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# MEROMORPHIC OPERATOR VALUED FUNCTIONS

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CONTENTS

# INTRODUCTION

Let X and Y be complex Banach spaces. The complex Banach space of all bounded linear operators from X into Y will be denoted by $L(X,Y)$. In this treatise we study locally holomorphic functions defined on an open subset of the complex plane $\mathbb{C}$ with values in $L(X,Y)$. In particular we are interested in poles of such functions.

Let A be a locally holomorphic function defined on a (deleted) neighbourhood of 0 with values in $L(X,Y)$. Suppose that the values of A are compact linear operators. Then it follows from Cauchy's integral formula that the coefficients of the Laurent expansion of A at 0 are compact too. In Chapter I we deal with the question whether this result remains true if compact is replaced by degenerate. A linear operator is said to be degenerate if its range is finite-dimensional. The answer to the above question turns out to be affirmative, provided that 0 is not an essential singularity of A; without this extra condition it is negative.

In Chapter II we consider a meromorphic function A. The values of this function are assumed to be semi-Fredholm operators from X into Y with complemented null space and range. This implies that for each $\lambda$ in the domain of A there exists an element $B(\lambda)$ of $L(Y,X)$ such that

$$A(\lambda) = A(\lambda)B(\lambda)A(\lambda), \quad B(\lambda) = B(\lambda)A(\lambda)B(\lambda).$$

Our main result is that under certain conditions on the poles of A the operators $B(\lambda)$ can be chosen in such a way that the function

$$\lambda \longmapsto B(\lambda)$$

is meromorphic.

Let T be a bounded linear operator on X and let m be a positive integer. We say that the complex number $\lambda_0$ is a pole of T of order m if $\lambda_0$ is a pole of order m of the locally holomorphic function

$$\lambda \longmapsto (\lambda I_X - T)^{-1}.$$

To characterize the poles of T, one can use the ascent $\alpha(T)$ and descent $\delta(T)$ of T. In fact it is known that 0 is a pole of T of order m if and only if $\alpha(T) = \delta(T) = m$. In Chapter III we generalize this result to a characterization of the poles of the resolvent of an arbitrary locally holomorphic function A defined on an open neighbourhood of 0 with values in $L(X,Y)$. By definition, the resolvent of A is the function

$$\lambda \longmapsto A(\lambda)^{-1}.$$

This function is defined on the set of all $\lambda$ in the domain of A such that $A(\lambda)$ is bijective.

The system of internal references we use is explained by the following example. Theorem 3.4 in Chapter II is referred to as Theorem II.3.4 if the reference is made outside Chapter II, and as Theorem 3.4 otherwise.

# CHAPTER I

## DEGENERATE FUNCTIONS

Let X and Y be complex Banach spaces, and let A be a locally holomorphic function defined on a (deleted) neighbourhood of 0 with values in $L(X,Y)$. Suppose that the values of A are degenerate linear operators, i.e., the ranges of these operators are finite-dimensional. In Section 2 of this chapter we prove that the coefficients of the Laurent expansion of A at 0 are degenerate, provided that 0 is not an essential singularity of A. In addition, we present an example to show that without this extra condition the theorem does not hold.

Section 1 of this chapter is of preliminary character. In Section 3 we prove a global representation theorem for a certain type of degenerate operator valued holomorphic function.

### 1. PRELIMINARIES

In this section we collect together some definitions and notations concerning operators and operator valued functions, and we present a preliminary lemma.

Often we shall use the symbols $+\infty$ and $-\infty$. The phrase *extended real number* refers to one of those symbols or a real number. Instead of $+\infty$ we sometimes write $\infty$. For the algebraic relations between the symbols $\pm\infty$ and the real numbers we refer to §0 in [17]. The expressions $+\infty + (-\infty)$ and $-\infty + \infty$ have no meaning. The phrase *extended integer* refers to an integer or one of the symbols $\pm\infty$.

Let E be a linear space. If E has finite dimension n, we shall write $\dim E = n$, otherwise $\dim E = +\infty$.

The null space and range of a linear operator T are denoted by $N(T)$ and $R(T)$ respectively. A linear operator is said to be *degenerate* if its range is finite-dimensional.

Let X be a complex Banach space. The normed conjugate of X is denoted by $X^*$. It is well-known that $X^*$ is a complex Banach space. We use the symbol $\langle f,x\rangle$ to designate the value $f(x)$ of a linear functional f on X at the point x.

Let X and Y be complex Banach spaces. The complex Banach space of all bounded linear operators from X into Y is denoted by $L(X,Y)$. Let T be an element of $L(X,Y)$. The conjugate operator of T is denoted by $T^*$. Thus $T^*$ is the bounded linear operator from $Y^*$ into $X^*$ given by

$$\langle T^*g,x\rangle = \langle g,Tx\rangle.$$

It is not difficult to see that T is degenerate if and only if the same is true for $T^*$.

Degenerate operators are easy to construct. Take f in $X^*$ and u in Y. Let $f \otimes u$ denote the linear operator from X into Y defined by

$$(f \otimes u)(x) = \langle f,x\rangle\, u.$$

Then $f \otimes u$ is bounded and degenerate, and hence any finite sum of operators of this type is a degenerate element of $L(X,Y)$. In fact each degenerate bounded linear operator can be written in such a form (see Section 27A in [8]).

By a *region* we mean a non-void connected open subset of the complex plane $\mathbb{C}$. A subset $\Gamma$ of a region G is called a *discrete* subset of G if $\Gamma$ has no accumulation points in G. A discrete subset of a region G is at most countable and its complement in G is again a region.

In this treatise we shall freely use the standard notions concerning Banach space valued locally holomorphic functions of one complex variable. For a fairly complete survey of these notions we refer to Section III.14 of [10]. Our definition of meromorphy differs slightly from the usual convention.

1.1. DEFINITION. A complex Banach space valued function f is said to be *meromorphic* on a region G if there exists a discrete subset $\Sigma$ of G such that

(i) f is holomorphic on $G \setminus \Sigma$;

(ii) each point of $\Sigma$ is either a pole or a removable singularity of f.

The points of the (unique) set $\Sigma$ are then called the *singular points* and those of $G \setminus \Sigma$ the *regular points of* f *in* G.

Let X and Y be complex Banach spaces, and let A be a function defined on a subset of the complex plane with values in $L(X,Y)$. Then the *conjugate* $A^*$ of A will be the function with values in $L(Y^*,X^*)$ given by

$$A^*(\lambda) = A(\lambda)^*,$$

where $\lambda$ is in the domain of A. Since the $^*$-operation is an injective continuous linear map, the function A is meromorphic on a region G if and only if the same is true for $A^*$. In that case the n-th coefficient of the Laurent expansion of $A^*$ at a point $\lambda_0$ of G is the conjugate of the n-th coefficient of the Laurent expansion of A at $\lambda_0$.

The operator valued function A is said to be *degenerate* if for each $\lambda$ in its domain the operator $A(\lambda)$ is degenerate. Clearly, A is degenerate if and only if $A^*$ is degenerate.

In the remainder of this section G is a region and Y is a complex Banach space. The following lemma will be used in Sections 2 and 3.

1.2. LEMMA. *Let* m *be a positive integer, and let* $u_1,\dots,u_m$ *be functions with values in* Y *and meromorphic on* G. *Further, let* $\lambda_0$ *be a regular point of* $u_1,\dots,u_m$ *in* G, *and suppose that* $u_1(\lambda_0),\dots,u_m(\lambda_0)$ *are linearly independent. Then there exist a discrete subset* $\Gamma$ *of* G *and holomorphic functions*

$$h_i : G \setminus \Gamma \to Y^* \qquad (i = 1,\dots,m)$$

*such that*

(i) $\lambda_0 \in G \setminus \Gamma$ *and* $u_1,\dots,u_m$ *are holomorphic on* $G \setminus \Gamma$;

(ii) *for each* $\lambda$ *in* $G \setminus \Gamma$, *the vectors* $u_1(\lambda),\dots,u_m(\lambda)$ *are linearly independent;*

(iii) $h_1,\dots,h_m$ *are meromorphic on* G;

(iv) *for each* $\lambda$ *in* $G \setminus \Gamma$

$$\langle h_i(\lambda),u_j(\lambda)\rangle = \delta_{ij} \qquad (i,j = 1,\dots,m),$$

*where* $\delta_{ij}$ *denotes the Kronecker delta.*

PROOF. Since $u_1(\lambda_0),\dots,u_m(\lambda_0)$ are linearly independent, there exist $g_1,\dots,g_m$ in $Y^*$ such that

$$\langle g_i,u_j(\lambda_0)\rangle = \delta_{ij} \qquad (i,j = 1,\dots,m).$$

For each regular point $\lambda$ of $u_1,\dots,u_m$ in G, let $F(\lambda)$ be the determinant of the matrix $(\langle g_i,u_j(\lambda)\rangle)$. Clearly, the complex function F is meromorphic on

G. Let $\Sigma$ be the set of singular points of F in G, and put

$$\Gamma = \{\lambda \in G \setminus \Sigma : F(\lambda) = 0\} \cup \Sigma.$$

Since F is non-zero and meromorphic on the region G, the set $\Gamma$ is a discrete subset of G. Further it is easily seen that (i) and (ii) hold.

Next, for each $\lambda$ in $G \setminus \Gamma$ we define $h_1(\lambda),\dots,h_m(\lambda)$ to be the solution in $Y^*$ of the equations

$$\sum_{j=1}^{m} \langle g_i, u_j(\lambda)\rangle X_j = g_i \qquad (i = 1,\dots,m).$$

Since $F(\lambda) \neq 0$ for each $\lambda$ in $G \setminus \Gamma$, this solution is uniquely determined and we can apply Cramer's theorem to show that the functions $h_1,\dots,h_m$ are holomorphic on $G \setminus \Gamma$, meromorphic on G and that (iv) holds. This proves the lemma.

Let $u_1,\dots,u_m$ be as in the preceding lemma, and let H be the set of all regular points $\lambda$ of $u_1,\dots,u_m$ in G such that $u_1(\lambda),\dots,u_m(\lambda)$ are linearly independent. It is clear that

$$G \setminus \Gamma \subset H$$

if $\Gamma$ is a discrete subset of G with the properties described in Lemma 1.2. The question arises whether $\Gamma$ can be chosen in such a way that $G \setminus \Gamma = H$. Using an unpublished result of K.-H.Förster and G. Garske about holomorphic one-sided inverses of Banach paraalgebra valued holomorphic functions (Theorem 12 in [12]), one can show that the answer to this question is affirmative, provided that H = G (see [5] for details). Without this extra condition the answer is unknown.

## 2. DEGENERATE MEROMORPHIC FUNCTIONS

In this section X and Y are complex Banach spaces. Further, A is a function with values in $L(X,Y)$ and meromorphic on a region G. The set of singular points of A in G is denoted by $\Sigma$.

2.1. LEMMA. *Let* n *be a non-negative integer, and let*

$$\Omega_n = \{\lambda \in G \setminus \Sigma : \dim R(A(\lambda)) \geqq n\}.$$

*Then* $\Omega_n = \emptyset$ *or* $G \setminus \Omega_n$ *is a discrete subset of* G.

PROOF. Observe that $G \setminus \Omega_0 = \Sigma$. Since A is meromorphic on G, the set $\Sigma$ is a discrete subset of G. Hence the lemma is true for n = 0.

Next let n be strictly positive. Suppose $\Omega_n \neq \emptyset$ and take $\lambda_0$ in $\Omega_n$. Choose $x_1,\dots,x_n$ in X such that $A(\lambda_0)x_1,\dots,A(\lambda_0)x_n$ are linearly independent, and define for i = 1,...,n the function $u_i$ by

$$u_i(\lambda) = A(\lambda)x_i \qquad (\lambda \in G \setminus \Sigma).$$

Then $u_1,\dots,u_n$ are holomorphic on $G \setminus \Sigma$ and meromorphic on G. Since $\lambda_0$ is a regular point of $u_1,\dots,u_n$ in G, and since $u_1(\lambda_0),\dots,u_n(\lambda_0)$ are linearly independent, we can apply Lemma 1.2 to show the existence of a discrete subset $\Gamma$ of G such that $\Sigma \subset \Gamma$ and $u_1(\lambda),\dots,u_n(\lambda)$ are linearly independent for each $\lambda$ in $G \setminus \Gamma$. Obviously, this implies that $G \setminus \Gamma$ is a subset of $\Omega_n$, and hence $G \setminus \Omega_n \subset \Gamma$. This shows that $G \setminus \Omega_n$ is a discrete subset of G, and the proof is complete.

The function A is said to be *degenerate meromorphic* on G if $A(\lambda)$ is degenerate for each $\lambda$ in $G \setminus \Sigma$.

2.2. THEOREM. *Suppose that* A *is degenerate meromorphic on* G. *Then the function*

$$\lambda \longmapsto \dim R(A(\lambda)) \qquad (\lambda \in G \setminus \Sigma) \tag{1}$$

*has a finite maximum,* m *say, and the set*

$$\Lambda = \{\lambda \in G \setminus \Sigma : \dim R(A(\lambda)) < m\}$$

*is a discrete subset of* G.

PROOF. Let for each non-negative integer n the set $\Omega_n$ be defined as in the preceding lemma. The hypotheses concerning A imply that

$$G = \bigcup_{n=0}^{\infty} (G \setminus \Omega_n).$$

The set G is a region and thus uncountable. Hence there exists a non-negative integer k such that $G \setminus \Omega_k$ is uncountable. By Lemma 2.1, this implies that $\Omega_k = \emptyset$, and so k is an upper bound for the function (1) on $G \setminus \Sigma$. But then, since its values are integers, the function (1) has a finite maximum.

It remains to prove that $\Lambda$ is a discrete subset of G. Since $\Omega_m$ is non-void, Lemma 2.1 shows that $G \setminus \Omega_m$ is a discrete subset of G. Clearly, $\Lambda$ is a subset of $G \setminus \Omega_m$, and hence the proof is complete.

2.3. PROPOSITION. *Suppose that* A *is degenerate meromorphic on* G *and not identically zero on* $G \setminus \Sigma$. *Let*

$$m = \max \{\dim R(A(\lambda)) : \lambda \in G \setminus \Sigma\}.$$

*Take* $\lambda_0$ *in* $G \setminus \Sigma$ *such that* $\dim R(A(\lambda_0)) = m$. *Then* m *is a positive integer and there exist a discrete subset* $\Gamma$ *of* G *and holomorphic functions*

$$f_i : G \setminus \Gamma \longrightarrow X^*, \quad u_i : G \setminus \Sigma \longrightarrow Y \qquad (i = 1,\dots,m)$$

*such that*

(i) $\lambda_0 \in G \setminus \Gamma$ *and* $\Sigma \subset \Gamma$;

(ii) *the functions* $f_1,\dots,f_m$ *and* $u_1,\dots,u_m$ *are meromorphic on* G;

(iii) *for each* $\lambda$ *in* $G \setminus \Gamma$, *the vectors* $u_1(\lambda),\dots,u_m(\lambda)$ *form a basis of* $R(A(\lambda))$;

(iv) *for each* $\lambda$ *in* $G \setminus \Gamma$, *the functionals* $f_1(\lambda),\dots,f_m(\lambda)$ *form a basis of* $R(A^*(\lambda))$;

(v) *for each* $\lambda$ *in* $G \setminus \Gamma$

$$A(\lambda) = \sum_{i=1}^{m} f_i(\lambda) \otimes u_i(\lambda).$$

PROOF. The preceding theorem shows that m exists and is finite. Since A is not identically zero on $G \setminus \Sigma$, we have $m \geq 1$. Choose $x_1,\dots,x_m$ in X such that the vectors $A(\lambda_0)x_1,\dots,A(\lambda_0)x_m$ form a basis of $R(A(\lambda_0))$, and define for $i = 1,\dots,m$ the function $u_i$ by

$$u_i(\lambda) = A(\lambda)x_i \qquad (\lambda \in G \setminus \Sigma).$$

Then $u_1,\dots,u_m$ are holomorphic on $G \setminus \Sigma$ and meromorphic on $G$. Thus $\lambda_0$ and the functions $u_1,\dots,u_m$ satisfy the conditions of Lemma 1.2. Choose $\Gamma$ and $h_1,\dots,h_m$ as in Lemma 1.2, and define for $i = 1,\dots,m$ the function $f_i$ by

$$f_i(\lambda) = A^*(\lambda)h_i(\lambda) \qquad (\lambda \in G \setminus \Gamma).$$

Then $\Gamma$ is a discrete subset of $G$ and (i) holds. Further, the functions $f_1,\dots,f_m$ are holomorphic on $G \setminus \Gamma$ and meromorphic on $G$. This proves (ii). Since for each $\lambda$ in $G \setminus \Gamma$ the vectors $u_1(\lambda),\dots,u_m(\lambda)$ are linearly independent, it follows that (iii) holds.

Let $\lambda \in G \setminus \Gamma$. To prove (iv) we observe that

$$\langle f_i(\lambda),x_j\rangle = \langle A^*(\lambda)h_i(\lambda),x_j\rangle = \langle h_i(\lambda),A(\lambda)x_j\rangle =$$

$$= \langle h_i(\lambda),u_j(\lambda)\rangle = \delta_{ij} \qquad (i,j = 1,\dots,m).$$

Hence $f_1(\lambda),\dots,f_m(\lambda)$ are linearly independent. Since

$$\dim R(A^*(\lambda)) = \dim R(A(\lambda)) = m,$$

we obtain that (iv) holds.

The proof of (v) is now standard (cf. the proof of Proposition 27A in [8]). Take $x$ in $X$. Because of (iii), there exist $\alpha_1,\dots,\alpha_m$ in $\mathbb{C}$ such that

$$A(\lambda)x = \sum_{j=1}^{m} \alpha_j u_j(\lambda).$$

Using the definition of the functions $f_1,\dots,f_m$ and Lemma 1.2(iv), we derive

$$\langle f_i(\lambda),x\rangle = \langle A^*(\lambda)h_i(\lambda),x\rangle = \langle h_i(\lambda),A(\lambda)x\rangle =$$

$$= \sum_{j=1}^{m} \alpha_j \langle h_i(\lambda),u_j(\lambda)\rangle = \alpha_i \qquad (i = 1,\dots,m).$$

Therefore

$$A(\lambda)x = \sum_{j=1}^{m} \langle f_j(\lambda),x\rangle\, u_j(\lambda) = [\sum_{j=1}^{m} f_j(\lambda) \otimes u_j(\lambda)](x).$$

This completes the proof.

Suppose that A is degenerate meromorphic on G and not identically zero on $G \setminus \Sigma$. Let $\Lambda$ be as in Theorem 2.2 and $\Gamma$ as in Proposition 2.3. It is clear that

$$\Lambda \cup \Sigma \subset \Gamma.$$

The question arises whether $\Gamma$ can be chosen in such a way that $\Lambda \cup \Sigma = \Gamma$. A partial solution of this problem will be given in the next section. In fact we shall prove that the answer to the above question is affirmative if $\Lambda$ and $\Sigma$ are both void.

2.4. LEMMA. *Let* u *and* f *be functions with values in* Y *and* $X^*$, *respectively. Suppose that* u *and* f *are meromorphic on* G. *Define for each regular point* $\lambda$ *of* u *and* f *in* G

$$T(\lambda) = f(\lambda) \otimes u(\lambda). \qquad (2)$$

*Then the function* T *is meromorphic on* G *and for each* $\lambda$ *in* G *the coefficients of the Laurent expansion of* T *at* $\lambda$ *are degenerate.*

PROOF. Take $\lambda$ in G, and let $u_n$ and $f_n$ denote the n-th coefficient of the Laurent expansion of u and f at $\lambda$, respectively. Then, for z in some deleted neighbourbood of $\lambda$, we have

$$T(z) = \sum_{n=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} (z-\lambda)^{n+m}(f_n \otimes u_m).$$

Since u and f are both meromorphic on G, there exists an integer k such that

$$f_n = 0, \quad u_n = 0 \quad (n < k).$$

This implies that

$$T(z) = \sum_{i=2k}^{+\infty} (z-\lambda)^i \Big( \sum_{n=k}^{i-k} f_n \otimes u_{i-n} \Big)$$

for z in some deleted neighbourhood of $\lambda$, and the lemma is proved.

We now come to the second main result of this section.

2.5. THEOREM. *Suppose that* A *is degenerate meromorphic on* G. *Then for each* $\lambda$ *in* G *the coefficients of the Laurent expansion of* A *at* $\lambda$ *are degenerate.*

PROOF. We may assume that A is not identically zero on $G \setminus \Sigma$. Then, by Proposition 2.3, there exists a discrete subset $\Gamma$ of G such that on $G \setminus \Gamma$ the function A can be written as a finite sum of functions of the form (2). The preceding lemma shows that for such functions the theorem holds. Hence the desired result is true in general.

The following example shows that the preceding theorem does not hold if the singularities of A are allowed to be essential.

2.6. EXAMPLE. Let X = Y be the sequence space $l_\infty$. For each positive integer n, let $e_n$ denote the element in $l_\infty$ with all coordinates zero except the n-th, which is equal to one. Put

$$u_n = \frac{1}{n!} e_n \qquad (n = 1,2,\dots).$$

Further, define for n = 1,2,... the linear functional $f_n$ on $l_\infty$ by

$$f_n(x) = \frac{1}{n!} x_n.$$

It is clear that $f_n \in l_\infty^*$. Consider the functions u and f defined on $\mathbb{C} \setminus \{0\}$ by

$$u(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n} u_{n+1}, \qquad f(\lambda) = \sum_{n=0}^{\infty} \lambda^{n} f_{n+1},$$

and let $A : \mathbb{C} \setminus \{0\} \longrightarrow L(X,Y)$ be given by

$$A(\lambda) = f(\lambda) \otimes u(\lambda).$$

Then A is a well-defined holomorphic function. Obviously, A is degenerate. In fact

$$\dim R(A(\lambda)) = 1 \qquad (\lambda \neq 0).$$

We shall prove that none of the coefficients of the Laurent expansion of A at 0 is degenerate.

Let m be an arbitrary integer, and let $A_m$ be the m-th coefficient of the Laurent expansion of A at 0. An easy computation shows that

$$A(\lambda)e_k = \sum_{i=0}^{\infty} \frac{1}{k!} \lambda^{-i+k-1} u_{i+1} \qquad (\lambda \neq 0;\ k = 1,2,3,\ldots).$$

This implies that for each positive integer k > m

$$A_m e_k = \frac{1}{k!} u_{k-m},$$

and hence

$$e_n = n! u_n \in R(A_m) \qquad (n = |m|+1,\ |m|+2,\ldots).$$

It follows that $R(A_m)$ is infinite-dimensional.

Suppose that there exists a non-negative integer k such that the set

$$\{\lambda \in G \setminus \Sigma : \dim R(A(\lambda)) \leqq k\} \tag{3}$$

has an accumulation point in G. Then we can apply Lemma 2.1 to show that

$$\dim R(A(\lambda)) \leqq k \qquad (\lambda \in G \setminus \Sigma),$$

and hence it follows that under this condition A is degenerate meromorphic on G. It is interesting to observe that this result is not necessarily true if in the hypothesis the set (3) is replaced by

$$\{\lambda \in G \setminus \Sigma : \dim R(A(\lambda)) < +\infty\}. \tag{4}$$

For a counterexample we refer to Example 5.14 in [4] (a similar example has appeared in [18]). However, if the set (4) is uncountable, then it can be shown that A is degenerate meromorphic on G. In fact this is done in the first part of the proof of Theorem 2.2.

The results of this section are taken from the author's interim report [4]. For the case when X = Y, Theorem 2.2 has also been proved by J.S. Howland (see Theorem 1 in [18]). Further Howland has shown that the derivative of a degenerate holomorphic function is again degenerate (see [18], Theorem 3 and its proof). One can use this result to give a short proof of Theorem 2.5. On the other hand it is clear that Theorem 2.5 can be used to get Howland's result about the derivative even for $X \neq Y$. The methods used in Howland's paper [18] differ considerably from those used here. For instance, the representation of a degenerate meromorphic function given in Proposition 2.3 does not appear in [18].

## 3. DEGENERATE HOLOMORPHIC FUNCTIONS

Let X be a complex Banach space, and let E be a function defined on a region G with values in the set of all linear subspaces of X. We say that E *has a holomorphic basis* if either

$$E(\lambda) = \{0\} \qquad (\lambda \in G)$$

or there exist a positive integer n and holomorphic functions

$$u_i : G \to X \qquad (i = 1,\ldots,n)$$

such that for each $\lambda$ in G the vectors $u_1(\lambda),\ldots,u_n(\lambda)$ form a basis of the subspace $E(\lambda)$. This concept has been introduced by P. Saphar in [38]. Saphar has shown that, in order to prove that E has a holomorphic basis, it suffices to show that E has this property locally (see Proposition 14 in [38]). We shall use this result to give a partial solution of the problem posed in the paragraph preceding Lemma 2.4.

In the remainder of this section X and Y are complex Banach spaces and A is a holomorphic function defined on a region G with values in $L(X,Y)$.

3.1. LEMMA. *Suppose that* A *is degenerate and that the function*

$$\lambda \longmapsto \dim R(A(\lambda)) \qquad (\lambda \in G)$$

*is constant. Then the subspace valued function*

$$\lambda \longmapsto R(A(\lambda)) \qquad (\lambda \in G) \tag{1}$$

*has a holomorphic basis.*

PROOF. We may assume that A is not identically zero on G. Then we can apply Proposition 2.3 to show that locally the function (1) has a holomorphic basis, and hence, by Saphar's result, this is true globally.

3.2. THEOREM. *Let* m *be a positive integer and suppose that*

$$\dim R(A(\lambda)) = m \qquad (\lambda \in G).$$

*Then there exist holomorphic functions*

$$f_i : G \longrightarrow X^*, \quad u_i : G \longrightarrow Y \qquad (i = 1,\dots,m)$$

*such that for each* $\lambda$ *in* G

$$A(\lambda) = \sum_{i=1}^{m} f_i(\lambda) \otimes u_i(\lambda). \tag{2}$$

PROOF. From Lemma 3.1 we know that there exist holomorphic functions

$$u_i : G \longrightarrow Y \qquad (i = 1,\dots,m)$$

such that for each $\lambda$ in G the vectors $u_1(\lambda),\dots,u_m(\lambda)$ form a basis of $R(A(\lambda))$. Let the functions $f_1,\dots,f_m$ from G into the product space $\mathbb{C}^X$ be defined by the formula

$$A(\lambda)x = \sum_{i=1}^{m} f_i(\lambda)(x)u_i(\lambda). \tag{3}$$

Then $f_1,\dots,f_m$ are well-defined. We need to show that $f_1,\dots,f_m$ are holomorphic functions with values in $X^*$.

Take $\lambda_0$ in G. One can apply Lemma 1.2 to show that there exist a discrete subset $\Gamma$ of G and holomorphic functions

$$h_i : G \setminus \Gamma \longrightarrow Y^* \qquad (i = 1,\dots,m)$$

such that $\lambda_0 \in G \setminus \Gamma$ and for each $\lambda$ in $G \setminus \Gamma$

$$\langle h_i(\lambda),u_j(\lambda)\rangle = \delta_{ij} \qquad (i,j = 1,\dots,m).$$

A simple computation shows that for each $\lambda$ in $G \setminus \Gamma$ and $i = 1,\dots,m$

$$f_i(\lambda)(x) = [A^*(\lambda)h_i(\lambda)](x) \qquad (x \in X).$$

Hence $f_1(\lambda),\ldots,f_m(\lambda)$ are in $X^*$ for each $\lambda$ in $G \setminus \Gamma$ and the functions $f_1,\ldots,f_m$ are holomorphic. From formula (3) we may conclude that (2) holds. This completes the proof.

Suppose that A satisfies the conditions of the preceding theorem, and let

$$f_i : G \longrightarrow X^*, \quad u_i : G \longrightarrow Y \qquad (i = 1,\ldots,m)$$

be holomorphic functions such that for each $\lambda$ in G formula (2) holds. Then it is not difficult to prove that for each $\lambda$ in G the vectors $u_1(\lambda),\ldots,u_m(\lambda)$ form a basis of $R(A(\lambda))$ and $f_1(\lambda),\ldots,f_m(\lambda)$ form a basis of $R(A^*(\lambda))$ (cf. Proposition 2.3(iii) and (iv)).

# CHAPTER II

## RIESZ-MEROMORPHIC FUNCTIONS

The first three sections of this chapter are of introductory character. In Sections 1 and 2 we collect together a number of results concerning Fredholm, semi-Fredholm, projective semi-Fredholm and compact operators. In Section 3 we introduce the notion of a Riesz-meromorphic function, we define the index of such a function, and we use a result of M. Ribaric and I. Vidav [37] to show that the resolvent of a Riesz-meromorphic function, if it exists, is Riesz-meromorphic. The last result is one of our main tools in Sections 4 and 5.

The definition of a Riesz-meromorphic function implies that the values of such a function are operators with a relative inverse. An operator T is said to have a relative inverse if there exists an operator S such that T = TST and S = STS. In Sections 4 and 5 we show that a Riesz-meromorphic function has a global Riesz-meromorphic relative inverse.

### 1. PRELIMINARIES

In this section we collect together a number of results concerning Fredholm, semi-Fredholm and compact operators, and we introduce the relevant terminology. Further we define the notion of a projective operator.

Let M be a linear subspace of a linear space E. The quotient space of E modulo M will be denoted by E/M. The elements of E/M are called the *cosets* of M (in E). The dimension of the quotient space E/M is called the *codimension* of M (in E). To designate this extended integer, we use the symbol codim M.

Let T be a linear operator from a linear space X into a linear space Y. The dimension of the null space N(T) of T is called the *nullity* of T; it will be denoted by n(T). The codimension of the range R(T) of T is called the *defect* of T; it will be denoted by d(T).

Suppose that X and Y are complex Banach spaces. Recall that $L(X,Y)$ denotes the complex Banach space of all bounded linear operators from X into Y. An element T of $L(X,Y)$ is called a *semi-Fredholm operator* if R(T) is closed and at least one of the extended integers n(T) and d(T) is finite. The set of all such operators is designated by $SF(X,Y)$. If $T \in SF(X,Y)$, then the extended integer ind(T) given by

$$\mathrm{ind}(T) = n(T) - d(T)$$

is well-defined; it is called the *index* of T. A *Fredholm operator* is a semi-Fredholm operator with a finite index. One can show that an element T of $L(X,Y)$ is Fredholm if and only if n(T) and d(T) are both finite (see Ch. VII in [34]). The set of all Fredholm operators from X into Y is denoted by $F(X,Y)$.

When X = Y, we shall write $L(X)$ for $L(X,Y)$, $SF(X)$ for $SF(X,Y)$, and $F(X)$ for $F(X,Y)$.

An element T of $L(X,Y)$ is said to be *compact* if the closure of the set

$$\{Tx : x \in X, \ \|x\| \leq 1\}$$

is a compact subset of Y. The set of all such operators will be denoted by $K(X,Y)$. We shall write $K(X)$ for $K(X,X)$. The set $K(X,Y)$ is a closed linear subspace of $L(X,Y)$, and it is not difficult to prove that each degenerate bounded linear operator from X into Y belongs to $K(X,Y)$.

Let $T \in L(X,Y)$ and $S \in L(Y,Z)$, where Z is a complex Banach space. It is well-known that the product ST is compact if at least one of the operators T and S is compact. It follows that $K(X)$ is a closed two-sided ideal in the complex Banach algebra $L(X)$.

Let $T \in L(X,Y)$. An operator S in $L(Y,X)$ is called a

(i) *left c-inverse* of T if $I_X - ST \in K(X)$;

(ii) *right c-inverse* of T if $I_Y - TS \in K(Y)$;

(iii) *c-inverse* of T if $I_X - ST \in K(X)$ and $I_Y - TS \in K(Y)$.

Here $I_X$ and $I_Y$ denote the identity operators on X and Y respectively. The operator T is called *left c-invertible* if T has a left c-inverse. Similarly, T is said to be *right c-invertible (c-invertible)* if T has a right c-inverse (c-inverse).

Throughout this chapter we shall freely use the following basic results concerning semi-Fredholm and compact operators (see, e.g., [13], [15], [23] and Ch. VII in [34] for details).

(a) For each extended integer k, the set

$$\{T \in SF(X,Y) : \mathrm{ind}(T) = k\}$$

is open in $L(X,Y)$, and hence $SF(X,Y)$ and $F(X,Y)$ are open subsets of $L(X,Y)$.

(b) Let $T \in SF(X,Y)$ and $S \in SF(Y,Z)$, where Z is a complex Banach space. Then $ST \in SF(X,Z)$ and

$$\mathrm{ind}(ST) = \mathrm{ind}(S) + \mathrm{ind}(T),$$

provided that the right hand side of this equation makes sense.
(c) If $T \in SF(X,Y)$ and $S \in K(X,Y)$, then $T + S \in SF(X,Y)$ and $\mathrm{ind}(T + S) = \mathrm{ind}(T)$.
(d) An element T of $L(X,Y)$ is a Fredholm operator with index 0 if and only if there exists a degenerate bounded linear operator S from X into Y such that T + S is bijective.
(e) The following statements are equivalent:

(j) $T \in F(X,Y)$;

(jj) T is c-invertible;

(jjj) T is both left and right c-invertible.

In the next section we shall show that the result (e) can be extended in a certain sense to a subclass of $SF(X,Y)$ which in general is strictly larger than $F(X,Y)$. In order to do this, we need the notion of a projective operator.

Let M be a linear subspace of a linear space E. A linear subspace N of E is called an *algebraic complement* of M (in E) if E is the direct sum of M and N, i.e.

$$E = M \oplus N.$$

If N is an algebraic complement of M, then dim N = codim M.

Let X be a complex Banach space. By a *topological complement* of a linear subspace M of X we mean an algebraic complement N of M such that N is closed. A subspace M of X is said to be *projective* (or *complemented)* if M is closed and has a topological complement. Observe that each finite-dimensional subspace of X is projective. The same holds for closed subspaces of finite codimension.

Let M be a projective subspace of X with a topological complement, N say. Define the linear operator P on X by

$$Px = \begin{cases} 0 & \text{if } x \in N, \\ x & \text{if } x \in M. \end{cases}$$

Then P is a well-defined bounded linear operator on X,

$$N(P) = N, \qquad R(P) = M,$$

and $P^2 = P$. We call P the *projection of* X *onto* M *along* N.

Conversely, let P be a linear operator on X such that $P^2 = P$. Then

$$X = N(P) \oplus R(P).$$

It is well-known that P is bounded if and only if N(P) and R(P) are closed, and, in that case, P is the projection of X onto R(P) along N(P).

The foregoing implies that a projection P of X is a bounded linear operator on X such that $P^2 = P$. Hence, if P is a projection of X, then the same is true for $I_X - P$. Observe that

$$N(I_X - P) = R(P), \qquad R(I_X - P) = N(P).$$

Degenerate projections of X are easy to construct. Let m be a positive integer. Take $x_1,\ldots,x_m$ in X and $f_1,\ldots,f_m$ in $X^*$ such that

$$\langle f_i, x_j\rangle = \delta_{ij} \qquad (i,j = 1,\ldots,m), \tag{1}$$

and define P by

$$P = \sum_{i=1}^{m} f_i \otimes x_i. \tag{2}$$

Then P is a projection of X onto the linear hull of the set $\{x_1,\ldots,x_m\}$ along the subspace

$$\bigcap_{i=1}^{m} \{x \in X : \langle f_i, x\rangle = 0\}.$$

From (1) it follows that $x_1,\ldots,x_m$ are linearly independent. Hence the range of P is m-dimensional. Conversely, if P is a projection such that dim R(P) = m, then there exist $x_1,\ldots,x_m$ in R(P) and $f_1,\ldots,f_m$ in $X^*$ satisfying (1) and such that (2) holds.

Let T be a bounded linear operator from the complex Banach space X into the complex Banach space Y. We say that T is *projective* if N(T) and R(T) are both projective. Clearly, any Fredholm operator from X into Y is projective. Further, it is easily seen that a degenerate element of $L(X,Y)$ is projective.

One can characterize the projective elements of $L(X,Y)$ in terms of relative inverses. An element S of $L(Y,X)$ is called a *relative inverse* of T if

$$T = TST, \qquad S = STS$$

(cf. [3] and [38]). The next two lemmas are given without proof. They are due to P. Saphar (see Propositions 10 and 12 in [38]). Combining these two lemmas one sees that an element T of $L(X,Y)$ is projective if and only if T has a relative inverse.

1.1. LEMMA. *Let* $T \in L(X,Y)$, *and suppose that* S *is a relative inverse of* T. *Then* ST *is the projection of* X *onto* R(S) *along* N(T) *and* TS *is the projection of* Y *onto* R(T) *along* N(S).

1.2. LEMMA. *Let* T *be a projective bounded linear operator from* X *into* Y, *and let* N *and* R *be topological complements of* N(T) *and* R(T) *respectively. Then there exists a unique relative inverse* S *of* T *such that* N(S) = R *and* R(S) = N.

## 2. PROJECTIVE SEMI-FREDHOLM OPERATORS

Let X and Y be complex Banach spaces. The set of all projective semi-Fredholm operators from X into Y will be denoted by $PSF(X,Y)$. Instead of $PSF(X,X)$, we shall write $PSF(X)$. Observe that $F(X,Y)$ is a subset of $PSF(X,Y)$. Further we note that a projective operator T is semi-Fredholm if and only if at least one of the extended integers n(T) and d(T) is finite.

In this section we shall show that certain results concerning the c-invertibility of Fredholm operators extend to the class $PSF(X,Y)$. Further we shall prove that the results (a), (b) and (c) mentioned in Section 1 remain true if $SF$ is replaced by $PSF$. For the case when X = Y (and X an arbitrary locally convex vector space), this has been done by A. Pietsch in [36]. See also [3] and [44]. Our methods are similar to those used by Pietsch.

2.1. LEMMA. *Let* $T \in L(X,Y)$. *Then*

(i) $T \in PSF(X,Y)$ *and* $\mathrm{ind}(T) < +\infty$ *if and only if there exists* S *in* $L(Y,X)$ *such that* $I_X - ST$ *is degenerate;*

(ii) $T \in PSF(X,Y)$ *and* $\mathrm{ind}(T) > -\infty$ *if and only if there exists* S *in* $L(Y,X)$ *such that* $I_Y - TS$ *is degenerate.*

PROOF. (i) Suppose that $T \in PSF(X,Y)$ and that $\mathrm{ind}(T) < +\infty$. Since T is projective, T has a relative inverse, S say. By Lemma 1.1, the operator ST is a projection of X along N(T). Thus the range of $I_X - ST$ is N(T). Since $\mathrm{ind}(T) < +\infty$, the dimension of N(T) is finite. This implies that $I_X - ST$ is degenerate. Hence S has the desired property.

Conversely, let S be an element of $L(Y,X)$ such that $I_X - ST$ is degenerate. Then TST - T is degenerate too, and hence TST - T is projective. Let V be a relative inverse of TST - T. Put

$$U = S - (ST - I_X)V(TS - I_Y).$$

Then $U \in L(Y,X)$ and TUT = T. From the last equation it easily follows that UTU is a relative inverse of T. So T is projective. Since N(T) is a subset of $R(I_X - ST)$, and since $I_X - ST$ is degenerate, the nullity of T is finite. Thus $T \in PSF(X,Y)$ and $\mathrm{ind}(T) < +\infty$. This proves (i).

(ii) The proof of (ii) is similar.

2.2. THEOREM. *Let* $T \in L(X,Y)$. *Then*

(i) $T \in PSF(X,Y)$ *and* $\mathrm{ind}(T) < +\infty$ *if and only if* T *is left c-invertible;*

(ii) $T \in PSF(X,Y)$ *and* $\mathrm{ind}(T) > -\infty$ *if and only if* T *is right c-invertible.*

PROOF. (i) The "only if part" is contained in Lemma 2.1(i). To prove the "if part", let U be a left c-inverse of T. Then $UT - I_X$ is a compact operator. Since $I_X$ is Fredholm, it follows that

$$UT = I_X + (UT - I_X)$$

is a Fredholm operator too. But then we can apply Lemma 2.1(i) to show that there exists V in $L(X)$ such that $I_X - V(UT)$ is degenerate. Observe that $VU \in L(Y,X)$ and that $I_X - (VU)T$ is degenerate. Hence, using Lemma 2.1

again, $T \in PSF(X,Y)$ and $\text{ind}(T) < +\infty$. This proves (i).

(ii) The proof of (ii) is similar.

For the case when $X = Y$, we obtain the following corollary (cf. [36], Theorems 3.3 and $3.3^{N,B}$).

2.3. COROLLARY. *Let* $T \in L(X)$ *and let* $\kappa$ *denote the canonical mapping of* $L(X)$ *onto the quotient algebra* $L(X)/K(X)$. *Then*

(i) $T \in F(X)$ *if and only if* $\kappa(T)$ *is invertible in* $L(X)/K(X)$;

(ii) $T \in PSF(X)$ *and* $\text{ind}(T) = -\infty$ *if and only if* $\kappa(T)$ *is left invertible, but not invertible, in* $L(X)/K(X)$;

(iii) $T \in PSF(X)$ *and* $\text{ind}(T) = +\infty$ *if and only if* $\kappa(T)$ *is right invertible, but not invertible, in* $L(X)/K(X)$.

Theorem 2.2 shows that the set $PSF(X,Y)$ is invariant under compact perturbations. By combining this with the result (c) mentioned in Section 1, we obtain the following theorem.

2.4. THEOREM. *Let* $T \in PSF(X,Y)$ *and* $S \in K(X,Y)$. *Then* $T + S \in PSF(X,Y)$ *and* $\text{ind}(T + S) = \text{ind}(T)$.

Next we show that the results (a) and (b) mentioned in Section 1 remain true if $SF$ is replaced by $PSF$.

2.5. THEOREM. *Let* $T \in PSF(X,Y)$ *and* $S \in PSF(Y,Z)$, *where* $Z$ *is a complex Banach space. Then* $ST \in PSF(X,Z)$ *and*

$$\text{ind}(ST) = \text{ind}(S) + \text{ind}(T), \tag{1}$$

*provided that the right hand side of this equation makes sense.*

PROOF. Suppose that the right hand side of (1) is well-defined. This means that either

$$\text{ind}(T) < +\infty, \quad \text{ind}(S) < +\infty \tag{2}$$

or

$$\text{ind}(T) > -\infty, \quad \text{ind}(S) > -\infty. \tag{3}$$

Assume that (2) holds. Then, by Theorem 2.2(i), the operators T and S are both left c-invertible. If U and V are left c-inverses of T and S respectively, then UV is a left c-inverse of ST. According to Theorem 2.2(i), this implies that $ST \in PSF(X,Z)$. By using Theorem 2.2(ii), formula (3) yields the same result. Thus $ST \in PSF(X,Z)$. The fact that (1) holds has already been mentioned in Section 1 (result (b)). This proves the theorem.

2.6. THEOREM. *For each extended integer* k, *the set*

$$\{T \in PSF(X,Y) : \mathrm{ind}(T) = k\}$$

*is open in* $L(X,Y)$.

PROOF. It suffices to show that $PSF(X,Y)$ is open in $L(X,Y)$. Take T in $PSF(X,Y)$, and suppose that $\mathrm{ind}(T) < +\infty$. (The case when $\mathrm{ind}(T) > -\infty$ can be treated similarly). By Theorem 2.2(i), there exists U in $L(Y,X)$ such that $I_X - UT$ is compact. Choose S in $L(X,Y)$ such that $||S||.||U|| < 1$. Then $I_X + US$ is invertible in $L(X)$. Hence

$$U(T + S) = (I_X + US) - (I_X - UT)$$

is c-invertible. Therefore T + S is left c-invertible, and thus, by Theorem 2.2(i), we have $T + S \in PSF(X,Y)$. This shows that $PSF(X,Y)$ is an open subset of $L(X,Y)$.

It can be shown that the conjugate of a projective semi-Fredholm operator is again a projective semi-Fredholm operator. The converse of this statement does not hold. Counterexamples have been constructed by A. Pietsch ([36], pp. 366,367). These examples also show that in general $PSF(X,Y)$ is a proper subset of $SF(X,Y)$.

## 3. RIESZ-MEROMORPHIC FUNCTIONS

In this section X and Y are complex Banach spaces. Let A be a function defined on a subset D of $\mathbb{C}$ with values in $L(X,Y)$. The set of all $\lambda$ in D such that $A(\lambda)$ is bijective is called the *resolvent set* of A; it will be denoted by Res[A]. The function $A^{-1}$ defined on Res[A] by

$$A^{-1}(\lambda) = A(\lambda)^{-1}$$

is called the *resolvent* of A. It is a function with values in $L(Y,X)$.

From operator theory we know that the set $GL(X,Y)$ of all bijective bounded linear operators from X into Y is open in $L(X,Y)$, and that the function

$$T \longmapsto T^{-1}$$

is a continuous map of $GL(X,Y)$ into $L(Y,X)$. Hence, if A is continuous on D, then Res[A] is open in the relative topology of D and $A^{-1}$ is continuous. In particular, if D is open and if A is locally holomorphic on D, then Res [A] is an open subset of $\mathbb{C}$. Further, in that case, $A^{-1}$ is locally holomorphic on Res[A].

The main goal of this section is to present a sufficient condition in order that a meromorphic Fredholm operator valued function A has a meromorphic resolvent $A^{-1}$. For that purpose we introduce the concept of a Riesz-point and that of a Riesz-meromorphic function.

Let $\lambda_0$ be a complex number such that A is holomorphic on some deleted neighbourhood of $\lambda_0$. Let $T_n$ denote the n-th coefficient of the Laurent expansion of A at $\lambda_0$. We say that $\lambda_0$ is a *Riesz-point* of A if the following conditions are satisfied:

(i) $\lambda_0$ is not an essential singularity of A, i.e., there exists an integer m such that

$$T_{-n} = 0 \qquad (n = m+1, m+2,\dots);$$

(ii) $T_{-n}$ is degenerate for each positive integer n;

(iii) $T_0$ is a projective semi-Fredholm operator.

Let $\lambda_0$ be a Riesz-point of A. Then the extended integer $\mathrm{ind}(T_0)$ is called the *index* of $\lambda_0$ (as a Riesz-point of A).

3.1. LEMMA. *Let* $\lambda_0$ *be a Riesz-point of* A *with index* k. *Then there exists a neighbourhood* U *of* $\lambda_0$ *such that each* $\lambda$ *in* U *is a Riesz-point of* A *with index* k.

PROOF. Let for $0 < |\lambda - \lambda_0| < r$

$$A(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n T_n + \sum_{n=1}^{m} (\lambda - \lambda_0)^{-n} T_{-n} \tag{1}$$

be the Laurent expansion of A at $\lambda_0$. Then $T_{-1},\ldots,T_{-m}$ are degenerate and $T_0$ is a projective semi-Fredholm operator with index k. Let

$$V_k = \{T \in PSF(X,Y) : \text{ind}(T) = k\}.$$

We know that $V_k$ is an open subset of $L(X,Y)$ (Theorem 2.6). Hence, since $T_0 \in V_k$, there exists $0 < \varepsilon < r$ such that

$$\sum_{n=0}^{\infty} (\lambda - \lambda_0)^n T_n \in V_k \qquad (|\lambda - \lambda_0| < \varepsilon).$$

Further, we know that $V_k$ is invariant under compact perturbations (Theorem 2.4). Our hypotheses imply that the operator

$$\sum_{n=1}^{m} (\lambda - \lambda_0)^{-n} T_{-n} \qquad (\lambda \neq \lambda_0)$$

is bounded and degenerate. In particular, it is compact. But then, using formula (1), it follows that

$$A(\lambda) \in V_k \qquad (0 < |\lambda - \lambda_0| < \varepsilon). \tag{2}$$

Take $U = \{\lambda : |\lambda - \lambda_0| < \varepsilon\}$. Since A is holomorphic on $U \setminus \{\lambda_0\}$, formula (2) implies that each point $\lambda$ in U is a Riesz-point of A with index k.

In the remainder of this section G will be a region. The function A is said to be *Riesz-meromorphic* on G if each point of G is a Riesz-point of A. Suppose that A is Riesz-meromorphic on G. Since G is a region, the preceding lemma implies the existence of a unique extended integer k such that each point of G is a Riesz-point of A with index k. We call k the *index* of A (on G). The term Riesz-meromorphic function has not appeared in literature before, but several authors have studied these functions (see [7], [14], [37] and [40]).

To illustrate our terminology we consider a special case. Let T be a bounded linear operator on the complex Banach space X. The spectrum and the resolvent of T will be denoted by $\sigma(T)$ and $R(\cdot;T)$ respectively. A point $\lambda_0$ of $\sigma(T)$ is called a *pole* of T of *order* n if $\lambda_0$ is a pole of order n of the locally holomorphic function

$$z \longmapsto R(z;T) \qquad (z \in \mathbb{C} \setminus \sigma(T)). \tag{3}$$

A pole $\lambda_0$ of T is said to be of *finite rank* if the spectral projection associated with T and $\lambda_0$ is degenerate.

Let $\lambda_0$ be a pole of T of finite rank and of order m. Let P be the spectral projection associated with T and $\lambda_0$. Consider the Laurent expansion:

$$R(z;T) = \sum_{n=0}^{\infty} (z - \lambda_0)^n S_n + \sum_{n=1}^{m} (z - \lambda_0)^{-n} S_{-n}.$$

From spectral theory (see Section 5.8 in [42]) we know that

$$S_{-n} = (T - \lambda_0 I_X)^{n-1} P \qquad (n = 1,\ldots,m)$$

and

$$(T - \lambda_0 I_X)S_0 = S_0(T - \lambda_0 I_X) = P - I_X.$$

Since P is degenerate, it follows that the operators $S_{-1},\ldots,S_{-m}$ are degenerate and that $S_0$ is a Fredholm operator with index 0. Hence $\lambda_0$ is a Riesz-point with index 0 of the function (3).

Clearly, each point in $\mathbb{C} \setminus \sigma(T)$ is also a Riesz-point with index 0 of the function (3). It is easy to see that the function (3) has no other Riesz-points. More precisely the set of Riesz-points of the function (3) is the union of $\mathbb{C} \setminus \sigma(T)$ and the set of all poles of T of finite rank.

In literature the term *Riesz operator* is used to denote a bounded linear operator T on X which has the property that each point of $\sigma(T) \setminus \{0\}$ is a pole of T of finite rank. In other words, T is a Riesz operator if and only if the function (3) is Riesz-meromorphic on $\mathbb{C} \setminus \{0\}$. Riesz operators can be characterized in terms of Fredholm operators (see [26]). In fact, T is a Riesz operator if and only if $\lambda I_X - T$ is Fredholm for each $\lambda \neq 0$, and this in turn is equivalent to the statement that the function

$$\lambda \longmapsto \lambda I_X - T \qquad (\lambda \in \mathbb{C}) \tag{4}$$

is Riesz-meromorphic on $\mathbb{C} \setminus \{0\}$.

Let G be $\mathbb{C} \setminus \{0\}$, and let A be the function (4). Clearly, G is a

region.
Further, $\mathrm{Res}[A] = \mathbb{C} \setminus \sigma(T)$ and $A^{-1} = R(.;T)$. The results of the preceding paragraphs show that the following statements are equivalent:

(i) T is a Riesz operator;

(ii) A is Riesz-meromorphic on G;

(iii) $A^{-1}$ is Riesz-meromorphic on G.

This justifies our terminology.

In this section we shall show that the equivalence of (ii) and (iii) mentioned above holds in general, provided that $A(z)$ is bijective for some regular point z of A in G. This result will be proved by using Theorem I in [37]. In the paper [37] M. Ribaric and I. Vidav introduced the concept of an essentially meromorphic function. We proceed with the definition of this notion.

In the remainder of this section A will be a meromorphic function on the region G with values in $L(X,Y)$, and $\Sigma$ will be the set of singular points of A in G. The function A is said to be *essentially meromorphic* on G if for each $\lambda$ in G the coefficients of the principal part of the Laurent expansion of A at $\lambda$ are degenerate and if $A(\mu)$ is compact for each $\mu$ in $G \setminus \Sigma$. Observe that Theorem I.2.5 shows that a degenerate meromorphic function is essentially meromorphic. In other words, if $A(\lambda)$ is degenerate for each $\lambda$ in $G \setminus \Sigma$, then A is essentially meromorphic on G.

Essentially meromorphic functions may be used to construct Riesz-meromorphic functions. Suppose X = Y and let A be essentially meromorphic on G. Then the function

$$\lambda \longmapsto I_X - A(\lambda) \qquad (\lambda \in G \setminus \Sigma)$$

is Riesz-meromorphic on G with index 0. This result is a special case of the following theorem.

3.2. THEOREM. *Let* A *and* B *be meromorphic functions on the region* G *with values in* $L(X,Y)$. *Suppose that* A *is Riesz-meromorphic on* G *with index* k *and that* B *is essentially meromorphic on* G. *Define for each regular point* $\lambda$ *of* A *and* B *in* G

$$T(\lambda) = A(\lambda) + B(\lambda). \qquad (5)$$

*Then the function* T *is Riesz-meromorphic on* G *with index* k.

PROOF. Clearly, T is meromorphic on G. Take $\lambda_0$ in G. Since the sum of two degenerate operators is degenerate, the coefficients of the principal part of the Laurent expansion of T at $\lambda_0$ are degenerate operators. Let $A_0$ and $B_0$ denote the "constant terms" in the Laurent expansions of A and B at $\lambda_0$, respectively. It remains to show that $A_0 + B_0$ is a projective semi-Fredholm operator with index k.

There exists a deleted neighbourhood U of $\lambda_0$ such that each $\lambda$ in U is a regular point of B. Thus

$$B(\lambda) \in K(X,Y) \qquad (\lambda \in U).$$

Since $K(X,Y)$ is a closed linear subspace of $L(X,Y)$, we can use the Cauchy integral formula to show that the coefficients of the Laurent expansion of B at $\lambda_0$ belong to $K(X,Y)$. In particular, $B_0$ is compact. Our hypotheses imply that $A_0 \in PSF(X,Y)$ and $\mathrm{ind}(A_0) = k$. Thus we can apply Theorem 2.4 to get the desired result.

The class of degenerate meromorphic functions is a subclass of the class of essentially meromorphic functions. So we have the following corollary.

3.3. COROLLARY. *Let* A *and* B *be meromorphic functions on the region* G *with values in* $L(X,Y)$. *Suppose that* A *is Riesz-meromorphic on* G *with index* k *and that* B *is degenerate meromorphic on* G. *Then the function* T *defined by formula* (5) *is Riesz-meromorphic on* G *with index* k.

We now come to the theorem announced above. The proof of this theorem turns out to be an immediate application of a result of M. Ribaric and I. Vidav [37]. For the special case when X = Y is a Hilbert space, the theorem was proved by P.M. Bleher [7]. The methods used by Bleher are similar to those employed by M. Ribaric and I. Vidav (cf. also [40]).

3.4. THEOREM. *Let* A *be Riesz-meromorphic on the region* G, *and suppose that* A(z) *is bijective for some regular point* z *of* A *in* G. *Then* $A^{-1}$ *is Riesz-meromorphic on* G *with index* 0.

PROOF. Let H be the set of all $\lambda$ in G such that $\lambda$ is a Riesz-point of $A^{-1}$

with index 0. We need to show that $H = G$. Since $G$ is a region, it is sufficient to prove that $H$ is a non-void open and closed (with respect to the relative topology) subset of $G$. Clearly $z \in H$, and hence $H \neq \emptyset$. Further, Lemma 3.1 implies that $H$ is open. Hence it remains to show that $H$ is closed in the relative topology of $G$.

Take $\lambda_0$ in the closure of $H$ with respect to the relative topology on $G$. We want to show that $\lambda_0 \in H$. We begin with two observations. First of all, since $A(z)$ is bijective, we may assume that $X = Y$. Secondly, for the same reason, the index of $\lambda_0$ as a Riesz-point of $A$ is zero.

Let for $0 < |\lambda - \lambda_0| < r$

$$A(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n T_n + \sum_{n=1}^{m} (\lambda - \lambda_0)^{-n} T_{-n}$$

be the Laurent expansion of $A$ at $\lambda_0$. Then $T_0$ is a Fredholm operator with index zero and $T_{-1},\ldots,T_{-m}$ are bounded and degenerate. The first fact implies the existence of a degenerate bounded linear operator $F$ on $X$ such that $T_0 + F$ is bijective. Define the holomorphic functions $A_1$ and $A_2$ by

$$A_1(\lambda) = -F + \sum_{n=1}^{m} (\lambda - \lambda_0)^{-n} T_{-n} \qquad (0 < |\lambda - \lambda_0| < r)$$

and

$$A_2(\lambda) = F + \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n T_n \qquad (|\lambda - \lambda_0| < r).$$

Then $A_2(\lambda_0)$ is bijective and

$$A(\lambda) = A_1(\lambda) + A_2(\lambda) \qquad (0 < |\lambda - \lambda_0| < r).$$

Since the operators $T_{-1},\ldots,T_{-m}$ and $F$ are degenerate, there exists a finite-dimensional subspace $W$ of $X$ such that

$$R(A_1(\lambda)) \subset W \qquad (0 < |\lambda - \lambda_0| < r).$$

Now we can apply Theorem I in [37] to show that there exists $\varepsilon > 0$ such that $A^{-1}$ is defined on the set $0 < |\lambda - \lambda_0| < \varepsilon$ and on this set $A^{-1}$ has the same properties as $A$. More precisely,

$$A^{-1}(\lambda) = B_1(\lambda) + B_2(\lambda) \qquad (0 < |\lambda - \lambda_0| < \varepsilon), \tag{6}$$

where $B_1$ is a degenerate meromorphic function on the region $|\lambda - \lambda_0| < \varepsilon$, the function $B_2$ is holomorphic on $|\lambda - \lambda_0| < \varepsilon$, and $B_2(\lambda_0)$ is bijective. Observe that $\lambda_0$ is a Riesz-point with index 0 of $B_2$. Hence, by Lemma 3.1, there exists $0 < \delta < \varepsilon$ such that $B_2$ is Riesz-meromorphic with index 0 on the region $|\lambda - \lambda_0| < \delta$. But then, using formula (6), Corollary 3.3 implies that on $|\lambda - \lambda_0| < \delta$ the function $A^{-1}$ is Riesz-meromorphic with index 0. In particular, $\lambda_0$ is a Riesz-point with index 0 of $A^{-1}$. Thus $\lambda_0 \in H$. This completes the proof.

We conclude this section with the following proposition, which will be used in the next two sections.

3.5. PROPOSITION. *Let* p *and* q *be extended integers such that* p + q *is defined. Let* A *be Riesz-meromorphic on* G *with index* p*, and let* B *be a Riesz-meromorphic function on* G *with index* q *and values in* $L(Y,Z)$, *where* Z *is a complex Banach space. Define for each regular point* $\lambda$ *of* A *and* B *in* G

$$T(\lambda) = B(\lambda)A(\lambda).$$

*Then the function* T *is Riesz-meromorphic on* G *with index* p + q.

PROOF. Clearly, T is meromorphic on G. Take $\lambda_0$ in G. By multiplying the Laurent expansions of A and B at $\lambda_0$, we obtain the Laurent expansion of T at $\lambda_0$. From this it is easy to see that the coefficients of the principal part of the Laurent expansion of T at $\lambda_0$ are degenerate. Let $T_0$, $A_0$ and $B_0$ denote the "constant terms" in the Laurent expansions of T, A and B at $\lambda_0$, respectively. From our hypotheses we know that $A_0$ and $B_0$ are projective semi-Fredholm operators with indices p and q respectively. Hence we can use Theorem 2.5 to show that $B_0A_0$ is a projective semi-Fredholm operator with index p + q. Now $T_0 - B_0A_0$ is a finite sum of degenerate elements of $L(X,Z)$. So Theorem 2.4 implies that $T_0 \in PSF(X,Z)$ and $\mathrm{ind}(T_0) = p + q$. This completes the proof.

## 4. RIESZ-MEROMORPHIC FUNCTIONS WITH FINITE INDEX

The main problem treated in this section concerns the existence of a (global) meromorphic relative inverse of a Riesz-meromorphic function with finite index.

Throughout this section X and Y are complex Banach spaces. Further, A is a function with values in $L(X,Y)$ and meromorphic on a region G. The set of singular points of A in G will be denoted by $\Sigma$.

Let the extended integers $m_n[A;G]$ and $m_d[A;G]$ be given by

$$m_n[A;G] = \min \{n(A(\lambda)) : \lambda \in G \setminus \Sigma\}$$

and

$$m_d[A;G] = \min \{d(A(\lambda)) : \lambda \in G \setminus \Sigma\}.$$

Further, let

$$H[A;G] = \{\lambda \in G \setminus \Sigma : n(A(\lambda)) = m_n[A;G],\ d(A(\lambda)) = m_d[A;G]\}.$$

Often we shall omit [A;G] in the symbols defined above.

In general it is not clear whether or not H will be non-void. However, if A is Riesz-meromorphic on G, then it is not difficult to show that $H \neq \emptyset$. Indeed, suppose that A is Riesz-meromorphic on G. If A has finite index k, then

$$n(A(\lambda)) - d(A(\lambda)) = \operatorname{ind}(A(\lambda)) = k \qquad (\lambda \in G \setminus \Sigma),$$

and hence

$$H = \{\lambda \in G \setminus \Sigma : n(A(\lambda)) = m_n\} = \{\lambda \in G \setminus \Sigma : d(A(\lambda)) = m_d\}.$$

If A has index $-\infty$, then $d(A(\lambda)) = +\infty$ for all $\lambda$ in $G \setminus \Sigma$, and so

$$H = \{\lambda \in G \setminus \Sigma : n(A(\lambda)) = m_n\}.$$

Finally, if A has index $+\infty$, then $n(A(\lambda)) = +\infty$ for all $\lambda$ in $G \setminus \Sigma$, and therefore

$$H = \{\lambda \in G \setminus \Sigma : d(A(\lambda)) = m_d\}.$$

From these expressions for H, it easily follows that H is non-void whenever A is Riesz-meromorphic on G.

4.1. THEOREM. *Suppose that A is Riesz-meromorphic on* G *with finite index* k. *Let* $\lambda_0 \in H[A;G]$, *and let* N *and* R *be topological complements of* $N(A(\lambda_0))$ *and* $R(A(\lambda_0))$ *respectively. Then the set* Γ, *given by*

$$\Gamma = G \setminus \{\lambda \in G \setminus \Sigma : X = N(A(\lambda)) \oplus N,\ Y = R(A(\lambda)) \oplus R\},$$

*is a discrete subset of* G *and there exists a holomorphic function* B *defined on* $G \setminus \Gamma$ *with values in* $L(Y,X)$ *such that*

(i) B *is Riesz-meromorphic on* G *with index* -k;

(ii) *for each* λ *in* $G \setminus \Gamma$, *we have* $N(B(\lambda)) = R$ *and* $R(B(\lambda)) = N$;

(iii) *for each* λ *in* $G \setminus \Gamma$, *the operator* $B(\lambda)$ *is a relative inverse of* $A(\lambda)$.

PROOF. The proof consists of three parts. The first two parts deal with the case when k = 0, the third with the general case. Observe that k = 0 is equivalent to the statement that $m_n = m_d$.

(I) Suppose $m_n = m_d = 0$. Then N = X and R = {0}, and hence $\lambda_0 \in \mathrm{Res}[A]$. Further, it follows that

$$\Gamma = \Sigma \cup (G \setminus \mathrm{Res}[A]).$$

According to Theorem 3.4 the resolvent $A^{-1}$ of A is Riesz-meromorphic on G with index 0. This implies that Γ is a discrete subset of G. If we define B to be the restriction of $A^{-1}$ to $G \setminus \Gamma$, then B is a holomorphic function with values in $L(Y,X)$ which meets the requirements.

(II) Suppose $m_n = m_d = m \geq 1$. Let $x_1,\ldots,x_m$ be a basis of $N(A(\lambda_0))$, and choose $f_1,\ldots,f_m$ in $X^*$ such that

$$\langle f_i, x_j \rangle = \delta_{ij} \qquad (i,j = 1,\ldots,m)$$

and

$$\langle f_i, x \rangle = 0 \qquad (x \in N;\ i = 1,\ldots,m).$$

The dimension of R is equal to the codimension of $R(A(\lambda_0))$, and hence R is m-dimensional. Let $y_1,\dots,y_m$ form a basis of R, and let

$$C = \sum_{i=1}^{m} f_i \otimes y_i.$$

Then $N(C) = N$ and $R(C) = R$. Define T on $G \setminus \Sigma$ by

$$T(\lambda) = A(\lambda) + C.$$

The function T is holomorphic on $G \setminus \Sigma$ and Riesz-meromorphic on G with index 0 (cf. Corollary 3.3). In particular, it follows that for each $\lambda$ in $G \setminus \Sigma$ the operator $T(\lambda)$ is a Fredholm operator with index 0. We shall show that

$$\mathrm{Res}[T] = G \setminus \Gamma. \tag{1}$$

Take $\lambda$ in Res[T]. Then $\lambda \in G \setminus \Sigma$ and

$$N(T(\lambda)) = \{0\}, \quad R(T(\lambda)) = Y.$$

From the definition of $T(\lambda)$ it is clear that

$$N(A(\lambda)) \cap N \subset N(T(\lambda)), \quad R(A(\lambda)) + R \supset R(T(\lambda)).$$

Hence

$$N(A(\lambda)) \cap N = \{0\}, \quad R(A(\lambda)) + R = Y.$$

Further,

$$n(A(\lambda)) \geqq m = \mathrm{codim}\, N, \quad d(A(\lambda)) \geqq m = \dim R.$$

This implies that

$$X = N(A(\lambda)) \oplus N, \quad Y = R(A(\lambda)) \oplus R,$$

and so $\lambda \in G \setminus \Gamma$. Thus Res[T] is a subset of $G \setminus \Gamma$. Conversely, take $\lambda$ in $G \setminus \Gamma$. Then $\lambda$ certainly belongs to $G \setminus \Sigma$. Let x be in the null space of

$T(\lambda)$. Then $A(\lambda)x = -Cx = C(-x)$. But

$$R(A(\lambda)) \cap R(C) = R(A(\lambda)) \cap R = \{0\},$$

for $\lambda \in G \setminus \Gamma$. Thus $A(\lambda)x = Cx = 0$, and hence $x \in N(A(\lambda)) \cap N(C)$. Since $N(C) = N$ and $N(A(\lambda)) \cap N = \{0\}$, this implies that $x = 0$. Thus $T(\lambda)$ is injective. From this and the fact that $T(\lambda)$ is a Fredholm operator with index 0, it follows that $T(\lambda)$ is also surjective. Hence $\lambda \in \text{Res}[T]$, and so $G \setminus \Gamma$ is a subset of Res[T]. This proves (1).

Let S denote the resolvent of T, i.e., $S(\lambda) = T(\lambda)^{-1}$ for each $\lambda$ in Res[T]. Since $\lambda_0 \in G \setminus \Gamma$, formula (1) shows that $\text{Res}[T] \neq \emptyset$, and hence we can apply Theorem 3.4 to prove that S is Riesz-meromorphic on G with index 0. Observe that $\Gamma$ is the set of singular points of S in G. Thus $\Gamma$ is a discrete subset of G.

Next we shall prove that for each $\lambda$ in $G \setminus \Gamma$

$$\langle f_i, S(\lambda)y_j\rangle = \delta_{ij} \qquad (i,j = 1,\dots,m). \tag{2}$$

Take $\lambda$ in $G \setminus \Gamma$. From the definition of $T(\lambda)$ it is clear that

$$T(\lambda)N(A(\lambda)) \subset R.$$

From (1) we see that $T(\lambda)$ is bijective, and hence

$$\dim T(\lambda)N(A(\lambda)) = \dim N(A(\lambda)) \geq m = \dim R.$$

This shows that

$$T(\lambda)N(A(\lambda)) = R. \tag{3}$$

Since $y_1,\dots,y_m$ form a basis of R, formula (3) implies that $S(\lambda)y_1,\dots,S(\lambda)y_m$ form a basis of $N(A(\lambda))$. Further, for $j = 1,\dots,m$,

$$y_j = T(\lambda)S(\lambda)y_j = CS(\lambda)y_j = \sum_{i=1}^{m} \langle f_i, S(\lambda)y_j\rangle\, y_i.$$

Since $y_1,\dots,y_m$ are linearly independent, the last formula yields (2).

Consider for $\lambda$ in $G \setminus \Gamma$ the operator

$$Q(\lambda) = \sum_{i=1}^{m} f_i \otimes S(\lambda)y_i .$$

The results of the preceding paragraph show that $Q(\lambda)$ is the projection of $X$ onto $N(A(\lambda))$ along $N$. Define on $G \setminus \Gamma$ the function $P$ by

$$P(\lambda) = I_X - Q(\lambda).$$

Since $S$ is meromorphic on $G$, the same is true for the function $Q$. In addition, $Q$ is degenerate. So it follows from Corollary 3.3 that $P$ is Riesz-meromorphic on $G$ with index 0. From the properties of $Q$ it follows that for each $\lambda$ in $G \setminus \Gamma$ the operator $P(\lambda)$ is the projection of $X$ onto $N$ along $N(A(\lambda))$.

Define the function $B$ on $G \setminus \Gamma$ by

$$B(\lambda) = P(\lambda)S(\lambda).$$

We shall prove that $B$ has the required properties. Since $S$ is holomorphic on $G \setminus \Gamma$, the same holds for the functions $Q$ and $P$, and hence it is also true for $B$. Further we know that $P$ and $S$ are Riesz-meromorphic on $G$ with index 0. So we can apply Proposition 3.5 to show that the same is true for the function $B$.

For each $\lambda$ in $G \setminus \Gamma$, we have

$$N(B(\lambda)) = S(\lambda)^{-1}N(P(\lambda)) = T(\lambda)N(A(\lambda)) = R$$

(see formula (3)) and

$$R(B(\lambda)) = P(\lambda)R(S(\lambda)) = P(\lambda)X = R(P(\lambda)) = N.$$

It remains to show that $B(\lambda)$ is a relative inverse of $A(\lambda)$ for each $\lambda$ in $G \setminus \Gamma$.

Let $\lambda \in G \setminus \Gamma$. Since the range of $P(\lambda)$ is $N$, we have $CP(\lambda) = 0$. Further, $A(\lambda)Q(\lambda) = 0$ because $Q(\lambda)$ maps $X$ onto $N(A(\lambda))$. From these observations it follows that

$$T(\lambda)P(\lambda) = (A(\lambda) + C)P(\lambda) = A(\lambda)P(\lambda)$$

and

$$A(\lambda)P(\lambda) = A(\lambda)(I_X - Q(\lambda)) = A(\lambda).$$

Combining these two equations, we obtain

$$S(\lambda)A(\lambda) = S(\lambda)T(\lambda)P(\lambda) = P(\lambda).$$

But then

$$A(\lambda)B(\lambda)A(\lambda) = A(\lambda)P(\lambda)S(\lambda)A(\lambda) = A(\lambda).$$

Similarly, using the fact that $P(\lambda)^2 = P(\lambda)$, we derive

$$B(\lambda)A(\lambda)B(\lambda) = P(\lambda)S(\lambda)A(\lambda)P(\lambda)S(\lambda) =$$

$$= P(\lambda)S(\lambda) = B(\lambda).$$

Thus $B(\lambda)$ is a relative inverse of $A(\lambda)$. This completes part (II) of the proof.

(III) Next we suppose that the index k is strictly positive. Let $Y_1$ be the complex Banach space given by

$$Y_1 = Y \times \mathbb{C}^k,$$

and let $A_1$ be the function with values in $L(X,Y_1)$ defined on $G \setminus \Sigma$ by

$$A_1(\lambda)x = (A(\lambda)x,0) \qquad (x \in X).$$

Then $A_1$ is holomorphic on $G \setminus \Sigma$ and Riesz-meromorphic on G with index 0. Further, $\lambda_0 \in H[A_1;G]$ and $\Gamma$ is the complement in G of the set

$$\{\lambda \in G \setminus \Sigma : X = N(A_1(\lambda)) \oplus N, Y_1 = R(A_1(\lambda)) \oplus (R \times \mathbb{C}^k)\}.$$

The results obtained in parts (I) and (II) of the present proof show that $\Gamma$ is a discrete subset of G and that there exists a holomorphic function $B_1$

defined on $G \setminus \Gamma$ with values in $L(Y_1,X)$ such that

(j) $B_1$ is Riesz-meromorphic on G with index 0;

(jj) for each $\lambda$ in $G \setminus \Gamma$, we have $N(B_1(\lambda)) = R \times \mathbb{C}^k$ and $R(B_1(\lambda)) = N$;

(jjj) for each $\lambda$ in $G \setminus \Gamma$, the operator $B_1(\lambda)$ is a relative inverse of $A_1(\lambda)$.

Let $\sigma$ be the canonical imbedding of Y in $Y_1$. Thus $\sigma(y) = (y,0)$ for each y in Y. Define the function B on $G \setminus \Gamma$ by

$$B(\lambda) = B_1(\lambda)\sigma.$$

Then B is holomorphic on $G \setminus \Gamma$ and B is Riesz-meromorphic on G with index -k (cf. Proposition 3.5). Thus B satisfies condition (i) of the theorem. A straightforward verification shows that B meets the other requirements too.

The case $k < 0$ can be treated similarly, the difference being that $A_1$ is defined to be a function with values in $L(X_1,Y)$, where $X_1$ is the complex Banach space given by

$$X_1 = X \times \mathbb{C}^{-k}.$$

This completes the proof of Theorem 4.1.

From Lemma 1.2 it follows that the function B appearing in Theorem 4.1 is unique. Further, Lemma 1.1 shows that for each $\lambda$ in $G \setminus \Gamma$ the operator $B(\lambda)A(\lambda)$ is the projection of X onto N along $N(A(\lambda))$, and that the operator $A(\lambda)B(\lambda)$ is the projection of Y onto $R(A(\lambda))$ along R.

A first step in the direction of Theorem 4.1 may be found in F.V. Atkinson's paper [3] (see also §4 in [15]). The special case when X = Y and

$$A(\lambda) = \lambda I_X - T \qquad (\lambda \in \mathbb{C}) \tag{4}$$

for some T in $L(X)$ has been studied by P. Saphar (see Lemma 3 and Theorem 1 in Chapter I of [39]).

Under the conditions of Theorem 4.1, the function A is holomorphic and Fredholm operator valued on $G \setminus \Sigma$. It has been proved by several authors that this implies that the set $\Delta[A;G]$, defined by

$$\Delta[A;G] = (G \setminus \Sigma) \setminus H[A;G],$$

is a discrete subset of $G \setminus \Sigma$ (see [2], [3], [11], [15], [30] and [41]; cf. also Theorem III.3.8). A slightly stronger result may be obtained from Theorem 4.1.

4.2. THEOREM. *Suppose that* A *is Riesz-meromorphic on* G *with finite index. Then* $\Delta[A;G]$ *is a discrete subset of* G.

PROOF. Let $\Gamma$ be as in the preceding theorem. Then

$$G \setminus \Gamma \subset H[A;G] \subset G \setminus \Delta[A;G]. \tag{5}$$

Hence $\Delta[A;G]$ is contained in the discrete subset $\Gamma$ of G. This implies the desired result.

Next we present an example showing that the first inclusion in (5) may be strict.

4.3. EXAMPLE. Let $X = Y$ be the two-dimensional space $\mathbb{C}^2$, let $G = \mathbb{C}$, and let

$$A(\lambda)(\alpha,\beta) = (\lambda^2\alpha + \lambda\beta + \alpha,\ \lambda^2\alpha + \lambda\beta + \alpha) \qquad (\alpha,\beta,\lambda \in \mathbb{C}).$$

Then A is holomorphic on $\mathbb{C}$ and the values of A are Fredholm operators with index 0. Thus A is Riesz-meromorphic on $\mathbb{C}$ with index 0. It is easily verified that

$$n(A(\lambda)) = d(A(\lambda)) = 1 \qquad (\lambda \in \mathbb{C}),$$

and hence $H[A;\mathbb{C}] = \mathbb{C}$. Let N and $\Gamma$ be as in Theorem 4.1. Then $N \neq \{0\}$ and

$$N(A(\lambda)) \cap N = \{0\} \qquad (\lambda \in \mathbb{C} \setminus \Gamma).$$

This implies that

$$\cup\ \{N(A(\lambda)) : \lambda \in \mathbb{C} \setminus \Gamma\} \neq \mathbb{C}^2.$$

Since for all complex numbers $\alpha$ and $\beta$ the quadratic equation

$$\alpha Z^2 + \beta Z + \alpha = 0 \tag{6}$$

has a solution in $\mathbb{C}$, we have

$$\cup \{N(A(\lambda)) : \lambda \in \mathbb{C}\} = \mathbb{C}^2.$$

Hence $\mathbb{C} \setminus \Gamma \neq \mathbb{C} = H[A;\mathbb{C}]$.

We conclude this section with a few remarks. Suppose that A is holomorphic and Fredholm operator valued on G. Further, suppose that the functions

$$\lambda \longmapsto n(A(\lambda)) \qquad (\lambda \in G)$$

and

$$\lambda \longmapsto d(A(\lambda)) \qquad (\lambda \in G)$$

are constant on G. In other words, suppose that H = G. In view of Theorem 4.1 the question arises whether there exists a holomorphic function T defined on G with values in $L(Y,X)$ such that for each $\lambda$ in G the operator $T(\lambda)$ is a relative inverse of $A(\lambda)$. Using the result of K.-H. Förster and G. Garske mentioned at the end of Section I.1 (i.e., Theorem 12 in [12]), one can show that the answer is affirmative (see [5] for details).

A second question is whether for each compact subset K of G there exists a (closed) subspace N of X such that

$$X = N(A(\lambda)) \oplus N \qquad (\lambda \in K). \tag{7}$$

P. Saphar has proved that the answer is positive if A is as in formula (4) (see Proposition 2 in Chapter I of [39]). In general, however, the answer is negative. To see this, let A be as in Example 4.3, and let K be the closed unit disc in $\mathbb{C}$. Since for all complex numbers $\alpha$ and $\beta$ the quadratic equation (6) has at least one solution in K, we have

$$\cup \{N(A(\lambda)) : \lambda \in K\} = \mathbb{C}^2.$$

This implies that, if N is a subspace of $X = \mathbb{C}^2$ such that (7) is satisfied, then $N = \{0\}$. But this contradicts the fact that $n(A(\lambda)) = 1$ for each $\lambda$ in the complex plane.

Finally, one may ask whether for each compact subset K of G, there exists a (closed) subspace R of Y such that

$$Y = R(A(\lambda)) \oplus R \qquad (\lambda \in K).$$

Again the answer is positive if A is of the form (4) (see Proposition 1 in Chapter I of [39]), and negative in the general case. It is not difficult to construct a counterexample.

## 5. RIESZ-MEROMORPHIC FUNCTIONS WITH INFINITE INDEX

The aim of this section is to prove an analogue of Theorem 4.1 for Riesz-meromorphic functions with infinite index. Here we restrict ourselves to the special case when X = Y. However, our results also hold for $X \neq Y$ (see the remark at the end of this section).

Throughout this section X will be a complex Banach space and G will be a region. The identity operator on X is denoted by I. Further, A will be a function with values in $L(X)$, and we shall suppose that A is meromorphic on G. The set of singular points of A in G will be denoted by $\Sigma$.

To get the desired analogue of Theorem 4.1, we need an extension of Theorem 3.4 to left [right] invertible operators. Such an extension may be obtained from the work of B. Gramsch (see Theorems 1, 11 and 12 in [14]). It is given in the next proposition.

5.1. PROPOSITION. *Suppose that* A *is Riesz-meromorphic on* G *with index* $-\infty[+\infty]$. *Let* $\lambda_0$ *be a regular point of* A *in* G, *and suppose that* $A(\lambda_0)$ *is injective [surjective]. Then there exists a discrete subset* $\Gamma$ *of* G *and a holomorphic function* S *defined on* $G \setminus \Gamma$ *with values in* $L(X)$ *such that*

(i) $\Sigma \subset \Gamma$ *and* $\lambda_0 \in G \setminus \Gamma$;

(ii) S *is Riesz-meromorphic on* G *with index* $+\infty[-\infty]$;

(iii) *for each* $\lambda$ *in* $G \setminus \Gamma$, *we have* $S(\lambda)A(\lambda) = I$ $[A(\lambda)S(\lambda) = I]$.

PROOF. We consider only the case when A has index $-\infty$; the other case can be treated similarly.

Let $\kappa$ denote the canonical mapping from $L(X)$ onto $L(X)/K(X)$, and define the function $\alpha$ on $G \setminus \Sigma$ by

$$\alpha(\lambda) = \kappa(A(\lambda)).$$

Then $\alpha$ is holomorphic on $G \setminus \Sigma$ and meromorphic on G. The function A is Riesz-meromorphic on G, and hence for each $\lambda$ in $\Sigma$ the coefficients of the principal part of the Laurent expansion of A at $\lambda$ are degenerate. This implies that the points of $\Sigma$ are removable singularities of the function $\alpha$. Let $\bar{\alpha}$ denote the holomorphic extension of $\alpha$ to the whole of G. Then the values of $\bar{\alpha}$ are left invertible, but not invertible, in $L(X)/K(X)$. This follows from Corollary 2.3 and the fact that A is Riesz-meromorphic on G with index $-\infty$.

According to a result of G.R. Allan (Corollary of Theorem 1 in [1]), there exists a holomorphic function $\tau$ defined on G with values in $L(X)/K(X)$ such that for each $\lambda$ in G

$$\tau(\lambda)\bar{\alpha}(\lambda) = \kappa(I).$$

In [14] B. Gramsch has shown that such a function $\tau$ may be lifted to a holomorphic function with values in $L(X)$. More precisely, there exists a holomorphic function $T : G \longrightarrow L(X)$ such that

$$\kappa(T(\lambda)) = \tau(\lambda) \qquad (\lambda \in G).$$

Further, since $A(\lambda_0)$ is injective, the function T can be chosen in such a way that $T(\lambda_0)A(\lambda_0)$ is bijective (see in [14] the proof of Theorem 1 and Remark 3). With this function T we shall construct the function S.

Firstly, consider the function C defined on $G \setminus \Sigma$ by

$$C(\lambda) = T(\lambda)A(\lambda) - I.$$

Since T is holomorphic and A is meromorphic on G, the function C is meromorphic on G. For each $\lambda$ in $\Sigma$ the coefficients of the principal part of the Laurent expansion of C at $\lambda$ are degenerate, because A has this property. Further, for each $\lambda$ in $G \setminus \Sigma$, we have

$$\kappa(C(\lambda)) = \kappa(T(\lambda)A(\lambda) - I) =$$

$$= \tau(\lambda)\alpha(\lambda) - \kappa(I) = 0.$$

This implies that $C(\lambda)$ is a compact operator for each $\lambda$ in $G \setminus \Sigma$. By combining these results, we obtain that C is essentially meromorphic on G.

Secondly, we consider the function B defined on $G \setminus \Sigma$ by

$$B(\lambda) = T(\lambda)A(\lambda) = C(\lambda) + I.$$

From Theorem 3.2 it follows that B is Riesz-meromorphic on G with index 0. In addition, T is chosen in such a way that $B(\lambda_0)$ is bijective. So we can apply Theorem 3.4 to show that the resolvent $B^{-1}$ of B is a Riesz-meromorphic function on G with index 0. In particular, it follows that $G \setminus \mathrm{Res}[B]$ is a discrete subset of G. Put

$$\Gamma = G \setminus \mathrm{Res}[B].$$

Observe that $\Sigma \subset \Gamma$ and $\lambda_0 \in G \setminus \Gamma$.

Next we define S on $G \setminus \Gamma$ by

$$S(\lambda) = B^{-1}(\lambda)T(\lambda).$$

Then S is holomorphic on $G \setminus \Gamma$, and for each $\lambda$ in $G \setminus \Gamma$ we have

$$S(\lambda)A(\lambda) = B^{-1}(\lambda)T(\lambda)A(\lambda) = B^{-1}(\lambda)B(\lambda) = I.$$

It remains to show that S satisfies condition (ii). In order to prove (ii), it suffices to show that the values of T are projective semi-Fredholm operators with index $+\infty$. Because then we know that T is a Riesz-meromorphic function on G with index $+\infty$, and we can use Proposition 3.5 and the fact that $B^{-1}$ is Riesz-meromorphic on G with index 0 to get (ii).

Take $\lambda$ in G. Observe that $\kappa(T(\lambda)) = \tau(\lambda)$ is a left inverse of $\bar{\alpha}(\lambda)$ in $L(X)/K(X)$. Since $\bar{\alpha}(\lambda)$ is not invertible in this algebra, it follows that $\kappa(T(\lambda))$ cannot have a left inverse in $L(X)/K(X)$. So $\kappa(T(\lambda))$ has a right inverse, namely $\bar{\alpha}(\lambda)$, but is not left invertible. But then it follows from Corollary 2.3(iii) that $T(\lambda)$ is a projective semi-Fredholm operator with index $+\infty$. This completes the proof.

5.2. THEOREM. *Suppose that* A *is Riesz-meromorphic on* G *with index* $-\infty$. *Let* $\lambda_0 \in H[A;G]$, *and let* N *be a topological complement of* $N(A(\lambda_0))$. *Then the set* $\Gamma_n$, *given by*

$$\Gamma_n = G \setminus \{\lambda \in G \setminus \Sigma : X = N(A(\lambda)) \oplus N\},$$

*is a discrete subset of* G. *Moreover, there exist a discrete subset* $\Gamma$ *of* G *and a holomorphic function* B *defined on* $G \setminus \Gamma$ *with values in* $L(X)$ *such that*

(i) $\lambda_0 \in G \setminus \Gamma$ *and* $\Sigma \subset \Gamma_n \subset \Gamma$;

(ii) B *is Riesz-meromorphic on* G *with index* $+\infty$;

(iii) *for each* $\lambda$ *in* $G \setminus \Gamma$, *the operator* $B(\lambda)$ *is a relative inverse of* $A(\lambda)$ *and* $R(B(\lambda)) = N$.

PROOF. The arguments are similar to those used in parts (I) and (II) of the proof of Theorem 4.1. Therefore we omit most of the details.

From the definition of $\Gamma_n$ it is clear that $\Sigma \subset \Gamma_n$. Let $m = m_n$. If $m = 0$, then $A(\lambda_0)$ is injective, and one can apply the preceding proposition to show that in this case the theorem holds. Therefore we may assume that m is strictly positive.

Let $x_1,\dots,x_m$ and $f_1,\dots,f_m$ be defined as in part (II) of the proof of Theorem 4.1. Since $d(A(\lambda_0)) = +\infty$, there exist $y_1,\dots,y_m$ in X being linearly independent modulo $R(A(\lambda_0))$. Define the function T on $G \setminus \Sigma$ by

$$T(\lambda) = A(\lambda) + \sum_{i=1}^{m} f_i \otimes y_i.$$

Then T is holomorphic on $G \setminus \Sigma$ and Riesz-meromorphic on G with index $-\infty$. Since $T(\lambda_0)$ is injective, Proposition 5.1 shows the existence of a discrete subset $\Gamma$ of G and a holomorphic function S defined on $G \setminus \Gamma$ with values in $L(X)$ such that

(j) $\Sigma \subset \Gamma$ and $\lambda_0 \in G \setminus \Gamma$;

(jj) S is Riesz-meromorphic on G with index $+\infty$;

(jjj) for each $\lambda$ in $G \setminus \Gamma$, we have $S(\lambda)T(\lambda) = I$.

The next step is to show that for each $\lambda$ in $G \setminus \Gamma$

$$\langle f_i, S(\lambda)y_j\rangle = \delta_{ij} \qquad (i,j = 1,\dots,m).$$

The proof of this is similar to that of the corresponding formula in part (II) of the proof of Theorem 4.1.

Define the function P on $G \setminus \Gamma$ by

$$P(\lambda) = I - \sum_{i=1}^{m} f_i \otimes S(\lambda)y_i.$$

Then for each $\lambda$ in $G \setminus \Gamma$ the operator $P(\lambda)$ is the projection of $X$ onto $N$ along $N(A(\lambda))$, and hence

$$X = N(A(\lambda)) \oplus N.$$

This implies that $\Gamma_n \subset \Gamma$. Thus $\Gamma_n$ is a discrete subset of $G$.

Finally, let $B$ be defined on $G \setminus \Gamma$ by

$$B(\lambda) = P(\lambda)S(\lambda).$$

Repeating the arguments used in part (II) of the proof of Theorem 4.1, one can show that $B$ meets the requirements of the theorem. This completes the proof.

Let $\Gamma$ and $B$ be as in the preceding theorem. Then for each $\lambda$ in $G \setminus \Gamma$ the operator $B(\lambda)A(\lambda)$ is the projection of $X$ onto $N$ along $N(A(\lambda))$, and $A(\lambda)B(\lambda)$ is a projection of $X$ onto $R(A(\lambda))$. This follows from Lemma 1.1.

5.3. THEOREM. *Suppose that* $A$ *is Riesz-meromorphic on* $G$ *with index* $+\infty$. *Let* $\lambda_0 \in H[A;G]$, *and let* $R$ *be an algebraic complement of* $R(A(\lambda_0))$. *Then the set* $\Gamma_d$, *given by*

$$\Gamma_d = G \setminus \{\lambda \in G \setminus \Sigma : X = R(A(\lambda)) \oplus R\},$$

*is a discrete subset of* $G$. *Moreover, there exist a discrete subset* $\Gamma$ *of* $G$ *and a holomorphic function* $B$ *defined on* $G \setminus \Gamma$ *with values in* $L(X)$ *such that*

(i) $\lambda_0 \in G \setminus \Gamma$ *and* $\Sigma \subset \Gamma_d \subset \Gamma$;

(ii) $B$ *is Riesz-meromorphic on* $G$ *with index* $-\infty$;

(iii) *for each* $\lambda$ *in* $G \setminus \Gamma$, *the operator* $B(\lambda)$ *is a relative inverse of* $A(\lambda)$ *and* $N(B(\lambda)) = R$.

PROOF. From the definition of $\Gamma_d$ it is clear that $\Sigma \subset \Gamma_d$. Let $m = m_d$. If $m = 0$, then $A(\lambda_0)$ is surjective, and one can apply Proposition 5.1 to show that in this case the theorem holds. Therefore we may assume that $m$ is strictly positive.

Let $y_1,\dots,y_m$ form a basis of R, and let $x_1,\dots,x_m$ in $N(A(\lambda_0))$ be linearly independent. Here we use that $n(A(\lambda_0)) = +\infty$ and that $d(A(\lambda_0)) = m$. Choose $f_1,\dots,f_m$ in $X^*$ such that

$$<f_i,x_j> = \delta_{ij} \qquad\qquad (i,j = 1,\dots,m),$$

and define the function T on $G \setminus \Sigma$ by

$$T(\lambda) = A(\lambda) + \sum_{i=1}^{m} f_i \otimes y_i.$$

Then T is holomorphic on $G \setminus \Sigma$ and Riesz-meromorphic on G with index $+\infty$. It is not difficult to prove that $T(\lambda_0)$ is surjective. Hence, by Proposition 5.1, there exist a discrete subset $\Gamma$ of G and a holomorphic function S defined on $G \setminus \Gamma$ with values in $L(X)$ such that

(j) $\Sigma \subset \Gamma$ and $\lambda_0 \in G \setminus \Gamma$;

(jj) S is Riesz-meromorphic on G with index $-\infty$;

(jjj) for each $\lambda$ in $G \setminus \Gamma$, we have $T(\lambda)S(\lambda) = I$.

Take $\lambda$ in $G \setminus \Gamma$. It is clear from (jjj) that $T(\lambda)$ is surjective. Hence we have

$$X = R(A(\lambda)) + R.$$

In addition, $d(A(\lambda)) \geq m = \dim R$. Both facts together imply that

$$X = R(A(\lambda)) \oplus R.$$

This shows that $\lambda \in G \setminus \Gamma_d$. Thus $\Gamma_d \subset \Gamma$, and it follows that $\Gamma_d$ is a discrete subset of G.

Next we shall show that for each $\lambda$ in $G \setminus \Gamma$

$$<f_i,S(\lambda)y_j> = \delta_{ij} \qquad\qquad (i,j = 1,\dots,m). \quad (1)$$

Take $\lambda$ in $G \setminus \Gamma$. For $j = 1,\dots,m$, we have

$$A(\lambda)S(\lambda)y_j = T(\lambda)S(\lambda)y_j - \sum_{i=1}^{m} <f_i,S(\lambda)y_j>\, y_i =$$

$$= y_j - \sum_{i=1}^{m} <f_i,S(\lambda)y_j>\, y_i.$$

This shows that for $j = 1,\dots,m$

$$y_j - \sum_{i=1}^{m} \langle f_i, S(\lambda)y_j\rangle\, y_i \in R(A(\lambda)) \cap R.$$

But $R(A(\lambda)) \cap R = \{0\}$, and hence

$$y_j = \sum_{i=1}^{m} \langle f_i, S(\lambda)y_j\rangle\, y_i \qquad (j = 1,\dots,m).$$

Since $y_1,\dots,y_m$ are linearly independent, the last formula imlies (1). Observe that (1) may be written as

$$\langle S^*(\lambda)f_i, y_j\rangle = \delta_{ij} \qquad (i,j = 1,\dots,m).$$

Consider the function Q defined on $G \setminus \Gamma$ by

$$Q(\lambda) = \sum_{i=1}^{m} S^*(\lambda)f_i \otimes y_i.$$

The result of the preceding paragraph shows that $Q(\lambda)$ is a projection of X onto R. We shall prove that the null space of $Q(\lambda)$ is $R(A(\lambda))$.

Take $\lambda$ in $G \setminus \Gamma$. If $Q(\lambda)y = 0$, then

$$\langle S^*(\lambda)f_i, y\rangle = \langle f_i, S(\lambda)y\rangle = 0 \qquad (i = 1,\dots,m),$$

hence

$$A(\lambda)S(\lambda)y = T(\lambda)S(\lambda)y - \sum_{i=1}^{m} \langle f_i, S(\lambda)y\rangle\, y_i = y,$$

and thus $y \in R(A(\lambda))$. Conversely, let $y \in R(A(\lambda))$. Since $Q(\lambda)$ is a projection, we have $y = z + w$ with z in $R(Q(\lambda)) = R$ and w in $N(Q(\lambda))$. But we have just proved that $N(Q(\lambda)) \subset R(A(\lambda))$. Therefore $z = y - w \in R(A(\lambda)) \cap R$. This implies that $z = 0$, and it follows that $y = w \in N(Q(\lambda))$. So we have proved that $N(Q(\lambda)) = R(A(\lambda))$.

Define the function B on $G \setminus \Gamma$ by setting

$$B(\lambda) = S(\lambda)(I - Q(\lambda)).$$

Then B is holomorphic on $G \setminus \Gamma$, and B is Riesz-meromorphic on G with index 0 (use Corollary 3.3 and Proposition 3.5, respectively). It remains to show that statement (iii) of the theorem holds.

Take $\lambda$ in $G \setminus \Gamma$. Since $S(\lambda)$ is injective, we have

$$N(B(\lambda)) = N(I - Q(\lambda)).$$

But $Q(\lambda)$ is a projection. Therefore $N(I - Q(\lambda)) = R(Q(\lambda)) = R$. This shows that $N(B(\lambda)) = R$. Observe that for each y in X

$$\begin{aligned} A(\lambda)S(\lambda)y &= T(\lambda)S(\lambda)y - \sum_{i=1}^{m} \langle f_i, S(\lambda)y\rangle\, y_i = \\ &= y - \sum_{i=1}^{m} \langle S^*(\lambda)f_i, y\rangle\, y_i = \\ &= (I - Q(\lambda))y. \end{aligned}$$

Hence $A(\lambda)S(\lambda) = I - Q(\lambda)$. Furthermore

$$(I - Q(\lambda))A(\lambda) = A(\lambda),$$

because $R(A(\lambda)) = N(Q(\lambda))$. Finally, since $Q(\lambda)$ is a projection, we have

$$B(\lambda)(I - Q(\lambda)) = S(\lambda)(I - Q(\lambda))^2 = S(\lambda)(I - Q(\lambda)) = B(\lambda).$$

From these equalities, it follows that

$$\begin{aligned} A(\lambda)B(\lambda)A(\lambda) &= A(\lambda)S(\lambda)(I - Q(\lambda))A(\lambda) = \\ &= (I - Q(\lambda))A(\lambda) = A(\lambda) \end{aligned}$$

and

$$\begin{aligned} B(\lambda)A(\lambda)B(\lambda) &= B(\lambda)A(\lambda)S(\lambda)(I - Q(\lambda)) = \\ &= B(\lambda)(I - Q(\lambda))^2 = B(\lambda). \end{aligned}$$

Thus $B(\lambda)$ is a relative inverse of $A(\lambda)$, and the proof is complete.

Let $\Gamma$ and B be as in the preceding theorem. One can use Lemma 1.1 to show that for each $\lambda$ in $G \setminus \Gamma$ the operator $B(\lambda)A(\lambda)$ is a projection of X along $N(A(\lambda))$, and that $A(\lambda)B(\lambda)$ is the projection of $X$ onto $R(A(\lambda))$ along $R$.

We can use the preceding results to prove an analogue of Theorem 4.2 for Riesz-meromorphic functions with arbitrary index (cf. [3], [11] and [30]).

5.4. THEOREM. *Suppose that* $A$ *is Riesz-meromorphic on* $G$. *Then* $\Delta[A;G]$ *is a discrete subset of* $G$.

PROOF. In view of Theorem 4.2 we may assume that the index of $A$ is infinite. We consider only the case when $A$ has index $+\infty$; the other case can be treated similarly.

Suppose that $A$ has index $+\infty$, and let $\Gamma_d$ be as in the preceding theorem. Then

$$G \setminus \Gamma_d \subset H[A;G] \subset G \setminus \Delta[A;G].$$

Hence $\Delta[A;G]$ is contained in the discrete subset $\Gamma_d$ of $G$. This implies the desired result.

In order to extend the preceding results to the case when $X \neq Y$, we only have to prove that Proposition 5.1 holds for Riesz-meromorphic functions with values in $L(X,Y)$, where $Y$ is a complex Banach space possibly different from $X$. The proof of this is based on the extension of Allan's theorem due to Förster and Garske (see Theorem 12 in [12]) and on the fact that the lifting result of Gramsch [14] is valid for holomorphic functions with values in any quotient space of a complex Banach space modulo a closed subspace.

# CHAPTER III

## POLES OF THE RESOLVENT

Let T be a bounded linear operator on a complex Banach space X. The complex number $\lambda_0$ is said to be a pole of T of order m if $\lambda_0$ is a pole of order m of the locally holomorphic function

$$\lambda \longmapsto (\lambda I_X - T)^{-1}.$$

Here the symbol $I_X$ denotes the identity operator on X. It is well-known that the poles of T can be characterized in terms of the ascent $\alpha(T)$ and descent $\delta(T)$ of T. In fact the following theorem holds: 0 is a pole of T of order m if and only if $\alpha(T) = \delta(T) = m$. One can view this result as a characterization of the poles of the resolvent of a particular kind of locally holomorphic function, namely the function

$$A(\lambda) = \lambda I_X - T \qquad (\lambda \in \mathbb{C}).$$

Then the question arises whether a similar result holds for an arbitrary locally holomorphic operator valued function A defined on an open neighbourhood of 0. In this chapter we show that this indeed is the case.

In order to get such a characterization for an arbitrary function A, we have to define a generalized ascent and descent. This is done in the first section of this chapter. The proof of the theorem for operators cited above is based on a certain decomposition of the underlying space. In Section 2 we prove that similar decompositions exist if A is of the form

$$A(\lambda) = T + \lambda S, \tag{1}$$

whereas in general this is not true. In Section 4 we prove that for our purposes it suffices to consider the case when A is of the form (1). Moreover we show that without loss of generality we may suppose that the operator S in (1) is injective and maps closed sets onto closed sets. This enables us to prove the desired characterization theorem (Theorem 5.2).

In Section 3 we introduce a stability number. This number plays an important role in the perturbation theory of Fredholm operator valued holomorphic functions. In Section 5 we employ this number to give some information about Fredholm operator valued holomorphic functions with infinite ascent or descent at 0.

In Section 6 we prove that in the commutative case there exist interesting relationships between the ascent and descent of a holomorphic function A at 0 and the ascent and descent of the zero coefficient $A_0$ of the Taylor expansion of A at 0. In Section 7 we use the results of Section 6 to obtain some more information about the relationships between the spectral properties of the operator $A_0$ and those of the function A.

## 1. GENERALIZED ASCENT AND DESCENT

Let U be a linear operator on a (complex) linear space E. Recall (see Section 5.41 in [42]) that the *ascent* of U is the extended integer $\alpha(U)$ given by

$$\alpha(U) = \min \{m : N(U^m) = N(U^{m+1})\}.$$

Similarly, the *descent* of U is the extended integer $\delta(U)$ defined by

$$\delta(U) = \min \{m : R(U^m) = R(U^{m+1})\}.$$

Here, as in the sequel, $\min \emptyset = +\infty$. From Lemmas 3.1 and 3.2 in [21] it follows that

$$\alpha(U) = \min \{m : N(U) \cap R(U^m) = \{0\}\} \tag{1}$$

and

$$\delta(U) = \min \{m : R(U) + N(U^m) = E\}. \tag{2}$$

The notions of ascent and descent play a role in spectral theory. In particular they can be used to characterize poles of bounded (and unbounded closed) linear operators (cf. [26]; see also Theorem 5.1 in Section 5 for more details). One of the main purposes of this section is to generalize these notions in such a way that they become applicable to locally holomorphic operator valued functions.

In the following E and F are (complex) linear spaces, and $T, T_1, T_2, \ldots$ are linear operators from E into F. The sequence $\{T_n\}_{n=1}^{\infty}$ will be denoted by $\mathcal{T}$, and we shall write $T_0$ instead of T whenever this is convenient. For $m = 0,1,2,\ldots$, let $H_m[T;\mathcal{T}]$ be the set of all x in E with the property that there exist $x_0,\ldots,x_m$ in E such that $x_0 = x$ and

$$\sum_{i=0}^{n} T_i x_{n-i} = 0 \qquad (n = 0,\ldots,m).$$

Further, let $H'_m[T;\mathcal{T}]$ be the set of all y in F with the property that there are $x_0,\ldots,x_m$ in E such that

$$\sum_{i=0}^{n} T_i x_{n-i} = \delta_{nm} y \qquad (n = 0,\ldots,m).$$

Recall that $\delta_{nm}$ denotes the Kronecker delta. It is easily verified that $\{H_m[T;\mathcal{T}]\}_{m=0}^{\infty}$ is a decreasing sequence of subspaces of E and that $\{H'_m[T;\mathcal{T}]\}_{m=0}^{\infty}$ is an increasing sequence of subspaces of F. Moreover,

$$H_0[T;\mathcal{T}] = N(T), \quad H'_0[T;\mathcal{T}] = R(T).$$

The extended integer $\alpha[T;\mathcal{T}]$, defined by

$$\alpha[T;\mathcal{T}] = \min \{m : H_m[T;\mathcal{T}] = \{0\}\},$$

will be called the *ascent of* T *relative to the sequence* $\mathcal{T}$; the extended integer $\delta[T;\mathcal{T}]$, given by

$$\delta[T;\mathcal{T}] = \min \{m : H'_m[T;\mathcal{T}] = F\},$$

will be called the *descent of* T *relative to the sequence* $\mathcal{T}$.

By way of illustration, we consider the particular case when E = F and

$$T_1 = I_E, \quad T_n = 0 \qquad (n = 2,3,\ldots),$$

where $I_E$ denotes the identity operator on E. Then we have for $m = 0,1,2,\ldots$

$$H_m[T;\mathcal{T}] = N(T) \cap R(T^m), \quad H'_m[T;\mathcal{T}] = R(T) + N(T^m),$$

and hence, using formulas (1) and (2), it follows that

$$\alpha[T;\mathcal{T}] = \alpha(T), \quad \delta[T;\mathcal{T}] = \delta(T).$$

This justifies our terminology.

Whenever this is convenient, we shall omit $[T;\mathcal{T}]$ in the symbols defined above. The following lemma will prove to be very useful. It may be viewed as an analogue of Lemma 3.5 in [21] and formula (2) in [33].

1.1. LEMMA. *Let* m *and* k *be non-negative integers. Then*

$$\dim H'_{m+k}/H'_m = \dim H_m/H_{m+k}.$$

PROOF. The statement is trivially true for $k = 0$. Therefore we assume that k is strictly positive. Observe that

$$\dim H'_{m+k}/H'_m = \sum_{i=0}^{k-1} \dim H'_{m+i+1}/H'_{m+i}$$

and

$$\dim H_m/H_{m+k} = \sum_{i=0}^{k-1} \dim H_{m+i}/H_{m+i+1}.$$

Thus it suffices to show that

$$\dim H'_{m+1}/H'_m = \dim H_m/H_{m+1}.$$

We shall prove that the quotient spaces appearing in this formula are linearly isomorphic.

For each x in $H_m$, let $[x]$ denote the element of $H_m/H_{m+1}$ containing x. Take y in $H'_{m+1}$. Then there exist $x_0,\dots,x_{m+1}$ in E satisfying the equations

$$\sum_{i=0}^{n} T_i X_{n-i} = \delta_{n,m+1}y \qquad (n = 0,\dots,m+1). \tag{3}$$

Observe that this implies that $x_0 \in H_m$. If $\bar{x}_0,\dots,\bar{x}_{m+1}$ in E form another solution of (3), then $x_0 - \bar{x}_0 \in H_{m+1}$, and so $[x_0] = [\bar{x}_0]$. Hence the function $\Phi$ from $H'_{m+1}$ into the quotient space $H_m/H_{m+1}$, given by

$$\Phi(y) = [x_0],$$

is well-defined. A straightforward argument shows that $\Phi$ is linear. We shall prove that $\Phi$ is surjective and that the null space of $\Phi$ is equal to $H'_m$. This will imply that $H'_{m+1}/H'_m$ and $H_m/H_{m+1}$ are linearly isomorphic.

Take $[x]$ in $H_m/H_{m+1}$. Since $x \in H_m$, there exist $x_0,\dots,x_m$ in E such that $x_0 = x$ and

$$\sum_{i=0}^{n} T_i x_{n-i} = 0 \qquad (n = 0,\dots,m).$$

Put $x_{m+1} = 0$ and define

$$y = \sum_{i=0}^{m+1} T_i x_{m+1-i}. \tag{4}$$

Then $x_0,\dots,x_{m+1}$ form a solution of the equations (3) with y as in (4). Hence $y \in H'_{m+1}$ and $\Phi(y) = [x_0] = [x]$. Thus $\Phi$ maps $H'_{m+1}$ onto the whole of $H_m/H_{m+1}$.

Next we prove that the null space of $\Phi$ is equal to $H'_m$. Take y in $H'_m$. Then there are $u_0,\dots,u_m$ in E such that

$$\sum_{i=0}^{n} T_i u_{n-i} = \delta_{nm} y \qquad (n = 0,\dots,m).$$

Put

$$x_0 = 0, \quad x_k = u_{k-1} \qquad (k = 1,\dots,m+1).$$

Then $x_0,\dots,x_{m+1}$ satisfy the equations (3), and hence $\Phi(y) = [x_0] = [0]$. Thus $H'_m$ is a subset of the null space of $\Phi$. Conversely, take y in the null space of $\Phi$. Since $y \in H'_{m+1}$, there exist $x_0,\dots,x_{m+1}$ in E satisfying (3). We know that $[x_0] = \Phi(y) = [0]$. Hence $x_0 \in H_{m+1}$. Choose $v_0,\dots,v_{m+1}$ in E such that $v_0 = x_0$ and

$$\sum_{i=0}^{n} T_i v_{n-i} = 0 \qquad (n = 0,\dots,m+1).$$

Then it is clear that

$$\sum_{i=0}^{n} T_i (x_{n-i} - v_{n-i}) = \delta_{n,m+1} y \qquad (n = 0,\dots,m+1).$$

Since $v_0 = x_0$, it follows that

$$\sum_{i=0}^{n-1} T_i(x_{n-i} - v_{n-i}) = \delta_{n,m+1}y \qquad (n = 1,\dots,m+1).$$

But this implies that $y \in H'_m$. Thus the null space of $\Phi$ is a subset of $H'_m$, and the proof is complete.

Let U be a linear operator on E. Under certain conditions on U the numbers $\alpha(U)$ and $\delta(U)$ are equal. For instance this is the case if $\alpha(U)$ and $\delta(U)$ are both finite (see Theorem 3.6 in [43] and Theorem 5.41-E in [42]). Another condition is that nullity $n(U)$ and defect $d(U)$ of U are both finite and equal (cf. statements (c) and (d) of Theorem 4.5 in [43]). The next two theorems extend these results.

1.2. THEOREM. *Suppose that* $\alpha[T;\mathcal{T}]$ *and* $\delta[T;\mathcal{T}]$ *are both finite. Then* $\alpha[T;\mathcal{T}] = \delta[T;\mathcal{T}]$.

PROOF. Put $k = \alpha[T;\mathcal{T}]$ and $m = \delta[T;\mathcal{T}]$. Then

$$H_k = H_{k+m} = \{0\}, \quad H'_m = H'_{m+k} = F.$$

Applying the preceding lemma, we obtain

$$\dim H_m/H_{m+k} = \dim H'_{m+k}/H'_m = \dim F/F = 0,$$

and so $H_m = H_{m+k} = \{0\}$. Hence $k \leq m$. The reverse inequality can be established in the same way. This proves the theorem.

1.3. THEOREM. *Let* $n(T) = d(T) < +\infty$. *Then* $\alpha[T;\mathcal{T}] = \delta[T;\mathcal{T}]$.

PROOF. Firstly, we show that $\alpha[T;\mathcal{T}] \leq \delta[T;\mathcal{T}]$. Without loss of generality we may suppose that $\delta[T;\mathcal{T}]$ is finite. Put $m = \delta[T;\mathcal{T}]$. Then

$$\dim H'_m/H'_0 = \dim F/R(T) = d(T) = n(T) < +\infty.$$

Applying Lemma 1.1, we get

$$\dim N(T)/H_m = \dim H_0/H_m = \dim H'_m/H'_0 = n(T) < +\infty.$$

This implies that $H_m = \{0\}$, and hence it follows that $\alpha[T;\mathcal{T}] \leq m = \delta[T;\mathcal{T}]$.

The reverse inequality can be proved similarly.

Using the same technique as in the proof of the preceding result, one can show that the following two statements are true:

(i) If $n(T) > d(T)$, then $\alpha[T;\mathcal{T}] = +\infty$.

(ii) If $n(T) < d(T)$, then $\delta[T;\mathcal{T}] = +\infty$.

This also extends a well-known result about the ascent and descent of a single linear operator.

## 2. DECOMPOSITIONS

Let U be a linear operator on a (complex) linear space E, let m be a non-negative integer, and suppose that $\alpha(U) = \delta(U) = m$. Then, as is well-known (cf., e.g., Section 5.41 in [42]), we have

$$E = N(U^m) \oplus R(U^m),$$

the restriction of U to $N(U^m)$ is a nilpotent linear operator on $N(U^m)$ of index of nilpotency m, and the restriction of U to $R(U^m)$ is a bijective linear operator on $R(U^m)$. The main goal of this section is to prove an analoguous result for the generalized ascent and descent. In order to get such a result, we need a number of sequences of subspaces similar to those defined in the preceding section.

Let $T = T_0$ and $\mathcal{T} = \{T_n\}_{n=1}^{\infty}$ be as in Section 1. For $m = 1,2,\ldots$, we define $N_m[T;\mathcal{T}]$ to be the set of all x in E with the property that there exist $x_0,\ldots,x_{m-1}$ in E such that $x_{m-1} = x$ and

$$\sum_{i=0}^{n} T_i x_{n-i} = 0 \qquad (n = 0,\ldots,m-1).$$

Further we put $N_0[T;\mathcal{T}] = \{0\}$. Then $\{N_m[T;\mathcal{T}]\}_{m=0}^{\infty}$ is an increasing sequence of subspaces of E.

For $m = 1,2,\ldots$, let $R_m[T;\mathcal{T}]$ be the set of all x in E with the property that there exist $x_0,\ldots,x_m$ in E such that $x_0 = x$ and

$$\sum_{i=0}^{n} T_i x_{n-i} = 0 \qquad (n = 1,\ldots,m).$$

Further we define $R_0[T;\mathcal{T}] = E$. Then $\{R_m[T;\mathcal{T}]\}_{m=0}^{\infty}$ is a decreasing sequence of subspaces of E. Observe that

$$H_m[T;\mathcal{T}] = N(T) \cap R_m[T;\mathcal{T}]$$

for each non-negative integer m.

For $m = 0,1,2,\ldots$, let $N'_m[T;\mathcal{T}]$ be the set of all y in F with the property that there exist $x_0,\ldots,x_m$ in E such that $x_m = 0$ and

$$\sum_{i=0}^{n} T_i x_{n-i} = \delta_{nm} y \qquad (n = 0,\ldots,m).$$

Then $\{N'_m[T;\mathcal{T}]\}_{m=0}^{\infty}$ is an increasing sequence of subspaces of F. Observe that $N'_0[T;\mathcal{T}] = \{0\}$ and that

$$H'_m[T;\mathcal{T}] = R(T) + N'_m[T;\mathcal{T}]$$

for each non-negative integer m.

Finally, we define the sequence $\{R'_m[T;\mathcal{T}]\}_{m=0}^{\infty}$. For $m = 1,2,\ldots$, let $R'_m[T;\mathcal{T}]$ be the set of all y in F with the property that there exist $x_0,\ldots,x_{m-1}$ in E such that

$$\sum_{i=0}^{n} T_i x_{n-i} = \delta_{n0} y \qquad (n = 0,\ldots,m-1).$$

Further we put $R'_0[T;\mathcal{T}] = F$. Then $\{R'_m[T;\mathcal{T}]\}_{m=0}^{\infty}$ is a decreasing sequence of subspaces of F.

Often we shall omit $[T;\mathcal{T}]$ in the symbols introduced above.

Consider the particular case when E = F and

$$T_1 = I_E, \quad T_n = 0 \qquad (n = 2,3,4,\ldots).$$

Then we have for $m = 0,1,2,\ldots$

$$N_m = N'_m = N(T^m), \quad R_m = R'_m = R(T^m).$$

In Section 1 we observed that in the present case

$$\alpha[T;\mathcal{T}] = \alpha(T), \quad \delta[T;\mathcal{T}] = \delta(T).$$

Hence, if $\alpha[T;\mathcal{T}] = \delta[T;\mathcal{T}] = m < +\infty$, then $\alpha(T) = \delta(T) = m < +\infty$, and thus

$$E = N(T^m) \oplus R(T^m).$$

In other words, if $\alpha[T;\mathcal{T}] = \delta[T;\mathcal{T}] = m < +\infty$, then in this particular case

$$E = N_m \oplus R_m, \quad F = N'_m \oplus R'_m.$$

In the next example we shall show that these decompositions do not hold in general.

2.1. EXAMPLE. Let $E = F$ be $\mathbb{C}^3$, and let the operators $T$, $T_1$ and $T_2$ be defined by

$$T(x_1,x_2,x_3) = (0,x_1,x_2),$$

$$T_1(x_1,x_2,x_3) = (0,0,x_3),$$

$$T_2(x_1,x_2,x_3) = (x_3,0,0).$$

We shall prove that in this case $\alpha[T;\mathcal{T}] = \delta[T;\mathcal{T}] = 2$. Further we shall show that

$$N_2 \cap R_2 \neq \{0\}, \quad N'_2 \cap R'_2 \neq \{0\},$$

and hence it will follow that in this case we do not have the desired decompositions.

Take $x$ in $R_2$. Then there exist $u$ and $v$ in $\mathbb{C}^3$ such that

$$Tu + T_1x = 0, \quad Tv + T_1u + T_2x = 0.$$

This means that

$$(0,u_1,x_3+u_2) = 0, \quad (x_3,v_1,u_3+v_2) = 0. \tag{1}$$

In particular, $x_3 = 0$. Conversely, let $x \in \mathbb{C}^3$ and suppose that $x_3 = 0$. Put $u = v = 0$. Then (1) is satisfied, and hence $x \in R_2$. This proves that

$$R_2 = \{x \in \mathbb{C}^3 : x_3 = 0\}. \tag{2}$$

In the same way one can show that $R_1 = \mathbb{C}^3$. From the definition of T, it is clear that

$$N(T) = \{x \in \mathbb{C}^3 : x_1 = x_2 = 0\}.$$

Thus $H_2 = N(T) \cap R_2 = \{0\}$ and $H_1 = N(T) \cap R_1 \neq \{0\}$. We conclude that $\alpha[T;T] = 2$.

Take y in $N_2'$. Then there exist x and u in $\mathbb{C}^3$ such that

$$Tx = 0, \quad Tu + T_1x = 0, \quad T_1u + T_2x = y.$$

In other words

$$(0,x_1,x_2) = 0, \quad (0,u_1,x_3+u_2) = 0, \quad (x_3,0,u_3) = y. \tag{3}$$

This implies that $y_2 = 0$. Conversely, let $y \in \mathbb{C}^3$ and suppose that $y_2 = 0$. Put $x = (0,0,y_1)$ and $u = (0,-y_1,y_3)$. Then (3) is satisfied, and so $y \in N_2'$. Hence

$$N_2' = \{y \in \mathbb{C}^3 : y_2 = 0\}. \tag{4}$$

In the same way one can show that

$$N_1' = \{y \in \mathbb{C}^3 : y_1 = y_2 = 0\}.$$

From the definition of T it is clear that

$$R(T) = \{y \in \mathbb{C}^3 : y_1 = 0\}.$$

Thus $H_2' = R(T) + N_2' = \mathbb{C}^3$ and $H_1' = R(T) + N_1' \neq \mathbb{C}^3$. We conclude that $\delta[T;T] = 2$.

Take x in $N_2$. Then there exists u in $\mathbb{C}^3$ such that

$$Tu = 0, \quad Tx + T_1u = 0.$$

This means that

$$(0,u_1,u_2) = 0, \quad (0,x_1,x_2+u_3) = 0. \tag{5}$$

In particular, $x_1 = 0$. Conversely, let $x \in \mathbb{C}^3$ and suppose that $x_1 = 0$. Put $u = (0,0,-x_2)$. Then (5) is satisfied, and so $x \in N_2$. Hence

$$N_2 = \{x \in \mathbb{C}^3 : x_1 = 0\}.$$

Combining this with (2), we see that $N_2 \cap R_2 \neq \{0\}$.

Take y in $R_2'$. Then there exist x and u in $\mathbb{C}^3$ such that

$$Tx = y, \quad Tu + T_1x = 0.$$

In other words

$$(0,x_1,x_2) = y, \quad (0,u_1,x_3+u_2) = 0. \tag{6}$$

This implies that $y_1 = 0$. Conversely, let $y \in \mathbb{C}^3$ and suppose that $y_1 = 0$. Put $x = (y_2,y_3,0)$ and $u = 0$. Then (6) is satisfied, and so $y \in R_2'$. Thus

$$R_2' = \{y \in \mathbb{C}^3 : y_1 = 0\}.$$

Combining this with (4), we see that $N_2' \cap R_2' \neq \{0\}$.

In the particular case when

$$T_n = 0 \qquad (n = 2,3,\ldots), \tag{7}$$

we can prove that direct sum decompositions exist. For a detailed investigation of this case we refer to the author's interim report [6]. Here we present only the main results of [6], which lead us directly to a decomposition theorem similar to the result mentioned at the beginning of this section. In Section 4 we shall show that for many purposes it suffices to consider the special case when (7) is satisfied. For an alternative definition of the sequences of subspaces $\{N_m\}$, $\{R_m\}$, $\{N_m'\}$ and $\{R_m'\}$ which can be used in this context, we refer to the remark at the end of this section.

In the remainder of this section it is assumed that (7) holds. The operator $T_1$ will be denoted by S.

2.2. PROPOSITION. *Let* m *be a non-negative integer. Then*

(i) $TN_{m+1} \subset N'_m$;

(ii) $TN_m \subset N'_m$;

(iii) $TR_m = R'_{m+1}$;

(iv) $TR_m \subset R'_m$;

(v) $SN_m = N'_m$;

(vi) $SR_m \subset R'_m$;

(vii) $R_m = S^{-1}R'_m$.

PROOF. Observe that $N_1 = N(T)$ and $R'_1 = R(T)$. Using this one easily verifies that the proposition is true for $m = 0$. In the following m is assumed to be strictly positive.

(i) Take x in $N_{m+1}$. Then there exist $x_0,\dots,x_m$ in E such that

$$x_m = x, \quad Tx_0 = 0, \quad Tx_i + Sx_{i-1} = 0 \qquad (i = 1,\dots,m).$$

It is clear that

$$Tx_0 = 0, \quad Tx_i + Sx_{i-1} = 0 \quad (i = 1,\dots,m-1), \quad Sx_{m-1} = -Tx.$$

Thus $-Tx \in N'_m$. Hence $Tx \in N'_m$, and the proof of (i) is complete.

(ii) Follows immediately from (i) and the fact that $N_m$ is a subset of $N_{m+1}$.

(iii) Take x in $R_m$ and put $y = Tx$. Since $x \in R_m$, there exist $x_0,\dots,x_m$ in E such that

$$x_0 = x, \quad Tx_i + Sx_{i-1} = 0 \qquad (i = 1,\dots,m). \qquad (8)$$

So we have

$$Tx_0 = y, \quad Tx_i + Sx_{i-1} = 0 \qquad (i = 1,\dots,m), \qquad (9)$$

and hence $y \in R'_{m+1}$. We conclude that $TR_m$ is a subset of $R'_{m+1}$.

To prove the reverse inclusion, take y in $R'_{m+1}$. Then there exist $x_0,\dots,x_m$ in E such that (9) is satisfied. In particular, the second part of formula (8) holds, and so $x_0 \in R_m$. Hence $y = Tx_0 \in TR_m$. This proves (iii).

(iv) Follows immediately from (iii) and the fact that $R'_{m+1}$ is a subset of $R'_m$.

(v) Take x in $N_m$ and put $y = Sx$. Since $x \in N_m$, there exist $x_0,\ldots,x_{m-1}$ in E such that $x_{m-1} = x$ and

$$Tx_0 = 0, \quad Tx_i + Sx_{i-1} = 0 \qquad (i = 1,\ldots,m-1). \tag{10}$$

It is clear that

$$Tx_0 = 0, \quad Tx_i + Sx_{i-1} = 0 \quad (i = 1,\ldots,m-1), \quad Sx_{m-1} = y. \tag{11}$$

Thus $y \in N'_m$. We conclude that $SN_m$ is a subset of $N'_m$.

Conversely, take y in $N'_m$. Then there exist $x_0,\ldots,x_{m-1}$ in E such that (11) is satisfied. In particular, formula (10) holds, and so $x_{m-1} \in N_m$. Hence $y = Sx_{m-1} \in SN_m$, and the proof of (v) is complete.

(vi) Take x in $R_m$. Then there exist $x_0,\ldots,x_m$ in E such that (8) is satisfied. So we have

$$Tx_1 = -Sx, \quad Tx_{i+1} + Sx_i = 0 \qquad (i = 1,\ldots,m-1). \tag{12}$$

It follows that $-Sx \in R'_m$. Hence $Sx \in R'_m$, and the proof of (vi) is complete.

(vii) As an immediate consequence of (vi), we have $R_m \subset S^{-1}R'_m$. To prove the reverse inclusion, take x in $S^{-1}R'_m$. Then $-Sx \in R'_m$, and hence there exist $x_1,\ldots,x_m$ in E such that (12) is satisfied. Put $x_0 = x$. Then formula (8) holds, and so $x \in R_m$. This proves (vii).

2.3. LEMMA. *Let* k *and* m *be non-negative integers. Suppose that* $N_k \cap R_m = \{0\}$. *Then* $N'_k \cap R'_m = \{0\}$.

PROOF. Take y in $N'_k \cap R'_m$. Since $y \in N'_k$, it follows from Proposition 2.2(v) that $y = Sx$ for some x in $N_k$. Using the fact that $y \in R'_m$, we see that $x \in S^{-1}R'_m$. By Proposition 2.2(vii), we have $S^{-1}R'_m = R_m$, and so $x \in R_m$. Thus $x \in N_k \cap R_m$. But $N_k \cap R_m = \{0\}$ by hypothesis. We conclude that $x = 0$. Hence $y = Sx = 0$, and the proof is complete.

2.4. PROPOSITION. *Suppose that* $\alpha[T;T] = m < +\infty$. *Then*

$$N_k \cap R_m = \{0\}, \quad N'_k \cap R'_m = \{0\}$$

*for* $k = 0,1,2,\ldots$ . *In particular*, $N_m \cap R_m = \{0\}$ *and* $N'_m \cap R'_m = \{0\}$.

PROOF. In view of the preceding lemma, it suffices to show that

$$N_k \cap R_m = \{0\} \tag{13}$$

for $k = 0,1,2,\ldots$ . We prove this by induction.

It is clear that formula (13) holds for $k = 0$. Let $k$ be a non-negative integer such that (13) is true. We shall prove that $N_{k+1} \cap R_m = \{0\}$.

Take $x$ in $N_{k+1} \cap R_m$. Since $x \in N_{k+1}$, there exist $u_0,\ldots,u_k$ in $E$ such that

$$u_k = x, \quad Tu_0 = 0, \quad Tu_i + Su_{i-1} = 0 \qquad (i = 1,\ldots,k). \tag{14}$$

Since $x \in R_m$, there exist $v_0,\ldots,v_m$ in $E$ such that

$$v_0 = x, \quad Tv_i + Sv_{i-1} = 0 \qquad (i = 1,\ldots,m). \tag{15}$$

Combining (14) and (15), we see that $u_0 \in H_{k+m}$. By assumption $\alpha[T;T] = m$, and so $H_{k+m} = \{0\}$. Hence $u_0 = 0$. This, together with (14), implies that $x \in N_k$. Since $x \in R_m$ too, we have $x \in N_k \cap R_m$. Using the induction hypothesis, we get $x = 0$. This proves the proposition.

2.5. LEMMA. *Let* $k$ *and* $m$ *be non-negative integers. Suppose that* $N'_m + R'_k = F$. *Then* $N_m + R_k = E$.

PROOF. Take $x$ in $E$. By hypothesis $F = N'_m + R'_k$, and hence we can write $Sx = y + z$ with $y$ in $N'_m$ and $z$ in $R'_k$. Proposition 2.2(v) shows that $y = Su$ for some $u$ in $N_m$. It is clear that $S(x - u) = z \in R'_k$, and so $x - u \in S^{-1}R'_k$. But $S^{-1}R'_k = R_k$ by Proposition 2.2(vii). Thus $x - u \in R_k$. Since $x = u + (x - u)$, it follows that $x \in N_m + R_k$. This proves the lemma.

2.6. PROPOSITION. *Suppose that* $\delta[T;T] = m < +\infty$. *Then*

$$N_m + R_k = E, \quad N'_m + R'_k = F$$

*for* $k = 0,1,2,\ldots$ . *In particular*, $N_m + R_m = E$ *and* $N'_m + R'_m = F$.

PROOF. In view of the preceding lemma, it suffices to show that

$$N'_m + R'_k = F \tag{16}$$

for $k = 0,1,2,\ldots$ . We prove this by induction.

It is clear that formula (16) holds for $k = 0$. Let $k$ be a non-negative integer such that (16) is true. We shall prove that $N'_m + R'_{k+1} = F$.

Take $y$ in $F$. By assumption $\delta[T;\mathcal{T}] = m$, and so $R(T) + N'_m = H'_m = F$. Hence we can write $y = Tx + z$, with $x \in E$ and $z \in N'_m$. Using the induction hypothesis, we see that there exist $v$ in $R'_k$ and $w$ in $N'_m$ such that $Sx = v + w$. From Proposition 2.2(v) we know that $w = Su$ for some $u$ in $N_m$. It is clear that $S(x - u) = v \in R'_k$, and so $x - u \in S^{-1}R'_k$. But $S^{-1}R'_k = R_k$ by Proposition 2.2(vii). Hence $x - u \in R_k$. Using Proposition 2.2(iii), we get $Tx - Tu \in TR_k = R'_{k+1}$. Since $Tx = y - z$, it follows that $y - (z + Tu) \in R'_{k+1}$. It remains to prove that $z + Tu \in N'_m$.

Recall that $z \in N'_m$ and $u \in N_m$. Hence $z + Tu$ belongs to the set $N'_m + TN_m$. But $TN_m$ is a subset of $N'_m$ by Proposition 2.2(ii). Since $N'_m$ is a linear subspace of $F$, it follows that $N'_m + TN_m = N'_m$. This completes the proof.

In the remainder of this section we shall suppose that

$$\alpha[T;\mathcal{T}] = \delta[T;\mathcal{T}] = m < +\infty.$$

Then, by Propositions 2.4 and 2.6, we have

$$E = N_m \oplus R_m, \quad F = N'_m \oplus R'_m.$$

We proceed by studying the restrictions of the operators $T$ and $S$ to the subspaces $N_m$ and $R_m$.

According to Proposition 2.2(ii), the operator $T$ maps $N_m$ into $N'_m$. Hence we may consider the restriction of $T$ to $N_m$ as a linear operator from $N_m$ into $N'_m$. This operator will be denoted by $T_N$. The operators $T_R : R_m \longrightarrow R'_m$ and $S_N : N_m \longrightarrow N'_m$ are defined similarly (cf. statements (iv) and (v) of Proposition 2.2). Observe that $S_N$ is surjective. This is immediate from Proposition 2.2(v).

2.7. PROPOSITION. *The operator* $S_N$ *is bijective.*

PROOF. The null space of $S_N$ is $N_m \cap N(S)$. From Proposition 2.2(vii) it is clear that $N(S) \subset R_m$. Thus the null space of $S_N$ is a subset of $N_m \cap R_m$. But $N_m \cap R_m = \{0\}$ by Proposition 2.4. Hence $S_N$ is injective. The surjectivity of $S_N$ has already been established above.

2.8. PROPOSITION. *The operator* $T_R$ *is bijective.*

PROOF. The null space of $T_R$ is $N(T) \cap R_m$. We know that $N(T) \cap R_m = H_m$. By assumption $\alpha[T;\mathcal{T}] = m$, and so $H_m = \{0\}$. This proves that $T_R$ is injective.

The range of $T_R$ is $TR_m$. Proposition 2.2(iii) shows that $TR_m = R'_{m+1}$. It remains to prove that

$$R'_{m+1} = R'_m. \tag{17}$$

First of all, we consider the case when $m = 0$. Then $\delta[T;\mathcal{T}] = 0$, and so $R(T) = H'_0 = F$. Hence $R'_1 = R(T) = F = R'_0$. This proves that (17) holds if $m = 0$.

Next we consider the case when m is strictly positive. Since $R'_{m+1} \subset R'_m$, it suffices to show that $R'_m$ is a subset of $R'_{m+1}$. Take y in $R'_m$. Then there exist $u_0,\ldots,u_{m-1}$ in E such that

$$Tu_0 = y,\ Tu_i + Su_{i-1} = 0 \qquad (i = 1,\ldots,m-1).$$

By assumption $\delta[T;\mathcal{T}] = m$, and so $H'_m = F$. In particular, $-Su_{m-1} \in H'_m$. Hence there exist $v_0,\ldots,v_m$ in E such that

$$Tv_0 = 0,\ Tv_i + Sv_{i-1} = 0 \ \ (i = 1,\ldots,m-1),\ Tv_m + Sv_{m-1} = -Su_{m-1}.$$

It is clear that

$$T(u_0 + v_0) = y,\ \ T(u_i + v_i) + S(u_{i-1} + v_{i-1}) = 0$$

for $i = 1,\ldots,m-1$, and

$$Tv_m + S(v_{m-1} + u_{m-1}) = 0.$$

Thus $y \in R'_{m+1}$, and the proof is complete.

Let M and N be (complex) linear spaces, and let U and V be linear operators from M into N. Suppose that V is bijective. We say that U is *nilpotent of index* m *relative to* V if $V^{-1}U$ is a nilpotent linear operator on M of index of nilpotency m.

2.9. PROPOSITION. *The operator* $T_N$ *is nilpotent of index* m *relative to* $S_N$.

PROOF. By Proposition 2.7, the operator $S_N$ is bijective. Let W denote the product of $T_N$ and $(S_N)^{-1}$. Thus W is the linear operator on $N_m$ defined by

$$W = (S_N)^{-1} \circ T_N.$$

We have to prove that W is nilpotent of index of nilpotency m.

First of all, we consider the case when m = 0. Then $N_m = \{0\}$, and so $W^k = 0$ for each non-negative integer k. This implies that W is nilpotent of index of nilpotency 0.

Next we consider the case when m is strictly positive. Using induction, we shall prove that

$$N(W^k) = N_k \tag{18}$$

for $k = 0,\dots,m$.

It is clear that (18) holds for k = 0. Further, the null space of W is equal to that of $T_N$. Hence $N(W) = N(T) \cap N_m$. Since $N(T) = N_1 \subset N_m$, we have $N(W) = N_1$. Thus formula (18) also holds for k = 1. Let k be a positive integer less than m such that (18) is true. We shall prove that $N(W^{k+1}) = N_{k+1}$.

Take x in $N_{k+1}$. Then there exist $x_0,\dots,x_k$ in E such that $x_k = x$ and

$$Tx_0 = 0, \quad Tx_i + Sx_{i-1} = 0 \quad (i = 1,\dots,k).$$

Observe that $x_k \in N_{k+1} \subset N_m$ and $x_{k-1} \in N_k \subset N_m$. Further,

$$T_N x_k + S_N x_{k-1} = Tx_k + Sx_{k-1} = 0,$$

and so

$$Wx_k + x_{k-1} = 0. \tag{19}$$

From the induction hypothesis we see that $x_{k-1} \in N(W^k)$. Thus $W^k x_{k-1} = 0$. Applying $W^k$ to both sides of the equation (19), we get $W^{k+1}x_k = 0$. Hence $x = x_k \in N(W^{k+1})$. This shows that $N_{k+1}$ is a subset of $N(W^{k+1})$.

To prove the reverse inclusion, take x in $N(W^{k+1})$. Then $-Wx \in N(W^k) = N_k$, and hence there exist $x_0,\ldots,x_{k-1}$ in E such that $x_{k-1} = -Wx$ and

$$Tx_0 = 0, \quad Tx_i + Sx_{i-1} = 0 \qquad (i = 1,\ldots,k-1). \tag{20}$$

Since $x_{k-1} = -Wx$, we have $Tx + Sx_{k-1} = 0$. Combining this with (20), we see that $x \in N_{k+1}$. Thus (18) holds for $k = 0,\ldots,m$.

From formula (18) we see that

$$N(W^{m-1}) = N_{m-1}, \quad N(W^m) = N_m. \tag{21}$$

The second part of (21) shows that $W^m = 0$. It remains to prove that $W^{m-1} \neq 0$. In view of the first part of (21), it suffices to show that $N_{m-1} \neq N_m$.

Suppose that $N_{m-1} = N_m$. We shall prove that this assumption leads to a contradiction.

First of all, we consider the case when $m = 1$. Then we have $N(T) = N_1 = N_0 = \{0\}$. This implies that $H_0 = \{0\}$, which contradicts the hypothesis concerning $\alpha[T;T]$.

Next we consider the case when $m \geqq 2$. Take x in $H_{m-1}$. Then there exist $u_0,\ldots,u_{m-1}$ in E such that

$$u_0 = x, \quad Tu_0 = 0, \quad Tu_i + Su_{i-1} = 0 \qquad (i = 1,\ldots,m-1).$$

It is clear that $u_{m-1} \in N_m$. By assumption $N_m = N_{m-1}$, and so $u_{m-1} \in N_{m-1}$. Hence there exist $v_0,\ldots,v_{m-2}$ in E such that

$$v_{m-2} = u_{m-1}, \quad Tv_0 = 0, \quad Tv_i + Sv_{i-1} = 0 \quad (i = 1,\ldots,m-2).$$

Define $x_0,\ldots,x_m$ in E by

$$x_i = \begin{cases} u_o & \text{for } i = 0, \\ u_i - v_{i-1} & \text{for } i = 1,\ldots,m-1, \\ 0 & \text{for } i = m. \end{cases}$$

Then $x_0 = u_0 = x$ and

$$Tx_0 = 0, \quad Tx_i + Sx_{i-1} = 0 \qquad (i = 1,\dots,m).$$

This implies that $x \in H_m$. Thus $H_{m-1}$ is a subset of $H_m$. By hypothesis $\alpha[T;\mathcal{T}] = m$, and so $H_m = \{0\}$. We conclude that $H_{m-1} = \{0\}$ too. Hence $m = \alpha[T;\mathcal{T}] \leqq m-1$, and we have a contradiction. This proves the proposition.

Summarizing, we have the following generalization of the decomposition result mentioned at the beginning of this section.

2.10. THEOREM. *Let* m *be a non-negative integer, and suppose that* $\alpha[T;\mathcal{T}] = \delta[T;\mathcal{T}] = m$, *where* $\mathcal{T}$ *is the sequence* $\{S,0,0,\dots\}$. *Then*

(i) $E = N_m[T;\mathcal{T}] \oplus R_m[T;\mathcal{T}]$ *and* $F = N'_m[T;\mathcal{T}] \oplus R'_m[T;\mathcal{T}]$;

(ii) S *maps* $R_m[T;\mathcal{T}]$ *into* $R'_m[T;\mathcal{T}]$;

(iii) S *maps* $N_m[T;\mathcal{T}]$ *into* $N'_m[T;\mathcal{T}]$ *and the restriction of* S *to* $N_m[T;\mathcal{T}]$, *considered as a linear operator from* $N_m[T;\mathcal{T}]$ *into* $N'_m[T;\mathcal{T}]$, *is bijective;*

(iv) T *maps* $R_m[T;\mathcal{T}]$ *into* $R'_m[T;\mathcal{T}]$ *and the restriction of* T *to* $R_m[T;\mathcal{T}]$, *considered as a linear operator from* $R_m[T;\mathcal{T}]$ *into* $R'_m[T;\mathcal{T}]$, *is bijective;*

(v) T *maps* $N_m[T;\mathcal{T}]$ *into* $N'_m[T;\mathcal{T}]$ *and the restriction of* T *to* $N_m[T;\mathcal{T}]$, *considered as a linear operator from* $N_m[T;\mathcal{T}]$ *into* $N'_m[T;\mathcal{T}]$, *is nilpotent of index* m *relative to the restriction of* S *to* $N_m[T;\mathcal{T}]$, *considered as a linear operator from* $N_m[T;\mathcal{T}]$ *into* $N'_m[T;\mathcal{T}]$.

In the particular case considered here, namely when $\mathcal{T}$ is the sequence $\{S,0,0,\dots\}$, we have

$$N_0 = \{0\}, \quad N_{m+1} = T^{-1}SN_m \qquad (m = 0,1,2,\dots). \qquad (22)$$

The second part of this formula will be used in the proof of Lemma 4.7. Here follows the proof of (22).

First of all, we note that $N_0 = \{0\}$ and $N_1 = N(T)$, and hence $N_1 = T^{-1}SN_0$. Next, let m be a positive integer, and take x in $N_{m+1}$. Then there exist $x_0,\dots,x_m$ in E such that

$$x_m = x, \quad Tx_0 = 0, \quad Tx_i + Sx_{i-1} = 0 \quad (i = 1,\dots,m).$$

It is clear that $x_{m-1} \in N_m$. Since $Tx = Tx_m = -Sx_{m-1}$, it follows that $Tx \in SN_m$. Hence $x \in T^{-1}SN_m$. We conclude that $N_{m+1}$ is a subset of $T^{-1}SN_m$. To prove the reverse inclusion, take x in $T^{-1}SN_m$. Then $Tx = Su$ for some u in $N_m$. Choose $x_0,\dots,x_{m-1}$ in E such that

$$x_{m-1} = u, \quad Tx_0 = 0, \quad Tx_i + Sx_{i-1} = 0 \quad (i = 1,\dots,m-1).$$

If we put $x_m = -x$, then we have $Tx_m + Sx_{m-1} = 0$. This implies that $-x \in N_{m+1}$. Hence $x \in N_{m+1}$, and the proof of (22) is complete.

In the same way one can prove that

$$R_0 = E, \quad R_{m+1} = S^{-1}TR_m \qquad (m = 0,1,2,\dots), \tag{23}$$
$$N'_0 = \{0\}, \quad N'_{m+1} = ST^{-1}N'_m \qquad (m = 0,1,2,\dots), \tag{24}$$
$$R'_0 = F, \quad R'_{m+1} = TS^{-1}R'_m \qquad (m = 0,1,2,\dots). \tag{25}$$

In the author's interim report [6], the formulas (22) up to and including (25) are employed as definitions. Some of the sequences of subspaces defined in this way have been used earlier by T. Kato ([24], page 288), M.A. Kaashoek ([20], page 453), R.K. Oliver ([33], pp. 364, 366) and S. Goldberg ([13], pp. 114, 115).

3. THE STABILITY NUMBER

Let $T = T_0$ and $\mathcal{T} = \{T_n\}_{n=1}^{\infty}$ be as in Section 1. The set of all x in E with the property that there exists a sequence $\{x_i\}_{i=0}^{\infty}$ in E such that $x_0 = x$ and

$$\sum_{i=0}^{n} T_i x_{n-i} = 0 \qquad (n = 0,1,2,\dots)$$

will be denoted by $H_\infty[T;\mathcal{T}]$. Observe that $H_\infty[T;\mathcal{T}]$ is a subspace of N(T). In fact

$$H_\infty[T;\mathcal{T}] \subset H_m[T;\mathcal{T}] \qquad (m = 0,1,2,\dots). \tag{1}$$

The extended integer $k[T;\mathcal{T}]$, defined by

$$k[T;\mathcal{T}] = \dim N(T)/H_\infty[T;\mathcal{T}],$$

is called the *stability number of* T *relative to the sequence* $\mathcal{T}$. This name will be explained at the end of this section.

Let

$$H[T;\mathcal{T}] = \bigcap_{m=0}^{\infty} H_m[T;\mathcal{T}], \quad H'[T;\mathcal{T}] = \bigcup_{m=0}^{\infty} H'_m[T;\mathcal{T}].$$

Observe that $H[T;\mathcal{T}]$ is a subspace of E contained in N(T), and that $H'[T;\mathcal{T}]$ is a subspace of F containing R(T). For brevity we shall often use the following notations

$$H_\infty = H_\infty[T;\mathcal{T}], \quad H = H[T;\mathcal{T}], \quad H' = H'[T;\mathcal{T}].$$

The aim of this section is to show that

$$k[T;\mathcal{T}] = \dim N(T)/H = \dim H'/R(T). \tag{2}$$

We begin by proving the second part of (2).

3.1. PROPOSITION. $\dim N(T)/H = \dim H'/R(T)$.

PROOF. Without loss of generality we may assume that at least one of the extended integers appearing in the statement is finite. Suppose that $\dim H'/R(T)$ is finite. Then, as is easily verified, there exists a non-negative integer p such that

$$H'_m = H'_p \qquad (m = p,p+1,\ldots).$$

Applying Lemma 1.1, we see that this implies that

$$H_m = H_p \qquad (m = p,p+1,\ldots).$$

Hence $H = H_p$ and $H' = H'_p$, and thus, again by Lemma 1.1,

$$\dim N(T)/H = \dim H_0/H_p = \dim H'_p/H'_0 = \dim H'/R(T).$$

A similar proof can be given when $\dim N(T)/H$ is finite.

Next we investigate the relationship between the sets $H_\infty$ and H. From formula (1) it is clear that

$$H_\infty \subset H. \tag{3}$$

This inclusion may be strict.

3.2. EXAMPLE. We shall construct an operator T on the sequence space $l_1$ such that

$$\{0\} \neq \bigcap_{m=1}^{\infty} R(T^m) \subset N(T). \tag{4}$$

This provides an example showing that the inclusion in (3) may be strict. Indeed, suppose that E = F, let $T_0 = T$ be an operator on E satisfying (4), let $T_1$ be the identity operator on E, and let

$$T_n = 0 \qquad (n = 2,3,\ldots).$$

Then, as was established in Section 1,

$$H_m = N(T) \cap R(T^m) \qquad (m = 0,1,2,\ldots).$$

Hence, by formula (4),

$$H = \bigcap_{m=1}^{\infty} R(T^m) \neq \{0\}.$$

Take x in $H_\infty$. Then there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in E such that $x_0 = x$ and

$$Tx_n + x_{n-1} = 0 \qquad (n = 1,2,\ldots).$$

Clearly, this implies that

$$x_1 \in \bigcap_{m=1}^{\infty} R(T^m) \subset N(T).$$

Hence $x = x_0 = -Tx_1 = 0$. This shows that $H_\infty = \{0\}$, and thus $H_\infty \neq H$.

The definition of the operator T is as follows. Let $\{i_m\}_{m=1}^{\infty}$ be the sequence of positive integers defined by

$$i_1 = 2, \quad i_{m+1} = i_m + m \qquad (m = 1,2,\dots).$$

Define T on $l_1$ by setting

$$(Tx)_n = \begin{cases} \sum_{m=2}^{\infty} x_{i_m - 1} & \text{for } n = 1, \\ 0 & \text{for } n = i_1, i_2, i_3, \dots\dots, \\ x_{n-1} & \text{otherwise.} \end{cases}$$

Then T is a well-defined linear operator on $l_1$. We shall show that T has the required properties.

For each positive integer n, let $e_n$ be the element in $l_1$ with all coordinates zero except the n-th, which is equal to one. From the definition of T it is clear that

$$Te_1 = 0.$$

Let V be the set $\{i_m : m = 1,2,\dots\}$. A straightforward argument shows that for $n = 2,3,\dots$

$$Te_n = \begin{cases} e_1 & \text{if } n+1 \in V, \\ e_{n+1} & \text{if } n+1 \notin V. \end{cases}$$

From this it follows that

$$T^m e_{i_m} = e_1 \qquad (m = 1,2,\dots).$$

Hence $e_1 \in \bigcap_{m=1}^{\infty} R(T^m)$. This proves the first part of (4).

Next we prove the second half of (4). From the definition of T it follows that for each x in $l_1$

$$(T^n x)_n = 0 \qquad (n = 2,3,\dots).$$

Hence, if $y \in \bigcap_{m=1}^{\infty} R(T^m)$, then $y_n = 0 \quad (n = 2,3,\dots)$, and thus $Ty = 0$. This shows that

$$\bigcap_{m=1}^{\infty} R(T^m) \subset N(T),$$

and the proof is complete.

We proceed by showing that $H_\infty = H$ whenever there exists a non-negative integer p such that

$$H_m = H_p \qquad (m = p,p+1,\dots). \tag{5}$$

In order to do this we need the following notations. For $k,m = 0,1,2,\dots$, let $H_m^{(k)}$ be the set of all $(u_0,\dots,u_k)$ in the product space $E^{k+1}$ with the property that there exist $x_0,\dots,x_{k+m}$ in E such that $(x_0,\dots,x_k) = (u_0,\dots,u_k)$ and

$$\sum_{i=0}^{n} T_i x_{n-i} = 0 \qquad (n = k,\dots,k+m).$$

Further, let

$$H^{(k)} = \bigcap_{m=0}^{\infty} H_m^{(k)}.$$

Identifying $E^1$ and E, we obtain

$$H_m^{(0)} = H_m \qquad (m = 0,1,2,\dots), \qquad H^{(0)} = H.$$

3.3. LEMMA. *Suppose that* (5) *is satisfied for some non-negative integer* p. *Let* k *be a non-negative integer, and let* $(u_0,\dots,u_k) \in H^{(k)}$. *Then there exists* x *in* E *such that* $(u_0,\dots,u_k,x) \in H^{(k+1)}$.

PROOF. Define for $m = 0,1,2,\dots\dots$ the subset $V_m$ of E by

$$V_m = \{x \in E : (u_0,\dots,u_k,x) \in H_m^{(k+1)}\}.$$

We need to prove that

$$\bigcap_{m=0}^{\infty} V_m \neq \emptyset. \tag{6}$$

First of all, we note that for $m = 0,1,2,\dots$ the set $V_m$ is non-void. This can be seen as follows. Since $H^{(k)}$ is a subset of $H_{m+1}^{(k)}$, we have $(u_0,\dots,u_k) \in H_{m+1}^{(k)}$. Choose $x_0,\dots,x_{k+m+1}$ in E such that $(x_0,\dots,x_k) = (u_0,\dots,u_k)$ and

$$\sum_{i=0}^{n} T_i x_{n-i} = 0 \qquad (n = k,\ldots,k+m+1).$$

Then $(u_0,\ldots,u_k,x_{k+1}) \in H_m^{(k+1)}$, and hence $x_{k+1} \in V_m$.

From the definitions it is clear that

$$H_{m+1}^{(k+1)} \subset H_m^{(k+1)} \qquad (m = 0,1,2,\ldots),$$

and so we have

$$V_{m+1} \subset V_m \qquad (m = 0,1,2,\ldots). \tag{7}$$

Next we shall prove that for any x in $V_m$

$$V_m = x + H_m \qquad (m = 0,1,2,\ldots). \tag{8}$$

Take x in $V_m$. Then $(u_0,\ldots,u_k,x) \in H_m^{(k+1)}$, and hence there exists a solution $x_0,\ldots,x_{k+m+1}$ in E of the equations

$$\sum_{i=0}^{n} T_i x_{n-i} = 0 \qquad (n = k+1,\ldots,k+m+1) \tag{9}$$

such that $(x_0,\ldots,x_k,x_{k+1}) = (u_0,\ldots,u_k,x)$. Let y be another element of $V_m$, and let $y_0,\ldots,y_{k+m+1}$ in E form a solution of (9) with $(y_0,\ldots,y_k,y_{k+1}) = (u_0,\ldots,u_k,y)$. It is clear that

$$\sum_{i=0}^{n} T_i(y_{n-i} - x_{n-i}) = 0 \qquad (n = k+1,\ldots,k+m+1).$$

Using the fact that $x_i = y_i$ for $i = 0,\ldots,k$, we obtain

$$\sum_{i=0}^{n-k-1} T_i(y_{n-i} - x_{n-i}) = 0 \qquad (n = k+1,\ldots,k+m+1).$$

This implies that $y_{k+1} - x_{k+1} \in H_m$. But $x_{k+1} = x$ and $y_{k+1} = y$, and so $y - x \in H_m$. This proves that $V_m$ is a subset of $x + H_m$.

Conversely, take z in $x + H_m$. Then $z - x \in H_m$, and hence there exist $v_0,\ldots,v_m$ in E such that $v_0 = z - x$ and

$$\sum_{i=0}^{n} T_i v_{n-i} = 0 \qquad (n = 0,\ldots,m).$$

Define $z_0,\dots,z_{k+m+1}$ in E by

$$z_i = \begin{cases} x_i & \text{for } i = 0,\dots,k, \\ x_i + v_{i-k-1} & \text{for } i = k+1,\dots,k+m+1. \end{cases}$$

A straightforward computation shows that $z_0,\dots,z_{k+m+1}$ form a solution of the equations (9). This implies that $(z_0,\dots,z_k,z_{k+1}) \in H_m^{(k+1)}$. Observe that $(z_0,\dots,z_k,z_{k+1}) = (u_0,\dots,u_k,z)$. Hence $z \in V_m$. This shows that $x + H_m$ is a subset of $V_m$, and the proof of (8) is complete.

Now we shall use the hypothesis that (5) is satisfied for some non-negative integer p. Let m be an integer with $m \geqq p$ and take x in $V_m$. Then x is also in $V_p$, for $V_m$ is a subset of $V_p$. Thus $V_m = x + H_m$ and $V_p = x + H_p$. But, since $m \geqq p$, we have $H_m = H_p$. Hence $V_m = V_p$ for $m = p,p+1,\dots$ . This, together with (7), implies that

$$\bigcap_{m=0}^{\infty} V_m = V_p.$$

Since $V_p$ is non-void, it follows that formula (6) holds, and the proof is complete.

3.4. PROPOSITION. *Suppose that*

$$H_m = H_p \qquad (m = p,p+1,\dots)$$

*for some non-negative integer* p. *Then* $H_\infty = H$.

PROOF. Since $H_\infty$ is a subset of H, it suffices to show that $H \subset H_\infty$. Take x in H and put $x_0 = x$. Then

$$x_0 \in H^{(0)},$$

and hence, by the preceding lemma, there exists $x_1$ in E such that

$$(x_0,x_1) \in H^{(1)}.$$

But then, using the same argument, there exists $x_2$ in E such that

$$(x_0, x_1, x_2) \in H^{(2)}.$$

Proceeding in this way, we obtain a sequence $\{x_k\}_{k=0}^{\infty}$ in E such that for $k = 0,1,2,\ldots$

$$(x_0, \ldots, x_k) \in H^{(k)}.$$

In particular,

$$(x_0, \ldots, x_k) \in H_0^{(k)} \qquad (k = 0,1,2,\ldots).$$

But this implies that

$$\sum_{i=0}^{k} T_i x_{k-i} = 0 \qquad (k = 0,1,2,\ldots),$$

and hence $x = x_0 \in H_\infty$. This completes the proof.

The preceding proposition says that $H_\infty = H$, provided that (5) is satisfied for some non-negative integer p. Observe that (5) is equivalent to

$$H'_m = H'_p \qquad (m = p,p+1,\ldots).$$

This is immediate from Lemma 1.1. It is easily verified that (5) holds for some non-negative integer p if dim N(T)/H (or, equivalently, dim H'/R(T)) is finite. This implies that the following corollary to Proposition 2.4 is true.

3.5. COROLLARY. *If* $\dim N(T)/H < +\infty$, *then* $H_\infty = H$.

We now come to the proof of the first equality in (2).

3.6. PROPOSITION. $k[T;\mathcal{T}] = \dim N(T)/H$.

PROOF. Since $H_\infty$ is contained in H, we have

$$k[T;\mathcal{T}] = \dim N(T)/H_\infty \geqq \dim N(T)/H.$$

By Corollary 3.5, this inequality cannot be strict. Hence the result follows.

Summarizing, we have the following theorem.

3.7. THEOREM. $k[T;\mathcal{T}] = \dim N(T)/H[T;\mathcal{T}] = \dim H'[T;\mathcal{T}]/R(T)$.

In the remainder of this section we shall explain the term "stability number". Further we shall introduce some notations concerning locally holomorphic operator valued functions.

Let X and Y be complex Banach spaces, and let A be a locally holomorphic function defined on an open neighbourhood of 0 with values in $L(X,Y)$. Denote by $A_n$ the n-th coefficient of the Taylor expansion of A at 0. Recall that

$$A_n = \frac{1}{n!} A^{(n)}(0) \qquad (n = 0,1,2,\ldots),$$

where $A^{(n)}$ denotes the n-th derivative of the function A. In particular, $A_0 = A(0)$. The stability number of $A_0$ relative to the sequence $\{A_n\}_{n=1}^{\infty}$ will be denoted by k[A]. It is called the *stability number of* A (*at* 0). The importance of this number appears from the following stability theorem.

3.8. THEOREM. *Suppose that* $A_0$ *has closed range and that* k[A] *is finite. Then there exists* $\delta > 0$ *such that*

(i) $R(A(\lambda))$ *is closed for* $|\lambda| < \delta$;

(ii) $n(A(\lambda)) = n(A_0) - k[A]$ *for* $0 < |\lambda| < \delta$;

(iii) $d(A(\lambda)) = d(A_0) - k[A]$ *for* $0 < |\lambda| < \delta$.

This theorem is a special case of a more general result due to K.-H. Förster (cf. Theorems 1 and 2 in [11]; for related results see [20], [24] and [30]). It will be used in Section 5.

We have already introduced the symbol k[A]. In the same way we define the symbols $H_m[A]$, $H'_m[A]$, $N_m[A]$, $N'_m[A]$, $R_m[A]$, $R'_m[A]$, $H_\infty[A]$, $H[A]$, $\alpha[A]$ and $\delta[A]$. Thus if $\mathcal{A}$ denotes the sequence $\{A_n\}_{n=1}^{\infty}$, then

$$H_m[A] = H_m[A_0;\mathcal{A}] \qquad (m = 0,1,2,\ldots),$$
$$H'_m[A] = H'_m[A_0;\mathcal{A}] \qquad (m = 0,1,2,\ldots),$$
$$N_m[A] = N_m[A_0;\mathcal{A}] \qquad (m = 0,1,2,\ldots),$$
$$N'_m[A] = N'_m[A_0;\mathcal{A}] \qquad (m = 0,1,2,\ldots),$$
$$R_m[A] = R_m[A_0;\mathcal{A}] \qquad (m = 0,1,2,\ldots),$$

$$R'_m[A] = R'_m[A_0;A] \qquad (m = 0,1,2,\ldots),$$
$$H_\infty[A] = H_\infty[A_0;A],$$
$$H[A] = H[A_0;A],$$
$$\alpha[A] = \alpha[A_0;A],$$
$$\delta[A] = \delta[A_0;A],$$
$$k[A] = k[A_0;A].$$

These notations will be used in Sections 4, 5 and 7. We call $\alpha[A]$ the *ascent* and $\delta[A]$ the *descent* of A (*at* 0).

4. LINEARIZATION

In this section X and Y are complex Banach spaces. Further, A is a locally holomorphic function defined on an open neighbourhood D of 0 with values in $L(X,Y)$. Our aim is to show that for many purposes it suffices to consider the case when A depends linearly on the complex variable $\lambda$.

We start with the construction of two auxiliary complex Banach spaces E and F. The linear space E is the subspace of the product space

$$\Pi = \prod_{k=0}^{\infty} X_k, \quad X_k = X \qquad (k = 0,1,2,\ldots) \tag{1}$$

consisting of all sequences $\xi = \{\xi_k\}_{k=0}^{\infty} \in \Pi$ such that

$$\sup \{||\xi_k|| : k = 0,1,2,\ldots\} < +\infty.$$

On E a norm is given by the formula

$$||\xi|| = \sup \{||\xi_k|| : k = 0,1,2,\ldots\}.$$

With this norm E is a complex Banach space. The complex Banach space F is defined in the same way as the space E, the difference being that in (1) we take $X_0 = Y$ instead of $X_0 = X$.

Next we define a "linearization" of the function A. Let r be a positive real number such that the closure of the open disc $\Delta_r$ with center 0 and radius r is a subset of D. Thus

$$\Delta_r = \{\lambda \in \mathbb{C} : |\lambda| < r\} \subset \{\lambda \in \mathbb{C} : |\lambda| \leqq r\} \subset D.$$

Further, let $A_n$ denote the n-th coefficient of the Taylor expansion of A at 0. Then

$$A(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n \qquad (\lambda \in \Delta_r)$$

and

$$\sum_{n=0}^{\infty} r^n \, ||A_n|| < +\infty. \tag{2}$$

Define the functions T and S from E into F by

$$T\xi = (A_0\xi_0, \xi_1, \xi_2, \ldots)$$

and

$$S\xi = (\sum_{n=1}^{\infty} r^{n-1} A_n \xi_{n-1},\ -r^{-1}\xi_0,\ -r^{-1}\xi_1, \ldots\ldots).$$

Then T and S are well-defined bounded linear operators from E into F. In the remainder of this section the holomorphic function

$$\lambda \longmapsto T + \lambda S \qquad (\lambda \in \mathbb{C})$$

will be denoted by L. We shall prove that there are several useful relations between the properties of the function A and those of the "linear" function L.

Take $\lambda$ in $\Delta_r$, $\xi$ in E and $\eta$ in F. Suppose that

$$\eta = (T + \lambda S)\xi. \tag{3}$$

Then we have

$$\eta_0 = A_0\xi_0 + \lambda \sum_{n=1}^{\infty} r^{n-1} A_n \xi_{n-1} \tag{4}$$

and

$$\eta_k = \xi_k - (\frac{\lambda}{r})\xi_{k-1} \qquad (k = 1,2,\ldots).$$

From the last formula it is easy to deduce that

$$\xi_k = (\frac{\lambda}{r})^k \xi_0 + \sum_{j=0}^{k-1} (\frac{\lambda}{r})^j \eta_{k-j} \qquad (k = 1,2,\dots). \tag{5}$$

For $k = 1,2,\dots$, let $\phi_k$ be the function from $\Delta_r$ into $L(F,X)$ defined by

$$\phi_k(\lambda)\eta = \sum_{j=o}^{k-1} (\frac{\lambda}{r})^j \eta_{k-j}.$$

Then (5) shows that

$$\xi_k = (\frac{\lambda}{r})^k \xi_0 + \phi_k(\lambda)\eta \qquad (k = 1,2,\dots). \tag{6}$$

Using this in (4), we derive

$$\eta_0 = A(\lambda)\xi_0 + \sum_{n=2}^{\infty} \lambda r^{n-1} A_n \phi_{n-1}(\lambda)\eta. \tag{7}$$

Define the function $\phi$ from $\Delta_r$ into $L(F,Y)$ by setting

$$\phi(\lambda)\eta = \eta_0 - \sum_{k=1}^{\infty} \lambda r^k A_{k+1} \phi_k(\lambda)\eta. \tag{8}$$

Observe that the function $\phi_k$ is holomorphic on $\Delta_r$ and that

$$||\phi_k(\lambda)|| \leq \frac{r}{r - |\lambda|} \quad (\lambda \in \Delta_r;\ k = 1,2,\dots).$$

This together with (2) implies that the series

$$\sum_{k=1}^{\infty} \lambda r^k A_{k+1} \phi_k(\lambda)$$

converges uniformly on each compact subset of $\Delta_r$. Hence the function $\phi$ given by (8) is well-defined and holomorphic on $\Delta_r$. From (7) it follows that

$$A(\lambda)\xi_0 = \phi(\lambda)\eta. \tag{9}$$

Conversely, it is not difficult to prove that (6) and (9) together imply that (3) holds. We summarize these results in the following lemma.

4.1. LEMMA. *Let* $\lambda \in \Delta_r$, $\xi \in E$ *and* $\eta \in F$. *Then*

$$\eta = L(\lambda)\xi = (T + \lambda S)\xi$$

*if and only if*

(i) $A(\lambda)\xi_0 = \phi(\lambda)\eta$,

(ii) $\xi_k = (\frac{\lambda}{r})^k \xi_0 + \phi_k(\lambda)\eta \qquad (k = 1,2,\ldots)$.

Let $\kappa$ denote the bounded linear operator from Y into F given by

$$\kappa y = (y,0,0,\ldots).$$

It is easily verified that

$$\phi(\lambda) \circ \kappa = I_Y \qquad (\lambda \in \Delta_r). \tag{10}$$

4.2. PROPOSITION. *Let* $\lambda \in \Delta_r$. *Then*

(i) $R(L(\lambda)) = \phi(\lambda)^{-1}[R(A(\lambda))]$;

(ii) $R(A(\lambda)) = \kappa^{-1}[R(L(\lambda))]$.

PROOF. It is immediate from Lemma 4.1 that

$$R(L(\lambda)) \subset \phi(\lambda)^{-1}[R(A(\lambda))].$$

Now suppose that $\eta \in F$ and $\phi(\lambda)\eta \in R(A(\lambda))$. Take $\xi_0$ in X such that $A(\lambda)\xi_0 = \phi(\lambda)\eta$, and put

$$\xi_k = (\frac{\lambda}{r})^k \xi_0 + \phi_k(\lambda)\eta \qquad (k = 1,2,\ldots).$$

Then

$$||\xi_k|| \leq ||\xi_0|| + \frac{r}{r - |\lambda|}\,||\eta|| \qquad (k = 0,1,2,\ldots),$$

and hence $\xi = \{\xi_k\}_{k=0}^{\infty} \in E$. Lemma 4.1 shows that $\eta = L(\lambda)\xi \in R(L(\lambda))$. This proves (i).

To prove (ii), we observe that, by (i),

$$\kappa^{-1}[R(L(\lambda))] = (\phi(\lambda) \circ \kappa)^{-1}[R(A(\lambda))].$$

But $\phi(\lambda) \circ \kappa = I_Y$. Hence (ii) holds, and the proof is complete.

For each $\lambda$ in $\Delta_r$, let the function $\psi(\lambda)$ from X into E be given by

$$\psi(\lambda)x = (x,\ \lambda r^{-1}x,\ \lambda^2 r^{-2}x, \ldots\ldots).$$

Then $\psi(\lambda)$ is a well-defined injective (bounded) linear operator from X into Y and

$$L(\lambda) \circ \psi(\lambda) = \kappa \circ A(\lambda) \qquad (\lambda \in \Delta_r). \tag{11}$$

4.3. PROPOSITION. *Let* $\lambda \in \Delta_r$. *Then* $N(L(\lambda)) = \psi(\lambda)[N(A(\lambda))]$.

PROOF. The inclusion $\psi(\lambda)[N(A(\lambda))] \subset N(L(\lambda))$ is immediately clear from formula (11). To prove the reverse inclusion, let $\xi$ be an element of $N(L(\lambda))$. Then Lemma 4.1 shows that

(i) $A(\lambda)\xi_0 = 0$,

(ii) $\xi_k = (\frac{\lambda}{r})^k \xi_0 \qquad (k = 1,2,\ldots)$.

From (i) we see that $\xi_0 \in N(A(\lambda))$, and according to (ii) we have $\psi(\lambda)\xi_0 = \xi$. Thus $N(L(\lambda)) \subset \psi(\lambda)[N(A(\lambda))]$, and the proof is complete.

We now come to the main results of this section.

4.4. THEOREM. *Let* $\lambda \in \Delta_r$. *Then*

(i) $A(\lambda)$ *has closed range if and only if* $L(\lambda)$ *has closed range;*

(ii) $n(A(\lambda)) = n(L(\lambda))$;

(iii) $d(A(\lambda)) = d(L(\lambda))$.

PROOF. Statement (i) follows from Proposition 4.2 and the continuity of the operators $\phi(\lambda)$ and $\kappa$. Statement (ii) is an immediate consequence of Proposition 4.3 and the injectivity of the operator $\psi(\lambda)$. The first part of Proposition 4.2 implies that $d(L(\lambda)) \leqq d(A(\lambda))$. The reverse inequality follows from the second part of the same lemma. This proves (iii).

4.5. THEOREM. *Let* n *be a positive integer. Then* 0 *is a pole of* $A^{-1}$ *of order* n *if and only if* 0 *is a pole of* $L^{-1}$ *of order* n.

PROOF. For $m = 0,1,2,\ldots$, let $\pi_m$ denote the bounded linear operator from E into X given by

$$\pi_m \xi = \xi_m.$$

Observe that a linear operator U from F into E is equal to the null operator if (and only if)

$$\pi_m \circ U = 0 \qquad (m = 0,1,2,\dots).$$

From Theorem 4.4 it follows that for $\lambda$ in $\Delta_r$ the operator $L(\lambda)$ is bijective if and only if the same is true for $A(\lambda)$. Hence, in order to prove the theorem, we may suppose that there exists $0 < \varepsilon < r$ such that $L(\lambda)$ and $A(\lambda)$ are both bijective for $0 < |\lambda| < \varepsilon$. But then we can apply Lemma 4.1 to show that for these values of $\lambda$

(i) $\pi_0 \circ L^{-1}(\lambda) = A^{-1}(\lambda) \circ \phi(\lambda)$,

(ii) $\pi_k \circ L^{-1}(\lambda) = (\frac{\lambda}{r})^k A^{-1}(\lambda) \circ \phi(\lambda) + \phi_k(\lambda) \quad (k = 1,2,\dots)$.

From (i) together with $\phi(\lambda) \circ \kappa = I_Y$ (see formula (10)), we obtain

(iii) $A^{-1}(\lambda) = \pi_0 \circ L^{-1}(\lambda) \circ \kappa$

for $0 < |\lambda| < \varepsilon$. The statements (i), (ii) and (iii) imply certain relations between the coefficients of the Laurent expansion of $A^{-1}$ at 0 and those of the Laurent expansion of $L^{-1}$ at 0. These relations, together with the observation made in the first paragraph of the present proof, imply the desired result.

The next theorem concerns the numbers $\alpha[A]$, $\delta[A]$ and $k[A]$. These numbers were introduced at the end of the preceding section.

4.6. THEOREM. (i) $\alpha[A] = \alpha[L]$;

(ii) $\delta[A] = \delta[L]$;

(iii) $k[A] = k[L]$.

PROOF. Recall that $A_n$ denotes the n-th coefficient of the Taylor expansion of A at 0.

(i) Let $\sigma$ be the linear operator from X into E defined by

$$\sigma x = (x,0,0,\dots\dots).$$

It is clear that $\sigma$ is injective. Hence, to prove (i), it suffices to show that

$$H_m[L] = \sigma H_m[A] \tag{12}$$

for $m = 0,1,2,\dots$ .

Observe that $\sigma = \psi(0)$. Using Proposition 4.3, it follows that

$$N(T) = \sigma N(A_0).$$

But $N(T) = H_0[L]$ and $N(A_0) = H_0[A]$. Thus (12) holds for $m = 0$.

Let $m$ be a positive integer. Take $x$ in $H_m[A]$. Then there exist $x_0,\dots,x_m$ in X such that $x_0 = x$ and

$$\sum_{i=0}^{n} A_i x_{n-i} = 0 \qquad (n = 0,\dots,m). \tag{13}$$

Define $\xi^0,\dots,\xi^m$ in E by

$$\xi_n^k = \begin{cases} r^{-n}x_{k-n} & \text{for } n = 0,\dots,k, \\ 0 & \text{for } n = k+1,k+2,\dots \end{cases} \qquad (k = 0,\dots,m). \tag{14}$$

It is clear that $\xi^0 = \sigma x_0 = \sigma x$. Further, a simple computation, based on formula (13), shows that

$$T\xi^0 = 0, \quad T\xi^i + S\xi^{i-1} = 0 \qquad (i = 1,\dots,m). \tag{15}$$

This implies that $\sigma x \in H_m[L]$. Thus $\sigma H_m[A]$ is a subset of $H_m[L]$.

Conversely, take $\xi$ in $H_m[L]$. Then there exist $\xi^0,\dots,\xi^m$ in E such that (15) is satisfied and $\xi^0 = \xi$. Define $x_0,\dots,x_m$ in X by

$$x_k = \xi_0^k \qquad (k = 0,\dots,m). \tag{16}$$

Since $T\xi^0 = 0$, we have

$$\xi^0 = (x_0,0,0,\dots\dots), \tag{17}$$

and so $\xi = \xi^0 = \sigma x_0$. Combining the equation $T\xi^1 + S\xi^0 = 0$ and formula (17), we obtain

$$\xi^1 = (x_1, r^{-1}x_0, 0, 0, \dots\dots).$$

Proceeding in this way, one proves that the elements $\xi^0,\dots,\xi^m$ are given by formula (14). But then, by computing the zero coordinates in formula (15), it follows that $x_0,\dots,x_m$ satisfy (13). Hence $x_0 \in H_m[A]$, and so $\xi = \sigma x_0 \in \sigma H_m[A]$. Thus $H_m[L]$ is a subset of $\sigma H_m[A]$, and the proof of (i) is complete.

(ii) Let $\pi$ be the linear operator from F into Y defined by

$$\pi\eta = \eta_0.$$

It is clear that $\pi$ is surjective. Hence, to prove (ii), it suffices to show that

$$H_m'[L] = \pi^{-1}H_m'[A] \tag{18}$$

for $m = 0,1,2,\dots$ .

Observe that $\pi = \phi(0)$. Using Proposition 4.2(i), it follows that

$$R(T) = \pi^{-1}R(A_0).$$

But $R(T) = H_0'[L]$ and $R(A_0) = H_0'[A]$. Thus (18) holds for $m = 0$.

Let m be a positive integer. Take $\eta$ in $\pi^{-1}H_m'[A]$. Then $\eta_0 = \pi\eta \in H_m'[A]$, and hence there exist $x_0,\dots,x_m$ in X such that

$$\sum_{i=0}^{n} A_i x_{n-i} = \delta_{nm}\eta_0 \qquad (n = 0,\dots,m). \tag{19}$$

Define $\xi^0,\dots,\xi^{m-1}$ in E by

$$\xi_n^k = \begin{cases} r^{-n}x_{k-n} & \text{for } n = 0,\dots,k, \\ 0 & \text{for } n = k+1,k+2,\dots . \end{cases} \qquad (k = 0,\dots,m-1). \tag{20}$$

Further, let $\xi^m \in E$ be given by

$$\xi_n^m = \begin{cases} x_m & \text{for } n = 0, \\ r^{-n}x_{m-n} + \eta_n & \text{for } n = 1,\dots,m, \\ \eta_n & \text{for } n = m+1, m+2,\dots . \end{cases} \tag{21}$$

A straightforward computation, based on formula (19), shows that

$$T\xi^0 = 0, \quad T\xi^i + S\xi^{i-1} = 0 \quad (i = 1,\dots,m-1), \quad T\xi^m + S\xi^{m-1} = \eta. \tag{22}$$

This implies that $\eta \in H'_m[L]$. Thus $\pi^{-1}H'_m[A]$ is a subset of $H'_m[L]$.

To prove the reverse inclusion, take $\eta$ in $H'_m[L]$. Then there exist $\xi^0,\dots,\xi^m$ in E such that (22) is satisfied. Define $x_0,\dots,x_m$ in X by formula (16). Using the same arguments as in the last paragraph of the proof of (i), one can show that the elements $\xi^0,\dots,\xi^{m-1}$ are given by formula (20). Combining this with the equation $T\xi^m + S\xi^{m-1} = \eta$, one sees that the element $\xi^m$ is given by formula (21). But then, again using (22), it follows that $x_0,\dots,x_m$ and $\eta_0$ satisfy (19). Hence $\pi\eta = \eta_0 \in H_m[A]$, and so $\eta \in \pi^{-1}H_m[A]$. We conclude that $H_m[L]$ is a subset of $\pi^{-1}H_m[A]$, and the proof of (ii) is complete.

(iii) Using the same arguments as in the proof of (i), one can show that

$$H_\infty[L] = \sigma H_\infty[A]. \tag{23}$$

Recall that

$$k[A] = \dim N(A_0)/H_\infty[A] = \dim H_0[A]/H_\infty[A]$$

and

$$k[L] = \dim N(T)/H_\infty[L] = \dim H_0[L]/H_\infty[L].$$

The desired result is now immediate from (12), (23) and the injectivity of the operator $\sigma$.

Define the function V on F by

$$V\eta = -r(\eta_1,\eta_2,\dots\dots).$$

Then V is a bounded linear operator from F onto E. From the definition of V and S it is clear that $VS = I_E$. This implies that S is injective and that the inverse of S, being the restriction of V to R(S), is continuous.

Let M be a closed subset of E, and take $\eta$ in the closure of SM. Then there exists a sequence $\{\xi^n\}_{n=1}^{\infty}$ in M such that

$$\lim_{n\to\infty} S\xi^n = \eta.$$

Observe that

$$V\eta = \lim_{n\to\infty} V(S\xi^n) = \lim_{n\to\infty} \xi^n.$$

Since M is closed, it follows that $V\eta \in M$. Further,

$$S(V\eta) = \lim_{n\to\infty} S\xi^n = \eta,$$

and hence $\eta \in SM$. We conclude that SM is a closed subset of F. This proves that S maps closed subsets of E onto closed subsets of F.

For later use we present two more lemmas. The first one deals with the subspaces $N_m[L]$ and $N'_m[L]$, the second with $R'_m[L]$. These subspaces were introduced at the end of the preceding section.

4.7. LEMMA. *Let* m *be a non-negative integer. Then* $N_m[L]$ *is a closed subspace of* E *and* $N'_m[L]$ *is a closed subspace of* F.

PROOF. From Proposition 2.2(v) we know that $N'_m[L] = SN_m[L]$. Since S maps closed sets onto closed sets, it suffices to show that $N_m[L]$ is closed. We prove this by induction.

By definition $N_0[L] = \{0\}$, and so $N_0[L]$ is closed. Let k be a non-negative integer such that $N_k[L]$ is closed. Then, on account of the property of S just mentioned, $SN_k[L]$ is closed too. Using the continuity of T, it follows that $T^{-1}SN_k[L]$ is closed. But $T^{-1}SN_k[L] = N_{k+1}[L]$ (cf. the remark at the end of Section 2). Hence $N_{k+1}[L]$ is closed, and the proof is complete.

4.8. LEMMA. *Let* m *be a non-negative integer. Then there exist a complex Banach space* Z *and a bounded linear operator* U *from* Z *into* F *such that the range of* U *is* $R'_m[L]$.

PROOF. Since $R'_0[L] = F$, we may assume that m is strictly positive. Let the sequence $\{M_k\}_{k=0}^{\infty}$ of subspaces of E be inductively defined by

$$M_0 = E, \quad M_k = T^{-1}SM_{k-1} \qquad (k = 1,2,\ldots\ldots).$$

Then $M_k$ is closed for each non-negative integer k. This follows by means of induction from the continuity of T and the fact that S maps closed sets onto closed sets.

Let W be the backwards shift on E multiplied by the scalar r. Thus W is the bounded linear operator on E given by

$$W(\xi_0,\xi_1,\ldots) = r(\xi_1,\xi_2,\ldots).$$

From the definition of T and S, one can easily deduce that, if $\xi$ and $\zeta$ are elements of E such that

$$T\xi + S\zeta = 0,$$

then $\zeta = W\xi$.

Define the bounded linear operator $T_m$ from E into F by

$$T_m = TW^{m-1}.$$

We shall prove that

$$T_m M_{m-1} = R_m'[L]. \tag{24}$$

Take $\eta$ in $R_m'[L]$. Then there exist $\xi^0,\ldots,\xi^{m-1}$ in E such that

$$T\xi^0 = \eta, \quad T\xi^i + S\xi^{i-1} = 0 \qquad (i = 1,\ldots,m-1).$$

Observe that $\xi^{m-1} \in M_{m-1}$. Further,

$$\xi^{i-1} = W\xi^i \qquad (i = 1,\ldots,m-1),$$

and hence $\xi^0 = W^{m-1}\xi^{m-1}$. Thus

$$\eta = T\xi^0 = TW^{m-1}\xi^{m-1} = T_m\xi^{m-1}.$$

Since $\xi^{m-1} \in M_{m-1}$, it follows that $\eta \in T_m M_{m-1}$. We conclude that $R_m'[L]$ is a subset of $T_m M_{m-1}$.

To prove the reverse inclusion, take $\xi$ in $M_{m-1}$. Then there exist

$\xi^0,\dots,\xi^{m-1}$ in E such that

$$\xi^0 = \xi, \quad S\xi^i + T\xi^{i-1} = 0 \qquad (i = 1,\dots,m-1).$$

Observe that $T\xi^{m-1} \in R_m'[L]$. Further,

$$\xi^i = W\xi^{i-1} \qquad (i = 1,\dots,m-1),$$

and hence $\xi^{m-1} = W^{m-1}\xi^0$. Thus

$$T_m\xi = T_m\xi^0 = TW^{m-1}\xi^0 = T\xi^{m-1}.$$

It follows that $T_m\xi \in R_m'[L]$. This proves (24).

Recall that E is a complex Banach space and that $M_{m-1}$ is a closed subspace of E. Hence $M_{m-1}$ is a complex Banach space too. The restriction of $T_m$ to $M_{m-1}$ is a bounded linear operator from $M_{m-1}$ into F. From (24) we know that the range of this operator is $R_m'[L]$. Thus, if we define Z to be $M_{m-1}$ and U to be the restriction of $T_m$ to $M_{m-1}$, then Z and U have the desired properties. This proves the lemma.

Several authors have studied operator polynomials by using a linearization method due to H. Wielandt (cf. [16], [25], [27], [29], [32] and [35]). K.-H. Förster [11] has extended Wielandt's method in such a way that it becomes applicable to operator power series. Independently, a similar extension has been carried out by G. Maibaum [28]. The method used here is a modification of those employed by Förster and Maibaum; Proposition 4.3, Theorem 4.4(ii) and Theorem 4.6(iii) are modifications of results appearing in [11]. The other results of this section seem to be new.

We conclude with a remark concerning Theorem 4.5. M.V. Pattabhiraman and P. Lancaster [35] and G. Maibaum [27] have linearized operator polynomials with bijective leading coefficient. They obtained results on poles similar to Theorem 4.5. For details, see Section 2 in [35] and Section 2.1 in [27].

5 CHARACTERIZATION OF POLES

Let T be a bounded linear operator on a complex Banach space Z. Recall that 0 is said to be a pole of T of order m if 0 is a pole of order m of the function

$$\lambda \longmapsto (\lambda I_Z - T)^{-1} \qquad (\lambda \in \mathbb{C} \setminus \sigma(T)).$$

The following theorem is known from spectral theory (cf. Section 2 in [26]; for earlier versions, see Section 5.8 in [42], Section 9 in [43] and Theorem I.25 in [9]).

5.1. THEOREM. *Let* m *be a positive integer. Then* 0 *is a pole of* T *of order* m *if and only if* $\alpha(T) = \delta(T) = m$.

Let A be the function defined by

$$A(\lambda) = \lambda I_Z - T \qquad (\lambda \in \mathbb{C}).$$

Then Theorem 5.1 says that 0 is a pole of order m of the resolvent $A^{-1}$ of A if and only if $\alpha[A] = \delta[A] = m$. The main purpose of this section is to prove that this result is true for an arbitrary locally holomorphic operator valued function A defined on an open neighbourhood of 0.

In the following X and Y are complex Banach spaces. Further, A is a locally holomorphic function defined on an open neighbourhood D of 0 with values in $L(X,Y)$.

5.2. THEOREM. *Let* m *be a positive integer. Then* 0 *is a pole of* $A^{-1}$ *of order* m *if and only if* $\alpha[A] = \delta[A] = m$.

PROOF. The proof consists of two parts. The first part deals with the "only if part", the second with the "if part" of the theorem.

(I) Suppose that 0 is a pole of $A^{-1}$ of order m. Let $A_k$ denote the k-th coefficient of the Taylor expansion of A at 0, and let $B_n$ denote the n-th coefficient of the Laurent expansion of $A^{-1}$ at 0. Since 0 is a pole of $A^{-1}$ of order m, we have

$$A^{-1}(\lambda) = \sum_{n=-m}^{+\infty} \lambda^n B_n$$

for $\lambda$ in some deleted neighbourhood U of 0. Observe that

$$A(\lambda)A^{-1}(\lambda) = I_Y, \quad A^{-1}(\lambda)A(\lambda) = I_X \qquad (\lambda \in U).$$

This implies that

$$\sum_{i=0}^{n} A_i B_{n-m-i} = \delta_{nm} I_Y \qquad (n = 0,\dots,m) \tag{1}$$

and

$$\sum_{i=0}^{n} B_{n-m-i} A_i = \delta_{nm} I_X \qquad (n = 0,\dots,m). \tag{2}$$

From (1) it is immediately clear that for each y in Y

$$\sum_{i=0}^{n} A_i B_{n-m-i} y = \delta_{nm} y \qquad (n = 0,\dots,m).$$

Hence $y \in H_m'[A]$ for each y in Y. Thus $H_m'[A] = Y$, and so $\delta[A] \leqq m$.

Further, it follows from (1) that for each y in Y

$$\sum_{i=0}^{n} A_i B_{n-m-i} y = 0 \qquad (n = 0,\dots,m-1).$$

This shows that $B_{-m} y \in H_{m-1}[A]$ for each y in Y. Since 0 is a pole of $A^{-1}$ of order m, the operator $B_{-m}$ is non-zero. Thus $H_{m-1}[A] \neq \{0\}$, and so $\alpha[A] \geqq m$.

Take x in $H_m[A]$. Then there are $x_0,\dots,x_m$ in X such that $x_0 = x$ and

$$\sum_{i=0}^{k} A_i x_{k-i} = 0 \qquad (k = 0,\dots,m).$$

Using (2), we infer

$$x = x_0 = \sum_{n=0}^{m} \delta_{nm} x_{m-n} = \sum_{n=0}^{m} \delta_{nm} I_X x_{m-n} =$$

$$= \sum_{n=0}^{m} \Big( \sum_{i=0}^{n} B_{n-m-i} A_i x_{m-n} \Big) =$$

$$= \sum_{k=0}^{m} B_{-k} \Big( \sum_{i=0}^{k} A_i x_{k-i} \Big) = 0.$$

Hence $H_m[A] = \{0\}$, and so $\alpha[A] \leqq m$.

Combining the preceding results, we obtain $\delta[A] \leqq m = \alpha[A]$. In particular, $\alpha[A]$ and $\delta[A]$ are both finite. By Theorem 1.2, this implies that $\alpha[A]$ and $\delta[A]$ are equal. We conclude that $\alpha[A] = \delta[A] = m$. This completes part (I) of the proof.

(II) Suppose that $\alpha[A] = \delta[A] = m$. Let the complex Banach spaces E and F, the operators T and S, and the function L be as in the preceding section. Recall that T and S are bounded linear operators from E into F and that

$$L(\lambda) = T + \lambda S \qquad (\lambda \in \mathbb{C}).$$

In view of Theorem 4.5, it suffices to show that 0 is a pole of order m of the resolvent $L^{-1}$ of L.

From statements (i) and (ii) of Theorem 4.6 we know that $\alpha[L] = \alpha[A]$ and $\delta[L] = \delta[A]$. Thus $\alpha[L] = \delta[L] = m$. By Theorem 2.10(i), this implies that

$$E = N_m[L] \oplus R_m[L], \qquad F = N'_m[L] \oplus R'_m[L]. \tag{3}$$

We shall prove that these decompositions are topological, i.e., we shall show that the subspaces $N_m[L]$, $R_m[L]$, $N'_m[L]$ and $R'_m[L]$ are closed.

From Lemma 4.7 we know that $N_m[L]$ and $N'_m[L]$ are both closed. Further, we know from Lemma 4.8 that there exist a complex Banach space Z and a bounded linear operator U from Z into F such that $R(U) = R'_m[L]$. Hence, using the second part of (3) and the fact that $N'_m[L]$ is closed, the range of U has a closed algebraic complement. This implies that R(U) is closed (see Theorem IV.1.12 in [13]; cf. also [22], page 276). Thus $R'_m[L]$ is closed. Using the continuity of S, it follows that $S^{-1}R'_m[L]$ is closed too. But $S^{-1}R'_m[L] = R_m[L]$ by Proposition 2.2(vii), and so $R_m[L]$ is closed. This proves that the decompositions in (3) are topological.

From statements (ii) and (iv) of Theorem 2.10 we know that both T and S map $R_m[L]$ into $R'_m[L]$. Let $T_R$ and $S_R$ denote the restrictions of T and S to $R_m[L]$. We consider $T_R$ and $S_R$ as linear operators from the complex Banach space $R_m[L]$ into the complex Banach space $R'_m[L]$. Observe that $T_R$ and $S_R$ are bounded. From Theorem 2.10(iv) we know that $T_R$ is bijective. Hence there exists $\varepsilon > 0$ such that $T_R + \lambda S_R$ is bijective for $|\lambda| < \varepsilon$.

From statements (iii) and (v) of Theorem 2.10 we know that both T and S map $N_m[L]$ into $N_m'[L]$. Let $T_N$ and $S_N$ denote the restrictions of T and S to $N_m[L]$. We consider $T_N$ and $S_N$ as linear operators from the complex Banach space $N_m[L]$ into the complex Banach space $N_m'[L]$. Observe that $T_N$ and $S_N$ are bounded. From statements (iii) and (v) of Theorem 2.10 we know that $S_N$ is bijective and that $T_N$ is nilpotent of index m relative to $S_N$. That is

$$(S_N^{-1}T_N)^{m-1} \neq 0, \quad (S_N^{-1}T_N)^m = 0. \tag{4}$$

For $\lambda \neq 0$, let $R_N(\lambda)$ be the bounded linear operator from $N_m'[L]$ into $N_m[L]$ given by

$$R_N(\lambda) = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{\lambda^n} (S_N^{-1}T_N)^{n-1}S_N^{-1}.$$

A straightforward computation, based on the second part of (4), shows that

$$R_N(\lambda)(T_N + \lambda S_N) = I_{N_m[L]}, \quad (T_N + \lambda S_N)R_N(\lambda) = I_{N_m'[L]} \quad (\lambda \neq 0).$$

Hence $T_N + \lambda S_N$ is bijective and

$$(T_N + \lambda S_N)^{-1} = R_N(\lambda) \qquad (\lambda \neq 0).$$

Let P be the projection of F onto $R_m'[L]$ along $N_m'[L]$. Since the decomposition of F given in (3) is topological, this projection is continuous. Combining the preceding results, we see that $L(\lambda)$ is bijective for $0 < |\lambda| < \varepsilon$ and that for these values of $\lambda$

$$L^{-1}(\lambda) = (T + \lambda S)^{-1} = J_N R_N(\lambda)(I_F - P) + J_R(T_R + \lambda S_R)^{-1}P.$$

Here $J_N$ and $J_R$ are the canonical embeddings of $N_m[L]$ and $R_m[L]$ into E. The (-m)-th coefficient of the Laurent expansion of $L^{-1}$ at 0 is

$$(-1)^{m-1}J_N(S_N^{-1}T_N)^{m-1}S_N^{-1}(I_F - P).$$

Hence, using the first part of (4), this coefficient is non-zero. It follows that 0 is a pole of $L^{-1}$ of order m, and part (II) of the proof is complete.

It is interesting to observe that in the definition of the subspaces $H_m[A]$ and $H'_m[A]$ only the first m+1 coefficients of the Taylor expansion of A at 0 appear. In the following we shall draw some conclusions from this observation.

Let B be a locally holomorphic function defined on an open neighbourhood of 0 with values in $L(X,Y)$, and let m be a positive integer. We say that B is an *m-th order approximation of* A *(at* 0*)* if

$$A_n = B_n \qquad (n = 0,\dots,m).$$

Here $A_n$ and $B_n$ denote the n-th coefficient of the Taylor expansion of A and B at 0, respectively. It is clear that, if B is an m-th order approximation of A, then, in turn, A is an m-th order approximation of B.

Suppose that B is an m-th order approximation of A. Then it is immediate from the observation made above that

$$H_k[A] = H_k[B], \quad H'_k[A] = H'_k[B] \qquad (k = 0,\dots,m).$$

Hence $\alpha[A] = m$ if and only if $\alpha[B] = m$, and $\delta[A] = m$ if and only if $\delta[B] = m$. Combining this with Theorem 5.2, we obtain the following result.

5.3. COROLLARY. *Let* m *be a positive integer, and let* B *be an* m*-th order approximation of* A. *Then* 0 *is a pole of* $A^{-1}$ *of order* m *if and only if* 0 *is a pole of* $B^{-1}$ *of order* m.

Next we present two results concerning the case when $A_0$ is a Fredholm operator. Here $A_0 = A(0)$.

5.4. PROPOSITION. *Suppose that* $A_0$ *is a Fredholm operator and that* $\mathrm{ind}(A_0) = 0$. *Further, suppose that* $0 < \alpha[A] < +\infty$. *Then*

(i) 0 *is a pole of* $A^{-1}$ *of order* $\alpha[A] = \delta[A]$;

(ii) 0 *is a Riesz-point of* $A^{-1}$.

PROOF. It is clear that 0 is a Riesz-point of A. Hence, by Lemma II.3.1, there exists a neighbourhood U of 0 such that each $\lambda$ in U is a Riesz-point of A. We may assume that U is a region. Then A is Riesz-meromorphic on U. Thus, by Theorem II.3.4, it suffices to show that (i) is true.

The hypotheses concerning $A_0$ imply that $\alpha[A] = \delta[A]$. This appears from Theorem 1.3. So we have

$$0 < \alpha[A] = \delta[A] < +\infty.$$

Hence, by Theorem 5.2, statement (i) is correct, and the proof is complete.

The preceding proposition remains true if $\alpha[A]$ is replaced by $\delta[A]$. This is immediate from the fact that $\alpha[A] = \delta[A]$, provided that $A_0$ is a Fredholm operator with index 0 (cf. Theorem 1.3).

The set Sp[A] of all $\lambda$ in D such that $A(\lambda)$ is not bijective is called the *spectrum* of A. Observe that Sp[A] is the complement in D of the resolvent set of A. Thus

$$Sp[A] = D \setminus Res[A].$$

It is clear that Sp[A] is closed in the relative topology of D.

5.5. PROPOSITION. *Suppose that* $A_0$ *is a Fredholm operator. Further, suppose that at least one of the extended integers* $\alpha[A]$ *and* $\delta[A]$ *is infinite. Then* 0 *is an interior point of* Sp[A].

PROOF. We consider only the case when $\alpha[A] = +\infty$; the other case can be treated similarly.

Since $N(A_0)$ is finite-dimensional, there exists a non-negative integer p such that

$$H_m[A] = H_p[A] \qquad (m = p, p+1, \ldots),$$

and hence it follows that

$$H[A] = \bigcap_{m=0}^{\infty} H_m[A] = H_p[A].$$

Since $\alpha[A] = +\infty$, we have $H_p[A] \neq \{0\}$, and so $H[A] \neq \{0\}$. Thus

$$\dim N(A_0)/H[A] < \dim N(A_0) = n(A_0).$$

By Theorem 3.7, we have

$$k[A] = \dim N(A_0)/H[A].$$

Hence $k[A] < n(A_0)$. Now one can apply Förster's perturbation theorem (Theorem 3.8) to show that $n(A(\lambda)) > 0$ for $\lambda$ in some deleted neighbourhood $V$ of 0. This shows that $\lambda \in Sp[A]$ for $\lambda$ in $V$. Our hypotheses imply that $0 \in Sp[A]$. Hence the proof is complete.

We conclude this section with two remarks. The first one concerns Proposition 5.5. Without the condition that $A_0$ is a Fredholm operator, this proposition does not hold. To see this, consider the case when $X = Y$ and

$$A(\lambda) = \lambda I_X - T \qquad (\lambda \in \mathbb{C}),$$

where $T$ is a quasi-nilpotent bounded linear operator on $X$ with $\alpha(T) = \delta(T) = +\infty$. Then

$$\alpha[A] = \alpha(T), \quad \delta[A] = \delta(T), \quad Sp[A] = \sigma(T).$$

Hence $\alpha[A] = \delta[A] = +\infty$ and 0 is an isolated point of $Sp[A]$. It is not difficult to construct an operator $T$ with the desired properties. Take for instance $X = l_\infty$ and

$$(Tx)_k = \frac{x_{k+1}}{k} \qquad (x \in l_\infty;\ k = 1,2,\ldots).$$

The second remark concerns Corollary 5.3. A first step in the direction of this result appears in [19]. In this paper J.S. Howland studied the case when $X = Y$ and $A$ is of the form

$$A(\lambda) = I_X + K(\lambda),$$

where the values of the function $K$ are compact linear operators on $X$. Among other things he proved that in this case 0 is a simple pole (i.e., a pole of order one) of $A^{-1}$ if and only if 0 is a simple pole of the resolvent of the function

$$\lambda \longmapsto A_0 + \lambda A_1 \qquad (\lambda \in \mathbb{C})$$

(see Corollary 3.3 in [19]). Here the operators $A_0$ and $A_1$ are the first two coefficients of the Taylor expansion of $A$ at 0.

## 6 ASCENT AND DESCENT IN THE COMMUTATIVE CASE

Throughout this section $T, T_1, T_2, \ldots$ are linear operators on a (complex) linear space E. The sequence $\{T_n\}_{n=1}^{\infty}$ will be denoted by $\mathcal{T}$, and we shall write $T_0$ instead of T whenever this is convenient. We say that T *commutes with the sequence* $\mathcal{T}$ if

$$T_n T = T T_n \qquad (n = 1,2,3,\ldots).$$

The aim of this section is to show that in the commutative case there are interesting relations between $\alpha[T;\mathcal{T}]$ and $\alpha(T)$, and also between $\delta[T;\mathcal{T}]$ and $\delta(T)$.

In the following we shall use the fact that, if T commutes with the sequence $\mathcal{T}$, then there exist linear operators $S_k^{(m)}$ on E ($m = 0,1,2,\ldots$; $k = 0,\ldots,m$) such that

$$S_0^{(m)} = T^m \qquad (m = 0,1,2,\ldots) \tag{1}$$

and

$$\sum_{i=0}^{n} T_i S_{n-i}^{(m)} = 0 \qquad (n = 1,\ldots,m;\ m = 1,2,\ldots). \tag{2}$$

The definition of these operators is somewhat complicated.

Observe that $S_1^{(m)},\ldots,S_m^{(m)}$ have to satisfy the following system of equations:

$$\sum_{i=1}^{n} T_{n-i} X_i = -T_n T^m \qquad (n = 1,\ldots,m). \tag{3}$$

Viewing (3) as a system of linear equations in complex numbers, one can write down the solution by using Cramer's rule. A careful examination of this solution leads to the following definition:

$$S_0^{(0)} = I_E,$$

$$S_k^{(m)} = T S_k^{(m-1)} \qquad (k = 0,\ldots,m-1;\ m = 1,2,\ldots),$$

$$S_m^{(m)} = -\sum_{i=1}^{m} T_i S_{m-i}^{(m-1)} \qquad (m = 1,2,\ldots).$$

It is clear that the operators $S_k^{(m)}$ defined in this manner satisfy (1). Further, a simple computation shows that (2) holds too, provided that T commutes with the sequence $\mathcal{T}$.

First we investigate the relationships between $\alpha[T;\mathcal{T}]$ and $\alpha(T)$.

6.1. LEMMA. *Let* m *be a non-negative integer. Suppose that* T *commutes with the sequence* $\mathcal{T}$. *Then* $N(T) \cap R(T^m) \subset H_m$.

PROOF. The statement is trivially true for m = 0. Therefore we assume that m is strictly positive.

Take x in $N(T) \cap R(T^m)$. Choose u in E such that $x = T^m u$ and define $x_0,\dots,x_m$ in E by

$$x_k = S_k^{(m)} u \qquad (k = 0,\dots,m).$$

Then $x_0 = S_0^{(m)} u = T^m u = x$ and

$$\sum_{i=0}^{n} T_i x_{n-i} = \sum_{i=0}^{n} T_i S_{n-i}^{(m)} u = 0 \qquad (n = 1,\dots,m).$$

Moreover, since $x \in N(T)$,

$$T_0 x_0 = Tx = 0.$$

Hence $x \in H_m$, and the proof is complete.

6.2. PROPOSITION. *Suppose that* T *commutes with the sequence* $\mathcal{T}$. *Then* $\alpha(T) \leqq \alpha[T;\mathcal{T}]$.

PROOF. By definition,

$$\alpha[T;\mathcal{T}] = \min \{m : H_m = \{0\}\}.$$

If m is a non-negative integer such that $H_m = \{0\}$, then, according to the preceding lemma, $N(T) \cap R(T^m) = \{0\}$ too. Now the desired result is immediate from formula (1) in Section 1.

The inequality in the preceding proposition may be strict. An example showing this will be given at the end of this section (Example 6.11).

Next we shall give a sufficient condition in order that $\alpha[T;\mathcal{T}]$ and $\alpha(T)$ coincide. We start with two lemmas.

6.3. LEMMA. *Let* m *be a non-negative integer, and let* $x_0,\ldots,x_m$ *be elements of* E *such that*

$$\sum_{i=0}^{n} T_i x_{n-i} = 0 \qquad (n = 0,\ldots,m). \tag{4}$$

*Suppose that* T *commutes with the sequence* $\mathcal{T}$. *Then*

$$T^n x_k = 0 \qquad (k = 0,\ldots,m;\ n = k+1,k+2,\ldots).$$

PROOF. Evidently, it suffices to show that

$$T^{k+1} x_k = 0 \tag{5}$$

for $k = 0,\ldots,m$. We prove this by induction.

From (4) it is clear that $Tx_0 = 0$, and hence (5) holds for $k = 0$. Let p be a non-negative integer less than m such that

$$T^{k+1} x_k = 0 \qquad (k = 0,\ldots,p). \tag{6}$$

From (4) we see that

$$Tx_{p+1} + T_1 x_p + \ldots.. + T_{p+1} x_0 = 0.$$

Applying $T^{p+1}$ to both sides of this equality and using the hypothesis that T commutes with the sequence $\mathcal{T}$, we obtain

$$T^{p+2} x_{p+1} + T_1 T^{p+1} x_p + \ldots.. + T_{p+1} T^{p+1} x_0 = 0.$$

By formula (6), the left hand side of this equation is equal to $T^{p+2}x_{p+1}$. Thus $T^{p+2}x_{p+1} = 0$. This completes the proof.

6.4. LEMMA. *Suppose that* T *commutes with the sequence* $\mathcal{T}$ *and that* $\alpha(T) = m < +\infty$. *Then* $H_m \subset N(T_1^m)$.

PROOF. Take x in $H_m$. Then there exist $x_0,\ldots,x_m$ in E such that $x_0 = x$ and

$$\sum_{i=0}^{n} T_i x_{n-i} = 0 \qquad (n = 0,\dots,m). \tag{7}$$

By induction we shall prove that for $k = 0,\dots,m$

$$T^{m-k}T_1^k x_{m-k} = 0. \tag{8}$$

According to the preceding lemma, we have

$$T^n x_k = 0 \qquad (k = 0,\dots,m;\ n = k+1,k+2,\dots). \tag{9}$$

From $T^{m+1}x_m = 0$ and $m = \alpha(T)$, it follows that $T^m x_m = 0$. Thus (8) holds for $k = 0$. Let p be a non-negative integer less than m such that

$$T^{m-p}T_1^p x_{m-p} = 0. \tag{10}$$

From (7) we see that

$$Tx_{m-p} + T_1 x_{m-p-1} + \dots\dots + T_{m-p}x_0 = 0.$$

Applying $T^{m-p-1}T_1^p$ to both sides of this equality and using the hypothesis that T commutes with the sequence $\mathcal{T}$, we obtain

$$T^{m-p}T_1^p x_{m-p} + T^{m-p-1}T_1^{p+1}x_{m-p-1} + \dots\dots + T_1^p T_{m-p}T^{m-p-1}x_0 = 0.$$

But (9) and (10) together imply that the left hand side of this equation is equal to $T^{m-p-1}T_1^{p+1}x_{m-p-1}$. Thus $T^{m-p-1}T_1^{p+1}x_{m-p-1} = 0$. This proves (8) for $k = 0,\dots,m$. In particular, we have shown that

$$T_1^m x_0 = T^0 T_1^m x_0 = 0.$$

Hence $x_0 \in N(T_1^m)$, and the proof is complete.

6.5. THEOREM. *Suppose that* T *commutes with the sequence* $\mathcal{T}$ *and that* $T_1$ *is injective. Then* $\alpha[T;\mathcal{T}] = \alpha(T)$.

PROOF. According to Proposition 6.2, we have $\alpha(T) \leqq \alpha[T;\mathcal{T}]$. Assume that this inequality is strict. Then it follows that $\alpha(T)$ is finite.

Let $m = \alpha(T)$. Then the preceding lemma shows that $H_m$ is a subset of $N(T_1^m)$. Since $T_1$ is injective, this implies that $H_m = \{0\}$. Hence $\alpha[T;\mathcal{T}] \leqq m = \alpha(T)$, which contradicts the assumption. This proves the theorem.

We proceed by studying the relationships between $\delta[T;\mathcal{T}]$ and $\delta(T)$.

6.6. LEMMA. *Let* m *be a non-negative integer. Suppose that* T *commutes with the sequence* $\mathcal{T}$. *Then* $H'_m \subset R(T) + N(T^m)$.

PROOF. The statement is trivially true for $m = 0$. Therefore we assume that m is strictly positive.

Take y in $H'_m$. Then there exist $x_0,\dots,x_m$ in E such that

$$\sum_{i=0}^{n} T_i x_{n-i} = 0 \qquad (n = 0,\dots,m-1)$$

and

$$y = T_0x_m + T_1x_{m-1} + \dots\dots + T_mx_0.$$

It follows from Lemma 6.3 that

$$T^n x_k = 0 \qquad (k = 0,\dots,m-1;\ n = k+1,k+2\dots).$$

But then, using the hypothesis that T commutes with the sequence $\mathcal{T}$,

$$T^m y = T^{m+1}x_m + T_1T^m x_{m-1} + \dots\dots + T_mT^m x_0 = T^{m+1}x_m.$$

Hence $T^m(y - Tx_m) = 0$, and so $y \in R(T) + N(T^m)$. This proves the lemma.

6.7. PROPOSITION. *Suppose that* T *commutes with the sequence* $\mathcal{T}$. *Then* $\delta(T) \leqq \delta[T;\mathcal{T}]$.

PROOF. By definition,

$$\delta[T;\mathcal{T}] = \min\{m : H'_m = E\}.$$

If m is a non-negative integer such that $H'_m = E$, then, according to the preceding lemma, $R(T) + N(T^m) = E$ too. Now the desired result is immediate from formula (2) in Section 1.

The inequality in the preceding proposition may be strict (see Example 6.11 at the end of this section). The next theorem gives a sufficient condition in order that $\delta[T;\mathcal{T}]$ and $\delta(T)$ coincide. It is an easy consequence of Proposition 6.7 and the following lemma. In the proof of this lemma we use the existence of a certain sequence $V_0, V_1, V_2, \ldots$ of linear operators on E. The elements of this sequence are inductively defined by the following formulas:

$$V_0 = V_1 = 0, \quad V_n = -T_1V_{n-1} - \sum_{i=2}^{n} T_iS_{n-i}^{(n-2)} \qquad (n = 2,3,\ldots).$$

Here $S_k^{(m)}$ $(m = 0,1,2,\ldots;\ k = 0,\ldots,m)$ are the operators defined in the third paragraph of this section. Recall that

$$S_k^{(m)} = TS_k^{(m-1)} \qquad (k = 0,\ldots,m-1;\ m = 1,2,\ldots)$$

and

$$S_m^{(m)} = -\sum_{i=1}^{m} T_iS_{m-i}^{(m-1)} \qquad (m = 1,2,\ldots). \tag{11}$$

Suppose that T commutes with the sequence $\mathcal{T}$. Then, using the above formulas, it is not difficult to prove that

$$TV_n + T_1TV_{n-1} = S_n^{(n)} + T_1S_{n-1}^{(n-1)} \qquad (n = 2,3,\ldots).$$

Thus

$$TV_n - S_n^{(n)} = -T_1\{TV_{n-1} - S_{n-1}^{(n-1)}\} \qquad (n = 2,3,\ldots),$$

and hence

$$TV_n - S_n^{(n)} = (-T_1)^{n-1}\{TV_1 - S_1^{(1)}\} \qquad (n = 1,2,\ldots).$$

Since $V_1 = 0$ and $S_1^{(1)} = -T_1$, this implies that for $n = 1,2,\ldots$

$$TV_n - S_n^{(n)} = (-1)^{n+1}T_1^n.$$

From the definitions it is clear that this equality also holds for $n = 0$.

It follows that

$$TV_n - S_n^{(n)} = (-1)^{n+1}T_1^n \qquad (n = 0,1,2,\dots), \qquad (12)$$

provided that T commutes with the sequence $\mathcal{T}$.

6.8. LEMMA. *Suppose that* T *commutes with the sequence* $\mathcal{T}$ *and that* $\delta(T) = m < +\infty$. *Then* $R(T_1^m) \subset H'_m$.

PROOF. We may assume that m is strictly positive.

Take y in $R(T_1^m)$ and choose x in E such that $y = T_1^m x$. Since $m = \delta(T)$, we have $E = R(T) + N(T^m)$, and hence there exist u and v in E such that

$$x = Tu + v, \quad T^m v = 0.$$

Define $x_0,\dots,x_m$ in E by

$$x_i = \begin{cases} S_i^{(m-1)}v & \text{for } i = 0,\dots,m-1, \\ V_m v + (-1)^{m+1}T_1^m u & \text{for } i = m. \end{cases}$$

Then, by formula (1),

$$Tx_0 = TS_0^{(m-1)}v = T^m v = 0,$$

and, by formula (2),

$$\sum_{i=0}^{n} T_i x_{n-i} = \sum_{i=0}^{n} T_i S_{n-i}^{(m-1)} v = 0 \qquad (n = 1,\dots,m-1).$$

Thus

$$\sum_{i=0}^{n} T_i x_{n-i} = 0 \qquad (n = 0,\dots,m-1).$$

Next we observe that

$$\sum_{i=0}^{m} T_i x_{m-i} = Tx_m + \sum_{i=1}^{m} T_i x_{m-i} =$$

$$= TV_m v + (-1)^{m+1}TT_1^m u + \sum_{i=1}^{m} T_i S_{m-i}^{(m-1)} v.$$

According to formula (11), the last term is equal to $-S_m^{(m)}v$. Thus

$$\sum_{i=0}^{m} T_i x_{m-i} = (TV_m - S_m^{(m)})v + (-1)^{m+1} T T_1^m u.$$

But then, using formula (12) and the fact that T commutes with $T_1$,

$$\sum_{i=0}^{m} T_i x_{m-i} = (-1)^{m+1} T_1^m v + (-1)^{m+1} T T_1^m u =$$

$$= (-1)^{m+1} T_1^m (v + Tu) =$$

$$= (-1)^{m+1} T_1^m x =$$

$$= (-1)^{m+1} y.$$

Thus $(-1)^{m+1}y \in H'_m$. Hence $y \in H'_m$, and the proof is complete.

6.9. THEOREM. *Suppose that* T *commutes with the sequence* $\mathcal{T}$ *and that* $T_1$ *is surjective. Then* $\delta[T;\mathcal{T}] = \delta(T)$.

PROOF. According to Proposition 6.7, we have $\delta(T) \leq \delta[T;\mathcal{T}]$. Assume that this inequality is strict. Then it follows that $\delta(T)$ is finite. Put $m = \delta(T)$. The preceding lemma shows that $R(T_1^m)$ is a subset of $H'_m$. Since $T_1$ is surjective, this implies that $H'_m = E$. Hence $\delta[T;\mathcal{T}] \leq m = \delta(T)$, which contradicts the assumption. This proves the theorem.

We conclude this section with two examples.

6.10. EXAMPLE. Let E be $\mathbb{C}^5$, and let

$$T_n = 0 \qquad (n = 2,3,\dots).$$

Further, let T and $T_1$ on $\mathbb{C}^5$ be given by

$$T(x_1,x_2,x_3,x_4,x_5) = (x_2,x_3,0,0,0)$$

and

$$T_1(x_1,x_2,x_3,x_4,x_5) = (x_5,x_4,x_3,x_2,x_1).$$

It is easily verified that $\alpha(T) = \delta(T) = 3$ and $\alpha[T;\mathcal{T}] = \delta[T;\mathcal{T}] = 2$. Observe that $T_1$ is bijective and that $T$ and $T_1$ do not commute. This example shows that the hypothesis in Propositions 6.2 and 6.7, and Theorems 6.5 and 6.9, that $T$ commutes with the sequence $\mathcal{T}$ cannot be omitted.

6.11. EXAMPLE. Let $E$ be non-trivial, let each of the operators $T, T_1, T_3, \ldots$ be equal to the null operator on $E$, and let $T_2$ be the identity operator on $E$. Then $\alpha(T) = \delta(T) = 1$ and $\alpha[T;\mathcal{T}] = \delta[T;\mathcal{T}] = 2$. Observe that $T$ commutes with the sequence $\mathcal{T}$. This example shows that the inequalities in Propositions 6.2 and 6.7 may be strict.

## 7. ISOLATED POINTS OF THE SPECTRUM IN THE COMMUTATIVE CASE

Throughout this section $X$ is a complex Banach space. Further, $A$ is a locally holomorphic function defined on an open neighbourhood $D$ of $0$ with values in $L(X)$. The n-th coefficient of the Taylor expansion of $A$ at $0$ will be denoted by $A_n$. In particular, $A_0 = A(0)$ and $A_1 = A'(0)$, where $A'$ denotes the derivative of the function $A$. We say that $A$ is *commutative at* $0$ if

$$A(\lambda)A_0 = A_0A(\lambda) \qquad (\lambda \in D). \tag{1}$$

It is not difficult to see that (1) implies that

$$A_nA_0 = A_0A_n \qquad (n = 1,2,\ldots). \tag{2}$$

If $D$ is connected, then (1) and (2) are equivalent. Observe that (2) means that $A_0$ commutes with the sequence $\{A_n\}_{n=1}^{\infty}$.

In this section we study the case when $A$ is commutative at $0$. Our aim is to get information about the relationships between the spectral properties of the operator $A_0$ and those of the function $A$.

7.1. THEOREM. *Suppose that* $A$ *is commutative at* $0$ *and that* $0$ *is a pole of* $A^{-1}$ *of order* $m$. *Then* $\alpha(A_0) = \delta(A_0) \leq m$.

PROOF. Since $A$ is commutative at $0$, one can apply Propositions 6.2 and 6.7 to show that

$$\alpha(A_0) \leq \alpha[A], \quad \delta(A_0) \leq \delta[A].$$

According to Theorem 5.2, we have $\alpha[A] = \delta[A] = m$, and so

$$\alpha(A_0) \leq m, \quad \delta(A_0) \leq m.$$

In particular, $\alpha(A_0)$ and $\delta(A_0)$ are both finite. Hence $\alpha(A_0) = \delta(A_0)$, and the proof is complete.

The next result is a straightforward application of Theorem 5.1 and the preceding theorem.

7.2. COROLLARY. *Suppose that* $A$ *is commutative at* 0 *and that* 0 *is a pole of* $A^{-1}$ *of order* $m$. *Then* 0 *is a pole of the operator* $A_0$ *of order not exceeding* $m$.

The inequality in Theorem 7.1 may be strict. Further, Theorem 7.1 and Corollary 7.2 do not hold without the condition that $A$ is commutative at 0. Examples showing this will be given at the end of this section.

Next we investigate the case when $A$ is commutative at 0 and $A_1 = A'(0)$ is bijective. Our first result is a generalization of Theorem 5.1.

7.3. THEOREM. *Suppose that* $A$ *is commutative at* 0 *and that* $A_1$ *is bijective. Let* $m$ *be a positive integer. Then* 0 *is a pole of* $A^{-1}$ *of order* $m$ *if and only if* $\alpha(A_0) = \delta(A_0) = m$.

PROOF. From Theorem 5.2 we know that 0 is a pole of $A^{-1}$ of order $m$ if and only if

$$\alpha[A] = \delta[A] = m. \tag{3}$$

Since $A$ is commutative at 0, formula (2) holds. Thus $A_0$ commutes with the sequence $\{A_n\}_{n=1}^{\infty}$. Further, by hypothesis, $A_1$ is both injective and surjective. Now one can apply Theorems 6.5 and 6.9 to show that $\alpha[A] = \alpha(A_0)$ and $\delta[A] = \delta(A_0)$. Hence (3) is equivalent to

$$\alpha(A_0) = \delta(A_0) = m.$$

This proves the theorem.

7.4. COROLLARY. *Suppose that* A *is commutative at* 0 *and that* $A_1$ *is bijective. Let* m *be a positive integer. Then* 0 *is a pole of* $A^{-1}$ *of order* m *if and only if* 0 *is a pole of the operator* $A_0$ *of order* m.

We proceed with a definition. The function A is said to be *commutative* (on D) if

$$A(\lambda)A(\mu) = A(\mu)A(\lambda) \qquad (\lambda,\mu \in D).$$

It is clear that, if A is commutative, then A is commutative at 0 too.

The case when A is commutative has been studied by L. Mittenthal [31]. Among other things, he proved that the conclusion of Theorem 7.3 holds, provided that A is commutative and $A_1$ is bijective (cf. Theorems 7 and 13 in [31]). The methods used in Mittenthal's paper [31] differ considerably from those used here. In particular, we have not used the generalized spectral theory developed in [31].

Next we present some results concerning isolated points of Sp[A]. We begin with the following lemma.

7.5. LEMMA. *Suppose that* A *is commutative at* 0 *and that* $A_1$ *is bijective. Further, suppose that* $\sigma(A_0) = \{0\}$. *Then* 0 *is an isolated point of* Sp[A].

PROOF. Recall that $A_n$ denotes the n-th coefficient of the Taylor expansion of A at 0. Define the function B by

$$B(\lambda) = A(\lambda) - A_0 \qquad (\lambda \in D).$$

Then there exists $\delta > 0$ such that

$$B(\lambda) = \lambda(A_1 + \lambda A_2 + \ldots\ldots) \qquad (|\lambda| < \delta).$$

Since $A_1$ is bijective, this implies that 0 is an isolated point of Sp[B].

Take $\lambda$ in Res[B]. Then $A(\lambda)$ is the sum of the bijective operator $B(\lambda)$ and the quasi-nilpotent operator $A_0$. Moreover, the operators $B(\lambda)$ and $A_0$ commute. It follows that $A(\lambda)$ is bijective. Hence Res[B] is a subset of Res[A].

Our hypotheses imply that $A_0$ is not bijective. Thus $0 \in$ Sp[A]. Combining this with the preceding results, we see that 0 is an isolated point of Sp[A]. This proves the lemma.

Let T be a linear operator on X, and let M be a subspace of X invariant under T. In the following we shall use the symbol $T|M$ to denote the restriction of T to M considered as a linear operator on M.

7.6. THEOREM. *Suppose that* A *is commutative at* 0 *and that* $A_1$ *is bijective. Further, suppose that* 0 *is an isolated point of* $\sigma(A_0)$. *Then* 0 *is an isolated point of* Sp[A].

PROOF. Let P be the spectral projection associated with $A_0$ and 0. Put $M = R(P)$ and $N = N(P)$. Then M and N are closed subspaces of X and $X = M \oplus N$. Further we know that M and N are both invariant under $A_0$, that $A_0|N$ is bijective and that

$$\sigma(A_0|M) = \{0\}. \tag{4}$$

Since A is commutative at 0, we have

$$A(\lambda)P = PA(\lambda) \qquad (\lambda \in D),$$

and hence

$$A(\lambda)M \subset M, \quad A(\lambda)N \subset N \qquad (\lambda \in D). \tag{5}$$

Define the functions $A_M$ and $A_N$ on D by

$$A_M(\lambda) = A(\lambda)|M, \quad A_N(\lambda) = A(\lambda)|N.$$

Then $A_M$ and $A_N$ are locally holomorphic functions with values in $L(M)$ and $L(N)$, respectively. From operator theory we know that $A(\lambda)$ is bijective if and only if $A_M(\lambda)$ and $A_N(\lambda)$ are both bijective. Thus

$$\text{Res}[A] = \text{Res}[A_M] \cap \text{Res}[A_N].$$

Since $A_N(0) = A_0|N$ is bijective, we have $0 \in \text{Res}[A_N]$. Hence, to complete the proof, it suffices to show that 0 is an isolated point of $\text{Sp}[A_M]$.

Observe that $A_M$ is commutative at 0. Let $B_0$ be the first coefficient of the Taylor expansion of $A_M$ at 0, and let $B_1$ be the second. Then $B_0 = A_0|M$ and hence, by formula (4),

$$\sigma(B_0) = \{0\}.$$

From (5) we see that M and N are both invariant under $A_1$. By hypothesis $A_1$ is bijective, and so $A_1|M$ is bijective too. Since $B_1 = A_1|M$, it follows that $B_1$ is bijective. Now Lemma 7.5 shows that 0 is an isolated point of $Sp[A_M]$, and the proof is complete.

We do not know whether the converse of the preceding theorem is true. However, using a method of L. Mittenthal (see the proof of Theorem 13 in [31]), we can prove the following partial converse.

7.7. THEOREM. *Suppose that* A *is commutative and that* $A_1$ *is bijective. Further, suppose that* 0 *is an isolated point of* Sp[A]. *Then* 0 *is an isolated point of* $\sigma(A_0)$.

PROOF. Define the bounded linear operator P on X by

$$P = \frac{1}{2\pi i} \int_{|\lambda|=r} A'(\lambda)A^{-1}(\lambda)d\lambda, \tag{6}$$

where r is a positive real number such that

$$\{\lambda \in \mathbb{C} : 0 < |\lambda| < 2r\} \subset \mathrm{Res}[A].$$

Since A is commutative and $A_1 = A'(0)$ is bijective, the operator P is a projection of X. This was proved by L. Mittenthal (cf. [31], pp. 122,123). Put M = R(P) and N = N(P). Then M and N are closed subspaces of X and $X = M \oplus N$. Using (6) and the commutativity of A, we see that $A_0$ and P commute. Hence M and N are both invariant under $A_0$. To prove the theorem, it suffices to show that $A_0|N$ is bijective and that $A_0|M$ is quasi-nilpotent.

Define the function L on Res[A] by

$$L(\lambda) = A'(\lambda)A^{-1}(\lambda).$$

It is clear that L is a locally holomorphic function with values in $L(X)$. Let $L_n$ denote the n-th coefficient of the Laurent expansion of L at 0. Then

$$L(\lambda) = \sum_{n=-\infty}^{+\infty} \lambda^n L_n \qquad (0 < |\lambda| < 2r).$$

It is easily verified that

$$L(\lambda)P = PL(\lambda) \qquad (\lambda \in \mathrm{Res}[A]).$$

This implies that

$$L_n P = PL_n \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Hence

$$L(\lambda)M \subset M, \quad L(\lambda)N \subset N \qquad (\lambda \in \mathrm{Res}[A])$$

and

$$L_n M \subset M, \quad L_n N \subset N \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Define the functions $L_M$ and $L_N$ on Res[A] by

$$L_M(\lambda) = L(\lambda)|M, \quad L_N(\lambda) = L(\lambda)|N.$$

Then $L_M$ and $L_N$ are locally holomorphic functions with values in $L(M)$ and $L(N)$, respectively. The Laurent expansions of $L_M$ and $L_N$ at 0 are given by

$$L_M(\lambda) = \sum_{n=-\infty}^{+\infty} \lambda^n (L_n|M), \quad L_N(\lambda) = \sum_{n=-\infty}^{+\infty} \lambda^n (L_n|N).$$

They certainly hold for $0 < |\lambda| < 2r$.

From Cauchy's integral formula, we know that

$$L_n = \frac{1}{2\pi i} \int_{|\lambda|=r} \lambda^{-(n+1)} A'(\lambda)A^{-1}(\lambda)d\lambda \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

It is clear that $P = L_{-1}$. Put $T = L_{-2}$. Thus

$$T = \frac{1}{2\pi i} \int_{|\lambda|=r} \lambda A'(\lambda)A^{-1}(\lambda)d\lambda.$$

Using Mittenthal's generalized operational calculus (see [31]), we infer

$$L_{-n} = T^{n-1}P = PT^{n-1} \qquad (n = 1,2,\dots).$$

This implies that

$$L_{-n}|M = (T|M)^{n-1}, \quad L_{-n}|N = 0 \qquad (n = 1,2,\dots).$$

Hence

$$L_M(\lambda) = \sum_{n=1}^{\infty} \lambda^{-n}(T|M)^{n-1} + \sum_{n=0}^{\infty} \lambda^n (L_n|M) \quad (0 < |\lambda| < 2r) \qquad (7)$$

and

$$L_N(\lambda) = \sum_{n=0}^{\infty} \lambda^n (L_n|N) \qquad (0 < |\lambda| < 2r).$$

From (7) we see that the series

$$\sum_{n=1}^{\infty} \lambda^{-n}(T|M)^{n-1}$$

is convergent for each $\lambda \neq 0$. Thus $T|M$ is quasi-nilpotent and

$$R(\lambda;T|M) = \sum_{n=1}^{\infty} \lambda^{-n}(T|M)^{n-1} \qquad (\lambda \neq 0).$$

Combining this with (7), we obtain

$$L_M(\lambda) = R(\lambda;T|M) + \sum_{n=0}^{\infty} \lambda^n (L_n|M) \quad (0 < |\lambda| < 2r).$$

Define the functions $A_M$ and $A_N$ on D by

$$A_M(\lambda) = A(\lambda)|M, \quad A_N(\lambda) = A(\lambda)|N.$$

It is easily verified that $A_M$ and $A_N$ are well-defined locally holomorphic functions with values in $L(M)$ and $L(N)$, respectively. From operator theory we know that $A(\lambda)$ is bijective if and only if $A_M(\lambda)$ and $A_N(\lambda)$ are both bijective, and in that case

$$A_M^{-1}(\lambda) = A^{-1}(\lambda)|M, \quad A_N^{-1}(\lambda) = A^{-1}(\lambda)|N.$$

Define the functions $A'_M$ and $A'_N$ on D by

$$A'_M(\lambda) = A'(\lambda)|M, \quad A'_N(\lambda) = A'(\lambda)|N.$$

Then $A'_M$ and $A'_N$ are well-defined locally holomorphic functions with values in $L(M)$ and $L(N)$, respectively. It is clear that

$$A'_M(\lambda)A_M^{-1}(\lambda) = L_M(\lambda), \quad A'_N(\lambda)A_N^{-1}(\lambda) = L_N(\lambda) \qquad (\lambda \in \mathrm{Res}[A]).$$

Thus

$$A'_M(\lambda)A_M^{-1}(\lambda) = R(\lambda;T|M) + \sum_{n=0}^{\infty} \lambda^n(L_n|M) \quad (0 < |\lambda| < 2r) \qquad (8)$$

and

$$A'_N(\lambda)A_N^{-1}(\lambda) = \sum_{n=0}^{\infty} \lambda^n(L_n|N) \qquad (0 < |\lambda| < 2r). \qquad (9)$$

Next we prove that $A_0|N$ is bijective. From (9) we see that

$$\lim_{\lambda\to 0} A'_N(\lambda)A_N^{-1}(\lambda) = L_0|N.$$

In addition,

$$\lim_{\lambda\to 0} A_N(\lambda) = A_N(0) = A_0|N.$$

Combining these results, we get

$$(L_0|N)(A_0|N) = \lim_{\lambda\to 0} A'_N(\lambda) = A'_N(0) = A_1|N.$$

By hypothesis $A_1$ is bijective, and so $A_1|N$ is bijective too. Since $L_0$ and $A_0$ commute, it follows that $A_0|N$ is bijective.

It remains to prove that $A_0|M$ is quasi-nilpotent. From (8) we deduce that for $0 < |\lambda| < 2r$

$$(\lambda I_M - T|M)A'_M(\lambda) = \{I_M + (\lambda I_M - T|M) \sum_{n=0}^{\infty} \lambda^n(L_n|M)\}A_M(\lambda).$$

Taking limits, we obtain

$$(T|M)(A_1|M) = \{(T|M)(L_0|M) - I_M\}(A_0|M).$$

Recall that $T|M$ is quasi-nilpotent. Since $L_0$ and $T$ commute, it follows that $(T|M)(L_0|M)$ is quasi-nilpotent too. Hence $(T|M)(L_0|M) - I_M$ is bijective, and so

$$A_0|M = \{(T|M)(L_0|M) - I_M\}^{-1}(T|M)(A_1|M). \tag{10}$$

Since the operators appearing in the right hand side of (10) commute with each other, it follows that $A_0|M$ is quasi-nilpotent. This proves the theorem.

We conclude this section with some examples. First of all, we deal with the condition that $A_1$ is bijective.

7.8. EXAMPLE. Let X be a non-trivial complex Banach space, and let

$$A(\lambda) = -\lambda T + \lambda^2 I_X \qquad (\lambda \in \mathbb{C}), \tag{11}$$

where T is some bounded linear operator on X. Clearly, A is commutative. Further,

$$Sp[A] = \sigma(T) \cup \{0\}$$

and

$$A^{-1}(\lambda) = \frac{1}{\lambda}(\lambda I_X - T)^{-1} \qquad (\lambda \in Res[A]).$$

Observe that $A_0 = 0$. Hence $\alpha(A_0) = \delta(A_0) = 1$.

Take in (11) the operator T to be the zero operator. Then $Sp[A] = \{0\}$ and

$$A^{-1}(\lambda) = \lambda^{-2} I_X \qquad (\lambda \neq 0).$$

In particular it follows that 0 is a pole of $A^{-1}$ of order 2. Since $\alpha(A_0) = \delta(A_0) = 1$, this choice of T shows that the inequality in Theorem

7.1 may be strict. It also shows that Theorem 7.3 does not remain true if the condition that $A_1$ is bijective is omitted.

Next we take T to be a non-nilpotent, quasi-nilpotent bounded linear operator on X. Then again Sp[A] = {0}, but now

$$A^{-1}(\lambda) = \sum_{n=0}^{\infty} \lambda^{-(n+2)}T^n \qquad (\lambda \neq 0).$$

Since T is non-nilpotent, it follows that 0 is an essential singularity of $A^{-1}$. Thus, in this way, we obtain another example showing that in Theorem 7.3 the condition that $A_1$ is bijective cannot be omitted.

Finally, we take X to be the sequence space $l_1$ and T will be the backwards shift on $l_1$, i.e.,

$$(Tx)_n = x_{n+1} \qquad (n = 1,2,\dots).$$

Then $Sp[A] = \{\lambda \in \mathbb{C} : |\lambda| \leqq 1\}$, and thus in this case 0 is not an isolated point of Sp[A]. Observe that 0 is an isolated point of $\sigma(A_0)$. Hence this choice of X and T shows that in Theorem 7.6 the condition that $A_1$ is bijective is not superfluous.

The following examples deal with the condition that A is commutative at 0.

7.9. EXAMPLE. Let X be the sequence space $l_\infty$, and let

$$A(\lambda) = A_0 + \lambda A_1 \qquad (\lambda \in \mathbb{C}),$$

where $A_0$ and $A_1$ are the bounded linear operators on $l_\infty$ defined by

$$A_0(x_1,x_2,\dots\dots) = (0,x_4,x_1,x_6,x_3,x_8,x_5,\dots\dots)$$

and

$$A_1(x_1,x_2,\dots\dots) = (x_2,0,0,\dots\dots).$$

These operators do not commute, and hence A is not commutative at 0. Clearly, $0 \in Sp[A]$. We shall prove that Sp[A] = {0} and that 0 is a simple pole of $A^{-1}$.

Define the bounded linear operators $S_0$ and $S_1$ on $l_\infty$ by

$$S_0(x_1,x_2,\dots\dots) = (x_3,0,x_5,x_2,x_7,x_4,x_9,\dots\dots)$$

and

$$S_1(x_1,x_2,\dots\dots) = (0,x_1,0,0,\dots\dots).$$

A straightforward computation shows that

$$(A_0 + \lambda A_1)(S_0 + \frac{1}{\lambda} S_1) = (S_0 + \frac{1}{\lambda} S_1)(A_0 + \lambda A_1) = I_X \quad (\lambda \neq 0),$$

and therefore

$$Sp[A] = \{0\}, \quad A^{-1}(\lambda) = S_0 + \frac{1}{\lambda} S_1 \qquad (\lambda \neq 0).$$

Since $S_1$ is non-zero, the last formula shows that 0 is a simple pole of $A^{-1}$.

It is easily verified that $\alpha(A_0) = \delta(A_0) = +\infty$. Hence it follows that the assumption in Theorem 7.1 and Corollary 7.2 that A is commutative at 0 cannot be omitted. Observe that the spectrum of the operator $A_0$ equals the closed unit disc in $\mathbb{C}$. Thus in this case 0 is not even an isolated point of $\sigma(A_0)$.

7.10. EXAMPLE. Let X be $\mathbb{C}^5$, and let

$$A(\lambda) = A_0 + \lambda A_1 \qquad (\lambda \in \mathbb{C}),$$

where $A_0$ and $A_1$ are the bounded linear operators on $\mathbb{C}^5$ defined by

$$A_0(x_1,x_2,x_3,x_4,x_5) = (x_2,x_3,0,0,0)$$

and

$$A_1(x_1,x_2,x_3,x_4,x_5) = (x_5,x_4,x_3,x_2,x_1).$$

Thus $A_0 = T$ and $A_1 = T_1$, where T and $T_1$ are the operators defined in Example 6.10. Clearly, $A_0$ is nilpotent with index of nilpotence 3. This implies that $\alpha(A_0) = \delta(A_0) = 3$. Further, $A_1$ is bijective. The operators $A_0$ and $A_1$ do not commute, and hence the function A is not commutative at 0. We shall prove that Sp[A] = {0} and that 0 is a pole of $A^{-1}$ of order 2.

Clearly, $0 \in$ Sp[A]. Observe that

$$A_1^2 = I_X, \quad (A_0A_1)^2 = (A_1A_0)^2 = 0.$$

Using these formulas one easily verifies that for each $\lambda \neq 0$ the operator $A(\lambda)$ is bijective and that

$$A^{-1}(\lambda) = \lambda^{-1}A_1 - \lambda^{-2}A_1A_0A_1 \qquad (\lambda \neq 0).$$

This implies that Sp[A] = {0}, and, since $A_1A_0A_1 \neq 0$, it follows that 0 is a pole of $A^{-1}$ of order 2.

This example shows that in Theorem 7.3. (and also in Theorem 7.1) the condition that A is commutative at 0 cannot be omitted.

7.11. EXAMPLE. Let X be the complex Banach space $l_\infty(\mathbb{Z})$ of all bounded functions $f : \mathbb{Z} \longrightarrow \mathbb{C}$ endowed with the supremum norm. Here $\mathbb{Z}$ denotes the set of all integers. Further, let

$$A(\lambda) = A_0 + \lambda A_1 \qquad (\lambda \in \mathbb{C}),$$

where $A_0$ and $A_1$ are the bounded linear operators on $l_\infty(\mathbb{Z})$ defined by

$$(A_0f)(n) = \begin{cases} f(n) & \text{for } n = 0,1,2,\ldots, \\ 0 & \text{for } n = -1,-2,\ldots, \end{cases}$$

and

$$(A_1f)(n) = f(n+1) \qquad (n \in \mathbb{Z}).$$

Then $\alpha(A_0) = \delta(A_0) = 1$, and thus 0 is a simple pole of the operator $A_0$. In particular, 0 is an isolated point of $\sigma(A_0)$. The operator $A_1$ is bijective. Since $A_0$ and $A_1$ do not commute, the function A is not commutative at 0. We shall prove that 0 is not an isolated point of Sp[A]. This will imply that in Theorem 7.6 the condition that A is commutative at 0 is not superfluous.

Observe that

$$A(\lambda) = A_1(A_1^{-1}A_0 + \lambda I_X) \qquad (\lambda \in \mathbb{C}).$$

This implies that $\mathrm{Sp}[A] = \sigma(-A_1^{-1}A_0)$. An easy computation shows that

$$[(A_1^{-1}A_0)f](n) = \begin{cases} f(n-1) & \text{for } n = 1,2,\dots, \\ 0 & \text{for } n = 0,-1,-2,\dots . \end{cases}$$

Let g be the element in $l_\infty(\mathbb{Z})$ defined by

$$g(0) = 1, \quad g(n) = 0 \qquad (0 \neq n \in \mathbb{Z}).$$

Then it is not difficult to prove that

$$g \notin R(A_1^{-1}A_0 + \lambda I_X) \qquad (|\lambda| < 1).$$

Thus the open unit disc in $\mathbb{C}$ is a subset of $\sigma(-A_1^{-1}A_0) = \mathrm{Sp}[A]$. This proves that 0 is not an isolated point of Sp[A]. Observe that this is another example showing that the hypothesis in Theorem 7.3 that A is commutative at 0 cannot be omitted.

REFERENCES

1. G.R. ALLAN, Holomorphic vector-valued functions on a domain of holomorphy.
J. London Math. Soc. 42 (1967), 509-513.

2. F.V. ATKINSON, A spectral problem for completely continuous operators.
Acta Math. Acad. Sci. Hungar. 3 (1952), 53-60.

3. F.V. ATKINSON, On relatively regular operators.
Acta Sci. Math. 15 (1953), 38-56.

4. H. BART, Meromorphic operator valued functions.
Rapport nr. 20, Wiskundig Seminarium der Vrije Universiteit, Amsterdam, 1971.

5. H. BART, Holomorphic relative inverses of operator valued functions. To appear.

6. H. BART, Generalized ascent and descent.
Rapport nr. 28, Wiskundig Seminarium der Vrije Universiteit, Amsterdam, 1972.

7. P.M. BLEHER, On operators depending meromorphically on a parameter.
Vestnik Moskov. Univ., Ser. I Mat. Meh. 24 (1969), no. 5, 30-36 [Russian].

8. R.A. BONIC, Linear functional analysis.
Gordon and Breach, New York, 1969.

9. S.R. CARADUS, Contributions to the theory of operators with meromorphic resolvents.
Doctoral dissertation, University of California, Los Angeles, January, 1965.

10. N. DUNFORD and J.T. SCHWARTZ, Linear operators, Part I.
Interscience, New York, 1967.

11. K.-H. FÖRSTER, Über lineare, abgeschlossene Operatoren, die analytisch von einem Parameter abhängen.
Math. Z. 95 (1967), 251-258.

12. K.-H. FÖRSTER and G. GARSKE, Holomorphie einseitiger Inverser von holomorphen Funktionen mit Werten in einer Banachparaalgebra.
Private communication (1972).

13. S. GOLDBERG, Unbounded linear operators.
McGraw-Hill, New York, 1966.

14. B. GRAMSCH, Meromorphie in der Theorie der Fredholmoperatoren mit Anwendungen auf elliptische Differentialoperatoren.
Math. Ann. 188 (1970), 97-112.

15. B. GRAMSCH and R. MEISE, Vorlesung über Fredholm operatoren.
Lecture Notes, Mainz, 1968.

16. K.P. HADELER, Eigenwerte von Operatorpolynomen.
Arch. Rational Mech. Anal. 20 (1965), 72-80.

17. P.R. HALMOS, Measure Theory.
D. van Nostrand, New York, 1950.

18. J.S. HOWLAND, Analyticity of determinants of operators on a Banach space.
Proc. Amer. Math. Soc. 28 (1971), 177-180.

19. J.S. HOWLAND, Simple poles of operator-valued functions.
J. Math. Anal. Appl. 36 (1971), 12-21.

20. M.A. KAASHOEK, Stability theorems for closed linear operators.
Indag. Math. 27 (1965), 452-466.

21. M.A. KAASHOEK, Ascent, descent, nullity and defect, a note on a paper by A.E. Taylor.
Math. Ann. 172 (1967), 105-115.

22. M.A. KAASHOEK and D.C. LAY, On operators whose Fredholm set is the complex plane.
Pacific J. Math. 21 (1967), 275-278.

23. T. KATO, Perturbation theory for linear operators.
Springer, Berlin-Heidelberg-New York, 1966.

24. T. KATO, Perturbation theory for nullity, deficiency, and other quantities of linear operators.
J. Analyse Math. 6 (1958), 261-322.

25. H. KUMMER, Zur praktischen Behandlung nichtlinearer Eigenwertaufgaben abgeschlossener linearer Operatoren.
Mitt. Math. Sem. Giessen, Heft 62 (1964).

26. D.C. LAY, Spectral analysis using ascent, descent, nullity and defect.
Math. Ann. 184 (1970), 197-214.

27. G. MAIBAUM, Über Scharen positiver Operatoren.
Math. Ann. 184 (1970), 238-256.

28. G. MAIBAUM, Ein Störungssatz für Scharen vollstetiger Operatoren.
Math. Nachr. 43 (1970), 313-319.

29. I. MAREK, On the polynomial eigenvalue problem with positive operators and location of the spectral radius.
Apl. Mat. 14 (1969), 146-159.

30. A.S. MARKUS, On holomorphic operator-functions.
Dokl. Akad. Nauk SSSR 119 (1958), 1099-1102 [Russian].

31. L. MITTENTHAL, Operator valued analytic functions and generalizations of spectral theory.
Pacific J. Math. 24 (1968), 119-132.

32. P.H. MÜLLER, Eine neue Methode zur Behandlung nichtlinearer Eigenwertaufgaben.
Math. Z. 70 (1959), 381-406.

33. R.K. OLIVER, Note on a duality relation of Kaashoek.
Indag. Math. 28 (1966), 364-368.

34. R.S. PALAIS, Seminar on the Atiyah-Singer index theorem.
Princeton University Press, 1965.

35. M.V. PATTABHIRAMAN and P. LANCASTER, Spectral properties of a polynomial operator.
Numer. Math. 13 (1969), 247-259.

36. A. PIETSCH, Zur Theorie der $\sigma$-Transformationen in lokalkonvexen Vektorräumen.
Math. Nachr. 21 (1960), 347-369.

37. M. RIBARIC and I. VIDAV, Analytic properties of the inverse $A^{-1}(z)$ of an analytic operator valued function $A(z)$.
Arch. Rational Mech. Anal. 32 (1969), 298-310.

38. P. SAPHAR, Contribution a l'étude des applications linéaires dans un espace de Banach.
Bull. Soc. Math. France 92 (1964), 363-384.

39. P. SAPHAR, Sur les applications linéaires dans un espace de Banach. II.
Ann. Sci. École Norm. Sup. (3) 82 (1965), 205-240.

40. R.T. SEELEY, Integral equations depending analytically on a parameter.
Indag. Math. 24 (1962), 434-442. Errata: Indag. Math. 25 (1963), 654.

41. B. SZ.-NAGY, On a spectral problem of Atkinson.
Acta Math. Acad. Sci. Hungar. 3 (1952), 61-66.

42. A.E. TAYLOR, Introduction to functional analysis.
Wiley and Sons, New York, 1967.

43. A.E. TAYLOR, Theorems on ascent, descent, nullity and defect of linear operators.
Math. Ann. 163 (1966), 18-49.

44. B. YOOD, Properties of linear transformations preserved under addition of a completely continuous transformation.
Duke Math. J. 18 (1951), 599-612.

AUTHOR INDEX

SUBJECT INDEX

NOTATION INDEX