{aligned} |(N_\alpha h)(z)| &= \left| \int_{-\infty}^{\infty} K_\alpha(z, t) h(t) dt \right| \leq \\ &\leq \left( \int_{-\infty}^{\infty} K_\alpha(z, t) |h(t)|^p dt \right)^{\frac{1}{p}} \left( \int_{-\infty}^{\infty} K_\alpha(z, t) dt \right)^{1/q}, \end{aligned}$$

where  $q$  is the conjugate exponent of  $p$ . Since

$$\int_{-\infty}^{\infty} K_{\alpha}(z,t) dt = (\cos \alpha)^{-\frac{1}{2}} \exp(-\pi z^2 \tanh \alpha) \quad (z \in \mathbb{C}),$$

we infer that

$$\|N_{\alpha} h\|_p^p = \int_{-\infty}^{\infty} |(N_{\alpha} h)(z)|^p dz \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{\alpha}(z,t) |h(t)|^p dt dz,$$

and an application of Fubini's theorem shows that  $\|N_{\alpha} h\|_p^p \leq \|h\|_p^p$ .

This implies that

$$\|N_{\alpha} f - f\|_p \leq \frac{2\varepsilon}{3} + \|N_{\alpha} b - b\|_p.$$

Now  $|(N_{\alpha} b)(z)| \leq M \exp(-\pi A z^2)$  ( $z \in \mathbb{R}$ ) for all  $\alpha$ ,  $0 < \alpha < 1$  with some  $M > 0$ ,  $A > 0$ , and  $(N_{\alpha} b)(z) \rightarrow b(z)$  ( $\alpha \downarrow 0$ ) except in the discontinuity points of  $b$  (this follows by writing  $K_{\alpha}(z,t)$  as  $(\sinh \alpha)^{-\frac{1}{2}} \exp(-\pi(z - t/\cosh \alpha)^2 \cos \alpha) \times \exp(-\pi t^2 \tanh \alpha)$ ). Hence, by Lebesgue's theorem on dominated convergence,  $\|N_{\alpha} b - b\|_p \rightarrow 0$  ( $\alpha \downarrow 0$ ). If  $\alpha > 0$  is sufficiently small, then  $\|N_{\alpha} b - b\|_p < \frac{\varepsilon}{3}$ , whence  $\|N_{\alpha} f - f\|_p < \varepsilon$ .  $\square$

REMARKS 1. The theorem does not hold if  $p = \infty$  (take  $f := \chi_z^1$ ).

2. If  $f \in S^+$  is continuous, then  $(N_{\alpha} f)(t) \rightarrow f(t)$  ( $\alpha \downarrow 0$ ) locally uniformly.

1.7. We summarize a number of properties of the operators  $N_{\alpha}$  ( $\alpha > 0$ ).

(i)  $(N_{\alpha} f, g) = (f, N_{\alpha} g)$  for  $\alpha > 0$ ,  $f \in S$ ,  $g \in S$  (cf. [B1], 6.5).

(ii) For every  $\alpha > 0$  and every  $p$  with  $1 \leq p \leq \infty$  there are positive numbers  $C_{\alpha p}$ ,  $A$  and  $B$  such that for every  $f \in L_p(\mathbb{R})$

$$|(N_{\alpha} f)(t)| \leq C_{\alpha p} \|f\|_p \exp(-\pi A(\operatorname{Re} t)^2 + \pi B(\operatorname{Im} t)^2) \quad (t \in \mathbb{C}).$$

This is a slight generalization of [B1], 6.3.

(iii) If  $f \in S$  and  $\alpha > 0$ , then there is at most one  $g \in S$  satisfying  $f = N_{\alpha} g$ . Also, if  $f \in S$ , then there exists an  $\alpha > 0$  and a  $g \in S$  with  $f = N_{\alpha} g$ . Moreover, if  $f \in S$ , and the positive numbers  $M$ ,  $A$  and  $B$  are such that

$$|f(t)| \leq M \exp(-\pi A(\operatorname{Re}t)^2 + \pi B(\operatorname{Im}t)^2) \quad (t \in \mathbb{C}),$$

then we can find an  $\alpha > 0$ ,  $C > 0$ ,  $A' > 0$ ,  $B' > 0$ , only depending on  $A$  and  $B$ , such that

$$|g(t)| \leq MC \exp(-\pi A'(\operatorname{Re}t)^2 + \pi B'(\operatorname{Im}t)^2) \quad (t \in \mathbb{C})$$

holds for the unique  $g$  with  $f = N_\alpha g$  (cf. [B1] 10.1).

(iv) As a linear operator of  $L_2(\mathbb{R})$ ,  $N_\alpha$  ( $\alpha > 0$ ) is a Hilbert-Schmidt operator. The eigenfunctions of  $N_\alpha$  are the Hermite functions  $\psi_k$  ( $k \in \mathbb{N}_0$ ) (cf. [B1], 27.6.3 for the normalization chosen), and the corresponding eigenvalues are  $e^{-(k+\frac{1}{2})\alpha}$  ( $k \in \mathbb{N}_0$ ). The set  $\{\psi_k \mid k \in \mathbb{N}_0\}$  forms a complete orthonormal set in  $L_2(\mathbb{R})$ , and  $\psi_k \in S$  for every  $k \in \mathbb{N}_0$ . We list some further properties of  $\psi_k$  ( $k \in \mathbb{N}_0$ ).

a) If  $\alpha > 0$ , then

$$K_\alpha(z, t) = \sum_{k=0}^{\infty} e^{-(k+\frac{1}{2})\alpha} \psi_k(z) \psi_k(t) \quad (z \in \mathbb{C}, t \in \mathbb{C}).$$

b) If  $f \in S$ , then there exists an  $\varepsilon > 0$  such that

$$(f, \psi_k) = O(e^{-k\varepsilon}) \quad (k \in \mathbb{N}_0).$$

On the other hand, if  $(c_k)_{k \in \mathbb{N}_0}$  is a sequence in  $\mathbb{C}$  with  $c_k = O(e^{-k\varepsilon})$  ( $k \in \mathbb{N}_0$ ) for some  $\varepsilon > 0$ , then  $f := \sum_{k=0}^{\infty} c_k \psi_k \in S$  and  $(f, \psi_k) = c_k$  ( $k \in \mathbb{N}_0$ ).

1.8. We give some other important linear operators of  $S$ .

(i) The Fourier transform  $F$  and its inverse  $F^*$ :

$$Ff := \int_{z \in \mathbb{C}} \int_{-\infty}^{\infty} e^{-2\pi izt} f(t) dt \quad (f \in S),$$

$$F^*f := \int_{z \in \mathbb{C}} \int_{-\infty}^{\infty} e^{2\pi izt} f(t) dt \quad (f \in S).$$

Cf. [B1], section 8 and 9.

(ii) The shift operators  $T_a$  and  $R_b$  ( $a \in \mathbb{C}$ ,  $b \in \mathbb{C}$ ):

$$T_a f := \prod_{z \in \mathbb{C}} f(z + a) \quad (f \in S),$$

$$R_b f := \prod_{z \in \mathbb{C}} e^{-2\pi i b z} f(z) \quad (f \in S).$$

The operator  $Z_\lambda$  ( $\lambda > 0$ ):

$$Z_\lambda f := \prod_{z \in \mathbb{C}} \lambda^{\frac{1}{2}} f(\lambda z) \quad (f \in S).$$

The operators  $P$  and  $Q$ :

$$P f := \prod_{z \in \mathbb{C}} \frac{f'(z)}{2\pi i} \quad (f \in S),$$

$$Q f := \prod_{z \in \mathbb{C}} z f(z) \quad (f \in S).$$

Cf. [B1], section 11 for relations involving these operators. We write  $\frac{d}{dz}$  for the operator  $2\pi i P$  (ordinary differentiation).

(iii) The operators  $T_g$  with  $g \in \mathbb{C}$  (cf. appendix 2,3), and the operators  $M_h$  with  $h \in M$  (cf. appendix 2, 7).

1.9. A *generalized function* is a mapping  $F = \prod_{\alpha > 0} F_\alpha$  of  $(0, \infty)$  into  $S$  such that  $N_\alpha F_\beta = F_{\alpha+\beta}$  ( $\alpha > 0$ ,  $\beta > 0$ ). It is sometimes convenient to write  $F(\alpha)$  or  $N_\alpha F$  instead of  $F_\alpha$  ( $\alpha > 0$ ). The set of all generalized functions is denoted by  $S^*$ . If  $\alpha > 0$ , then  $N_\alpha(S^*)$  denotes the set  $\{F(\alpha) \mid F \in S^*\}$ .

1.10. If  $F \in S^*$ ,  $g \in S$ , then we define the inner product  $(F, g)$  as follows. Write  $g = N_\alpha h$  with some  $\alpha > 0$ ,  $h \in S$  (cf. 1.7 (iii)) and put  $(F, g) := (F_\alpha, h)$ . This number depends only on  $F$  and  $g$  (cf. [B1], section 18). Similarly, we define  $(g, F)$ . We have  $(N_\alpha F, g) = (F, N_\alpha g)$  for  $F \in S^*$ ,  $g \in S$ ,  $\alpha > 0$ .

If  $F \in S^*$ , then we have for every  $\varepsilon > 0$

$$(F, \psi_k) = 0 \quad (e^{k\varepsilon}) \quad (k \in \mathbb{N}_0),$$

and

$$F_\varepsilon = \sum_{k=0}^{\infty} (F, \psi_k) N_\varepsilon \psi_k.$$

On the other hand, if  $(c_k)_{k \in \mathbb{N}_0}$  is a sequence in  $\mathbb{C}$  with  $c_k = O(e^{k\varepsilon})$  ( $k \in \mathbb{N}_0$ ) for every  $\varepsilon > 0$ , then  $F := \bigcup_{\alpha > 0} \sum_{k=0}^{\infty} c_k N_\alpha \psi_k \in S^*$ , and  $(F, \psi_k) = c_k$  ( $k \in \mathbb{N}_0$ ). It follows that  $F = 0$  if  $N_\alpha F = 0$  for some  $\alpha > 0$ .

If  $F \in S^*$ ,  $g \in S$ , then

$$(F, g) = \sum_{k=0}^{\infty} (F, \psi_k) (\psi_k, g).$$

Cf. [B1], 27.6.3.

1.11. We give some examples of generalized functions.

(i) Let  $f \in S^+$ . The embedding of  $f$  ( $\text{emb}(f)$ ) is defined as  $\text{emb}(f) := \bigcup_{\alpha > 0} N_\alpha f$ . Cf. [B1], section 20. Note that if  $f \in S^+$ ,  $g \in S$

$$(\text{emb}(f), g) = \int_{-\infty}^{\infty} f(t) \overline{g(t)} dt.$$

If  $f \in L_1(\mathbb{R})$ ,  $\text{emb}(f) = 0$  then  $f = 0$  (a.e.). This follows from theorem 1.6 by noting that  $N_\alpha(\text{emb}(f)) = 0$  ( $\alpha > 0$ ). With this result we can prove that  $f = 0$  (a.e.) if  $f \in S^+$ ,  $\text{emb}(f) = 0$ . For let  $g := \bigcup_z \exp(-\pi z^2)$ . Now  $g \cdot f \in L_1(\mathbb{R})$  by 1.5, and  $\text{emb}(g, f) = g \cdot \text{emb}(f) = 0$  (cf. appendix 2,6), whence  $g \cdot f = 0$  (a.e.), so  $f = 0$  (a.e.).

(ii) For  $b \in \mathbb{C}$ , the "delta function at  $b$ " is defined as  $\delta_b := \bigcup_{\alpha > 0} \bigcup_{t \in \mathbb{C}} K_\alpha(t, b)$ . We have  $(g, \delta_b) = g(\bar{b})$  for  $g \in S$ .

(iii) For  $a \in \mathbb{C}$ , the generalized function  $e_a$  is defined as  $e_a := \text{emb}(\bigcup_{t \in \mathbb{R}} e^{-2\pi i a t})$ . We have  $(g, e_a) = (F^* g)(\bar{a})$  for  $g \in S$ .

1.12. We next define convergence in the space  $S$  ( $S$ -convergence). If  $(f_n)_{n \in \mathbb{N}}$  is a sequence in  $S$ , then we write  $f_n \xrightarrow{S} 0$  if there are positive numbers  $A$  and  $B$  such that

$$f_n(t) \exp(\pi A(\operatorname{Re} t)^2 - \pi B(\operatorname{Im} t)^2) \rightarrow 0$$

uniformly in  $t \in \mathbb{C}$ . If  $f \in S$  and  $(f_n)_{n \in \mathbb{N}}$  is a sequence in  $S$ , then we write  $f_n \xrightarrow{S} f$  if  $f_n - f \xrightarrow{S} 0$ . In a similar way we define  $S$ -convergence of series.

In [B1], 23.1 the following theorem is proved. If  $(f_n)_{n \in \mathbb{N}}$  is a sequence in  $S$  then  $f_n \xrightarrow{S} 0$  if and only if there exists an  $\alpha > 0$  and a sequence  $(g_n)_{n \in \mathbb{N}}$  in  $S$  such that  $f_n = N_\alpha g_n$  ( $n \in \mathbb{N}$ ),  $g_n \xrightarrow{S} 0$ .

1.13. The following criteria for  $S$ -convergence are sometimes useful.

**THEOREM.** Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence in  $S$ , and assume that there is an  $M > 0$   $A > 0$ ,  $B > 0$  such that

$$|f_n(t)| \leq M \exp(-\pi A(\operatorname{Re} t)^2 + \pi B(\operatorname{Im} t)^2) \quad (n \in \mathbb{N}, t \in \mathbb{C}).$$

Then the three following statements are equivalent.

- (i)  $f_n \xrightarrow{S} 0$ .
- (ii)  $f_n \rightarrow 0$  (i.e.  $(f_n, g) \rightarrow 0$  ( $n \rightarrow \infty$ ) for every  $g \in L_2(\mathbb{R})$ ).
- (iii)  $f_n(t) \rightarrow 0$  ( $n \rightarrow \infty$ ) for every  $t \in \mathbb{R}$ .

**PROOF.** It follows from 1.7 (iii) that there is an  $\varepsilon > 0$  and an  $M > 0$  such that for every  $n \in \mathbb{N}$  there is an  $h_n \in S$  with  $f_n = N_\varepsilon h_n$ ,  $\|h_n\| \leq M$ . Hence  $|(f_n, \psi_k)| \leq M e^{-k\varepsilon}$  ( $k \in \mathbb{N}_0$ ,  $n \in \mathbb{N}$ ).

Obviously (i)  $\Rightarrow$  (iii). Also, by Lebesgue's theorem on dominated convergence, (iii)  $\Rightarrow$  (ii). We show that (ii)  $\Rightarrow$  (i). Assume that  $f_n \rightarrow 0$ . Write  $\varepsilon = \alpha + \beta$  with  $\alpha > 0$ ,  $\beta > 0$ , and let  $g_n := N_\beta h_n$  ( $n \in \mathbb{N}$ ). Then  $|(g_n, \psi_k)| \leq M e^{-k\beta}$  ( $k \in \mathbb{N}_0$ ,  $n \in \mathbb{N}$ ), and  $(g_n, \psi_k) = e^{(k+\frac{1}{2})\alpha} (f_n, \psi_k) \rightarrow 0$  if  $n \rightarrow \infty$  ( $k \in \mathbb{N}_0$ ). It easily follows that  $\|g_n\|^2 = \sum_{k=0}^{\infty} |(g_n, \psi_k)|^2 \rightarrow 0$  if  $n \rightarrow \infty$ . Hence  $f_n = N_\alpha g_n \xrightarrow{S} 0$  by 1.7 (ii).  $\square$

**EXAMPLE.** If  $f \in S$ , then  $\sum_{k=0}^n (f, \psi_k) \psi_k \xrightarrow{S} f$ .

1.14. The space  $S$  may be identified with the space  $S_{\frac{1}{2}}$  studied in [GSII], IV, §2.3. This space consists of all complex-valued functions  $\varphi$  defined on the reals that have derivatives of any order, and that satisfy inequalities of the type

$$(*) \quad |x^k \varphi^{(q)}(x)| \leq C A^k B^q k^{\frac{1}{2}k} q^{\frac{1}{2}q} \quad (x \in \mathbb{R}, k \in \mathbb{N}_0, q \in \mathbb{N}_0)$$

where  $C > 0$ ,  $A > 0$ ,  $B > 0$  depend on  $\varphi$ . It is proved in [GSII], IV, §2.3 that such a  $\varphi$  is the restriction to  $\mathbb{R}$  of an element of  $S$ . Conversely, if  $\varphi \in S$ , then the restriction of  $\varphi$  to  $\mathbb{R}$  satisfies inequalities of the type (\*), and belongs to  $S_{\frac{1}{2}}$  ([GS2], IV, §7.5).

Convergence in  $S_{\frac{1}{2}}$  is defined as follows. A sequence  $(\varphi_n)_{n \in \mathbb{N}}$  is said to converge to zero if  $\varphi_n^{(q)}(x) \rightarrow 0$  locally uniformly in  $x \in \mathbb{R}$  for every  $q \in \mathbb{N}_0$ , and if there are numbers  $C > 0$ ,  $A > 0$ ,  $B > 0$  such that the inequalities in (\*) hold for all  $\varphi = \varphi_n$  ( $n \in \mathbb{N}_0$ ).

It can be proved (by a careful inspection of the proofs in [GSII], IV, §2.3 and [GSII], IV, §7.5) that a sequence  $(\varphi_n)_{n \in \mathbb{N}}$  converges to zero in  $S_{\frac{1}{2}}$  if and only if the sequence of analytic continuations converges to zero in  $S$ .

It is proved in [Wł], VIII, §29,3 that  $S_{\frac{1}{2}}$  can be regarded as a nuclear space (cf. also [Mi]); we shall give an alternative proof of this fact in section 2 of this appendix.

1.15. Now we define convergence in the space  $S^*$  ( $S^*$ -convergence). If  $(F_n)_{n \in \mathbb{N}}$  is a sequence in  $S^*$ , then we write  $F_n \xrightarrow{S^*} 0$  if  $N_\alpha F_n \xrightarrow{S} 0$  for every  $\alpha > 0$ . If  $F \in S^*$  and  $(F_n)_{n \in \mathbb{N}}$  is a sequence in  $S^*$ , then we write  $F_n \xrightarrow{S^*} F$  if  $F_n - F \xrightarrow{S^*} 0$ . In a similar way we define  $S^*$ -convergence of series.

In [B1], 24.4 the following theorem is proved. If  $(F_n)_{n \in \mathbb{N}}$  is a sequence in  $S^*$ , then there is an  $F \in S^*$  with  $F_n \xrightarrow{S^*} F$  if and only if the sequence  $((F_n, g))_{n \in \mathbb{N}}$  is convergent for every  $g \in S$ .

EXAMPLE. If  $F \in S^*$ , then  $\text{emb}(\sum_{k=1}^n (F, \psi_k) \psi_k) \xrightarrow{S^*} F$ .

1.16. The following theorem expresses continuity of  $(, )$  as a mapping of  $S^* \times S$  in  $\mathbb{C}$ .

THEOREM. Let  $F \in S^*$ , and let  $(F_n)_{n \in \mathbb{N}}$  be a sequence in  $S^*$  with  $F_n \xrightarrow{S^*} F$ . Let  $f \in S$ , and let  $(f_n)_{n \in \mathbb{N}}$  be a sequence in  $S$  with  $f_n \xrightarrow{S} f$ . Then we have  $(F_n, f_n) \rightarrow (F, f)$  ( $n \rightarrow \infty$ ).

PROOF. Let  $\alpha > 0$ ,  $g \in S$ ,  $g_n \in S$  be such that  $f = N_\alpha g$ ,  $f_n = N_\alpha g_n$  ( $n \in \mathbb{N}$ ),  $g_n \xrightarrow{S} g$  (cf. 1.12). Now we have for  $n \in \mathbb{N}$

$$|(F_n, f_n) - (F, f)| \leq |(F_n, f_n - f)| + |(F_n - F, f)|;$$

the first term is at most  $\|N_\alpha F\| \|g_n - g\|$ , and the second one is at most  $\|N_\alpha F - N_\alpha F\| \|f\|$ . We easily infer from the definitions 1.12 and 1.15 that  $(F_n, f_n) - (F, f) \rightarrow 0$  ( $n \rightarrow \infty$ ).  $\square$

1.17. We devote some attention to smooth functions and generalized functions of  $n$  complex variables ( $n \in \mathbb{N}$ ). The previous definitions and theorems can be formulated and proved (with proper modifications) for the  $n$ -dimensional case. The class  $S^n$ , e.g., is defined as the set of all complex-valued functions  $f$ , analytic in its  $n$  variables, for which there exist positive numbers  $M, A$  and  $B$  such that

$$|f(t_1, \dots, t_n)| \leq M \exp(-\pi A \sum_{k=1}^n (\operatorname{Re} t_k)^2 + \pi B \sum_{k=1}^n (\operatorname{Im} t_k)^2)$$

for  $(t_1, \dots, t_n) \in \mathbb{C}^n$ .

As an example of a smooth function of  $n$  variables we have

$$f_1 \otimes \dots \otimes f_n := \bigcup_{(t_1, \dots, t_n) \in \mathbb{C}^n} f_1(t_1) \cdot \dots \cdot f_n(t_n),$$

where  $f_1 \in S, \dots, f_n \in S$ .

On  $S^n$  we take the ordinary inner product of  $L_2(\mathbb{R}^n)$ , and denote it by  $(, )_n$  (or  $(, )$  if confusion is excluded).

The classes  $S^{n+}$  and  $S^{n*}$  (of embeddable and generalized functions respectively) are introduced in a similar way (the smoothing operator  $N_{\alpha, n}$  ( $\alpha > 0$ ) is an integral operator with kernel

$$K_{\alpha, n} = \bigcup_{(z_1, \dots, z_n; t_1, \dots, t_n)} \prod_{i=1}^n K_\alpha(z_i, t_i).$$

If  $f \in S^{n+}$ , we write  $\operatorname{emb}(f)$  for the generalized function  $\bigcup_{\alpha > 0} N_{\alpha, n} f$ .



As an example of a generalized function of  $n$  variables we have

$$F_1 \otimes \dots \otimes F_n := \bigcup_{\alpha > 0} F_1(\alpha) \otimes \dots \otimes F_n(\alpha),$$

where  $F_1 \in S^*$ ,  $\dots$ ,  $F_n \in S^*$ .

The  $n$ -dimensional version of 1.7 (iv) b) reads as follows. If  $f \in S^n$ , then there exists an  $\varepsilon > 0$  such that  $(f, \psi_{k,n}) = 0$  ( $e^{-\varepsilon(k_1 + \dots + k_n)}$ ) ( $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$ ). Here  $\psi_{k,n}$  denotes  $\psi_{k_1} \otimes \dots \otimes \psi_{k_n}$  for  $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$ . Also, if  $(c_k)_{k \in \mathbb{N}_0^n}$  is a multi-sequence in  $\mathbb{C}$  with  $c_k = 0$  ( $e^{-\varepsilon(k_1 + \dots + k_n)}$ ) ( $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$ ) for some  $\varepsilon > 0$ , then the function  $f := \sum_{k \in \mathbb{N}_0^n} c_k \psi_{k,n}$  is an element of  $S^n$ , and  $(f, \psi_{k,n}) = c_k$  ( $k \in \mathbb{N}_0^n$ ). We have a similar result for generalized functions of  $n$  variables (cf. 1.10).

## 2. CONVERGENCE AND TOPOLOGY IN THE SPACES $S$ AND $S^*$

2.1. We introduce a number of topologies on  $S$  and  $S^*$ . These topologies turn out to be compatible with the notions of sequential convergence in  $S$  and  $S^*$  (cf. 1.12 and 1.15).

2.2. DEFINITION. The *weak topology* on  $S$  is the linear topology generated by all sets of the form  $\{f \in S \mid (f, F) \in O\}$ , where  $F \in S^*$  and  $O$  is an open subset of  $\mathbb{C}$ . The *weak topology* on  $S^*$  is the linear topology generated by all sets of the form  $\{F \in S^* \mid (F, f) \in O\}$  where  $f \in S$  and  $O$  is an open subset of  $\mathbb{C}$ .

It is easy to see that both  $S$  and  $S^*$  are locally convex topological Hausdorff spaces with the respective weak topologies.

If  $(F_n)_{n \in \mathbb{N}}$  is a sequence in  $S^*$ , then  $F_n \xrightarrow{S^*} 0$  if and only if  $F_n \rightarrow 0$  in weak sense (cf. 1.15). The corresponding statement holds for the space  $S$ ; the proof of it is postponed until 2.5.

2.3. We introduce on  $S^*$  a stronger topology. This topology is associated with a countable number of inner product norms on  $S^*$ .

DEFINITION. For  $n \in \mathbb{N}$  we define  $(\cdot, \cdot)^{(n)} : S^* \times S^* \rightarrow \mathbb{C}$  by

$$(F, G)^{(n)} := (F(\frac{1}{n}), G(\frac{1}{n})) \quad (F \in S^*, G \in S^*),$$

where  $(\cdot, \cdot)$  denotes the ordinary inner product in  $L_2(\mathbb{R})$  (recall that

$F(\alpha) = F_\alpha = N_\alpha F$  for  $F \in S^*$ ,  $\alpha > 0$ ). The corresponding norm is denoted by  $\| \cdot \|^{(n)}$ . The *strong topology* on  $S^*$  is the linear topology generated by all sets of the form  $\{F \in S^* \mid \|F\|^{(n)} < \varepsilon\}$  where  $n \in \mathbb{N}$ ,  $\varepsilon > 0$ .

It is not hard to prove that  $S^*$  is a Fréchet space with the strong topology: if we put

$$d(F, G) := \sum_{n=1}^{\infty} \frac{\|F - G\|^{(n)}}{1 + \|F - G\|^{(n)}} 2^{-n} \quad (F \in S^*, G \in S^*),$$

then  $(S^*, d)$  is a complete metric space, and the metric topology coincides with the strong topology.

We further note that the strong topology has a countable base, viz.  $\{\{F \in S^* \mid \|F - F_0\|^{(n)} < r_0\} \mid n \in \mathbb{N}, r_0 \in \mathbb{Q}, F_0 \in T\}$  with  $T$  the set of all embeddings of finite linear combinations of Hermite functions with rational coefficients.

It is easy to see that the strong topology on  $S^*$  is really stronger than the weak topology.

It follows from 1.14 that weak and strong sequential convergence in  $S^*$  are equivalent. Hence, strong sequential convergence is equivalent to  $S^*$ -convergence.

2.4. Other properties of  $S^*$  with the strong topology are perfectness (cf. [GSII], Ch. I, § 6.1), and nuclearity (cf. [GW], Ch. I, § 3.2). The latter property is of interest for us, and we shall therefore say what nuclearity means for our case and present a detailed proof of nuclearity of  $S^*$ .

Denote by  $\bar{S}_n$  the completion of  $S^*$  with respect to the norm  $\| \cdot \|^{(n)}$  ( $n \in \mathbb{N}$ ). Then  $\bar{S}_n$  is a Hilbert space in which the canonical embedding of  $S^*$  is dense (with respect to  $\| \cdot \|^{(n)}$ ). Since  $\| \cdot \|^{(m)} \leq \| \cdot \|^{(n)}$  we can regard  $\bar{S}_n$  as a subset of  $\bar{S}_m$  ( $m \in \mathbb{N}$ ,  $n \in \mathbb{N}$ ,  $m \leq n$ ). If we denote the canonical mapping of  $S^*$  into  $\bar{S}_n$  by  $T_n$  and the canonical mapping of  $\bar{S}_n$  into  $\bar{S}_m$  by  $T_m^n$ , then both  $T_n$  and  $T_m^n$  are continuous, and  $T_m^n = T_m^n T_n$  ( $n \in \mathbb{N}$ ,  $m \in \mathbb{N}$ ,  $m \leq n$ ).

In order to prove nuclearity of  $S^*$  with the strong topology, it suffices to show that for every  $m \in \mathbb{N}$  the mapping  $T_m^{m+1}$  is nuclear. We shall therefore prove that there exist orthonormal sequences  $(f_k)_{k \in \mathbb{N}_0}$  and  $(g_k)_{k \in \mathbb{N}_0}$  in  $\bar{S}_{m+1}$  and  $\bar{S}_m$  respectively (of course with respect to the corresponding inner products), and a sequence  $(\lambda_k)_{k \in \mathbb{N}_0}$  with  $\lambda_k > 0$ ,  $\sum_{k=0}^{\infty} \lambda_k < \infty$  such that

$$T_m^{m+1} F = \sum_{k=0}^{\infty} \lambda_k (F, f_k)^{(m+1)} g_k \quad (F \in \bar{S}_{m+1}).$$

Let  $m \in \mathbb{N}$ , and define

$$f_k := e^{(k+\frac{1}{2})/(m+1)} T_{m+1} \psi_k \quad (k \in \mathbb{N}_0),$$

$$g_k := e^{(k+\frac{1}{2})/m} T_m \psi_k \quad (k \in \mathbb{N}_0).$$

It is not hard to prove that  $(f_k)_{k \in \mathbb{N}_0}$  and  $(g_k)_{k \in \mathbb{N}_0}$  are orthonormal sequences in  $\bar{S}_{m+1}$  and  $\bar{S}_m$  respectively, and we have for every  $F \in S^*$

$$\begin{aligned} T_m^{m+1} T_{m+1} F &= T_m F = \sum_{k=0}^{\infty} (F, \psi_k) T_m \psi_k = \\ &= \sum_{k=0}^{\infty} (T_{m+1} F, f_k)^{(m+1)} \exp((k + \frac{1}{2})((m+1)^{-1} - m^{-1})) g_k. \end{aligned}$$

Since  $T_{m+1}(S^*)$  is dense in  $\bar{S}_{m+1}$ , the above formula extends to the entire space  $\bar{S}_{m+1}$ . Now convergence of  $\sum_{k=0}^{\infty} \exp((k + \frac{1}{2})((m+1)^{-1} - m^{-1}))$  shows nuclearity of  $S^*$ .

2.5. We are now able to prove the following result on  $S$ -convergence.

**THEOREM.** Let  $(f_m)_{m \in \mathbb{N}}$  be a sequence in  $S$ . Then  $f_m \xrightarrow{S} 0$  if and only if  $f_m \rightarrow 0$  weakly.

**PROOF.** It is seen at once from 1.10 and 1.13 that  $f_m \xrightarrow{S} 0$  implies  $(F, f_m) \rightarrow 0$  ( $m \rightarrow \infty$ ) for every  $F \in S^*$ . Hence  $f_m \xrightarrow{S} 0$  implies  $f_m \rightarrow 0$  weakly.

Now assume that  $f_m \rightarrow 0$  weakly. Then  $((F, f_m))_{m \in \mathbb{N}}$  is a bounded sequence for every  $F \in S^*$ . Hence  $S^* = \bigcup_{K=1}^{\infty} V_K$ , where

$$V_K := \{F \in S^* \mid |(F, f_m)| \leq K \quad (m \in \mathbb{N})\} \quad (K \in \mathbb{N}).$$

As  $S^*$ , with the strong topology, is a space of second category (this follows from 2.3), we can find a strongly open set  $U$  and a  $K_0 \in \mathbb{N}$  such that  $V_{K_0}$  is dense in  $U$ . This  $U$  contains a set  $\{F \in S^* \mid \|F - F_0\|^{(n)} < \delta_0\}$

with some  $\delta_0 > 0$ ,  $F_0 \in S^*$ ,  $n \in \mathbb{N}$ . It is not hard to see that every  $F \in S^*$  with  $\|F - F_0\|^{(n)} \leq \frac{\delta_0}{2}$  satisfies  $|(F, f_m)| \leq K_0$  for every  $m \in \mathbb{N}$ . Hence we can find a constant  $C$ , viz.  $2\delta_0^{-1}(\sup_{m \in \mathbb{N}} |(F_0, f_m)| + K_0)$ , such that

$$(*) \quad |(F, f_m)| \leq C \|F\|^{(n)} \quad (m \in \mathbb{N}, F \in S^*).$$

Now take  $F = \delta(t)$  (with  $t \in \mathbb{C}$ ) in (\*). We then get (cf. 1.11 (ii))

$$\begin{aligned} |f_m(t)|^2 &\leq C^2 \|N_{n-1}^{-1} \delta(t)\|^2 = C^2 (N_{2n-1}^{-1} \delta(t))(\bar{t}) \leq \\ &\leq MC^2 \exp(-\pi A(\operatorname{Re} t)^2 + \pi B(\operatorname{Im} t)^2) \end{aligned}$$

for some  $M > 0$ ,  $A > 0$ ,  $B > 0$  independent of  $t \in \mathbb{C}$ ,  $m \in \mathbb{N}$ . We have by weak convergence  $f_m(t) = (f_m, \delta(\bar{t})) \rightarrow 0$  ( $m \rightarrow \infty$ ) for every  $t \in \mathbb{C}$ . Hence, by theorem 1.13,  $f_m \xrightarrow{S} 0$ .  $\square$

2.6. We shall introduce a topology on  $S$  by considering  $S$  as the union of the subspaces  $S_n^* := N_{n-1}^{-1}(S^*)$  ( $n \in \mathbb{N}$ ). For  $n \in \mathbb{N}$  we introduce the inner products  $(\cdot, \cdot)_{n\ell}$  ( $\ell \in \mathbb{N}$ ) on  $S_n^*$  by

$$(F, G)_{n\ell} := (F(-\frac{1}{n} + \frac{1}{\ell}), G(-\frac{1}{n} + \frac{1}{\ell})) \quad (F \in S_n^*, G \in S_n^*).$$

It is obvious how to define  $F(-\frac{1}{n} + \frac{1}{\ell})$  for  $F \in S_n^*$ ,  $\ell \in \mathbb{N}$ . Now  $S_n^*$  is a nuclear space.

Let  $n \leq m$ . It is obvious that  $S_n^* \subset S_m^*$ , and that the topology on  $S_n^*$  is finer than the topology on  $S_n^*$  induced by the topology on  $S_m^*$ . Since  $S = \bigcup_{n=1}^{\infty} S_n^*$ , it makes sense to endow  $S$  with the inductive limit topology, i.e., the finest locally convex topology  $\tau$  on  $S$  such that for every  $n \in \mathbb{N}$  the topology on  $S_n^*$  is finer than the topology on  $S_n^*$  induced by  $\tau$  (cf. [FW], § 23,2).

It is not hard to see that  $\tau$  is finer than the weak topology on  $S$ . Hence, if  $f_m \in S$  ( $m \in \mathbb{N}$ ) and  $f_m \rightarrow 0$  in the sense of  $\tau$ , then  $f_m \rightarrow 0$  weakly. On the other hand, if  $f_m \in S$  ( $m \in \mathbb{N}$ ) and  $f_m \rightarrow 0$  weakly, then (by 2.5)  $f_m \xrightarrow{S} 0$ . Hence  $f_m \rightarrow 0$  in the sense of the topology of one of the spaces  $S_n^*$  (cf. 1.1.12). Since the latter topology is finer than the one on  $S_n^*$  induced by  $\tau$ , we conclude that  $f_m \rightarrow 0$  in the sense of  $\tau$ .

We finally remark that  $S$  (with the topology  $\tau$ ) is a nuclear space according to the definition used in [FW], §27,1 (this follows from 2.4 and [FW], § 27, Satz 2.8).

### 3. CONTINUOUS LINEAR FUNCTIONALS OF $S$ AND $S^*$

3.1. In this section we study continuous linear functionals of  $S$  and  $S^*$ . It will turn out to be of no concern which one of the topologies of section 2 on  $S$  (or  $S^*$ ) is taken to define continuity. We also consider bi-linear functionals defined on the product space  $S \times S$  that are continuous in both variables separately, and we shall prove the kernel theorem and a theorem on convergence in  $S^{2*}$ .

3.2. DEFINITION. A linear functional  $L$  of  $S$  is called *continuous* if  $Lf_n \rightarrow 0$  ( $n \rightarrow \infty$ ) for every sequence  $(f_n)_{n \in \mathbb{N}}$  in  $S$  with  $f_n \xrightarrow{S} 0$ . A linear functional  $L$  of  $S^*$  is called *continuous* if  $LF_n \rightarrow 0$  ( $n \rightarrow \infty$ ) for every sequence  $(F_n)_{n \in \mathbb{N}}$  in  $S^*$  with  $F_n \xrightarrow{S^*} 0$ .

3.3. THEOREM. Let  $L$  be a linear functional of  $S$ . Then continuity of  $L$  (in the sense of definition 3.2) is equivalent to each of the three following statements.

- (i) There is an  $F \in S^*$  such that  $Lf = (f, F)$  ( $f \in S$ ).
- (ii)  $L$  is continuous with respect to the weak topology.
- (iii)  $L$  is continuous with respect to the topology  $\tau$  (cf. 2.6).

PROOF. (i) Assume that  $L$  is continuous in the sense of definition 3.2. It is easy to see that  $\bigcup_{f \in L_2(\mathbb{R})} LN_\alpha f$  is a continuous linear functional of  $L_2(\mathbb{R})$  for every  $\alpha > 0$ . Hence, for every  $\alpha > 0$ , there is a  $C_\alpha > 0$  such that  $|LN_\alpha f| \leq C_\alpha \|f\|$  ( $f \in L_2(\mathbb{R})$ ). According to [B1], theorem 22.2 (note that  $L$  is quasi-bounded; cf. [B1], section 22) there is an  $F \in S^*$  such that  $Lf = (f, F)$  for every  $f \in S$ .

It is seen at once from 1.16 that  $L$  is continuous in the sense of definition 3.2 if there is an  $F \in S^*$  such that  $Lf = (f, F)$  for every  $f \in S$ .

(ii) It follows from (i) and 2.5 that  $L$  is continuous in the sense of definition 3.2 if and only if  $L$  is weakly continuous.

(iii) Let  $L$  be continuous in the sense of definition 3.2. It follows from (ii) that  $L$  is weakly continuous, and hence continuous with respect to  $\tau$  (since  $\tau$  is finer than the weak topology).

On the other hand, if  $L$  is continuous with respect to  $\tau$ , then  $Lf_n \rightarrow 0$

$(n \rightarrow \infty)$  for every sequence  $(f_n)_{n \in \mathbb{N}}$  in  $S$  with  $f_n \xrightarrow{S} 0$  (since  $S$ -convergence of a sequence is equivalent to convergence in the sense of  $\tau$ ). Hence,  $L$  is continuous in the sense of definition 3.2.  $\square$

**COROLLARY.** If  $(L_n)_{n \in \mathbb{N}}$  is a sequence of continuous linear functionals of  $S$  such that  $(L_n f)_{n \in \mathbb{N}}$  converges for every  $f \in S$ , then  $\bigcup_{f \in S} \lim_{n \rightarrow \infty} L_n f$  is a continuous linear functional of  $S$ . To show this, let  $n \in \mathbb{N}$  and take  $F_n \in S^*$  such that  $L_n f = (f, F_n)$  ( $f \in S$ ). The sequence  $(F_n)_{n \in \mathbb{N}}$  converges in  $S^*$ -sense to an  $F \in S^*$  (cf. 1.15). Hence  $\lim_{n \rightarrow \infty} L_n f = (f, F)$  for  $f \in S$ .

**3.4. THEOREM.** Let  $L$  be a linear functional of  $S^*$ . Continuity of  $L$  (in the sense of definition 3.2) is equivalent to each of the three following statements.

- (i) There is an  $f \in S$  such that  $LF = (F, f)$  ( $F \in S^*$ ).
- (ii)  $L$  is continuous with respect to the weak topology.
- (iii)  $L$  is continuous with respect to the strong topology.

**PROOF.** (i) If  $L$  is continuous in the sense of 3.2, then, by [J1], appendix 1, theorem 3.7, there exists an  $f \in S$  with  $LF = (F, f)$  ( $F \in S^*$ ).

If  $f \in S$ , then  $\bigcup_{F \in S^*} (F, f)$  is continuous in the sense of 3.2 by 1.16.

(ii) It follows from (i) and equivalence of sequential weak convergence and  $S^*$ -convergence that  $L$  is continuous in the sense of 3.2 if and only if  $L$  is continuous with respect to the weak topology.

(iii) Since  $S^*$  is a Fréchet space with the strong topology,  $L$  is strongly continuous if and only if  $LF_n \rightarrow 0$  ( $n \rightarrow \infty$ ) for every sequence  $(F_n)_{n \in \mathbb{N}}$  with  $F_n \rightarrow 0$  strongly. By 2.3 we conclude that  $L$  is strongly continuous if and only if  $LF_n \rightarrow 0$  ( $n \rightarrow \infty$ ) for every sequence  $(F_n)_{n \in \mathbb{N}}$  with  $F_n \xrightarrow{S^*} 0$ .  $\square$

**COROLLARY.** If  $(L_n)_{n \in \mathbb{N}}$  is a sequence of continuous linear functionals of  $S^*$  such that  $(L_n F)_{n \in \mathbb{N}}$  converges for every  $F \in S^*$ , then  $\bigcup_{F \in S^*} \lim_{n \rightarrow \infty} L_n F$  is a continuous linear functional of  $S^*$ . This follows from a slight generalization of theorem 2.5 (cf. the proof of 3.3, corollary).

**3.5.** We next consider bi-linear functionals defined on  $S \times S$ . A mapping  $B$  of  $S \times S$  into  $\mathbb{C}$  is called bi-linear if it is linear in each variable separately. The following lemma will be needed in the proofs of theorem 3.6 and 3.7.

**LEMMA.** Let  $B_n$  be a bi-linear functional defined on  $S \times S$ , and assume that  $B_n$  is continuous in each variable separately ( $n \in \mathbb{N}$ ). Assume that  $(B_n(f, g))_{n \in \mathbb{N}}$  converges for every  $f \in S$ ,  $g \in S$ . Then there exists exactly

one  $F \in S^{2*}$  such that  $(f \otimes g, F) = \lim_{n \rightarrow \infty} B_n(f, g)$ .

PROOF. Let  $\alpha > 0$ , and consider for every  $g \in L_2(\mathbb{R})$  the collection

$\{ \bigcup_{f \in L_2(\mathbb{R})} |B_n(N_\alpha f, N_\alpha g)| \mid n \in \mathbb{N} \}$  of bounded linear functionals of  $L_2(\mathbb{R})$ .

For every  $f \in L_2(\mathbb{R})$  there is an  $M > 0$  such that  $|B_n(N_\alpha f, N_\alpha g)| \leq M$  ( $n \in \mathbb{N}$ ).

By the Banach-Steinhaus theorem there is an  $M_g > 0$  such that

$$|B_n(N_\alpha f, N_\alpha g)| \leq M_g \|f\| \quad (n \in \mathbb{N}, f \in L_2(\mathbb{R})).$$

Next consider the collection  $\{ \bigcup_{g \in L_2(\mathbb{R})} |B_n(N_\alpha f, N_\alpha g)| \mid f \in L_2(\mathbb{R}), \|f\| = 1, n \in \mathbb{N} \}$  of bounded linear functionals of  $L_2(\mathbb{R})$ . According to what was proved above these functionals are pointwise bounded. Hence, again by the Banach-Steinhaus theorem, there is an  $M > 0$  such that

$$|B_n(N_\alpha f, N_\alpha g)| \leq M \|g\| \quad (n \in \mathbb{N}, f \in L_2(\mathbb{R}), \|f\| = 1, g \in L_2(\mathbb{R})).$$

We now proved that to every  $\alpha > 0$  there exists an  $M > 0$  such that

$$|B_n(N_\alpha f, N_\alpha g)| \leq M$$

for every  $n \in \mathbb{N}$ ,  $f \in L_2(\mathbb{R})$ ,  $g \in L_2(\mathbb{R})$  with  $\|f\| = \|g\| = 1$ . In particular we get

$$(*) \quad \forall \varepsilon > 0 \quad \exists M > 0 \quad \forall n \in \mathbb{N} \quad [ |B_n(\psi_k, \psi_\ell)| \leq M e^{(k+\ell)\varepsilon} \quad (k \in \mathbb{N}_0, \ell \in \mathbb{N}_0) ].$$

Next define  $c_{k\ell} := \lim_{n \rightarrow \infty} B_n(\psi_k, \psi_\ell)$  ( $k \in \mathbb{N}_0, \ell \in \mathbb{N}_0$ ). Then  $c_{k\ell} = 0$  ( $e^{(k+\ell)\varepsilon}$ ) ( $k \in \mathbb{N}_0, \ell \in \mathbb{N}_0$ ) for every  $\varepsilon > 0$ . Hence

$$F := \bigcup_{\alpha > 0} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \bar{c}_{k\ell} N_\alpha \psi_k \otimes N_\alpha \psi_\ell$$

is an element of  $S^{2*}$  (cf. 1.17).

Let  $f \in S$ ,  $g \in S$ . We shall show that  $(f \otimes g, F) = \lim_{n \rightarrow \infty} B_n(f, g)$ . For  $n \in \mathbb{N}$  we have (by continuity of  $B_n$  with respect to the first variable)

$$B_n(f, g) = B_n \left( \sum_{k=0}^{\infty} (f, \psi_k) \psi_k, g \right) = \sum_{k=0}^{\infty} (f, \psi_k) B_n(\psi_k, g).$$

Also (by continuity of  $B_n$  with respect to the second variable)

$$B_n(f, g) = \sum_{k=0}^{\infty} (f, \psi_k) \left( \sum_{\ell=0}^{\infty} B_n(\psi_k, \psi_\ell) (g, \psi_\ell) \right).$$

Rewriting the repeated series as a double one (cf. (\*)) and using

$(f, \psi_k) (g, \psi_\ell) = (f \otimes g, \psi_k \otimes \psi_\ell)$  ( $k \in \mathbb{N}_0, \ell \in \mathbb{N}_0$ ), we get

$$B_n(f, g) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} B_n(\psi_k, \psi_\ell) (f \otimes g, \psi_k \otimes \psi_\ell).$$

If we take  $n \rightarrow \infty$ , then we find by (\*)

$$\lim_{n \rightarrow \infty} B_n(f, g) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} c_{k\ell} (f \otimes g, \psi_k \otimes \psi_\ell) = (f \otimes g, F)$$

(also cf. 1.10).

It is obvious that there is at most one  $F \in S^{2*}$  with  $(f \otimes g, F) = \lim_{n \rightarrow \infty} B_n(f, g)$  for every  $f \in S, g \in S$ .  $\square$

REMARK. In a similar way as the above lemma the following fact can be proved. Let  $R$  be a Banach space, and let  $B_n : S \times S \rightarrow R$  be a bi-linear mapping, continuous in each variable separately ( $n \in \mathbb{N}$ ). Assume that  $(B_n(f, g))_{n \in \mathbb{N}}$  converges for every  $f \in S, g \in S$ . Then there is a continuous linear mapping  $B : S^2 \rightarrow R$  such that  $B(f, g) = \lim_{n \rightarrow \infty} B_n(f, g)$  for every  $f \in S, g \in S$ .

3.6. The kernel theorem is an immediate consequence.

THEOREM. Let  $B$  a bi-linear functional defined on  $S \times S$ , and assume that  $B$  is continuous in each variable separately. Then there exists exactly one  $F \in S^{2*}$  such that  $(f \otimes g, F) = B(f, g)$  for every  $f \in S, g \in S$ .

PROOF. Take  $B_n = B$  for  $n \in \mathbb{N}$  in theorem 3.5.  $\square$

REMARK. Let  $F \in S^{2*}$ . Then the bi-linear functional  $\Psi_{[f, g] \in S \times S} (f \otimes g, F)$  is continuous in each variable separately.

3.7. Another consequence of lemma 3.5 is the following theorem.

THEOREM. Let  $(F_n)_{n \in \mathbb{N}}$  be a sequence in  $S^{2*}$ . Then  $(F_n)_{n \in \mathbb{N}}$  is  $S^{2*}$ -convergent if and only if  $\lim_{n \rightarrow \infty} (f \otimes g, F_n)$  exists for every  $f \in S, g \in S$ .

PROOF. Assume that  $\lim_{n \rightarrow \infty} (f \otimes g, F_n)$  exists for every  $f \in S, g \in S$ . By lemma 3.5 and 3.6, remark we can find an  $F \in S^{2*}$  such that  $(f \otimes g, F) =$



$= \lim_{n \rightarrow \infty} (f \otimes g, F_n)$  for  $f \in S$ ,  $g \in S$ . We have as in the proof of lemma 3.5

$$\forall \varepsilon > 0 \exists M > 0 \forall n \in \mathbb{N} [ |(\psi_k \otimes \psi_\ell, F_n)| \leq M e^{(k+\ell)\varepsilon} \quad (k \in \mathbb{N}_0, \ell \in \mathbb{N}_0) ].$$

From this it is not hard to prove that  $(h, F_n - F) \rightarrow 0$  ( $n \rightarrow \infty$ ) for every  $h \in S^2$ . The theorem easily follows from the fact that  $F_n \xrightarrow{S^{2*}} F$  if and only if  $F_n \rightarrow F$  weakly.  $\square$

#### 4. CONTINUOUS LINEAR OPERATORS OF $S$ AND $S^*$

4.1. We quote the main results of [J1], appendix 1, section 2 and 3 about linear operators of  $S$  and  $S^*$ . Again, it will turn out to be of no concern which one of the topologies of section 2 on  $S$  (or  $S^*$ ) is taken to define continuity. In the theory of continuous linear operators of  $S^*$  an important role is played by linear mappings of  $S$  into  $S$  with an adjoint that also maps  $S$  into  $S$ .

4.2. DEFINITION. A linear operator  $T$  of  $S$  is called *continuous* if  $Tf_n \xrightarrow{S} 0$  for every sequence  $(f_n)_{n \in \mathbb{N}}$  in  $S$  with  $f_n \xrightarrow{S} 0$ . A linear operator  $T$  of  $S^*$  is called *continuous* if  $TF_n \xrightarrow{S^*} 0$  for every sequence  $(F_n)_{n \in \mathbb{N}}$  in  $S^*$  with  $F_n \xrightarrow{S^*} 0$ .

4.3. DEFINITION. A linear operator  $T$  of  $S$  is said to *have an adjoint* if for every  $g \in S$  there is a  $g^* \in S$  such that  $(Tf, g) = (f, g^*)$  for  $f \in S$ .

If  $T$  has an adjoint, then the  $g^*$  of the above definition is uniquely determined by  $T$  and depends linearly on  $g \in S$ . Hence  $T^* := \bigcup_{g \in S} g^*$  is a linear operator of  $S$ , called the *adjoint* of  $T$ .

4.4. The main results of [J1], appendix 1, section 2 are listed in the following theorem.

THEOREM. Let  $T$  be a linear operator of  $S$ . The following statements are equivalent.

- (i)  $T$  is continuous.
- (ii)  $\bigcup_{f \in S} (Tf)(t)$  is a continuous linear functional of  $S$  for every  $t \in \mathbb{C}$ .
- (iii)  $\bigcup_{f \in S} (Tf, g)$  is a continuous linear functional of  $S$  for every  $g \in S$ .
- (iv)  $Tf_n$  is pointwise bounded for every sequence  $(f_n)_{n \in \mathbb{N}}$  in  $S$  with  $f_n \xrightarrow{S} 0$ .

- (v)  $TN_\alpha$  is a bounded linear operator of  $L_2(\mathbb{R})$  for every  $\alpha > 0$ .
- (vi) For every  $\alpha > 0$  there exists a  $\beta > 0$  and a bounded linear operator  $T_1$  of  $L_2(\mathbb{R})$  such that  $TN_\alpha = N_\beta T_1$ .
- (vii)  $TN_\alpha$  has an adjoint for every  $\alpha > 0$ .

For the proof of this theorem we refer to [J1], appendix 1, 2.2 through 2.9.

4.5. We give some further results on continuous linear operators of  $S$ .

**THEOREM.** Let  $T$  be a linear operator of  $S$ . Then continuity of  $T$  (in the sense of definition 4.2) is equivalent to each of the three following statements.

- (i)  $\bigcup_{F \in S^*} (Tf, F)$  is a continuous linear functional of  $S$  for every  $F \in S^*$ .
- (ii)  $T$  is continuous with respect to the weak topology.
- (iii)  $T$  is continuous with respect to the topology  $\tau$  of 2.6.

**PROOF.** (i) If  $T$  is continuous in the sense of definition 4.2, then  $(Tf_n, F) \rightarrow 0$  ( $n \rightarrow \infty$ ) for every sequence  $(f_n)_{n \in \mathbb{N}}$  in  $S$  with  $f_n \xrightarrow{S} 0$  ( $F \in S^*$ ). Hence  $\bigcup_{F \in S^*} (Tf, F)$  is a continuous linear functional of  $S$  ( $F \in S^*$ ).

On the other hand if  $\bigcup_{F \in S^*} (Tf, F)$  is a continuous linear functional for every  $F \in S^*$ , and if  $(f_n)_{n \in \mathbb{N}}$  is a sequence in  $S$  with  $f_n \xrightarrow{S} 0$ , then  $(Tf_n, F) \rightarrow 0$  for every  $F \in S^*$ , whence  $Tf_n \xrightarrow{S} 0$  by theorem 2.5.

(ii) It is trivial that  $T$  is continuous with respect to the weak topology if and only if  $\bigcup_{F \in S^*} (Tf, F)$  is a continuous linear functional of  $S$  for every  $F \in S^*$ .

(iii)  $(S, \tau)$  is the inductive limit of the Fréchet subspaces  $S_n^*$  ( $n \in \mathbb{N}$ ). Since every Fréchet space is a bornological space (cf. [FW], §11, 4.2),  $S$  is also a bornological space (cf. [FW], §23, 2.9). Hence  $T$  is continuous with respect to  $\tau$  if and only if  $Tf_n \rightarrow 0$  in the sense of  $\tau$  for every sequence  $(f_n)_{n \in \mathbb{N}}$  in  $S$  with  $f_n \rightarrow 0$  in the sense  $\tau$  (cf. [FW], §11, 3.2). Now sequential convergence in the sense of  $\tau$  is the same as  $S$ -convergence (cf. 2.6). Hence  $T$  is continuous with respect to  $\tau$  if and only if  $T$  is continuous in the sense of definition 4.2.  $\square$

4.6. **DEFINITION.** Let  $T$  be a linear operator of  $S$ . Then the operators  $\bar{T}$  and  $\underline{T}$  are defined respectively by

$$\bar{T}f := \overline{Tf} \quad , \quad \underline{T}f = (Tf)_- \quad (f \in S),$$

where for  $g \in S$

$$\bar{g} := \bigcup_{z \in \mathbb{C}} \overline{g(z)}, \quad g_- := \bigcup_{z \in \mathbb{C}} g(-z).$$

(Note that  $\bar{g} \in S$ ,  $g_- \in S$  for  $g \in S$ .)

4.7. THEOREM. (i) If  $T$  is a continuous linear operator of  $S$ , then so are  $\bar{T}$  and  $T_-$ .

(ii) If  $T$  is a linear operator of  $S$  with an adjoint, then  $T$  is continuous, and so is  $T^*$ . Furthermore  $\bar{T}$  and  $T_-$  have adjoints, and  $(\bar{T})^* = \overline{(T^*)}$ ,  $(T_-)^* = (T^*)_-$ .

PROOF. The first statement of (ii) follows from 4.4 (vii); the others are trivial.  $\square$

4.8. All linear operators of  $S$  occurring in practice (e.g. those of 1.7) are continuous. If one accepts the axiom of choice, one can give an example of a linear operator of  $S$  that is not continuous (cf. also [B1], 27.22).

The following theorem gives some closedness properties of the class of continuous linear operators of  $S$ .

THEOREM. Let  $(T_n)_{n \in \mathbb{N}}$  be a sequence of continuous linear operators of  $S$ .

(i) If  $N \in \mathbb{N}$ , and  $P$  is a polynomial in  $N$  variables, then  $P(T_1, \dots, T_N)$  is a continuous linear operator of  $S$ .

(ii) Assume that for every  $f \in S$  there is a  $Tf \in S$  such that  $T_n f \rightarrow Tf$  pointwise. Then  $T$  is a continuous linear operator of  $S$ .

(iii) Assume that for every  $f \in S$  there is a  $Tf \in S$  such that  $\text{emb}(T_n f) \xrightarrow{S^*} \text{emb}(Tf)$  (cf. 1.11 (i)). Then  $T$  is a continuous linear operator of  $S$ .

PROOF. By [J1], appendix 1, 2.11 and 2.12 we only have to prove (iii). Let  $g \in S$ . Now  $(Tf, g) = \lim_{n \rightarrow \infty} (T_n f, g)$  for every  $f \in S$ , whence  $\bigcup_{f \in S} (Tf, g)$  is a continuous linear functional of  $S$  (cf. 3.3, corollary). It follows from theorem 4.4 (iii) that  $T$  is continuous.  $\square$

REMARK. It can be proved that the inverse of a continuous linear bijection of  $S$  is continuous.

4.9. As to continuous linear operators of  $S^n$  ( $n \in \mathbb{N}$ ) the following remarks are in order. The previous definitions and theorems hold (with the proper modifications) also for the case of linear operators of  $S^n$ . Let us restrict

ourselves for notational convenience to the case  $n = 2$ .

An important example of a continuous linear operator of  $S^2$  is the operator  $T_1^{(1)}$  (where  $T_1$  is a continuous linear operator of  $S$ ), defined by

$$T_1^{(1)} f := \int_{(z_1, z_2) \in \mathbb{C}^2} T_1 \left( \int_{t \in \mathbb{C}} f(t, z_2) \right) (z_1) \quad (f \in S^2).$$

Similarly, the operator  $T_2^{(2)}$  (where  $T_2$  is a continuous linear operator of  $S$ ), defined by

$$T_2^{(2)} f := \int_{(z_1, z_2) \in \mathbb{C}^2} T_2 \left( \int_{t \in \mathbb{C}} f(z_1, t) \right) (z_2) \quad (f \in S^2),$$

is a continuous linear operator of  $S^2$ . It follows from a 2-dimensional version of 4.8 (i) that  $T_1^{(1)} T_2^{(2)}$  (often denoted by  $T_1 \otimes T_2$  and called the *tensor product* of  $T_1$  and  $T_2$ ) is a continuous linear operator of  $S^2$ . We also have  $T_1^{(1)} T_2^{(2)} = T_2^{(2)} T_1^{(1)}$ . For proofs we refer to [J1], appendix 1, 2.13.

The operator  $Z_U$  defined by  $Z_U f = \int_{(x, y)} f\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right)$  for  $f \in S$  is also an example of a continuous linear operator of  $S^2$ , and so is  $F^{(2)} Z_U$  (this example is closely related to the Wigner distribution).

4.10. We next consider continuous linear operators of  $S^*$ . The proof of the following theorem is almost the same as the one of theorem 4.5; we omit it.

**THEOREM.** Let  $T$  be a linear operator of  $S^*$ . Then continuity of  $T$  (in the sense of definition 4.2) is equivalent to each of the three following statements.

- (i)  $\int_{F \in S^*} (TF, f)$  is a continuous linear functional of  $S^*$  for every  $f \in S$ .
- (ii)  $T$  is continuous with respect to the weak topology.
- (iii)  $T$  is continuous with respect to the strong topology.

**COROLLARY.** If  $T$  is a continuous linear operator of  $S^*$ , then there is a continuous linear operator  $\tilde{T}$  of  $S$  such that  $(TF, g) = (F, \tilde{T}g)$  for  $F \in S^*$ ,  $g \in S$ . To show this, let  $g \in S$ . By (i) of the above theorem and 3.4 (i) there is exactly one  $f_g \in S$  with  $(TF, g) = (F, f_g)$  for  $F \in S^*$ . It is easy to show that  $f_g$  depends linearly and continuously on  $g$ . Hence,  $\tilde{T} := \int_{g \in S} f_g$  satisfies the requirements.

4.11. Theorem 3.2 of [J1], appendix 1, section 3, reads as follows.

THEOREM. If  $T$  is a linear operator of  $S$  with an adjoint  $T^*$ , then it is possible to extend  $T$  to a continuous linear operator  $T_1$  of  $S^*$  such that

$$\begin{aligned} T_1(\text{emb}(f)) &= \text{emb}(Tf) & (f \in S), \\ (T_1 F, g) &= (F, T^* g) & (F \in S^*, g \in S). \end{aligned}$$

Furthermore, such an extension is unique.

We shall denote the extended operator by  $T$  again.

There is a certain converse of this theorem (cf. [J1], appendix 1, 3.8): if  $T$  is a linear operator of  $S$  that is extended to a continuous linear operator  $T_1$  of  $S^*$  such that  $T_1(\text{emb}(f)) = \text{emb}(Tf)$  ( $f \in S$ ), then  $T$  has an adjoint.

4.12. DEFINITION. Let  $T$  be a linear operator of  $S^*$ . Then the operators  $\bar{T}$  and  $T_-$  are defined respectively by

$$\bar{T}F := \overline{TF}, \quad T_- F = (TF)_- \quad (F \in S^*),$$

where for  $G \in S^*$  (cf. 4.6)

$$\bar{G} := \bigcup_{\alpha > 0} \bar{G}_\alpha, \quad G_- := \bigcup_{\alpha > 0} (G_\alpha)_-.$$

(Note that  $\bar{G} \in S^*$ ,  $G_- \in S^*$ , and that  $(\bar{G}, g) = \overline{(G, g)}$ ,  $(G_-, g) = (G, g_-)$  for  $G \in S^*$ ,  $g \in S$ .)

4.13. THEOREM. (i) If  $T$  is a continuous linear operator of  $S^*$ , then so are  $\bar{T}$  and  $T_-$ .

(ii) If  $T$  is the extension of a linear operator  $T_1$  of  $S$  with an adjoint, then  $\bar{T}$  and  $T_-$  are the extensions of  $\bar{T}_1$  and  $(T_1)_-$  respectively (cf. 4.7 (ii)).

PROOF. Almost trivial.  $\square$

4.14. The proof of the following theorem is almost the same as the one of (i) and (iii) of theorem 4.8; we omit it.

THEOREM. Let  $(T_n)_{n \in \mathbb{N}}$  be a sequence of continuous linear operators of  $S^*$ .

(i) If  $N \in \mathbb{N}$ , and  $P$  is a polynomial in  $N$  variables, then  $P(T_1, \dots, T_N)$  is a continuous linear operator of  $S^*$ .

(ii) Assume that for every  $F \in S^*$  there is a  $TF \in S^*$  such that  $T_n F \xrightarrow{S^*} TF$ . Then  $T$  is a continuous linear operator of  $S^*$ .

REMARK. It can be proved that the inverse of a continuous linear bijection of  $S^*$  is continuous.

4.15. EXAMPLE. The operators  $N_a$ ,  $F$ ,  $T_a$  ( $a \in \mathbb{C}$ ),  $R_b$  ( $b \in \mathbb{C}$ ),  $Z_\lambda$  ( $\lambda > 0$ ),  $P$ ,  $Q$  have adjoints, so they can be extended to continuous linear operators of  $S^*$ . The same holds for the convolution and multiplication operators of appendix 2.

We note that relations between linear operators of  $S$  with an adjoint remain valid after extension to  $S^*$ . E.g., we have  $QF = FP$  for every  $F \in S^*$ .

4.16. We make the following remarks about linear operators of  $S^{n^*}$  ( $n \in \mathbb{N}$ ). The definitions and theorems of 4.10 through 4.15 can be given (with proper modifications) for the case of linear operators of  $S^{n^*}$ . For notational convenience we shall restrict ourselves to the case  $n = 2$ .

An important example of a linear operator of  $S^2$  with an adjoint is the operator  $T_1^{(1)}$  (cf. 4.9), where  $T_1$  is a linear operator of  $S$  with an adjoint. Then  $(T_1^{(1)})^* = (T_1^*)^{(1)}$ , and a similar thing holds for  $T_2^{(2)}$ , where  $T_2$  is a linear operator of  $S$  with an adjoint. We note that  $T_1^{(1)} T_2^{(2)} = T_2^{(2)} T_1^{(1)}$  holds for the extended operators; we write  $T_1 \otimes T_2$  instead of  $T_1^{(1)} T_2^{(2)}$  or  $T_2^{(2)} T_1^{(1)}$ . As an example we mention that every  $F \in S^{2^*}$  is differentiable in both variables, and that the order of differentiation is immaterial:  $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$ . For the proofs we refer to [J1], appendix 1, 3.12.

As examples of extendable linear operators of  $S^2$  we further mention  $Z_U$  and  $F^{(2)}_{Z_U}$  (cf. the end of 4.9).

## 5. $S^*$ AS A MEASURE SPACE

5.1. DEFINITION.  $\Lambda^*$  is the  $\sigma$ -algebra on  $S^*$  generated by the weak topology.

5.2. It is useful to have alternative descriptions of the  $\sigma$ -algebra  $\Lambda^*$ . Let  $\Lambda_1^*$  be the  $\sigma$ -algebra on  $S^*$  generated by the strong topology, and let  $\Lambda_2^*$  be the  $\sigma$ -algebra on  $S^*$  generated by all sets of the form  $\{F \mid F_\alpha(t) \in O\}$ , where  $\alpha > 0$ ,  $t \in \mathbb{C}$ ,  $O$  an open set in  $\mathbb{C}$ .

THEOREM.  $\Lambda^* = \Lambda_1^* = \Lambda_2^*$ .

PROOF. (i) Let  $A$  be a strongly open set in  $S^*$  of the form  $\{F \in S^* \mid \|N_\alpha(F - F_0)\| < \varepsilon\}$ , where  $\alpha > 0$ ,  $\varepsilon > 0$ ,  $F_0 \in S^*$ . This  $A$  consists of all  $F \in S^*$  with  $\sum_{k=0}^{\infty} |(F - F_0, \psi_k)|^2 e^{-(2k+1)\alpha} < \varepsilon^2$ . Since the mapping  $\bigcup_{F \in S^*} \sum_{k=0}^{\infty} |(F - F_0, \psi_k)|^2 e^{-(2k+1)\alpha}$  is  $\Lambda^*$ -measurable, we conclude that  $A \in \Lambda^*$ .

Now let  $B$  be any strongly open set in  $S^*$ . Since the strong topology of  $S^*$  has a countable base of sets of the form  $\{F \mid \|N_\alpha(F - F_0)\| < \varepsilon\}$  with  $\alpha > 0$ ,  $\varepsilon > 0$ ,  $F_0 \in S^*$  (cf. 2.3), we easily infer that  $B \in \Lambda^*$ . Hence  $\Lambda_1^* \subset \Lambda^*$ .

On the other hand, the strong topology of  $S^*$  is finer than the weak topology, whence  $\Lambda^* \subset \Lambda_1^*$ . Therefore  $\Lambda_1^* = \Lambda^*$ .

(ii) Let  $\alpha > 0$ ,  $t \in \mathcal{T}$ , and let  $O$  be an open set in  $\mathcal{C}$ . Since  $F_\alpha(t) = (F, (\delta(\bar{t}))_\alpha)$  ( $F \in S^*$ ) and  $(\delta(\bar{t}))_\alpha \in S$ , we infer that  $\bigcup_{F \in S^*} F_\alpha(t)$  is  $\Lambda^*$ -measurable, whence  $\{F \mid F_\alpha(t) \in O\} \in \Lambda^*$ . Therefore  $\Lambda_2^* \subset \Lambda^*$ .

Now let  $\alpha > 0$ ,  $\varepsilon > 0$ ,  $F_0 \in S^*$ . The set  $A := \{F \mid \|N_\alpha(F - F_0)\| < \varepsilon\}$  consists of all  $F \in S^*$  with

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=-N}^N |(N_\alpha(F - F_0))(\frac{k}{N})|^2 < \varepsilon^2.$$

Hence  $A$  is  $\Lambda_2^*$ -measurable. It follows as in (i) that  $\Lambda_1^* \subset \Lambda_2^*$ , whence  $\Lambda^* = \Lambda_1^* = \Lambda_2^*$ . □

REMARK. If  $n \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_n \in S$ ,  $B \in \mathcal{B}(\mathcal{C}^n)$ , then  $\{F \mid ((F, f_1), \dots, (F, f_n)) \in B\}$  is called a *Borel cylinder*. It follows easily from the above theorem that  $\Lambda^*$  equals the  $\sigma$ -algebra generated by all Borel cylinders.

5.3. We note that  $S^*$  is a Polish space with the strong topology (i.e.  $S^*$  is a separable space for which there exists a metric such that the metric topology is the strong topology, and such that the space is complete; cf. 2.3). Hence  $S^*$  is a Lusin space with both the weak topology and the topology  $\tau_1$  generated by all sets of the form  $\{F \mid F_\alpha(t) \in O\}$  with  $\alpha > 0$ ,  $t \in \mathcal{T}$ ,  $O$  an open set in  $\mathcal{C}$  (cf. [S], Ch. II, section 1, definition 2). This implies, by [S], Ch. II, section 3, theorem 9, that  $S^*$  is a Radon space with respect to the strong topology, the weak topology and the topology  $\tau_1$  (i.e. every finite measure defined on  $(S^*, \Lambda^*)$  is inner regular with respect to each of the three topologies).

5.4. THEOREM. Let  $T$  be a continuous linear operator of  $S^*$ . Then  $T^{-1}(\Lambda^*) \subset \Lambda^*$ .

PROOF. The class  $C$  of all elements  $A$  of  $\Lambda^*$  with  $T^{-1}(A) \in \Lambda^*$  is a  $\sigma$ -algebra. If  $A$  is a set of the form  $\{F \mid ((F, f_1), \dots, (F, f_n)) \in B\}$  with  $n \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_n \in S$ ,  $B \in \mathcal{B}(C^n)$ , then

$$T^{-1}(A) = \{F \mid ((F, \tilde{T}f_1), \dots, (F, \tilde{T}f_n)) \in B\},$$

where  $\tilde{T}$  is as in 4.10, corollary. Hence  $T^{-1}(A) \in \Lambda^*$ . Since  $\Lambda^*$  is generated by the collection of all sets  $A$  of the above type, we obtain  $C = \Lambda^*$ .  $\square$

5.5. We give some examples of measurable subsets of  $S^*$ .

(i) The set  $\text{emb}(L_p(\mathbb{R}))$  with  $1 \leq p < \infty$ . We first show that  $F \in \text{emb}(L_p(\mathbb{R}))$  if and only if  $\lim_{n \rightarrow \infty, m \rightarrow \infty} \|N_{n-1}F - N_{m-1}F\|_p = 0$  ( $\|\cdot\|_p$  denotes ordinary  $p$ -norm).

If  $F = \text{emb}(f)$  with  $f \in L_p(\mathbb{R})$ , then we infer from theorem 1.6 that

$$\lim_{n \rightarrow \infty, m \rightarrow \infty} \|N_{n-1}F - N_{m-1}F\|_p = \lim_{n \rightarrow \infty, m \rightarrow \infty} \|N_{n-1}f - N_{m-1}f\|_p = 0.$$

Conversely, let  $F \in S^*$ , and assume that  $\lim_{n \rightarrow \infty, m \rightarrow \infty} \|N_{n-1}F - N_{m-1}F\|_p = 0$ . There is an  $f \in L_p(\mathbb{R})$  such that  $\lim_{n \rightarrow \infty} \|N_{n-1}F - f\|_p = 0$ . We obtain for  $g \in S$

$$(F, g) = \lim_{n \rightarrow \infty} (N_{n-1}F, g) = \int_{-\infty}^{\infty} f(t) \overline{g(t)} dt,$$

whence  $F = \text{emb}(f)$ .

We thus have  $F \in \text{emb}(L_p(\mathbb{R}))$  if and only if

$$\forall k \in \mathbb{N} \exists N \in \mathbb{N} \forall n \in \mathbb{N}, m \in \mathbb{N}, \left[ \int_{-\infty}^{\infty} |(N_{n-1}F)(t) - (N_{m-1}F)(t)|^p dt < k^{-1} \right].$$

The proof of measurability of  $\text{emb}(L_p(\mathbb{R}))$  will be completed if we can show that  $\{F \mid \int_{-\infty}^{\infty} |(N_{n-1}F)(t) - (N_{m-1}F)(t)|^p dt < k^{-1}\}$  is a measurable set for every  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}$ ,  $k \in \mathbb{N}$ . We note that for  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}$ ,  $F \in S^*$

$$\begin{aligned} & \int_{-\infty}^{\infty} |(N_{n-1}F)(t) - (N_{m-1}F)(t)|^p dt = \\ & = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{\ell=-L/2}^{L/2} \left| (N_{n-1}F)\left(\frac{\ell}{L}\right) - (N_{m-1}F)\left(\frac{\ell}{L}\right) \right|^p, \end{aligned}$$



and the sums on the right hand side are measurable as functions of  $F \in S^*$ .

(ii) The set  $\text{emb}(S^+)$ . It is not hard (but somewhat laborious) to show that  $F \in \text{emb}(S^+)$  if and only if  $g_k \cdot F \in \text{emb}(L_1(\mathbb{R}))$  for every  $k \in \mathbb{N}$ . Here  $g_k$  denotes  $\int_z \exp(-\pi k^{-1} z^2)$  for  $k \in \mathbb{N}$  (cf. appendix 2,6 for the definition of  $g_k \cdot F$ ). Now  $\text{emb}(S^+) = \bigcap_{k=1}^{\infty} T_k^{-1}(\text{emb}(L_1(\mathbb{R})))$  where  $T_k$ , defined as

$$T_k F := g_k \cdot F \quad (F \in S^*),$$

is a continuous linear operator of  $S^*$  for  $k \in \mathbb{N}$ . Measurability of  $\text{emb}(S^+)$  follows from theorem 5.4 and (i).

(iii) The set  $\text{emb}(L_{\infty}(\mathbb{R}))$ . It can be proved that  $F \in \text{emb}(L_{\infty}(\mathbb{R}))$  if and only if  $F \in \text{emb}(S^+)$ , and  $(N_{n-1}F)(t)$  is bounded in  $n \in \mathbb{N}$ ,  $t \in \mathbb{R}$ . The latter condition can also be formulated as

$$\exists M \in \mathbb{N} \quad \forall n \in \mathbb{N} \quad \forall q \in \mathbb{Q} \quad [ |(N_{n-1}F)(q)| \leq M ].$$

From this measurability of  $\text{emb}(L_{\infty}(\mathbb{R}))$  easily follows.

(iv) The set  $\text{emb}(C^+)$ , where  $C^+$  denotes the set of all continuous elements of  $S^+$ . It is not hard (but somewhat laborious) to prove that  $F \in \text{emb}(C^+)$  if and only if  $F \in \text{emb}(S^+)$  and  $\lim_{k \rightarrow \infty} (N_{k-1}F)(t)$  exists locally uniformly in  $t \in \mathbb{R}$ . The latter condition can also be formulated as: for every  $A \in \mathbb{N}$ ,  $n \in \mathbb{N}$  there is a  $K \in \mathbb{N}$  such that

$$\forall_{k \in \mathbb{N}, \ell \in \mathbb{N}, \forall_{q \in \mathbb{Q}, |q| \leq A} \quad [ |(N_{k-1}F)(q) - (N_{\ell-1}F)(q)| < n^{-1} ].$$

$$k > K, \ell > K$$

From this measurability of  $\text{emb}(C^+)$  easily follows.

(v) The set  $\text{emb}(C_n^+)$ , where  $n \in \mathbb{N}$  and  $C_n^+$  is the set of all elements of  $S^+$  whose  $n^{\text{th}}$  derivative exists everywhere on  $\mathbb{R}$  and belongs to  $C^+$ . It can be proved that  $F \in \text{emb}(C_n^+)$  if and only if  $P^n F \in \text{emb}(C^+)$ . Since  $P^n$  maps  $S^*$  continuously into  $S^*$  we infer from theorem 5.4 that  $\text{emb}(C_n^+)$  is measurable. Similarly, the set  $\text{emb}(C_{\infty}^+)$  (of all elements of  $S^+$  whose derivatives exist up to any order and belong to  $S^+$ ) is a measurable set in  $S^*$ .

(vi) The set  $\text{emb}(A^+)$ , where  $A^+$  denotes the set of all elements of  $S^+$  which have an analytic continuation to the entire complex plane. It can be proved that  $F \in \text{emb}(A^+)$  if and only if  $F \in \text{emb}(S^+)$  and  $\lim_{n \rightarrow \infty} (N_{n-1}F)(t)$  exists locally uniformly in  $\mathbb{C}$ . Measurability of  $\text{emb}(A^+)$  is proved as in (iv).

(vii) The set  $\text{emb}(S)$ . We have  $F \in \text{emb}(S)$  if and only if there is an  $n \in \mathbb{N}$ ,  $M \in \mathbb{N}$  with  $|(F, \psi_k)| \leq M \exp(-n^{-1}k)$  ( $k \in \mathbb{N}_0$ ). Measurability of  $\text{emb}(S)$  is easily proved.

5.6. The space  $S$  endowed with the inductive limit topology  $\tau$  of 2.6, is a nuclear space, and  $S^*$  can be regarded as its dual (cf. theorem 3.3). There is a large amount of literature on cylindrical measures defined on the Borel cylinders of the dual of a nuclear space. Classical results are Minlos' theorem about  $\sigma$ -additivity of cylindrical measures, and the theorem of Sazonov-Badrikian about the relation between Radon measures and positive definite functions (cf. [S], Ch. III, §3, theorem 2 and Ch. III, §4, theorem 3; cf. also [Sche], section 3).

We observe that the methods of the proof of theorem 1.1.15 can be used to derive a version of Minlos' theorem for  $S^*$ . To go into some more detail let  $P = \{P(f_1, \dots, f_n) \mid n \in \mathbb{N}, f_1 \in S, \dots, f_n \in S\}$  be a cylindrical measure on the Borel cylinders of  $S^*$ . So  $P(f_1, \dots, f_n)$  is a probability measure on  $\mathbb{T}^n$  for every  $n \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_n \in S$ , and if  $m \in \mathbb{N}$ ,  $g_1 \in S, \dots, g_m \in S$ ,  $A \in \mathcal{B}(\mathbb{T}^n)$ ,  $B \in \mathcal{B}(\mathbb{T}^m)$ , then

$$(P(f_1, \dots, f_n))(A) = (P(g_1, \dots, g_m))(B)$$

whenever  $\{F \mid ((F, f_1), \dots, (F, f_n)) \in A\} = \{F \mid ((F, g_1), \dots, (F, g_m)) \in B\}$ . Now Minlos' theorem states that, under certain continuity conditions, there exists a probability measure  $P^*$  on  $(S^*, \mathcal{A}^*)$  such that

$$P^* (\{F \mid ((F, f_1), \dots, (F, f_n)) \in A\}) = (P(f_1, \dots, f_n))(A) \text{ for every } n \in \mathbb{N},$$

$f_1 \in S, \dots, f_n \in S$ ,  $A \in \mathcal{B}(\mathbb{T}^n)$ . We sketch the proof of this fact. Apply the Daniell-Kolmogorov theorem ([T], 2.3, theorem 2) to the consistent set

$\mathcal{P}_\Psi := \{P(\psi_{k_1}, \dots, \psi_{k_j}) \mid j \in \mathbb{N}, k_1 \in \mathbb{N}_0, \dots, k_j \in \mathbb{N}_0\}$  of probability measures ( $\psi_k$  denotes as usual the  $k^{\text{th}}$  Hermite function for  $k \in \mathbb{N}_0$ ) to obtain a probability measure  $P$  on the space  $\mathbb{T}^{\mathbb{N}_0}$  (with ordinary product  $\sigma$ -algebra)

such that  $(P(\psi_{k_1}, \dots, \psi_{k_j}))(A) = P(\{\omega = (\omega_k)_{k \in \mathbb{N}_0} \in \mathbb{T}^{\mathbb{N}_0} \mid (\omega_{k_1}, \dots, \omega_{k_j}) \in A\})$  for  $j \in \mathbb{N}$ ,  $k_1 \in \mathbb{N}_0, \dots, k_j \in \mathbb{N}_0$ ,  $A \in \mathcal{B}(\mathbb{T}^j)$ . It can be shown (cf. the proof of 1.1.16) that, under fairly weak continuity conditions,

$$P(\{\omega = (\omega_k)_{k \in \mathbb{N}_0} \in \mathbb{T}^{\mathbb{N}_0} \mid \forall \varepsilon > 0 [\omega_k = 0 (e^{k\varepsilon}) (k \in \mathbb{N}_0)]\}) = 1,$$

and that for every  $n \in \mathbb{N}$ ,  $f_1 \in S, \dots, f_n \in S$  the probability measure on  $\mathbb{T}^n$  generated by the distribution function of

$$\psi_{\omega} = (\omega_k)_{k \in \mathbb{N}_0} \sum_{k=0}^{\infty} \omega_k ((\psi_k, f_1), \dots, (\psi_k, f_n))$$

equals  $P(f_1, \dots, f_n)$ . The proof of Minlos' theorem can be completed now in the same way as the proof of theorem 1.1.15.

## APPENDIX 2

CONVOLUTION THEORY IN  $S$  AND  $S^*$ 

1. This appendix gives the relevant notions and theorems on convolution theory in  $S$  and  $S^*$ . Further information about this theory can be found in [J2].

2. DEFINITION. If  $F \in S^*$ , then we define  $T_F$  by putting

$$T_F f := \bigcup_{t \in \mathbb{C}} (T_t f, F_-)$$

for  $f \in S$  (cf. appendix 1, 4.12;  $T_t$  denotes for  $t \in \mathbb{C}$  the ordinary shift operator of appendix 1, 1.8(ii)). If  $F = \text{emb}(f)$  with  $f \in S$ , then we write  $T_f$  instead of  $T_{\text{emb}(f)}$ .

To avoid confusion with the shift operators, we shall always denote convolution operators by  $T_f, T_g, T_h, \dots, T_F, T_G, T_H, \dots$  with  $f \in S, g \in S, h \in S, \dots, F \in S^*, G \in S^*, H \in S^*, \dots$ , whereas shift operators are denoted by  $T_a, T_b, T_c, \dots, T_p, T_q, \dots, T_t, T_s, \dots, T_x, T_y, T_z, \dots$  with  $a \in \mathbb{C}, b \in \mathbb{C}, c \in \mathbb{C}, \dots, p \in \mathbb{C}, q \in \mathbb{C}, \dots, t \in \mathbb{C}, s \in \mathbb{C}, \dots, x \in \mathbb{C}, y \in \mathbb{C}, z \in \mathbb{C}, \dots$ .

3. DEFINITION. The class  $C$  is defined as the set of all  $F \in S^*$  such that  $T_F$  maps  $S$  into  $S$ .

4. EXAMPLES. (i)  $\text{emb}(S) \subset C \subset S^*$ .

(ii) If  $a \in \mathbb{C}$ ,  $F = \delta_a$ , then  $T_F = T_{-a}$ , whence  $\delta_a \in C$ .

(iii) If  $f$  is an integrable function defined on  $\mathbb{R}$  with a compact support, then  $\text{emb}(f) \in C$ .

5. THEOREM. Let  $F \in C, G \in C, H \in C$ . Then

(i)  $F_- \in C, \bar{F} \in C, \bar{F}_- \in C$  (cf. appendix 1, 4.12).

(ii)  $T_F$  is a continuous linear operator of  $S$ , and  $\bar{T}_F = T_{\bar{F}}$ ,  $(T_F)_- = T_{F_-}$ ,  $(\bar{T}_F)_- = \bar{T}_{F_-}$  (cf. appendix 1, 4.12).

(iii)  $T_F$  has an adjoint (cf. appendix 1, 4.3), viz.  $T_{\bar{F}_-}$ .

Extend  $T_F, T_G, T_H$  to continuous linear operators of  $S^*$  according to appendix 1, 4.11. Then

(iv)  $T_F G = T_{\bar{G}} \bar{F} \in C$ .

$$(iv) T_F G = T_G \bar{F} \quad .$$

$$(v) T_F T_G = T_{T_F G} = T_{T_G F} = T_G T_F.$$

PROOF. Cf. [J2], section 3. □

6. DEFINITION. Let  $h: \mathbb{C} \rightarrow \mathbb{C}$  satisfy  $\forall_{\epsilon > 0} [\int_z h(z) \exp(-\pi \epsilon z^2) \in S]$ . We define the multiplication operator  $M_h$  by putting

$$M_h f = \int_z h(z) f(z)$$

for  $f \in S$ .

If  $h$  is as in the above definition, then  $M_h$  can be extended in the familiar way (cf. appendix 1, 4.11) to a continuous linear operator of  $S^*$  (again denoted by  $M_h$ ). We shall often write  $h.F$  instead of  $M_h F$  if  $F \in S^*$ .

7. DEFINITION. Let  $M$  be the class of all generalized functions  $F$  for which there exists an  $h$  as in definition 6 such that  $F = \text{emb}(h)$ . On  $M$  we define the mapping  $\text{emb}^{-1}$  by putting  $\text{emb}^{-1}(F) = h$  if  $F \in M$ ,  $F = \text{emb}(h)$  and if  $h$  is as in definition 6.

8. THEOREM. Let  $F \in S^*$ . Then  $F \in C$  if and only if  $F.F \in M$ . If  $F \in C$ ,  $G \in S^*$ , then  $F(T_F G) = \text{emb}^{-1}(F.F).FG$ .

PROOF. This is [J2], 4.5 and 4.6. □

9. THEOREM. If  $g \in S$ ,  $G \in S^*$ , then  $g.G \in C$ . If  $f \in S$ ,  $F \in S^*$ , then  $\text{emb}(T_F f) \in M$ , and  $T_F F = \text{emb}(T_F f)$ .

PROOF. This is [J2], 5.3, 5.4 and 5.5. □

10. DEFINITION. Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence in  $C$ ,  $f \in C$ . We write  $f_n \xrightarrow{C} 0$  if  $T_{f_n} g \xrightarrow{S} 0$  for every  $g \in S$ ; we write  $f_n \xrightarrow{C} f$  if  $f_n - f \xrightarrow{C} 0$ .

11. EXAMPLES. (i) If  $(f_n)_{n \in \mathbb{N}}$  is an  $S$ -convergent sequence in  $S$ , then  $(\text{emb}(f_n))_{n \in \mathbb{N}}$  is a  $C$ -convergent sequence in  $C$ . If  $(g_n)_{n \in \mathbb{N}}$  is a  $C$ -convergent sequence in  $C$ ; then  $(g_n)_{n \in \mathbb{N}}$  is an  $S^*$ -convergent sequence in  $S^*$ .

(ii) If  $f \in C$ , then  $\text{emb}(N_\alpha f) \xrightarrow{C} f$  if  $\alpha \downarrow 0$  (we have of course a similar definition of  $C$ -convergence for this case as in 10). Cf. [J2], lemma 3.6 for a proof.

(iii) Let  $h \in C$ ,  $(\text{emb}^{-1}(Fh))(0) = 1$ . For  $\lambda > 0$  define  $V_\lambda h$  as the generalized function satisfying  $(V_\lambda h, f) = (h, \int_x f(x/\lambda))$  for  $f \in S$  (cf. appendix 1,

theorem 3.3). Then  $V_\lambda \xrightarrow{\mathbb{C}} \delta_0$  ( $\lambda \rightarrow \infty$ ). As special cases we have  $\text{emb}(k_\varepsilon) \xrightarrow{\mathbb{C}} \delta_0$  ( $\varepsilon \rightarrow \infty$ ),  $\text{emb}(h_\tau) \xrightarrow{\mathbb{C}} \delta_0$  ( $\tau \downarrow 0$ ). Here  $k_\varepsilon := \prod_z \varepsilon^{\frac{1}{2}} \exp(-\pi \varepsilon z^2)$  ( $\varepsilon > 0$ ),  $h_\tau := \frac{1}{2\tau} \chi_{(-\tau, \tau)}$  ( $\tau > 0$ ). Cf. [J2], 5.10, example (iv).

## APPENDIX 3

## THE WIGNER DISTRIBUTION FOR SMOOTH AND GENERALIZED FUNCTIONS

In this appendix the Wigner distribution for elements of  $S$  and  $S^*$  is studied. In section 1 we give definitions and formulas about the Wigner distribution for smooth functions. Almost all formulas given there can be found in [B1], section 12 and 14. In section 2 we study Wigner distributions of generalized functions and time-frequency convolution operators.

## 1. THE WIGNER DISTRIBUTION FOR SMOOTH FUNCTIONS

1.1. DEFINITION. Let  $f_1 \in S$ ,  $f_2 \in S$ . The *Wigner distribution*  $W(f_1, f_2) = \int_{(q,p) \in \mathbb{C}^2} W(q,p; f_1, f_2)$  of  $f_1$  and  $f_2$  is defined by putting

$$W(q,p; f_1, f_2) = \int_{-\infty}^{\infty} e^{-2\pi i p t} f_1(q + \frac{1}{2}t) \overline{f_2(q - \frac{1}{2}t)} dt$$

for  $q \in \mathbb{C}$ ,  $p \in \mathbb{C}$ . If  $f_1 = f_2 = f$  then we call  $W(f, f)$  the *Wigner distribution* of  $f$ .

It can be proved that  $W(f_1, f_2) \in S^2$  if  $f_1 \in S$ ,  $f_2 \in S$  (cf. [B1], section 13).

We shall often consider the Wigner distribution for real values of its arguments  $q$  and  $p$ . Then the variables  $q$  and  $p$  are referred to as *time* and *frequency variable* respectively, and in this connection  $\mathbb{R}^2$  is called the *time-frequency plane* or *phase plane*.

1.2. We list some useful formulas involving Wigner distributions. Let  $f_1 \in S$ ,  $f_2 \in S$ ,  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$ .

- (i)  $W(a,b; f_1, f_2) = \overline{W(a,b; f_2, f_1)}$ .
- (ii)  $W(q,p; T_a R_b f_1, T_a R_b f_2) = W(q,p; R_b T_a f_1, R_b T_a f_2) = W(q+a, p+b; f_1, f_2)$  for  $q \in \mathbb{R}$ ,  $p \in \mathbb{R}$ .
- (iii)  $W(a,b; Ff_1, Ff_2) = W(-b, a; f_1, f_2)$ .
- (iv)  $\int_{-\infty}^{\infty} e^{2\pi i p b} W(a,p; f_1, f_2) dp = f_1(a + \frac{1}{2}b) \overline{f_2(a - \frac{1}{2}b)}$ .
- (v)  $F^{(1)} * F^{(2)} * W(f_1, f_2) = \int_{(q,b)} e^{2\pi i q a} f_1(a + \frac{1}{2}b) \overline{f_2(a - \frac{1}{2}b)} da$ .
- (vi)  $F^{(1)} W(f_1, f_2) = \int_{(a,b)} (Ff_1)(b + \frac{1}{2}a) \overline{(Ff_2)(b - \frac{1}{2}a)}$ .

$$\begin{aligned}
\text{(vii)} \quad & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(q,p; f_1, f_2) dq dp = (f_1, f_2). \\
\text{(viii)} \quad & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(q,p; f_1, f_2) \overline{W(q,p; f_3, f_4)} dq dp = (f_1, f_3) \overline{(f_2, f_4)} \text{ for } f_3 \in S, \\
& f_4 \in S.
\end{aligned}$$

The proofs of these formulas (except those of (v) and (vi)) can be found in [B1], section 12 and 14, where also a number of formulas about the first and second order moments of Wigner distributions are given. Formula (v) is proved by taking inverse Fourier transform in (iv) with respect to the variable  $a$ , and formula (vi) follows from (iv) applied to  $Ff_1$  and  $Ff_2$  and (iii).

The right hand side in (v) is often called the *(cross) ambiguity function* of  $f_1$  and  $f_2$ , and it plays an important role in branches of engineering like geometrical optics, Fourier optics and radar analysis. We refer to [P], [PH], [Re] and [St] for applications and further properties of ambiguity functions.

1.3. Let  $f_1 \in S$ ,  $f_2 \in S$ . The number  $W(q,p; f_1, f_2)$  (with fixed values of  $q \in \mathbb{R}$ ,  $p \in \mathbb{R}$ ) depends on  $f_1$  and  $f_2$  like an inner product (this is not necessarily positive definite). However, if  $f \in S$ , then certain averages over the phase plane of  $W(f, f)$  are non-negative. We mention

$$\text{(ix)} \quad \int_{R_{a,b}} W(q,p; f, f) dq dp = \int_a^b |f(t)|^2 dt,$$

$$\text{(x)} \quad \int_{T_{a,b}} W(q,p; f, f) dq dp = \int_a^b |(Ff)(t)|^2 dt.$$

Here  $R_{a,b} = \{(q,p) \mid a \leq q \leq b, p \in \mathbb{R}\}$ ,  $T_{a,b} = \{(q,p) \mid q \in \mathbb{R}, a \leq p \leq b\}$  for  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$ ,  $a < b$ . Also,

$$\text{(xi)} \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(q,p; f, f) \overline{W(q-a, p-b; g, g)} dq dp = \left| \frac{1}{2} W(\frac{1}{2}a, \frac{1}{2}b; f, g_-) \right|^2$$

for  $g \in S$ ,  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$  (cf. 4.6 for the definition of  $g_-$ ).

Formula (ix) is proved in [B1], 27.14 and (x) is an immediate consequence of (ix) and 1.2 (iii); formula (xi) follows from 1.2 (ii) and 1.2 (viii).



As a consequence of (xi) we have for  $\gamma > 0$

$$(xii) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{2\pi}{\gamma}(q-a)^2 - 2\pi\gamma(p-b)^2\right) W(q,p;f,f) dq dp \geq 0.$$

This follows from (xi) by taking  $g = \int_x \left(\frac{2}{\gamma}\right)^{\frac{1}{2}} \exp(-\pi\gamma^{-1}x^2)$  for  $\gamma > 0$ . See [B1], 27.12.2.1 and [B2] for further inequalities involving Wigner distributions.

1.4. It has advantages to rewrite the Wigner distribution in such a way that time and frequency occur symmetrically. If  $f_1 \in S$ ,  $f_2 \in S$ , then we define the function  $V(f_1, f_2) = \int_{(x,y) \in \mathbb{C}^2} V(x,y;f_1, f_2)$  by

$$V(x,y;f_1, f_2) = \int_{-\infty}^{\infty} e^{-2\pi i y t} f_1\left(\frac{x+t}{\sqrt{2}}\right) \overline{f_2\left(\frac{\bar{x}-t}{\sqrt{2}}\right)} dt$$

for  $x \in \mathbb{C}$ ,  $y \in \mathbb{C}$ .

Note that  $V(f_1, f_2) = F^{(2)} Z_U(f_1 \otimes \bar{f}_2)$  (cf. appendix 1, 4.9), and that

$$V(x,y;f_1, f_2) = \frac{1}{\sqrt{2}} W\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}; f_1, f_2\right)$$

for  $x \in \mathbb{C}$ ,  $y \in \mathbb{C}$ .

We have by [B1], theorem 16.1 for every  $\alpha > 0$

$$V(N_\alpha f_1, N_\alpha f_2) = N_\alpha^{(1)} N_\alpha^{(2)} V(f_1, f_2).$$

1.5. The Wigner distribution of two functions can be interpreted as a time-frequency convolution. We give the following definition (cf. appendix 2,2).

DEFINITION. If  $g \in S^*$ , then the mapping  $S_g$  is defined by

$$S_g f = \int_{(a,b) \in \mathbb{C}^2} (e^{\pi i a b} T_a R_b f, g)$$

for  $f \in S$ . In case  $g \in S$ , we write  $S_g$  instead of  $S_{\text{emb}(g)}$ .

We have for  $f \in S$ ,  $g \in S$

$$(S_g f)(a,b) = \frac{1}{2} W\left(\frac{1}{2}a, \frac{1}{2}b; f, g\right)$$

if  $a \in \mathbb{C}$ ,  $b \in \mathbb{C}$ . We can now write 1.3 (xi) as

$$T_{W(g_-, g_-)} W(f, f) = S_g f \cdot \overline{S_g f}$$

for  $f \in S$  (cf. appendix 2, 2 and [J2], 5.11; the dot denotes pointwise multiplication).

1.6. We give some further results about time-frequency convolution operators (we omit the proofs).

a) If  $g \in S^*$ ,  $f \in S$ , then  $S_g f \in \text{emb}^{-1}(M^2)$  (cf. [J2], 5.11).

The proof of this fact uses the same kind of arguments as the proof of [J2], theorem 5.5.

b) If  $g \in S^*$ , then  $S_g$  maps  $S$  into  $S^2$  if and only if  $g \in \text{emb}(S)$ .

## 2. THE WIGNER DISTRIBUTION FOR GENERALIZED FUNCTIONS

2.1. DEFINITION. If  $F_1 \in S^*$ ,  $F_2 \in S^*$ , then we define the generalized function  $V(F_1, F_2)$  by

$$V(F_1, F_2) = \bigcup_{\alpha > 0} V(N_\alpha F_1, N_\alpha F_2).$$

(It follows from 1.4 and appendix 1, 1.17 that  $V(F_1, F_2) \in S^{2*}$ .) The Wigner distribution  $W(F_1, F_2)$  is defined by

$$W(F_1, F_2) = Z_{\sqrt{2}}^{(1)} Z_{\sqrt{2}}^{(2)} V(F_1, F_2)$$

(cf. appendix 1, 1.8 (ii), 1.17, 4.15 and 4.16).

We note that  $V(F_1, F_2) = F^{(2)} Z_U(F_1 \otimes \bar{F}_2)$  for  $F_1 \in S^*$ ,  $F_2 \in S^*$ . Here  $F^{(2)} Z_U$  is the extension to  $S^{2*}$  according to the 2-dimensional version of appendix 1, theorem 4.11. This can be proved by showing that  $(V(F_1, F_2), N_{\alpha, 2} g) = (V(N_\alpha F_1, N_\alpha F_2), g)$  equals  $(F^{(2)} Z_U(F_1 \otimes \bar{F}_2), N_{\alpha, 2} g) = (F^{(2)} Z_U(N_\alpha F_1 \otimes \overline{N_\alpha F_2}), g)$  for every  $\alpha > 0$ ,  $g \in S^2$ .

2.2. We next extend the time-frequency convolution operators  $S_g$  (with  $g \in S$ ) to continuous linear mappings of  $S^*$  into  $S^{2*}$  (this can be done as well if  $g \in S^*$ , but we only meet cases with  $g \in S$  in this thesis).

It is easy to show that  $S_g$  (as a mapping of  $S$  into  $S^2$ ) has an adjoint,

i.e. there is a linear mapping  $S_g^*$  of  $S^2$  into  $S$  such that

$$(S_g f, h)_2 = (f, S_g^* h) \quad (f \in S, h \in S^2).$$

Now  $S_g$  can be extended to a continuous linear mapping (again denoted by  $S_g$ ) of  $S^{*g}$  into  $S^{2*}$  with the aid of a version of appendix 1, theorem 4.11.

2.3. We describe a direct way to extend the operator  $S_g$ . Define the mapping  $S'_g$  by

$$S'_g F = \int_{(a,b)} (e^{\pi i a b} T_a R_b F, g)$$

for  $F \in S^*$ . Then we have  $S_g F = \text{emb}(S'_g F)$  for  $F \in S^*$  (it is readily seen from 1.6 a) that  $S'_g F \in \text{emb}^{g^{-1}}(M^2)$ ). We sketch the proof of this fact. The equality  $S_g F = \text{emb}(S'_g F)$  holds if  $F \in \text{emb}(S)$ . To handle the general case, take  $F \in S^*$  and note that  $F \xrightarrow{S^*} F_\alpha$  ( $\alpha \downarrow 0$ ), whence  $S_g F \xrightarrow{S^{2*}} S_g F_\alpha$  ( $\alpha \downarrow 0$ ). It suffices to show that  $\text{emb}(S'_g F_\alpha) \xrightarrow{S^{2*}} \text{emb}(S'_g F)$  ( $\alpha \downarrow 0$ ). This may be done by proving inequalities for  $|(e^{\pi i a b} T_a R_b F_\alpha, g)|$  (where  $a \in \mathbb{T}$ ,  $b \in \mathbb{T}$ ,  $\alpha > 0$ ) of the type occurring in [J2], lemma 5.2 (cf. also the proof of [J2], theorem 5.5).

2.4 The following theorem generalizes 1.3 (xi).

**THEOREM.** If  $F \in S^*$ ,  $g \in S$ , then

$$T_{W(g_-, g_-)} W(F, F) = \text{emb}(S'_g F \cdot \overline{S'_g F}).$$

**PROOF.** The above equality holds if  $F \in \text{emb}(S)$ . The general case is handled as in 2.3.  $\square$

**REMARK.** The above theorem may be used to show that  $F \in \text{emb}(S)$  in case  $F \in S^*$ ,  $W(F, F) \in \mathcal{C}^2$ . For assume  $F \in S^*$ ,  $W(F, F) \in \mathcal{C}^2$ . Then  $T_{W(g, g)} W(F, F) \in S^2$  for every  $g \in S$  and hence  $S'_g F \in S^2$  for every  $g \in S$ . This implies by 1.6 b) that  $F \in \text{emb}(S)$ .

## APPENDIX 4

## TWO THEOREMS ON GENERALIZED FUNCTIONS OF SEVERAL VARIABLES

In this appendix we prove two theorems on generalized functions of several variables. The first one deals with generalized functions that are invariant under translations in one or more variables. The second theorem is concerned with generalized functions of positive type.

## 1. TRANSLATION INVARIANCE OF GENERALIZED FUNCTIONS

1.1. DEFINITION. For  $n \in \mathbb{N}$  we define  $H_n := \text{emb}(\bigcup_{(t_1, \dots, t_n) \in \mathbb{R}^n} 1)$ . If  $n = 1$  we usually write  $H$  instead of  $H_1$ .

Note that  $\bigcup_{(t_1, \dots, t_n) \in \mathbb{R}^n} 1 \in S^{n+}$  for  $n \in \mathbb{N}$ .

1.2. THEOREM. Let  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}_0$ , and let  $F \in S^{(n+m)*}$ . Assume that  $T_{a_1}^{(1)} \dots T_{a_n}^{(n)} F = F$  for every  $a_1 \in \mathbb{R}, \dots, a_n \in \mathbb{R}$ . If  $m > 0$ , there exists exactly one  $G \in S^{m*}$  with  $F = H_n \otimes G$  (= tensor product of  $H_n$  and  $G$ , compare appendix 1, 1.17). If  $m = 0$  there exists exactly one  $c \in \mathbb{C}$  with  $F = c H_n$ .

PROOF. Assume  $m > 0$ . If  $f \in S^{n+m}$  then  $T_{\vec{f}} F$  has the form  $H_n \otimes G_{\vec{f}}$  where  $G_{\vec{f}} \in M^m$  (cf. appendix 2, 9 and [J2], 5.11). Let  $(f_k)_{k \in \mathbb{N}}$  be a sequence in  $S^{n+m}$  with  $\text{emb}(f_k) \xrightarrow{C^{n+m}} \otimes_{\ell=1}^{n+m} \delta_0$  (cf. appendix 2, 10 and 11). Now  $T_{\vec{f}_k} F \xrightarrow{S^{(n+m)*}} F$ , and the sequence  $(G_{\vec{f}_k})_{k \in \mathbb{N}}$  is  $S^{m*}$ -convergent. To see this, take an  $h \in S^n$  with  $\int_{\mathbb{R}^n} h(x) dx = 1$ , and note that for every  $g \in S^m$

$$(G_{\vec{f}_k}, g) = (H_n \otimes G_{\vec{f}_k}, h \otimes g) = (T_{\vec{f}_k} F, h \otimes g) \rightarrow (F, h \otimes g)$$

if  $k \rightarrow \infty$ . Hence (cf. appendix 1, 1.15 and 1.17) the sequence  $(G_{\vec{f}_k})_{k \in \mathbb{N}}$  is  $S^{m*}$ -convergent. If we denote the  $S^{m*}$ -limit by  $G$ , then

$$F = \lim_{k \rightarrow \infty} T_{\vec{f}_k} F = \lim_{k \rightarrow \infty} H_n \otimes G_{\vec{f}_k} = H_n \otimes G$$

(the limit is in  $S^{(n+m)*}$ -sense). Hence  $F$  has the required form. It is trivial that there is at most one  $G \in S^{m*}$  with  $F = H_n \otimes G$ . Hence the theorem is proved for  $m > 0$ .

The case  $m = 0$  can be handled similarly.  $\square$

## 2. GENERALIZED FUNCTIONS OF POSITIVE TYPE

2.1. Let  $n \in \mathbb{N}$ .

DEFINITION. An  $F \in S^{n*}$  is said to be of *positive type* (notation  $F \geq 0$ ) if  $(F, f) \geq 0$  for every  $f \in S^n$  with  $f(x) \geq 0$  ( $x \in \mathbb{R}^n$ ).

2.2. We shall prove the following theorem on functions of positive type.

THEOREM. If  $F \in S^{n*}$  and  $F \geq 0$ , then there exists exactly one measure  $P$  on the Borel sets of  $\mathbb{R}^n$  such that

$$(F, f) = \int_{\mathbb{R}^n} \bar{f} \, dP \quad (f \in S^n).$$

This  $P$  satisfies

$$\int_{\mathbb{R}^n} \exp(-\pi \varepsilon |x|^2) \, dP(x) < \infty$$

for every  $\varepsilon > 0$  ( $|x|^2 = x_1^2 + \dots + x_n^2$  for  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ ).

REMARK. If  $P$  is a measure on the Borel sets of  $\mathbb{R}^n$  such that

$$\int_{\mathbb{R}^n} \exp(-\pi \varepsilon |x|^2) \, dP(x) < \infty$$

for every  $\varepsilon > 0$ , then the generalized function  $F$ , determined by

$$(F, f) = \int_{\mathbb{R}^n} \bar{f} \, dP \quad (f \in S^n),$$

is of positive type.

2.3. Before giving the proof of the above theorem we need some preparations.

We denote for  $\gamma \in \mathbb{R}$  by  $G_\gamma$  the function  $\prod_{x=(x_1, \dots, x_n) \in \mathbb{R}^n} \exp(-\pi \gamma \sum_{i=1}^n x_i^2)$ .

LEMMA. Let  $\gamma > 0$  and let  $P_j$  be a measure on the Borel set of  $\mathbb{R}^n$  with  $\int_{\mathbb{R}^n} G_{\gamma+\varepsilon} \, dP_j < \infty$  for every  $\varepsilon > 0$  ( $j = 1, 2$ ). If for every

$f \in V := \{G_\gamma \cdot h \mid h \in S^n\}$  we have

$$\int_{\mathbb{R}^n} f dP_1 = \int_{\mathbb{R}^n} f dP_2$$

then  $P_1 = P_2$ .

PROOF. Let  $\varepsilon > 0$ , and define  $\mu_j$  by

$$\mu_j(A) := \int_A G_{\gamma+\varepsilon} dP_j \quad (A \in \mathcal{B}(\mathbb{R}^n))$$

for  $j = 1, 2$ . Then  $\mu_j$  is a finite measure on  $\mathcal{B}(\mathbb{R}^n)$  for  $j=1, 2$ . If we denote  $(t, x) = t_1 x_1 + \dots + t_n x_n$  for  $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ ,  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , then  $\int_x \exp(i(t, x)) G_{\gamma+\varepsilon}(x) \in V$  for every  $t \in \mathbb{R}^n$ . Hence

$$\begin{aligned} \int_{\mathbb{R}^n} \exp(i(t, x)) d\mu_1(x) &= \int_{\mathbb{R}^n} \exp(i(t, x)) G_{\gamma+\varepsilon}(x) dP_1(x) = \\ &= \int_{\mathbb{R}^n} \exp(i(t, x)) G_{\gamma+\varepsilon}(x) dP_2(x) = \int_{\mathbb{R}^n} \exp(i(t, x)) d\mu_2(x) \end{aligned}$$

for every  $t \in \mathbb{R}^n$ . We easily conclude from [T], 2.5, theorem 1 that  $\mu_1 = \mu_2$ . It is not hard to show now that  $P_1 = P_2$ .  $\square$

2.4. The proof of theorem 2.2 uses a second lemma.

LEMMA. Let  $R \in S^{n+}$  (cf. appendix 1, 1.17), and assume that  $R$  is real-valued, non-decreasing and continuous from the right in  $\mathbb{R}^n$ . Let  $P$  denote the Lebesgue-Stieltjes measure on the Borel sets of  $\mathbb{R}^n$  generated by  $R$ . If  $D_n$  denotes  $\frac{\partial^n}{\partial x_1 \dots \partial x_n}$ , then we have

$$(f, D_n(\text{emb}(R))) = \int_{\mathbb{R}^n} f dP \quad (f \in S^n).$$

Here  $D_n$  has been extended according to the  $n$ -dimensional version of appen-

dix 1, theorem 4.11.

PROOF. Let  $f \in S^n$ . We have by definition and the fact that  $D_n^* = (-1)^n D_n$

$$(f, D_n(\text{emb}(R))) = (-1)^n (D_n f, \text{emb}(R)) = (-1)^n \int_{\mathbb{R}^n} (D_n f)(x) R(x) dx.$$

If  $A > 0$ , then

$$(*) \quad \int_{C_A} (D_n f)(x) R(x) dx = f(x) R(x) \Big|_{x=(-A, \dots, -A)}^{x=(A, \dots, A)} + (-1)^n \int_{C_A} f dP,$$

where  $C_A$  denotes the  $n$ -cube  $\{x \in \mathbb{R}^n \mid -A < x_i \leq A\}$ . It follows from monotonicity of  $R$  and from the fact that  $R \in S^{n+}$  that

$$\forall \varepsilon > 0 [R(x) = 0 (G_{-\varepsilon}(x)) \quad (x \in \mathbb{R}^n)].$$

If we let  $A \rightarrow \infty$  in  $(*)$ , we obtain

$$\int_{\mathbb{R}^n} (D_n f)(x) R(x) dx = (-1)^n \int_{\mathbb{R}^n} f dP,$$

whence  $(f, D_n(\text{emb}(R))) = \int_{\mathbb{R}^n} f dP$ .  $\square$

2.5. We now give the proof of theorem 2.2. Let  $\gamma > 0$  be fixed, and define for  $\varepsilon > 0$  the function

$$F_{\varepsilon, \gamma} := \varepsilon^{n/2} G_{\gamma} \cdot \text{emb}^{-1}(T_{G_{\varepsilon}} F)$$

(cf. appendix 2, 7 and [J2], 5.11). Now  $F_{\varepsilon, \gamma} \in S^n$ ,  $\text{emb}(F_{\varepsilon, \gamma}) \xrightarrow{S^{2*}} G_{\gamma} \cdot F$  if  $\varepsilon \rightarrow \infty$  (cf. appendix 2, 11(iii)) and  $F_{\varepsilon, \gamma}(x) \geq 0$  if  $x \in \mathbb{R}^n$ . We further define for  $\varepsilon > 0$

$$H_{\varepsilon, \gamma} := \bigcup_{x \in \mathbb{R}^n} \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} F_{\varepsilon, \gamma}(t_1, \dots, t_n) dt_1 \dots dt_n.$$

This  $H_{\varepsilon, \gamma}$  is real-valued, non-decreasing and continuous in  $\mathbb{R}^n$ , and we infer from

$$(*) \quad 0 \leq H_{\varepsilon, \gamma}(x) \leq (G_{\gamma}, \varepsilon^{n/2} T_G F) \rightarrow (G_{\gamma}, F) \quad (\varepsilon \rightarrow \infty)$$

for  $x \in \mathbb{R}^n$  that  $H_{\varepsilon, \gamma}(x)$  is bounded in  $\varepsilon \geq 1$  and  $x \in \mathbb{R}^n$ .

We appeal to the Helly compactness theorem ([T], 4.1, theorem 2) to conclude that there exists a sequence  $(\varepsilon_k)_{k \in \mathbb{N}}$  with  $\varepsilon_k \rightarrow \infty$  such that

$(H_{\varepsilon_k, \gamma})_{k \in \mathbb{N}}$  is a weakly convergent sequence. This means that there exists a function  $H_{\gamma}: \mathbb{R}^n \rightarrow \mathbb{R}$ , non-decreasing and continuous from the right, such that  $H_{\varepsilon_k, \gamma}(x) \rightarrow H_{\gamma}(x)$  ( $k \rightarrow \infty$ ) at every continuity point  $x$  of  $H_{\gamma}$ .

Denote by  $P_{\gamma}$  and  $P_{\varepsilon_k, \gamma}$  the Lebesgue-Stieltjes measures generated by  $H_{\gamma}$  and  $H_{\varepsilon_k, \gamma}$  ( $k \in \mathbb{N}$ ) respectively. Let  $f \in S$ . We show that  $\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f dP_{\varepsilon_k, \gamma} = \int_{\mathbb{R}^n} f dP_{\gamma}$ . Therefore write  $\gamma = \gamma_1 + \gamma_2$  with  $\gamma_1 > 0$ ,  $\gamma_2 > 0$ . We have for  $k \in \mathbb{N}$ ,  $T > 0$

$$\begin{aligned} \int_{|x| \geq T} dH_{\varepsilon_k, \gamma}(x) &= \int_{|x| \geq T} G_{\gamma_1}(x) dH_{\varepsilon_k, \gamma_2}(x) \leq \\ &\leq \exp(-\pi \gamma_1 n T^2) (G_{\gamma_2}, F) \end{aligned}$$

(cf. (\*)). It follows that the left hand side tends to zero if  $T \rightarrow \infty$  uniformly in  $k \in \mathbb{N}$ . We conclude from [T], 4.1, theorem 4 that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f dP_{\varepsilon_k, \gamma} = \int_{\mathbb{R}^n} f dP_{\gamma}$$

Applying lemma 2.4 we obtain

$$\begin{aligned} (G_{\gamma}, F, f) &= \lim_{k \rightarrow \infty} (\text{emb}(F_{\varepsilon_k, \gamma}), f) = \lim_{k \rightarrow \infty} (D_n(\text{emb}(H_{\varepsilon_k, \gamma})), f) = \\ &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f dP_{\varepsilon_k, \gamma} = \int_{\mathbb{R}^n} f dP_{\gamma} \end{aligned}$$

for  $f \in S^n$ .

Let  $\gamma_1 > 0$ ,  $\gamma_2 > 0$ . We shall show that the measures  $P_1$  and  $P_2$ , defined by  $P_1(A) := \int_A G_{-\gamma_1} dP_{\gamma_1}$  and  $P_2(A) := \int_A G_{-\gamma_2} dP_{\gamma_2}$  respectively for  $A \in \mathcal{B}(\mathbb{R}^n)$ , are equal. If  $\gamma := \max(\gamma_1, \gamma_2)$ , then we have (with the notation of lemma 2.3) for every  $f \in V$



$$(F, f) = (G_{\gamma_j} \cdot F, G_{-\gamma_j} \cdot f) = \int_{\mathbb{R}^n} G_{-\gamma_j} \cdot \bar{f} \, dP_{\gamma_j} = \int_{\mathbb{R}^n} \bar{f} \, dP_j$$

for  $j = 1, 2$ . Hence  $\int_{\mathbb{R}^n} \bar{f} \, dP_1 = \int_{\mathbb{R}^n} \bar{f} \, dP_2$  ( $f \in V$ ), so  $P_1 = P_2$  by lemma 2.3.

If we put now  $P(A) := \int_A G_{-\gamma} \, dP_{\gamma}$  for every Borel set  $A$  in  $\mathbb{R}^n$  with some  $\gamma > 0$ , then it is easily seen from the foregoing that

$$(**) \quad (F, f) = \int_{\mathbb{R}^n} \bar{f} \, dP \quad (f \in S^n).$$

This shows the existence of a  $P$  with the required properties (it is trivial that  $\int_{\mathbb{R}^n} \exp(-\pi \varepsilon |x|^2) \, dP(x) < \infty$  for every  $\varepsilon > 0$ ).

Uniqueness of a  $P$  as in theorem 2.2 follows from lemma 2.3 (applied with  $\gamma = 0$ ).



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## INDEX OF SYMBOLS

In this index  $f$  is a smooth function,  $F$  is a generalized function,  $T$  is a continuous linear operator of  $S$  or  $S^*$ , and  $X$  is a generalized stochastic process. The numbers between parentheses refer to the page on which the symbols are introduced.

$S, S^n$ ( $n \in \mathbb{N}$ )	: spaces of smooth functions (114,122)
$S^+, S^{n+}$ ( $n \in \mathbb{N}$ )	: spaces of embeddable functions (114, 122)
$S^*, S^{n*}$ ( $n \in \mathbb{N}$ )	: spaces of generalized functions (118, 122)
$C, C^n$ ( $n \in \mathbb{N}$ )	: spaces of elements of convolution class (142)
$M, M^n$ ( $n \in \mathbb{N}$ )	: spaces of elements of multiplication class (143)
$V$	: (51)
$W, W_1$	: Wiener classes (86,88)
$W^*, W_1^*$	: generalized Wiener classes (94,101)
$S_{\Omega,p}, S_{\Omega,p}^n$ ( $1 \leq p \leq \infty, n \in \mathbb{N}$ )	: spaces of smooth stochastic processes (11)
$S_{\Omega,p}^*, S_{\Omega,p}^{n*}, \tilde{S}_{\Omega,p}^*, \tilde{S}_{\Omega,p}^{n*}$ ( $1 \leq p \leq \infty, n \in \mathbb{N}$ )	: spaces of generalized stochastic processes (10,12)
$S_{\Omega,p}^+, S_{\Omega,p}^{n+}, \bar{S}_{\Omega,p}^+, \bar{S}_{\Omega,p}^{n+}$ ( $1 \leq p \leq \infty, n \in \mathbb{N}$ )	: spaces of embeddable stochastic processes (25,30)
$\tau$	: inductive limit topology of $S$ (126)
$\Lambda^*$	: $\sigma$ -algebra of measurable sets in $S^*$ (136)
$N_\alpha, N_{\alpha,n}$ ( $\alpha > 0, n \in \mathbb{N}$ )	: smoothing operators (114,122)
$\text{emb}, \text{emb}^{-1}$	: (inverse) embedding operator (119,143)
$F$	: Fourier transform (117)
$F^*$	: inverse Fourier transform (117)
$T_a$ ( $a \in \mathbb{C}$ )	: time shift (118)
$R_b$ ( $b \in \mathbb{C}$ )	: frequency shift (118)
$V_a$ ( $a \in \mathbb{R}^2$ )	: time-frequency shift (69)

$Z_\lambda$ ( $\lambda > 0$ )	: dilatation operator (118)
$P, \frac{d}{dz}$	: differentiation operators (118)
$Q$	: multiplication by $x$ (118)
$T_g, T_G$ ( $g \in S$ or $C, G \in S^*$ )	: convolution operators (142)
$M_h$ ( $h \in M$ )	: multiplication operator (143)
$Z_A$ ( $A$ a real orthogonal matrix)	: transformation of variables operator (for the case $A = U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ ): 134)
$S_g$ ( $g \in S^*$ ), $S'_g$ ( $g \in S$ )	: time-frequency convolution operators (147,149)
$\bar{f}, f_-, \bar{F}, F_-$	: (133,135)
$T^*$	: adjoint operator (131)
$\bar{T}, T_-$	: (132,135)
*	: ordinary convolution (101)
$\otimes$	: tensor product (20,122,123,134,136)
$\psi_k$ ( $k \in \mathbb{N}_0$ )	: $k^{\text{th}}$ Hermite function (117)
$\delta_a$ ( $a \in \mathbb{C}$ )	: $\delta$ -function at $a$ (119)
$e_a$ ( $a \in \mathbb{C}$ )	: $\int_t e^{-2\pi i a t}$ (119)
$H$	: $\int_t 1$ (150)
$h_\tau$ ( $\tau > 0$ )	: $\frac{1}{2\tau} \chi_{(-\tau, \tau)}$ (144)
$k_\varepsilon$ ( $\varepsilon > 0$ )	: $\int_t \varepsilon^{\frac{1}{2}} \exp(-\pi \varepsilon t^2)$ (144)
$g_\gamma$ ( $\gamma > 0$ )	: $\int_t \left(\frac{2}{\gamma}\right)^{\frac{1}{2}} \exp(-\pi \gamma^{-1} t^2)$ (65)
$G_\gamma(a, b)$ ( $a \in \mathbb{R}, b \in \mathbb{R}, \gamma > 0$ )	: Gabor function at $(a, b)$ (68)
$K_\alpha$ ( $\alpha > 0$ )	: $\int_{(z, t)} \left(\frac{1}{\sinh \alpha}\right)^{\frac{1}{2}} \exp\left(\frac{-\pi}{\sinh \alpha} ((z^2 + t^2) \cdot \cosh \alpha - 2zt)\right)$ (114)
$\varphi_g$ ( $g \in W$ ), $\Phi_G$ ( $G \in W^*$ )	: autocorrelation functions (86,96)
$\sigma_g$ ( $g \in W$ )	: spectrum (87)
$\sigma'_g$ ( $g \in W$ ), $F\Phi_G$ ( $G \in W^*$ )	: spectral density functions (87,98)
$W(f, f), W(f, f), V(F, F), V(F), W(F, F)$	: Wigner distribution (145,147,148,106)
$E_X$	: expectation function (32)



$\underline{R}_X$	: autocorrelation function (33)
$\underline{C}_X$	: autocovariance function (33)
$\underline{F}^{(2)}_{Z_U \underline{R}_X \underline{V}_X}$	: (expected) Wigner distributions (33)
$\underline{P}_X^*$	: associated measure on $\Lambda^*$ (18)



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