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Realization of continuous-time positive linear systems

J.M. van den Hof

CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

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Abstract

Positive linear systems are used in biomathematics, economics, and other research areas. For discrete-time positive linear systems, part of the realization problem has been solved. In this paper the solution of the corresponding problem for continuous-time positive linear systems will be presented, which can be deduced from that of the discrete-time case by a transformation. Sufficient and necessary conditions for the existence of a positive realization are presented. To solve the problem of minimality, the solution of the factorization of positive matrices is needed. © 1997 Elsevier Science B.V.

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1. Introduction

The purpose of this paper is to present results on the realization of continuous-time positive linear systems.

Positive linear systems are used in biomathematics, economics, chemometrics, and other research areas. A *finite-dimensional positive linear system* is a linear dynamic system in which the input, state, and output space are spaces over the positive real numbers. Systems in this class are useful models in biomathematics, where they are called *linear compartmental systems*, see [10]. The identification problem for this class of systems is unsolved. No conditions are known which are both necessary and sufficient for global structural identifiability of such systems [6]. These conditions may be based on realization theory for positive linear systems and they are investigated further in this paper. An early reference on structural identifiability is [1].

In this paper time-invariant finite-dimensional continuous-time positive linear systems will be treated. A positive realization of a given positive impulse response function is a positive linear system, such that its impulse response function equals the given one. A positive realization of an impulse response function is said to be minimal if the state space as a vector space over the positive real numbers is of minimal dimension. The positive realization problem is to show existence of a positive realization of a positive impulse response function and to classify all minimal positive realizations. For the discrete-time case, necessary and sufficient conditions for the existence of a positive realization in terms of polyhedral cones were presented in [7], i.e., a positive realization exists if and only if there exists a backward-shift invariant polyhedral cone containing the cone spanned by the columns of the Hankel matrix. In the same paper a sufficient condition for minimality was stated (if the positive rank of the Hankel matrix equals the dimension of the state space, then the system is minimal). This condition is more restricting than the reachability/observability condition, but still not necessary. In [8] a necessary and sufficient condition has been derived, using positive system rank. For the problem of minimality, techniques of the theory of positive linear algebra and polyhedral cones are used. Examples of references on positive linear systems are [2,4,12–16]. We will see that the continuous-time case

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can be deduced from the discrete-time case by transformation. Important is that for the result on minimality *all* transformations to discrete-time systems have to be considered. This makes the continuous-time case more difficult and worthwhile to study, as shown in Section 4. There an example is presented to show that it is not sufficient to consider only one transformation.

The outline of this paper is as follows. The problem is formulated in Section 2. In Section 3 the existence of a positive realization is proven. In Section 4 results on the characterization of minimality are presented.

2. Problem formulation

In this section notation is introduced and the problem is posed.

The set $\mathbb{R}_+ = [0, +\infty)$ is called the set of the *positive real numbers*. Let $\mathbb{Z}_+ = \{1, 2, \ldots\}$ denote the set of positive integers, $\mathbb{Z}_n = \{1, \ldots, n\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$. Denote by $\mathbb{R}_+^n$ the set of n-tuples of the positive real numbers. The set $\mathbb{R}_+^{n\times m}$ will be called the set of *positive matrices* of size $n \times m$. Note that $\mathbb{R}_+^n$ is not a vector space over $\mathbb{R}$ because it does not admit an inverse with respect to addition. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be a Metzler matrix if all its off-diagonal elements are in $\mathbb{R}_+$, see [11]. Metzler matrices can be characterized as follows.

Proposition 2.1. *A matrix $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix if and only if there exists an $\alpha \in \mathbb{R}$ satisfying $(A + \alpha I) \in \mathbb{R}_+^{n\times n}$.*

Definition 2.1. Consider a continuous-time linear dynamic system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^k$, $t \in T = [t_0, \infty)$. Eq. (1) is said to describe a *(continuous-time) positive linear system* if for all $x_0 \in \mathbb{R}_+^n$ and for all $u(t) \in \mathbb{R}_+^m$, $t \in T$, we have $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+^k$ for $t \in T$.

The following proposition presents a characterization of continuous-time positive linear systems.

Proposition 2.2. *A continuous-time linear dynamic system of the form* (1) *is a positive linear system if and only if*

$$B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{k\times n}, \quad D \in \mathbb{R}_+^{k\times m} \quad \text{and} \quad A \text{ is a Metzler matrix.}$$

Proof. Suppose $u(t) = 0$ for all $t \in T$. For $i \in \mathbb{Z}_n$, $x_i(t) \geqslant 0$ if and only if $\dot{x}_i \geqslant 0$ whenever $x_i = 0$ and $x_j \geqslant 0$ for all $j \neq i$. This is equivalent to $a_{ij} \geqslant 0$ for all $j \neq i$. Now the conditions for B, C, and D follow. □

Consider the impulse response function $W : [0, \infty) \to \mathbb{R}_+^{k\times m}$ of the system (1), given by

$$W(0) = D; \quad W(t) = Ce^{At}B, \quad t > 0.$$

$e^{At} \geqslant 0$ if and only if A is a Metzler matrix. Indeed, if A is a Metzler matrix, there exists an $\alpha \in \mathbb{R}$ satisfying $A + \alpha I \in \mathbb{R}_+^{n\times n}$. From this it follows that $e^{(A+\alpha I)t} \in \mathbb{R}_+^{n\times n}$ for all $t \geqslant 0$, and the relation

$$e^{At} = e^{(A+\alpha I)t}e^{-\alpha t}$$

implies $e^{At} \in \mathbb{R}_+^{n\times n}$ for all $t \geqslant 0$. The other way round, if $e^{At} \geqslant 0$, then $e^{At}x_0 \in \mathbb{R}_+^n$ whenever $x_0 \in \mathbb{R}_+^n$ for all $t \geqslant t_0$, so $\dot{x} = Ax$ implies $x(t) \geqslant 0$ whenever $x_0 \geqslant 0$. It follows that A is a Metzler matrix. So for continuous-time positive linear systems, besides B, C, and D, also e^{At} is a positive matrix for $t \geqslant 0$, which implies $W(t) \in \mathbb{R}_+^{k\times m}$ for all $t \geqslant 0$. On the other hand, the Markov parameters corresponding to $W(t)$ are not necessarily positive. However, for $\alpha \in \mathbb{R}$ satisfying $A + \alpha I \in \mathbb{R}_+^{n\times n}$,

$$D, \quad C(A + \alpha I)^{j-1}B, \quad j = 1, 2, \ldots$$

are elements of $\mathbb{R}_+^{k\times m}$. This follows from Proposition 2.1. It can be shown that these matrices are the Markov parameters corresponding to the impulse response function $e^{\alpha t}W(t)$. This fact will be used in the sequel.

Problem 2.1. The *continuous-time positive realization problem* for a positive impulse response function.

a. Formulate necessary and sufficient conditions for the existence of a continuous-time positive linear system such that the impulse response function of this system equals the given impulse response function. If such a system exists, it is called a *positive realization* of the given impulse response function.
b. Determine the minimal dimension of the state space of a positive realization. If the state space of a positive realization is minimal, this realization is called a *minimal positive realization.*
c. Classify all minimal positive realizations of the given impulse response function.
d. If two positive realizations of the same impulse response function are minimal, then indicate the relation between them.

A positive linear system is called a *minimal positive linear system* if it is a minimal positive realization of its impulse response function.

The solution of this problem does not follow from the realization theory of ordinary linear systems. As can be seen from the example given by (4), minimality of a positive linear system does not imply minimality of an ordinary linear system. An important concept for positive linear systems is the *positive rank.* For completeness its definition is given below, together with some more important notions.

Definition 2.2. A positive matrix $M\in\mathbb{R}_+^{n\times n}$ is said to be a *monomial matrix* if every row and every column contains exactly one strictly positive element.

Definition 2.3. Let $k,m\in\mathbb{Z}_+$, $m<k$. A positive matrix $A\in\mathbb{R}_+^{k\times m}$ is said to be *part of a monomial* in $\mathbb{R}_+^{k\times k}$ if there exists a $B\in\mathbb{R}_+^{k\times(k-m)}$ such that

$$(A\ \ B)$$

is a monomial in $\mathbb{R}_+^{k\times k}$. A positive matrix $C\in\mathbb{R}_+^{m\times k}$ is also said to be part of a monomial if C^{T} is part of a monomial as defined above.

It follows that A is part of a monomial if and only if either A contains exactly one strictly positive element in every column and at most one strictly positive element in every row (case $m<k$), or A contains at most one strictly positive element in every column and exactly one strictly positive element in every row (case $m>k$). With slight abuse of terminology, a monomial is sometimes also called part of a monomial, just to make the nomenclature easier.

A property of monomials is that they are the only positive matrices whose inverses are again positive matrices.

Definition 2.4. Let $A\in\mathbb{R}_+^{k\times m}$ for $k,m\in\mathbb{Z}_+$. If $A=0$, the *positive rank* of A is defined to be 0. The *positive rank* of the matrix $A\neq 0$ is defined as the least integer $n\in\mathbb{Z}_+$ for which there exists a factorization

$$A=BC, \tag{2}$$

with $B\in\mathbb{R}_+^{k\times n}$ and $C\in\mathbb{R}_+^{n\times m}$. Let pos-rank($A$) denote this integer.

A *positive matrix factorization* of A is any factorization of A of the form (2) for arbitrary $n\in\mathbb{Z}_+$. A *minimal positive matrix factorization* of A is any positive matrix factorization of A in which $n=$ pos-rank(A). A is said to be *strictly factorizable* if there exists a positive matrix factorization of the form (2), with $n\leqslant\min\{k,m\}$, in which neither B nor C is part of a monomial.

It follows that a matrix $A\in\mathbb{R}_+^{k\times m}$ is not strictly factorizable if and only if any factorization of the form (2), with $n\leqslant\min\{k,m\}$, is such that either B or C is part of a monomial.

3. Existence of a positive realization

In this section necessary and sufficient conditions for the existence of a positive realization of a continuous-time positive impulse response function will be presented. The discrete-time case has been treated in [7,9]. In those papers the convex cone analysis is used. The reader is referred to those papers for explanation of the terminology.

Definition 3.1. A continuous-time impulse response function $W:\mathbb{R}_+ \to \mathbb{R}_+^{k\times m}$ is said to be *a Metzler impulse response function* if there exists an $\alpha\in\mathbb{R}$ such that the Markov parameters corresponding to $e^{\alpha t}W(t)$,

$$M_\alpha(0)=W(0);\quad M_\alpha(j)=\frac{d^{j-1}}{dt^{j-1}}e^{\alpha t}W(t)|_{t=0},\ \ j=1,2,\dots$$

are positive matrices.

For the existence of a positive realization, the following result can be stated. Let $T=\mathbb{N}$, $Y=\mathbb{R}_+^k$. Let σ denote the *backward shift operator*

$$(\sigma y)(t)=y(t+1),\qquad \text{for } y:T\to Y.$$

A cone $C_1\subseteq\mathbb{R}_+^\infty$ is said to be *backward shift invariant* if $y_1\in C_1$ implies $\sigma y_1\in C_1$.

Theorem 3.1. *Let $T=\mathbb{R}_+$, $Y=\mathbb{R}_+^k$, $U=\mathbb{R}_+^m$. Consider a continuous-time positive impulse response function $W:T\to\mathbb{R}_+^{k\times m}$. There exists a positive linear system*

$$\dot x(t)=Ax(t)+Bu(t),\quad x(t_0)=x_0,$$
$$y(t)=Cx(t)+Du(t),$$

such that the impulse response function of this system equals W if and only if W is a Metzler impulse response function and there exists a set $C_1\subseteq\mathbb{R}_+^\infty$ satisfying
1. *C_1 is a polyhedral cone;*
2. $\mathrm{cone}(H_\alpha)\subseteq C_1$;
3. *C_1 is backward shift invariant,*

with $H_\alpha=(M_\alpha(1)^T\ \ M_\alpha(2)^T\ \ M_\alpha(3)^T\ \ \cdots)^T$ for $\alpha\in\mathbb{R}$ satisfying $M_\alpha(j)\in\mathbb{R}_+^{k\times m}$ for all $j\in\mathbb{Z}_+$.

Compared to the discrete-time case, the condition of W being a Metzler impulse response function has to be added.

Proof of Theorem 3.1. ($\Rightarrow$) Assume W is the impulse response function of the positive linear system

$$\dot x(t)=Ax(t)+Bu(t),\quad x(t_0)=x_0,$$
$$y(t)=Cx(t)+Du(t),$$

with $X=\mathbb{R}_+^n$, $B\in\mathbb{R}_+^{n\times m}$, $C\in\mathbb{R}_+^{k\times n}$, $D\in\mathbb{R}_+^{k\times m}$, and $A\in\mathbb{R}^{n\times n}$ a Metzler matrix, for $n\in\mathbb{Z}_+$. It follows that there exists an $\alpha\in\mathbb{R}$ satisfying $A+\alpha I\in\mathbb{R}_+^{n\times n}$. Then

$$M_\alpha(0)=W(0)=D,$$
$$M_\alpha(j)=\frac{d^{j-1}}{dt^{j-1}}e^{\alpha t}W(t)|_{t=0}=\frac{d^{j-1}}{dt^{j-1}}e^{\alpha t}Ce^{At}B|_{t=0}=\frac{d^{j-1}}{dt^{j-1}}Ce^{(A+\alpha I)t}B|_{t=0}$$
$$=C(A+\alpha I)^{j-1}B,\quad j=1,2,\dots$$

with $M_\alpha(j) \in \mathbb{R}_+^{k\times m}$, since $A+\alpha I \in \mathbb{R}_+^{n\times n}$. So W is a Metzler impulse response function. This provides an A_α $(=A+\alpha I)$, B, C, and D satisfying

$$M_\alpha(0) = D,$$
$$M_\alpha(j) = CA_\alpha^{j-1}B, \quad j=1,2,\ldots,$$

and with Theorem 4.4 in [7], 1–3 follow.

($\Leftarrow$) Because W is a Metzler impulse response function there exists an $\alpha \in \mathbb{R}$ such that for all $j \in \mathbb{Z}_+$, $M_\alpha(j) \in \mathbb{R}_+^{k\times m}$. Step (a)–(f) in the proofs of Theorems 4.1 and 4.4 in [7] provide $\tilde{A} \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{k\times n}$, and $D \in \mathbb{R}_+^{k\times m}$ satisfying

$$M_\alpha(0) = D,$$
$$M_\alpha(j) = C\tilde{A}^{j-1}B, \quad j=1,2,\ldots.$$

$M_\alpha(j)$ are the Markov parameters corresponding to $\tilde{W}(t) = C e^{\tilde{A}t}B$, and because W is a Metzler impulse response function, $\alpha \in \mathbb{R}$ satisfies $\tilde{W}(t) = e^{\alpha t}W(t)$. So with $A = \tilde{A} - \alpha I$, there exists a positive linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0,$$
$$y(t) = Cx(t) + Du(t),$$

with $W(t) = Ce^{At}B$ for $t > 0$, and $W(0) = D$. □

4. Characterization of minimality

In this section results on the characterization of minimality for the continuous-time case are presented and turn out to be related to the results for the discrete-time case, see [7]. The problem is to derive sufficient and necessary conditions for a continuous-time positive linear system to be a minimal positive realization of the impulse response function. As in [7], attention is restricted to the positive rank and extremal cones. About the positive rank in relation to a continuous-time positive linear system the following can be said.

Consider a positive linear system (A,B,C) with $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{k\times n}$, and $(A+\alpha I) \in \mathbb{R}_+^{n\times n}$ for some $\alpha \in \mathbb{R}$. Let $A_\alpha = A + \alpha I$. For $p,q \in \mathbb{Z}_+$, define $H_\alpha(p,q)$ to be the Hankel matrix

$$H_\alpha(p,q) = \begin{pmatrix} CB & CA_\alpha B & \cdots & CA_\alpha^{q-1}B \\ CA_\alpha B & CA_\alpha^2 B & & \vdots \\ \vdots & & \ddots & \vdots \\ CA_\alpha^{p-1}B & \cdots & \cdots & CA_\alpha^{p+q-2}B \end{pmatrix}. \tag{3}$$

Proposition 4.1. *Consider a positive linear system (A,B,C) with $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{k\times n}$, and $A \in \mathbb{R}^{n\times n}$ a Metzler matrix. For all $\alpha \in \mathbb{R}$ satisfying $A+\alpha I \in \mathbb{R}_+^{n\times n}$ and for every $p,q \in \mathbb{Z}_+$,* pos-rank$(H_\alpha(p,q)) \leqslant n$.

Proof. Analogous to discrete-time case, Proposition 5.9 in [7]. □

Below the relation between minimal discrete-time positive linear systems and continuous-time positive linear systems is presented.

Theorem 4.2. *Let the continuous-time positive linear system (A,B,C) be given as above. (A,B,C) is a minimal continuous-time positive linear system if and only if $(A+\beta I,B,C)$ is a minimal discrete-time positive linear system for all $\beta \in \mathbb{R}$ satisfying $A+\beta I \in \mathbb{R}_+^{n\times n}$.*

Proof. ($\Rightarrow$) Assume (A,B,C) is a minimal continuous-time positive linear system. Suppose there exists a $\beta \in \mathbb{R}$ such that $(A+\beta I,B,C)$ is a discrete-time positive linear system that is not minimal. Then there exists a discrete-time positive linear system $(\tilde{A},\tilde{B},\tilde{C})$, with $\tilde{A} \in \mathbb{R}_+^{n_1 \times n_1}$, $\tilde{B} \in \mathbb{R}_+^{n_1 \times m}$, $\tilde{C} \in \mathbb{R}_+^{k \times n_1}$, for $n_1 < n$, with the same impulse response function as $(A+\beta I,B,C)$. But then $(\tilde{A}-\beta I,\tilde{B},\tilde{C})$ is a continuous-time positive linear system with state-space dimension n_1 and the same impulse response function as (A,B,C), so (A,B,C) is not minimal. This is a contradiction. So $(A+\beta I,B,C)$ is a minimal discrete-time positive linear system for all $\beta \in \mathbb{R}$ satisfying $A+\beta I \in \mathbb{R}_+^{n \times n}$.

($\Leftarrow$) Assume $(A+\beta I,B,C)$ is a minimal discrete-time positive linear system for all $\beta \in \mathbb{R}$ satisfying $A+\beta I \in \mathbb{R}_+^{n \times n}$. Suppose (A,B,C) is not a minimal continuous-time positive linear system. Then there exists a continuous-time positive linear system $(\hat{A},\hat{B},\hat{C})$, with $\hat{A} \in \mathbb{R}^{n_1 \times n_1}$ a Metzler matrix, $\hat{B} \in \mathbb{R}_+^{n_1 \times m}$, $\hat{C} \in \mathbb{R}_+^{k \times n_1}$, for $n_1 < n$, with the same impulse response function as (A,B,C). Since $\hat{A}$ is, like A, a Metzler matrix, there exists an $\alpha \in \mathbb{R}$, satisfying both $\hat{A}+\alpha I \in \mathbb{R}_+^{n_1 \times n_1}$ and $A+\alpha I \in \mathbb{R}_+^{n \times n}$. So $(\hat{A}+\alpha I,\hat{B},\hat{C})$ is a discrete-time positive linear system with the same impulse response function as $(A+\alpha I,B,C)$, but with a smaller state space dimension, hence $(A+\alpha I,B,C)$ is not minimal. Contradiction. It follows that (A,B,C) is a minimal continuous-time positive linear system. $\square$

If $(A+\beta I,B,C)$ is a minimal discrete-time positive linear system for only one $\beta \in \mathbb{R}$ satisfying $A+\beta I \in \mathbb{R}_+^{n \times n}$, this is not sufficient for (A,B,C) to be minimal as continuous-time positive linear system, as the following example shows.

Example 4.1. Consider the continuous-time positive linear system of the form (1) with

$$A=\begin{pmatrix} -0.8 & 0.25 & 0 & 0 \\ 1 & -0.8 & 0 & 0 \\ 0 & 0.39 & -0.8 & 0.8 \\ 0 & 0 & 0.8 & -0.8 \end{pmatrix}, \qquad B=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad C=(1 \;\; 1.1 \;\; 0 \;\; 2).$$

For $\alpha = 0.8$, $A+\alpha I = A_\alpha \in \mathbb{R}_+^{4 \times 4}$. The discrete-time positive linear system (A_α,B,C) is minimal. To see this, consider the transfer function

$$C(\lambda I - A_\alpha)^{-1}B = \frac{\lambda^3 + 1.1\lambda^2 - 0.64\lambda - 0.8}{(\lambda^2-0.25)(\lambda^2-0.64)} = \frac{\lambda^2+1.6\lambda+0.16}{(\lambda+0.5)(\lambda^2-0.64)}.$$

Suppose there exists a positive realization of order 3. Then there should exist a matrix $\tilde{A} \in \mathbb{R}_+^{3 \times 3}$ with eigenvalues 0.8, -0.8, and -0.5. Because $\tilde{A} \in \mathbb{R}_+^{3 \times 3}$, trace$(\tilde{A}) \geqslant 0$. But trace$(\tilde{A})$ equals the sum of the eigenvalues of $\tilde{A}$, which is $0.8+(-0.8)+(-0.5) = -0.5$. This is a contradiction. So (A_α,B,C) is a minimal discrete-time positive linear system. Now consider the Hankel matrix

$$H_\alpha(4,4) = \begin{pmatrix} CB & CAB & CA^2B & CA^3B \\ CAB & CA^2B & CA^3B & CA^4B \\ CA^2B & CA^3B & CA^4B & CA^5B \\ CA^3B & CA^4B & CA^5B & CA^6B \end{pmatrix} = \begin{pmatrix} 1 & 1.1 & 0.25 & 0.899 \\ 1.1 & 0.25 & 0.899 & 0.0625 \\ 0.25 & 0.899 & 0.0625 & 0.62411 \\ 0.899 & 0.0625 & 0.62411 & 0.015625 \end{pmatrix}.$$

The claim is that $H_\alpha(4,4)$ has positive rank 4. This will be proven in the appendix.

But, also for $\beta = 1.6$, $A+\beta I = A_\beta \in \mathbb{R}_+^{4 \times 4}$. Now (A_β,B,C) is a discrete-time positive linear system, which is *not* minimal. Indeed, the discrete-time positive linear system $(\hat{A},\hat{B},\hat{C})$, with

$$\hat{A} = \begin{pmatrix} 0 & 0 & 0 \\ 1.6 & 1.6 & 0 \\ 0.3 & 0 & 0.3 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{C} = (1 \;\; 1 \;\; 1),$$

is a discrete-time positive linear system with the same impulse response function as (A_β, B, C). It is minimal, since it is minimal as linear system, i.e., $(\widehat{A}, \widehat{B})$ is reachable and $(\widehat{A}, \widehat{C})$ is observable. So $(\widehat{A} - \beta I, \widehat{B}, \widehat{C})$, with

$$\widehat{A} - \beta I = \begin{pmatrix} -1.6 & 0 & 0 \\ 1.6 & 0 & 0 \\ 0.3 & 0 & -1.3 \end{pmatrix}, \qquad \widehat{B} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \widehat{C} = (1 \;\; 1 \;\; 1),$$

is a minimal continuous-time positive linear system for (A, B, C).

So, while (A_α, B, C) is a minimal discrete-time positive linear system, and pos-rank$(H_\alpha(4,4)) = 4$, for $\alpha = 0.8$, (A, B, C) is *not* a minimal continuous-time positive linear system.

Note that with Proposition 4.1, pos-rank$(H_\beta(p,q)) \leqslant 3$ for all $p, q \in \mathbb{Z}_+$, so

$$\text{pos-rank}(H_\beta(4,4)) \leqslant 3 < 4 = \text{pos-rank}(H_\alpha(4,4)).$$

To show that there exists a continuous-time positive linear system that is *not* minimal as an ordinary linear system, but is minimal as a positive linear system, consider the example in [15]. Let

$$A = \begin{pmatrix} -2 & 0 & 0 & 1 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad C = (1 \;\; 1 \;\; 0 \;\; 1) \tag{4}$$

be a continuous-time positive linear system. Note that this system is not minimal as an ordinary linear system. In [15] it has been shown that the system is minimal as a continuous-time positive linear system. Another way of showing this is using the theory of this section as follows. With Theorem 4.2 and Example 4.1 it is not sufficient to check whether $(A + 2I, B, C)$ is minimal as a discrete-time positive linear system. For (A, B, C) to be a minimal continuous-time positive linear system, it has to be shown that $(A + \beta I, B, C)$ is minimal as a discrete-time positive linear system for all $\beta \in \mathbb{R}$ satisfying $A + \beta I \in \mathbb{R}_+^{4\times 4}$. Consider for (4) the discrete-time positive linear systems $(A + \beta I, B, C)$ for arbitrary $\beta \geqslant 2$. The poles of the transfer function $C(\lambda I - A)^{-1}B$ are $\{\beta - 1, \beta - 2 + i, \beta - 2 - i\}$. If $\{\beta - 1, \beta - 2 + i, \beta - 2 - i\}$ were the eigenvalues of a positive matrix $A \in \mathbb{R}_+^{3\times 3}$, then Eq. (4.2.1) in [3] must hold. For $k = 1$ and $m = 2$ this equation reads

$$(\lambda_1 + \lambda_2 + \lambda_3)^2 \leqslant 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2), \tag{5}$$

in which $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues. Substituting

$$\lambda_1 = \beta - 1, \qquad \lambda_2 = \beta - 2 + i, \qquad \lambda_3 = \beta - 2 - i,$$

it can be seen that (5) does not hold for any $\beta \in \mathbb{R}$. So there does not exist a positive matrix $A \in \mathbb{R}_+^{3\times 3}$ with eigenvalues $\{\beta - 1, \beta - 2 + i, \beta - 2 - i\}$ for any $\beta \in \mathbb{R}$. It follows that $(A + \beta I, B, C)$ is a minimal discrete-time positive linear system for all $\beta \geqslant 2$, so with Theorem 4.2 (A, B, C) is a minimal continuous-time positive linear system.

This section will be closed with an analogue for Proposition 5.10 in [7].

Proposition 4.3. *Let $(A, B, C) \in (\mathbb{R}^{n\times n} \times \mathbb{R}_+^{n\times m} \times \mathbb{R}_+^{k\times n})$ be a continuous-time positive linear system. If there exist $p, q \in \mathbb{Z}_+$ such that for all $\alpha \in \mathbb{R}$ satisfying $A + \alpha I \in \mathbb{R}_+^{n\times n}$,* pos-rank$(H_\alpha(p,q)) = n$, *then (A, B, C) is a minimal positive linear system.*

Proof. This follows from Theorem 4.2 above and Proposition 5.10 in [7]. □

Note that the positive rank of $H_\alpha(p,q)$ has to be determined for *all* $\alpha \in \mathbb{R}$ satisfying $A + \alpha I \in \mathbb{R}_+^{n\times n}$, which makes the problem even more difficult than the problem for the discrete-time case.

5. Conclusions

As in the discrete-time case, the condition of reachability and observability is only sufficient for a continuous-time positive linear system to be minimal, but not necessary, as has been shown by the example described by (4).

Proposition 4.3 only presents a sufficient condition for minimality, but in [8] a necessary and sufficient condition has been derived. For continuous-time positive linear systems it comes down to the following: a continuous-time positive linear system (A,B,C) is a minimal positive linear system if and only if the *positive system rank* of $H_\alpha(p,q)$ equals n for all $\alpha \in \mathbb{R}$ satisfying $A+\alpha I \in \mathbb{R}_+^{n\times n}$. For the definition of positive system rank and further details, the reader is referred to the above-mentioned paper.

Appendix. Proof of pos-rank$(H_\alpha(4,4))=4$ in Example 4.1

For the proof a result from [8] is needed. For completeness this result will be stated below, in Theorem A.1

Definition A.1. A finite set of vectors in $\mathbb{R}_+^k$, say $\{v_1,\ldots,v_m\} \subset \mathbb{R}_+^k$, is said to be *positively dependent* if there exists an $i \in \mathbb{Z}_m$ such that v_i can be written as a nonnegative linear combination of $\{v_j,\ j\in\mathbb{Z}_m,\ j\neq i\}$, or

$$v_i = \sum_{j=1,j\neq i}^{m} \lambda_j v_j, \quad \text{in which } \lambda_j \in \mathbb{R}_+ \quad \text{for } j\in\mathbb{Z}_m,\ j\neq i.$$

It is said to be *positively independent* otherwise.

Definition A.2. A finite set of vectors (nonempty, not all zero) $\{v_1,\ldots,v_m\} \subset \mathbb{R}_+^k$, is said to be a *frame* of the polyhedral cone $C \subseteq \mathbb{R}_+^k$ if the conditions
1. the set $\{v_1,\ldots,v_m\}$ is positively independent;
2. the set $\{v_1,\ldots,v_m\}$ spans the cone C;

both hold. The integer m is said to be the *size* of the frame. Let $k,\ m\in\mathbb{Z}_+,\ m\leqslant k$. Denote the set of polyhedral cones with a frame of size m as

$$C_{k,m} = \{C \subseteq \mathbb{R}_+^k \mid C \text{ is polyhedral cone, with a frame of } m \text{ vectors}\}.$$

The following definition and propositions come from [5]. They are needed for Definition A.4.

Definition A.3. Let C be a polyhedral cone in $\mathbb{R}_+^k$ of dimension m. Then C has one m-facet, itself, and no r-facets for $r>m$. If $r<m$, then F is an r-facet of C if
1. F is a subcone of an $(r+1)$-facet G;
2. $F \subseteq \partial G$;
3. no subcone of G contained in ∂G properly contains F;
4. $F \neq \emptyset$.

Denote by $\mathscr{F}_r(C)$ the set of r-facets of C.

Define on $C_{k,m}$ an order relation by inclusion of cones. The notion of extremal cone is defined below.

Definition A.4. Let $k,m\in\mathbb{Z}_+,\ m\leqslant k$. A cone in $C_{k,m}$ is said to be an *extremal cone* if it is a maximal element in $C_{k,m}\backslash\mathscr{F}_m(\mathbb{R}_+^k)$ with respect to the order relation; denote

$$CE_{k,m} = \{C \in C_{k,m}\backslash\mathscr{F}_m(\mathbb{R}_+^k) \mid C \text{ extremal cone}\}.$$

Note that $\mathbb{R}_+^k$ is the only maximal element in $C_{k,k}$, and $\mathscr{F}_k(\mathbb{R}_+^k)=\{\mathbb{R}_+^k\}$.

Theorem A.1. *Consider $A\in\mathbb{R}_+^{k\times m}$. If* pos-rank$(A)=n$, *then there exist a $B\in\mathbb{R}_+^{k\times n}$ and a $C\in\mathbb{R}_+^{n\times m}$ such that $A=BC$ and* cone$(B)\in CE_{k,n}\cup\mathscr{F}_n(\mathbb{R}_+^k)$.

As in Section 3.4 of [3], the following notation will be used. For $A\in\mathbb{R}_+^{k\times m}$, let a_j denote the jth column of A. a_j^* denotes the $\{0,1\}$ vector defined by $a_{ij}^*=1$ if $a_{ij}>0$ and $a_{ij}^*=0$ if $a_{ij}=0$.

Theorem A.2. *Let $A\in\mathbb{R}_+^{k\times m}$, with $k\geqslant m$. Let $1\leqslant i,j\leqslant m$. If $a_i^*\geqslant a_j^*$, then A is strictly factorizable.*

Proof. The proof is analogous to the proof of Theorem 3.4.19 in [3]. □

Corollary A.3. *If $A\in CE_{k,m}\cup\mathscr{F}_m(\mathbb{R}_+^k)$, with $k\geqslant m$, i.e., if A is not strictly factorizable, then A contains a zero and a strictly positive element in every column, and a strictly positive element in every row. It also contains a zero in at least m rows.*

Consider the matrix

$$H_\alpha(4,4)=\begin{pmatrix} 1 & 1.1 & 0.25 & 0.899 \\ 1.1 & 0.25 & 0.899 & 0.0625 \\ 0.25 & 0.899 & 0.0625 & 0.62411 \\ 0.899 & 0.0625 & 0.62411 & 0.015625 \end{pmatrix}.$$

Suppose pos-rank$(H_\alpha(4,4))=3$. Then with Theorem A.1 there exist a $B\in\mathbb{R}_+^{4\times 3}$ and a $C\in\mathbb{R}_+^{3\times 4}$ such that $H_\alpha(4,4)=BC$ and cone$(B)\in CE_{4,3}\cup\mathscr{F}_3(\mathbb{R}_+^4)$. If cone$(B)\in CE_{4,3}\cup\mathscr{F}_3(\mathbb{R}_+^4)$, then from Corollary A.3 it follows that B contains at least one zero and one nonzero element in every column, and in at least 3 rows a zero. Note that, with r_1, r_2, r_3, and r_4 denoting the four rows of $H_\alpha(4,4)$, that

$$\tfrac{8}{25}r_1+\tfrac{16}{25}r_2=\tfrac{1}{2}r_3+r_4.$$

So this relation should also hold for the rows of B. With a post-multiplication by a monomial $M\in\mathbb{R}_+^{3\times 3}$, B can contain a one in every column. So B has, without loss of generality, one of the following forms:

$$B_1=\begin{pmatrix} 0 & \frac{25}{8}b+\frac{25}{16} & 1 \\ \frac{25}{32}a+\frac{25}{16} & 0 & c \\ a & 1 & 0 \\ 1 & b & \frac{16}{25}c+\frac{8}{25} \end{pmatrix},\qquad B_2=\begin{pmatrix} 0 & 1 & c \\ \frac{25}{32}a+\frac{25}{16} & b & 1 \\ a & 0 & \frac{16}{25}c+\frac{32}{25} \\ 1 & \frac{16}{25}b+\frac{8}{25} & 0 \end{pmatrix},$$

$$B_3=\begin{pmatrix} 0 & \frac{25}{8}b+\frac{25}{16} & c \\ \frac{25}{32}a+\frac{25}{16} & 0 & 1 \\ a & 1 & \frac{16}{25}c+\frac{32}{25} \\ 1 & b & 0 \end{pmatrix},\qquad B_4=\begin{pmatrix} \frac{25}{8}a+\frac{25}{16} & 1 & c \\ 0 & b & 1 \\ 1 & 0 & \frac{16}{25}c+\frac{32}{25} \\ a & \frac{16}{25}b+\frac{8}{25} & 0 \end{pmatrix},$$

with $a\geqslant 0$, $b\geqslant 0$, $c\geqslant 0$. Since $H_\alpha(4,4)=BC$, C can be calculated by $C=B^*H_\alpha(4,4)$, with B^* a left inverse of B. Consider $C_1=B_1^*H_\alpha(4,4)$. All elements of C_1 must be positive. Consider the third and the fourth element of the third row of C_1.

$$C_{1(3,3)}=\frac{1.95034375a+0.3046875+3.4006875ab-0.390625b}{a+2+4acb+2ac},$$

$$C_{1(3,4)}=\frac{0.048828125a-0.15234375-1.70034375ab-3.9006875b}{a+2+4acb+2ac}.$$

Now

$$C_{1(3,3)} \geqslant 0 \quad \text{if and only if } b \leqslant \frac{1.95034375a + 0.3046875}{0.390625 - 3.4006875a} =: x, \quad \text{for } a \neq \frac{0.390625}{3.4006875},$$

$$C_{1(3,4)} \geqslant 0 \text{ if and only if } b \leqslant \frac{0.048828125a - 0.15234375}{1.70034375a + 3.9006875} =: y.$$

So $b \leqslant \min\{x, y\}$. But $0 \leqslant a \leqslant 1$ implies $y < 0$, and $a \geqslant 1$ implies $x < 0$, thus for all $a \geqslant 0$ there holds $b < 0$. This contradicts that $H_\alpha(4,4)$ can be written as $B_1 C_1$ with B_1 and C_1 positive matrices and B_1 given above.

Now consider the second and fourth element of the first row of $C_2 = B_2^* H_\alpha(4,4)$.

$$C_{2(1,2)} = \frac{0.39cb + 0.32c - 2.816b - 1.158}{2acb + ac + 2c + 4},$$

$$C_{2(1,4)} = \frac{0.0975cb + 0.08c - 2.30144b - 1.08822}{2acb + ac + 2c + 4}.$$

Now

$$C_{2(1,2)} \geqslant 0 \text{ if and only if } b \leqslant \frac{0.32c - 1.158}{2.816 - 0.39c} =: x, \quad \text{for } c \neq \frac{2.816}{0.39},$$

$$C_{2(1,4)} \geqslant 0 \text{ if and only if } b \leqslant \frac{0.08c - 1.08822}{2.30144 - 0.0975c} =: y, \quad \text{for } c \neq \frac{2.30144}{0.0975}.$$

So $b \leqslant \min\{x, y\}$. But $0 \leqslant c \leqslant 10$ implies $y < 0$, and $c \geqslant 10$ implies $x < 0$, thus for all $c \geqslant 0$ there holds $b < 0$.

Now consider the third and fourth element of the third row of $C_3 = B_3^* H_\alpha(4,4)$.

$$C_{3(3,3)} = \frac{-1.70034375ab + 0.1953125b - 0.15234375 - 0.975171875}{acb + 2cb + 4b + 2},$$

$$C_{3(3,4)} = \frac{0.850171875ab + 1.95034375b + 0.076171875 - 0.0244140625a}{acb + 2cb + 4b + 2}.$$

Now

$$C_{3(3,3)} \geqslant 0 \text{ if and only if } a \leqslant \frac{0.1953125b - 0.15234375}{0.975171875 + 1.70034375b} =: x,$$

$$C_{3(3,4)} \geqslant 0 \text{ if and only if } a \leqslant \frac{1.95034375b + 0.076171875}{0.0244140625 - 0.850171875b} =: y, \quad \text{for } b \neq \frac{0.0244140625}{0.850171875}.$$

So $a \leqslant \min\{x, y\}$. But $0 \leqslant b \leqslant 0.1$ implies $x < 0$, and $b \geqslant 0.1$ implies $y < 0$, thus for all $b \geqslant 0$ there holds $a < 0$.

For the last possibility, consider the first and third element of the first row of $C_4 = B_4^* H_\alpha(4,4)$.

$$C_{4(1,1)} = \frac{0.39cb - 0.704c + 1.28b - 1.158}{2b + 1 + 2acb + 4ab},$$

$$C_{4(1,3)} = \frac{0.0975cb - 0.57536c + 0.32b - 1.08822}{2b + 1 + 2acb + 4ab}.$$

Now

$$C_{4(1,1)} \geqslant 0 \text{ if and only if } c \leqslant \frac{1.28b - 1.158}{0.704 - 0.39b} =: x, \quad \text{for } b \neq \frac{0.704}{0.39},$$

$$C_{4(1,3)} \geqslant 0 \text{ if and only if } c \leqslant \frac{0.32b - 1.08822}{0.57536 - 0.0975b} =: y, \quad \text{for } b \neq \frac{0.57536}{0.0975}.$$

So $c \leqslant \min\{x, y\}$. But $0 \leqslant b \leqslant 3$ implies $y < 0$, and $b \geqslant 3$ implies $x < 0$, thus for all $b \geqslant 0$ there holds $c < 0$.

Conclusion: $H_\alpha(4,4)$ cannot be written as BC with $B \in \mathbb{R}_+^{4\times 3}$, $C \in \mathbb{R}_+^{3\times 4}$, so pos-rank$(H_\alpha(4,4)) = 4$. $\square$

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