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RANDOM WALKS WITH STATIONARY INCREMENTS AND RENEWAL THEORY

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CONTENTS

PREFACE

The present work is a slightly revised version of my thesis, written under the supervision of my promotor dr. P.J. Holewijn and copromotor dr. A.A. Balkema.

The treatment of renewal theory, given here, is an interplay of two subjects: Palm theory and coupling. Palm theory is used to describe a limit distribution in renewal theory and a coupling technique is used to prove convergence results.

Dr. P.J. Holewijn induced me to study renewal theory. I benefited from discussions with dr. A.J. Lawrance on Palm theory and with dr. I. Meilijson on coupling and ergodic theory. During a long period I discussed my research with dr. A.A. Balkema. This resulted in considerable improvements of the book, both in proofs and presentation.

I thank the Mathematisch Centrum for the opportunity to publish this monograph in the series Mathematical Centre Tracts and all those at the Mathematisch Centrum who have contributed to its technical realization.

Amsterdam, June 1979 Henry Berbee

CHAPTER 0

PRELIMINARIES

0.0. INTRODUCTION

Renewal theory is one of the main areas of the theory of random walks with stationary, independent increments. In this monograph we want to study renewal theory for random walks with stationary increments. The independence assumption will be relaxed to certain forms of asymptotic independence. In this more general context some of the techniques that were useful if the increments are independent cannot be applied any more. Also some of the traditional theorems, such as Blackwell's theorem, lose their central position. As a consequence we have to determine the new problems that interest us in this more general theory and we want to develop the techniques with which they can be solved. The first part of this introduction indicates in which direction we are looking for answers to the questions sketched above. The second part of the introduction summarizes the results that we obtain.

Let $(S_n)_{n\geq 0}$ be a random walk with stationary, independent increments, started in $\{0\}$. Renewal theory discusses the asymptotic behaviour of the *renewal measure*

$$H(B) := \sum_{n\geq 0} P(S_n \in B), \quad B \in \mathcal{B}^1,$$

and some related topics in random walk theory. One of its main results is Blackwell's theorem. Assume that the increments of the random walk are strictly positive with finite expectation μ. Under these conditions the renewal measure is locally finite. Suppose that the distribution F of the increments is *nonlattice*, i.e. F is not concentrated on a discrete lattice $L_d := d\mathbb{Z}$, $d > 0$. Then Blackwell's theorem asserts that for any $b > 0$

$$\lim_{t\to\infty} H(t,t+b] = \frac{1}{\mu}\, b. \tag{0.0.1}$$

There is a slightly weaker result that is illuminating from our point of view. We associate the renewal measure H to the random walk $(S_n)_{n\geq 0}$ started in {0}. To the random walk $(S_n-h)_{n\geq 0}$ we associate the measure $T_h H$ defined by

$$T_h H(B) := \sum_{n\geq 0} P(S_n - h \in B), \quad B \in \mathcal{B}^1.$$

Here $T_h H$ is the translation of H over a distance h. As a consequence of Blackwell's theorem we have for any $b > 0$

(0.0.2) $$\lim_{t\to\infty} |T_h H(t,t+b] - H(t,t+b]| = 0.$$

This limit relation expresses that the measures $T_h H$ and H are asymptotically the same. In other words, we derived a property that might be called "loss of memory" about the initial position of the random walk. This loss of memory plays an important role in our discussions.

Loss of memory properties, expressed in terms of the random walk are also known. We can mention

(0.0.3) $$\lim_{n\to\infty} \| P_{S_n} - P_{S_n+h} \| = 0, \quad h \text{ real},$$

where $\|\cdot\|$ denotes total variation. This limit relation is valid for a random walk with stationary, independent increments if the distribution F of the increments has a density with respect to the Lebesgue measure ℓ. It is even valid under a slightly stronger condition. Call F *spread out* if some convolution F^{n*}, $n \geq 1$, is not singular with respect to ℓ. If F is spread out then the limit relation (0.0.3) holds too. We can prove that (0.0.3) holds if and only if F is spread out (see Section 5.3). Apparently it is natural to consider (0.0.3) under the spread out condition: these two properties are equivalent.

For distributions F concentrated on the rational numbers (0.0.3) is not valid. There is a simple way to see this. If h is irrational then P_{S_n} and P_{S_n+h} are mutually singular and hence the total variation expression in (0.0.3) equals 2. However, a result like (0.0.2) may be valid for such distributions F. The reason is that we did not use the total variation metric in (0.0.2). In Chapter 1 we shall see that by weakening the metric in (0.0.3) we can obtain a more generally valid loss of memory property.

A loss of memory result like (0.0.3) above can be proved by means of a coupling technique. We argue as follows. Suppose we succeed to construct a random walk $(S_n')_{n\geq 0}$, distributed as $(S_n)_{n\geq 0}$ such that

$$(0.0.4) \qquad S_n' = S_n + h, \qquad n \geq \tau,$$

where τ is a finite random time. By a simple inequality (see Lemma 1.1.1)

$$\| P_{S_n'} - P_{S_n+h} \| \leq 2\, P(S_n' \neq S_n+h), \qquad n \geq 1,$$

and hence for $n \to \infty$

$$\| P_{S_n} - P_{S_n+h} \| \leq 2\, P(\tau > n) \to 0,$$

so (0.0.3) holds. The problem that we did not consider above is the construction of the random walk $(S_n')_{n\geq 0}$. This is possible if F is spread out. In Chapter 1 we consider problems of this type. There we discuss a coupling technique due to ORNSTEIN [1969] and we prove loss of memory results and also Blackwell's theorem.

One of the reasons for the importance of Blackwell's theorem in the theory of random walks with independent, stationary increments, is that this theorem can be used to prove various other limit results in renewal theory (see SMITH [1958] or FELLER [1969]). The independence assumption has a crucial role in the proof of these results. Our aim is to relax the independence assumption. Then Blackwell's theorem can no longer be used to obtain these limit results. Instead we are forced to prove more detailed results than (0.0.1) to get a sufficiently rich theory. To this purpose we prove limit results for point processes.

Define the point process N_t^+, $t \geq 0$, on $(0,\infty)$ by

$$(0.0.5) \qquad N_t^+(B) := \sum_{n\geq 0} \chi_B(S_n - t),$$

where $B \subset (0,\infty)$ is any Borel set. The renewal measure H can be expressed in terms of these point processes because

$$H(t;t+h] = EN_t^+(0,h]$$

for positive h and t. Therefore, convergence results for N_t^+, $t \to \infty$, yield more detailed information than convergence results for the renewal measure H.

It is quite well possible to obtain also for point processes loss of memory properties by means of coupling. Suppose that the increments of the random walk $(S_n)_{n\geq 0}$ are strictly positive and assume that (0.0.4) holds.

Define $(N_t^+)'$ in terms of $(S_n')_{n\geq 0}$ analogous to 0.0.5. On the set $\{t > S_\tau\}$ we have

$$N_t^+ = (N_{t+h}^+)'.$$

For the distribution of these point processes we have the inequality

$$\| P_{N_t^+} - P_{(N_{t+h}^+)'} \| \leq 2\, P(N_t^+ \neq (N_{t+h}^+)')$$

and thus we get

$$\| P_{N_t^+} - P_{N_{t+h}^+} \| \leq 2\, P(S_\tau \geq t) \to 0$$

for $t \to \infty$. We can use this convergence result to obtain for example (0.0.2). Apparently if we succeed to construct (0.0.4), it is possible to obtain a strong convergence theorem. We already mentioned that if the distribution F of the increments is spread out and the increments are a stationary, independent sequence, (0.0.4) can be obtained. However, without the independence assumption it will be necessary to develop different methods to prove the required convergence theorems. For technical reasons we do not construct random walks $(S_n)_{n\geq 0}$ and $(S_n')_{n\geq 0}$ with property (0.0.4) in the dependent case. Nevertheless this property gives a good idea of the direction in which we try to find a solution of our renewal theoretic problems.

Blackwell's theorem contains a nonlattice condition. If the increments of the random walk are independent, this nonlattice condition can be formulated in terms of the distribution F of the increments. The reason is that if the increments are independent, F determines the distribution of the entire random walk. However, without the independence assumption this is no longer true and we also have to adapt the definition of nonlattice. A similar problem arises for the spread out condition. These problems will be treated in the last chapters.

The dependence of the increments can cause quite unexpected phenomena. For example for random walks with dependent increments transience does not imply that the renewal measure is locally finite. In Section 2.1 we shall discuss some examples of random walks with dependent, stationary increments.

In this monograph we study random walks with stationary increments and discrete time parameter. Only the first chapter assumes independence of the increments. Later on we assume some form of asymptotic independence. As we proceed with our study we require stronger asymptotic independence assumptions.

Most of our random walk results are formulated in terms of point processes. We only consider metric topologies and most of our convergence theorems use the total variation metric. The last three chapters form the main part of the monograph. There we refine the coupling technique and prove the main limit theorems.

Chapter 1 has an introductory character. It discusses coupling for random walks with stationary, independent increments and proves some well known results such as Blackwell's theorem.

Chapter 2 discusses some general properties of random walks. We only assume that the increments of the random walk are stationary and do not impose any independence assumption. Suppose $(S_n)_{n\geq 0}$ is a random walk started in $\{0\}$. It is often more natural to formulate the results in terms of an extension $(S_n)_{n\in\mathbb{Z}}$ of this random walk. This extended process satisfies

$$(0.0.6) \qquad S_0 = 0, \quad \xi_n = S_n - S_{n-1}, \qquad n \in \mathbb{Z},$$

where the increments $(\xi_n)_{n\in\mathbb{Z}}$ form a stationary sequence of real random variables. Such an extended process can always be constructed and has a unique distribution. In order not to overburden the notation we shorten $(\xi_n)_{n\in\mathbb{Z}}, (S_n)_{n\in\mathbb{Z}},\ldots$ to $\xi_{\mathbb{Z}}, S_{\mathbb{Z}},\ldots$.

Define for any Borel set B on the real line

$$(0.0.7) \qquad N_0(B) := \sum_{n\in\mathbb{Z}} \chi_B(S_n).$$

If $N_0(B)$ is finite for any bounded Borel set B, the random walk $S_{\mathbb{Z}}$ is called *transient*. In that case we consider N_0 as a point process with values in a measurable space $(\mathcal{N},\mathcal{D})$, where $\mathcal{N}$ is the set of locally finite, integer valued measures on the real line and $\mathcal{D}$ is some suitable σ-field on $\mathcal{N}$ (see Section 0.3).

Section 2.1 contains some simple results for transient random walks and also discusses some examples. Section 2.2 centers around an inequality, given by KAPLAN [1955], that states that for any positive h we have

$$EN_0(t,t+h] \leq EN_0(-h,h)$$

uniformly for all real t. This inequality expresses that the expected number of points of the random walk on an interval with length h is dominated by the expected value of the number of points on an interval (-h,h), centered at the origin. We give a simplified proof of this inequality. The main result of Section 2.2 is a related inequality. For any positive h and real t we have

$$P(N_0(t,t+h] \geq p) \leq 2\ P(N_0(-h,h) \geq p)$$

for any integer p. This inequality determines a bound on the size of point clusters on an interval of length h in terms of the distribution of the number of points on an interval (-h,h) centered at the origin. The inequality is used in integrability problems. The proofs of the inequalities above are combinatorial and therefore deviate from the main line of our investigations.

In Section 2.3 we discuss transience and recurrence of a random walk. We prove that it is possible to split up the probability space into two disjoint sets, I_t, the set of transience, and I_r, the set of recurrence. On the set of transience N_0 is locally finite, i.e. $N_0(B)$ is finite if B is bounded, and on the set of recurrence each point S_n is a limit point of the random walk, i.e. for each open neighbourhood O of S_n one has the equality $N_0(O) = \infty$. A proof that such a splitting of the probability space in a set of transience and a set of recurrence can be given, leads, quite unexpectedly to an elaborate argument that can be found in Section 2.3.

Chapter 3 forms the background for the renewal theoretic results of later chapters. Most results in this chapter are known or close to known results. Suppose that the random walk $S_{\mathbb{Z}}$ has stationary, strictly positive increments with finite expectation. Define for real t the point process N_t on the real line by

$$(0.0.8) \qquad N_t(B) := \sum_{n\in\mathbb{Z}} \chi_B(S_n-t), \qquad B \in \mathcal{B}^1.$$

The point process N_t is a translation of the point process N_0 defined by (0.0.7) over a distance t. Our aim is to derive convergence theorems for N_t, $t \to \infty$. In Section 3.1 we describe the distribution of a point process N that arises under weak conditions on the random walk as the limit process of N_t, $t \to \infty$. Section 3.2 discusses convergence of N_t to N for $t \to \infty$. One of the results shows that if the increments of the random walk form an ergodic, stationary process then

$$(0.0.9) \qquad \lim_{t\to\infty} \frac{1}{t}\int_0^t P(N_s \in D)\,ds = P(N\in D), \qquad D \in \mathcal{D}.$$

Thus under the weak requirement of ergodicity we already have Cesaro convergence of N_t to N. The result above can be strengthened somewhat: The process of increments is ergodic if and only if

$$(0.0.10) \qquad \lim_{t\to\infty} \frac{1}{t} \int_0^t P(N_s \in D \mid N_0 \in D_0)\,ds = P(N \in D), \qquad D \in \mathcal{D},$$

for any $D_0 \in \mathcal{D}$ for which the set $\{N_0 \in D_0\}$ has positive probability. The limit relation in (0.0.10) does not only express Cesaro convergence of N_t to N but also formulates a weak form of asymptotic independence between the sets $\{N_t \in D\}$ and $\{N_0 \in D_0\}$ for $t \to \infty$. In Chapter 6 we shall prove a result similar to (0.0.10) but for a stronger convergence concept than Cesaro convergence.

Chapter 4 forms a short interlude in our discussion of random walks and renewal theory. Its main aim is to discuss a concept of asymptotic independence for stationary processes, called weak Bernoulli. To define this concept we use a measure of dependence between random variables. Let X and Y be random variables (vectors) on the same probability space. The *dependence* of X and Y is defined as

$$\perp(X,Y) := \tfrac{1}{2}\| P_{X,Y} - P_X \times P_Y \|.$$

Clearly $\perp(X,Y)$ vanishes if and only if X and Y are independent. A stationary sequence $X_{\mathbb{Z}}$ of random variables is called *weak Bernoulli* if

$$(0.0.11) \qquad \lim_{n\to\infty} \perp((X_k)_{k\le 0}, (X_k)_{k\ge n}) = 0,$$

i.e. if the dependence between the past $(X_k)_{k\le 0}$ and the far future $(X_k)_{k\ge n}$ vanishes asymptotically. This condition of asymptotic independence is equivalent to

$$\lim_{n\to\infty} \| P_{(X_k)_{k\ge n} \mid (X_k)_{k\le 0}} - P_{(X_k)_{k\ge n}} \| = 0 \text{ a.s.}$$

For a Markov dependent process this simplifies to

$$\lim_{n\to\infty} \| P_{X_n \mid X_0} - P_{X_n} \| = 0 \text{ a.s.}$$

The first part of Chapter 4 contains some technical results concerning coupling and its relation with the measure of dependence defined above. At the end of Chapter 4 we characterize the weak Bernoulli property by coupling.

We prove that a stationary process $X_{\mathbb{Z}}$ is weak Bernoulli if and only if there exists a probability space with processes $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$, distributed as $X_{\mathbb{Z}}$, such that $(X''_n)_{n\leq 0}$ and $X'_{\mathbb{Z}}$ are independent and

$$X''_n = X'_n \quad \text{for } n \geq \tau,$$

where τ is a finite random time. We already saw an example of the use of coupling in renewal theory. In view of the coupling characterization of weak Bernoulli processes given above, it will hardly be surprising that we can use these processes in our generalization of renewal theory. In Chapter 6 they will have an important role.

In Chapters 5 and 6 we change our point of view slightly. We suppose that there is given a stationary sequence $X_{\mathbb{Z}}$ with values in a Borel space Γ, and assume that the increments of the random walk $S_{\mathbb{Z}}$ are given in terms of $X_{\mathbb{Z}}$ by

$$\xi_n = f(X_n), \quad n \in \mathbb{Z},$$

where f is a real measurable function on Γ. We shall say that the random walk $S_{\mathbb{Z}}$ is *controlled* by $X_{\mathbb{Z}}$. It is no restriction to assume that such a controlling process $X_{\mathbb{Z}}$ exists: in case it is not given explicitly we can always take $X_{\mathbb{Z}}$ to be the process of increments of the random walk. The reason that we study random walks controlled by a stationary process is that often the properties of the random walk are given in terms of this controlling process. It is also possible that $X_{\mathbb{Z}}$ has some useful property that is not available for the process of increments. For example $X_{\mathbb{Z}}$ might be Markov dependent. It is also possible to study semi-Markov processes in terms of random walks controlled by a stationary sequence.

In Chapter 5 we study for a random walk $S_{\mathbb{Z}}$, controlled by a stationary sequence $X_{\mathbb{Z}}$, the asymptotic behaviour of the distribution of S_n for $n \to \infty$. Section 5.3 is concerned with the limit relation

$$\lim_{n\to\infty} \| P_{S_n} - P_{S_n+h} \| = 0, \quad h \text{ real.} \tag{0.0.12}$$

Section 5.2 considers a similar, slightly weaker limit relation. The limit relation above expresses loss of memory about the initial position of the random walk. It implies that for any bounded interval I on the real line

$$\lim_{n\to\infty} P(S_n \in I) = 0$$

and thus describes a property that might be called the spreading behaviour of S_n on the real line for $n \to \infty$. The asymptotic independence condition that we impose in Chapter 5 is still weaker than (0.0.11). To obtain (0.0.12) we have to adapt the spread out condition. The proof of the results of Chapter 5 is prepared in Section 5.1. There we reduce the study of the limit relations to the investigations of sequences of independent, identically distributed (i.i.d.) random vectors. It is then possible to obtain (0.0.12) by coupling techniques.

The major difficulty of the theory in the last chapters lies in the reduction to the i.i.d. case. To this purpose we use an approximation method. It would go too far to discuss in this introduction the problems that we get involved with and therefore we have to refer at this point to Section 5.1. To overcome these problems we are obliged to make an assumption on the process $X_{\mathbb{Z}}$ (condition (5.1.3)). This assumption is satisfied in some important cases: if the sequence $X_{\mathbb{Z}}$ consists of countably valued random variables and if the sequence $X_{\mathbb{Z}}$ is Markov dependent.

In Chapter 6 we want to describe the convergence of N_t to N for $t \to \infty$. In particular we want to strengthen (0.0.9). Suppose that the random walk $S_{\mathbb{Z}}$ has strictly positive increments with finite expectation. We assume that the random walk $S_{\mathbb{Z}}$ is controlled by a weak Bernoulli process $X_{\mathbb{Z}}$. Under some additional conditions we prove in Section 6.3 that

$$(0.0.13) \qquad \lim_{t\to\infty} \| P_{N_t^+} - P_{N^+} \| = 0,$$

where N_t^+ and N^+ are the restrictions to $(0,\infty)$ of N_t and N. Blackwell's theorem can be obtained as a corollary. We obtain an even stronger result. Let N_t^- be the restriction of N_t to $(-\infty,0]$. We have

$$(0.0.14) \qquad \lim_{t\to\infty} \| P_{N_t^+ | N_0^-} - P_{N^+} \| = 0 \text{ a.s.}$$

This limit relation does not only strengthen (0.0.13), but it also expresses asymptotic independence of the past N_0^- and the future N_t^+ for $t \to \infty$. The proofs of this and other results are prepared in the first two sections of Chapter 6.

In Section 6.4 we consider two special cases, where it is possible to give a more complete treatment of the subject. We study the countable case, where $X_{\mathbb{Z}}$ is a sequence of countably valued random variables and the Markov case, where the sequence $X_{\mathbb{Z}}$ is Markov dependent. We obtain necessary and sufficient conditions for the validity of limit relations like (0.0.14) in terms of the properties of the process $X_{\mathbb{Z}}$. Especially the countable case seems to be neglected in the literature.

The limit theorems of Chapters 3 and 6 describe convergence to a stationary point process N (or N^+). In Section 6.5 we study the mixing properties of this limit process. The results we obtain in this direction are related to the theory of special flows, a subject studied in ergodic theory. There is a close relationship between special flows and renewal theory as it has been discussed in this monograph.

0.1. NOTATIONS AND CONVENTIONS

The k-dimensional Euclidean space is denoted by $\mathbb{R}^k$, $k \geq 1$. Unless otherwise stated we assume it is provided with the Borel σ-field $\mathcal{B}^k$, generated by its Euclidean topology. A *Borel space* is a measurable space $(\Gamma,\mathcal{T})$, for which there exists a 1-1 bimeasurable mapping from Γ onto a measurable subset of the real line. Often we write Γ instead of $(\Gamma,\mathcal{T})$. We assume throughout that the random variables that we consider have their values in a Borel space. As a consequence, if X and Y are random variables we can always select a regular version of the conditional distribution $P_{Y|X}$ of Y given X. The class of Borel spaces is fairly large. Each Euclidean space is a Borel space and moreover each countable product of Borel spaces is again a Borel space. So if $X := (X_n)_{n\geq 1}$ consists of random variables with values in a Borel space then X itself has its values in a Borel space.

The set of natural numbers {1,2,...} is denoted by $\mathbb{N}$ and the set of integers by $\mathbb{Z}$. To denote random vectors we use an uncommon but short notation. Suppose X_n, $n \in \mathbb{Z}$, is a sequence of random variables with values in a Borel space $(\Gamma,\mathcal{T})$. Let L be an integer set. We write $X_L := (X_\ell)_{\ell\in L}$. The random vector X_L is considered as a random variable with values in the Cartesian product $(\Gamma^L,\mathcal{T}^L) := \prod_{\ell\in L} (\Gamma,\mathcal{T})$. If k is an integer we write $X_{L+k} := (X_{\ell+k})_{\ell\in L}$ and consider X_{L+k} as a random variable with values in Γ^L. These notations are also used if X_n is defined for $n \in \mathbb{N}$ instead of $n \in \mathbb{Z}$, but then we require that $L \subset \mathbb{N}$ and $k \geq 0$. Using these notations stationarity of the process $X_{\mathbb{N}}$ can be described as the property that $X_{\mathbb{N}}$ is distributed as $X_{\mathbb{N}+1}$. The *tail σ-field* of a process $X_{\mathbb{N}}$ is the σ-field consisting of the events that are $X_{\mathbb{N}+n}$-measurable for all $n \geq 0$.

If μ is a signed measure on a measurable space we denote by μ^+ its positive and by $\mu^- := (-\mu)^+$ its negative part. The *total variation* $\|\mu\|$ of μ is defined as the sum of the total mass of μ^- and μ^+. The minimum $\mu \wedge \nu$ and the maximum $\mu \vee \nu$ of two nonnegative measures μ and ν are given by

$$\mu \wedge \nu = \mu - (\mu-\nu)^+, \quad \mu \vee \nu = \mu + (\nu-\mu)^+. \tag{0.1.1}$$

The indicator function χ_A of a set A is defined as the function with value 1 on A and value 0 on A^c. If events A and B satisfy $P(A \backslash B) = 0$ we write $A \subset B$ a.s. We write $A = B$ a.s. if both $A \subset B$ a.s. and $B \subset A$ a.s. If to each element A in a σ-field $\mathcal{A}$ there corresponds a set B in a σ-field $\mathcal{B}$ such that $A = B$ a.s. then we write $\mathcal{A} \subset \mathcal{B}$ a.s.

0.2. ERGODIC THEORY AND STATIONARY PROCESSES

Ergodic theorists use concepts such as measure preserving transformations and partitions where probabilists make use of expressions such as stationary processes and random variables. First we introduce the language of ergodic theory. Then we translate the corresponding concepts into the terminology of probability theory.

Let $(\Omega,\mathcal{A},P)$ be a probability space and T a measurable mapping from Ω onto itself. Suppose T is *measure preserving*, i.e. $P(T^{-1}A) = P(A)$ for all $A \in \mathcal{A}$. A measurable set A is called *invariant* if $T^{-1}A = A$ and *a.s. invariant* if $T^{-1}A = A$ a.s. If all invariant sets have probability 0 or 1 we call T *ergodic*.

A process $X_{\mathbb{Z}}$ with values in a Borel space $(\Gamma,\mathcal{T})$ is called *stationary* if $(X_n)_{n\in\mathbb{Z}}$ and $(X_{n+1})_{n\in\mathbb{Z}}$ have the same distribution. The connection with the setting above can be given as follows. Let $(\Omega,\mathcal{A})$ be the Cartesian product $(\Gamma^{\mathbb{Z}},\mathcal{T}^{\mathbb{Z}}) := \prod_{n\in\mathbb{Z}} (\Gamma,\mathcal{T})$ and take P to be the distribution of $X_{\mathbb{Z}}$. Let T be the *shift transformation* on $\Gamma^{\mathbb{Z}}$ defined by

$$(T\gamma)_n := \gamma_{n+1}, \quad n \in \mathbb{Z}, \ \gamma \in \Gamma^{\mathbb{Z}}.$$

The *(a.s.) invariant sets* for $X_{\mathbb{Z}}$ are the events of the form $\{X_{\mathbb{Z}} \in A\}$ with $A \in \mathcal{A}$ (a.s.) invariant under T. The process $X_{\mathbb{Z}}$ is called *ergodic* if the invariant sets have probability 0 or 1.

A process $X_{\mathbb{N}}$ with values in a Borel space $(\Gamma,\mathcal{T})$ is called *stationary* if $P_{X_{\mathbb{N}}} = P_{X_{\mathbb{N}+1}}$. Take $(\Omega,\mathcal{A})$ to be $\prod_{n\in\mathbb{N}} (\Gamma,\mathcal{T})$, let $P := P_{X_{\mathbb{N}}}$ and let the *shift transformation* T on $\Gamma^{\mathbb{N}}$ be defined by

$$(T\gamma)_n := \gamma_{n+1}, \quad n \in \mathbb{N}.$$

Define (a.s.) invariant sets and ergodicity of $X_{\mathbb{N}}$ as above. By using the

Kolmogorov extension theorem we can extend $X_{\mathbb{N}}$ to a stationary process $X_{\mathbb{Z}}$ with a uniquely determined distribution (see BREIMAN [Proposition 6.5]).

The following result is known as the *individual ergodic theorem*.

THEOREM 0.2.1. *Let* $\xi_{\mathbb{N}}$ *be a stationary sequence of real random variables with* $E\xi_1^- < \infty$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \xi_k = E(\xi_1 | J_\xi) \quad a.s.,$$

where J_ξ *is the* σ*-field of invariant sets for* $\xi_{\mathbb{N}}$.

PROOF. See BREIMAN [Theorem 6.28] for the proof in case $E|\xi_1| < \infty$. Apply this result and a truncation argument to get the theorem in case $E\xi_1 = \infty$. □

References to books, for example to Breiman's book on probability, contain the name of the author and, between brackets, a location in the book. If necessary, also its date of publication is mentioned. References to articles mention the name of the author and the date of publication.

0.3. POINT PROCESSES

Several of our results are given their clearest formulation in terms of point processes. This section introduces some of the notations that are involved. First we discuss point processes on the real line. Because some useful examples can be covered by assigning "marks" to the points of a point process, we also discuss marked point processes on the real line. Literature on point processes can be found in RIPLEY [1976] and KALLENBERG [1976].

A point process on the real line can be defined in the following way. Let N be the set of integer valued measures m on the real line that are finite on bounded intervals. Suppose N is provided with the σ-field $\mathcal{D}$ generated by the mappings

$$m \to m(B), \quad B \in \mathcal{B}^1.$$

A measurable mapping N on a probability space with values in $(N,\mathcal{D})$ is called a *point process on the real line*. We allow also that N has its values up to a null set in the measurable space $(N,\mathcal{D})$. The *distribution* of N is the measure P_N on $(N,\mathcal{D})$ defined by

$$P_N(D) := P(N \in D), \quad D \in \mathcal{D}.$$

The *intensity measure* λ of N is defined by

$$\lambda(B) := EN(B), \quad B \in \mathcal{B}^1.$$

A real number x is called a *point* of $m \in \mathcal{N}$ if $m\{x\} > 0$ and a *multiple point* if $m\{x\} > 1$. Let $\mathcal{N}_1 \subset \mathcal{N}$ be the set of measures $m \in \mathcal{N}$ without multiple points. A point process is called *simple* if it has its values up to a null set in $\mathcal{N}_1$. Identify the measures $m \in \mathcal{N}_1$ with their support. With this identification a simple point process becomes a *random set*.

Define the *translation* T_t, t real, on the real line by

$$T_t x := x - t, \quad x \text{ real}.$$

This determines a translation of the elements $m \in \mathcal{N}$ by

$$T_t m(B) := m(T_t^{-1}B), \quad B \in \mathcal{B}^1.$$

If f is a function on the real line, define

$$T_t f(x) := f(T_t x), \quad x \text{ real},$$

and note that

$$T_t \chi_B = \chi_{T_t^{-1}B}, \quad B \in \mathcal{B}^1.$$

The point process N is called *stationary* if the translation T_t as a mapping on $\mathcal{N}$ for each real t is measure preserving on $(\mathcal{N},\mathcal{D},P_N)$. A measurable set $D \in \mathcal{D}$ is *invariant* under translations if $T_t^{-1}D = D$ for all real t. The point process N is called *ergodic* if each set $\{N \in D\}$ with D invariant under translations has probability 0 or 1. If N is stationary then

$$\text{(0.3.1)} \qquad \{N = 0\} \cup \{N(-\infty,0] = N(0,\infty) = \infty\}$$

has probability 1 by Poincaré's recurrence principle (see HALMOS [1956,p.10] or BREIMAN [Proposition 6.38]). If N is stationary its intensity measure has the form $c\ell$ with ℓ the Lebesgue measure and $c \in [0,\infty]$ a constant, called the *intensity* (see HALMOS [1950, XI.60]). In that case the intensity equals the expected value of the random variable $\lim_{t\to\infty} \frac{1}{t} N(0,t]$, as follows from a simple application of the ergodic theorem.

PROPOSITION 0.3.1. *$(\mathcal{N},\mathcal{D})$ is a Borel space.*

PROOF. Each measure $m \in \mathcal{N}$ can be represented as

$$m(B) = \sum_{n\in\mathbb{Z}} \chi_B(u_n), \qquad B \in \mathcal{B}^1,$$

where

$$\ldots \le u_{-1} \le u_0 \le 0 < u_1 \le \ldots$$

Here we assume that u_n has its values in the extended real line $\overline{\mathbb{R}}^1 := [-\infty,\infty]$. Thus also finite measures m can be represented in this way. We have obtained an invertible mapping from $\mathcal{N}$ onto a measurable subset of $\prod_{n\in\mathbb{Z}} \overline{\mathbb{R}}^1$. It is easily seen that this mapping and its inverse are measurable. Hence $(\mathcal{N},\mathcal{D})$ is a Borel space. □

A well known topology on the set of distributions $\mathcal{P}$ of point processes on the real line is the weak topology with respect to the vague topology on $\mathcal{N}$. However, a stronger topology is more useful for the limit problems that we discuss. A definition of this topology is given as follows. If $P \in \mathcal{P}$ and ν is an absolutely continuous probability measure on the real line define $\nu * P$ by

$$(0.3.2) \qquad \nu * P(D) := \int_{\mathbb{R}^1} T_t P(D)\, d\nu(t),$$

where $T_t P(D) := P(T_t^{-1} D)$, t real. Define the pseudometric d_ν on $\mathcal{P}$ by

$$(0.3.3) \qquad d_\nu(P_1,P_2) := \| \nu*P_1 - \nu*P_2 \|$$

and let

$$(0.3.4) \qquad d(P_1,P_2) := \sum_{n\ge 1} 2^{-n}\, d_{\nu_{1/n}}(P_1,P_2),$$

where ν_ε, $\varepsilon > 0$, is the homogeneous distribution on $(0,\varepsilon)$. In the appendix we show that d is a metric on $\mathcal{P}$. We also prove that the topology induced by d is weaker than the topology introduced by the total variation metric on $\mathcal{P}$ and that the d-topology is stronger than the weak topology on $\mathcal{P}$ with respect to the vague topology on $\mathcal{N}$. KALLENBERG [1976] discusses this weak topology.

The reader will be able to reformulate the definitions stated above, for point processes on $(0,\infty)$, and also for point processes on $\mathbb{Z}$ or $\mathbb{N}$. The restriction $\bar{N}$ of a point process N on the real line to an interval I is defined as the point process

$$\bar{N}(B) := N(B)$$

for all Borel sets $B \subset I$.

To define a marked point process only a few changes are necessary. Let $(\Gamma,\mathcal{T})$ be a Borel space, to be called the *mark space*. If m is a measure on $\mathbb{R}^1 \times \Gamma$ define the measure m^s on the real line by

$$m^s(B) := m(B \times \Gamma), \quad B \in \mathcal{B}^1.$$

Let $\mathcal{N}$ be the set of all integer valued measures m on $\mathbb{R}^1 \times \Gamma$ such that m^s is finite on bounded intervals. Let the σ-field $\mathcal{D}$ on $\mathcal{N}$ be generated by the mappings

$$m \to m(C)$$

on $\mathcal{N}$ with $C \subset \mathbb{R}^1 \times \Gamma$ measurable. A *marked point process* N on the real line with mark space Γ is a measurable mapping on a probability space with values in $(\mathcal{N},\mathcal{D})$. We also call N a marked point process on $\mathbb{R}^1 \times \Gamma$. The translation T_t on $\mathbb{R}^1 \times \Gamma$ is defined by

$$T_t(x,\gamma) := (x-t,\gamma).$$

Define the translation of measures and functions on $\mathbb{R}^1 \times \Gamma$ as above for point processes on the real line. Let also stationarity and ergodicity be defined as above and introduce d_ν and d on the set $\mathcal{P}$ of distributions of marked point processes as above with $\nu * P$ defined by (0.3.2). If N is a marked point process on $\mathbb{R}^1 \times \Gamma$ let its *projection* N^s on the real line be defined as

$$N^s(B) := N(B \times \Gamma), \quad B \in \mathcal{B}^1.$$

If N is stationary define its *intensity* as the intensity of N^s. N is called *simple* if N^s is simple.

For marked point processes on an interval the definitions can be given similarly as above. The translation T_tN, $t \geq 0$, of a marked point process N on $(0,\infty) \times \Gamma$ is defined as the marked point process on $(0,\infty) \times \Gamma$, given by

$$T_tN(B) := N(T_t^{-1}B)$$

with $B \subset (0,\infty) \times \Gamma$ measurable.

CHAPTER 1

RENEWAL THEORY AND COUPLING; AN INTRODUCTION

The renewal theorem for random walks with stationary, independent increments can be derived in various ways. In recent years some attention is given to proofs of this theorem that make use of coupling techniques. These coupling techniques play an important role in this monograph. Because they are relatively unknown it seems desirable to show here how the renewal theorem can be derived using such a coupling technique. We discuss the merits of this coupling technique from several points of view and at the end of the chapter we indicate where a possibility exists to apply coupling techniques also for random walks with dependent increments.

The renewal theorem in the lattice case was stated and proved by ERDÖS, FELLER and POLLARD [1949]. Their proof was immediately generalized by BLACKWELL [1948,1953] who gave the nonlattice analogue of the theorem. The final version of the renewal theorem without conditions on positivity of increments and existence of expectation was given by FELLER and OREY [1961]. At present many different methods exist to prove the renewal theorem. FELLER [1971, Chapter XI] gives a simple proof using a lemma due to CHOQUET and DENY [1960]. This approach was useful to obtain renewal theorems for semi-Markov chains (see KESTEN [1974]) and for random walks on groups (see REVUZ [1975]). FELLER and OREY [1961] presented a Fourier analytic method of proof that could be used to obtain results on speed of convergence in renewal theorems (see SMITH [1966] and STONE and WAINGER [1967]). Apart from analytic methods of proof there are also methods of a more probabilistic nature. We describe a coupling technique that was given in ORNSTEIN [1969, Theorem 0.7], where it is developed to study the asymptotic behaviour of the convolutions of a distribution

Let us first describe what is meant by coupling. Suppose $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$ are sequences of random variables with values in the same Borel space. Let K be an integer set. We say that X' and X" are *coupled over* K if

$$P(X'_K = X''_K) = 1.$$

A similar definition can be given for sequences $X'_{\mathbb{N}}$ and $X''_{\mathbb{N}}$ for $K \subset \mathbb{N}$. To describe asymptotic properties we use the concept successful coupling. We say the processes X' and X" are *successfully coupled* if

$$\lim_{n\to\infty} P(X'_{\mathbb{N}+n} = X''_{\mathbb{N}+n}) = 1.$$

By the lemma below we may conclude that if X' and X" are successfully coupled, then

$$(1.1.1) \qquad \|P_{X'_{\mathbb{N}+n}} - P_{X''_{\mathbb{N}+n}}\| \le 2\, P(X'_{\mathbb{N}+n} \neq X''_{\mathbb{N}+n}) \to 0$$

for $n \to \infty$. So successful coupling implies that the processes $X'_{\mathbb{N}}$ and $X''_{\mathbb{N}}$ asymptotically have the same distribution.

<u>LEMMA 1.1.1</u>. *Let* Y' *and* Y" *be random variables with values in the same Borel space. Then we have*

$$\delta(Y',Y") := \tfrac{1}{2}\|P_{Y'}-P_{Y"}\| \le P(Y'\neq Y").$$

<u>PROOF</u>. Observe that for all measurable sets B

$$\mu(B) := P(Y'\in B, Y'=Y") \le P(Y'\in B).$$

A similar relation holds with Y' and Y" interchanged. Hence

$$\|P_{Y'}-P_{Y"}\| = \|(P_{Y'}-\mu) - (P_{Y"}-\mu)\| \le 2\|P_{Y'}-\mu\| = 2 \quad P(Y'\neq Y"). \qquad \square$$

The lemma above shows that there is a narrow relationship between coupling and certain expressions using total variation. In Sections 4.1 and 4.2 we study this relationship more closely. Among other results we prove that if two distributions $P_{Y'}$ and $P_{Y"}$ on the same Borel space are prescribed, we can construct a probability space with random variables Y' and Y", having the prescribed distributions, such that the inequality in the lemma above is satisfied as an equality.

In general, there are several techniques to prove that a coupling is successful. The proof of the theorem below forms an example. This proof uses the so-called Ornstein coupling. The theorem discusses the spreading behaviour of a random walk and is formulated by DOBRUSHIN [1956].

We need the following concepts. Define a *lattice* L_d by

$$\begin{aligned} (1.1.2) \qquad L_d &:= \{0\}, & d &= \infty, \\ &:= d\mathbb{Z}, & 0 &< d < \infty, \\ &:= \mathbb{R}^1, & d &= 0. \end{aligned}$$

If $0 < d \le \infty$ the lattice L_d is called *discrete*. The smallest lattice on which a distribution F on the real line is concentrated, is called its *minimal lattice*. In case it is discrete, F is called *lattice*. Let L_d be the smallest lattice such that F is concentrated on a non centered lattice $c + L_d$, c real. It is called the *minimal weak lattice* of F and if it is discrete F is called *weakly lattice*. The first paragraph of the proof below shows that the minimal weak lattice exists.

THEOREM 1.1.2 (DOBRUSHIN). *Let F be a probability distribution with a discrete minimal weak lattice* L_d. *Suppose* $(S_n)_{n\ge 0}$ *is a random walk with independent, F-distributed increments with* $S_0 \equiv 0$. *Then for all* $h \in L_d$

$$\lim_{n\to\infty} \| P_{S_n} - P_{S_n+h} \| = 0.$$

PROOF. First we describe the minimal weak lattice width d in an alternative way. The distribution of F is by assumption discrete. Consider

$$d_1 := \text{g.c.d. } \{x_2 - x_1 : F\{x_1\}, F\{x_2\} > 0\}.$$

By the definition of the weak lattice each element of the set above is contained in L_d. If $F\{c\} > 0$ then it is easily seen that F is concentrated on $c + L_d$. These two assertions together imply that $d = d_1$.

Let $X_0, X_1, \ldots, X_0', X_1', \ldots$ be independent random variables, X_0 and X_0' having arbitrary distributions G and G', concentrated on L_d and let the other random variables have distribution F. First we consider the case that F has finite mean. The differences

$$\Delta_k := X_k' - X_k, \quad k \ge 1,$$

have a symmetric distribution with finite, vanishing mean, and by the definition of d_1, with minimal lattice $L_{d_1} = L_d$. The differences

$$S_n' - S_n = X_0' - X_0 + \sum_{k=1}^{n} \Delta_k, \quad n \ge 0,$$

form a random walk on L_d with increments with vanishing expectation and therefore the meeting time τ of S_n', $n \ge 0$, and S_n, $n \ge 0$, defined by

$$\tau := \inf\{n \geq 0: S'_n = S_n\}$$

is defined with probability 1 as a finite random variable by the Chung Fuchs theorem (see BREIMAN [Section 3.7]). The random time τ is a Markov time for (S_n, S'_n), $n \geq 0$. Define a new process $(S''_n)_{n\geq 0}$ by

$$S''_n := S_n, \quad 0 \leq n \leq \tau,$$
$$:= S'_n, \quad n > \tau.$$

Write $S''_n := \sum_{k=0}^n X''_k$, $n \geq 0$. Note that $(S''_n)_{n\geq 0}$ is formed out of $(S_n)_{n\geq 0}$ by replacing the increments X_k, $k > \tau$, by X'_k, $k > \tau$. Using the Markov property it is easily seen that this replacement does not affect the distribution. So $(S''_n)_{n\geq 0}$ is distributed as $(S_n)_{n\geq 0}$. Note that S'_n and S''_n are coupled for $n > \tau$, and hence

$$\| P_{S''_n} - P_{S'_n} \| \leq 2\, P(\tau \geq n) \to 0$$

for $n \to \infty$. Because S''_n and S_n have the same distribution we have

$$\lim_{n\to\infty} \| G * F^{n*} - G' * F^{n*} \| = 0.$$

This implies the assertion for distributions F with finite mean.

The proof above uses that (X_k, X'_k), $k \geq 1$, is distributed as a pair of independent, F-distributed random variables. However, the proof also works if (X_k, X'_k), $k \geq 1$, have a common bivariate distribution with marginals F, such that L_d is the minimal lattice of the distribution of Δ_k.

If the mean of F is not defined as a finite number the coupling proof above fails at one point: It is not clear that the expected value of Δ_k exists. To repair this we use the remark in the preceding paragraph. Write $F = F_m + F^m$, where F_m is the restriction of F to $(-m,m)$. Choose m so large that the minimal weak lattice of $H_m := \frac{1}{\|F_m\|} F_m$ is L_d. Let the common distribution of (X_k, X'_k), $k \geq 1$, be given by

$$P_{X_k, X'_k} = \left(\frac{1}{\|F_m\|} F_m \times F_m\right) + F^m_d = \|F_m\|\; H_m \times H_m + F^m_d.$$

Here F^m_d is the measure concentrated on the diagonal of $\mathbb{R}^2$ that has marginals F^m. It is easily checked that the pair (X_k, X'_k), $k \geq 1$, has marginals F. Moreover, on $A := \{|X_k| \geq m\}$ we have $X_k = X'_k$ and on A^c the pair (X_k, X'_k) is $H_m \times H_m$-distributed. Hence, Δ_k vanishes on A and Δ_k is on A^c the difference of independent, H_m-distributed random variables. Hence, the distribution of Δ_k

has minimal lattice L_d. Also $|\Delta_k| \leq 2m$ and $E\Delta_k = 0$ exists. So with this choice of $P_{X_k,X_k'}$ the coupling proof can be used again. □

The coupling in the proof above is shown to be successful by means of the Chung Fuchs theorem on recurrence of a random walk. The same successful coupling can be used to derive the renewal theorem for the lattice case.

THEOREM 1.1.3 (FELLER, ERDŐS, POLLARD). *Let* F *be a distribution on* $\{0,1,\ldots\}$ *with minimal lattice* $\mathbb{Z}$ *and finite, positive mean* μ. *Suppose that* S_n, $n \geq 0$, *is a random walk with independent,* F*-distributed increments such that* $S_0 \equiv 0$. *Then the renewal measure* $H(B) := \sum_{n\geq 0} P(S_n \in B)$, $B \in \mathcal{B}^1$, *satisfies*

$$\lim_{n\to\infty} H\{n\} = \frac{1}{\mu}.$$

PROOF. Suppose first that $\mathbb{Z}$ is the minimal weak lattice of F. We use the coupling construction in the proof of Theorem 1.1.2. Let G be degenerate at $\{0\}$ and define G' to be the so-called *survivor distribution*, satisfying

$$G'\{i\} := \frac{F(i,\infty)}{\mu}, \qquad i \geq 0.$$

Denote by N the point process on $\{0,1,\ldots\}$ counting the occurrences of the S_n-points, i.e.

$$N(B) := \sum_{n\geq 0} \chi_B(S_n)$$

for $B \subset \{0,1,\ldots\}$. Let N, N' and N" be the corresponding point processes for the S-, S'- and S"-processes, defined in the proof of Theorem 1.1.2.

First we show that with this definition, N' is stationary. Let σ be the first entrance time of $(S_n')_{n\geq 0}$ into $[1,\infty)$ and consider $\bar{S}_n := S_{\sigma+n}' - 1$, $n \geq 0$. Observe that $\bar{S}_0$ is distributed as S_0':

$$\begin{aligned} P(S_\sigma'=k) &= P(X_0'=k) + \sum_{j=1}^{\infty} P(X_0'=X_1'=\ldots=X_{j-1}'=0,\ X_j'=k) \\ &= \frac{1}{\mu} F(k,\infty) + \sum_{j=1}^{\infty} \frac{F(0,\infty)}{\mu} F(0)^{j-1} F\{k\} = \frac{F(k-1,\infty)}{\mu}, \quad k \geq 1. \end{aligned}$$

Using that σ is a Markov time for $(S_n')_{n\geq 0}$ it is easily shown that $(S_n')_{n\geq 0}$ is distributed as $(\bar{S}_n)_{n\geq 0}$. It follows that N' is a stationary point process on $\{0,1,2,\ldots\}$. Remark also that

$$EN'\{k\} = P(N'\{k\} > 0) \sum_{j=1}^{\infty} F\{0\}^j.$$

By stationarity and the definition of the survivor distribution we obtain $EN'\{k\} = \frac{1}{\mu}$, $k \geq 0$.

We now apply the coupling construction in the proof of Theorem 1.1.2. Note that on the set $\{n: n > S'_\tau = S''_\tau\}$ the point processes N' and N" coincide. Therefore,

$$\| P_{N''\{k\}} - P_{N'\{k\}} \| \leq 2\, P(k \leq S'_\tau) \to 0$$

for $k \to \infty$ by Lemma 1.1.1. Because N and N" are equally distributed

$$H\{k\} - \frac{1}{\mu} = EN''\{k\} - EN'\{k\}$$

$$= (P(N''\{k\} > 0) - P(N'\{k\} > 0)) \left(\sum_{j=1}^{\infty} F\{0\}^j \right) \to 0$$

for $k \to \infty$. This proves the assertion if $\mathbb{Z}$ is the minimal weak lattice of F.

If F has minimal lattice $\mathbb{Z}$ but minimal weak lattice $L_d \subsetneq \mathbb{Z}$ we employ a simple trick used in MEILIJSON [1975]. We insert an atom at {0} in F, i.e. we replace F by $\bar{F} := p\, \delta_0 + (1-p)F$, where $0 < p < 1$, and δ_0 is the probability measure degenerate at {0}. Observe that $\bar{F}$ has minimal weak lattice $\mathbb{Z}$. We can express the renewal measure H associated to F in the renewal measure $\bar{H}$ associated to $\bar{F}$. This can be done as follows. We construct a sequence of independent, $\bar{F}$-distributed increments. Generate repeatedly and independently with probability p a zero and with probability 1-p an F-distributed random variable. Clearly the waiting time from the n-th to the (n+1)-th generation of an F-distributed random variable is geometrically distributed with parameter p and expectation $\frac{1}{1-p}$. Furthermore, the waiting times and the F-distributed increments are independent. It follows easily that

$$\bar{H} = \frac{1}{1-p} H.$$

By the first part of the proof we have, because $\bar{F}$ has minimal lattice $\mathbb{Z}$,

$$H\{n\} = (1-p)\bar{H}\{n\} \to (1-p) \cdot \frac{1}{\bar{\mu}} = \frac{1}{\mu}.$$

Here we used that the mean $\bar{\mu}$ of $\bar{F}$ is $\bar{\mu} = (1-p)\mu$. □

Apart from some technical details the proof above is an application of the Ornstein coupling described in the proof of Theorem 1.1.2. The best known application of a coupling argument in the field of Markov chains with a countable state space is the classical coupling. The idea goes back to DOEBLIN [1937] and is given in the proof of the theorem below. The theorem considers a Markov chain with countable state space. Consult PITMAN [1974] for more details.

THEOREM 1.1.4. *Let $X_{\mathbb{N}}$ be an irreducible, positive recurrent, aperiodic Markov chain started with arbitrary initial distribution. There is an invariant distribution π such that*

$$\lim_{n\to\infty} \| P_{X_n} - \pi \| = 0.$$

PROOF. As is well known, for irreducible Markov chains positive recurrence is equivalent with the existence of an invariant distribution. Let π be invariant for the Markov chain $X_{\mathbb{N}}$. Suppose that $X'_{\mathbb{N}}$ is a Markov chain, independent of $X_{\mathbb{N}}$, with the same transition probabilities as $X_{\mathbb{N}}$ and with initial distribution π. Note that (X_n, X'_n), $n \in \mathbb{N}$, is an irreducible, aperiodic Markov chain with an invariant distribution $\pi\times\pi$. Hence (X_n, X'_n), $n \in \mathbb{N}$, is positive recurrent and, therefore, the meeting time

$$\tau := \inf\{n \in \mathbb{N}: \ X_n = X'_n\}$$

is defined as a finite random variable with probability 1. The process $X''_{\mathbb{N}}$ defined by

$$\begin{aligned} X''_n &:= X_n, \quad 1 \le n \le \tau, \\ &:= X'_n, \quad n > \tau, \end{aligned}$$

is distributed as $X_{\mathbb{N}}$, as can be seen by using the Markov property. Because also $X'_{\mathbb{N}}$ and $X''_{\mathbb{N}}$ are successfully coupled we have for $n \to \infty$

$$\| P_{X_n} - \pi \| = \| P_{X''_n} - P_{X'_n} \| \le 2\, P(\tau \ge n) \to 0. \qquad \square$$

It is well known (see PITMAN [1974]) that Theorem 1.1.3 can be derived from Theorem 1.1.4. So the classical coupling yields another proof of the renewal theorem. Let us compare the classical and the Ornstein coupling.

First we investigate the Ornstein coupling. Consider the meeting time τ of the processes S_n and S'_n, $n \ge 1$, occurring in the proof of Theorem 1.1.2. We can see as follows that $E\tau = \infty$. Consider the random walk $\sum_{k=1}^{n} \Delta_k$, $n \ge 1$.

Because $E\Delta_k = 0$ we have by FELLER [XII.2] that the expected entrance time of this random walk into $(0,\infty)$ is infinite. Similarly this holds for the entrance time into $(-\infty,0)$. Hence if G and G' are not degenerate at the same point, that is if with positive probability $S_0'-S_0$ is positive or negative, then the entrance time τ of $\sum_{k=1}^{n} \Delta_k$ into the point $-(S_0'-S_0)$ has infinite expectation, i.e. $E\tau = \infty$. As a consequence the Ornstein coupling is a very slow coupling device. However, it can be used for distributions F that have infinite mean. It is interesting to compare it at this point with the classical coupling.

For the classical coupling it is possible under conditions on moments of recurrence times to give estimates for the speed of convergence of $P(\tau \geq n)$ to 0 for $n \to \infty$. In this way PITMAN [1974] and LINDVALL [1977b] obtain estimates for the asymptotic behaviour of the renewal measure. Clearly in this respect the classical coupling is more useful than the Ornstein coupling. The classical coupling has a disadvantage. An example in FREEDMAN [p.45] shows that null recurrent Markov chains cannot be studied using the classical coupling. The classical coupling cannot be used to give a proof for the ergodic theorem for Markov chains due to OREY [1962] for the null recurrent case. However, by means of the Ornstein coupling Orey's theorem can be proved also in the null recurrent case. In that case one uses the Ornstein coupling for distributions with infinite mean.

If a distribution F on the real line is not concentrated on a discrete lattice, then F is called *nonlattice*. If F is not weakly lattice, then we call F *strongly nonlattice*. The result below is the continuous analogue of Theorems 1.1.2 and 1.1.3. Part (i) is due to KERSTAN and MATTHES [1965] and part (ii) is Blackwell's theorem. The method of proof is again the Ornstein coupling, but now we use neighbourhood recurrence of the random walk.

The measure ν occurring in assertion (i) of the theorem below has only a technical significance: It provides a smoothing of the distributions and without it, conclusion (i) does not hold (see also Example 3.2.6).

THEOREM 1.1.5. *Let* F *be a probability distribution on the real line. Suppose* S_n, $n \geq 0$, *is a random walk with independent,* F*-distributed increments and with* $S_0 \equiv 0$.

(i) *Let* F *be strongly nonlattice. For all absolutely continuous probability measures* ν *on the real line and for any real* h *we have*

$$\lim_{n\to\infty} \| \nu * P_{S_n} - \nu * P_{S_n+h} \| = 0.$$

(ii) *Let* F *be a nonlattice distribution concentrated on* $[0,\infty)$ *with finite positive mean* μ. *The renewal measure* $H(B) := \sum_{n\geq 0} P(S_n \in B)$, $B \in \mathcal{B}^1$, *satisfies for every positive* h

$$\lim_{t\to\infty} H(t,t+h] = \frac{h}{\mu}.$$

PROOF of (i). A real number x is called *point of increase* of F if $F(x-\varepsilon,x+\varepsilon) > 0$ for all positive ε. Remark that the set

$$\{y-x:\ x \text{ and } y \text{ are points of increase of } F\}$$

is not contained in any discrete lattice because F is strongly nonlattice. First suppose F has finite mean. Let $X_0, X_1, \ldots, X_0', X_1', \ldots$ be independent, X_0 and X_0' having arbitrary distributions G and G' and suppose the other random variables have distribution F. Remark that $\Delta_k = X_k' - X_k$, $k \geq 1$, has expectation 0 and is nonlattice. Let $S_n := \sum_{k=0}^n X_k$, $n \geq 0$, and $S_n' = \sum_{k=0}^n X_k'$, $n \geq 0$, and observe that by the Chung Fuchs theorem

$$S_n' - S_n = S_0' - S_0 + \sum_{k=1}^{n} \Delta_k, \qquad n \geq 0,$$

is a neighbourhood recurrent random walk. Let η be positive. It follows that

$$\tau := \inf\{n \geq 0:\ |S_n' - S_n| < \eta\}$$

is finite a.s. As in the proof of Proposition 3.1.2 it follows that $S_n'' := \sum_{k=0}^n X_k''$, $n \geq 0$, with

$$\begin{aligned} X_k'' &:= X_k, & 0 \leq k \leq \tau, \\ &:= X_k', & k > \tau, \end{aligned}$$

is distributed as S_n, $n \geq 0$. Furthermore, by the definition of τ

$$S_n'' = S_n' + D, \qquad n \geq \tau, \tag{1.1.3}$$

with $|D| < \eta$ a.s. Hence for any Borel set $B \in \mathcal{B}^1$ we have on $\{n \geq \tau\}$

$$\begin{aligned} &\left|\frac{1}{\varepsilon}\int_0^\varepsilon \chi_{T_t B}(S_n'')\,dt - \frac{1}{\varepsilon}\int_0^\varepsilon \chi_{T_t B}(S_n')\,dt\right| \\ \leq &\left|\frac{1}{\varepsilon}\int_0^\varepsilon \chi_B(S_n'+D+t)\,dt - \frac{1}{\varepsilon}\int_0^\varepsilon \chi_B(S_n'+t)\,dt\right| \leq 2\,\frac{|D|}{\varepsilon} \leq 2\,\frac{\eta}{\varepsilon}\,. \end{aligned}$$

Let ν_ε be the homogeneous distribution on $(0,\varepsilon)$. The expectation of the terms occurring in the difference are $\nu_\varepsilon * P_{S''_n}(B)$ and $\nu_\varepsilon * P_{S'_n}(B)$. It follows that by the definition of total variation

$$\| \nu_\varepsilon * P_{S''_n} - \nu_\varepsilon * P_{S'_n} \| \leq 2 \ (P(n < \tau) + 2 \ \frac{\eta}{\varepsilon}) \to 4 \ \frac{\eta}{\varepsilon}$$

for $n \to \infty$. Because η is arbitrary this implies (i) for $\nu = \nu_\varepsilon$. By the remark to inequality A.1 we obtain (i) for general ν (see the appendix). If F has no finite mean one uses the technique described at the end of the proof of Theorem 1.1.2.

PROOF of (ii). Suppose for the moment that F is strongly nonlattice and consider the proof of (i). Take G to be degenerate at {0} and let G' be the *survivor distribution* for the nonlattice case, i.e. the distribution with density

$$f(x) := \frac{1 - F(x)}{\mu}, \quad x > 0,$$

$$:= 0 \quad , \quad x \leq 0,$$

with respect to the Lebesgue measure ℓ. Define for S_n, $n \geq 0$, the point process N by

$$N(B) := \sum_{n \geq 0} \chi_B(S_n), \quad B \in \mathcal{B}^1,$$

and let N' and N" be defined similarly for the S'- and S"-processes. The intensity measure of N' is given by $G' * H$. By using Laplace transforms it is easily shown that this measure coincides with $\frac{1}{\mu}\ell$ on $(0,\infty)$. By (1.1.3) we have because $|D| < \eta$

$$\begin{aligned} |N''(t,t+h] - N'(t,t+h]| &\leq N'(t-\eta,t+\eta) + N'(t+h-\eta,t+h+\eta) \\ &\quad + N'(t,t+h]\chi_{(t-\eta,\infty)}(S'_\tau) + N''(t,t+h)\chi_{(t-\eta,\infty)}(S''_\tau). \end{aligned} \tag{1.1.4}$$

Consider the third term in the right-hand side of (1.1.4). It is easy to see that N'(t,t+h] is stochastically less or equal than N[0,h). Furthermore, EN[0,h) is finite (compare FELLER [VI.6 and VI.10]). By choosing n sufficiently large in

$$EN'(t,t+h]\chi_{(t-\eta,\infty)}(S'_\tau) \le EN'(t,t+h]\chi_{[n,\infty)}(N'(t,t+h])$$
$$+ nP(S'_\tau \in (t-\eta,\infty))$$

the first term on the right can be made arbitrarily small. Then take t sufficiently large to make the second term arbitrarily small. This estimate can also be given for the last term in (1.1.4). Because N" is distributed as N and N' has intensity measure $\frac{1}{\mu}\ell$ on $(0,\infty)$ we obtain from (1.1.4)

$$\limsup_{t\to\infty} \left|EN(t,t+h] - \frac{h}{\mu}\right| \le \frac{2\eta}{\varepsilon} + \frac{2\eta}{\varepsilon} + 0 + 0$$

and because η is arbitrary positive this proves (ii). In case F is nonlattice we insert an atom at $\{0\}$ and argue as in the proof of Theorem 1.1.3. □

NOTE to Theorem 1.1.5(i). The limit property of Theorem 1.1.5(i) is also sufficient for F to be strongly nonlattice. To see this observe that if F is concentrated on a discrete displaced lattice $c+L_d$, $d > 0$, then $\nu_\varepsilon * P_{S_n}$ is concentrated on

$$\{x+y: 0 < y < \varepsilon,\ x \in nc + L_d\}.$$

If ε is small enough then for a suitably chosen h the measures $\nu_\varepsilon * P_{S_n}$ and $\nu_\varepsilon * P_{S_n+h}$ are mutually singular, thus contradicting the limit property.

NOTE to Theorem 1.1.5(ii). LINDVALL [1977] gave another coupling proof for the Blackwell theorem. He proves the existence of a properly defined meeting time by using the Hewitt-Savage 0-1 law.

We did not construct a successful coupling of S'_n and S''_n: for $n \ge \tau$ the random walks run parallel at a short but possibly nonvanishing distance. Thus we did not obtain

(1.1.5) $$\lim_{n\to\infty} \| P_{S_n} - P_{S_n+h} \| = 0.$$

However, it is possible to prove this limit relation, but under stronger conditions than those used in the theorem above. HERMANN [1965] shows that if some convolution F^{n*}, $n \ge 1$, is non singular with respect to the Lebesgue measure, then (1.1.5) holds. Chapter 5 is concerned with a generalization of this result for random walks with dependent increments.

An application of the Ornstein coupling to random walks with dependent increments causes some problems: both the Markov property and the recurrence properties of a random walk with independent, identically distributed (i.i.d.) increments are used. However, in later chapters we shall see that still much can be done.

A subject studied in Chapter 4 is successful coupling. Suppose that there are given two processes $X_{\mathbb{N}}$ and $X'_{\mathbb{N}}$ with values in the same Borel space. We show in Section 4.3 that there exists a probability space with processes $\bar{X}_{\mathbb{N}}$ and $\bar{X}'_{\mathbb{N}}$, distributed as $X_{\mathbb{N}}$ and $X'_{\mathbb{N}}$ respectively, such that

$$P(\bar{X}_{\mathbb{N}+n} \neq \bar{X}'_{\mathbb{N}+n}) = \delta(X_{\mathbb{N}+n}, X'_{\mathbb{N}+n}), \qquad n \geq 0,$$

so such that the inequality in Lemma 1.1.1 is satisfied as an equality, uniformly in $n \geq 0$. Hence, if

$$\lim_{n\to\infty} \| P_{X_{\mathbb{N}+n}} - P_{X'_{\mathbb{N}+n}} \| = 0$$

then

$$\lim_{n\to\infty} P(\bar{X}_{\mathbb{N}+n} \neq \bar{X}'_{\mathbb{N}+n}) = 0,$$

so then the processes $\bar{X}_{\mathbb{N}}$ and $\bar{X}'_{\mathbb{N}}$ are successfully coupled. Apparently successful couplings can be constructed under quite general circumstances.

The Ornstein coupling uses that the increments of the random walk are stationary, independent (so i.i.d.). We want to relax the independence to asymptotic independence. In Section 5.1 we show that even if the increments of the random walk are dependent, something as an i.i.d. property can be regained. Suppose that $X_{\mathbb{Z}}$ is a stationary, weak Bernoulli process. Let there be given an integer set $L^* := \{1,\ldots,\ell\}$ and an integer $m \geq 1$. We show that it is possible to construct a process $\bar{X}_{\mathbb{Z}}$ such that

$$P(X_{\mathbb{Z}} \neq \bar{X}_{\mathbb{Z}})$$

is small and such that the process $\bar{X}_{\mathbb{Z}}$ has the property that the m random vectors

$$\bar{X}_{L^*+k_j}, \qquad 1 \leq j \leq m,$$

are i.i.d. Here $k_1,\ldots,k_m$ are integers that are constructed such that $k_m >> \ldots >> k_1$. The process $\bar{X}_{\mathbb{Z}}$, constructed in this way, is in general not stationary. In Chapter 5 we show how this approximation property of $X_{\mathbb{Z}}$

can be used to obtain results as (1.1.5) for random walks with dependent increments.

The applications of coupling, mentioned above are only concerned with renewal theory. Coupling techniques are also used in other fields of probability theory. A broad field of applications of coupling is the study of Markovian lattice interactions. Here coupling was introduced by VASERSHTEIN [1969] and DOBRUSHIN [1971]. See LIGGETT [1977] for a survey of the literature.

CHAPTER 2

RANDOM WALKS WITH STATIONARY INCREMENTS

2.1. TRANSIENT RANDOM WALKS

Suppose $\xi_{\mathbb{N}}$ is a stationary sequence of real random variables. The process $(S_n)_{n\geq 0}$, defined by

(2.1.1) $$S_0 := 0; \quad S_n := \xi_1 + \dots \xi_n, \quad n \geq 1,$$

is called a *random walk with stationary increments* $\xi_{\mathbb{N}}$. If $\xi_{\mathbb{Z}}$ is a stationary sequence of real random variables we define a process $S_{\mathbb{Z}}$ by requiring

(2.1.2) $$S_0 := 0; \quad \xi_n = S_n - S_{n-1}, \; n \in \mathbb{Z}.$$

The process $S_{\mathbb{Z}}$ is called a *random walk with stationary increments* $\xi_{\mathbb{Z}}$. Though a definition of S_n for both positive and negative n looks uncommon, it frequently is useful. Define for any Borel set $B \in \mathcal{B}^1$

(2.1.3) $$N(B) := \sum_{n\in\mathbb{Z}} \chi_B(S_n).$$

In general, N(B) is a random variable with values in the extended real line. Define the *set of transience* I_t of $S_{\mathbb{Z}}$ by

$$I_t := \{N \text{ is finite on all bounded intervals}\}.$$

The complement I_r of I_t is called the *set of recurrence* of $S_{\mathbb{Z}}$. We often assume that $S_{\mathbb{Z}}$ is *transient*, i.e. I_t has probability 1. In that case N can be seen as a point process on the real line. We use the concepts sets of transience and recurrence also for random walks $(S_n)_{n\geq 0}$, with the obvious meaning.

Several results, known for random walks with stationary, independent increments are also valid without the assumption of independence. In this chapter we discuss some of these results, but we also give examples of results that cannot be generalized. In Section 2.3 we show that an important property involving transience and recurrence can also be obtained without independence; if 0 is a limit point of $(S_n)_{n\geq 0}$ then each S_k is a limit point of $(S_n)_{n\geq 0}$; if 0 is not a limit point of $(S_n)_{n\geq 0}$ then S_k is not a limit point of $(S_n)_{n\geq 0}$ for each k. However, other properties related to transience and recurrence do not generalize: In the independent case the interval finiteness of the intensity measure of N is equivalent with transience, but this is not true in the general case (see Example 2.2.7). For results that hold in the general case, the proof is often quite different from the corresponding proof that uses independence.

In this section we discuss some simple results for transient random walks. Suppose $S_{\mathbb{Z}}$ is a transient random walk with stationary increments. The increments may assume both positive and negative values. We prove the existence of a random walk $\bar{S}_{\mathbb{Z}}$ with stationary increments $\bar{\xi}_{\mathbb{Z}}$ that are nonnegative, such that $\bar{N}$ defined by

$$(2.1.4) \qquad \bar{N}(B) := \sum_{n\in\mathbb{Z}} \chi_B(\bar{S}_n), \quad B \in \mathcal{B}^1,$$

coincides with N a.s. There are several relations between $\xi_{\mathbb{Z}}$ and $\bar{\xi}_{\mathbb{Z}}$. Ergodicity of $\xi_{\mathbb{Z}}$ implies ergodicity of $\bar{\xi}_{\mathbb{Z}}$ and if $E\xi_1$ exists as a finite number then $E\xi_1 = E\bar{\xi}_1$. The main argument that we use in the proofs is a rearrangement of the points S_n of the point process N. This argument is already known. KAPLAN [1955] and DALEY and OAKES [1974] use it in similar results. Note that the result described above does not fit in the context of random walks with stationary, independent increments: if $\xi_{\mathbb{Z}}$ consists of independent random variables, then this is not necessarily true for $\bar{\xi}_{\mathbb{Z}}$.

At the end of the section we discuss some other results for transient random walks and we present some examples. In Proposition 2.1.9 we give a criterion for transience.

First we discuss a reformulation of stationarity.

PROPOSITION 2.1.1. *Let* $\xi_{\mathbb{Z}}$ *be a sequence of real random variables and let* $S_{\mathbb{Z}}$ *be defined by* (2.1.2). *Define the process* $S_{\mathbb{Z}}^{(k)}$ *by*

$$S_n^{(k)} := S_{n+k} - S_k, \quad n \in \mathbb{Z}.$$

The process $\xi_{\mathbb{Z}}$ *is stationary if and only if* $S_{\mathbb{Z}}^{(k)}$ *is distributed as* $S_{\mathbb{Z}}$ *for all integers* k.

PROOF. $S_{\mathbb{Z}}$ and $\xi_{\mathbb{Z}}$ mutually determine each other by (2.1.2). A similar relationship holds for $S_{\mathbb{Z}}^{(k)}$ and its increments $\xi_{\mathbb{Z}}^{(k)}$. Hence $\xi_{\mathbb{Z}}$ is distributed as $\xi_{\mathbb{Z}}^{(k)}$ if and only if $S_{\mathbb{Z}}$ is distributed as $S_{\mathbb{Z}}^{(k)}$. □

We need a side result concerning the range of $S_{\mathbb{Z}}$. It is easily seen that the condition of the lemma is satisfied if the random walk $S_{\mathbb{Z}}$ is transient.

LEMMA 2.1.2. *Let* $S_{\mathbb{Z}}$ *be a random walk with stationary increments. If for* $n \to \infty$ *(or for* $n \to -\infty$*)*

$$|S_n| \xrightarrow{P} \infty$$

then apart from a null set

$$\inf_{n\in\mathbb{Z}} S_n = -\infty, \quad \sup_{n\in\mathbb{Z}} S_n = \infty.$$

PROOF. The extended real random variable

$$\zeta_k := \sup_{n\in\mathbb{Z}} (S_{n+k} - S_k)$$

is nonnegative and satisfies $\zeta_0 = \zeta_k + S_k$. Hence

$$\{|S_k| > c\} \subset \{\zeta_0 > c\} \cup \{\zeta_k > c\}$$

and because $\{\zeta_0 < \infty\} = \{\zeta_k < \infty\}$ we have

$$P(|S_k| > c,\ \zeta_0 < \infty\} \le P(c < \zeta_0 < \infty\} + P(c < \zeta_k < \infty\}$$

$$= 2\ P(c < \zeta_0 < \infty\}.$$

Here the equality follows because by Proposition 2.1.1 the distribution of ζ_k does not depend on k. Let $k \to \infty$. Because $|S_k| \xrightarrow{P} \infty$ we obtain

$$P(\zeta_0 < \infty) \le 2\ P(c < \zeta_0 < \infty).$$

Because c is arbitrary it follows that $\{\zeta_0 < \infty\}$ is a null set and therefore $\sup_{n\in\mathbb{Z}} S_n = \infty$ a.s. An application of the same argument for $S'_{\mathbb{Z}} := (-S_n)_{n\in\mathbb{Z}}$ yields that $\inf_{n\in\mathbb{Z}} S_n = -\infty$ a.s. □

We want to give the definition of a process that will be called the rearrangement of $S_{\mathbb{Z}}$. To this purpose we first consider a sequence $s_{\mathbb{Z}}$ of real numbers. We assume that the sequence $s_{\mathbb{Z}}$ has the property that only a finite number of its elements are situated on each finite interval and that infinitely many elements occur in both $(-\infty,0]$ and $(0,\infty)$. We construct an index sequence $t_{\mathbb{Z}}$, forming a permutation of $\mathbb{Z}$, with $t_0 = 0$ and such that

(2.1.5) $$\ldots \le s_{t_{-1}} \le s_{t_0} \le s_{t_1} \le \ldots \quad .$$

To this purpose we add to each index n an index m, called its *successor*, in the following way: If for some $i > n$ we have $s_i = s_n$, we let m be the smallest of these i. Otherwise consider the smallest s_i in $s_{\mathbb{Z}}$ that is larger than s_n. If this determines i uniquely we take $m = i$ and otherwise we let m be the smallest of the indices i. In this way we constructed to each n its successor m. We write $n \alpha m$. To each n there exists also a unique "predecessor" m, such that $m \alpha n$. Thus if we take $t_0 = 0$ we can construct a sequence $t_{\mathbb{Z}}$ such that

(2.1.6) $$\ldots \alpha\ t_{-1}\ \alpha\ t_0\ \alpha\ t_1\ \alpha \ldots,$$

while (2.1.5) is satisfied. It is easy to see that each s_n occurs in (2.1.5) and that each n occurs once in (2.1.6). Thus we constructed $t_{\mathbb{Z}}$, depending on $s_{\mathbb{Z}}$, satisfying to our requirements.

Note that if $S_{\mathbb{Z}}$ is transient, then by Lemma 2.1.2 the requirements mentioned above for $s_{\mathbb{Z}}$ are met by $S_{\mathbb{Z}}$ with probability 1.

DEFINITION 2.1.3. The process $\bar{S}_{\mathbb{Z}}$, with increments $\bar{\xi}_{\mathbb{Z}}$, is called the *rearrangement* of a transient random walk $S_{\mathbb{Z}}$ with stationary increments $\xi_{\mathbb{Z}}$, if $\bar{S}_n = S_{\tau_n}$, $n \in \mathbb{Z}$, where, with probability 1, $\tau_{\mathbb{Z}}$ is a permutation of $\mathbb{Z}$ such that $\tau_0 = 0$ and

$$\ldots \le S_{\tau_{-1}} \le S_{\tau_0} \le S_{\tau_1} \le \ldots,$$

while $\tau_{n-1} < \tau_n$ if $S_{\tau_{n-1}} = S_{\tau_n}$, $n \in \mathbb{Z}$.

Note that if $\bar{S}_{\mathbb{Z}}$ is the rearrangment of $S_{\mathbb{Z}}$ then the point process $\bar{N}$ defined by (2.1.4) coincides with N defined by (2.1.3) with probability 1.

THEOREM 2.1.4. *Let* $S_{\mathbb{Z}}$ *be a transient random walk with stationary increments. Suppose its rearrangement* $\bar{S}_{\mathbb{Z}}$ *and* $\tau_{\mathbb{Z}}$ *are as in Definition* 2.1.3, *and have processes of increments* $\bar{\xi}_{\mathbb{Z}}$ *and* $\nu_{\mathbb{Z}}$. *Then we have*

(i) *the process* $S^k_{\mathbb{Z}}$ *defined by*

$$s_n^k := s_{\tau_k+n} - s_{\tau_k},\ n \in \mathbb{Z},$$

is distributed as $S_{\mathbb{Z}}$ *for all integers* k;

(ii) *the process* $((\bar{\xi}_n, \nu_n))_{n\in\mathbb{Z}}$ *is stationary.*

PROOF of (i). Let $s_{\mathbb{Z}} := (s_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers satisfying the requirements mentioned in the introduction to Definition 2.1.3. The proof is based on the following simple observation. For any k we have $-k \alpha 0$ for the sequences $s_{\mathbb{Z}}$ if and only if $0 \alpha k$ for the sequence $(s_{n-k} - s_{-k})_{n\in\mathbb{Z}}$.

Observe that for an arbitrary Borel set $B \in \mathcal{B}^{\mathbb{Z}}$ we have

$$P(S^1_{\mathbb{Z}} \in B) = \sum_{k\in\mathbb{Z}} P(\tau_1 = k,\ (S_{n+k} - S_k)_{n\in\mathbb{Z}} \in B)$$

$$= \sum_{k\in\mathbb{Z}} P(0 \alpha k \text{ for } S_{\mathbb{Z}},\ (S_{n+k} - S_k)_{n\in\mathbb{Z}} \in B).$$

Because $(\xi_{n+k})_{n\in\mathbb{Z}}$ is distributed as $(\xi_n)_{n\in\mathbb{Z}}$ we have

$$P(S^1_{\mathbb{Z}} \in B) = \sum_{k\in\mathbb{Z}} P(0 \alpha k \text{ for } (S_{n-k} - S_{-k})_{n\in\mathbb{Z}},\ S_{\mathbb{Z}} \in B)$$

$$= \sum_{k\in\mathbb{Z}} P(-k \alpha 0 \text{ for } S_{\mathbb{Z}},\ S_{\mathbb{Z}} \in B) = P(S_{\mathbb{Z}} \in B).$$

This proves (i) for $k = 1$. Similarly the assertion follows for other k.

PROOF of (ii). Consider again the sequence $s_{\mathbb{Z}}$. Let $t_{\mathbb{Z}}$ be the index sequence constructed in the introduction to Definition 2.1.3 and let the mapping ψ be defined by

$$\psi(s_{\mathbb{Z}}) := ((s_{t_n})_{n\in\mathbb{Z}}, t_{\mathbb{Z}}).$$

The index sequence belonging to

$$s^1_{\mathbb{Z}} := (s_{t_1+n} - s_{t_1})_{n\in\mathbb{Z}}$$

is $t^1_{\mathbb{Z}} := (t_{n+1} - t_1)_{n\in\mathbb{Z}}$ and therefore

$$\psi(s^1_{\mathbb{Z}}) = ((s_{t_{n+1}} - s_{t_1})_{n\in\mathbb{Z}}, t^1_{\mathbb{Z}}).$$

Because of (i) we have $\psi(S_{\mathbb{Z}}) \overset{d}{=} \psi(S^1_{\mathbb{Z}})$ and by the definition of $\bar{\xi}_{\mathbb{Z}}$ and $\nu_{\mathbb{Z}}$ assertion (ii) follows. □

COROLLARY 2.1.5. *If* $S_{\mathbb{Z}}$ *is a transient random walk with stationary increments then its rearrangement* $\bar{S}_{\mathbb{Z}}$ *is also a random walk with stationary increments. The point processes* N *and* $\bar{N}$ *coincide with probability* 1.

PROOF. Obvious from the theorem above and Definition 2.1.3. □

To study relationship between $E\xi_1$ and $E\bar{\xi}_1$, we use that

$$\lim_{t\to\infty} \frac{1}{t} N(0,t] = \lim_{t\to\infty} \frac{1}{t} \bar{N}(0,t]$$

together with the following lemma.

LEMMA 2.1.6. *Let* s_n, $n \geq 0$ *be a sequence of real numbers such that*

$$\lim_{n\to\infty} \frac{s_n}{n} = \mu,$$

with $0 \leq \mu \leq \infty$. *Let* $n_{s,t} := \#\{n \geq 0:\ s < s_n \leq t\}$. *Then we have*

$$\lim_{t\to\infty} \frac{n_{0,t}}{t} = \frac{1}{\mu}, \qquad 0 < \mu < \infty,$$

$$= 0, \qquad \mu = \infty,$$

$$\lim_{t\to\infty} \frac{n_{-t,t}}{t} = \infty, \qquad \mu = 0.$$

PROOF. For $0 < \mu < \infty$ the proof is carried on in the following way. Let $\varepsilon > 0$ be arbitrary. The number κ of points (n,s_n) outside the cone

$$\{(n,x):\ (\mu-\varepsilon)n \leq x \leq (\mu+\varepsilon)n,\ n \geq 0\}$$

is given by

$$\kappa := \#\{n \geq 0:\ s_n < (\mu-\varepsilon)n\} + \#\{n \geq 0:\ s_n > (\mu+\varepsilon)n\}.$$

By our assumptions on s_n, $n \geq 0$ it follows that κ is finite. Obviously we have the inclusion

$$\{n \geq 0:\ s_n \leq t\} \cup \{n \geq 0:\ s_n > (\mu+\varepsilon)n\} \supset \{n \geq 0:\ (\mu+\varepsilon)n \leq t\}$$

and as a consequence

(2.1.7) $$\kappa + \#\{n \geq 0:\ s_n \leq t\} \geq \#\{n \geq 0:\ (\mu+\varepsilon)n \leq t\}.$$

Furthermore, the inclusion

$$\{n \geq 0:\ s_n \leq t\} \subset \{n \geq 0:\ (\mu-\varepsilon)n \leq t\} \cup \{n \geq 0:\ s_n < (\mu-\varepsilon)n\}$$

holds and hence

$$\#\{n\geq 0:\ s_n \leq t\} \leq \#\{n\geq 0:\ (\mu-\varepsilon)n \leq t\} + \kappa. \tag{2.1.8}$$

The assumptions on s_n, $n\geq 0$, also imply that $\#\{n\geq 0:\ s_n < 0\}$ is finite. Combination of (2.17) and (2.1.8) yields

$$\frac{1}{\mu+\varepsilon} \leq \liminf_{t\to\infty} \frac{n_{0,t}}{t} \leq \limsup_{t\to\infty} \frac{n_{0,t}}{t} \leq \frac{1}{\mu-\varepsilon}$$

for $0 < \varepsilon < \mu$. So we have proved the assertion for $0 < \mu < \infty$.
For $\mu = \infty$ the lemma is proved using (2.1.8), while in the argument $(\mu-\varepsilon)$ has to be replaced by any arbitrarily large number.

For $\mu = 0$ the assertion is proved using

$$\{n\geq 0:\ \varepsilon n \leq t\} \subset \{n\geq 0:\ |s_n| \geq \varepsilon n\} \cup \{n\geq 0:\ |s_n| < t\}.$$

The last set on the right contains $n_{-t,t}$ elements and by our assumptions the set in the middle is finite, so

$$\liminf_{t\to\infty} \frac{n_{-t,t}}{t} \geq \frac{1}{\varepsilon}$$

for all $\varepsilon > 0$. □

PROPOSITION 2.1.7. *Let $S_{\mathbb{Z}}$ be a transient random walk with stationary increments $\xi_{\mathbb{Z}}$. Suppose $\bar{S}_{\mathbb{Z}}$ is its rearrangement. If $E|\xi_1|$ if finite then the process of increments $\bar{\xi}_{\mathbb{Z}}$ of $\bar{S}_{\mathbb{Z}}$ satisfies*

$$|E(\xi_1|J_\xi)| = E(\bar{\xi}_1|J_{\bar{\xi}}) > 0 \text{ a.s.},$$

where J_ξ and $J_{\bar{\xi}}$ are the invariant σ-fields of $\xi_{\mathbb{Z}}$ and $\bar{\xi}_{\mathbb{Z}}$. In particular, if $\xi_{\mathbb{Z}}$ is ergodic $E|\xi_1| = E\bar{\xi}_1$.

PROOF. First we show $E(\bar{\xi}_1|J_{\bar{\xi}})>0$ a.s. By the general properties of conditional expectation we have, because $\bar{\xi}_n \geq 0$ a.s.

$$\{\bar{\xi}_n > 0\} \subset \{E(\bar{\xi}_n|J_{\bar{\xi}}) > 0\} \text{ a.s.}$$

By stationarity we have $E(\bar{\xi}_n|J_{\bar{\xi}}) = E(\bar{\xi}_1|J_{\bar{\xi}})$ a.s., so

$$\bigcup_{n\in\mathbb{Z}} \{\xi_n > 0\} \subset \{E(\bar{\xi}_1|J_{\bar{\xi}}) > 0\} \text{ a.s.}$$

By taking complements it follows that

$$\{E(\bar{\xi}_1|J_{\bar{\xi}}) = 0\} \subset \{\forall n: \bar{\xi}_n = 0\} = \{\forall n: S_{\tau_n} = 0\} \text{ a.s.}$$

Because $S_{\mathbb{Z}}$ is transient the last set has probability 0, so we have $E(\bar{\xi}_1|J_{\bar{\xi}}) > 0$ a.s.

The relation between the conditional expectations of ξ_1 and $\bar{\xi}_1$ is a consequence of the property that $N = \bar{N}$ a.s. By the ergodic theorem

$$\lim_{n\to\infty} \frac{S_{\tau_n}}{n} = E(\bar{\xi}_1|J_{\bar{\xi}}) \text{ a.s.}, \quad \lim_{n\to\infty} \frac{S_{\tau_{-n}}}{n} = -E(\bar{\xi}_1|J_{\bar{\xi}}) \text{ a.s.}$$

By Lemma 2.1.6

$$\lim_{t\to\infty} \frac{\bar{N}(0,t)}{t} = \lim_{t\to\infty} \frac{\bar{N}(-t,0)}{t} = \frac{1}{E(\bar{\xi}_1|J_{\bar{\xi}})} < \infty \text{ a.s.} \tag{2.1.9}$$

Another application of the ergodic theorem shows that with probability 1

$$\lim_{n\to\infty} \frac{S_n}{n} = E(\xi_1|J_{\xi}) \text{ a.s.}, \quad \lim_{n\to\infty} \frac{S_{-n}}{n} = -E(\xi_1|J_{\xi}) \text{ a.s.} \tag{2.1.10}$$

Hence on the set $A = \{|E(\xi_1|J_{\xi})| > 0\}$ we have by Lemma 2.1.6

$$\lim_{t\to\infty} \frac{N(0,t)}{t} = \lim_{t\to\infty} \frac{N(-t,0)}{t} = \frac{1}{|E(\xi_1|J_{\xi})|} \text{ a.s.}$$

So on A we have by (2.1.9) that $E(\bar{\xi}_1|J_{\bar{\xi}}) = |E(\xi_1|J_{\xi})|$. The complement of A is a null set. This is proved by noting that on A^c we have by Lemma 2.1.6 and (2.1.10)

$$\lim_{t\to\infty} \frac{N(-t,t)}{t} = \infty \text{ a.s.}$$

This contradicts with (2.1.9) unless $P(A^c) = 0$. If $\xi_{\mathbb{Z}}$ is ergodic we clearly have $|E\xi_1| = E(\bar{\xi}_1|J_{\bar{\xi}})$ a.s. This proves the proposition. □

As a consequence of the next proposition it follows that $\bar{\xi}_{\mathbb{Z}}$ is ergodic if $\xi_{\mathbb{Z}}$ is ergodic.

<u>PROPOSITION 2.1.8</u>. *Suppose* $S_{\mathbb{Z}}$ *is a transient random walk with stationary increments* $\xi_{\mathbb{Z}}$. *Let its rearrangement* $\bar{S}_{\mathbb{Z}}$ *have increments* $\bar{\xi}_{\mathbb{Z}}$. *Then* $J_{\bar{\xi}} \subset J_{\xi}$ *a.s., where* J_{ξ} *and* $J_{\bar{\xi}}$ *are the invariant σ-fields of* $\xi_{\mathbb{Z}}$ *and* $\bar{\xi}_{\mathbb{Z}}$.

PROOF. Suppose $s_{\mathbb{Z}}$ is a sequence of real numbers of which only a finite number are situated in each finite interval and of which infinitely many occur in both $(-\infty,0]$ and $(0,\infty)$. Denote the increments of $s_{\mathbb{Z}}$ by $x_{\mathbb{Z}} := (s_n - s_{n-1})_{n\in\mathbb{Z}}$ and let $\bar{R}$ be the set of sequences $x_{\mathbb{Z}}$ obtained in this way. We consider two mappings on $\bar{R}$. The first is the shift transformation T given by

$$T\, x_{\mathbb{Z}} := (x_{n+1})_{n\in\mathbb{Z}}.$$

To define the second, $\tilde{T}$, let $t_{\mathbb{Z}}$ be the sequence of indices defined in the introduction of Definition 2.1.3. Let $\tilde{T}$ be given by

$$\tilde{T}\, x_{\mathbb{Z}} := T^{t_1}\, x_{\mathbb{Z}}.$$

It is useful to denote

$$\phi(x_{\mathbb{Z}}) := (s_{t_n} - s_{t_{n-1}})_{n\in\mathbb{Z}}.$$

By the definition of $\tilde{T}$ we have on the set $\bar{R}$

$$T\, \phi(x_{\mathbb{Z}}) = \phi(\tilde{T}\, x_{\mathbb{Z}})$$

and by the definition of t_k

$$\tilde{T}^k\, x_{\mathbb{Z}} = T^{t_k}\, x_{\mathbb{Z}}.$$

Hence

$$T^k\, \phi(x_{\mathbb{Z}}) = \phi(\tilde{T}^k\, x_{\mathbb{Z}}) = \phi(T^{t_k}\, x_{\mathbb{Z}}).$$

Let k run along the integers. Because $t_{\mathbb{Z}}$ permutes $\mathbb{Z}$ the sequence t_k runs along the integers too. We obtain our key equality

$$(2.1.11) \qquad \{T^k\, \phi(x_{\mathbb{Z}}):\ k \in \mathbb{Z}\} = \{\phi T^k(x_{\mathbb{Z}}):\ k \in \mathbb{Z}\}.$$

Suppose the set $B \subset \bar{R}$ is *shift invariant*, i.e.

$$x_{\mathbb{Z}} \in B \Longleftrightarrow T\, x_{\mathbb{Z}} \in B,$$

or equivalently

$$\exists k:\ T^k\, x_{\mathbb{Z}} \in B \Longleftrightarrow \forall k:\ T^k\, x_{\mathbb{Z}} \in B.$$

By using (2.1.11) it is easily seen that the set $\phi^{-1}(B)$ is also shift invariant.

The proof is now carried on as follows. Because $S_{\mathbb{Z}}$ is transient, $\xi_{\mathbb{Z}} \in \bar{R}$ a.s. Suppose that A is a.s. invariant for $\bar{\xi}_{\mathbb{Z}}$, i.e. there is a shift invariant set $B \in \mathcal{B}^{\mathbb{Z}}$ such that

$$A = \{\bar{\xi}_{\mathbb{Z}} \in B\} \text{ a.s.}$$

Because $\bar{\xi}_{\mathbb{Z}} = \phi(\xi_{\mathbb{Z}})$ it follows that

$$A = \{\bar{\xi}_{\mathbb{Z}} \in B \cap \bar{R}\} = \{\xi_{\mathbb{Z}} \in \phi^{-1}(B \cap \bar{R})\} \text{ a.s.,}$$

with $\phi^{-1}(B\cap\bar{R})$ shift invariant. This proves that A is also a.s. invariant for $\xi_{\mathbb{Z}}$. □

The next proposition gives a criterion for transience of a random walk in terms of its behaviour around the origin. In Sections 2.2 and 2.3 we shall meet other examples of global properties that are determined by the behaviour at the origin.

PROPOSITION 2.1.9. *Suppose* $S_{\mathbb{Z}}$ *is a random walk with stationary increments and let* N *be defined by* (2.1.3). *If on some open interval* I *of the origin*

$$N(I) < \infty \text{ a.s.}$$

then this holds for any bounded interval, i.e. $S_{\mathbb{Z}}$ *is transient.*

PROOF. We may suppose that for a symmetric interval $I = (-\varepsilon,\varepsilon)$, $\varepsilon > 0$, we have $N(-\varepsilon,\varepsilon) < \infty$ a.s. Define

$$A_k := \{N(S_k-\varepsilon, S_k+\varepsilon) < \infty\}.$$

Write A_{k+1} as

$$A_{k+1} := \{\#\{n: S_n \in (S_{k+1}-\varepsilon, S_{k+1}+\varepsilon)\} < \infty\}$$

$$= \{\#\{n: S_n-S_1 \in (S_{k+1}-S_1-\varepsilon, S_{k+1}-S_1+\varepsilon\} < \infty\}.$$

By Proposition 2.1.1 it follows that $P(A_{k+1}) = P(A_k)$ for all k, and because by our assumption $P(A_0) = 1$ we have $P(A_k) = 1$ for all k. So with the exception of a null set we have for all k

$$N(S_k-\varepsilon, S_k+\varepsilon) < \infty.$$

Therefore we may conclude that $S_{\mathbb{Z}}$ is transient. □

The remaining part of the section contains some examples. The most well known example of a random walk with stationary increments is a random walk where the increments are both independent and stationary. For these processes we can prove:

PROPOSITION 2.1.10. *Let* $S_{\mathbb{Z}}$ *be a transient random walk with independent, stationary increments. Suppose its rearrangement* $\bar{S}_{\mathbb{Z}}$ *has increments* $\bar{\xi}_{\mathbb{Z}}$. *Then* $E\bar{\xi}_1$ *is finite if and only if* $E\xi_1$ *exists and is finite.*

PROOF. Suppose that $E\xi_1$ exists and is finite. In that case we have to prove that $E\bar{\xi}_1$ is finite. We may assume that $E\xi_1 \neq 0$ for otherwise this would contradict transience of the random walk (see FELLER [VI.10]). We can assume that $0 < E\xi_1 < \infty$, because the negative case can be treated similarly. By FELLER [XII.2] the first ascending record of $S_{\mathbb{Z}}$ has finite expectation. The definition of $\bar{\xi}_1$ shows that this record dominates $\bar{\xi}_1$, so $E\bar{\xi}_1 < \infty$.

The converse is proved by using the renewal theorem for distributions without expectation. Suppose that $E\bar{\xi}_1 < \infty$ but that $E\xi_1$ is not defined. FELLER [XI.9] asserts that for each bounded interval I

$$\lim_{t\to\pm\infty} E \sum_{n\geq 0} \chi_I(S_n + t) = 0.$$

Hence for the point process N defined by (2.1.3) we have

$$\lim_{t\to\pm\infty} E\, T_t N(I) = 0,$$

where T_t is the translation on the real line. Thus for the Cesaro average we have

$$\lim_{t\to\infty} \frac{EN(-t,t)}{2t} = 0. \tag{2.1.12}$$

By BREIMAN [Corollary 6.33] the process $\xi_{\mathbb{Z}}$ is ergodic, so by Proposition 2.1.8 also $\bar{\xi}_{\mathbb{Z}}$ is ergodic. By the ergodic theorem we have for the rearranged sequence $\bar{S}_n = S_{\tau_n}$

$$\lim_{n\to\infty} \frac{S_{\tau_n}}{n} = -\lim_{n\to\infty} \frac{S_{\tau_{-n}}}{n} = \frac{1}{E\bar{\xi}_1} \text{ a.s.}$$

and by Lemma 2.1.6

$$\lim_{t\to\infty} \frac{N(-t,t)}{2t} = \frac{1}{E\bar{\xi}_1} \text{ a.s.}$$

By Fatou's lemma and the finiteness of $E\bar{\xi}_1$

$$\liminf_{t\to\infty} \frac{EN(-t,t)}{2t} \geq \frac{1}{E\bar{\xi}_1} > 0.$$

This contradicts (2.1.12), so $E\xi_1$ has to exist and by Proposition 2.1.7 the expectation $|E\xi_1| = E\bar{\xi}_1$ exists as a finite number. □

EXAMPLE 2.1.11. Let $X_{\mathbb{N}}$ be a Markov chain on a Borel space Γ started with an invariant probability measure. As is mentioned in Section 0.2 the process can be extended to a stationary process $X_{\mathbb{Z}}$. Let f be a real valued measurable function on Γ. The process $\xi_{\mathbb{Z}}$ defined by

$$\xi_n := f(X_n), \quad n \in \mathbb{Z},$$

is stationary and determines by (2.1.2) a random walk $S_{\mathbb{Z}}$ with stationary increments. Using a terminology of Chapter 5, we shall say that $S_{\mathbb{Z}}$ is controlled by $X_{\mathbb{Z}}$. In Section 6.4 we study the properties of this random walk.

EXAMPLE 2.1.12 (superposition on a trend). Let $((\xi'_n,\xi''_n))_{n\in\mathbb{Z}}$ be a stationary sequence of pairs (ξ'_n,ξ''_n) of real random variables. Let $S'_{\mathbb{Z}}$ be the random walk with increments $\xi'_{\mathbb{Z}}$. We call $S'_{\mathbb{Z}}$ a *trend* if $S'_n(\omega)$ is monotone as a function of n for almost every ω. The process $S_{\mathbb{Z}}$ defined by

$$S_n := S'_n + \xi''_n - \xi''_0, \quad n \in \mathbb{Z}, \tag{2.1.13}$$

is called a *superposition on a trend*. $S_{\mathbb{Z}}$ is a random walk with stationary increments

$$\xi_n = \xi'_n + \xi''_n - \xi''_{n-1}, \quad n \in \mathbb{Z}.$$

The next proposition shows that many random walks are superpositions on a trend.

PROPOSITION 2.1.13. *A random walk* $S_{\mathbb{Z}}$ *with stationary increments, such that*

$$\lim_{n\to\infty} S_n = \infty \ a.s.$$

is a superposition on a trend.

PROOF. The process

$$\zeta_n := \inf_{m\geq n} S_m$$

is nonascending, but may differ from 0 if n = 0. Define $S'_n := \zeta_n - \zeta_0$, $n \in \mathbb{Z}$, and take $\xi'_{\mathbb{Z}}$ to be the process of increments of $S'_{\mathbb{Z}}$. Note that

$$S_n = S'_n + (S_n - \zeta_n) - (S_0 - \zeta_0)$$

has the form (2.1.13) if we define $\xi''_n := S_n - \zeta_n$, $n \in \mathbb{Z}$. We still have to prove that $((\xi'_n, \xi''_n))_{n\in\mathbb{Z}}$ is stationary. Let $\xi_{\mathbb{Z}}$ be the process of increments of $S_{\mathbb{Z}}$ and observe that

$$\begin{aligned} \xi'_n &= \inf_{m\geq n} (S_m - S_{n-1}) - \inf_{m\geq n-1} (S_m - S_{n-1}) \\ &= \inf_{m\geq n} \sum_{k=n}^{m} \xi_k - \min(0, \inf_{m\geq n} \sum_{k=n}^{m} \xi_k), \end{aligned}$$

$$\begin{aligned} \xi''_n &= S_n - \inf_{m\geq n} S_m \\ &= -\inf_{m>n} \sum_{k=n+1}^{m} \xi_k. \end{aligned}$$

Hence (ξ'_n, ξ''_n) can be written as a function of $(\xi_{n-1}, \xi_n, \xi_{n+1}, \ldots)$ not depending on n. It follows that $((\xi'_n, \xi''_n))_{n\in\mathbb{Z}}$ is stationary. □

A random walk $S_{\mathbb{Z}}$ with stationary, independent increments $\xi_{\mathbb{Z}}$ has important symmetry properties. The symmetry makes it possible to use the duality principle (see FELLER [XII.2]). The duality principle makes use of the fact that the process $\xi_{\mathbb{Z}}$ is *exchangeable*, i.e. for each permutation $(k_m, \ldots, k_n)$ of $(m, \ldots, n)$ the random vectors $(\xi_{k_m}, \ldots, \xi_{k_n})$ and $(\xi_m, \ldots, \xi_n)$ are equally distributed. Because of this property the random walk $S_{\mathbb{Z}}$ is distributed as the random walk $\tilde{S}_{\mathbb{Z}}$ with increments $\tilde{\xi}_n := \xi_{-n}$, $n \in \mathbb{Z}$.

If one only assumes that the increments $\xi_{\mathbb{Z}}$ of the random walk are stationary, then there still are several symmetry properties. For example the process $\tilde{S}_{\mathbb{Z}}$ with $\tilde{S}_0 = 0$ and with increments $\tilde{\xi}_n = \xi_{-n}$, $n \in \mathbb{Z}$, is a random walk with stationary increments. For each n the random variables S_n and $\tilde{S}_n$ are equally distributed. By DOOB [Chapter X] each invariant event of $\xi_{\mathbb{N}}$ differs at most a null set from an invariant event of $\tilde{\xi}_{\mathbb{N}}$ and vice versa. However, there is also asymmetry: The invariant sets

$$\{\lim_{n\to\infty} S_n = \infty\} \quad \text{and} \quad \{\lim_{n\to-\infty} S_n = -\infty\}$$

do not always have equal probability. Below we present an example of such a phenomenon. Clearly the process of increments of the random walk in the example is not exchangeable.

EXAMPLE 2.1.14 (random walks and asymmetry). Our example will have the property that $\lim_{n\to\infty} S_n = \infty$ a.s., while $\lim_{n\to-\infty} S_n$ does not exist a.s. We construct a superposition on a trend as in Example 2.1.12. Let $\xi''_{\mathbb{Z}}$ consist of independent random variables, distributed as the absolute value of a Cauchy distributed random variable. Let $\xi'_n = 1$ for all n and take $S_{\mathbb{Z}}$ and $S'_{\mathbb{Z}}$ as in Example 2.1.12. We have $S'_n = n$, $n \in \mathbb{Z}$, and because $\xi''_n \geq 0$ a.s.

$$(2.1.14) \qquad P(\lim_{n\to\infty} S_n = \infty) = P(\liminf_{n\to-\infty} S_n = -\infty) = 1.$$

With a simple application of the Borel Cantelli lemma applied to

$$A_n := \{S_n > M\} = \{\xi''_n > \xi''_0 + M-n\}, \qquad n < 0,$$

one proves that for all real M

$$P(\limsup_{n\to-\infty} A_n) = 1$$

and hence $\limsup_{n\to-\infty} S_n = \infty$ a.s. Together with (2.1.14) it follows that the random walk $S_{\mathbb{Z}}$ has the required properties.

2.2. POINT CLUSTERS AND THE BEHAVIOUR AT THE ORIGIN

Suppose $S_{\mathbb{Z}}$ is a transient random walk with stationary increments. Let N be defined by (2.1.3). We want to investigate the number of points $N(t,t+h]$ lying in an interval with length $h > 0$. Especially we want to find an estimate for the probability that a large number of points occur in the interval $(t,t+h]$, i.e. we estimate the "cluster size". Our main result is

$$(2.2.1) \qquad P(N(t,t+h] \geq p) \leq 2\, P(N(-h,h) \geq p), \qquad p > 0,$$

which, heuristically speaking, expresses that the size of point clusters occurring at a point t of the real line is dominated by the size of point clusters at the origin. Our investigation departs from an inequality given by KAPLAN [1955] for the renewal measure.

The *renewal measure* H is defined as

$$(2.2.2) \qquad H(B) := \sum_{n\geq 0} P(S_n \in B), \quad B \in \mathcal{B}^1.$$

The *symmetrized renewal measure* H^s is defined as the intensity measure of N, i.e.

$$H^s(B) := \sum_{n\in\mathbb{Z}} P(S_n \in B), \quad B \in \mathcal{B}^1.$$

Note that H^s is symmetric around the origin, because by stationarity each random variable S_n is distributed as $-S_{-n}$, $n \in \mathbb{Z}$. If the increments of the random walk are stationary and independent then the following inequality is valid (see FELLER [VI.10, Theorem 1])

$$H(t,t+h] \leq H(-h,h)$$

for positive h and real t. Its proof is a simple application of the Markov property. KAPLAN [1955] gave an analogue of this inequality for random walks with stationary increments. Its most elegant presentation is given in terms of the symmetrized renewal measure as

$$H^s(t,t+h] \leq H^s(-h,h)$$

with t real and h positive real. Its proof is based on a combinatorial lemma.

The importance of inequality (2.2.1) can be judged from an application that we shall give in an integrability problem. Let us consider the global renewal theorem. This theorem considers the limit behaviour of $\frac{1}{t}H(0,t]$ (see Theorem 2.2.6 below). By means of the ergodic theorem one easily obtains that

$$(2.2.3) \qquad \frac{1}{t} N(0,t]$$

converges a.s. for $t \to \infty$. The question that interests us now, is whether the expected value also converges. Kaplan obtains from his inequality that $\frac{1}{t} H^s(0,t] = \mathcal{O}(1)$ for $t \to \infty$, if H^s is bounded on a neighbourhood of the origin. Convergence of this expression can be obtained if the uniform integrability of (2.2.3) is proved. We achieve this by using (2.2.1). There is also another approach to this problem. DALEY [1971] gives a proof of the global renewal theorem using Palm theory. A similar problem is discussed in DELASNERIE [1977] for Blackwell's theorem (see Corollary 3.2.4).

The section begins with a proof of the inequalities mentioned above. Then we prove the global renewal theorem and finally we investigate the relationship between transience of a random walk and the finiteness of the renewal measure on bounded sets.

THEOREM 2.2.1 (KAPLAN). *Suppose* $S_{\mathbb{Z}}$ *is a transient random walk with stationary increments. The point process* N *defined by* (2.1.3) *satisfies*

$$EN(t,t+h] \le EN(-h,h)$$

with t *real and* h *positive.*

LEMMA 2.2.2. *Let* X *and* Y *be independent random variables with a common distribution function* F. *We have*

$$P(Y-X \in (t,t+h]) \le P(Y-X \in (-h,h)).$$

PROOF of Theorem 2.2.1. By Corollary 2.1.5 we may suppose without restricting generality that the increments of $S_{\mathbb{Z}}$ are nonnegative. The symmetry of the measure $H^s(B) := EN(B)$, $B \in \mathcal{B}^1$, implies that we only have to prove the theorem for $t \ge 0$. Observe that

$$N^{(n)} := \sum_{k=-n}^{-1} N(S_k + t,\ S_k + t + h], \qquad n \ge 1,$$

are the partial sums of a stationary sequence of random variables. An application of the ergodic theorem shows

$$E \lim_{n\to\infty} \frac{N^{(n)}}{n} = EN(t,t+h].$$

Similarly one proves that

$$N_0^{(n)} := \sum_{k=-n}^{-1} N(S_k-h,\ S_k+h)$$

satisfies

$$E \lim_{n\to\infty} \frac{N_0^{(n)}}{n} = EN(-h,h).$$

Apparently our task is to prove that

$$E \lim_{n\to\infty} \frac{N^{(n)}}{n} \le E \lim_{n\to\infty} \frac{N_0^{(n)}}{n}.$$

We want to apply Lemma 2.2.2. Let $F = F_\omega$ be the probability measure

$$F(B) := \frac{1}{n} \sum_{i=-n}^{-1} \chi_B(S_i(\omega)), \qquad B \in \mathcal{B}^1.$$

Then Lemma 2.2.2 yields

$$\frac{1}{n^2} M^{(n)} \le \frac{1}{n^2} M_0^{(n)},$$

where

$$M_0^{(n)} := \#\{(i,j): -n \le i,j \le -1,\ S_j - S_i \in (-h,h)\}$$

and

$$M^{(n)} := \#\{(i,j): -n \le i,j \le -1,\ S_j - S_i \in (t,t+h]\}.$$

Compare $M^{(n)}$ with

$$N^{(n)} = \#\{(i,j): -n \le i \le -1,\ j \in \mathbb{Z},\ S_j - S_i \in (t,t+h]\}.$$

Because $t \ge 0$ we may suppose that $i < j$ in the expression for $N^{(n)}$ above. We obtain that

$$N^{(n)} - M^{(n)} = \#\{(i,j): -n \le i \le -1,\ j \ge 0,\ S_j - S_i \in (t,t+h]\}$$

is constant for n sufficiently large. Comparison of $N_0^{(n)}$ with $M_0^{(n)}$ is simpler: Note that $N_0^{(n)} \ge M_0^{(n)}$. We can now obtain from $M^{(n)} \le M_0^{(n)}$ that

$$\lim_{n\to\infty} \frac{N^{(n)}}{n} \le \lim_{n\to\infty} \frac{N_0^{(n)}}{n} \text{ a.s.}$$

The theorem follows by taking expectations. □

PROOF of lemma 2.2.2. We first prove the lemma for distributions F that are concentrated on a finite set of points $A := \{x_1, \dots, x_n\}$ and have mass $\frac{1}{n}$ in each of these points. The statement of the lemma translates as follows. Let $C_0 \subset A \times A$ be the set of points lying in the strip

$$\{(x,y): y-x \in (-h,h)\}$$

and $C \subset A \times A$ the set of points in the strip

$$\{(x,y): y-x \in (t,t+h]\}.$$

The assertion of the lemma can be restated as $\#C \le \#C_0$. We prove it for

$t > 0$ and use induction on n. Consider

$$n(x_i) := \#\{k: (x_i,x_k) \in C_0\}, \qquad 1 \le i \le n,$$

and define

$$(2.2.4) \qquad m := \max_{1\le i\le n} n(x_i).$$

Remark that the maximum number of elements x_j in any interval $(x,x+h]$ is dominated by $n(x_i)$ for each $x_i \in (x,x+h]$, so this maximum is at most m.

Let j be the smallest index for which the maximum (2.2.4) is attained. We shall remove all pairs containing an element x_j from C and C_0. By the definition of m the number of elements in C_0 of the form (x_i,x_j) or (x_j,x_i), $1 \le i \le n$, is equal to $2m-1$. By our remark to (2.2.4) the number of elements in C of the form (x_j,x_i), $1 \le i \le n$, is not more than m. By the choice of j as the smallest index for which the maximum (2.2.4) is attained the number of elements in C of the form (x_i,x_j), $1 \le i \le n$, is at most $m-1$. So the number of elements removed from C is at most $2m-1$ and therefore does not exceed the number of elements removed from C_0. By our induction argument we have $\#C \le \#C_0$ for the new sets C and C_0. Hence this inequality has to be valid for the original sets C and C_0 too. This proves for $t > 0$ the assertion, in case F is a discrete distribution consisting of a finite number of atoms with equal mass. By continuity the assertion follows also for $t = 0$ and by applying a reflection around zero we obtain the assertion for $t < 0$.

If F is arbitrary, let F_n, $n \ge 1$, converge weakly to F. By the properties of weak convergence it is easy to see that the inequality of the lemma is valid for a set of (t,h) that is dense in $\mathbb{R} \times (0,\infty)$ and by continuity arguments on the distribution function of $X-Y$ the inequality follows. □

The combinatorics used in the proof of the lemma above is due to BALKEMA [1977]. KAPLAN [1955] gives an example that shows that his inequality is sharp in a suitable sense. We shall discuss inequality (2.2.1) now. With the exception of the combinatorics the proof is parallel to the proof above.

THEOREM 2.2.3. *Suppose* $S_{\mathbb{Z}}$ *is a transient random walk with stationary increments and let* N *be defined by* (2.1.3). *For any real* t *and real positive* h *we have*

$$P(N(t,t+h] \ge p) \le 2\, P(N(-h,h) \ge p),$$

where p *is any positive integer.*

The combinatorial lemma cannot be formulated as simply as before.

LEMMA 2.2.4. *Let* m *be a finite, integer valued measure on the real line. Define for any* $B \in \mathcal{B}^1$

$$m_p(B) := 1, \quad \textit{if } m(B) \geq p,$$
$$:= 0, \quad \textit{else,}$$

where p *is any positive integer. Then we have for* $h > 0$

$$\int m_p(x+t,\ x+t+h]dm(x) \leq 2 \int m_p(x-h,\ x+h)dm(x).$$

In the proof of the lemma we restate its assertion in terms of a counting problem.

PROOF of Theorem 2.2.3. The proof is parallel to the proof of Theorem 2.2.1. The probabilities in the inequality are written as expectations of limits of Cesaro averages. The proof is reduced to a comparison of

$$M^{(n)} := \#\{i: \#\{j: S_j \in (S_i+t,\ S_i+t+h]\} \geq p\}$$

and

$$M_0^{(n)} := \#\{i: \#\{j: S_j \in (S_i-h,\ S_i+h)\} \geq p\}.$$

Here the indices i and j are running along $-n,\ldots,-2,-1$. We use Lemma 2.2.4 and choose m as the measure for which m(B) equals the number of $(S_i(\omega))_{i=-n}^{-1}$ in each Borel set B. By using Lemma 2.2.4 we obtain $M^{(n)} \leq 2M_0^{(n)}$ and as before in Theorem 2.2.1 we derive the inequality from this observation. □

PROOF of lemma 2.2.4. We represent m by a sequence $x_1 \leq \ldots \leq x_n$ that satisfies

$$m(B) = \sum_{i=1}^{n} \chi_B(x_i), \qquad B \in \mathcal{B}^1.$$

Note that $x_1,\ldots,x_n$ are uniquely determined by m. We call this sequence of real numbers the *points* of m. Let t be real. Say that x_i has a *distant cluster* if $(x_i+t,\ x_i+t+h]$ contains at least p points x_j, and x_i has a *close cluster* if $(x_i-h,\ x_i+h)$ contains at least p points x_j. Define

$$C_0 := \{i: x_i \text{ has a close cluster}\},$$

$$C := \{i: x_i \text{ has a distant cluster}\}.$$

We claim

$$\#(C - C_0) \le \#C_0. \tag{2.2.5}$$

Note that this implies the assertion of the lemma, which can be restated as $\#C \le 2\,\#C_0$.

We write $i\,R\,j$ (or $(i,j) \in R$) if j belongs to the distant cluster of i, i.e. $x_j \in (x_i+t,\ x_i+t+h]$, and $i \in C\setminus C_0$. If $i\,R\,j$ then we have $j \in C_0$ and hence R is a subset of $(C\setminus C_0) \times C_0$. We obtain the inequality

$$\#R \ge p\,\#(C\setminus C_0), \tag{2.2.6}$$

because for each $i \in C\setminus C_0$ the interval $(x_i+t,\ x_i+t+h]$ contains at least p points x_j (these belong to the distant cluster of x_i). We also have the inequality

$$\#R \le (p-1)\,\#C_0. \tag{2.2.7}$$

This is proved as follows. Fix some $j \in C_0$ such that $i\,R\,j$ for some i. Then j is in the distant cluster of i. Let also j be in the distant cluster of x_k. We necessarily have $|x_i - x_k| < h$. Let $i_1 < \dots < i_r$ be the sequence of all indices k such that j is in their distant cluster. If $r \ge p$ then x_i would have a close cluster, i.e. $i \in C_0$. However, this would contradict $i\,R\,j$ by the second statement in the definition of R. So $r \le p-1$ and to each $j \in C_0$ there correspond at most $p-1$ indices i with $i\,R\,j$. This implies (2.2.7). The two inequalities (2.2.6) and (2.2.7) together yield (2.2.5) and hence the assertion of the lemma follows. □

Theorem 2.2.3 has a useful corollary.

<u>COROLLARY 2.2.5</u>. *Suppose* $S_{\mathbb{Z}}$ *is a transient random walk with stationary increments and let* N *be defined by* (2.1.3). *If* $EN(-h,h) < \infty$ *for some* $h = h_0 > 0$, *then the family of random variables* $N(t,t+h]$, t *real, is uniformly integrable for all* h. *This assertion holds also for a transient random walk* $(S_n)_{n\ge 0}$ *with* N *replaced by*

$$N^+(B) := \sum_{n\ge 0} \chi_B(S_n), \quad B \in \mathcal{B}^1. \tag{2.2.8}$$

PROOF. First we consider the random walk $S_{\mathbb{Z}}$. Observe that by Theorem 2.2.3 we have $E\phi(N(t,t+h]) \leq 2\,E\phi(N(-h,h))$ for any nonnegative nondescending function ϕ. Choose the function ϕ to be

$$\begin{aligned} \phi(x) &:= x, \quad x \geq p, \\ &:= 0, \quad x < p. \end{aligned}$$

We obtain

$$\int_{N(t,t+h]\geq p} N(t,t+h]\,dP \leq \int_{N(-h,h)\geq p} N(-h,h)\,dP,$$

uniformly in t. By our assumptions the right-hand side converges to 0 if $p \to \infty$ for $h = h_0$. Hence the sequence

$$N(t,t+h], \qquad t \in \mathbb{R}^1,$$

is uniformly integrable for $h = h_0$, and then also for any $h > 0$.

To prove the assertion for N^+ we argue as follows. Observe that because S_n is distributed as $-S_{-n}$

$$EN(-h,h) = \sum_{n\in\mathbb{Z}} P(S_n \in (-h,h)) \leq 2\,EN^+(-h,h) - 1 < \infty$$

for $h = h_0$. By Proposition 2.1.9 we know that $S_{\mathbb{Z}}$ is transient. Hence the assertion of the corollary holds for N and therefore also for N^+. □

As an application we prove the global renewal theorem as a direct consequence of the ergodic theorem. The proof by DALEY [1971] uses Palm theory and subadditive functions. The detour along (2.2.1) makes the present proof longer than if the integrability question would be settled with Palm theory.

THEOREM 2.2.6. *Suppose* $(S_n)_{n\geq 0}$ *is a transient random walk with stationary increments* $\xi_{\mathbb{N}}$. *Let the renewal measure* H, *given by* (2.2.2), *be finite on a neighbourhood of the origin. If* $E\xi_1 \leq \infty$ *exists and* $E(\xi_1|J_\xi) > 0$ *a.s., where* J_ξ *is the invariant σ-field of* $\xi_{\mathbb{N}}$, *then*

$$\lim_{t\to\infty} \frac{1}{t} H(0,t] = E\,\frac{1}{E(\xi_1|J_\xi)}\ .$$

Here we have to read $\frac{1}{\infty} := 0$.

PROOF. By the ergodic theorem

$$\lim_{n\to\infty} \frac{S_n}{n} = E(\xi_1 | J_\xi) > 0 \text{ a.s.}$$

So the sequence $(S_n)_{n\geq 0}$ is transient. By Lemma 2.1.6 we have

$$\lim_{t\to\infty} \frac{N^+(0,t]}{t} = \zeta := \frac{1}{E(\xi_1 | J_\xi)} \text{ a.s.},$$

where N^+ is defined by (2.2.8). By BREIMAN [Proposition 5.19] it is sufficient to show that $\frac{1}{t} N^+(0,t]$ is uniformly integrable to obtain the assertion.

Because $EN^+(-h,h) < \infty$ for some $h > 0$ the sequence

$$N^+(t,t+h], \quad t \text{ real},$$

is by Proposition 2.2.5 uniformly integrable for all $h > 0$. By Fatou's lemma and Theorem 2.2.1

$$E\zeta \leq \liminf_{t\to\infty} E \frac{N^+(0,t]}{t} \leq EN^+(-1,1) < \infty.$$

Write $A_k := \{N^+(k,k+1] \geq p\}$, $k \geq 0$, and note that on A_k^c we have $\frac{1}{t} N^+(k,k+1] \leq \frac{p}{t}$. Hence

$$\int_{N^+(0,t]\geq pt} \frac{N^+(0,t]}{t} dP$$

$$\leq \sum_{k=0}^{[t]} \int_{A_k} \frac{N^+(k,k+1]}{[t]} dP + \sum_{k=0}^{[t]} \int_{N^+(0,t]\geq pt} \frac{p}{t} \chi_{A_k^c} dP.$$

We obtain

$$\limsup_{t\to\infty} \int_{N^+(0,t]\geq pt} \frac{N^+(0,t]}{t} dP$$

$$\leq \sup_x \int_{N^+(x,x+1]\geq p} N^+(x,x+1] dP + p \limsup_{t\to\infty} P(N^+(0,t]\geq pt).$$

Because the term on the right is at most $p\, P(\zeta \geq p)$ we obtain by the finiteness of $E\zeta$ and the uniform integrability of $N^+(x,x+1]$, $x \in \mathbb{R}^1$, that the right-hand side of the last inequality is arbitrarily small if p is arbitrarily large. □

Above we have seen that several results that are known for random walks with independent, stationary increments have an analogue for random walks without this independence requirement. However, there is an important exception. For random walks with independent increments transience is equivalent with finiteness of the renewal measure on bounded sets (see FELLER [VI, 10]). This result does not hold for random walks with stationary increments.

EXAMPLE 2.2.7. Let $X_{\mathbb{N}}$ be a positive recurrent, irreducible Markov chain with countable state space Γ, started with an invariant probability measure. The process $X_{\mathbb{N}}$ is stationary and according to Section 0.2 can be extended to a stationary process $X_{\mathbb{Z}}$. Let $\gamma \in \Gamma$ be an element of the state space and let f be the indicator function of $\{\gamma\}$. Let $S_{\mathbb{Z}}$ be the random walk with increments $(f(X_n))_{n\in\mathbb{Z}}$.

Suppose that τ is the first entrance time after 0 of the state γ, i.e.

$$\tau := \inf\{n > 0: X_n = \gamma\}.$$

Let $(p_j)_{j\geq 1}$ be the conditional distribution of τ given $\{X_0 = \gamma\}$. We shall suppose that this distribution has finite first and infinite second moment. Observe that

$$\begin{aligned} P(\tau = j) &= \sum_{i=0}^{\infty} P(X_{-i}=\gamma, X_{-i+1}\neq\gamma,\ldots,X_{j-1}\neq\gamma, X_j=\gamma) \\ &= \sum_{i=0}^{\infty} p_{i+j}\, P(X_0=\gamma) \end{aligned}$$

by the stationarity of the Markov chain. By our assumptions on $(p_j)_{j\geq 1}$ we have

$$\begin{aligned} E\tau &= \sum_{j=1}^{\infty} j\, P(\tau=j) = P(X_0=\gamma) \sum_{j=0}^{\infty} j \sum_{i=j}^{\infty} p_i \\ &= P(X_0=\gamma) \sum_{i=0}^{\infty} \tfrac{1}{2}\, i(i+1) p_i = \infty. \end{aligned}$$

In terms of the random walk $S_{\mathbb{Z}}$ this means for the symmetrical renewal measure H^S that $H^S\{0\} = \infty$. But because γ is recurrent we also know that $S_{\mathbb{Z}}$ is transient.

Further discussions on the finiteness of the renewal measure and its connections with Palm theory can be found in NEVEU [1976, Proposition II.24] and DELASNERIE [1977].

2.3. RECURRENCE AND TRANSIENCE OF A RANDOM WALK

In Section 2.1 we defined sets of transience and recurrence for a random walk with stationary increments. In the present section we shall study this subdivision somewhat closer. Especially the concept of recurrence will be clarified. For random walks with stationary, independent increments the subject has been thoroughly studied (see BREIMAN [Section 3.7]). The following result gives a clear picture. In Corollary 2.3.4 we shall see what remains if we do not assume independence. Let the lattice L_d be defined by (1.1.2).

PROPOSITION 2.3.1. *Let* $(S_n)_{n\geq 0}$ *be a random walk with stationary, independent increments and let* N *be given by*

$$(2.3.1) \qquad N(B) := \sum_{n\geq 0} \chi_B(S_n), \qquad B \in \mathcal{B}^1.$$

There are two possibilities:

(i) *for all bounded intervals* I

$$N(I) < \infty \text{ a.s,}$$

(ii) *there is a lattice* L_d, $0 \leq d \leq \infty$, *such that for each interval* I

$$\begin{aligned} N(I) &= 0 \text{ a.s.} \quad &&\text{if } I \cap L_d = \phi, \\ &= \infty \text{ a.s.} \quad &&\text{else.} \end{aligned}$$

The random walk is called transient or recurrent, depending on the occurrence of case (i) *or case* (ii) *respectively.*

PROOF. See BREIMAN (Corollary 3.36). □

The arguments used in the proof of the proposition above depend on the Markov property. It follows from this result that in the independent case the set of transience has probability 0 or 1. However, this 0-1 property does not hold if the independence assumption is not satisfied. The following example illustrates this.

EXAMPLE 2.3.2. We consider a mixture of probability measures. Let $(S_n)_{n\geq 0}$ be the coordinate process on the measurable space $\prod_{n=0}^{\infty} (\mathbb{R}^1, \mathcal{B}^1)$. Let P_1 and P_2 be probability measures on this measurable space such that $(S_n)_{n\geq 0}$ is a random walk with stationary, independent increments under P_1 and P_2 and is transient under P_1 and recurrent under P_2. Then under $P := \frac{1}{2}P_1 + \frac{1}{2}P_2$ the process $(S_n)_{n\geq 0}$ is a random walk with stationary increments for which both the set of transience and of recurrence have probability $\frac{1}{2}$.

The example shows that we should not characterize a process as transient or recurrent. To overcome this difficulty we have introduced in Section 2.1 the notions of sets of recurrence and transience of a random walk. It will be shown that under the condition of ergodicity we can regain the 0-1 property.

The concept of transience is placed in a different light if we consider the assertion of Proposition 2.1.9. If $S_{\mathbb{Z}}$ is a random walk and N is defined by (2.1.3) then the random walk is transient if and only if for any open neighbourhood I of the origin $N(I) < \infty$ a.s. This suggests that the behaviour at the origin determines transience and recurrence of a random walk. We show that the sets

$$(2.3.2)\qquad \begin{aligned} J_r &:= \{0 \text{ is a limit point of } (S_n)_{n\geq 0}\}, \\ J_t &:= J_r^c, \end{aligned}$$

coincide a.s. with the sets of recurrence and transience of a random walk. In fact we shall show something more. With probability 1 each point S_n, $n \geq 0$ is a limit point of the random walk as soon as 0 is a limit point of the random walk. To formulate another property we introduce the following notations. Let $s := (s_n)_{n\geq 0}$ be a sequence of real numbers. Define the set $A_r(s)$ of right limit points of s in the sequence s as

$$A_r(s) := \{s_n\colon \text{there is a subsequence } s_{n_1} > s_{n_2} > \dots \to s_n\}.$$

The set $A_\ell(s)$ of left limit points of s in the sequence s is defined similarly. Let $A_\infty(s)$ be the set of infinitely multiple points of s in s, so

$$A_\infty(s) := \{s_n\colon s_m = s_n \text{ for infinitely many } m\}.$$

<u>THEOREM 2.3.3</u>. *Let* $S := (S_n)_{n\geq 0}$ *be a random walk with stationary increments. Define* J_r *and* J_t *by* (2.3.2). *Then apart from a null set* J_r *and* J_t *coincide with the sets of recurrence and transience* I_r *and* I_t *of* S. *Apart from a null set on* I_t *no point* S_n, $n \geq 0$, *is a limit point of* S *and on* I_r *each point* S_n, $n \geq 0$, *is a limit point of* S, *possibly an infinitely multiple point or else both right and left limit point of* S, *i.e. for all* $n \geq 0$

$$I_r = \{S_n \in A_\infty(S) \cup (A_r(S) \cap A_\ell(S))\} \text{ a.s.}$$

If $S_{\mathbb{Z}}$ is a random walk with stationary increments then the sets of recurrence of $(S_n)_{n\geq 0}$ and $(S_{-n})_{n\geq 0}$ and also the sets of transience of these random walks coincide apart from a null set.

In this more general context we can obtain as the result corresponding to Proposition 2.3.1:

<u>COROLLARY 2.3.4</u>. *Let $(S_n)_{n\geq 0}$ be a random walk with stationary, ergodic increments and let N be defined by (2.3.1). There are two possibilities:*

(i) *for all bounded intervals* I

$$N(I) < \infty \text{ a.s;}$$

(ii) *for all intervals* I

$$\{N(I) = 0\} \cup \{N(I) = \infty\}$$

has probability 1.

<u>PROOF of Corollary 2.3.4</u>. Observe that the set

$$\{N \text{ is finite on bounded intervals}\}$$

is an invariant set for the process of increments $\xi_{\mathbb{N}}$ and hence has probability 0 or 1 by the ergodicity assumption. Suppose it has probability 0. We shall prove that (ii) holds. By the definition of the recurrent set I_r we know that $P(I_r) = 1$. Let I be an interval, open or closed. By Theorem 2.3.3 we know that apart from a null set if some $S_n \in I$, it is an infinitely multiple point or both right and left limit point. Therefore, for all n

$$\{S_n \in I\} \subset \{N(I) = \infty\} \text{ a.s.,}$$

from which the assertion is easily inferred. □

The proof of Theorem 2.3.3 will be prepared in several steps. Throughout the rest of this section we shall make some assumptions about the space on which the random variables are defined. By the arguments given in Section 0.2 we may assume that $\xi_{\mathbb{Z}}$ is the coordinate process on $\mathbb{R}^{\mathbb{Z}}$. We assume that the probability measure P is defined on $\prod_{n\in\mathbb{Z}} (\mathbb{R}^1, \mathcal{B}^1)$ and is invariant under the shift transformation T on $\mathbb{R}^{\mathbb{Z}}$ given by

$$(Tx)_n := x_{n+1}, \quad n \in \mathbb{Z}.$$

The process $S_{\mathbb{Z}}$ is determined by

$$S_0(x) := 0, \quad x_n = S_n(x) - S_{n-1}(x), \quad n \in \mathbb{Z},$$

where $x \in \mathbb{R}^{\mathbb{Z}}$.

The proof is based on the following idea. Suppose we want to show that on a subset $\Omega \subset \mathbb{R}^{\mathbb{Z}}$ the random walk $S_{\mathbb{Z}}$ is transient, i.e. $\Omega \subset I_t$ a.s. We show in Proposition 2.3.8 that it is already sufficient to prove transience for certain subsequences of $S_{\mathbb{Z}}$. Thus a proof that Ω is included in the set of transience is considerably simplified by this proposition. This result is used in Proposition 2.3.9 to prove transience for a special class of random walks. Then the theorem is proved by means of a lemma.

The following proposition and its corollary serves as a preparation for the proof of Proposition 2.3.8. The argument is narrowly related to the proof of Theorem 2.1.4.

PROPOSITION 2.3.5. *Let* T *be the shift transformation. Suppose on a measurable subset* $\Omega_1 \subset \mathbb{R}^{\mathbb{Z}}$ *having positive probability, there exists a bijection* $\tilde{T}$ *of* Ω_1 *onto itself, given by*

$$\tilde{T}x = T^{f(x)}x, \quad x \in \Omega_1,$$

where f *is a measurable, integer valued function on the restriction of* $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\mathbb{Z}})$ *to* Ω_1. *Let* T *be measure preserving under* P. *Then* $\tilde{T}$ *is measure preserving under* $P_{\Omega_1}(\cdot) := P(\cdot|\Omega_1)$.

PROOF. Let the inverse of $\tilde{T}$ on Ω_1 be given by

$$\tilde{T}^{-1}x = T^{g(x)}x, \quad x \in \Omega_1.$$

The relation between f and g is described by

$$f(x) = k \iff g(T^k x) = -k.$$

Let $A \subset \Omega_1$ be a measurable set. Because $\tilde{T}$ is a bijection of Ω_1 onto itself we have

$$P(\tilde{T}^{-1} A) = \sum_{k \in \mathbb{Z}} P(f(x) = k,\ T^k x \in A,\ x \in \Omega_1).$$

Because T^k is measure preserving we have, using the relation between f and g,

$$P(\tilde{T}^{-1} A) = \sum_{k \in \mathbb{Z}} P(g(x) = -k,\ x \in A,\ T^{-k} x \in \Omega_1) = P(A).$$

Because A and $\tilde{T}^{-1} A$ are included in Ω_1 we obtain

$$P(A|\Omega_1) = P(\tilde{T}^{-1}A|\Omega_1)$$

for all measurable sets $A \subset \Omega_1$. □

COROLLARY 2.3.6. *Under the assumptions of Proposition* 2.3.5, *let* $(\tau_n)_{n\in\mathbb{Z}}$ *on* Ω_1 *be defined as a sequence with increments* $(f\tilde{T}^{n-1})_{n\in\mathbb{Z}}$, *i.e.*

$$\tau_0(x) := 0,$$

$$\tau_{n+1}(x) = \tau_n(x) + f(\tilde{T}^n x), \quad n \in \mathbb{Z},$$

for $x \in \Omega_1$. *Then* $(S_{\tau_n})_{n\in\mathbb{Z}}$ *is a random walk with stationary increments under* P_{Ω_1}.

PROOF. Define on Ω_1 the process $S := (S_n)_{n\in\mathbb{Z}}$ and observe that

$$S\tilde{T}^k = (S_{\tau_k+n} - S_{\tau_k})_{n\in\mathbb{Z}}$$

for each integer k. By Proposition 2.3.5 the process $(S\tilde{T}^k)_{k\in\mathbb{Z}}$ is stationary under P_{Ω_1}, so the increments $(S_{\tau_{k+1}} - S_{\tau_k})_{k\in\mathbb{Z}}$ are a stationary process.□

Central in the proof of Theorem 2.3.3 is the proposition below. It extends an argument applied by KAPLAN [1955] to investigate transience for a class of random walks discussed in Example 2.1.11.

In our formulations we use the notion of a random integer set, i.e. a random set on the integers $\mathbb{Z}$, as defined in Section 0.3. Note that the realizations of such a random set are integer sets $K \subset \mathbb{Z}$. Define for $K \subset \mathbb{Z}$ the set

$$K - k := \{j-k: j \in \mathbb{Z}\}$$

for all integers k. First we discuss an example.

EXAMPLE 2.3.7. Suppose a random integer set M satisfies for all $x \in \mathbb{R}^{\mathbb{Z}}$

$$M(Tx) = M(x) - 1.$$

Define $A := \{0 \in M\}$ and observe that

$$M(x) = \{n \in \mathbb{Z}: T^n x \in A\}.$$

Let $\bar{A}$ be the set of all $x \in \mathbb{R}^{\mathbb{Z}}$ for which $T^n x \in A$ for infinitely many positive and infinitely many negative n. Let $\underline{A}$ be the set of all x for which $T^n x \in A$ for some n. By Poincaré's recurrence theorem (see HALMOS [1956, p.10] or BREIMAN [Proposition 6.38] the sets $\underline{A}$ and $\bar{A}$ coincide a.s. and therefore $\bar{A}$ coincides a.s. with $\{M \neq \phi\}$.

Let $(s_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers and let $K \subset \mathbb{Z}$. We say that the sequence $(s_n)_{n\in K}$ is *transient* if on each bounded interval there occurs only a finite number of elements of this sequence.

<u>PROPOSITION 2.3.8</u>. *Let M be a random integer set defined on* $\mathbb{R}^{\mathbb{Z}}$ *satisfying for all* $x \in \mathbb{R}^{\mathbb{Z}}$

$$M(Tx) = M(x) - 1.$$

If S_M *is transient a.s. then*

$$\{M \neq \phi\} \subset \{S_{\mathbb{Z}} \textit{ is transient}\} \textit{ a.s.}$$

<u>PROOF</u>. We apply Proposition 2.3.5. Define A, $\bar{A}$ and $\underline{A}$ as in Example 2.3.7 and let $\Omega_1 := A \cap \bar{A}$. Because $\underline{A} = \bar{A}$ a.s. and $A \subset \underline{A}$ we have $A = \Omega_1$ a.s. Consider for $x \in \Omega_1$ the set M(x) and write its elements as

$$\dots < \tau_{-1}(x) < \tau_0(x) = 0 < \tau_1(x) < \dots .$$

The mapping

$$\tilde{T}x := T^{\tau_1(x)}x, \quad x \in \Omega_1,$$

clearly is a bijection onto Ω_1 and by Proposition 2.3.5 is measure preserving under $P_{\Omega_1}(\cdot) := P(\cdot|\Omega_1)$. As in the proof of Corollary 2.3.6 we derive that the sequence $(S\tilde{T}^k)_{k\in\mathbb{Z}}$ with $S := (S_n)_{n\in\mathbb{Z}}$ is stationary and thus $(S_{\tau_k})_{k\in\mathbb{Z}}$ is a random walk with stationary increments. Because S_M is transient P-a.s. the random walk $(S_{\tau_k})_{k\in\mathbb{Z}}$ is P_{Ω_1}-a.s. transient and by Lemma 2.1.2 the range of this random walk exceeds every positive and negative bound P_{Ω_1}-a.s. We shall describe a covering of the real line. Define $(S^+_{\tau_k})_{k\in\mathbb{Z}}$ as follows. If for some $j > k$ we have $S_{\tau_k} = S_{\tau_j}$, take $S^+_{\tau_k} := S_{\tau_k}$ and otherwise let

$$S^+_{\tau_k} := \inf\{S_{\tau_n} : S_{\tau_n} > S_{\tau_k},\ n \in \mathbb{Z}\}.$$

With this definition the real line is covered by

$$[S_{\tau_k}, S^+_{\tau_k}), \quad k \in \mathbb{Z}, \tag{2.3.3}$$

and each point of $\mathbb{R}$ is covered exactly once. With E_{Ω_1} denoting expectation with respect to P_{Ω_1} we write

$$E_{\Omega_1} N[0,S^+_{\tau_0}) = \sum_{k\in\mathbb{Z}} E_{\Omega_1} \#\{S_n \in [0,S^+_{\tau_0}): \tau_k \leq n < \tau_{k+1}\}.$$

Using that $(S\tilde{T}^k)_{k\in\mathbb{Z}}$ is a stationary sequence, we obtain

$$E_{\Omega_1} N[0,S^+_{\tau_0}) = \sum_{k\in\mathbb{Z}} E_{\Omega_1} \#\{S_j - S_{\tau_{-k}} \in [0,S^+_{\tau_{-k}} - S_{\tau_{-k}}): 0 \leq j < \tau_1\}$$

$$= \sum_{k\in\mathbb{Z}} E_{\Omega_1} \#\{S_j \in [S_{\tau_{-k}}, S^+_{\tau_{-k}}): 0 \leq j < \tau_1\}.$$

Because (2.3.3) covers each element of $\mathbb{R}$ exactly once

$$E_{\Omega_1} N[0,S^+_{\tau_0}) = E_{\Omega_1} \tau_1.$$

Note that because P is invariant under T and $\bar{A}$ is invariant, the measure $P(\cdot|\bar{A})$ is invariant under T. By Kac's identity (see BREIMAN [Proposition 6.38])

$$E_{\Omega_1}(\tau_1) = 1/P(A|\bar{A})$$

and hence for k = 0

$$N[S_{\tau k}, S^+_{\tau k}) < \infty \quad P_{\Omega_1}\text{-a.s.}$$

By the stationarity of $(S\tilde{T}^k)_{k\in\mathbb{Z}}$ this holds for all integers k. Hence we have

$$\Omega_1 \subset \{S_{\mathbb{Z}} \text{ is transient}\} \ P\text{-a.s.}$$

Because T is measure preserving and the set on the right-hand side is invariant under T one easily deduces that for all n

$$T^n\Omega_1 \subset \{S_{\mathbb{Z}} \text{ is transient}\} \ P\text{-a.s.}$$

The union of the sets on the left-hand side is $\bar{A}$. According to Poincaré's recurrence principle $\bar{A}$ coincides P-a.s. with $\{M \neq \emptyset\}$. This proves the proposition. □

We apply the proposition for a subclass of random walks with points that are isolated from the right in a particular sense.

PROPOSITION 2.3.9. *Let* ε *be a positive random variable that is invariant under* T. *Suppose that*

$$\#\{n > 0 : S_n \in [0,\varepsilon)\} = 0.$$

Then $S_{\mathbb{Z}}$ *is a transient random walk.*

PROOF. By using stationarity it is easily seen that we may strengthen the assumption to

$$\#\{n > k : S_n \in [S_k, S_k+\varepsilon)\} = 0$$

for all integers k. The proof will be split up in a nonprobabilistic and a probabilistic part.

Part 1. Let $\Omega_1 \subset \mathbb{R}^{\mathbb{Z}}$ be the subset of all $x \in \mathbb{R}^{\mathbb{Z}}$ such that $s := (S_n(x))_{n\in\mathbb{Z}}$ has the following properties. There is a number e, depending on x, such that for all m

(j) $\nexists n > m : s_n \in [s_m, s_m+e)$

(jj) $\exists n > m : s_n \in (s_m-e, s_m)$

(jjj) $\exists n < m : s_n \in (s_m, s_m+e)$.

An example of such a sequence s is given by

$$\begin{aligned} s_n &:= 1 + \frac{1}{n-1}, \quad && n < 0, \\ &:= 0 \quad , && n = 0, \\ &:= -1 + \frac{1}{n+1}, && n > 0. \end{aligned}$$

Choose some fixed $x \in \Omega_1$ and consider s. We shall construct descending subsequence(s) out of s that satisfy some suitable properties. Similar to what we did in Section 2.1 we shall introduce here a notion of successor of an index. Because of (j) all elements s_n are different. Consider

(2.3.4) $\qquad \sup\{s_n : n > m,\ s_n < s_m\}.$

By (jj) the set in (2.3.4) is not empty and by (j) the maximum is attained at some uniquely determined index n, that we shall call the *successor* of m with respect to s. By using the conditions (j) and (jjj) we can also show that to each n there exists an index m such that n is successor to m. This index m is uniquely determined as the index m for which

$$\inf\{s_m : m < n,\ s_m > s_n\}$$

is attained as a minimum. As a consequence we can form index sequences

$$\ldots, m_{-1}, m_0, m_1, \ldots ,$$

where m_{n+1} is successor to m_n, $n \in \mathbb{Z}$. Observe that if m_0 is prescribed the sequence is uniquely determined. Each subsequence $(s_{m_n})_{n\in\mathbb{Z}}$ is descending with stepsizes between 0 and -e and by (j) every two subsequences can be "separated" in the following way. There is a number c such that all elements of one subsequence are above c and all elements of the other one are below c. Hence if there is a subsequence for which $\lim_{n\to\infty} s_{m_n} = -\infty$ and $\lim_{n\to-\infty} s_{m_n} = \infty$ then this subsequence is, apart from a renumbering, the only one and contains every element of s. Let $x \in \Omega_1$ and take $(\tau_n(x))_{n\in\mathbb{Z}}$ to be the subsequence with $m_0 = m_0(x)$ chosen as 0. The mapping $\tilde{T}$ defined by

$$\tilde{T}x := T^{\tau_1(x)} x, \quad x \in \Omega_1,$$

is a bijection of Ω_1 onto itself.

Part 2. Define

$$A_1 := \{\nexists n > 0: S_n \in (-\varepsilon, 0)\}$$

and let

$$M_1(x) := \{n \in \mathbb{Z}: T^n x \in A_1\}.$$

Because S_{M_1} contains only elements S_m, for which there is no $n > m$ such that $S_n \in (S_m-\varepsilon, S_m+\varepsilon)$ we easily see that S_{M_1} is transient. Hence by Proposition 2.3.8

$$\{M_1 \neq \phi\} \subset \{S_{\mathbb{Z}} \text{ is transient}\} \text{ a.s.}$$

Define

$$A_2 := \{\nexists n < 0: S_n \in (0, \varepsilon)\}$$

and let

$$M_2(x) := \{n \in \mathbb{Z}: T^n x \in A_2\}.$$

Because S_{M_2} contains only elements S_m for which $S_n \notin [S_m, S_m+\varepsilon)$ if $m \neq n$, we can conclude that S_{M_2} is transient. Hence

$$\{M_2 \neq \phi\} \subset \{S_{\mathbb{Z}} \text{ is transient}\} \text{ a.s.}$$

Observe that Ω_1, as defined in part 1, is given by $\Omega_1 = \{M_1 = M_2 = \phi\}$. By

Corollary 2.3.6 the process $(S_{\tau_n})_{n\in\mathbb{Z}}$ defined on Ω_1 is a random walk with stationary increments under $P_{\Omega_1}(\cdot) := P(\cdot|\Omega_1)$. Because the increments are negative the ergodic theorem assures us that

$$\lim_{n\to\infty} \frac{S_{\tau_n}}{n} = \lim_{n\to-\infty} \frac{S_{\tau_n}}{n} < 0 \quad P_{\Omega_1}\text{-a.s.}$$

and hence

$$\lim_{n\to\infty} S_{\tau_n} = -\lim_{n\to-\infty} S_{\tau_n} = \infty \quad P_{\Omega_1}\text{-a.s.}$$

By part 1 the sequences $(S_{\tau_n})_{n\in\mathbb{Z}}$ and $(S_n)_{n\in\mathbb{Z}}$ coincide P_{Ω_1}-a.s., so also

$$\Omega_1 \subset \{S_{\mathbb{Z}} \text{ is transient}\} \text{ a.s.} \qquad \square$$

<u>LEMMA 2.3.10</u>. *Let* N^+ *be the point process determined by* $(S_n)_{n>0}$ *and define*

$$A_0 := \{\lim_{\delta\downarrow 0} N^+[0,\delta) < \infty\}.$$

The random set M *defined by*

$$M_0(x) := \{n \in \mathbb{Z}: T^n x \in A_0\}, \qquad x \in \mathbb{R}^{\mathbb{Z}},$$

satisfies

$$\{M_0 \neq \phi\} \subset \{S_{\mathbb{Z}} \textit{ is transient}\} \textit{ a.s.}$$

<u>PROOF</u>. Let

$$A_1 := \{\lim_{\delta\downarrow 0} N^+[0,\delta) = 0\} \text{ and } M_1(x) := \{n \in \mathbb{Z}: T^n x \in A_1\}.$$

The inclusion

$$\text{(2.3.5)} \qquad \{M_1 \neq \phi\} \subset \{M_0 \neq \phi\}$$

is immediate. To prove that also $\supset$ holds in (2.3.5), suppose that $T^n x \in A_0$ for some n. Hence there are only a finite number of S_m, $m > n$, such that $S_m \in [S_n, S_n+\delta)$ for a finite $\delta > 0$. If for none of these S_m, $m > n$, holds $S_m = S_n$ then clearly $T^n x \in A_1$. Else let m' be the largest m with $S_m = S_n$, $m > n$, and observe that $T^{m'} x \in A_1$. This shows that we also have $\supset$ in (2.3.5). Define h(x) as the length of the largest set $[0,\delta)$, $\delta > 0$, containing no points S_n, $n > 0$, i.e.

$$h(x) := \sup\{\delta \geq 0: N^+[0,\delta) = 0\}.$$

Define for $x \in \mathbb{R}^{\mathbb{Z}}$

$$\varepsilon(x) := \tfrac{1}{2}(1 \wedge \sup\{h(T^n x): n \in \mathbb{Z}\}).$$

Observe that the random variable ε is invariant under T and define

$$M_2(x) := \{n \in \mathbb{Z}: h(T^n x) \geq \varepsilon(x) > 0\}$$

Notice that

$$\{M_2 \neq \phi\} = \{\varepsilon > 0\} = \{M_1 \neq \phi\} = \{M_0 \neq \phi\}$$

and by the invariance of ε under T

$$M_2(Tx) = M_2(x) - 1.$$

Because $\{M_2 \neq \phi\}$ is an invariant set, one easily shows that

$$P(\cdot \mid M_2 \neq \phi)$$

is invariant under T. Observe that all points in S_{M_2} are separated by distances of at least ε. Hence S_{M_2} is transient and by Proposition 2.3.9

$$P(S_{\mathbb{Z}} \text{ is transient} \mid M_2 \neq \phi) = 1.$$

As a result of $\{M_2 \neq \phi\} = \{M_0 \neq \phi\}$ the lemma is proved. □

PROOF of Theorem 2.3.3. We shall use Lemma 2.3.10. Let $S^+ := (S_n)_{n>0}$ and $S^- := (S_n)_{n<0}$. Define

$$G_{r\pm} := \bigcap_{n\in\mathbb{Z}} \{S_n \in A_\infty(S^\pm) \cup A_r(S^\pm)\}.$$

First we prove

$$G^c_{r+} \subset \{S_{\mathbb{Z}} \text{ is transient}\} \text{ a.s.}$$

Let N^+ be the point process determined by S^+ and observe that

$$\{0 \notin A_\infty(S^+) \cup A_r(S^+)\} = \{\lim_{\delta\downarrow 0} N^+[0,\delta) < \infty\}.$$

Denote this set by A_0. Observe that

$$x \in \{S_k \notin A_\infty(S^+) \cup A_r(S^+)\}$$

is equivalent with $T^k x \in A_0$ and hence

$$G_{r+}^{c} = \{M_0 \neq \phi\}$$

with M_0 defined as

$$M_0(x) := \{n: T^n x \in A_0\}.$$

By Lemma 2.3.10 we have

$$G_{r+}^{c} = \{M_0 \neq \phi\} \subset \{S_{\mathbb{Z}} \text{ is transient}\} \text{ a.s.}$$

A similar argument applied to $(S_{-n})_{n\in\mathbb{Z}}$ shows

$$G_{r-}^{c} \subset \{S_{\mathbb{Z}} \text{ is transient}\} \text{ a.s.}$$

This argument applied to $(-S_n)_{n\in\mathbb{Z}}$ shows that for

$$G_{\ell\pm} := \bigcap_{n\in\mathbb{Z}} \{S_n \in A_\infty(S^\pm) \cup A_\ell(S^\pm)\}$$

we also have

$$G_{\ell\pm}^{c} \subset \{S_{\mathbb{Z}} \text{ is transient}\} \text{ a.s.}$$

Hence the set

$$(G_{r+} \cap G_{\ell+} \cap G_{r-} \cap G_{\ell-}) \cup \{S_{\mathbb{Z}} \text{ is transient}\}$$

has probability 1. The assertions of the theorem follow as an immediate consequence. □

NOTES to the Proof of Theorem 2.3.3.

1° The proof of $I_r = J_r$ a.s. uses the order properties of the real line. Does a similar property hold for random walks in the plane?

2° If Theorem 2.3.3 is formulated for $S_{\mathbb{Z}}$ instead of for $(S_n)_{n\geq 0}$ its proof simplifies considerably. In that case Proposition 2.3.9 is superfluous.

3° Note that Proposition 2.1.9 is a consequence of Theorem 2.3.3.

EXAMPLE 2.3.11. Let $X := (X_n)_{n\geq 0}$ be a sequence of stationary, independent random variables on the real line. Define $\xi_n := X_n - X_{n-1}$, $n \in \mathbb{N}$, and let $S := (S_n)_{n\geq 0}$ be a random walk with stationary increments $\xi_{\mathbb{N}}$, i.e.

$$S_n = X_n - X_0, \quad n \geq 0.$$

We investigate the set of limit points of S. If $s := (s_n)_{n \geq 0}$ is a sequence of real numbers, define the *set of limit points* of s by

$$L(s) := \{x \in \mathbb{R}^1 : \text{ there is a subsequence } s_{n_k} \to x \text{ for } k \to \infty\}.$$

Clearly L(s) is a closed set. If x is a *point of increase* of the distribution F of X_n, $n \geq 0$, i.e. if for each $\varepsilon > 0$

$$F(x-\varepsilon, x+\varepsilon) > 0,$$

then $X_n \in (x-\varepsilon, x+\varepsilon)$ i.o. with probability 1. Let C be the set of points of increase of F. The set C is closed and satisfies $P(X_n \notin C) = 0$, $n \geq 0$. We obtain from the properties of C mentioned above that

$$L(S) = L(X) - X_0 = C - X_0 \text{ a.s.}$$

CHAPTER 3

PALM THEORY

3.1. EXISTENCE OF A LIMIT DISTRIBUTION IN RENEWAL THEORY

Let $S_{\mathbb{Z}}$ be a transient random walk with stationary, strictly positive increments. Define a point process N_0 on the real line by

$$(3.1.1) \qquad N_0(B) := \sum_{n\in\mathbb{Z}} \chi_B(S_n), \qquad B \in \mathcal{B}^1.$$

Note that N_0 and $S_{\mathbb{Z}}$ mutually determine each other. Let T_t, t real, be the translation on the real line, defined as in Section 0.3 and write

$$N_t := T_t N_0.$$

Suppose that the distribution of N_t converges for $t \to \infty$ in some sense. The present section is meant to provide a description of the limiting distribution Q. The limiting distribution Q will be expressed in terms of the distribution Q_0 of N_0. We thus obtain a mapping $Q_0 \to Q$ between two subclasses of distributions of point processes on the real line. A closer study of this mapping will show that it is invertible. Most of the results that are obtained here are already known. They are part of the so-called Palm theory originated by PALM [1943]. However, our approach is somewhat uncommon. It has the advantage that it yields immediately a formula for the limit distribution Q in the renewal theoretic convergence problem sketched above. There is also a disadvantage: the present approach is suitable for point processes on the real line and also for marked point processes on the real line. However, our approach is not suitable for point processes on more general spaces, because we use the order properties of the real line. A review of other methods in Palm theory can be found at the end of this section.

The first proposition contains an aspect that is possibly unknown in the literature. It considers random walks without the restriction that the increments are positive, i.e. the two-sided case. This result has an interesting aspect: Renewal theory frequently deals with the two-sided case by considering an imbedded random walk consisting of record values (see BLACKWELL [1953] and also KESTEN [1974]). Here we succeed in giving a definition of the limiting distribution Q without the use of records. In Chapter 6 we apply this result to get a generalized version of Blackwell's theorem and other results. It would have been possible to generalize the other propositions to cover the two-sided case also. This would however lead us out of the field of point processes and does not seem to yield a better understanding of the theory. Therefore, with the exception of the first proposition, we assume that the random walks have strictly positive increments.

Let the measurable space $(N,\mathcal{D})$ be defined as in Section 0.3. In formula (3.1.3) we use the convention $\int_a^b = -\int_b^a$. In the first proposition we consider for arbitrary $D \in \mathcal{D}$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \chi_D(T_s N_0)\,ds. \tag{3.1.2}$$

PROPOSITION 3.1.1. *Let the point process* N_0 *be defined by* (3.1.1) *with* $S_{\mathbb{Z}}$ *a random walk with stationary, ergodic increments, such that* $ES_1 \in (0,\infty)$. *Denote the distribution of* N_0 *by* Q_0. *Then the limit in* (3.1.2) *exists a.s. and equals*

$$Q(D) := \frac{1}{ES_1} E \int_0^{S_1} \chi_D(T_s N_0)\,ds. \tag{3.1.3}$$

As a function in D *the measure* Q *is a probability measure describing the distribution of a stationary, ergodic point process* N *with finite intensity* $\frac{1}{ES_1}$.

PROOF. Clearly Q is a signed measure with mass at most $E|S_1|/ES_1$. Define for $t > 0$

$$M_t^+ := \inf\{n \geq 1\colon S_n \geq t\},$$

$$M_t^- := \sup\{n \geq 1\colon S_n \leq t\}.$$

By the ergodic theorem $\lim_{n\to\infty} \frac{S_n}{n} = ES_1$ a.s. and hence $\lim_{n\to\infty} S_n = \infty$ a.s. Therefore, the random variables M_t^+ and M_t^- are properly defined. Remark that

$$S_{M_t^-} \le t \le S_{M_t^+}.$$

An application of $\lim_{n\to\infty} \frac{S_n}{n} = ES_1$ a.s. to this inequality yields

$$\limsup_{t\to\infty} \frac{M_t^-}{t} \le \frac{1}{ES_1}, \qquad \liminf_{t\to\infty} \frac{M_t^+}{t} \ge \frac{1}{ES_1} \text{ a.s.}$$

Because $M_t^+ \le M_t^- +1$ it follows that $\lim_{t\to\infty} \frac{M_t^\pm}{t} = \frac{1}{ES_1}$ a.s. Using another application of the ergodic theorem it follows that

$$\frac{1}{n}\int_0^{S_n} \chi_D(T_sN_0)\,ds = \frac{1}{n}\sum_{k=1}^n \int_{S_{k-1}}^{S_k} \chi_D(T_sN_0)\,ds = \frac{1}{n}\sum_{k=1}^n g(T^k\xi_{\mathbb{Z}})$$

converges a.s. Here $\xi_{\mathbb{Z}}$ is the process of increments of $S_{\mathbb{Z}}$ and T is the shift transformation on $\mathbb{R}^{\mathbb{Z}}$. Because $\xi_{\mathbb{Z}}$ is ergodic, the limit for $n \to \infty$ of the expression above coincides a.s. with

$$E\int_0^{S_1} \chi_D(T_sN_0)\,ds.$$

Observe the inequality

$$\frac{M_t^-}{t}\cdot\frac{1}{M_t^-}\int_0^{S_{M_t^-}} \chi_D(T_sN_0)\,ds \le \frac{1}{t}\int_0^t \chi_D(T_sN_0)\,ds \le \frac{M_t^+}{t}\cdot\frac{1}{M_t^+}\int^{M_t^+} \chi_D(T_sN_0)\,ds.$$

Using this inequality and the results above it follows that (3.1.2) equals $Q(D)$ a.s. Because (3.1.2) is nonnegative for all D it follows that Q is a nonnegative measure that has mass $Q(N) = 1$. Observe that (3.1.2) does not change if D is replaced by $T_s^{-1}D$ for any real s. So Q is invariant under translations. Because $\lim_{n\to\infty} \frac{S_n}{n} = ES_1$ a.s., we have $\lim_{t\to\infty} \frac{1}{t} N_0(0,t) = \frac{1}{ES_1}$ a.s. by Lemma 2.1.6. So apart from a null set for any t

$$T_tN_0 \in D_0 := \{m \in N\colon \lim_{t\to\infty} \frac{1}{t} m(0,t) = \frac{1}{ES_1}\}.$$

Hence $Q(D_0) = 1$, so a point process N with distribution Q has intensity $\frac{1}{ES_1}$. Let $D \in \mathcal{D}$ be invariant under all translations T_t. If for some ω the limit (3.1.2) is not 0 then for some and hence for all t we have $T_tN_0(\omega) \in D$. So for this ω the limit (3.1.2) equals 1. Hence (3.1.2) has its values in $\{0,1\}$ and therefore $Q(D) = 0$ or 1. So a point process N with distribution Q is ergodic.

NOTES to Proposition 3.1.1.

1° In case the increments of the random walk are positive with distribution F one easily calculates the distribution of the smallest positive point W of N as

$$P(W \le t) = \int_0^t \frac{1}{\mu}(1 - F(x))dx, \quad t \ge 0.$$

This distribution is called the survivor distribution (see FELLER [XI.4]). The point process N described in Proposition 3.1.1 can be identified as the "steady state" to which a renewal process with independent increments converges.

2° If the increments of $S_{\mathbb{Z}}$ are not ergodic but are stationary with $ES_1 \in (0,\infty)$, then (3.1.3) also describes the distribution Q of a stationary point process N with intensity $\frac{1}{ES_1}$. To prove this let $S^* := \lim_{n\to\infty} \frac{1}{n} S_n$ and define $\rho(D)$ by (3.1.2). Using the arguments in the proof of the proposition, we obtain in this case that $Q_0(D) = \frac{1}{ES^*} ES^* \rho(D)$. The properties of N mentioned above are easily checked.

3° The proposition can be given for marked point processes too: Assume that $(\Gamma,\mathcal{T})$ is a measurable space. Define the marked point process

$$N_0(B) := \sum_{n\in\mathbb{Z}} \chi_B(S_n,X_n), \qquad B \in \mathcal{B}^1\times\Gamma,$$

where $S_0 := 0$ and $(S_n-S_{n-1},X_n)_{n\in\mathbb{Z}}$ is a stationary, ergodic sequence. Assume that ES_1 is finite and positive. Then the assertions about (3.1.2) and Q mentioned in the proposition hold. The proof given above applies also in this more general case.

In the rest of this section we only consider random walks $S_{\mathbb{Z}}$ with strictly positive increments. We also assume that the stationary point processes N on the real line that we consider, are simple and satisfy

$$P(N \ne 0) = 1.$$

By (0.3.1) it is equivalent to assume

$$P(N(-\infty,0] = N(0,\infty) = \infty) = 1.$$

These assumptions are satisfied for the process N defined in the proposition above, as can be easily seen from (3.1.3) and our assumption on $S_{\mathbb{Z}}$.

Suppose the stationary point process N satisfies to these assumptions. We can enumerate its points to obtain a sequence $U_{\mathbb{Z}}$ satisfying with probability 1

$$(3.1.4)\qquad N(B) = \sum_{n\in\mathbb{Z}} \chi_{U_n}(B), \qquad B \in \mathcal{B}^1,$$

$$\ldots < U_{-1} < U_0 < 0 \le U_1 < \ldots .$$

Proposition 3.1.1 describes how to define to a given point process N_0 of the form (3.1.1) with distribution Q_0, a stationary point process N with distribution Q. The next proposition deals with a converse. Consider for arbitrary $D \in \mathcal{D}$

$$(3.1.5)\qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_D(T_{U_k} N).$$

PROPOSITION 3.1.2. *Let* N *be a simple, stationary, ergodic point process on the real line with intensity* $\lambda \in (0,\infty)$. *Denote its distribution by* Q. *Then the limit* (3.1.5) *is defined a.s. and equals*

$$(3.1.6)\qquad Q_0(D) := \frac{1}{EN(0,1]}\, E \sum_{U_n \in (0,1]} \chi_D(T_{U_n} N).$$

As a function in D *the measure* Q_0 *is a probability measure, describing the distribution of a point process* N_0 *that can be written as* (3.1.1), *with* $S_{\mathbb{Z}}$ *a random walk with stationary, ergodic, strictly positive increments with finite expectation* $ES_1 = 1/\lambda$.

PROOF. Following MATTHES [1963], remark that $\lambda\, Q_0(D)$ is the intensity of the thinned process N_D obtained by removing all points U_k from N, except for the points with $T_{U_k} N \in D$. It is easily seen that because N is ergodic, N_D is ergodic too. Hence by the ergodic theorem

$$\lim_{t\to\infty} \frac{1}{t} N_D(0,t] = \lambda\, Q_0(D) \text{ a.s.}$$

The argument we follow below is similar to what we did in the proof of Proposition 3.1.1. Note that N assumes its values with probability 1 in the set

$$\mathcal{N}_\infty := \{m \in \mathcal{N}\colon m \text{ is simple, } m(-\infty,0] = M(0,\infty) = \infty\},$$

because N is simple, ergodic and has positive intensity. Using that

$\lim_{t\to\infty} \frac{1}{t} N(0,t] = \lambda$ a.s. a simple argument shows $\lim_{n\to\infty} \frac{1}{n} U_n = \frac{1}{\lambda}$ a.s. Observe that

$$\frac{1}{n} \sum_{k=1}^{n} \chi_D(T_{U_k} N) = \frac{1}{n} N_D[0,U_n].$$

Applying the Cesaro convergence properties of N_D and U_n obtained above, one proves that the limit in (3.1.5) equals $Q_0(D)$ a.s. From the expression (3.1.6) it is clear that Q_0 is a measure and by the definition of intensity Q_0 is a probability measure. Because $N \in \mathcal{N}_\infty$ a.s. one obtains, using (3.1.5) or (3.1.6), that $Q_0(\mathcal{N}_\infty) = 1$. Similarly it follows that Q_0 has mass 1 on the set $D_0 := \{m \in \mathcal{N}\colon m\{0\} = 1\}$. Let $\mathcal{N}_0 := \mathcal{N}_\infty \cap D_0$ and let $\mathcal{D}_0$ be the restriction of $\mathcal{D}$ to $\mathcal{N}_0$. We may assume that N_0 is the identity on $(\mathcal{N}_0, \mathcal{D}_0, Q_0)$. Construct a sequence of ascending random variables $S_{\mathbb{Z}}$ with $S_0 = 0$ such that

$$N_0(B) = \sum_{n\in\mathbb{Z}} \chi_B(S_n), \qquad B \in \mathcal{B}^1.$$

By using that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_D(T_{U_k} N) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_D(T_{U_{k+1}} N)$$

it follows that the sets $\{N_0 \in D\}$ and $\{T_{S_1} N_0 \in D\}$ have the same probability for all D. Hence $S_{\mathbb{Z}}$ has stationary increments. If D_1 is invariant for the process of increments of $S_{\mathbb{Z}}$ then D_1 satisfies

$$D_1 = \{T_{S_n} N_0 \in D_1\}, \qquad n \in \mathbb{Z}.$$

Clearly in that case for a given ω we have $\chi_{D_1}(T_{U_n} N) = 0$ for all $n \in \mathbb{Z}$, or else $\chi_{D_1}(T_{U_n} N) = 1$ for all $n \in \mathbb{Z}$. Hence the limit (3.1.5) has its values in $\{0,1\}$. Above we saw that this limit equals $Q_0(D_1)$ a.s. Hence if D_1 is invariant $Q_0(D_1) = 0$ or 1. This proves ergodicity. To see that $ES_1 = \frac{1}{\lambda}$ observe that by stationarity N has its values in

$$D := \{m \in \mathcal{N}_\infty\colon \lim_{t\to\infty} \frac{1}{t} m(0,t] = \lambda\}$$

with probability 1. Hence by (3.1.6) we get $Q_0(D) = 1$. By the ergodic theorem and Lemma 2.1.6 we also have

$$ES_1 = \lim_{n\to\infty} \frac{1}{n} S_n = \lim_{t\to\infty} \frac{t}{N_0(0,t]} \text{ a.s.}$$

Therefore, using $Q_0(D) = 1$ we obtain $ES_1 = \frac{1}{\lambda}$. $\square$

NOTE to Proposition 3.1.2. If $N \neq 0$ a.s. is not ergodic, but is simple, stationary with finite intensity λ, then (3.1.6) also describes the distribution Q_0 of a point process N_0 that can be written as (3.1.1) with $S_{\mathbb{Z}}$ a random walk with stationary, strictly positive increments with finite expectation $\frac{1}{\lambda}$. To prove this let $\Lambda^* := \lim_{t\to\infty} \frac{1}{t} N(0,t]$ and define $\psi(D)$ by (3.1.5). Using the arguments above we obtain in this case that $Q_0(D) = \frac{1}{E\Lambda^*} E\Lambda^*\psi(D)$. The other properties of N_0 mentioned above, are now easily checked.

The propositions above describe mappings $Q_0 \to Q$ and $Q \to Q_0$. The following theorem shows that these mappings are mutually inverse.

Let $\mathcal{Q}_0$ be the set of distributions Q_0 of point processes N_0 on the real line given by (3.1.1), with $S_{\mathbb{Z}}$ a random walk with stationary, strictly positive increments having finite expectation. Let $\mathcal{Q}$ be the set of distributions Q of point processes N on the real line that are simple, stationary, a.s. nonvanishing and have finite intensity.

THEOREM 3.1.3. *Let* $\mathcal{Q}_0$ *and* $\mathcal{Q}$ *be defined as in the paragraph above. We use the notations* (3.1.1) *and* (3.1.4). *The definitions* (3.1.3) *and* (3.1.6) *describe mutually inverse mappings on* $\mathcal{Q}_0$ *and* $\mathcal{Q}$.

PROOF. Let θ and θ' be the mappings described by (3.1.3) and (3.1.6), respectively. By the notes to the propositions above the mappings are properly defined. The proof is split up in two parts.

Part 1. $\theta\circ\theta'$ is the identity map. To show this suppose that $Q_0 = \theta' Q$ for arbitrary $Q \in \mathcal{Q}$, so

$$(3.1.7) \qquad Ef(N_0) = \frac{1}{\lambda} E \sum_{U_n \in (0,1]} f(T_{U_n} N),$$

where $\lambda = EN(0,1]$ and f is nonnegative measurable. Choose f such that

$$f(N_0) = \frac{1}{ES_1} \int_0^{S_1} \chi_D(T_s N_0)\,ds$$

for any $D \in \mathcal{D}$. Equation (3.1.7) becomes

$$\theta Q_0(D) = \frac{1}{\lambda} E \sum_{U_n \in (0,1]} \frac{1}{ES_1} \int_{U_n}^{U_{n+1}} \chi_D(T_s N)\,ds$$

$$= \frac{1}{\lambda ES_1} E \int_{U_1}^{U_{N(0,1]+1}} \chi_D(T_s N)\,ds.$$

Because T_1N is distributed as N

$$\int_1^{U_{N(0,1]+1}} \chi_D(T_sN)\,ds \overset{d}{=} \int_0^{U_1} \chi_D(T_sN)\,ds.$$

Hence

$$\theta Q_0(D) = \frac{1}{\lambda ES_1} E\int_0^1 \chi_D(T_sN)\,ds = \frac{1}{\lambda ES_1} Q(D), \qquad D \in \mathcal{D},$$

so $\lambda ES_1 = 1$ and $\theta\theta' Q = Q$.

Part 2. $\theta'\circ\theta$ is the identity map. To show this suppose $Q = \theta Q_0$ for $Q_0 \in \mathcal{Q}_0$, so

$$Ef(N) = \frac{1}{ES_1} E\int_0^{S_1} f(T_sN_0)\,ds,$$

with f nonnegative measurable. Choose f such that

$$f(N) = \frac{1}{\lambda} \sum_{U_k\in(0,1]} \chi_D(T_{U_k}N)$$

for any $D \in \mathcal{D}$, so

$$\theta' Q(D) = \frac{1}{ES_1} E\int_0^{S_1} \frac{1}{\lambda} \sum_{S_n\in(s,s+1]} \chi_D(T_{S_n}N_0)\,ds.$$

Because $T_{S_n}N_0$ is distributed as N_0 we get

$$\theta' Q(D) = \frac{1}{\lambda ES_1} \sum_{n\in\mathbb{Z}} E\int_0^{S_{-n+1}-S_{-n}} \chi_{(s,s+1]}(-S_{-n})\chi_D(N_0)\,ds$$

$$= \frac{1}{\lambda ES_1} E\sum_n \int_{S_{-n}}^{S_{-n+1}} \chi_{(-1,0]}(s)\chi_D(N_0)\,ds$$

$$= \frac{1}{\lambda ES_1} Q_0(D).$$

This implies $\theta'\theta Q_0 = Q_0$. □

Let N be distributed as $Q \in \mathcal{Q}$. The following proposition, due to MATTHES [1963] shows that we can consider Q_0 as the distribution of N, conditioned to the occurrence of a point at 0. The measure Q_0 is called the *Palm measure* of Q.

PROPOSITION 3.1.4. *Let* N *be a stationary, simple point process on the real line, a.s. nonvanishing and having finite intensity. Let* Q_δ *be the distribution of* $T_{U_1}N$*, given* $U_1 < \delta$*, for some* $\delta > 0$*, where* $U_{\mathbb{Z}}$ *is defined by* (3.1.4). *The distribution* Q_0 *defined by* (3.1.6) *satisfies*

$$\lim_{\delta \downarrow 0} \|Q_\delta - Q_0\| = 0.$$

PROOF. See MATTHES [1963] or LEADBETTER [1972]. □

The proposition above gives the connection with *Palm theory*. The aim of Palm theory is to define for a point process N on some arbitrary space, the distribution of N conditoned to the occurrence of a point at some given fixed element of the space. The theory can be traced back to PALM [1943-44] who is concerned with streams of telephone calls. KHINTCHINE [1960] and SLIVNYAK [1962-66] made Palm's proofs rigorous and extended his results. They introduce conditional probabilities for point processes N on the real line by means of ratios

$$\lim_{\delta \downarrow 0} P(N(a,b] = j \mid N(-\delta,0] \geq 1).$$

RYLL-NARDZEWSKI [1961] defined Q_0 using the Radon-Nikodym and the Fubini theorem. His paper also gives a 1-1 correspondence for point processes on the real line similar to Theorem 3.1.3. Earlier this 1-1 correspondence was given by KAPLAN [1955], who departed, as we did, from a renewal theoretic point of view. Ryll-Nardzewski's defining method can also be used for point processes on more general spaces, as is done by MECKE [1967], KUMMER and MATTHES [1970] and PAPANGELOU [1974]. Another approach, suitable for point processes on the real line (or marked point processes) was given by MATTHES [1963] who defined

$$Q_0(D) := \frac{\lambda_D}{\lambda}$$

with λ_D and λ the intensities of N_D and N, as used in the proof of Proposition 3.1.2. Several authors gave reviews of Palm theory. We can mention DALEY and VERE JONES [1972], KERSTAN, MATTHES and MECKE [1974], DE SAM LAZARO and MEYER [1975] and NEVEU [1977].

3.2. PALM THEORY AND LIMIT BEHAVIOUR

The results that will interest us here and also in Chapter 6 are concerned with the convergence of $N_t := T_t N_0$ for $t \to \infty$ in distribution to a limit point process N, where N_0 is defined by (3.1.1). In this section we discuss three different types of convergence that can be obtained under the following three conditions of asymptotic independence:

(i) ergodicity of the increments of the random walk;

(ii) mixing of N;

(iii) triviality of the right tail σ-field of N.

The limit result we obtain in connection with (i) is the simplest and from our point of view the most interesting. It is given in Proposition 3.2.2. In this proposition we impose assumptions on the random walk and obtain a renewal theoretic result. The conditions (ii) and (iii) are not formulated immediately in terms of random walks and it is in general difficult to prove them. In Section 6.5 we indicate assumptions on the random walk under which (ii) and (iii) can be obtained.

Most of the propositons in this section are close to known results. In particular we can mention DELASNERIE [1977] who proves Blackwell's theorem out of (ii) (see Corollary 3.2.4).

Let N be a stationary point process on the real line, with distribution Q, defined on the measurable space $(\mathcal{N},\mathcal{D})$ (see Section 0.3). First we shall discuss three well known conditions of asymptotic independence on N. The first condition is *ergodicity*, defined in Section 0.3. Equivalent with ergodicity is

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t Q(D_0 \cap T_s^{-1} D_1)\,ds = Q(D_0)Q(D_1)$$

for all sets $D_0, D_1 \in \mathcal{D}$. This can be proved by means of the ergodic theorem for continuous time (see DE SAM LAZARO and MEYER [1975]). The second condition of asymptotic independence is called mixing. The point process N is said to be *mixing* if

$$\lim_{t\to\infty} Q(D_0 \cap T_t^{-1} D_1) = Q(D_0)Q(D_1)$$

for all sets $D_0, D_1 \in \mathcal{D}$. Clearly mixing is stronger than ergodicity. We define an even stronger concept of asymptotic independence. Let the sub-σ-fields $\mathcal{D}_s^t \subset \mathcal{D}$, $-\infty \le s < t \le \infty$, be induced by the mappings

$$m \to m(B)$$

on N_1, where B is any Borel set contained in (s,t). The point process N has *trivial right* (or *left*) *tail σ-field* if $\bigcap_t \mathcal{D}_t^\infty$ (or $\bigcap_t \mathcal{D}_{-\infty}^t$) contains only sets with probability 0 or 1 under Q. This condition implies mixing of N. To see this apply the next lemma and an approximation argument (compare SMORODINSKY [chapter VII]).

LEMMA 3.2.1. *A point process* N *with distribution* Q *has trivial right tail σ-field* $\mathcal{D}^\infty := \bigcap_t^\infty \mathcal{D}_t^\infty$ *if and only if for all* $D_0 \in \mathcal{D}$

$$\sup_{D \in \mathcal{D}_t^\infty} |Q(D \cap D_0) - Q(D)Q(D_0)| \to 0$$

for $t \to \infty$.

PROOF. Note that for $D \in \mathcal{D}_t^\infty$

$$Q(D \cap D_0) - Q(D)Q(D_0) = \int_D Q(D_0|\mathcal{D}_t^\infty) - Q(D_0)\,dQ.$$

The expression above is maximized for $D = D^+ := \{Q(D_0|\mathcal{D}_t^\infty) > Q(D_0)\}$ and minimized for $D = (D^+)^c$. Hence

$$g_t := \sup_{D \in \mathcal{D}_t^\infty} |Q(D \cap D_0) - Q(D)Q(D_0)| = \tfrac{1}{2} \int |Q(D_0|\mathcal{D}_t^\infty) - Q(D_0)|\,dQ.$$

Using that the left-hand side is monotone, together with a martingale theorem (see BREIMAN [5.24]) we obtain

$$\lim_{t\to\infty} g_t = \tfrac{1}{2} \int |Q(D_0|\mathcal{D}^\infty) - Q(D_0)|\,dQ.$$

If $\mathcal{D}^\infty$ is trivial the right-hand side vanishes. Else some $D_0 \in \mathcal{D}^\infty$ does not have the 0-1 property, in which case the right-hand side clearly is strictly positive. This proves the lemma. □

In the convergence results below we use the following notation. If Q_0 is a probability measure on $(N,\mathcal{D})$ we write Q_t for the measure

$$Q_t(D) = Q_0(T_t^{-1}D), \quad D \in \mathcal{D}.$$

Similarly we denote $Q_t(\cdot|D_0)$ for the measure $T_t Q_0(\cdot|D_0)$, where D_0 is any set with positive Q_0-measure.

First we consider ergodicity. The result that we obtain is close to SLIVNYAK [1962-66] who also considers Cesaro convergence properties.

PROPOSITION 3.2.2. *Let $S_{\mathbb{Z}}$ be a random walk with stationary, strictly positive increments having finite expectation. Let Q_0 be the distribution of N_0 defined by* (3.1.1). *The process of increments of $S_{\mathbb{Z}}$ is ergodic if and only if*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t Q_s(D|D_0)\,ds = Q(D), \qquad D \in \mathcal{D},$$

for all $D_0 \in \mathcal{D}$ with positive Q_0-measure, where Q is defined by (3.1.3).

PROOF. To prove the only if-part observe that if the increments of $S_{\mathbb{Z}}$ are ergodic, then by Proposition 3.1.1 we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \chi_D(T_s N_0)\,ds = Q(D) \text{ a.s.}$$

The convergence above holds P-a.s. and therefore also $P(\cdot|N_0 \in D_0)$-a.s. The left-hand side is bounded by 1. By the bounded convergence theorem we may take expectations with respect to the conditional measure. Hence the limit relation in the proposition above follows. To prove the if-part let $D_0 \in \mathcal{D}$ be a set that is invariant under T_t, t real. If $Q_0(D_0) > 0$ then by the limit relation in the assertion of the proposition and the invariance of D_0 under T

$$Q(D_0) = \lim_{t\to\infty} \frac{1}{t} \int_0^t Q_s(D_0|D_0)\,ds = Q_0(D_0|D_0) = 1.$$

If $Q_0(D_0) = 0$, consider the complement of D_0 and argue as above to obtain $Q(D_0^c) = 1$. It follows that each translation invariant set D_0 has Q-measure 0 or 1. Hence N is ergodic. By Propositions 3.1.1 and 3.1.2 and the 1 - 1 correspondence given in Theorem 3.1.3 it follows that N is ergodic if and only if the process of increments of $S_{\mathbb{Z}}$ is ergodic. Hence the if-part of the proposition follows. □

Another Cesaro limit result is given in Chapter 2. It is the global renewal theorem (Theorem 2.2.6). This theorem discusses the limit behaviour of the intensity measure of N_0.

DELASNERIE [1977] discusses results for mixing point processes N. Blackwell's theorem can be studied from this point of view. Using different methods we derive related results. A central position in our approach is taken by Proposition 3.1.4. With this approach, only a few changes are required to obtain also a limit result for the case where N has a trivial right tail σ-field.

The proposition below considers $\nu * Q_t(\cdot|D_0)$ instead of $Q_t(\cdot|D_0)$. The first measure is a "smoothed" version of the second one. As is shown in Example 3.2.6 the proposition does not hold without this smoothing. The limit relation that is obtained in the following proposition may look uncommon. However, the note following Proposition A.2 shows that the limit relation implies weak convergence of $Q_t(\cdot|D_0)$ to $Q(\cdot)$ with respect to the vague topology on $\mathcal{N}$. This type of convergence is better known.

PROPOSITION 3.2.3. *Let* N *be a stationary, mixing, simple point process on the real line with finite, positive intensity. Suppose* Q *is the distribution of* N, *and* Q_0 *is defined by* (3.1.6). *Then for all* D_0 *with positive* Q_0*-measure*

$$\lim_{t\to\infty} \nu * Q_t(D|D_0) = Q(D), \qquad D \in \mathcal{D},$$

for any absolutely continuous probability measure ν *on the real line.*

PROOF. Because by stationarity $\nu * Q = Q$, we have to show

$$\nu * Q_t(D|D_0) - \nu * Q(D) \to 0 \quad \text{for } t \to \infty.$$

Using an inequality, analogous to A.1, it is easy to see that it is sufficient to prove the limit property only for $\nu = \nu_\varepsilon$, $\varepsilon > 0$, the homogeneous distribution on $(0,\varepsilon)$.

We have to show that

$$g_t := \frac{1}{\varepsilon}\int_0^\varepsilon Q_0(T_{t+s}^{-1} D \cap D_0)\,ds$$

converges to $Q(D)Q_0(D_0)$ for $t \to \infty$. Proposition 3.1.4 is our main tool. By the definition of Q_δ stated in that proposition, we have

$$\lim_{\delta\downarrow 0} \|Q_0 - Q_\delta\| = 0$$

and therefore

$$(3.2.1) \qquad g_t - E\left(\frac{1}{\varepsilon}\int_0^\varepsilon \chi_D(T_{U_1+t+s} N)\,ds\ \chi_{D_0}(T_{U_1} N)\,\middle|\,U_1 < \delta\right) \to 0 \qquad \text{for } \delta \downarrow 0.$$

Define

$$C := \{T_{U_1} N \in D_0,\ U_1 < \delta\}.$$

Observe the inequality

$$(3.2.2)\qquad \left|\frac{1}{\varepsilon}\int_0^{\varepsilon} \chi_D(T_{U_1+t+s}\ N)\,ds - \frac{1}{\varepsilon}\int_0^{\varepsilon}\chi_D(T_{t+s}\ N)\,ds\right| \le \frac{2U_1}{\varepsilon}$$

and note that on the set $C \supset \{U_1 < \delta\}$ an upper bound for the right-hand side is $\frac{2\delta}{\varepsilon}$. With the definition of C and using this upper bound in (3.2.2) we obtain from (3.2.1)

$$(3.2.3)\qquad g_t - E\left(\frac{1}{\varepsilon}\int_0^{\varepsilon}\chi_D(T_{t+s}\ N)\chi_C\ ds\right)/P(U_1 < \delta) \to 0$$

for $\delta \downarrow 0$. By the definition of mixing and by the stationarity of N

$$P(\{T_{t+s}N \in D\} \cap C) \to P_N(D)P(C) = Q(D)P(C)$$

for $t \to \infty$. Hence by (3.2.3)

$$(3.2.4)\qquad g_t - Q(D)P(C)/P(U_1 < \delta)$$

is arbitrarily small as $\delta \downarrow 0$ and $t \to \infty$. Furthermore, for $\delta \downarrow 0$ we have by Proposition 3.1.4 that

$$P(C)/P(U_1 < \delta)$$

is arbitrarily close to $Q_0(D_0)$. Hence we obtain from (3.2.4) for $t \to \infty$

$$g_t - Q(D)Q_0(D_0) \to 0,$$

as was to be proved. □

The corollary below gives in a conditional form, a limit property as in Blackwell's theorem under the requirement of mixing. In an unconditional form it is due to Neveu (see DELASNERIE [1977]) and can be proved under slightly weaker conditions.

COROLLARY 3.2.4. *Let N be a stationary, mixing, simple point process with finite, positive intensity λ. Suppose Q is the distribution of N, and Q_0 is defined by* (3.1.6). *Let N_0 be distributed as Q_0 and suppose the (symmetric) renewal measure*

$$H(B) := EN_0(B), \quad B \in \mathcal{B}^1,$$

is finite on a neighbourhood of the origin. For any $D_0 \in \mathcal{D}$ *with* $P(N_0 \in D_0) > 0$

$$\lim_{t\to\infty} E(N_0(t,t+h] \mid N_0 \in D_0) = \lambda h, \quad h > 0.$$

PROOF. We apply Proposition 3.2.3 with $\nu = \nu_\varepsilon$, $\varepsilon > 0$, the homogeneous distribution on $(0,\varepsilon)$ and the uniform integrability of $N_0(t,t+h]$, t real, proved in Corollary 2.2.5. By the uniform integrability we have for arbitrary small $\delta > 0$, if t and p are large enough

$$\sum_{n>p} P(N_0(t,t+h] \geq n) < \delta. \tag{3.2.5}$$

Let $0 < \varepsilon < h$ be arbitrary and observe

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \sum_{n=1}^{p} P(N_0(s,s+h-\varepsilon] \geq n \mid N_0 \in D_0)\,ds$$

$$\leq \sum_{n=1}^{p} P(N_0(t,t+h] \geq n \mid N_0 \in D_0)$$

$$\leq \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \sum_{n=1}^{p} P(N_0(s-\varepsilon,s+h+\varepsilon] \geq n \mid N_0 \in D_0)\,ds.$$

The last term converges by Proposition 3.2.3 for $t \to \infty$ to

$$\sum_{n=1}^{p} P(N(-\varepsilon,h+\varepsilon] \geq n).$$

Because the intensity measure of N is $\lambda\ell$ with ℓ the Lebesgue measure, this sum is arbitrarily close to $(h+2\varepsilon)\lambda$ for p large. The left term is, for t and p large enough, arbitrarily close to $\lambda(h-\varepsilon)$. Because ε is arbitrary, the middle term is arbitrarily close to λh for t and p large enough. The assertion follows by combining this with (3.2.5). □

The next proposition considers the strongest of the three conditions of asymptotic independence. If $\bar{Q}$ is the distribution of any point process $\bar{N}$ on the real line, we denote by $\bar{Q}^+$ the distribution of the restriction to $(0,\infty)$ of the point process $\bar{N}$.

PROPOSITION 3.2.5. *Let* N *be a stationary, simple point process on the real line with trivial right tail* σ*-field and finite, positive intensity. Suppose*

Q is the distribution of N*, and Q_0 is defined by* (3.1.6). *Then for all* D_0 *with positive Q_0-measure*

$$Q_t(\cdot|D_0)^+ \to Q^+(\cdot)$$

with respect to the metric d *defined by* (0.3.4).

PROOF. Consider the proof of Proposition 3.2.3. By checking the argumentation in this proof it can be seen easily that the convergence

$$g_t - Q(D)Q_0(D_0) \to 0,$$

for $t \to \infty$ holds uniformly in $D \in \mathcal{D}_0^\infty$. Note that at the step where we formerly used mixing, now an application of Lemma 3.2.1 is used. The assertion of the proposition follows immediately from this uniform convergence and definitions (0.3.3) and (0.3.4). □

In the proposition above we cannot use the total variation metric, but instead we have to be satisfied with a convergence result of this type for smoothed versions of the probability measures. The example below illustrates why this has to be done.

EXAMPLE 3.2.6. Let F be a nonlattice distribution, concentrated on $(0,\infty)$ with finite mean. Let $S_{\mathbb{Z}}$ be a random walk with independent, F-distributed increments and define N_0 by (3.1.1). Let N be distributed as the measure Q defined by (3.1.3). From TOTOKI [1970] it follows that N is a mixing point process.

Suppose that F is concentrated on the rational numbers $\mathbb{Q}$. Let $D_0 := N$ and define D to be the set of integer valued Radon measures on $\mathbb{R}^1$ that are concentrated on $\mathbb{Q}$. Note that

$$\begin{aligned} Q_t(D|D_0) &= 1, \quad \text{if } t \in \mathbb{Q}, \\ &= 0, \quad \text{else.} \end{aligned}$$

Obviously we cannot prove

$$Q_t(\cdot|D_0) \to Q(\cdot)$$

for convergence on each set $D \in \mathcal{D}$, or for convergence in total variation. Therefore, we had to "smooth" the measure $Q_t(\cdot|D_0)$ in Proposition 3.2.3.

The conditions on N in the last two propositions, mixing and trivial right tail σ-field, are not always easy to check. TOTOKI [1970] needs Blackwell's theorem in its usual form (Theorem 1.1.5(ii)) to prove a result that implies that N is mixing. Apparently in this context Corollary 3.2.4 is not very useful to prove Blackwell's theorem in its usual form. We are left with the question how to prove the mixing condition on N under assumptions on the random walk. This problem occupies us in the remaining chapters. In Section 6.5 we obtain results in this direction and in particular we generalize Totoki's result.

CHAPTER 4

A MEASURE OF DEPENDENCE, COUPLING AND WEAK BERNOULLI PROCESSES

The renewal theoretic results derived in Chapter 1, use an independence assumption. To be able to weaken this assumption we study in this chapter $\perp(X,Y)$, a measure of dependence between random variables X and Y. We also investigate a concept of asymptotic independence for processes, that is called weak Bernoulli. We show that this concept of asymptotic independence is in a certain sense a coupling property. This seems to be promising because such coupling properties enabled us in Chapter 1 to derive renewal theorems. The first half of this chapter is concerned with the study of the dependence between two random variables, while the latter half considers the dependence relation for sequences of random variables.

4.1. A MEASURE OF DEPENDENCE

In this section we quantify the dependence between two random variables X and Y, defined on the same probability space, by a number $\perp(X,Y)$. We investigate this measure of dependence in the first three propositions. In Section 4.2 we consider the relation between $\perp(X,Y)$ and coupling. The last part of Section 4.1 contains some technical results, needed in Chapters 5 and 6 for error estimates.

We assume that the random variables that we consider have their values in a Borel space. As a consequence, if X and Y are random variables on a probability space, there exists a regular version of the conditional distribution $P_{Y|X}$, i.e. $P_{Y|X}(B)$ is a probability measure as a function in B for fixed ω, and is X-measurable for fixed B (see BREIMAN [Theorem 4.34]). The assumption mentioned above is not very restrictive. Every real random vector obviously has its value in a Borel space and because a finite or countable product of Borel spaces is again a Borel space it follows that each random vector that consists of a finite or countable number of random variables with values a Borel space, has its values in a Borel space too.

In general a Borel space is not provided with an algebraic or topological structure. However, the set of probability distributions on a Borel space can be topologized by means of the total variation metric.

In the first two sections of this chapter most results are formulated for real random variables. Since every Borel space is isomorphic with a measurable Borel set on the real line (in the sense that there is a measurable bijection), the results are automatically valid for random variables with values in a Borel space.

Define for random variables X and Y with values in the same Borel space

$$\delta(X,Y) := \tfrac{1}{2}\|P_X - P_Y\|.$$

Note that δ is bounded by 1 and that δ is not a metric on the space of random variables but on the space of their probability distributions. The random variables X and Y may be defined on different probability spaces. We have

$$(4.1.1) \qquad \delta(X,Y) = \|(P_X - P_Y)^+\| = \|(P_X - P_Y)^-\| = 1 - \|P_X \wedge P_Y\|.$$

If X and Y are random variables on a common probability space with values in possibly different Borel spaces, we define

$$\perp(X,Y) := \tfrac{1}{2}\|P_{X,Y} - P_X \times P_Y\|.$$

The number $\perp(X,Y)$ is a *measure of dependence*. It vanishes if and only if X and Y are independent and it is bounded by 1.

PROPOSITION 4.1.1. *If* X *and* Y *are real random variables, then*

$$\perp(X,Y) = \tfrac{1}{2}E\|P_{Y|X} - P_Y\|.$$

PROOF. Define for sets $B \subset \mathbb{R}^2$

$$B(x) := \{y\colon (x,y) \in B\}.$$

Choose a regular conditional distribution $P_{Y|X=x}$. Using REVUZ [Lemma 1.5.3] it follows that there exists a measurable set $B \subset \mathbb{R}^2$ such that

$$P_{Y|X=x} \begin{array}{ll} \geq P_Y & \text{on } B(x), \\ \leq P_Y & \text{on } B(x)^c. \end{array}$$

Hence $(B(x), B(x)^c)$ is a Hahn decomposition for $P_{Y|X=x} - P_Y$. By (4.1.1)

$$\perp(X,Y) = P_{X,Y}(B) - P_X \times P_Y(B)$$
$$= \int P_{Y|X=x}(B(x)) - P_Y(B(x))\,dP_X(x)$$
$$= \tfrac{1}{2}E\| P_{Y|X} - P_Y\|. \qquad \square$$

PROPOSITION 4.1.2. *If* X *and* Y *are real random variables and* f *is a real measurable function on the real line, then*

$$\perp(X,f(Y)) \le \perp(X,Y).$$

PROOF. By the definition of total variation we have for any real pair of random variables Y and Y'

$$(4.1.2) \qquad \|P_{f(Y)} - P_{f(Y')}\| \le \|P_Y - P_{Y'}\|.$$

Similarly we have

$$\|P_{f(Y)|X} - P_{f(Y)}\| \le \|P_{Y|X} - P_Y\|.$$

By Proposition 4.1.1 this implies the assertion. $\square$

If f is invertible we have

$$\perp(X,Y) = \perp(X,f(Y)).$$

This is proved by applying the proposition above twice.

We also define the conditional dependence

$$\perp_Y(X,Z) := \tfrac{1}{2}\|P_{X,Z|Y} - P_{X|Y} \times P_{Z|Y}\|.$$

Here $P_{X,Z|Y}$ denotes a regular version of the conditional distribution of (X,Z) given Y. The measure of dependence $\perp_Y(X,Z)$ vanishes a.s. if and only if the triple (X,Y,Z) is *Markovian*, i.e. if X and Z are independent, given Y. The following proposition shows that under a certain type of conditioning the dependence does not increase very much.

PROPOSITION 4.1.3. *Let* f *be a real measurable function on the real line and let* X *and* Y *be real random variables. Then*

$$E\perp_{f(Y)}(X,Y) \le 2\perp(X,Y).$$

PROOF. By the triangle inequality we have

$$E\|P_{X|Y,Z} - P_{X|Z}\| \le E\|P_{X|Y,Z} - P_X\| + E\|P_X - P_{X|Z}\|$$

and hence by Proposition 4.1.1

$$E\perp_Z(X,Y) \le \perp(X,(Y,Z)) + \perp(X,Z).$$

The assertion follows if we choose $Z = f(Y)$ and apply Proposition 4.1.2. □

In Chapters 5 and 6 we often approximate the distribution of a triple (X_1,X_2,X_3) by a triple (Y_1,Y_2,Y_3) that is Markovian. The following proposition estimates the error made in such an approximation.

PROPOSITION 4.1.4. *Let* $X := (X_1,X_2,X_3)$ *and* $Y := (Y_1,Y_2,Y_3)$ *be triples of real random variables. If* Y *is Markovian, then*

$$\delta(X,Y) \le E\perp_{X_2}(X_1,X_3) + \delta(X_2,Y_2) + \delta((X_1,X_2),(Y_1,Y_2)) + \delta((X_2,X_3),(Y_2,Y_3)).$$

To prove this proposition we need two simple lemmas. In their formulation we make use of the following notation. Let π be a finite measure on the real line and let $P_1,\ldots,P_n$ be transition probabilities on the real line. Define the measure $H := \pi\times P_1\times\ldots\times P_n$ on $\mathbb{R}^{n+1}$ by

$$H(B_1\times\ldots\times B_{n+1}) = \int_{B_{n+1}} P_1(x_1,B_1)\ldots P_n(x_1,B_n)\,d\pi(x_1),$$

where $B_1,\ldots,B_{n+1}$ are Borel sets on the real line. Suppose that also ρ is a finite measure and that $R_1,\ldots,R_n$ are transition probabilities on the real line. Define

$$\mu_k := \pi\times P_1\times\ldots\times P_k - \rho\times R_1\times\ldots\times R_k, \qquad 0\le k\le n.$$

LEMMA 4.1.5. *If* $P_n = R_n$ *then* $\|\mu_n\| = \|\mu_{n-1}\|$.

PROOF. Let $B\subset\mathbb{R}^{n+1}$ be measurable and define for $x\in\mathbb{R}^n$

$$B(x) := \{x_{n+1}\in\mathbb{R}^1:\ (x,x_{n+1})\in B\}.$$

Using this notation we have

$$\pi\times P_1\times\ldots\times P_n(B) = \int P_n(x_1,B(x))\,d\pi\times P_1\times\ldots\times P_{n-1}(x),$$

where we write $x_1 := (x)_1$. A similar equality holds for $\rho\times R_1\times\ldots\times R_n$.

Writing

$$f(x) := P_n(x_1, B(x)) = R_n(x_1, B(x))$$

we obtain

$$\mu_n(B) = \int f \, d\mu_{n-1}.$$

This implies

$$\|\mu_n^+\| = \sup_{B \in \mathcal{B}^{n+1}} \mu_n(B) \le \sup_{0 \le f \le 1} \int f \, d\mu_{n-1} = \|\mu_{n-1}^+\|.$$

Because the reverse inequality is trivially valid this implies the assertion. □

LEMMA 4.1.6. $\|(\pi \vee \rho) \times P_1 - (\pi \vee \rho) \times R_1\| \le \|\pi \times P_1 - \rho \times R_1\| + \|\pi - \rho\|$.

PROOF. The first term ν_1 in the difference

$$(4.1.3) \qquad (\pi \vee \rho) \times P_1 - (\pi \vee \rho) \times R_1$$

can be written as

$$\nu_1 = \pi \times P_1 + (\rho - \pi)^+ \times P_1.$$

A similar expression can be given for the second term ν_2 in (4.1.3). Hence

$$\nu_1 - \nu_2 = \pi \times P_1 - \rho \times R_1 + (\rho - \pi)^+ \times P_1 - (\pi - \rho)^+ \times R_1.$$

It follows that

$$(\nu_1 - \nu_2)^+ \le (\pi \times P_1 - \rho \times R_1)^+ + (\rho - \pi)^+ \times P_1.$$

Together with a similar estimate for $(\nu_1 - \nu_2)^-$ this implies the assertion of the lemma. □

PROOF of Proposition 4.1.4. Let P_i and R_i, $i = 1,3$ be transition probabilities on the real line defined by

$$P_i(x,B) := P(X_i \in B \mid X_2 = x), \quad R_i(x,B) := P(Y_i \in B \mid Y_2 = x).$$

Let π and ρ be the distributions of X_2 and Y_2, respectively.

Because Y is a Markov triple and by the triangle inequality

$$(4.1.4)\qquad \|P_X - P_Y\| = \|P_{X_2,X_1,X_3} - \rho\times R_1\times R_3\|$$

$$\leq \|P_{X_2,X_1,X_3} - \pi\times P_1\times P_3\| + \|\pi\times P_1\times P_3 - \rho\times R_1\times R_3\|$$

$$\leq 2\, E\bot_{X_2}(X_1,X_3) + \|\pi\times P_1\times P_3 - \rho\times R_1\times P_3\| + \|\rho\times R_1\times P_3 - \rho\times R_1\times R_3\|.$$

The last inequality is a consequence of the definition of $\bot_{X_2}$ and the triangle inequality. Observe that the last term in (4.1.4) equals

$$\|\rho\times P_3\times R_1 - \rho\times R_3\times R_1\| = \|\rho\times P_3 - \rho\times R_3\|$$

by Lemma 4.1.5. From (4.1.4) we obtain

$$\|P_X - P_Y\| \leq 2\, E\bot_{X_2}(X_1,X_3) + \|\pi\times P_1 - \rho\times R_1\| + \|\rho\times P_3 - \rho\times R_3\|$$

$$\leq 2\, E\bot_{X_2}(X_1,X_3) + \|\pi\times P_1 - \rho\times R_1\|$$

$$+ \|\pi\times P_3 - \rho\times R_3\| + \|\pi - \rho\|.$$

In the last inequality we used Lemma 4.1.6. The resulting inequality, multiplied by $\frac{1}{2}$, is easily translated into the assertion of the proposition. □

In Chapters 5 and 6 we also need some error estimates for distributions of pairs of random variables. Suppose $X := (X_1,X_2)$ and $Y := (Y_1,Y_2)$ are pairs of real random variables. From Lemma 4.1.6 we obtain

$$(4.1.5)\qquad \int \tfrac{1}{2}\|P_{X_2|X_1=x} - P_{Y_2|Y_1=x}\|\, dP_{X_1} \vee P_{Y_1}(x) \leq \delta(X,Y) + \delta(X_1,Y_1)$$

Note that the equality holds if X_1 and Y_1 have the same distribution. From Lemma 4.1.5 with $n = 1$ we can conclude that if for some suitable regular version of the conditional distribution holds

$$(4.1.6')\qquad P_{X_2|X_1=x} = P_{Y_2|Y_1=x}$$

then

$$(4.1.6'')\qquad \delta(X,Y) = \delta(X_1,Y_1).$$

4.2. COUPLING AND A MEASURE OF DEPENDENCE

Let X and Y be real random variables on the same probability space. We say that X and Y are *partially coupled* with probability p if

$$P(X=Y) = p.$$

Suppose there are given distributions P_X and P_Y on the real line. We want to construct random variables X and Y on the same probability space, under the condition that their distributions are P_X and P_Y, respectively, in such a way that the probability p of partial coupling is as large as possible. We shall show that p can be maximized. If this probability p is maximal, we say that X and Y are *maximally coupled*. In this section we also discuss other problems of this type. Corollary 4.2.5 is the main result in this section. It will be frequently used in the sequel.

The first two propositions are taken from SCHWARZ [1978].

PROPOSITION 4.2.1. *Let P_X and P_Y be probability distributions on the real line. There exists a joint distribution $P^*_{X,Y}$ with the following properties:*

(i) *the marginals of $P^*_{X,Y}$ are P_X and P_Y;*

(ii) *among the distributions $P_{X,Y}$ for which* (i) *holds, the distribution $P^*_{X,Y}$ maximizes* $P(X=Y)$. *The maximum equals $\|P_X \wedge P_Y\|$.*

PROOF. Suppose the distribution $P_{X,Y}$ on $\mathbb{R}^2$ has marginals P_X and P_Y. Let μ_Δ be the restriction of $P_{X,Y}$ to the diagonal Δ of $\mathbb{R}^2$ and define μ to be the projection of μ_Δ on the real line. Because μ is dominated by both P_X and P_Y, μ is also dominated by $P_X \wedge P_Y$. Hence we have

$$(4.2.1) \qquad P(X=Y) = \|\mu_\Delta\| = \|\mu\| \le \|P_X \wedge P_Y\|.$$

We have to select a distribution $P^*_{X,Y}$ for which the equality holds. Let μ^*_Δ be the measure on the diagonal Δ of $\mathbb{R}^2$ with marginals $\mu^* := P_X \wedge P_Y$. Let $p := \|\mu^*\|$ and define

$$P^*_{X,Y} := \mu^*_\Delta, \qquad \text{if } p = 1,$$

$$:= \frac{1}{1-p}(P_X - \mu^*) \times (P_Y - \mu^*) + \mu^*_\Delta, \quad \text{else.}$$

By (0.1.1) this distribution has marginals P_X and P_Y. The total mass concentrated on the diagonal is at least $\|\mu^*_\Delta\| = \|P_X \wedge P_Y\|$. Hence with the choice

$P_{X,Y} := P^*_{X,Y}$ the equality holds in (4.2.1). □

Thus we have shown that the equality in

$$P(X=Y) \le \|P_X \wedge P_Y\|$$

can be attained, or differently, by (4.1.1), the equality in

$$P(X \ne Y) \ge \delta(X,Y)$$

can be attained.

A similar result can be given for the number $\perp(X,Y)$. If X,Y and Y' are real random variables with Y' independent of X and distributed as Y, then we prove below that

$$P(Y \ne Y') \ge \perp(X,Y).$$

Also here the equality can be attained.

PROPOSITION 4.2.2. *Let $P_{X,Y}$ be a probability distribution on $\mathbb{R}^2$. There exists a probability distribution $P^*_{X,Y,Y'}$ on $\mathbb{R}^3$ such that*

(i) $P^*_{X,Y} = P_{X,Y}$ *and* $P^*_{Y'} = P^*_{Y'|X} = P_Y$;

(ii) *among the distributions $P_{X,Y,Y'}$ for which* (i) *holds, the distribution $P^*_{X,Y,Y'}$ minimizes $P(Y \ne Y')$. The minimum equals $\perp(X,Y)$.*

PROOF. If $P_{X,Y,Y'}$ satisfies (i) then by the first proposition

$$P(Y=Y'|X) \le \|P_{Y|X} \wedge P_{Y'|X}\| = \|P_{Y|X} \wedge P_Y\|. \tag{4.2.2}$$

By (4.1.1) and Proposition 4.1.1 we have

$$P(Y=Y') \le E\|P_{Y|X} \wedge P_Y\| = 1 - \perp(X,Y)$$

and hence

$$P(Y \ne Y') \ge \perp(X,Y). \tag{4.2.3}$$

Following the argument in the proof of the first proposition we can construct a probability measure $P(x,\cdot)$ on $\mathbb{R}^2$, with marginals $P_{Y|X=x}$ and P_Y, while the mass on the diagonal Δ of $\mathbb{R}^2$ is maximized. This can be done such that $P(x,B)$ for each $B \in \mathcal{B}^2$ is measurable in x. Define the distribution $P^*_{X,Y,Y'}$ on $\mathbb{R}^3$ by requiring that

$$P^*_{X,Y,Y'}(B_1 \times B_2 \times B_3) = \int_{B_1} P(x, B_2 \times B_3)\, dP_X(x), \qquad B_i \in \mathcal{B}^1.$$

Because $P^*(Y=Y'|X)$ is chosen maximal, the equality holds in (4.2.2) and hence in (4.2.3). This proves (ii). To prove (i), note that the choice of the marginals of $P(x,\cdot)$ implies that

$$P^*_{X,Y}(B_1 \times B_2 \times \mathbb{R}^1) = \int_{B_1} P_{Y|X=x}(B_2)\, dP_X(x) = P_{X,Y}(B_1 \times B_2),$$

$$P^*_{X,Y'}(B_1 \times \mathbb{R}^1 \times B_3) = \int_{B_1} P_Y(B_3)\, dP_X(x) = P_X(B_1) P_Y(B_3). \qquad \square$$

We call the probability space $(\Omega',\mathcal{A}',P')$ an *extension* of the probability space $(\Omega,\mathcal{A},P)$ if there is a measurable mapping Π from Ω' onto Ω. If X is a random variable on $(\Omega,\mathcal{A},P)$, we identify the random variable $X' := X \circ \Pi$ on the extended space with X and say that the random variable X on the extended space is a random variable on the *original* space. The example below shows how to extend a given probability space with a new random variable U, that is independent of all random variables on the original space and has some prescribed distribution.

EXAMPLE 4.2.3. Provide the unit interval $(0,1]$ with its Borel σ-field $\mathcal{B}$ and define on $((0,1],\mathcal{B})$ the Lebesgue measure ℓ_0. Consider the probability space

$$(\Omega',\mathcal{A}',P') := (\Omega,\mathcal{A},P) \times ((0,1],\mathcal{B},\ell_0).$$

Let Π and U be the projections on the first and second coordinate of Ω'. The probability space is thus said to be extended with a random variable U that is homogeneously distributed on $(0,1]$.

EXTENSION LEMMA 4.2.4. *Consider on a probability space* $(\Omega,\mathcal{A},P)$ *a random variable* X *with values in a Borel space* Γ. *Suppose* $P^*_{X,Y}$ *is a distribution on a Cartesian product of Borel spaces* $\Gamma \times \Gamma'$ *such that* $P^*_X = P_X$. *The probability space* $(\Omega,\mathcal{A},P)$ *can be extended with a random variable* Y' *with values in* Γ' *such that* $P_{X,Y'} = P^*_{X,Y}$, *while the dependence structure is not affected, in the sense that*

$$P_{Z|X,Y'} = P_{Z|X}$$

for any random variable Z *on the original probability space, with values in a Borel space.*

PROOF. If $\Gamma' = \mathbb{R}^1$ we argue as follows. Define the transition probability

$$P^*(x,B) := P^*_{Y|X=x}(B),$$

where $P_{Y|X=x}$ is a regular conditional distribution. The distribution function

$$F_x(t) := P^*(x,(-\infty,t]), \quad -\infty < t < \infty,$$

has a right continuous inverse F_x^{-1}.

Consider the probability space (Ω',A',P') defined in Example 4.2.3 and define

$$Y' := F_X^{-1}(U).$$

Thus we defined the extension with Y'. We still have to prove that this extension has the required properties. The simultaneous distribution of (X,Y') is $P^*_{X,Y}$ because

$$P(X\le s,\ Y'\le t) = P(X\le s,\ U\le F_X(t)) = \int_{(-\infty,s]} P(U \le F_X(t)\,|X=x)\,dP_X(x)$$
$$= \int_{(-\infty,s]} F_x(t)\,dP^*_X(x) = P^*_{X,Y}((-\infty,s]\times(-\infty,t]).$$

To prove that the extension did not affect the dependence structure we use that each random variable on the original space is independent of U. Hence if Z is a random variable on the original space, then (Z,X) and U are independent. This yields that

$$P_{Z|X,U} = P_{Z|X}.$$

Because (X,Y') is (X,U)-measurable this implies that

$$P_{Z|X,Y'} = P_{Z|X}.$$

This completes the proof if $\Gamma' = \mathbb{R}^1$. If Γ' is not the real line we know that the Borel space Γ' is isomorphic to a measurable subset of the real line and thus the construction of the required extension is reduced to the case that we already considered. □

COROLLARY 4.2.5. *Suppose on a probability space there is defined a pair* (X,Y) *of random variables, with values in Borel spaces. The probability space can be extended with a random variable* Y', *independent of* X *and distributed as* Y *such that*

$$P(Y\neq Y') = \perp(X,Y),$$

while the dependence structure is not affected in the sense that

(4.2.4) $$P_{Z|X,Y,Y'} = P_{Z|X,Y}$$

for any random variable Z defined on the original probability space, with values in a Borel space.

PROOF. Choose $P^*_{X,Y,Y'}$ as in Proposition 4.2.2. The restriction to real random variables in that proposition is clearly superfluous, because of the isomorphy of a Borel space with a measurable subset of the real line. Use the extension lemma above, to extend the probability space with a random variable Y' such that

$$P^*_{X,Y,Y'} = P_{X,Y,Y'}.$$

The extension thus obtained has the required properties. □

NOTE. The property (4.2.4) implies that for any pair of random variables Z and Z' on the original space with values in arbitrary Borel spaces holds

$$P_{Z,Z'|X,Y,Y'} = P_{Z,Z'|X,Y}$$

and hence

$$P_{Z|X,Y,Y',Z'} = P_{Z|X,Y,Z'}.$$

Thus the measure of dependence

$$\perp(Z,(Z',X,Y)) = \perp(Z,(Z',X,Y,Y'))$$

is not affected by the extension with Y'.

4.3. SUCCESSFUL COUPLING

If X and X' are random variables with values in the same Borel space, then

$$P(X\neq X') \geq \delta(X,X').$$

If the equality occurs, the random variables are called maximally coupled. In the preceding section we investigated how to construct maximally coupled random variables. In the present section we discuss a similar problem for sequences of random variables. In particular our results yield successful couplings for sequences of random variables.

Problems of maximal coupling for Markov sequences are investigated in GRIFFEATH [1975,1976] and PITMAN [1976]. It is interesting to observe, that a slight change of formulation of their results yields similar results that are, however, also valid for arbitrary sequences. In this new setting the Markov dependence appears to be inessential.

The main result in this section is Theorem 4.3.3. It relates a Cesaro convergence property to a successful coupling of processes. In the following section this result will form an important tool.

Theorem 4.3.1 is taken from GRIFFEATH [1976]. Its proof can be found there. Suppose $X_{\mathbb{N}}$ is a Markov chain on a Borel space Γ with stationary transition probabilities. If X_1 has distribution μ we denote by P_μ the distribution of $X_{\mathbb{N}}$. The distribution μ is called the *initial distribution*.

THEOREM 4.3.1. *Suppose μ and ν are initial distributions for the Markov chain $X_{\mathbb{N}}$ described above. There exists a probability space with processes $\bar{X}_{\mathbb{N}}$ and $\bar{X}'_{\mathbb{N}}$, distributed as P_μ and P_ν respectively, such that $\bar{X}_n$ and $\bar{X}'_n$ are maximally coupled and moreover*

$$1 - P(\bar{X}_k = \bar{X}'_k,\ k \ge n) = \delta(\bar{X}_n, \bar{X}'_n), \qquad n \ge 1.$$

It will be clear that the Markov property has an important role in the theorem above. By (4.1.6) it implies that

$$\delta(\bar{X}_n, \bar{X}'_n) = \delta((\bar{X}_k)_{k \ge n}, (\bar{X}'_k)_{k \ge n}), \qquad n \ge 1.$$

Using this property it can be seen that the result below, valid for arbitrary sequences of random variables, implies the maximal coupling theorem for Markov chains, given above. See GOLDSTEIN [1979] for a direct proof.

THEOREM 4.3.2. *Let $X_{\mathbb{N}}$ and $X'_{\mathbb{N}}$ be sequences of random variables with values in a Borel space Γ. There exists a probability space with processes $\bar{X}_{\mathbb{N}}$ and $\bar{X}'_{\mathbb{N}}$, marginally distributed as $X_{\mathbb{N}}$ and $X'_{\mathbb{N}}$ respectively, such that for all $n \ge 0$ the random vectors $\bar{X}_{\mathbb{N}+n}$ and $\bar{X}'_{\mathbb{N}+n}$ are maximally coupled.*

PROOF. Let $\mathcal{T}$ be the σ-field on the Borel space Γ. Consider the transition probability on $\Gamma^{\mathbb{N}} \times \mathcal{T}^{\mathbb{N}}$ defined by

$$P(x,B) := \chi_B(Tx), \qquad x \in \Gamma^{\mathbb{N}},\ B \in \mathcal{T}^{\mathbb{N}},$$

where $Tx := (x_{n+1})_{n \in \mathbb{N}}$. Observe that $Y_n := X_{\mathbb{N}+n}$, $n \ge 0$, and $Y'_n := X'_{\mathbb{N}+n}$, $n \ge 0$, are Markov chains on the Borel space $\Gamma^{\mathbb{N}}$ with transition probabilities as defined before. Apply to $(Y_n)_{n\geq 0}$ and $(Y'_n)_{n\geq 0}$ the maximal coupling theorem for Markov chains given above. This yields the required result. □

Both theorems above are useful to construct successful couplings (see Chapter 1); Note that by the definition of maximal coupling the processes $\bar{X}_{\mathbb{N}}$ and $\bar{X}'_{\mathbb{N}}$ are successfully coupled if an only if for $n \to \infty$

$$P(\bar{X}_{\mathbb{N}+n} \neq \bar{X}'_{\mathbb{N}+n}) = \tfrac{1}{2}\| P_{X_{\mathbb{N}+n}} - P_{X'_{\mathbb{N}+n}} \| \to 0.$$

The remaining part of this section is involved with a proof of the following relation between a Cesaro convergence property and a particular kind of successful coupling.

<u>THEOREM 4.3.3</u>. *Let $X_{\mathbb{N}}$ and $X'_{\mathbb{N}}$ be sequences of random variables with values in a Borel space Γ. The limit relation*

$$\lim_{n\to\infty} \Big\| \frac{1}{n}\sum_{i=0}^{n-1} P_{X_{\mathbb{N}+i}} - \frac{1}{n}\sum_{i=0}^{n-1} P_{X'_{\mathbb{N}+i}} \Big\| = 0 \tag{4.3.1}$$

is a necessary and sufficient condition for the existence of a probability space with processes $\bar{X}_{\mathbb{N}}$ and $\bar{X}'_{\mathbb{N}}$, marginally distributed as $X_{\mathbb{N}}$ and $X'_{\mathbb{N}}$ respectively, such that for some nonnegative, integer valued random variables σ_1 and σ_2

$$\bar{X}_{n+\sigma_1} = \bar{X}'_{n+\sigma_2}, \qquad n \in \mathbb{N}.$$

<u>PROOF</u>. The necessity of (4.3.1) is a consequence of Proposition 4.3.4 below. To prove that (4.3.1) is a sufficient condition, assume that

$$\delta_n := \tfrac{1}{2}\Big\| \frac{1}{n}\sum_{i=0}^{n-1} P_{X_{\mathbb{N}+i}} - \frac{1}{n}\sum_{i=0}^{n-1} P_{X'_{\mathbb{N}+i}} \Big\| \to 0$$

for $n \to \infty$. Suppose σ is an integer valued random variable, homogeneously distributed on $\{0,\ldots,n-1\}$ and independent of $X_{\mathbb{N}}$. Let σ' be a random variable, independent of $X'_{\mathbb{N}}$ and distributed as σ. We can rewrite δ_n as

$$\delta_n = \tfrac{1}{2}\| P_{X_{\mathbb{N}+\sigma}} - P_{X'_{\mathbb{N}+\sigma'}} \|.$$

Using Proposition 4.2.1 and Extension lemma 4.2.4, this equality might be used to construct a probability space such that, with the notations used in the statement of the theorem,

$$P(\bar{X}_{\mathbb{N}+\sigma_1} = \bar{X}'_{\mathbb{N}+\sigma_2}) = 1 - \delta_n .$$

Here the right-hand side is close to 1 for n large. However, we want the right-hand side to coincide with 1 and therefore we argue somewhat differently.

Let $\xi_{\mathbb{N}}$ be a sequence of independent random variables, independent of $X_{\mathbb{N}}$ with $P(\xi_n=0) = P(\xi_n=1) = \frac{1}{2}$, $n \geq 1$, and let $\sigma_{\mathbb{N}}$ be defined by

(4.3.2) $$\sigma_n := \xi_1 + 2\xi_2 + \ldots + 2^{n-1}\xi_n, \quad n \geq 1.$$

Let $\xi'_{\mathbb{N}}$ be independent of $X'_{\mathbb{N}}$, distributed as $\xi_{\mathbb{N}}$ and define $\sigma'_{\mathbb{N}}$ similar to $\sigma_{\mathbb{N}}$ above. Because the random variables σ_n are homogeneously distributed on $\{0,1,\ldots,2^n-1\}$ we have

$$\delta_{2^n} = \tfrac{1}{2}\| P_{X_{\mathbb{N}+\sigma_n}} - P_{X'_{\mathbb{N}+\sigma'_n}} \|, \quad n \geq 0,$$

and because $\xi_{\mathbb{N}}$ and $\xi'_{\mathbb{N}}$ are independent of $X_{\mathbb{N}}$ and $X'_{\mathbb{N}}$ respectively, we even have

$$\delta_{2^n} = \tfrac{1}{2}\| P_{Y_n} - P_{Y'_n} \|,$$

where

$$Y_n := (X_{\mathbb{N}+\sigma_n}, \xi_{\mathbb{N}+n}) \quad \text{and} \quad Y'_n := (X'_{\mathbb{N}+\sigma'_n}, \xi'_{\mathbb{N}+n}).$$

Note that the random variables $Y_{n+1}, Y_{n+2}, \ldots$ can be expressed into Y_n, i.e. $Y_{\mathbb{N}+n} = f(Y_n)$ and similarly $Y'_{\mathbb{N}+n} = f(Y'_n)$. Using (4.1.2) we obtain

$$\tfrac{1}{2}\| P_{Y_{\mathbb{N}+n}} - P_{Y'_{\mathbb{N}+n}} \| \leq \tfrac{1}{2}\| P_{Y_n} - P_{Y'_n} \| = \delta_{2^n} \to 0$$

for $n \to \infty$. By Theorem 4.3.2 there exists a probability space with processes $\bar{Y}_{\mathbb{N}} := (\bar{X}_{\mathbb{N}}, \bar{\xi}_{\mathbb{N}})$ and $\bar{Y}'_{\mathbb{N}} := (\bar{X}'_{\mathbb{N}}, \bar{\xi}'_{\mathbb{N}})$, distributed as $Y_{\mathbb{N}}$ and $Y'_{\mathbb{N}}$ respectively, such that the random variables $\bar{Y}_{\mathbb{N}+n}$ and $\bar{Y}'_{\mathbb{N}+n}$ are maximally coupled for $n \geq 1$. Hence

$$P(\bar{Y}_{\mathbb{N}+n} \neq \bar{Y}'_{\mathbb{N}+n}) = \delta(Y_{\mathbb{N}+n}, Y'_{\mathbb{N}+n}) \to 0$$

for $n \to \infty$. Therefore the random time τ defined by

$$\tau := \inf\{n \geq 0: \bar{Y}_n = \bar{Y}'_n\}$$

is finite with probability 1 and so for $n = \tau$

$$\bar{X}_{\mathbb{N}+\bar{\sigma}_n} = \bar{X}'_{\mathbb{N}+\bar{\sigma}'_n},$$

where $\bar{\sigma}_{\mathbb{N}}$ and $\bar{\sigma}'_{\mathbb{N}}$ are defined in terms of $\bar{\xi}_{\mathbb{N}}$ and $\bar{\xi}'_{\mathbb{N}}$ analogous to (4.3.2). Hence we have

$$\bar{X}_{\mathbb{N}+\bar{\sigma}_\tau} = \bar{X}'_{\mathbb{N}+\bar{\sigma}'_\tau}. \qquad \square$$

PROPOSITION 4.3.4. *Let* $X_{\mathbb{N}}$ *and* $X'_{\mathbb{N}}$ *be sequences of random variables with values in a Borel space* Γ. *Suppose that for any* $\varepsilon > 0$ *there exists a probability space with processes* $\bar{X}_{\mathbb{N}}$ *and* $\bar{X}'_{\mathbb{N}}$, *marginally distributed as* $X_{\mathbb{N}}$ *and* $X'_{\mathbb{N}}$ *respectively, such that for some nonnegative, integer valued random variables* σ_1 *and* σ_2

$$P(\bar{X}_{\mathbb{N}+\sigma_1} = \bar{X}'_{\mathbb{N}+\sigma_2}) \geq 1-\varepsilon.$$

Then the Cesaro convergence (4.3.1) *holds.*

PROOF. Let P and P' be the distribution of $X_{\mathbb{N}}$ and $X'_{\mathbb{N}}$ respectively. We shall decompose P and P' as

$$(4.3.3) \qquad P = P_\varepsilon + \sum_{k,\ell\geq 0} P_{k,\ell} \quad \text{and} \quad P' = P'_\varepsilon + \sum_{k,\ell\geq 0} P'_{k,\ell}.$$

The measures $P_{k,\ell}$ and $P'_{k,\ell}$ are defined as follows. Let $A := \{X_{\mathbb{N}+\sigma_1} = X_{\mathbb{N}+\sigma_2}\}$ and define

$$P_{k,\ell}(B) := P(\{X_{\mathbb{N}} \in B\} \cap \{\sigma_1=k,\ \sigma_2=\ell\} \cap A),$$

$$P'_{k,\ell}(B) := P(\{X'_{\mathbb{N}} \in B\} \cap \{\sigma_1=k,\ \sigma_2=\ell\} \cap A),$$

for measurable sets $B \subset \Gamma^{\mathbb{N}}$. We can now define P_ε and P'_ε by requiring (4.3.3). The total mass of the $P_{k,\ell}$-measures is P(A) and hence

$$\|P_\varepsilon\| = 1 - P(A) \leq \varepsilon.$$

Similarly we obtain $\|P'_\varepsilon\| \leq \varepsilon$.

In the argument below we need some new notations. Let T be the shift transformation on $\Gamma^{\mathbb{N}}$; For measures R on $\Gamma^{\mathbb{N}}$ we denote by T^nR the measure defined by

$$(4.3.4) \qquad T^nR(B) := R(T^{-n}B)$$

for measurable sets $B \subset \Gamma^{\mathbb{N}}$.

Using the definition of A one easily observes that

$$(4.3.5)\qquad T^k P_{k,\ell} = T^\ell P'_{k,\ell} \qquad k,\ell \geq 0.$$

The idea of the proof is to substitute (4.3.3) into the expression in (4.3.1) and then to use (4.3.5) to get an estimate of the expression in (4.3.1). In the signed measure

$$(4.3.6)\qquad \sum_{k,\ell\geq 0} \Big(\sum_{i=0}^{n-1} T^i P_{k,\ell} - \sum_{i=0}^{n-1} T^i P'_{k,\ell} \Big)$$

the terms of form $T^{i+k}P_{k,\ell}$ are cancelled by the terms $T^{i+\ell}P'_{k,\ell}$. Using this observation it follows that the terms $T^i P_{k,\ell}$, $0 \leq i < n$, that are not cancelled in this way have their index i in

$$\{i: 0 \leq i < k\} \qquad \text{if } \ell \leq k,$$

$$\{i: 0 \leq i < k \ \text{ or } \ i+\ell-k \geq n\} \qquad \text{if } \ell > k.$$

The total variation of the positive part of (4.3.6) is therefore at most

$$\sum_{k,\ell\geq 0} (k \wedge n + (\ell-k)^+ \wedge n) \| P_{k,\ell} \| = o(n)$$

for $n \to \infty$. A similar estimate holds for the negative part of (4.3.6) and hence

$$\Big\| \frac{1}{n} \sum_{i=0}^{n-1} T^i P - \frac{1}{n} \sum_{i=0}^{n-1} T^i P' \Big\|$$

$$\leq \frac{1}{n} \sum_{i=0}^{n-1} \| T^i (P_\varepsilon - P'_\varepsilon) \| + \frac{1}{n} \cdot o(n) \leq 2\varepsilon + o(1), \qquad n \to \infty.$$

Because ε is an arbitrary positive number, (4.3.1) follows. □

4.4. WEAK BERNOULLI PROCESSES

In Chapter 1 we used coupling as our main tool to prove results in renewal theory. To generalize these theorems for random walks with stationary increments, we want to determine a class of stationary processes that can be characterized by coupling properties such as used in Chapter 1. For this purpose we introduce a class of stationary processes, the weak Bernoulli processes, that are determined by a certain asymptotic independence property. We compare this class of processes with the better known classes of ϕ- and α-mixing processes. The section contains several examples of weak Bernoulli processes. From the point of view of Chapter 1 an important result is Theorem 4.4.7 that characterizes the weak Bernoulli processes by a coupling property. This seems to be promising to get an extension of the results in Chapter 1.

The main results in Chapter 6 are renewal theorems for weak Bernoulli processes. In Sections 6.4 and 6.5 we shall see that it is more natural to study these renewal theorems for a slightly larger class of processes, the Cesaro weak Bernoulli processes. The second half of this chapter studies these processes and characterizes them in Theorem 4.4.9 by a coupling property, very similar to the coupling property described in Theorem 4.4.7 for weak Bernoulli processes.

Let $X_{\mathbb{Z}}$ be a stationary sequence of random variables with values in a Borel space Γ. The process $X_{\mathbb{Z}}$ is called *weak Bernoulli* if

(4.4.1) $$\perp_n := \perp(X_{\mathbb{N}^c}, X_{\mathbb{N}+n})$$

satisfies $\lim_{n\to\infty} \perp_n = 0$. According to Proposition 4.1.2 this requirement is equivalent to

$$\lim_{n\to\infty} E\| P_{X_{\mathbb{N}+n}|X_{\mathbb{N}^c}} - P_{X_{\mathbb{N}+n}} \| = 0,$$

or to

(4.4.2) $$\lim_{n\to\infty} \| P_{X_{\mathbb{N}+n}|X_{\mathbb{N}^c}} - P_{X_{\mathbb{N}+n}} \| = 0 \text{ a.s.}$$

The equivalence of the last two limit relations follows because the total variation expression in (4.4.2) is by (4.1.2) nonascending in n.

The concept weak Bernoulli was suggested by Kolmogorov under the name *completely regular* (see VOLKONSKI and ROZANOV [1959]). Later it was used in connection with the isomorphism problem in ergodic theory (see FRIEDMAN and ORNSTEIN [1970]). From the point of view of ergodic theory the concept weak Bernoulli is not entirely natural because the weak Bernoulli property is not preserved under the isomorphism concept used in ergodic theory (see SMORODINSKY [1971]). However, from our probabilistic point of view it is quite satisfactory. This will be obvious from Theorem 4.4.3, but even more from the results in Sections 6.4 and 6.5.

Let us first compare the weak Bernoulli property with other conditions of asymptotic independence. Define for a stationary process $X_{\mathbb{Z}}$ the measure

$$\mu_n := P_{X_{\mathbb{N}^c}, X_{\mathbb{N}+n}} - P_{X_{\mathbb{N}^c}} \times P_{X_{\mathbb{N}}}, \qquad n \geq 1.$$

Let α_n and ϕ_n, $n \geq 1$, be the smallest real numbers such that

$$\mu_n(B \times B') \leq \alpha_n,$$

$$\mu_n(B \times B') \leq \phi_n \, P_{X_{\mathbb{N}}}(B'),$$

holds for all measurable sets B on $\Gamma^{\mathbb{N}^c}$ and B' on $\Gamma^{\mathbb{N}}$. The sequence $X_{\mathbb{Z}}$ is called *α-mixing* if $\lim_{n\to\infty} \alpha_n = 0$ and *φ-mixing* if $\lim_{n\to\infty} \phi_n = 0$.

PROPOSITION 4.4.1. $\alpha_n \leq \perp_n \leq \phi_n$, $n \geq 1$.

PROOF. Because μ_n is a difference of probability measures, we have

$$\perp_n = \tfrac{1}{2}\|\mu_n\| = \sup \mu_n(B),$$

where the supremum is taken over all measurable sets $B \subset \Gamma^{\mathbb{N}^c} \times \Gamma^{\mathbb{N}}$. Hence the first inequality holds. To prove the second inequality, consider the measurable space $\Gamma^{\mathbb{N}^c}$ provided with the probability measure $P_{X_{\mathbb{N}^c}}$. Let $\bar{\phi}_n$ be the L_∞-norm of

$$g_n(x) := \sup(P_{X_{\mathbb{N}+n} | X_{\mathbb{N}^c}=x}(B) - P_{X_{\mathbb{N}+n}}(B)),$$

where the supremum is taken over all measurable sets B on $\Gamma^{\mathbb{N}}$. It is easily seen that $\phi_n = \bar{\phi}_n$, $n \geq 1$. Furthermore, by Proposition 4.1.1 we have $\perp_n = Eg(X_{\mathbb{N}^c})$ and hence $\perp_n \leq \phi_n$. □

It follows that weak Bernoulli is implied by φ-mixing and implies α-mixing. BRADLEY [1978] gives an example of an α-mixing sequence that is not weak Bernoulli. The following result is well known.

PROPOSITION 4.4.2. *If a stationary sequence* $X_{\mathbb{Z}}$ *is α-mixing, it has trivial right (or left) tail σ-field, and therefore is also ergodic.*

PROOF. Remark that if A_m is $X_{\mathbb{N}^c+m}$-measurable and A is a right tail event (so is $X_{\mathbb{N}+m-n}$-measurable), then

$$\beta_m := |P(A_m \cap A) - P(A_m)P(A)| \leq \alpha_n$$

for each $n \geq 1$. Because $\lim_{n\to\infty} \alpha_n = 0$ it follows that $\beta_m = 0$, $m \geq 1$. Choose a sequence A_m, $m \ge 1$, such that $P(A \Delta A_m) \to 0$ for $n \to \infty$. It follows that $P(A) - P(A)^2 = 0$ and hence each right tail event A has probability 0 or 1. A similar argument holds for left tail events. □

Using OREY [1971, Proposition 1.4.3] or REVUZ [1974, Proposition 6.2.4] it can be easily seen from (4.4.2) that the properties trivial tail σ-field, α-mixing and weak Bernoulli coincide if $X_{\mathbb{Z}}$ is a Markov sequence. In that case by REVUZ [1974, Section 6.3] the property φ-mixing implies exponentially fast convergence, i.e. $\phi_n \le ab^n$, $n \ge 1$, for some positive a with $0 < b < 1$.

Let us now discuss some examples of weak Bernoulli processes.

EXAMPLE 4.4.3. A stationary sequence $X_{\mathbb{Z}}$ of independent, real random variables is obviously weak Bernoulli.

EXAMPLE 4.4.4 (Markov chains). Let $X_{\mathbb{N}}$ be an irreducible, positive recurrent, aperiodic Markov chain with a countable state space Γ. Suppose that $X_{\mathbb{N}}$ is started with the invariant probability measure π of the Markov chain and extend $X_{\mathbb{N}}$ to a stationary process $X_{\mathbb{Z}}$. Using the Markov property it follows that on $\{X_0=\gamma\}$

$$P_{X_{\mathbb{N}+n}|X_n,X_0} = P_{X_{\mathbb{N}+n}|X_n}$$

and hence by (4.1.6)

$$\| P_{X_{\mathbb{N}+n},X_n|X_0=\gamma} - P_{X_{\mathbb{N}+n},X_n} \| = \| P_{X_n|X_0=\gamma} - P_{X_n} \| .$$

By a well known convergence theorem for Markov chains (see BREIMAN [Theorem 7.38]) the right-hand side converges to 0 for $n \to \infty$ and hence

$$\| P_{X_{\mathbb{N}+n}|X_{\mathbb{N}^c}} - P_{X_{\mathbb{N}+n}} \| \to 0 \text{ a.s.}$$

It follows that the process $X_{\mathbb{Z}}$ is weak Bernoulli.

EXAMPLE 4.4.5 (continued fraction transformation). Each irrational number ω in the unit interval I has a continued fraction expansion

$$\omega = \cfrac{1}{X_1(\omega) + \cfrac{1}{X_2(\omega) + \dots}} ,$$

where the numbers $X_n(\omega) \in \mathbb{N}$, $n \ge 1$, are defined as follows. Let

$T\omega := (\frac{1}{\omega}) \bmod 1$. Then $X_1(\omega) = \frac{1}{\omega} - T\omega$ and $X_{n+1}(\omega) = X_n(T\omega) = \ldots = X_1(T^n\omega)$. Provide the unit interval with the Borel σ-field $\mathcal{B}(I)$. Let ℓ_I be the Lebesgue measure on the unit interval. Then $X_{\mathbb{N}}$ is a sequence of ℓ_I-a.s. defined random variables. If the unit interval is provided of Gauss's measure μ, given by

$$\mu(B) = \frac{1}{\log 2}\int_B \frac{1}{1+x}\,dx, \qquad B \in \mathcal{B}(I),$$

then $X_{\mathbb{N}}$ is a stationary process on $(I,\mathcal{B}(I),\mu)$, as is proved in BILLINGSLEY [Section 1.4]. Extend $X_{\mathbb{N}}$ to a stationary process $X_{\mathbb{Z}}$. It is known that $X_{\mathbb{Z}}$ is ϕ-mixing with exponentially decreasing ϕ_n, $n \to \infty$, (see IOSIFESCU [1978]) and hence $X_{\mathbb{Z}}$ is weak Bernoulli.

<u>EXAMPLE 4.4.6</u> (chains with complete connections). Let $X_{\mathbb{Z}}$ be a stationary sequence of random variables with values in a finite set Γ. Suppose there is a strictly positive, measurable function g on $\Gamma^{\mathbb{N}^c}$ such that

$$P(X_0=\gamma \mid X_{-1},\ldots,X_{-n}) \to g(\gamma,X_{-1},X_{-2},\ldots) \text{ a.s.}$$

for $n \to \infty$. LEDRAPPIER [1976] proves that if g satisfies

$$\sum_{n\geq 0} \sup \left|\log \frac{g(\bar{\gamma})}{g(\bar{\bar{\gamma}})}\right| < \infty,$$

where the supremum is taken over all $\bar{\gamma},\bar{\bar{\gamma}} \in \Gamma^{\mathbb{N}^c}$ with $\bar{\gamma}_k = \bar{\bar{\gamma}}_k$, $-n \leq k \leq 0$, then $X_{\mathbb{Z}}$ is weak Bernoulli.

Ledrappier's paper, mentioned above, contains also other examples. It discusses a lattice model studied in statistical mechanics, that describes a particle system on the integers $\mathbb{Z}$. If a site $n \in \mathbb{Z}$ is occupied then $X_n := 1$ and else $X_n := 0$. There can be defined a distribution of $X_{\mathbb{Z}}$, called a Gibbs measure, such that under some continuity conditions on this distribution, it can be proved that $X_{\mathbb{Z}}$ is weak Bernoulli. LEDRAPPIER [1976] mentions also other papers that discuss weak Bernoulli processes.

<u>THEOREM 4.4.7</u>. *A stationary sequence $X_{\mathbb{Z}}$ of random variables with values in a Borel space Γ is weak Bernoulli if and only if there exists a probability space with processes $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$ marginally distributed as $X_{\mathbb{Z}}$, such that $X'_{\mathbb{N}^c}$ and $X''_{\mathbb{Z}}$ are independent and $X'_{\mathbb{N}}$ and $X''_{\mathbb{N}}$ are successfully coupled.*

<u>PROOF</u>. If there exists a probability space as described in the theorem, then

because of the independence of $X'_{\mathbb{N}^c}$ and $X''_{\mathbb{Z}}$ and by Lemma 1.1.1

$$\perp(X_{\mathbb{N}^c}, X_{\mathbb{N}+n}) = \tfrac{1}{2}\| P_{X'_{\mathbb{N}^c}, X'_{\mathbb{N}+n}} - P_{X'_{\mathbb{N}^c}, X''_{\mathbb{N}+n}} \|$$

$$\leq P(X'_{\mathbb{N}+n} \neq X''_{\mathbb{N}+n}) \to 0$$

for $n \to \infty$.

To prove the converse, assume that $X_{\mathbb{Z}}$ is weak Bernoulli, i.e. (4.4.2) holds. We can select a regular conditional distribution

$$\mu_x(B) := P_{X_{\mathbb{N}} | X_{\mathbb{N}^c}=x}(B),$$

such that with the notation (4.3.4)

$$\delta_n := \tfrac{1}{2}\| T^n \mu_x - P_{X_{\mathbb{N}+n}} \| \to 0$$

for $n \to \infty$, up to a $P_{X_{\mathbb{N}^c}}$-null set. It is no restriction to assume that this holds for all x. By Theorem 4.3.2 we can construct a pair $(X'_{\mathbb{N}}, X''_{\mathbb{N}})$ of random vectors, marginally distributed as μ_x and $P_{X_{\mathbb{N}}}$ such that for each $n \geq 0$ the random vectors $X'_{\mathbb{N}+n}$ and $X''_{\mathbb{N}+n}$ are maximally coupled, i.e.

$$P_x(X'_{\mathbb{N}+n} \neq X''_{\mathbb{N}+n}) = \delta_n, \quad n \geq 0.$$

Because $\delta_n \to 0$ for $n \to \infty$ the processes $X'_{\mathbb{N}}$ and $X''_{\mathbb{N}}$ are successfully coupled. Note that the probability measure P_x thus constructed, depends on x.

We have to define a probability space $(\Omega, \mathcal{A}, P)$ on which also $X'_{\mathbb{N}^c}$ and $X''_{\mathbb{N}^c}$ are defined. Using Lemma 4.2.4 we can extend the probability space with a random vector $X''_{\mathbb{N}^c}$ such that $(P_x)_{X''_{\mathbb{Z}}} = P_{X_{\mathbb{Z}}}$. Also the new probability space depends on x. Using that the random variables have their values in a Borel space it is possible to show that

$$(P_x)_{X'_{\mathbb{N}^c}, X''_{\mathbb{Z}}}(B)$$

is measurable in x for each measurable set B. Note that the requirements

$$P_{X'_{\mathbb{N}^c}} = P_{X_{\mathbb{N}^c}} \quad \text{and} \quad P_{X'_{\mathbb{N}}, X''_{\mathbb{Z}} | X'_{\mathbb{N}^c}=x} = (P_x)_{X'_{\mathbb{N}^c}, X''_{\mathbb{Z}}}$$

determine a simultaneous distribution

$$P_{X'_{\mathbb{N}}, X''_{\mathbb{Z}}, X'_{\mathbb{N}^c}}$$

on processes $(X'_{\mathbb{Z}}, X''_{\mathbb{Z}})$ on a probability space, say $(\Omega, \mathcal{A}, P)$. Because with these requirements the random vector $X'_{\mathbb{N}}$, given $X'_{\mathbb{N}^c}=x$, has conditional distribution $P_{X_{\mathbb{N}}|X_{\mathbb{N}^c}=x}$, it follows that $X'_{\mathbb{Z}}$ is distributed as $X_{\mathbb{Z}}$. We also have

$$P_{X''_{\mathbb{Z}}|X'_{\mathbb{N}^c}=x} = (P_x)_{X''_{\mathbb{Z}}} = P_{X_{\mathbb{Z}}}$$

and hence $X'_{\mathbb{N}^c}$ and $X''_{\mathbb{Z}}$ are independent. Note also that $X''_{\mathbb{Z}}$ is distributed as $X_{\mathbb{Z}}$. Because $X'_{\mathbb{N}}$ and $X''_{\mathbb{N}}$ are successfully coupled, the probability space (Ω, A, P) satisfies the requirements. □

<u>NOTE</u>. By the if-part in the proof of the theorem above we have

$$P(X'_{\mathbb{N}+n} \neq X''_{\mathbb{N}+n}) \geq \perp_n, \qquad n \geq 0.$$

Apparently by the second part of the proof we can construct $(X'_{\mathbb{Z}}, X''_{\mathbb{Z}})$ such that the equality holds: The probability at the left equals

$$\int \delta_n(x) \, d\, P_{X'_{\mathbb{N}^c}}(x) = \perp_n, \qquad n \geq 0.$$

Here we apply Proposition 4.1.1 with $\delta_n = \delta_n(x)$ defined in the proof above.

Another class of processes for which a coupling property almost similar as above holds, are the Cesaro weak Bernoulli processes. Let $X_{\mathbb{Z}}$ be a stationary sequence of random variables with values in a Borel space. The process $X_{\mathbb{Z}}$ is called *Cesaro weak Bernoulli* if

$$\text{(4.4.3)} \qquad \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} P_{X_{\mathbb{N}^c}, X_{\mathbb{N}+k}} - P_{X_{\mathbb{N}^c}} \times P_{X_{\mathbb{N}}} \right\| = 0.$$

A weak Bernoulli process is Cesaro weak Bernoulli. To see this apply the following proposition and note that (4.4.4) follows from (4.4.2).

<u>PROPOSITION 4.4.8</u>. *The process* $X_{\mathbb{Z}}$ *described above is Cesaro weak Bernoulli if and only if*

$$\text{(4.4.4)} \qquad \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} P_{X_{\mathbb{N}+k}|X_{\mathbb{N}^c}} - P_{X_{\mathbb{N}}} \right\| = 0 \text{ a.s.}$$

<u>PROOF</u>. The proof is divided into two parts.

<u>Part 1</u>. Let Z_n be defined as the expression in (4.4.4). We show first that EZ_n is the expression in (4.4.3). Let σ_n be an integer valued random variable, homogeneously distributed on $\{0,\ldots,n-1\}$ and independent of $X_{\mathbb{Z}}$.

Note that

$$Z_n = \| P_{X_{\mathbb{N}+\sigma_n} | X_{\mathbb{N}^c}} - P_{X_{\mathbb{N}}} \| .$$

By stationarity $X_{\mathbb{N}}$ is distributed as $X_{\mathbb{N}+\sigma_n}$ and by Proposition 4.1.1

$$EZ_n = \perp(X_{\mathbb{N}^c}, X_{\mathbb{N}+\sigma_n}).$$

It is now easily seen that EZ_n equals the expression in (4.4.3) and hence $X_{\mathbb{Z}}$ is Cesaro weak Bernoulli if and only if $\lim_{n\to\infty} EZ_n = 0$. Thus we have to prove that $\lim_{n\to\infty} EZ_n = 0$ if and only if $\lim_{n\to\infty} Z_n = 0$ a.s. The if-part of this assertion is a consequence of the dominated convergence theorem. To prove the only if-part we argue as follows. By stationarity

$$Z_{n+m} = \| \frac{1}{n+m} \sum_{i=0}^{n+m-1} (P_{X_{\mathbb{N}+i} | X_{\mathbb{N}^c}} - P_{X_{\mathbb{N}+i}}) \|$$

$$\leq \frac{n}{n+m} Z_n + \frac{m}{n+m} \| \frac{1}{m} \sum_{i=0}^{m-1} (P_{X_{\mathbb{N}+n+i} | X_{\mathbb{N}^c}} - P_{X_{\mathbb{N}+n+i}}) \| .$$

The total variation expression at the right-hand side equals

$$\| P_{X_{\mathbb{N}+n+\sigma_m} | X_{\mathbb{N}^c}} - P_{X_{\mathbb{N}+n+\sigma_m}} \|$$

and because $X_{\mathbb{N}+n+\sigma_m}$ can be expressed as a function of $X_{\mathbb{N}+\sigma_m}$, the expression above is bounded by

$$\| P_{X_{\mathbb{N}+\sigma_m} | X_{\mathbb{N}^c}} - P_{X_{\mathbb{N}+\sigma_m}} \| = Z_m .$$

Therefore we have the inequality

$$Z_{n+m} \leq \frac{n}{n+m} Z_n + \frac{m}{n+m} Z_m .$$

Suppose $\lim_{n\to\infty} EZ_n = 0$. By Fatou's lemma this implies that $\liminf Z_n = 0$ a.s. In the second part of the proof we shall show that this implies $\lim_{n\to\infty} Z_n = 0$ a.s. Then this will prove the only if-part of the theorem.

Part 2. Let a_n, $n \geq 1$, be nonnegative numbers, satisfying

(4.4.5) $$a_{n+m} \leq \frac{n}{n+m} a_n + \frac{m}{n+m} a_m , \quad n,m \geq 1 .$$

Suppose that $\liminf a_n = 0$. We shall prove that $\lim_{n\to\infty} a_n = 0$. Let $\varepsilon > 0$ be arbitrary and choose k so large that $a_k < \frac{1}{2}\varepsilon$. Each number n can be written as $n = sk + r$ with $0 \le r < k$. The inequality (4.4.5) implies that a_1 dominates all a_n, $n \ge 1$. Hence

$$a_n = a_{sk+r} \le \frac{sk}{n} a_k + \frac{r}{n} a_r \le \tfrac{1}{2}\varepsilon + \frac{k}{n} a_1.$$

Here if n is large enough $a_n < \varepsilon$. It follows that $\lim_{n\to\infty} a_n = 0$. □

An equivalent formulation of the concept Cesaro weak Bernoulli can be given in terms of coupling.

THEOREM 4.4.9. *A stationary sequence $X_{\mathbb{Z}}$ of random variables with values in a Borel space Γ is Cesaro weak Bernoulli if and only if there is a probability space with processes $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$, marginally distributed as $X_{\mathbb{Z}}$, with $X'_{\mathbb{N}^c}$ and $X''_{\mathbb{Z}}$ independent such that there are nonnegative, integer valued random variables σ_1 and σ_2 for which*

$$X'_{n+\sigma_1} = X''_{n+\sigma_2}, \qquad n \in \mathbb{N}.$$

In the proof of the theorem above we need the following simple lemma.

LEMMA 4.4.10. *If $Y_{\mathbb{N}}$ is a sequence of random variables with values in a Borel space Γ, and $\sigma \ge 0$ is an integer valued random variable, then*

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{i=0}^{n-1} P_{Y_{\mathbb{N}+\sigma+i}} - \frac{1}{n}\sum_{i=0}^{n-1} P_{Y_{\mathbb{N}+i}} \right\| = 0.$$

PROOF. For $n \ge 1$ we have

$$\left| \sum_{i=0}^{n-1} P(Y_{\mathbb{N}+k+i} \in B,\ \sigma = k) - \sum_{i=1}^{n-1} P(Y_{\mathbb{N}+i} \in B,\ \sigma = k) \right|$$
$$\le ((n-k) \vee 0) P(\sigma = k).$$

Using this inequality and the definition of total variation we obtain for $n \to \infty$

$$\left\| \sum_{i=0}^{n-1} P_{Y_{\mathbb{N}+\sigma+i}} - \sum_{i=0}^{n-1} P_{Y_{\mathbb{N}+i}} \right\| \le 2 \sum_{k=0}^{m} (n-k) P(\sigma = k) = o(n). \qquad \square$$

PROOF of Theorem 4.4.9. First we prove the if-part. Let $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$ be processes as in the assertion of the theorem. It follows that

$$\| P_{X'_{\mathbb{N}+\sigma_1} | X'_{\mathbb{N}^c}} - P_{X''_{\mathbb{N}+\sigma_2} | X'_{\mathbb{N}^c}} \| = 0 \text{ a.s.}$$

and hence by Lemma 4.4.10

$$\lim_{n\to\infty} \| \frac{1}{n} \sum_{i=0}^{n-1} P_{X'_{\mathbb{N}+i} | X'_{\mathbb{N}^c}} - \frac{1}{n} \sum_{i=0}^{n-1} P_{X''_{\mathbb{N}+i} | X'_{\mathbb{N}^c}} \| = 0 \text{ a.s.}$$

Because $X'_{\mathbb{N}^c}$ and $X''_{\mathbb{Z}}$ are independent this yields (4.4.4) and hence $X_{\mathbb{Z}}$ is Cesaro weak Bernoulli.

To prove the converse we use (4.4.4). Select a regular conditional distribution

$$\mu_x(B) := P_{X_{\mathbb{N}} | X_{\mathbb{N}^c} = x}.$$

By stationarity we have $P_{X_{\mathbb{N}}} = \frac{1}{n} \sum_{i=0}^{n-1} P_{X_{\mathbb{N}+i}}$. Because of (4.4.4), using notation (4.3.4)

$$\lim_{n\to\infty} \| \frac{1}{n} \sum_{i=0}^{n-1} T^i \mu_x - \frac{1}{n} \sum_{i=0}^{n-1} P_{X_{\mathbb{N}+i}} \| = 0,$$

up to a $P_{X_{\mathbb{N}^c}}$-null set. It is no restriction to assume that this property holds for all x. We can now argue as in the corresponding part of the proof of Theorem 4.4.7. However, instead of Theorem 4.3.2 we use Theorem 4.3.3 to construct the successful coupling. □

EXAMPLE 4.4.11 (periodic Markov chains). Let $X_{\mathbb{N}}$ be an irreducible, positive recurrent Markov chain with countable state space Γ. We do not assume that the Markov chain is aperiodic. Extend $X_{\mathbb{N}}$ to a stationary process $X_{\mathbb{Z}}$. Using BREIMAN [Section 7.9] we can prove that

$$\lim_{n\to\infty} \| \frac{1}{n} \sum_{i=0}^{n-1} P_{X_{i+1} | X_0} - P_{X_0} \| = 0 \text{ a.s.}$$

and by using the Markov property as in Example 4.4.4 we obtain

$$\lim_{n\to\infty} \| \frac{1}{n} \sum_{i=0}^{n-1} P_{X_{\mathbb{N}+i} | X_{\mathbb{N}^c}} - P_{X_{\mathbb{N}}} \| = 0 \text{ a.s.}$$

Hence the process $X_{\mathbb{Z}}$ is Cesaro weak Bernoulli.

NOTE to Sections 4.1 and 4.2: Prof.dr. J. Fabius remarked that there is a large literature on measures of dependence. The measure of dependence $\perp(X,Y)$, studied in the report by SCHWARZ [1978], is investigated earlier in SILVEY [1964]*) and already in the fifties Fréchet has paid attention to the problem of constructing a simultaneous distribution $P^*_{X,Y}$ with prescribed marginals.

*) S.D. SILVEY, *On a measure of information*, Ann. Math. Stat. 35, 1157-1166.

CHAPTER 5

THE SPREADING BEHAVIOUR OF S_n

In this chapter we want to derive limit relations of the type

$$\lim_{n\to\infty} \| P_{S_n} - P_{S_n+h} \| = 0, \quad h \text{ real,} \tag{5.0.1}$$

for random walks with asymptotically independent increments. In Section 5.1 we show how to reduce the proof of results of this type to coupling problems for sequences of independent, identically distributed (i.i.d.) random vectors. In Section 5.3 we prove (5.0.1) and in Section 5.2 we prove under weaker conditions, a similar, weaker limit relation. Section 5.2 makes use of the Ornstein coupling and Section 5.3 utilizes the maximal coupling theorem for Markov chains.

In Chapters 5 and 6 we assume that the random walk $S_{\mathbb{Z}}$ is given in terms of a stationary sequence $X_{\mathbb{Z}}$ of random variables with values in a Borel space Γ. A real valued process $S_{\mathbb{Z}}$ is called a random walk *controlled* by $X_{\mathbb{Z}}$ if there is a measurable real function f defined on Γ, such that $S_{\mathbb{Z}}$ is given by

$$S_0 = 0; \quad S_n - S_{n-1} = f(X_n), \quad n \in \mathbb{Z}. \tag{5.0.2}$$

Clearly because $X_{\mathbb{Z}}$ is stationary, $S_{\mathbb{Z}}$ is a random walk with stationary increments.

If $S_{\mathbb{Z}}$ is any random walk and no controlling process $X_{\mathbb{Z}}$ is specified, we can always take $X_{\mathbb{Z}}$ to be the process of increments of $S_{\mathbb{Z}}$. This process controls $S_{\mathbb{Z}}$: take f the identity on the real line. In the sequel we only discuss random walks together with a controlling process. We have already seen that this is not a restriction. The reason for discussing controlled random walks is that in several examples the properties of the random walk are given in terms of $X_{\mathbb{Z}}$ and not immediately in terms of $S_{\mathbb{Z}}$. Also $X_{\mathbb{Z}}$ might have some useful property that is not available for $S_{\mathbb{Z}}$. One might think of Markov dependence (see Example 2.1.11). Also semi-Markov processes can be fitted into this framework (see Section 6.4).

EXAMPLE 5.0.1. Suppose $Y_{\mathbb{Z}}$ is a stationary, Markov dependent sequence with values in a Borel space Γ_0. Let $\xi_{\mathbb{Z}}$ be a real valued process such that ξ_n, given $Y_{\mathbb{Z}}$ and $(\xi_k)_{k\neq n}$, only depends on (Y_n, Y_{n+1}). Then (ξ_n, Y_n), $n \in \mathbb{Z}$, is called a *semi-Markov chain*. Define

$$\Gamma := \Gamma_0 \times \mathbb{R}^1 \times \Gamma_0, \quad X_n := (Y_n, \xi_n, Y_{n+1})$$

and let f on Γ be the projection on the second coordinate. Then $S_{\mathbb{Z}}$, determined by (5.0.2), is a random walk controlled by the Markov process $X_{\mathbb{Z}}$.

5.1. DEPENDENCE STRUCTURE AND COUPLINGS FOR RANDOM WALKS

Central in our discussion of random walks with dependent increments is the use of condition (5.1.3). The first half of this section discusses this condition. Theorem 5.1.7 is the main result of this section. It shows how the random walk problems of this section can be reduced to the study of i.i.d. random vectors. The second half of this section is concerned with a proof of this theorem.

We discuss random walks $S_{\mathbb{Z}}$ controlled by a stationary sequence $X_{\mathbb{Z}}$, i.e. (5.0.2) holds. If K is a finite integer set, we sometimes use the notation

$$(5.1.1) \qquad S^K := \sum_{k\in K} f(X_k).$$

In Chapter 1 we obtained results for the spreading behaviour of random walks with stationary, independent increments. The main tool that we used in the proofs was a coupling technique. In this chapter we want to relax independence to some form of asymptotic independence. To be able to use also in this context a coupling technique we have to get a good understanding of the dependence structure of the processes that are to be coupled. In fact we are only able to use a coupling approach if we impose condition (5.1.3) on the controlling process $X_{\mathbb{Z}}$. We show that the condition is satisfied if $X_{\mathbb{Z}}$ is a countably valued sequence or if $X_{\mathbb{Z}}$ is a Markov dependent sequence.

The asymptotic independence condition that we require on $X_{\mathbb{Z}}$ in this section is

$$(5.1.2) \qquad \lim_{n\to\infty} \perp(X_{L'}, X_{L''+n}) = 0$$

for all finite subsets L' and L" contained in $\mathbb{Z}$. If $X_{\mathbb{Z}}$ is countably valued then (5.1.2) is equivalent with a concept called mixing in ergodic theory (see the note to Lemma 5.1.4). The coupling constructions that we want to make, use (5.1.2) together with the following condition

$$(5.1.3) \qquad \lim_{L \to \mathbb{Z}} \perp_{X_{L \setminus K}} (X_K, X_{L^c}) = 0$$

for all finite subsets $K \subset \mathbb{Z}$, where the limit is taken along the directed set of finite subsets $L \subset \mathbb{Z}$, partially ordered by inclusion. Loosely speaking the condition requires that for a sufficiently large finite integer set L, the random vectors X_K and X_{L^c} are approximately independent, given $X_{L \setminus K}$. Condition (5.1.3) is studied in the first two propositions of this section.

Before presenting detailed proofs, we first sketch the background of ideas on which the theory in this chapter is founded. In Theorem 1.1.2 we derived the limit relation

$$\lim_{n \to \infty} \| P_{S_n} - P_{S_n + h} \| = 0$$

by means of coupling. Here we want to use a similar approach. In Theorem 1.1.2 we argued as follows. Let $X_{\mathbb{Z}}$ be the process of increments of $S_{\mathbb{Z}}$. We constructed processes $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$, distributed as $X_{\mathbb{Z}}$, such that for large n with large probability

$$(5.1.4) \qquad \begin{aligned} S''_j - S'_j &= 0, \quad j \le 0, \\ S''_j - S'_j &= h, \quad j > n, \end{aligned}$$

for some prescribed real h, while also

$$X''_j = X'_j, \qquad j \notin K_n,$$

where $K_n := \{1,\dots,n\}$ and $S'_{\mathbb{Z}}$ and $S''_{\mathbb{Z}}$ are the random walks with increments $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$, respectively.

The result that shows how we can use coupling in our generalized context is Theorem 5.1.7. To get some feeling for the kind of answer that this theorem gives, consider a real valued Markov sequence $X_{\mathbb{Z}}$ and let $S_{\mathbb{Z}}$ be the random walk with increments $X_{\mathbb{Z}}$. Consider for some integer k the distribution

$$F := P_{X_k | X_{k-1} = \gamma^-, X_{k+1} = \gamma^+}$$

and assume that F is not degenerate. Observe that we can construct random variables X'_k and X''_k that are, given $\{X_{k-1}=\gamma^-, X_{k+1}=\gamma^+\}$, F-distributed such that the distribution of the difference $X''_k - X'_k$ is not degenerate. Arguing in this way we can construct for any integer k triples (X_{k-1}, X'_k, X_{k+1}) and $(X_{k-1}, X''_k, X_{k+1})$ such that

$$P_{X'_k|(X_j)_{j\neq k}} = P_{X''_k|(X_j)_{j\neq k}} = P_{X_k|X_{k-1},X_{k+1}}, \tag{5.1.5}$$

while the difference $X''_k - X'_k$ may be nonvanishing. We shall say that we "*vary*" X_k, while X_{k-1} and X_{k+1} *remain fixed*.

Take a sequence $k_1 < \ldots < k_m$ of positive integers such that the triples $(X_{k_j-1}, X_{k_j}, X_{k_j+1})$ are (approximately) independent. This is possible by condition (5.1.2). We can then vary X_{k_j}, i.e. we can construct a pair (X'_{k_j}, X''_{k_j}) such that (5.1.5) holds for $k = k_j$, $1 \le j \le m$. It is important to note that, by the Markov property and the independence of the triples $(X_{k_j-1}, X_{k_j}, X_{k_j+1})$, $1 \le j \le m$, we can vary X_{k_j} independently for the different k_j, $1 \le j \le m$. Suppose for some real h it is possible to achieve

$$\sum_{j=1}^{m} (X''_{k_j} - X'_{k_j}) = h$$

with large probability. Then we can obtain (5.1.4). To see this, define $X'_j := X_j$ and $X''_j := X_j$, $j \notin \{k_1,\ldots,k_m\}$, and let $S'_{\mathbb{Z}}$ and $S''_{\mathbb{Z}}$ be defined as usual in terms of $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$. We have

$$\begin{aligned} S''_n - S'_n &= \sum_{j=1}^{m} (X''_{k_j} - X'_{k_j}) = h, \quad && n \ge k_m, \\ &= 0 \quad , && n \le 0, \end{aligned}$$

and hence (5.1.4) holds. Similar as in Theorem 1.1.2 we can derive the required limit relation (5.0.1). Thus our problem is how to vary X_{k_j}, $1 \le j \le m$, such that $\sum_{j=1}^m (X''_{k_j} - X'_{k_j}) = h$ with large probability. Note that this is a problem concerning the properties of the sequence of triples $(X_{k_j-1}, X_{k_j}, X_{k_j+1})$, $1 \le j \le m$, so the properties of a sequence of independent random vectors. This problem can be solved by methods such as the Ornstein coupling (see Proposition 5.2.7).

Theorem 5.1.7 formulates the insight above. It indicates the question on sequences of i.i.d. random vectors that has to be solved. The theorem is not only valid for Markov sequences $X_{\mathbb{Z}}$ but it applies also to more general processes. To understand how this more general result is obtained, we argue as follows. Instead of a sequence of integers $k_1,\dots,k_m$ we consider a sequence of finite integer sets $K_1,\dots,K_m$. We want to vary the random vectors X_{K_j}, $1 \le j \le m$. To this purpose we have to overcome the following problems:

(i) Let $X'_n := X_n$, $X''_n := X_n$, $n \notin \bigcup_j K_j$. How can we guarantee that $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$ are distributed as $X_{\mathbb{Z}}$?

(ii) Can we vary X_{K_j}, $1 \le j \le m$, independently for the different K_j, $1 \le j \le m$?

The first question is related to (5.1.5). Above we settled these questions by using the asymptotic independence assumption (5.1.2) and the Markov property. If the argument above is traced back one will observe that we only used the following consequence of the Markov property:

$$P_{X_{K_j} | X_{L_j \setminus K_j}, X_{L_j^c}} = P_{X_{K_j} | X_{L_j \setminus K_j}},$$

where $K_j := \{k_j\}$ and $L_j \setminus K_j = \{k_j-1, k_j+1\}$. The idea is to replace the Markov property by condition (5.1.3). This condition requires that if the sets $L_j \supset K_j$ are chosen large, then X_{K_j} and $X_{L_j^c}$ are approximately independent, given $X_{L_j \setminus K_j}$. We can again give a positive answer to questions (i) and (ii) above. So we can vary the X_{K_j}, independently for the different sets K_j, $1 \le j \le m$, while $X_{L_j \setminus K_j}$, $1 \le j \le m$, remain fixed. Again our problem is, how to vary X_{K_j}, $1 \le j \le m$, such that $\sum_{j=1}^m (X''_{K_j} - X'_{K_j}) = h$ with large probability. To this purpose we have to investigate the sequence X_{L_j}, $1 \le j \le m$, of (approximately) independent random vectors.

The idea above is formulated in Theorem 5.1.7. This theorem only requires the conditions (5.1.2) and (5.1.3). Condition (5.1.3) is obviously valid for Markov sequences $X_{\mathbb{Z}}$. In Proposition 5.1.2 we show that it is also satisfied for countably valued sequences $X_{\mathbb{Z}}$.

PROPOSITION 5.1.1. *Let $X_{\mathbb{Z}}$ be a sequence of random variables with values in a Borel space Γ. Define for finite integer sets* $K \subset L$

$$f_K(L) := E \perp_{X_{L \setminus K}} (X_K, X_{L^c}).$$

We have

$$f_K(L') \le 2\, f_K(L) \quad \textit{if } K \subset L \subset L',$$

$$f_K(L) \le 2\, f_{K'}(L) \quad \textit{if } K \subset K' \subset L.$$

PROOF. Observe that if $L \subset L'$ then, given $X_{L'\backslash K}$, the random vectors X_{L^c} and $X_{L'^c}$ mutually determine each other. Hence

$$\perp_{X_{L'\backslash K}}(X_K, X_{L'^c}) = \perp_{X_{L'\backslash K}}(X_K, X_{L^c}).$$

Take expectations and apply Proposition 4.1.3 to get

$$f_K(L') = E\perp_{X_{L'\backslash K}}(X_K, X_{L^c}) \leq 2\, E\perp_{X_{L\backslash K}}(X_K, X_{L^c}) = 2f_K(L).$$

Similarly we have

$$f_K(L) = E\perp_{X_{L\backslash K}}(X_{K'}, X_{L^c}) \leq 2f_{K'}(L). \qquad \square$$

NOTE. As a consequence of the first inequality of the proposition we have

$$\lim_{L\to\mathbb{Z}} f_K(L) = 0 \quad \text{or else} \quad \liminf_{L\supset K} f_K(L) > 0.$$

Hence (5.1.3) is equivalent with the validity of

$$\lim_{\ell\to\infty} E\perp_{X_{L\backslash K}}(X_K, X_{L^c}) = 0$$

with L of the form $L := \{-\ell,\ldots,\ell\}$, for any finite integer set K. Using the second inequality it follows that we only have to require this limit relation for sets K of the form $K := \{-k,\ldots,k\}$, $k \geq 1$. If $X_{\mathbb{Z}}$ is stationary we can restrict to requiring the limit relation above for sets K of the form $K := \{1,\ldots,k\}$, $k \geq 1$.

PROPOSITION 5.1.2. *Let $X_{\mathbb{Z}}$ be a sequence of random variables with values in a Borel space Γ. If $X_{\mathbb{Z}}$ is a Markov dependent sequence, or if $X_{\mathbb{Z}}$ is a countably valued sequence, then* (5.1.3) *is satisfied.*

PROOF. First suppose that $X_{\mathbb{Z}}$ is Markov dependent. Take $K := \{-k,\ldots,k\}$ and let $L \supset \{-k-1,\ldots,k+1\}$. Observe that

$$f_K(L) = E\perp_{X_{L\backslash K}}(X_K, X_{L^c}) = 0.$$

This proves (5.1.3).

Let us now consider the countably valued case. Assume Γ is countable. It is sufficient to prove

$$\lim_{n\to\infty} f_K(L_n) = 0 \tag{5.1.6}$$

with $L_n := \{-n,\ldots,n\}$, $n \geq 1$, for any finite integer set K. Suppose that

X_K has its values in the countable set $\{\gamma_1,\gamma_2,\ldots\}$. By Proposition 4.1.1 and because $K^c = L^c \cup (L\setminus K)$ we have

$$f_K(L) = E\| (P_{X_K|X_{L\setminus K}} - P_{X_K|X_{K^c}})^+\|$$

$$= \tfrac{1}{2} \sum_{k\geq 1} E|P(X_K=\gamma_k|X_{L\setminus K}) - P(X_K=\gamma_k|X_{K^c})|.$$

By a martingale theorem (see BREIMAN [Theorem 5.21]) we have

$$\lim_{n\to\infty} E|P(X_K=\gamma_k|X_{L_n\setminus K}) - P(X_K=\gamma_k|X_{K^c})| = 0,$$

with L_n as defined above. The expression above is dominated by $2\,P(X_K=\gamma_k)$. Sum over k and apply the dominated convergence theorem to exchange limit and summation. We obtain (5.1.6). □

The process $X_{\mathbb{Z}}$ in the example below does not satisfy (5.1.3).

EXAMPLE 5.1.3. Let ζ be a random variable, homogeneously distributed on $[0,1)$ and let $\alpha \in [0,1)$ be some irrational number. Define a stationary sequence $Y_{\mathbb{Z}}$ by

$$Y_n := 1 \quad \text{if } (\zeta+n\alpha) \bmod 1 \in [0,\tfrac{1}{2}),$$
$$:= 0 \quad \text{else.}$$

An important property of this process is that ζ is determined by $(Y_k)_{k<0}$, up to a null set. Hence the process $Y_{\mathbb{Z}}$ is *deterministic,* in the sense that Y_{n+1} is, up to a null set, $(Y_k)_{k\leq n}$-measurable. In particular

$$P_{Y_0|(Y_k)_{k<0}} = \delta_{\{Y_0\}} \text{ a.s.,}$$

with $\delta_{\{Y_0\}}$ the probability measure degenerate at $\{Y_0\}$. Because α is irrational, there is for each set $L := \{-\ell,\ldots,\ell\}$ some γ such that

$$p := P(Y_0=1 \mid Y_{L\setminus\{0\}}=\gamma)$$

satisfies $0 < p < 1$, where $\{Y_{L\setminus\{0\}}=\gamma\}$ has positive probability. Because $Y_{\mathbb{Z}}$ is a countably valued sequence, this process satisfies (5.1.3) by the proposition above.

Let $X^i_{\mathbb{Z}}$, $i \geq 1$, be a sequence of independent processes, distributed as $Y_{\mathbb{Z}}$. Define $X_n := (X^i_n)_{i\geq 1}$, $n \in \mathbb{Z}$. We show that $X_{\mathbb{Z}}$ does not satisfy (5.1.3). First note that if μ_i and ν_i, $i \geq 1$, are probability measures on $\{0,1\}$, with the measures μ_i degenerate at $\{k_i\}$, $i \geq 1$, then the product measures satisfy

$$\|\prod_{i\geq 1} \mu_i - \prod_{i\geq 1} \nu_i\| = 2\|(\prod_{i\geq 1} \mu_i - \prod_{i\geq 1} \nu_i)^+\|$$

$$\geq 2(\prod_{i\geq 1} \mu_i(\{k_i\}) - \prod_{i\geq 1} \nu_i(\{k_i\}))$$

$$= 2(1 - \prod_{i\geq 1} \nu_i(\{k_i\})).$$

Let L be a finite integer set containing $\{0\}$ and consider the expression

$$\|P_{X_0|X_{\{0\}^c}} - P_{X_0|X_{L\setminus\{0\}}}\| = \|\prod_{i\geq 1} \delta_{X_0^i} - \prod_{i\geq 1} P_{X_0^i|X^i_{L\setminus\{0\}}}\|.$$

Observe that $X^i_{L\setminus\{0\}} \equiv \gamma$ for infinitely many $i \geq 1$ a.s. Hence

$$\|P_{X_0|X_{\{0\}^c}} - P_{X_0|X_{L\setminus\{0\}}}\| \geq 2(1 - [p\vee(1-p)]^\infty) = 2 \text{ a.s.}$$

and therefore, using Proposition 4.1.1

$$\perp_{X_{L\setminus\{0\}}}(X_0, X_{L^c}) = 2$$

for all finite integer sets $L \supset \{0\}$. Thus (5.1.3) is violated for $X_{\mathbb{Z}}$.

As a first step in the investigation of the dependence structure of $X_{\mathbb{Z}}$ we prove the following lemma. It shows that we can replace sections of $X_{\mathbb{Z}}$ by new, mutually independent sections, such that the process changes only slightly.

LEMMA 5.1.4. *Let $X_{\mathbb{Z}}$ be a sequence of random variables with values in a Borel space Γ such that (5.1.2) is satisfied. Let there be given a finite integer set L^* and a positive integer m. For each $\varepsilon > 0$ there exists a Γ-valued process $\tilde{X}_{\mathbb{Z}}$ for which $\delta(X_{\mathbb{Z}}, \tilde{X}_{\mathbb{Z}}) < \varepsilon$ such that the following requirements are satisfied. There are integer sets $L_j := k_j + L^*$, $1 \leq j \leq m$, with $k_1 < \ldots < k_m$, such that $\tilde{X}_{L_j}$, $1 \leq j \leq m$, are m independent random vectors distributed as X_{L_j}, $1 \leq j \leq m$.*

PROOF. We prove the assertion above by induction on m. The process $\tilde{X}_{\mathbb{Z}}$ will be defined on the same probability space as $X_{\mathbb{Z}}$. Furthermore, we show that $\tilde{X}_{\mathbb{Z}}$ satisfies

(5.1.7) $\quad \perp(Z_1,(Z_2,X_L)) = \perp(Z_1,(Z_2,X_L,\tilde{X}_L)),$

where $L := \bigcup_{j=1}^{m} L_j$ and Z_1 and Z_2 are arbitrary, $X_{\mathbb{Z}}$-measurable random variables. Remark that (5.1.7) expresses that the dependence structure is not affected by the introduction of the new random variables. The assertion is trivial for m = 1: Take k_1 arbitrary and $\tilde{X}_{\mathbb{Z}} := X_{\mathbb{Z}}$. Now suppose that the assertion of the proposition and (5.1.7) hold for $m \geq 1$. We have numbers $k_1 < \ldots < k_m$ and a process $\tilde{X}_{\mathbb{Z}}$ as mentioned in the assertion of the proposition, such that (5.1.7) holds. Choose k_{m+1} so large that with $L_{m+1} := k_{m+1} + L^*$

$$\perp(X_{L_{m+1}},X_L) < \varepsilon - P(X_{\mathbb{Z}} \neq \tilde{X}_{\mathbb{Z}}),$$

where $L = \bigcup_{j=1}^{m} L_j$. Using Corollary 4.2.5 we can extend the probability space with a vector $\bar{X}_{L_{j+1}}$, distributed as $X_{L_{j+1}}$ and independent of $(X_L,\tilde{X}_L)$ such that

$$P(X_{L_{m+1}} \neq \bar{X}_{L_{m+1}}) = \perp(X_{L_{m+1}},(X_L,\tilde{X}_L)).$$

By (5.1.7) the right-hand side equals $\perp(X_{L_{m+1}},X_L)$. Let $\bar{X}$ be coupled with $\tilde{X}$ over L^c_{m+1}. This defines $\bar{X}_{\mathbb{Z}}$. Clearly $\bar{X}_{L_{m+1}}$ is independent of $\bar{X}_L$, so $\bar{X}_{L_j}$, $1 \leq j \leq m+1$, are m+1 independent random vectors. Note that

$$P(X_{\mathbb{Z}} \neq \bar{X}_{\mathbb{Z}}) \leq P(X_{\mathbb{Z}} \neq \tilde{X}_{\mathbb{Z}}) + P(X_{L_{m+1}} \neq \bar{X}_{L_{m+1}}) < \varepsilon.$$

By the note to Corollary 4.2.5 and by (5.1.7)

$$\begin{aligned}\perp(Z_1,(Z_2,X_{L\cup L_{m+1}},\bar{X}_{L\cup L_{m+1}})) &= \perp(Z_1,(Z_2,X_{L\cup L_{m+1}},\bar{X}_L)) \\ &= \perp(Z_1,(Z_2,X_{L\cup L_{m+1}})).\end{aligned}$$

This shows that the induction statement holds for m + 1. □

<u>NOTE</u>. Let $(\Omega,\mathcal{A},P)$ be a probability space and T a measure preserving mapping from Ω onto itself. The mapping T is called *mixing* if for $A_1,A_2 \in \mathcal{A}$

$$\lim_{n\to\infty} P(A_1 \cap T^{-n}A_2) = P(A_1)P(A_2).$$

Suppose $X_{\mathbb{Z}}$ is a stationary process with values in a countable space Γ. Let the σ-field $\mathcal{T}$ consist of all subsets of Γ. Consider the probability space

$(\Gamma^{\mathbb{Z}}, \mathcal{T}^{\mathbb{Z}}, P_{X_{\mathbb{Z}}})$ and take T to be the shift transformation on $\Gamma^{\mathbb{Z}}$. By using that each set $A \in \mathcal{T}^{\mathbb{Z}}$ can be approximated arbitrarily well by cilinder sets, one observes that T is mixing if and only if (5.1.2) holds.

The following theorem is the most detailed result in this section on the dependence structure of $X_{\mathbb{Z}}$. It proves the existence of a "nice" approximation $\tilde{X}_{\mathbb{Z}}$ of $X_{\mathbb{Z}}$ and it is fundamental to the coupling construction in Theorem 5.1.7.

THEOREM 5.1.5. *Let $X_{\mathbb{Z}}$ be a stationary sequence of random variables with values in a Borel space Γ, such that* (5.1.2) *and* (5.1.3) *are satisfied. Let there be given a finite integer set K^* and a positive integer m. For each $\varepsilon > 0$ there exists a Γ-valued process $\tilde{X}_{\mathbb{Z}}$ for which $\delta(X_{\mathbb{Z}}, \tilde{X}_{\mathbb{Z}}) < \varepsilon$, such that the following requirements are satisfied. There is an integer set $L^* \supset K^*$ and a sequence of integers $k_1 < \ldots < k_m$ such that, with the notations*

$$L_j := k_j + L^*, \quad K_j := k_j + K^*, \quad 1 \le j \le m,$$
$$L := \bigcup_{j=1}^{m} L_j, \quad K := \bigcup_{j=1}^{m} K_j,$$

the process $\tilde{X}_{\mathbb{Z}}$ satisfies:

(i) $\tilde{X}_{L_j}$, $1 \le j \le m$, *are independent random vectors distributed as X_{L_j}*, $1 \le j \le m$;

(ii) *given $\tilde{X}_{L\setminus K}$ the random vectors $\tilde{X}_{L^c}, \tilde{X}_{K_1}, \ldots, \tilde{X}_{K_m}$ are independent.*

PROOF. Select a finite integer set $L^* \supset K^*$ such that

$$\text{(5.1.8)} \qquad E \perp_{X_{L^*\setminus K^*}} (X_{K^*}, X_{(L^*)^c}) < \frac{\varepsilon}{m}.$$

By Lemma 5.1.4 there exist integers $k_1 < \ldots < k_m$ such that for a process $\bar{X}_{\mathbb{Z}}$ holds $\delta(X_{\mathbb{Z}}, \bar{X}_{\mathbb{Z}}) < \frac{1}{2}\varepsilon$ while, with the notations used above, $\bar{X}_{L_j}$, $1 \le j \le m$, are independent random vectors, distributed as X_{L_j}. Define $\tilde{X}$ to be coupled with $\bar{X}$ over K^c and construct $\tilde{X}_K$, independent of $\tilde{X}_{L^c}$, given $\tilde{X}_{L\setminus K}$, such that

$$\text{(5.1.9)} \qquad P_{\tilde{X}_{K_1}, \ldots, \tilde{X}_{K_m} | \tilde{X}_{L\setminus K}} = \prod_{j=1}^{m} P_{\bar{X}_{K_j} | \bar{X}_{L_j\setminus K_j}}.$$

This can be done by using Lemma 4.2.4. It follows that, given $\tilde{X}_{L\setminus K}$, the random vectors $\tilde{X}_{K_1}, \ldots, \tilde{X}_{K_m}, \tilde{X}_{L^c}$ are independent. Because $\bar{X}_{L_j\setminus K_j} = \tilde{X}_{L_j\setminus K_j}$,

$1 \le j \le m$, are independent random vectors, it follows easily from (5.1.9) that $\tilde{X}_{L_j} = \tilde{X}_{K_j \cup (L_j \setminus K_j)}$, $1 \le j \le m$, are independent random vectors. By applying Proposition 4.1.4 inductively we shall prove $\delta(X_{\mathbb{Z}}, \tilde{X}_{\mathbb{Z}}) < \varepsilon$. Use

$$P_{\tilde{X}_K | \tilde{X}_{K^c}} = P_{\tilde{X}_K | \tilde{X}_{L \setminus K}}$$

and (5.1.8) to obtain that for any j with $1 \le j \le m$

$$P_{\tilde{X}_{K_j} | \tilde{X}_{(L\setminus K) \cup K_1 \cup \ldots \cup K_{j-1}}} = P_{\tilde{X}_{K_j} | \tilde{X}_{L_j \setminus K_j}} .$$

Hence by Proposition 4.1.4

$$\delta(X_{K^c \cup K_1 \cup \ldots \cup K_j}, \tilde{X}_{K^c \cup K_1 \cup \ldots \cup K_j})$$

$$\le E \perp_{X_{L_j \setminus K_j}} (X_{K_j}, X_{K^c \cup K_1 \cup \ldots \cup K_{j-1}}) + \delta(X_{L_j \setminus K_j}, \tilde{X}_{L_j \setminus K_j})$$

$$+ \delta(X_{K^c \cup K_1 \ldots \cup K_{j-1}}, \tilde{X}_{K^c \cup K_1 \cup \ldots \cup K_{j-1}}) + \delta(X_{L_j}, \tilde{X}_{L_j}) .$$

Because $X_{L_j} \stackrel{d}{=} \tilde{X}_{L_j}$ the second and fourth terms vanish. The first term on the right is less than $\frac{\varepsilon}{m}$ by (5.1.8). Hence by an inductive application of this inequality we obtain

$$\delta(X_{\mathbb{Z}}, \tilde{X}_{\mathbb{Z}}) \le m \cdot \frac{\varepsilon}{m} + \delta(X_{K^c}, \tilde{X}_{K^c}) < \varepsilon . \qquad \square$$

The process $\tilde{X}_{\mathbb{Z}}$ constructed in the theorem above, satisfying conditions (i) and (ii) in the theorem, will be called the *window-frame process*, with *windows* K_j and *frames* $L_j \setminus K_j$. This terminology can be explained as follows: In Theorem 5.1.7 below we "vary" the values on the windows, while the values on the frames remain fixed (see the introduction of this section).

Let us first introduce some notations. Suppose there is given a distribution F on $\mathbb{R}^1 \times \Gamma_0$ with Γ_0 a Borel space. Let $(T,Y) := ((T_j,Y_j))_{j=1}^m$ be a random vector consisting of m independent, F-distributed increments. We can formulate limit properties of the random vector (T,Y) for $m \to \infty$ by means of a class $\mathcal{G}_F^m$ of distributions on the real line. This class is defined as follows. Let (T',Y') and (T",Y") be random vectors, marginally distributed as (T,Y), such that Y' = Y" while T' and T" are arbitrary. Let $\mathcal{G}_F^m$ be the class of distributions G of the partial sums

$$\sum_{j=1}^{m} \Delta_j, \quad \text{where } \Delta_j := T''_j - T'_j, \quad 1 \le j \le m,$$

for all random vectors (T',Y') and (T",Y") that satisfy the conditions above.

The following example shows how this class of distributions can be used to obtain limit properties.

EXAMPLE 5.1.6. Let $S_m := T_1 + \ldots + T_m$ and define S'_m and S''_m similarly in terms of T' and T". If the distribution G is defined as above then by Lemma 1.1.1

$$\delta(S_m, S_m - h) \le P(S'_m \neq S''_m - h) = P(\Delta_1 + \ldots + \Delta_m \neq h) = 1 - G(\{h\}).$$

Hence if for any $\varepsilon > 0$ the sets $\mathcal{G}_F^m$ contain for all sufficiently large m elements G with $G(\{h\}) > 1-\varepsilon$, then we have

$$\lim_{m\to\infty} \| P_{S_m} - P_{S_m - h} \| = 0.$$

We even have something more. Because $Y'_m = Y''_m$ we have by Lemma 1.1.1

$$\delta((S_m, Y_m), (S_m - h, Y_m)) \le 1 - P(S'_m = S''_m - h, Y'_m = Y''_m) = 1 - G(\{h\}).$$

So for the simultaneous distribution of (S_m, Y_m) we have

$$\lim_{m\to\infty} \| P_{S_m, Y_m} - P_{S_m - h, Y_m} \| = 0.$$

The convergence problems in this chapter will be translated into questions concerning the elements of $\mathcal{G}_F^m$. For this reformulation the following theorem is fundamental.

THEOREM 5.1.7. *Let* $\tilde{X}_{\mathbb{Z}}$ *be the window-frame process constructed in Theorem 5.1.5. Suppose* f *is a measurable function and let* F *be the common distribution of the independent random vectors*

$$(\tilde{S}^{K_j}, \tilde{X}_{L_j \setminus K_j}), \quad 1 \le j \le m,$$

where $\tilde{S}^{K_j}$ *is defined using the notation* (5.1.1). *If* $G \in \mathcal{G}_F^m$ *then there is a probability space with processes* $X'_{\mathbb{Z}}$ *and* $X''_{\mathbb{Z}}$, *marginally distributed as* $\tilde{X}_{\mathbb{Z}}$ *and coupled over the complement of* $K := K_1 \cup \ldots \cup K_m$, *such that* G *is the distribution of* $S''^K - S'^K$.

PROOF. Let $\tilde{X}_{\mathbb{Z}}$ be constructed as in Theorem 5.1.5. Define for $n \notin K$

$$(5.1.10) \qquad X'_n := \tilde{X}_n, \qquad X''_n := \tilde{X}_n.$$

The random vector (T,Y) defined by

$$(T,Y) := ((\tilde{S}^{K_j}, \tilde{X}_{L_j \setminus K_j}))_{j=1}^{m}$$

consists of m independent, F-distributed components by property (i) of Theorem 5.1.5. If $G \in \mathcal{G}_F^m$ there are random vectors (T',Y') and (T",Y"), distributed as (T,Y), with Y' = Y", such that G is the distribution of $\sum_{j=1}^{m} (T''_j - T'_j)$. By Lemma 4.2.4 the probability space can be extended with a random vector (S',S") that is, given Y, independent of the other random variables, such that (S',S",Y) is distributed as (T',T",Y'). By respectively this conditional independence and by (ii) in Theorem 5.1.5 we have the equalities

$$P_{S' | \tilde{X}_{L\setminus K}, \tilde{X}_{L^c}} = P_{T|Y} = P_{\tilde{S} | \tilde{X}_{L\setminus K}, \tilde{X}_{L^c}},$$

where $\tilde{S} := T$. It follows that $(\tilde{S}, \tilde{X}_{K^c})$ is distributed as (S', X'_{K^c}) and as (S'', X''_{K^c}). Construct (X'_K, X''_K) such that

$$(5.1.11) \qquad P_{X'_K, X''_K | S'=s', S''=s'', \tilde{X}_{K^c}=x} = P_{\tilde{X}_K | \tilde{S}=s', \tilde{X}_{K^c}=x} \times P_{\tilde{X}_K | \tilde{S}=s'', \tilde{X}_{K^c}=x}.$$

Then both $(X'_K, S', \tilde{X}_{K^c})$ and $(X''_K, S'', \tilde{X}_{K^c})$ are distributed as $(\tilde{X}_K, \tilde{S}, \tilde{X}_{K^c})$ and by (5.1.10), both $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$ are distributed as $\tilde{X}_{\mathbb{Z}}$. Because by (5.1.11) the j-th components of respectively S' and S" coincide a.s. with S'^{K_j} and S''^{K_j} and (S',S") is distributed as (T',T"), it follows that $S''^{K} - S'^{K}$ has distribution G. □

Theorem 5.1.7 can be used to obtain (5.0.1) by means of the following argument. Suppose we succeed to show $\mathcal{G}_F^m$, $m \geq 1$, has the property mentioned in Example 5.1.6., i.e. for each h there exists, for m sufficiently large, an element $G \in \mathcal{G}_F^m$ such that G({h}) is close to 1. Using Theorem 5.1.7 we can argue as follows. If n > sup K then

$$(5.1.12) \qquad S''_n - S'_n = S''^{K} - S'^{K}$$

has distribution G. As in Example 5.1.6 we conclude that

$$\lim_{m\to\infty} \| P_{\tilde{S}_m} - P_{\tilde{S}_m+h} \| = 0$$

and because $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$ is approximately distributed as $X_{\mathbb{Z}}$, we obtain, using (5.1.12)

$$\lim_{n\to\infty} \| P_{S_n} - P_{S_n+h} \| = 0.$$

Apparently we reduced the study of the limit relation above to the investigation of $\mathcal{G}_F^m$, so to the investigation of sequences of i.i.d. random vectors.

The class of probability distributions $\mathcal{G}_F^m$ has some useful and easily verified properties.

LEMMA 5.1.8. *$\mathcal{G}_F^m$ is increasing in* $m \ge 1$.

PROOF. Let $G \in \mathcal{G}_F^m$. Suppose (T'_j, Y_j), $1 \le j \le m$, and (T''_j, Y_j), $1 \le j \le m$, are both sequences of m independent, F-distributed random variables, such that $G \in \mathcal{G}_F^m$ is the distribution of

$$\sum_{j=1}^{m} (T''_j - T'_j).$$

Let (T_{m+1}, Y_{m+1}) be F-distributed and independent of the aforementioned random variables. Define $T'_{m+1} := T_{m+1}$ and $T''_{m+1} := T_{m+1}$. Then G is the distribution of

$$\sum_{j=1}^{m} (T''_j - T'_j) = \sum_{j=1}^{m+1} (T''_j - T'_j),$$

so $G \in \mathcal{G}_F^{m+1}$. □

LEMMA 5.1.9. *Let* F *and* $\bar{F}$ *be distributions of pairs of random variables* (T_1, Y_1) *and* $(T_1, \bar{Y}_1)$, *where* T_1 *is real and* Y_1 *and* $\bar{Y}_1$ *have their values in Borel spaces. If* $\bar{Y}_1$ *is* Y_1*-measurable, then*

$$\mathcal{G}_F^m \subset \mathcal{G}_{\bar{F}}^m, \quad m \ge 1.$$

PROOF. Suppose $G \in \mathcal{G}_F^m$. Then G is the distribution of $\sum_{j=1}^m T''_j - T'_j$, where both

$$(T'_j, Y_j), \quad 1 \le j \le m, \quad \text{and} \quad (T''_j, Y_j), \quad 1 \le j \le m,$$

are sequences of m independent, F-distributed pairs of random variables. There is a measurable function f such that $\bar{Y}_1 = f(Y_1)$. Let $\bar{Y}_j := f(Y_j)$. Then both

$$(T'_j, \bar{Y}_j),\quad 1 \le j \le m, \quad \text{and} \quad (T''_j, \bar{Y}_j),\quad 1 \le j \le m,$$

are independent, $\bar{F}$-distributed random variables. Hence $G \in \mathcal{G}^m_{\bar{F}}$. □

5.2. LOSS OF MEMORY IN A STRONGLY NONLATTICE RANDOM WALK

Suppose $S_{\mathbb{Z}}$ is a random walk controlled by a stationary sequence $X_{\mathbb{Z}}$. In this section we study limit relations such as

$$(5.2.1) \qquad \lim_{n\to\infty} \| \nu * P_{S_n} - \nu * P_{S_n + h} \| = 0$$

for all absolutely continuous probability measures ν on the real line. We have seen in Chapter 1 that if the increments are independent, then necessary and sufficient for the validity of this limit relation is that the distribution of the increments is strongly nonlattice. It will be possible to give an analogue of this result for random walks $S_{\mathbb{Z}}$, controlled by a stationary sequence $X_{\mathbb{Z}}$.

To this purpose we need a generalization of the concept strongly nonlattice. For a definition of nonlattice the reader is referred to Section 6.2. Define the *lattice* L_d with *lattice width* d by (1.1.2) and define (c) mod d for real c as the smallest nonnegative element of $c + L_d$. Let $S_{\mathbb{Z}}$ be a random walk controlled by a stationary sequence $X_{\mathbb{Z}}$. The random walk $S_{\mathbb{Z}}$ is called *weakly lattice* with respect to $X_{\mathbb{Z}}$ if there is a number $d \in (0,\infty]$ such that for all $n \ge 1$ the random variable $C_n := (S_n) \bmod d$ is, up to a null set, $X_{K_n^c}$-measurable, where $K_n := \{1,\dots,n\}$. Equivalently we may require for $d = \infty$ that S_n is, up to a null set, $X_{K_n^c}$-measurable for $n \ge 1$, and for $d \in (0,\infty)$ that $e^{2\pi i\, S_n/d}$ is, up to a null set, $X_{K_n^c}$-measurable for $n \ge 1$. The random walk $S_{\mathbb{Z}}$ is called *strongly nonlattice* with respect to $X_{\mathbb{Z}}$ if the weakly lattice condition is not satisfied.

EXAMPLE 5.2.1.

(i) Let $S_{\mathbb{Z}}$ be a random walk controlled by its own process of increments $X_{\mathbb{Z}}$ and suppose that $X_{\mathbb{Z}}$ is a sequence of independent, F-distributed random variables. If the weakly lattice condition above is satisfied for some $d \in (0,\infty]$ then the random variable $C_n := (S_n) \bmod d$ is both X_{K_n}- and $X_{K_n^c}$-measurable, up to a null set. Hence C_n is a constant with probability 1. A distribution F is called weakly lattice if F is concentrated on a set $c + L_d$, $d > 0$, or, equivalently, if for all $n \geq 1$ the convolutions F^{n*} are concentrated on $nc + L_d$, $d > 0$. It is now easy to see that $S_{\mathbb{Z}}$ is weakly lattice (with respect to its process of increments) if and only if F is weakly lattice.

(ii) Let $X_{\mathbb{Z}}$ be *deterministic*, i.e. each random variable X_{n+1} is, up to a null set, $X_{\mathbb{N}^c+n}$-measurable. Clearly S_n is, up to a null set, $X_{\mathbb{N}^c}$-measurable, so is certainly $X_{K_n^c}$-measurable, $n \geq 1$. So each random walk controlled by $X_{\mathbb{Z}}$ is weakly lattice with $d = \infty$.

(iii) Suppose $X_{\mathbb{Z}}$ is Markov dependent and has its values in a Borel space. Note that S_n, given (X_0,X_{n+1}), is independent of $X_{K_n^c}$. Hence $S_{\mathbb{Z}}$ is weakly lattice with respect to $X_{\mathbb{Z}}$, if for some $d \in (0,\infty]$ the random variable $(S_n) \bmod d$ is, up to a null set, (X_0,X_{n+1})-measurable. Suppose $X_{\mathbb{Z}}$ is stationary, real valued, such that its transition probability is given by

$$P(x,B) = \int_B f(x,y)\,dy, \quad B \in \mathcal{B}^1,$$

with f strictly positive and measurable on $\mathbb{R}^2$. Let $S_{\mathbb{Z}}$ be a random walk with increments $X_{\mathbb{Z}}$. With these definitions the conditional distribution of S_1, given (X_0,X_2), is with probability 1 absolutely continuous with respect to the Lebesgue measure. Clearly then $S_{\mathbb{Z}}$ is strongly nonlattice with respect to $X_{\mathbb{Z}}$.

Before stating the main result of this section, Theorem 5.2.3, we investigate the concept strongly nonlattice. The following lemma is helpful.

LEMMA 5.2.2. *Let* (T,Y) *be an* $\mathbb{R}^1 \times \Gamma_0$*-valued random variable, with* Γ_0 *a Borel space. Then the set* $\mathcal{D}$ *of all* $d \in [0,\infty]$ *such that* $(T) \bmod d$ *is, up to a null set,* Y*-measurable, has a largest element* $\bar{d}$.

PROOF. Consider a regular conditional distribution $P_{T|Y}$ and let $c(Y) + L_{d(Y)}$ be the smallest displaced lattice on which this conditional distribution is concentrated. If $(T) \bmod d$ is, up to a null set, Y-measurable then $P_{T|Y}$ is concentrated on a set $c'(Y)+L_d \supset c(Y)+L_{d(Y)}$ a.s. Therefore the set $\mathcal{D}$ has the form

$$\mathcal{D} = \{d \in [0,\infty]: L_d \supset L_{d(Y)} \text{ a.s.}\}.$$

If $\mathcal{D} \cap (0,\infty)$ contains two mutually prime numbers d' and d", then $\mathcal{D} = [0,\infty]$. Else if $\mathcal{D} \cap (0,\infty)$ is not empty, then $\mathcal{D}$ has a largest element $\bar{d} \in (0,\infty)$ and $d \in \mathcal{D}$ vanishes or divides $\bar{d}$. In case $\mathcal{D} \cap (0,\infty)$ is empty we have $\mathcal{D} = \{0\}$. The assertion of the lemma follows. □

<u>NOTE</u>. The set $\mathcal{D}$ has the form $\{d \in [0,\infty]: L_d \supset L_{\bar{d}}\}$.

Suppose K is a finite integer set and let S^K be defined by (5.1.1). Let d_K be the largest $d \in [0,\infty]$ such that $(S^K) \bmod d$ is, up to a null set, X_{K^c}-measurable. By the lemma above d_K is properly defined. Suppose $K' \subset K$ is a nonempty integer set and observe that

$$(S^{K'}) \bmod d_K = (S^K - S^{K\setminus K'}) \bmod d_K$$

is, up to a null set, $X_{(K')^c}$-measurable. By the definition of $d_{K'}$ and the note to the lemma above d_K divides $d_{K'}$, so $d_K \le d_{K'}$. Thus we proved the existence of the limit

$$d_\infty := \lim_{K\to\mathbb{Z}} d_K,$$

where the limit is taken along the directed set of finite integer sets K, partially ordered by inclusion. The lattice L_{d_∞} is called the *minimal weak lattice* of $S_{\mathbb{Z}}$ with respect to $X_{\mathbb{Z}}$, while d_∞ is the largest $d \in [0,\infty]$ such that for all $n \ge 1$, $(S_n) \bmod d$ is, up to a null set, $X_{K_n^c}$-measurable, where $K_n := \{1,\ldots,n\}$.

The next part of this section is concerned with a proof of (5.2.1) for strongly nonlattice random walks. The result is given the form of an equivalence between the strongly nonlattice condition and a limit property. Its proof can be found at the end of this section and is a remote application of the Ornstein coupling. A refinement of the limit relation of the theorem is given in the note to Proposition 6.1.5.

<u>THEOREM 5.2.3</u>. *Let* $S_{\mathbb{Z}}$ *be a random walk controlled by a stationary sequence* $X_{\mathbb{Z}}$ *with values in a Borel space. Suppose* $X_{\mathbb{Z}}$ *satisfies* (5.1.2) *and* (5.1.3). *If the random walk* $S_{\mathbb{Z}}$ *is strongly nonlattice with respect to* $X_{\mathbb{Z}}$, *the following limit relation holds. For any absolutely continuous probability measure* ν *on the real line and any real* h

$$\lim_{n\to\infty} E\| \nu * P_{S_n | X_{K_n^c}} - \nu * P_{S_n + h | X_{K_n^c}} \| = 0,$$

where $K_n := \{1,\ldots,n\}$. *Furthermore this limit property implies that* $S_{\mathbb{Z}}$ *is strongly nonlattice with respect to* $X_{\mathbb{Z}}$.

COROLLARY 5.2.4. *If* $X_{\mathbb{Z}}$ *satisfies the assumptions of Theorem* 5.2.3 *and* $S_{\mathbb{Z}}$ *is strongly nonlattice with respect to* $X_{\mathbb{Z}}$, *then for any absolutely continuous probability measure* ν *on the real line and any real* h

$$\lim_{n\to\infty} \| \nu * P_{S_n} - \nu * P_{S_n + h} \| = 0.$$

Consider ν as the distribution of a random variable Z, independent of $X_{\mathbb{Z}}$. Then we can write (for example) $\nu * P_{S_n} = P_{S_n + Z}$. If the limit relations above are rewritten using this notation, then the corollary is a simple consequence of the following lemma.

LEMMA 5.2.5. *Let* T,T' *and* Y *be random variables,* T *and* T' *real and* Y *with values in a Borel space. Then we have*

$$\|P_T - P_{T'}\| \le E\|P_{T|Y} - P_{T'|Y}\|.$$

PROOF. Using (4.1.2) and (4.1.5) we obtain

$$\begin{aligned}\|P_T - P_{T'}\| &\le \|P_{T,Y} - P_{T',Y}\| \\ &= E\|P_{T|Y} - P_{T'|Y}\|.\end{aligned}$$

□

At the end of the preceding section a connection was given between the process $X_{\mathbb{Z}}$ and a class $\mathcal{G}_F^m$ of distributions on the real line, defined in terms of sequences of i.i.d. random vectors. The proposition below uses the Ornstein coupling to study this class $\mathcal{G}_F^m$. In the proof of Theorem 5.2.3 we translate the knowledge of $\mathcal{G}_F^m$ thus obtained, into the limit relation of Theorem 5.2.3.

The following lemma implies that the number d_K, used in the definition of the minimal weak lattice above, can be seen as the minimal lattice width of a certain distribution on the real line.

LEMMA 5.2.6. *Let* (T,Y) *be an* $\mathbb{R}^1 \times \Gamma_0$*-valued random variable with distribution* F, *where* Γ_0 *is a Borel space. Let* $\bar{d}$ *be the largest* $d \in [0,\infty]$ *such that*

$$(T) \bmod d$$

is, up to a null set, Y*-measurable. Then* $\bar{d}$ *is the minimal lattice width of a distribution* H *on the real line, given by*

$$H(B) := E\, P_{T|Y} * P_{-T|Y}(B), \qquad B \in \mathcal{B}^1.$$

PROOF. Take some $d \in (0,\infty)$. Define the characteristic function

$$\phi_{T|Y}(u) = E(e^{iuT}|Y), \qquad u \in \mathbb{R}^1.$$

The assertion that (T) mod d is, up to a null set, Y-measurable, is equivalent with

$$|\phi_{T|Y}(2\pi d)| = 1 \text{ a.s.}$$

On the other hand H is concentrated on L_d if and only if

$$\phi_{T|Y}(2\pi d)\bar{\phi}_{T|Y}(2\pi d) = 1 \text{ a.s.}$$

These two equivalences imply the assertion of the lemma. □

The core in the proof of the limit relation of Theorem 5.2.3 is formed by the proposition below. The proof uses the Ornstein coupling for random walks.

PROPOSITION 5.2.7. *Let* (T,Y) *be* F*-distributed and define* $\bar{d}$ *as in the proposition above. Suppose an open interval* I *contains an element of* $L_{\bar{d}}$*. Let* ε *be arbitrary positive. If* m *is large enough the set* G_F^m *contains an element* G *such that*

$$G(I) > 1-\varepsilon.$$

PROOF. First suppose that T has finite expectation. At the end of the proof we indicate what has to be changed in case $E|T| = \infty$.

Construct a sequence of independent random vectors (T_j', T_j'', Y_j), $j \geq 1$, with distribution

$$P_{T',T'',Y}(B_1\times B_2\times C) = \int_C P_{T|Y=y}(B_1)P_{T|Y=y}(B_2)\,dP_Y(y).$$

Note that both (T',Y) and (T",Y) have distribution F and that T' and T" are independent, given Y. We investigate the random walk

$$Z_n := \sum_{j=1}^{n} \Delta_j, \quad n \geq 1, \quad \text{with} \quad \Delta_j := T''_j - T'_j, \quad j \geq 1.$$

This random walk has independent increments with expectation 0. By Lemma 5.2.6 the distribution H of Δ_j has minimal lattice $L_{\bar{d}}$. Let the random time τ be the first $n \geq 1$ for which $Z''_n - Z'_n \in I$. By the Chung Fuchs theorem (see BREIMAN [Section 3.7]) τ is finite with probability 1. Define

$$\begin{aligned} T^*_j &:= T''_j \quad 0 \leq j \leq \tau, \\ &:= T'_j \quad j > \tau \quad . \end{aligned}$$

By using that τ is a Markov time for the sequence (T'_j, T''_j, Y_j), $j \geq 1$, it follows that (T^*_j, Y_j), $j \geq 1$, is also a sequence of independent, F-distributed random variables. The distribution G_m of $Z^*_m - Z'_m = \sum_{j=1}^m (T^*_j - T'_j)$ is by definition an element of $\mathcal{G}^m_F$ and because

$$G_m(I) = P(\tau \leq m) \to 1$$

for $m \to \infty$, the assertion of the proposition follows.

If $E|T| = \infty$, we choose the distribution of the independent sequence of random vectors (T'_j, T''_j, Y_j), $j \geq 1$, slightly different: We use a truncation. Let c be some positive number, to be specified later and define

$$Q_{T|Y=y}(\cdot) := P_{T|Y=y}(\cdot \cap (-c,c)),$$

$$\bar{Q}_{T|Y=y} := P_{T|Y=y} - Q_{T|Y=y}$$

and let (T'_j, T''_j, Y_j), $j \geq 1$, have distribution

$$P_{T',T'',Y}(B_1 \times B_2 \times C) =$$

$$\int_C (Q_{T|Y=y}(B_1) Q_{T|Y=y}(B_2) + \bar{Q}_{T|Y=y}(B_1 \cap B_2)) dP_Y(y).$$

With these definitions (T',Y) and (T'',Y) are both F-distributed and if we denote $A := \{|T'| > c, |T''| > c\}$, then on A we have $T' = T''$ and on A^c, given Y, the random variables T' and T'' are independent. It will be clear that for $c \to \infty$ we have $P(A) \to 0$ and also

$$\| P_{T''-T'} - H \| \to 0,$$

where H is defined as in Lemma 5.2.6. Hence if d_c is the minimal lattice width of $P_{T''-T'}$, then $d_c \downarrow \bar{d}$ for $c \to \infty$. It follows that if we choose c sufficiently large, then $I \cap L_{d_c} \neq \phi$. The rest of the proof for the case $E|T| = \infty$ is identical with the proof in the finite case. □

In the course of the proof of Theorem 5.2.3 we need a simple lemma of a technical nature, that asserts that if T"-T' is small, then $P_{T''}$ and $P_{T'}$ are close in a suitable sense.

LEMMA 5.2.8. *Suppose* T' *and* T" *are real random variables. If* ν_ε *is the homogeneous distribution on* $(0,\varepsilon)$ *and* $\rho = |T''-T'|$ *then*

$$\|\nu_\varepsilon * P_{T'} - \nu_\varepsilon * P_{T''}\| \leq 2\, E(1 \wedge \tfrac{\rho}{\varepsilon}).$$

PROOF. Observe that for any Borel set $B \in \mathcal{B}^1$

$$\frac{1}{\varepsilon}\int_0^\varepsilon \chi_B(T'-t)\,dt - \frac{1}{\varepsilon}\int_0^\varepsilon \chi_B(T''-t)\,dt \leq \frac{\rho}{\varepsilon}.$$

Hence $\mu := \nu_\varepsilon * P_{T'} - \nu_\varepsilon * P_{T''}$ satisfies

$$\mu(B)^+ \leq E(1 \wedge \tfrac{\rho}{\varepsilon})$$

and because $\|\mu\| = 2 \sup_B \mu(B)^+$ the assertion of the lemma follows. □

PROOF of Theorem 5.2.3. First we prove the limit relation. A demonstration of the much easier second assertion of the theorem is postponed to the end of the proof. By using A.1 it follows that it is sufficient to prove the limit relation for distributions $\nu = \nu_\varepsilon$, so for the homogeneous distribution on $(0,\varepsilon)$, $\varepsilon > 0$.

Choose arbitrary real h and positive δ. We have to prove that for n large enough

$$(5.2.2)\qquad E\|\nu_\varepsilon * P_{S_n|X_{K_n^c}} - \nu_\varepsilon * P_{S_n+h|X_{K_n^c}}\| < \delta,$$

if $S_{\mathbb{Z}}$ is strongly nonlattice with respect to $X_{\mathbb{Z}}$, i.e. if $\lim_{K \to \mathbb{Z}} d_K = 0$, where d_K is defined in the introduction of this section. Choose the set $K^* := \{1,\ldots,k^*\}$ so large that $d := d_{K^*} < \frac{1}{8}\delta\varepsilon$. The interval $I := (h-d,h+d)$ has a nonempty intersection with the lattice L_d. Let F_1 be the distribution of

$$(S^{K^*},\, X_{K^{*c}}).$$

By Proposition 5.2.7 we can choose m so large that $G \in \mathcal{G}^m_{F_1}$ satisfies

$$G(I) > 1 - \tfrac{1}{8}\delta.$$

Construct the window-frame process $\tilde{X}_{\mathbb{Z}}$ with m windows, such that

$$\delta(X_{\mathbb{Z}},\tilde{X}_{\mathbb{Z}}) = \tfrac{1}{2}\| P_{X_{\mathbb{Z}}} - P_{\tilde{X}_{\mathbb{Z}}} \| < \tfrac{1}{16}\delta.$$

Note that $X_{L^*\setminus K^*}$ is $X_{K^{*c}}$-measurable. Let F be the distribution of $(S^{K^*}, X_{L^*\setminus K^*})$. By Lemma 5.1.9 the set $\mathcal{G}^m_F$ contains $\mathcal{G}^m_{F_1}$, so $G \in \mathcal{G}^m_F$. Construct $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$ as in Theorem 5.1.7. Then the processes $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$ are distributed as $\tilde{X}_{\mathbb{Z}}$, and are coupled over the complement of the union K of the windows, while $S''^K - S'^K$ is distributed as G. Let $n > \sup K$. So $K_n = \{1,\dots,n\} \supset K$ and the random variable

$$\rho := |S''_n - S'_n - h| = |S''^K - S'^K - h|$$

is distributed as $T_h G$ and satisfies

$$\text{(5.2.3)} \qquad P_\rho(-d,d) > 1 - \tfrac{1}{8}\delta.$$

This inequality forms the clue to the proof of (5.2.2).

Because X' and X" are coupled over K_n^c and are equally distributed

$$P_{X'_{\mathbb{Z}} | X'_{K_n^c}} = P_{X''_{\mathbb{Z}} | X''_{K_n^c}}$$

and so

$$P_{S'_n | X'_{K_n^c}} = P_{S''_n | X''_{K_n^c}}.$$

By the definition of ρ and Lemma 5.2.6, applied to the conditional distributions above, this implies that

$$\| \nu_\varepsilon * P_{S'_n | X'_{K_n^c}} - \nu_\varepsilon * P_{S''_n + h | X''_{K_n^c}} \| \le 2\, E(1 \wedge \tfrac{\rho}{\varepsilon} | X'_{K_n^c}).$$

Because $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$ are distributed as $\tilde{X}_{\mathbb{Z}}$ and by (5.2.3)

$$\text{(5.2.4)} \qquad E\| \nu_\varepsilon * P_{\tilde{S}_n | \tilde{X}_{K_n^c}} - \nu_\varepsilon * P_{\tilde{S}_n + h | \tilde{X}_{K_n^c}} \|$$

$$\le 2\, E(1 \wedge \tfrac{\rho}{\varepsilon}) \le 2(\tfrac{1}{8}\delta + \tfrac{1}{8}\delta) = \tfrac{1}{2}\delta.$$

Thus we have obtained inequality (5.2.2) in terms of $\tilde{X}_{\mathbb{Z}}$. We still have to translate it in terms of $X_{\mathbb{Z}}$. Consider first the left term in (5.2.4) and apply (4.1.5) to obtain

$$\int \tfrac{1}{2}\| \nu_\varepsilon * P_{\tilde{S}_n | \tilde{X}_{K_n^c}=x} - \nu_\varepsilon * P_{S_n | X_{K_n^c}=x} \| \, dP_{X_{K_n^c}}(x)$$

$$\leq \delta(\tilde{X}_{\mathbb{Z}}, X_{\mathbb{Z}}) + \delta(\tilde{X}_{K_n^c}, X_{K_n^c}) < 2 \cdot \frac{1}{16}\delta = \frac{1}{8}\delta.$$

A similar estimate holds for the second term of the difference in (5.2.4). We obtain from (5.2.4), using these estimates,

$$E\| \nu_\varepsilon * P_{S_n | X_{K_n^c}} - \nu_\varepsilon * P_{S_n + h | X_{K_n^c}} \| < 2 \cdot \frac{\frac{1}{8}\delta}{\frac{1}{2}} + \tfrac{1}{2}\delta = \delta.$$

This proves the first assertion. To prove the second assertion, assume that the limit relation holds, but that $S_{\mathbb{Z}}$ is weakly lattice with respect to $X_{\mathbb{Z}}$. There exists a positive number d such that (S_n) mod d is, up to a null set, $X_{K_n^c}$-measurable for all $n \geq 1$. Let ν be the homogeneous distribution on $(0, \frac{1}{4}d)$. If $h := \frac{1}{2}d$ we can choose n so large that

$$\text{(5.2.5)} \qquad E\| \nu * P_{S_n | X_{K_n^c}} - \nu * P_{S_n + h | X_{K_n^c}} \| < \tfrac{1}{2}.$$

On the other hand with the exception of a null set

$$P_{S_n | X_{K_n^c}}$$

is concentrated on $C_n + L_d$, where $C_n = (S_n)$ mod d is, up to a null set, measurable with respect to $X_{K_n^c}$. By the choice of h and ν it follows that the two measures occurring as the terms in the difference of (5.2.5) are with probability 1 mutually singular. This contradicts (5.2.5) and hence $S_{\mathbb{Z}}$ is strongly nonlattice with respect to $X_{\mathbb{Z}}$. □

5.3. LOSS OF MEMORY IN A SPREAD OUT RANDOM WALK

A probability measure μ on the real line is called *spread out* if for some $n \geq 1$ the convolution μ^{n*} is not singular with respect to the Lebesgue measure ℓ. Suppose $S_{\mathbb{Z}}$ is a random walk with independent, μ-distributed increments. HERMANN [1965] proves that if μ is spread out, then we have for any real h the limit relation

$$\lim_{n\to\infty} \| P_{S_n} - P_{S_n+h} \| = 0.$$

Also the converse of this assertion is valid: if for any real h this limit relation holds, then necessarily the distribution μ is spread out (see also STAM [1967]). In this section we prove a similar result for controlled random walks with dependent increments.

We mentioned in the introduction that this limit relation for random walks with independent, stationary increments might be proved by means of a coupling construction (see (0.0.4)). We discuss this in the note following on Corollary 5.3.7.

Let us first discuss a generalization of the concept spread out. Let $S_{\mathbb{Z}}$ be a random walk, controlled by a stationary sequence $X_{\mathbb{Z}}$. The random walk $S_{\mathbb{Z}}$ is called *spread out* with respect to $X_{\mathbb{Z}}$ if for some $n \geq 1$ with positive probability the conditional distribution

$$P_{S_n | X_{K_n^c}}, \quad \text{where} \quad K_n := \{1,\dots,n\},$$

is not singular with respect to the Lebesgue measure ℓ.

<u>EXAMPLE 5.3.1</u>. Suppose $S_{\mathbb{Z}}$ is a random walk with stationary, independent increments $X_{\mathbb{Z}}$. The spread out condition above is satisfied if and only if for some $n \geq 1$ the distribution μ of the increments satisfies the requirement that

$$\mu^{n*} = P_{S_n | X_{K_n^c}}$$

is nonsingular with respect to the Lebesgue measure ℓ. In other words, the random walk $S_{\mathbb{Z}}$ is spread out with respect to $X_{\mathbb{Z}}$ if and only if the distribution μ of its increments is spread out.

Another example of a spread out random walk is Example 5.2.1 (iii). The following theorem is the analogue of the limit result for random walks with stationary, independent increments, mentioned in the first paragraph of this section. This theorem is the main result of this section.

<u>THEOREM 5.3.2</u>. *Let* $S_{\mathbb{Z}}$ *be a random walk controlled by a stationary sequence* $X_{\mathbb{Z}}$ *with values in a Borel space. Suppose* $X_{\mathbb{Z}}$ *satisfies* (5.1.2) *and* (5.1.3). *If* $S_{\mathbb{Z}}$ *is spread out with respect to* $X_{\mathbb{Z}}$ *then*

(5.3.1) $$\lim_{n\to\infty} E\| P_{S_n | X_{K_n^c}} - P_{S_n+h | X_{K_n^c}} \| = 0, \quad h \text{ real},$$

where $K_n := \{1,\ldots,n\}$, $n \geq 1$. *Conversely, the limit property implies that* $S_{\mathbb{Z}}$ *is spread out with respect to* $X_{\mathbb{Z}}$.

The proof of the theorem above can be found at the end of the section. The limit relation of the theorem is refined in Proposition 6.1.5. As a consequence of Lemma 5.2.5 we can obtain the following corollary.

COROLLARY 5.3.3. *If* $X_{\mathbb{Z}}$ *satisfies the assumptions of Theorem* 5.3.2 *and* $S_{\mathbb{Z}}$ *is spread out with respect to* $X_{\mathbb{Z}}$, *then for any real* h

$$\lim_{n\to\infty} \| P_{S_n} - P_{S_n+h} \| = 0.$$

The proof of (5.3.1) will be carried out by applying Theorem 5.1.7. First we have to study the class $\mathcal{G}_F^m$ mentioned in that theorem. With the help of some lemmas it will be possible to reduce the study of this class $\mathcal{G}_F^m$ to the following simple property of a random walk with i.i.d. increments. This property is already known, as we noted in the first paragraph of this section. The proof below is based on symmetry considerations.

PROPOSITION 5.3.4. *Let* μ *be the homogeneous distribution on* $(-1,1)$. *Then for any real* h

$$\lim_{n\to\infty} \| \delta_h * \mu^{n*} - \delta_{-h} * \mu^{n*} \| = 0,$$

where δ_x *is the probability measure degenerate at* $\{x\}$.

PROOF. It suffices to prove the result for $h > 0$. Let $(S_n)_{n\geq 0}$ be a random walk with independent, μ-distributed increments started at h. Define τ to be the first entrance time of this random walk into $(-\infty,0]$, i.e.

$$\tau := \inf\{n \geq 0: S_n \leq 0\}.$$

Because μ has vanishing mean, τ is finite with probability 1. Define the measure

$$H_n(B) := P(S_n \in B,\ n > \tau), \qquad B \in \mathcal{B}^1,$$

and let G_n^+ be the positive part of

$$G_n := \delta_h * \mu^{n*} - \delta_{-h} * \mu^{n*}.$$

We prove by induction that

$$(5.3.2) \qquad G_n^+ \le H_n, \quad n \ge 0.$$

Because τ is finite a.s. this will imply the assertion by

$$\|G_n\| = 2\|G_n^+\| \le 2\|H_n\| = 2\, P(\tau > n) \to 0$$

for $n \to \infty$. For $n = 0$ the induction statement is obvious: both G_0^+ and H_0 coincide with δ_h. Suppose (5.3.2) holds for $n \ge 0$. To prove that (5.3.2) holds for $n = n+1$ we first show that G_{n+1}^+ is concentrated on $(0,\infty)$. The measure H_n clearly is concentrated on $(0,\infty)$. The induction assumption (5.3.2) implies therefore that G_n^+ is concentrated on $(0,\infty)$. Let g_n be the density of $\mu * G_n^+$ with respect to the Lebesgue measure ℓ. If $x \ge 0$ we have

$$(5.3.3) \qquad g_n(x) = \int \chi_{(-1,1)}(x-t)\, dG_n^+(t)$$

$$\ge \int \chi_{(-1,1)}(-x-t)\, dG_n^+(t) = g_n(-x).$$

By symmetry G_n^- is the reflection of G_n^+ with respect to 0 on the real line. Remark that (5.3.3) expresses that on $[0,\infty)$ the density of $\mu * G_n^+$ dominates the density of $\mu * G_n^-$. Therefore the measure

$$G_{n+1} = \mu * G_n^+ - \mu * G_n^-$$

has density

$$g_{n+1}(x) := g_n(x) - g_n(-x) \ge 0$$

on $[0,\infty)$ and by symmetry

$$g_{n+1}(x) := -(g_n(-x) - g_n(x)) \le 0$$

on $(-\infty,0]$. It follows that G_{n+1}^+ is concentrated on $(0,\infty)$ and moreover

$$G_{n+1}^+(B) \le \mu * G_n^+(B \cap (0,\infty)).$$

To prove the induction statement observe that with $X_{n+1} := S_{n+1} - S_n$

$$H_{n+1}(B) = P(S_n + X_{n+1} \in B \cap (0,\infty),\ \tau > n)$$

$$= \mu * H_n(B \cap (0,\infty))$$

$$\geq \mu * G_n^+(B \cap (0,\infty)) \geq G_{n+1}^+(B),$$

where the first inequality is justified by the induction assumption. This proves (5.3.2) for $n = n+1$. □

By using a scaling and a translation it follows from the proposition above that for any homogeneous distribution μ on some interval I

$$\lim_{n\to\infty} \| \mu^{n*} - \delta_h * \mu^{n*} \| = 0$$

for any real h. The maximal coupling theorem for Markov chains makes it possible to translate this limit relation in the following coupling property.

LEMMA 5.3.5. *Suppose* μ *is a homogeneous distribution on an interval* I *on the real line. Let* h *be any real number. There exists a probability space with random walks* $(S'_n)_{n\geq 0}$ *and* $(S''_n)_{n\geq 0}$ *with independent,* μ*-distributed increments and started in* 0 *such that*

$$S'_n,\quad n \geq 0, \qquad \textit{and} \qquad S''_n + h,\quad n \geq 0,$$

are successfully coupled.

PROOF. The processes S'_n, $n \geq 0$, and S''_n+h, $n \geq 0$, that have to be constructed, are Markov chains with transition probability

$$P(x,B) := F(T_x B), \quad B \in \mathcal{B}^1, \quad x \text{ real},$$

and initial distributions δ_0 and δ_h respectively. Theorem 4.3.1 and the limit relation above imply the validity of the lemma. □

The following lemma investigates the relation between the spread out condition and homogeneous distributions. The idea on the background is the following. If μ is a probability measure that is not singular with respect to the Lebesgue measure ℓ, then μ dominates a substochastic nonvanishing measure μ_1, having a bounded density f with respect to ℓ. Then μ^{2*} dominates

μ_1^{2*}. The measure μ_1^{2*} has by the dominated convergence theorem a continuous density

$$f^{2*}(x) = \int f(t)f(x-t)dx.$$

So if μ is any Lebesgue nonsingular probability measure then μ^{2*} dominates a measure of the form $c\ell_I$, where c is a positive constant and ℓ_I the restriction of the Lebesgue measure ℓ to an interval I.

LEMMA 5.3.6. *Let* F *be the distribution of two independent random vectors* (T_1,Y_1) *and* (T_2,Y_2), *whose components* T_i *and* Y_i *have their values on the real line and in a Borel space* Γ *respectively. Suppose with positive probability* $P_{T_1|Y_1}$ *is nonsingular with respect to the Lebesgue measure* ℓ. *Then there is a positive constant* c *and an interval* I *such that with positive probability*

$$P_{T_1+T_2|Y_1,Y_2} \geq c\ell_I.$$

PROOF. Select a measurable function f on $\mathbb{R}^1 \times \Gamma$ such that $f(\cdot,y_1)$ is a density with respect to ℓ of the Lebesgue continuous part of $P_{T1|Y1=y1}$. By REVUZ [Lemma 1.5.3] we may assume that f is bimeasurable. Define g on $\mathbb{R}^1 \times \Gamma^2$ by

$$g(t,z) = \int f(s,y_1)(f(t-s,y_2) \wedge 1)ds, \quad \text{where} \quad z = (y_1,y_2).$$

Define $Z := (Y_1,Y_2)$ and note that

$$P_{T_1+T_2|Z=z} \geq g(\cdot,z)\ell.$$

Using the dominated convergence theorem we obtain that g(t,z) for fixed z is continuous in t. By the independence assumption and because with positive probability, $f(\cdot,Y_1)$ and $f(\cdot,Y_2)$ are not equivalent to a Lebesgue null function, it follows that $g(\cdot,Z)$ is not equivalent to a Lebesgue null function with positive probability.

We can now choose J×B, with J a bounded open interval in $\mathbb{R}^1$ and $B \subset \Gamma^2$ a measurable set with $P(Z \in B) > 0$, such that $g(\cdot,z) \neq 0$ on J for each $z \in B$. Let $B_n \subset B$ be the set of all $z \in B$ such that on an interval $I_z \subset J$ with length at least $\frac{1}{n}$ holds

$$g(\cdot,z) \geq \frac{1}{n} \chi_{I_z}(\cdot).$$

Because for fixed z the function g(t,z) is continuous in t, it follows that $B_n \uparrow B$ for $n \to \infty$, so for some $n \geq 1$ we have $P(Z \in B_n) > 0$. Choose numbers $t_1 \leq \ldots \leq t_m$ such that $J = (t_1, t_m)$ and $0 < t_{j+1} - t_j < \frac{1}{2n}$, $1 \leq j \leq m$. Define the sets $B_n^j \subset B_n$ by

$$B_n^j := \{z \in B_n : g(\cdot, z) \geq \frac{1}{n} \text{ on } (t_j, t_{j+1})\}, \qquad 1 \leq j \leq m.$$

By the definition of B_n we have $B_n = \bigcup_{j=1}^m B_n^j$ and because $P(Z \in B_n) > 0$ for some $j = j_0$

$$P(Z \in B_n^{j_0}) > 0.$$

If we choose $c = \frac{1}{n}$ and $I := (t_{j_0}, t_{j_0+1})$ the assertion of the lemma follows.□

<u>COROLLARY 5.3.7</u>. *The probability space in the preceding lemma can be extended with a random variable* θ *with values in* $\{0,1\}$, *such that the set* $\{\theta=1\}$ *has positive probability and on this set*

$$P_{T_1+T_2 | Y_1, Y_2, \theta} = \mu,$$

where μ *is the homogeneous distribution on the interval* I.

<u>PROOF</u>. Let $U := T_1+T_2$ and $Z := (Y_1, Y_2)$. By the lemma above the set

$$C := \{z \in \Gamma^2 : P_{U|Z=z} \geq c\ell_I\}$$

has positive P_Z-measure. Define a random variable θ with values in $\{0,1\}$, such that on $\{Z \notin C\}$ we have $\theta = 0$ a.s. and on $\{Z \in C\}$

$$P(\theta = 1, U \in B | Z) = c\ell_I(B), \qquad B \in \mathcal{B}^1.$$

By the definition of C this is possible. The set $\{\theta=1\}$ has probability $c\ell(I)P(Z \in C) > 0$ and on $\{\theta=1\} = \{\theta=1, Z \in C\}$

$$P_{U|Z,\theta} = \mu. \qquad \square$$

In the note below we show that for random walks with stationary, independent increments the limit relation (0.0.3) can be obtained by means of the coupling property (0.0.4) (see also ORNSTEIN [1969]). A similar argument will be used in Proposition 5.3.8.

<u>NOTE</u>. Suppose $(S_n)_{n\geq 0}$ is a random walk with stationary, independent increments $\xi_{\mathbb{N}}$. Let us assume that P_{ξ_1} dominates $c\ell_I$ with $c > 0$ and ℓ_I the restriction of the Lebesgue measure ℓ to an open interval I. We show that it is possible to construct for any real h random walks $(S_n')_{n\geq 0}$ and $(S_n'')_{n\geq 0}$ such that

$$S_n', \quad n \geq 0, \quad \text{and} \quad S_n''+h, \quad n \geq 0,$$

are successfully coupled. Construct $\{0,1\}$-valued random variables θ_i, $i \geq 1$, such that (θ_i, ξ_i), $i \geq 1$, is a sequence of i.i.d. random vectors with $P_{\xi_i | \theta_i} = \mu$ on $\{\theta_i = 1\}$, where μ is the homogeneous distribution on the interval I. Construct, independent of this sequence, random walks $(T_n')_{n\geq 0}$ and $(T_n'')_{n\geq 0}$, such as in Lemma 5.3.5, with increments $\eta_{\mathbb{N}}'$ and $\eta_{\mathbb{N}}''$ respectively. Hence

$$T_n', \quad n \geq 0, \quad \text{and} \quad T_n''+h, \quad n \geq 0,$$

are successfully coupled. Let

$$\tau_0 := 0, \quad \tau_n := \inf\{k > \tau_{n-1}: \theta_k = 1\}, \quad n \geq 1.$$

Define $\xi_{\mathbb{N}}'$ and $\xi_{\mathbb{N}}''$ by $\xi_i' = \xi_i'' = \xi_i$ if $\theta_i = 0$ and take

$$\xi_{\tau_i}' := \eta_i', \quad i \geq 1, \quad \xi_{\tau_i}'' := \eta_i'', \quad i \geq 1.$$

It is easily checked that $\xi_{\mathbb{N}}'$ and $\xi_{\mathbb{N}}''$ are distributed as $\xi_{\mathbb{N}}$. Note that the random walks $(S_n')_{n\geq 0}$ and $(S_n'')_{n\geq 0}$ with increments $\xi_{\mathbb{N}}'$ and $\xi_{\mathbb{N}}''$ satisfy

$$S_{\tau_n}'' - S_{\tau_n}' = T_n'' - T_n', \quad n \geq 0,$$

and hence

$$S_n', \quad n \geq 0, \quad \text{and} \quad S_n''+h, \quad n \geq 0,$$

are successfully coupled, i.e. (0.0.4) holds. Arguing as in Section 0.0 we obtain

$$\lim_{n\to\infty} \| P_{S_n} - P_{S_n+h} \| = 0$$

for random walks with i.i.d. increments $\xi_{\mathbb{N}}$, satisfying $P_{\xi_1} \geq c\ell_I$.

If the distribution P_{ξ_1} is spread out, there always exists some $m \geq 1$

such that $G := P_{S_m}$ is not Lebesgue singular and then $G^{2*} \geq c\ell_I$ for some positive c and open interval I. If $n = 2km+\ell$, $k,\ell \geq 1$, we have

$$\|P_{S_n} - P_{S_n+h}\| \leq \|G^{2k*} - T_{-h}G^{2k*}\|,$$

where the translation T_{-h} is defined as usual. We saw already that the right-hand side converges to 0 for $k \to \infty$. Hence the limit relation (0.0.3) holds in general for spread out random walks with stationary, independent increments. Using the coupling theorem for Markov chains (Theorem 4.3.1) we can construct (0.0.4) also in this slightly more general case.

To derive (5.3.1) we apply Theorem 5.1.7 and the knowledge on $\mathcal{G}_F^m$ contained in the proposition below.

PROPOSITION 5.3.8. *Let* F *be the distribution of a random vector* (T,Y) *with* T *real valued and* Y *with values in a Borel space* Γ. *Suppose that with positive probability* $P_{T|Y}$ *is not singular with respect to the Lebesgue measure. Let* h *be any real number. If* ε *is arbitrary positive then, if* m *is large enough, the set* $\mathcal{G}_F^m$ *contains an element* G *such that*

$$G(\{h\}) > 1-\varepsilon.$$

PROOF. Let ε be arbitrary positive. We have to construct sequences (T_j', Y_j), $1 \leq j \leq m$, and (T_j'', Y_j), $1 \leq j \leq m$, both distributed as sequences of independent, F-distributed random variables, such that the distribution G of

$$\sum_{j=1}^{m} (T_j' - T_j'')$$

has mass at least $1-\varepsilon$ in $\{h\}$, if m is large.

To construct these sequences we use Lemma 5.3.5. Corollary 5.3.7 provides the link between F-distributed random variables and homogeneously distributed random variables.

Construct a sequence of independent random vectors $X_n := (U_n, Z_n, \theta_n)$, $n \geq 1$, distributed as the random vector $(T_1+T_2, (Y_1,Y_2), \theta)$, defined in the corollary. Define random times

$$\tau_0 := 0, \quad \tau_n := \inf\{k > \tau_{n-1}: \theta_k = 1\}, \qquad n \geq 1.$$

The process $\tau_{\mathbb{N}}$ is a sequence of regeneration times for $X_{\mathbb{N}}$. Moreover,

$$P_{U_n, X_{\mathbb{N}+n} | \tau_n = k, Z_i, i \le n, \theta_i, U_i, i<n} = \mu \times P_{X_{\mathbb{N}}},$$

where μ is the homogeneous distribution on the interval I mentioned in Corollary 5.3.7. Construct the random walks $(S'_n)_{n\ge 0}$ and $(S''_n)_{n\ge 0}$ of Lemma 5.3.5, independent of $X_{\mathbb{N}}$. Let the processes of increments of these random walks be $\xi'_{\mathbb{N}}$ and $\xi''_{\mathbb{N}}$ respectively. In the sequence $U_{\mathbb{N}}$ we replace the U-values at the regeneration times τ_k, $k \ge 1$, by the ξ'- and ξ''-values. We obtain processes $U'_{\mathbb{N}}$ and $U''_{\mathbb{N}}$ defined by

$$\begin{aligned} U'_n &:= \xi'_k \qquad U''_n := \xi''_k, \qquad && n = \tau_k,\ k \ge 1, \\ U'_n &:= U_n \qquad U''_n := U_n, && \text{else.} \end{aligned}$$

Because $\xi'_{\mathbb{N}}$ is a sequence of independent, μ-distributed random variables, independent of $X_{\mathbb{N}}$, we have

$$P_{\xi'_k, X_{\mathbb{N}+n} | \tau_n = k, Z_i, i \le n, \theta_i, U_i, i<n, \xi_i, i<k} = \mu \times P_{X_{\mathbb{N}}}.$$

It is now easily seen that the process (U'_n, Z_n, θ_n), $n \ge 1$, is distributed as (U_n, Z_n, θ_n), $n \ge 1$, so as independent, $P_{T_1+T_2,(Y_1,Y_2),\theta}$-distributed random variables. The same assertion holds for (U''_n, Z_n, θ_n), $n \ge 1$.

Using these sequences we shall construct (T'_n, Y_n), $n \ge 1$, and (T''_n, Y_n), $n \ge 1$. Write $Z_n := (Y_{2n-1}, Y_{2n})$, $n \ge 1$. By Lemma 4.2.4 we construct random variables (T'_{2n-1}, T'_{2n}), $n \ge 1$, such that

$$P_{(T'_{2n-1}, T'_{2n}, U'_n, Z_n)_{n\ge 1}} = \prod_{n\ge 1} P_{T_1, T_2, T_1+T_2, (Y_1, Y_2)}.$$

As a consequence we have $U'_n = T'_{2n-1} + T'_{2n}$ a.s. and furthermore

$$(T'_{2n-1}, T'_{2n}, Y_{2n-1}, Y_{2n}), \qquad n \ge 1,$$

is a sequence of independent, P_{T_1,T_2,Y_1,Y_2}-distributed random variables, or differently, (T'_n, Y_n), $n \ge 1$, is a sequence of independent, F-distributed random variables. Similarly we construct (T''_{2n-1}, T''_{2n}), $n \ge 1$, such that

$$P_{(T''_{2n-1}, T''_{2n}, U''_n, Z_n)_{n\ge 1}} = \prod_{n\ge 1} P_{T_1, T_2, T_1+T_2, (Y_1, Y_2)}.$$

This yields a sequence (T''_n, Y_n), $n \ge 1$, of independent, F-distributed random variables such that $U_n = T''_{2n-1} + T''_{2n}$, $n \ge 1$.

By Lemma 5.3.5 we have if k is larger than some random time

$$h = \sum_{j=1}^{k} (\xi_j' - \xi_j'') = \sum_{j=1}^{k} (U'_{\tau_j} - U''_{\tau_j})$$

and because $U_n'' = U_n'$ for $n \neq \tau_k$ for each k we have for n large enough

$$h = \sum_{j=1}^{n} (U_j' - U_j'') = \sum_{j=1}^{2n} (T_j' - T_j'').$$

Hence for m large enough $\mathcal{G}_F^{2m}$ contains an element G such that

$$G(\{h\}) = P(\sum_{j=1}^{2m} (T_j' - T_j'') = h)$$

is arbitrarily close to 1. Because $\mathcal{G}_F^m$ is increasing in m this implies the assertion of the proposition. □

PROOF of Theorem 5.3.2. The proof of the limit relation is parallel to the corresponding part of the proof of Theorem 5.2.3 in the previous section. We want to derive that for any positive δ, if n is large

(5.3.4) $$E\| P_{S_n|X_{K_n^c}} - P_{S_n+h|X_{K_n^c}} \| < \delta.$$

The proof can be sketched as follows: Pursue the argument in the proof of Theorem 5.2.3. Instead of constructing $G \in \mathcal{G}_F^m$ such that $G(I) > 1-\frac{1}{8}\delta$ with $I \ni h$, we can achieve now that $G(\{h\}) > 1-\frac{1}{8}\delta$. Therefore the proof simplifies: we do not have to apply the smoothing by convoluting with ν_ε. Instead we obtain as the inequality that corresponds to (5.2.4)

$$E\| P_{\tilde{S}_n|\tilde{X}_{K_n^c}} - P_{\tilde{S}_n+h|\tilde{X}_{K_n^c}} \| \le 2\, P(\rho \neq 0) \le \tfrac{1}{2}\delta.$$

The application of (4.1.5) to this inequality yields, similarly as we did previously, the required assertion (5.3.4).

A proof that the limit property implies that $S_{\mathbb{Z}}$ is spread out with respect to $X_{\mathbb{Z}}$ is given as follows. Choose n so large that on a set B with positive Lebesgue measure

(5.3.5) $$E\, \tfrac{1}{2}\| P_{S_n|X_{K_n^c}} - P_{S_n+h|X_{K_n^c}} \| \le \tfrac{1}{2}, \quad h \in B,$$

where $K_n := \{1,\dots,n\}$. Suppose $P_{S_n|X_{K_n^c}}$ is with probability 1 singular with respect to the Lebesgue measure ℓ. Then there is a measurable set $N \subset \mathbb{R}^1 \times \mathbb{R}^{K_n^c}$ such that, using the notation $N(x) := \{t \in \mathbb{R}^1 : (t,x) \in \mathbb{R}^1\}$,

$$\ell(N(X_{K_n^c})) = 0 \text{ a.s.}, \quad P(S_n \in N(X_{K_n^c}) | X_{K_n^c}) = 1 \text{ a.s.}$$

Remark that

$$\iint \chi_{N(x)}(t-h)\,dh\, dP_{S_n, X_{K_n^c}}(t,x) = 0. \tag{5.3.6}$$

According to Fubini this integral equals

$$\int P(S_n \in N(X_{K_n^c}) + h)\,dh.$$

This number exceeds by (5.3.5)

$$\int_B \tfrac{1}{2} dh = \tfrac{1}{2}\ell(B) > 0,$$

thus contradicting (5.3.6). As a consequence $S_{\mathbb{Z}}$ is spread out with respect to $X_{\mathbb{Z}}$. □

The results in Chapter 5 have an important limitation, that can be explained as follows. The two main results, Theorems 5.2.3 and 5.3.2 are derived under the condition (5.1.2) that requires

$$\lim_{n\to\infty} \perp(X_L, X_{L'+n}) = 0, \tag{5.3.7}$$

for any pair of finite integer sets L and L'. By Proposition 4.1.1 and the stationarity of $X_{\mathbb{Z}}$ this expression is equivalent to

$$\lim_{n\to\infty} E\| P_{X_{L'+n}|X_L} - P_{X_{L'}} \| = 0.$$

It seems attractive to give our limit results and especially Corollaries 5.2.4 and 5.3.3 a similar form. In case of Corollary 5.3.3 one might think of

$$\lim_{n\to\infty} E\| P_{S_n+h|X_L} - P_{S_n} \| = 0 \tag{5.3.8}$$

for every real h. Observe that by applying Lemma 5.2.5, we can obtain under the conditions of Theorem 5.3.2

$$\lim_{n\to\infty} E\| P_{S_n+h|X_L} - P_{S_n|X_L} \| = 0.$$

This result is, however, different from (5.3.8). The difficulty in deriving (5.3.8) can be clarified in the following way. The condition (5.3.7) is an asymptotic independence condition formulated in terms of fixed finite integer sets L and L', not depending on n. However, S_n is a function of X_{K_n}, $K_n := \{1,...,n\}$, so depends on an increasing number of X-variables as n increases. This makes it questionable whether (5.3.8) can be proved using only (5.1.2), (5.1.3) and the spread out condition. However, (5.3.8) can be derived if we replace (5.1.2) by the condition that $X_{\mathbb{Z}}$ is weak Bernoulli. This will be done in Chapter 6.

CHAPTER 6

RENEWAL THEORY

Let $S_{\mathbb{Z}}$ be a random walk controlled by a stationary sequence $X_{\mathbb{Z}}$ that satisfies (5.1.3) and has its values in a Borel space Γ. Chapter 6 studies renewal theory for this random walk $S_{\mathbb{Z}}$ under a strong assumption of asymptotic independence: In most of our results we assume that $X_{\mathbb{Z}}$ is weak Bernoulli.

Section 6.1 studies loss of memory of a stronger type than in Chapter 5. The results in this section bring us already close to the renewal theoretic theorems of Section 6.3. These renewal theorems are obtained from the loss of memory results of Section 6.1 by using a formula from Palm theory.

The renewal theorems of this chapter can be divided into two types: theorems for nonlattice and for spread out random walks. The nonlattice concept will be defined in Section 6.2. In our approach, renewal theory for nonlattice random walks is more difficult than for spread out random walks. The reason is that we do not obtain in Section 6.1 loss of memory results for nonlattice random walks, but for the slightly smaller class of strongly nonlattice random walks. Section 6.2 studies the gap between these two classes of random walks. For spread out random walks this difficulty does not arise and renewal theory for these random walks does not use Section 6.2.

Section 6.3 contains renewal theory for random walks controlled by a weak Bernoulli process. Assume that ES_1 exists as a finite positive number. For simplicity we assume in this summary that $S_{\mathbb{Z}}$ has strictly positive increments. Define the marked point process N_0 on the real line with marks in Γ by

$$N_0(B) := \sum_{n\in\mathbb{Z}} \chi_B(S_n, X_n),$$

where $B \subset \mathbb{R}^1 \times \Gamma$ is any measurable set. Define $N_t := T_t N_0$ and let N_t^+ and N_t^- be the restriction of N_t to $(0,\infty) \times \Gamma$ and $(-\infty,0] \times \Gamma$ respectively. We prove that if the random walk $S_{\mathbb{Z}}$ is spread out, then

$$\lim_{t\to\infty} \| P_{N_t^+} - P_{N^+} \| = 0$$

for some stationary, marked point process N^+ on $(0,\infty) \times \Gamma$. There even holds a stronger assertion:

$$\lim_{t\to\infty} \| P_{N_t^+ | N_0^-} - P_{N^+} \| = 0 \text{ a.s.}$$

This limit relation expresses loss of memory of N_t^+ for the past N_0^-. For nonlattice random walks a similar limit relation holds, using a weaker convergence concept. Blackwell's theorem is obtained as a corollary.

In Section 6.4 we restrict our attention to special classes of processes $X_{\mathbb{Z}}$, for which it is possible to obtain more complete results. The most interesting is the case where we assume that $X_{\mathbb{Z}}$ is a countably valued sequence. In this case it is possible to indicate necessary and sufficient conditions in terms of $X_{\mathbb{Z}}$ for the validity of the limit relation, obtained in Section 6.3 for nonlattice random walks. We also consider the case of Markov dependent sequences $X_{\mathbb{Z}}$. There is an extensive literature in this direction. We show that renewal theorems for semi-Markov chains can be obtained from our results and we give a survey of the literature.

Let N be a stationary, marked point process on the real line with marks in Γ. In Section 6.5 we investigate conditions under which N is *weak Bernoulli*, i.e.

$$\lim_{t\to\infty} \perp(N^-, (T_t N)^+) = 0,$$

where $(\cdot)^-$ and $(\cdot)^+$ denote restrictions to $(-\infty,0] \times \Gamma$ and $(0,\infty) \times \Gamma$ respectively. The results in this direction are related to the theory of mixing properties of special flows, a subject studied in ergodic theory. At the end of Section 6.5 we discuss this relationship.

6.1. WEAK BERNOULLI PROCESSES AND LOSS OF MEMORY

In this section we assume throughout that $S_{\mathbb{Z}}$ is a random walk controlled by a stationary sequence $X_{\mathbb{Z}}$ of random variables with values in a Borel space Γ. Moreover we suppose that $X_{\mathbb{Z}}$ satisfies condition (5.1.3).

In Chapter 5 we derived loss of memory results like

$$\lim_{n\to\infty} \| P_{S_n} - P_{S_n+h} \| = 0, \quad h \text{ real.}$$

It was shown that limit relations of this form were closely connected with the property "spread out" and the property "strongly nonlattice". In this section we consider a different kind of loss of memory. We want to obtain

$$\lim_{n\to\infty} E\| P_{S_n | X_{\mathbb{N}^c}} - P_{S_n} \| = 0,$$

where $\mathbb{N}^c := \{\ldots,-1,0\}$, so we discuss loss of memory of the past $X_{\mathbb{N}^c}$ of the controlling process $X_{\mathbb{Z}}$. The condition on asymptotic independence that we require in this section to derive such a limit relation is much heavier than condition (5.1.2) used in the preceding chapter. We suppose that the process $X_{\mathbb{Z}}$ is weak Bernoulli. In the present context this assumption is most clearly formulated as

$$\lim_{n\to\infty} E\| P_{X_{\mathbb{N}+n} | X_{\mathbb{N}^c}} - P_{X_{\mathbb{N}+n}} \| = 0.$$

The following two theorems are the main results of this section. They bring us close to the renewal theory in Section 6.3. In fact we need only a simple Palm theoretic consideration to derive our renewal theorems from the present loss of memory results.

A proof of both theorems below can be found at the end of this section.

THEOREM 6.1.1. *Let* $S_{\mathbb{Z}}$ *be a random walk controlled by a stationary sequence* $X_{\mathbb{Z}}$, *satisfying* (5.1.3). *The following two statements are equivalent:*

(i) *the process* $X_{\mathbb{Z}}$ *is weak Bernoulli and* $S_{\mathbb{Z}}$ *is spread out with respect to* $X_{\mathbb{Z}}$;

(ii) *the random walk* $S_{\mathbb{Z}}$, *controlled by* $X_{\mathbb{Z}}$, *satisfies*

$$\lim_{n\to\infty} E\| P_{S_n+h, X_{\mathbb{N}+n} | X_{\mathbb{N}^c}} - P_{S_n, X_{\mathbb{N}+n}} \| = 0, \quad h \textit{ real}.$$

A comparison of this theorem with Theorem 5.3.2 shows an interesting difference. In Theorem 5.3.2 we construct an equivalence of the spread out condition with a limit relation. The asymptotic independence condition is not mentioned in this equivalence relation. The theorem above states that the limit property (ii) is equivalent with both the spread out condition and the condition of asymptotic independence.

A variation on the same theme is the following limit theorem. If ν is an absolutely continuous probability measure on the real line and Q is a measure on $\mathbb{R}^1 \times \Gamma_1$, with Γ_1 a Borel space, we write

$$\nu * Q(B) = \int Q(T_t^{-1} B) d\nu(t),$$

for any measurable set $B \subset \mathbb{R}^1 \times \Gamma_1$.

THEOREM 6.1.2. *Let* $S_{\mathbb{Z}}$ *be a random walk controlled by a stationary sequence* $X_{\mathbb{Z}}$, *satisfying* (5.1.3). *The following two statements are equivalent:*

(i) *the process* $X_{\mathbb{Z}}$ *is weak Bernoulli and* $S_{\mathbb{Z}}$ *is strongly nonlattice with respect to* $X_{\mathbb{Z}}$;

(ii) *for any absolutely continuous probability measure* ν *on the real line*

$$\lim_{n\to\infty} E\| \nu * P_{S_n+h, X_{\mathbb{N}+n} | X_{\mathbb{N}^c}} - \nu * P_{S_n, X_{\mathbb{N}+n}} \| = 0, \quad h \textit{ real}.$$

To prepare the proofs of these theorems we shall refine the limit relations of Theorems 5.2.3 and 5.3.2. These limit relations are given in Chapter 5 for L_1-convergence. It is possible to give them also for a.s.-convergence and even such that the convergence is uniform on compact sets. This result will be described in Proposition 6.1.5 and a note to this proposition. A simple corollary will then pave the way to the proofs of the theorems above.

Define for real h the random variable

$$\phi_n(h) := \| P_{S_n+h | X_{K_n^c}} - P_{S_n | X_{K_n^c}} \|,$$

where $K_n := \{1,\dots,n\}$, $n \geq 1$. If we consider ϕ_n as a function in h we obtain the inequality

$$(6.1.1) \qquad \phi_n(h_1+h_2) \leq \phi_n(h_1) + \phi_n(h_2)$$

for real h_1 and h_2. The functions ϕ_n satisfy also another interesting property:

PROPOSITION 6.1.3. *The sequence* $\phi_n(h)$, $n \geq 1$, *is a reverse submartingale, i.e.*

$$E(\phi_n(h) | \phi_m(h),\ m > n) \geq \phi_{n+1}(h), \quad n \geq 1.$$

PROOF. Because $S_{\mathbb{Z}}$ is controlled by $X_{\mathbb{Z}}$, the increment $\xi_{n+1} := S_{n+1} - S_n$ is a function of X_{n+1}. Hence

$$\phi_n(h) = \| P_{S_{n+1}+h | X_{K_n^c}} - P_{S_{n+1} | X_{K_n^c}} \|$$

and because $K_{n+1}^{c} \subset K_{n}^{c}$ we have, using Lemma 5.2.5 for conditional distributions, given $X_{K_{n+1}^{c}}$,

$$\| P_{S_{n+1}+h|X_{K_{n+1}^{c}}} - P_{S_{n+1}|X_{K_{n+1}^{c}}} \| \le E(\phi_n(h)|X_{K_{n+1}^{c}}).$$

Because $\phi_m(h)$, $m > n$, is $X_{K_{n+1}^{c}}$-measurable, this implies the assertion. □

In Proposition 6.1.5 we strengthen the limit relation used in (5.3.2). In its proof we need a consequence of property (6.1.1) that is obtained from the following lemma.

LEMMA 6.1.4. *Suppose* $g\colon \mathbb{R}^1 \to [0,\infty)$ *is a measurable function that satisfies for real* h_1 *and* h_2

$$g(h_1+h_2) \le g(h_1)+g(h_2).$$

Let ε *and* b *be positive numbers such that* $0 < \varepsilon < b$ *and define*

$$B(\varepsilon) := \{h \in [-b,b]\colon g(h) > \varepsilon\}.$$

If we have

$$\ell(B(\tfrac{1}{4}\varepsilon)) \le \tfrac{1}{4}\varepsilon,$$

with ℓ *the Lebesgue measure, then*

$$B(\varepsilon) = \emptyset.$$

PROOF. Let $c \in [\frac{1}{2}b,b]$ and note that the set

$$\{x \in [0,c]\colon x \notin B(\tfrac{1}{4}\varepsilon),\ c-x \notin B(\tfrac{1}{4}\varepsilon)\}$$

has positive Lebesgue measure and thus is not empty. Select a number y in this set and note that

$$g(c) \le g(y)+g(c-y) \le \tfrac{1}{2}\varepsilon.$$

Hence g is dominated by $\frac{1}{2}\varepsilon$ on $[\frac{1}{2}b,b]$. Similarly one proves that g is dominated by $\frac{1}{2}\varepsilon$ on $[-b,-\frac{1}{2}b]$. Because each $h \in (-\frac{1}{2}b,\frac{1}{2}b)$ can be written as sum $h = h_1+h_2$ of elements out of these two sets, it follows that g is dominated by ε on $(-\frac{1}{2}b,\frac{1}{2}b)$. □

PROPOSITION 6.1.5. *If for each real* h *holds*

$$\lim_{n\to\infty} E\,\phi_n(h) = 0, \tag{6.1.2}$$

then for each compact set B

$$\lim_{n\to\infty} \sup_{h\in B} \phi_n(h) = 0 \text{ a.s.}$$

PROOF. Define

$$\phi_\infty(h) := \limsup_{n\to\infty} \phi_n(h).$$

Because ϕ_n is a reverse submartingale, nonnegative and bounded by 2, we have by a martingale theorem (see CHUNG [1974, Theorem 9.4.7])

$$\phi_n(h) \to \phi_\infty(h) \quad \text{for } n \to \infty$$

for a.s.-convergence and for convergence in L_1-norm. Using (6.1.2) it follows that for each real h we have $\phi_\infty(h) = 0$ a.s. The set

$$\mathcal{H} := \{h\colon \phi_\infty(h) = 0\}$$

satisfies, by Fubini's theorem,

$$E\,\ell(\mathcal{H}^c) = \int P(\phi_\infty(h) > 0)\,d\ell(h) = 0.$$

Here ℓ is the Lebesgue measure. Observe that (6.1.1) holds also for $n = \infty$. Using Lemma 6.1.4 we obtain that $\phi_\infty \equiv 0$ a.s. Hence we have for all real h

$$\lim_{n\to\infty} \phi_n(h) = 0 \text{ a.s.}$$

and therefore, for $n \to \infty$,

$$B_n(\varepsilon) := \{h\colon \forall m \geq n\colon \phi_m(h) > \varepsilon\} \downarrow \emptyset \text{ a.s.}$$

If B is a compact set, choose a real number b so large that $[-b,b] \supset B$. Because

$$\lim_{n\to\infty} \ell(B_n(\varepsilon) \cap [-b,b]) = 0 \text{ a.s.}$$

we obtain by Lemma 6.1.4 that if n is larger than some a.s.-finite random number, then

$$B_n(\varepsilon) \cap B \subset B_n(\varepsilon) \cap [-b,b] = \phi. \qquad \square$$

NOTE. It follows that in (5.3.1) one might also use instead of L_1-convergence

$$\lim_{n\to\infty} \sup_{|h|<b} \| P_{S_n+h|X_{K_n^c}} - P_{S_n|X_{K_n^c}} \| = 0 \text{ a.s.,} \quad b \text{ real.}$$

A similar property holds for

$$\phi_n^\nu(h) := \| \nu * P_{S_n+h|X_{K_n^c}} - \nu * P_{S_n|X_{K_n^c}} \|,$$

where ν is a probability measure on the real line. Also for these functions the inequality (6.1.1) holds and for fixed h the sequence $\phi_n^\nu(h)$, $n \geq 1$, is a reverse submartingale. Hence Proposition 6.1.5 holds also for ϕ_n^ν, $n \geq 1$, instead of ϕ_n, $n \geq 1$. It follows that the limit relation of Theorem 5.2.3 is equivalent with

$$\lim_{n\to\infty} \sup_{|h|\leq b} \| \nu * P_{S_n+h|X_{K_n^c}} - \nu * P_{S_n|X_{K_n^c}} \| = 0 \text{ a.s.,} \quad b \text{ positive,}$$

for any absolutely continuous probability measure ν on the real line.

COROLLARY 6.1.6. *Suppose* K *is a finite integer set. Let* U *be a real,* X_K-*measurable random variable. If* (6.1.2) *holds, then*

$$\lim_{n\to\infty} \| P_{S_n+U|X_{K_n^c}} - P_{S_n|X_{K_n^c}} \| = 0 \text{ a.s.,}$$

where $K_n := \{1,\ldots,n\}$.

PROOF. Suppose that $m \geq 1$ is so large that $K \subset \mathbb{N}^c \cup K_m$. By the stationarity of $X_{\mathbb{Z}}$ and (6.1.2) we have for each compact set B

$$\lim_{n\to\infty} \sup_{h\in B} \| P_{S_n-S_m+h|X_{(K_n\setminus K_m)^c}} - P_{S_n-S_m|X_{(K_n\setminus K_m)^c}} \| = 0 \text{ a.s.}$$

First suppose that $U \in B$ a.s. Then by the limit property above and because U is $X_{(K_n\setminus K_m)^c}$-measurable,

$$(6.1.3) \qquad Y_n := \| P_{S_n-S_m+U|X_{(K_n\setminus K_m)^c}} - P_{S_n-S_m|X_{(K_n\setminus K_m)^c}} \| \to 0 \text{ a.s.}$$

for $n \to \infty$. Because S_m is also $X_{(K_n\setminus K_m)^c}$-measurable, we have

$$Y_n = \| P_{(S_n - S_m + U) + S_m | X_{(K_n \setminus K_m)^c}} - P_{(S_n - S_m) + S_m | X_{(K_n \setminus K_m)^c}} \| .$$

By Lemma 5.2.5, applied to the conditional distribution, given $X_{K_n^c}$, we have

$$\| P_{S_n + U | X_{K_n^c}} - P_{S_n | X_{K_n^c}} \| \le E(Y_n | X_{K_n^c}) .$$

Because Y_n is bounded, the right-hand side converges to a.s. for $n \to \infty$ by the bounded convergence theorem and (6.1.3). This proves the assertion for bounded U.

If U is unbounded, note that for each $b > 0$

$$Z_n := \| P_{U + S_n | X_{K_n^c}} - P_{S_n | X_{K_n^c}} \| \le 2\, P(|U| > b | X_{K_n^c}) + V_n^b ,$$

where $\lim_{n\to\infty} V_n^b = 0$ a.s. by what we already proved. Hence

$$\limsup_{n\to\infty} Z_n \le 2 \lim_{n\to\infty} P(|U| > b | X_{K_n^c}) = 2Y_\infty(b) \text{ a.s.}$$

by a martingale theorem (see BREIMAN [Theorem 5.24]), where $Y_\infty(b)$ has expectation $P(|U| > b)$ and is monotone in b. Therefore $\lim_{b\to\infty} Y_\infty(b) = 0$ a.s. and so the assertion follows by letting $b \to \infty$ in the inequality above.□

PROOF of Theorem 6.1.1. First we show that (ii) implies (i). Obviously (ii) implies that $X_{\mathbb{Z}}$ is weak Bernoulli. Using the triangle inequality it follows that

$$E \| P_{S_n + h, X_{\mathbb{N}+n} | X_{\mathbb{N}^c}} - P_{S_n, X_{\mathbb{N}+n} | X_{\mathbb{N}^c}} \| \to 0$$

for $n \to \infty$ with h arbitrary real. By an application of (4.1.5) to conditional distributions, given $X_{\mathbb{N}^c \cup (\mathbb{N}+n)}$, it follows that the expression above equals

$$E \| P_{S_n + h | X_{\mathbb{N}^c \cup (\mathbb{N}+n)}} - P_{S_n | X_{\mathbb{N}^c \cup (\mathbb{N}+n)}} \| .$$

Hence also this expression converges to zero for $n \to \infty$. By Theorem 5.3.2 the random walk $S_{\mathbb{Z}}$ is spread out with respect to $X_{\mathbb{Z}}$. Hence (ii) implies (i).

To prove the converse note that it is sufficient to show that

$$Z_n := \| P_{S_n, X_{\mathbb{N}+n} | X_{\mathbb{N}^c}} - P_{S_n, X_{\mathbb{N}+n}} \|$$

converges to 0 in L_1-norm. This follows from the inequality

$$E\| P_{S_n+h, X_{\mathbb{N}+n} | X_{\mathbb{N}^c}} - P_{S_n, X_{\mathbb{N}+n}} \|$$

$$\leq E\| P_{S_n+h, X_{\mathbb{N}+n} | X_{\mathbb{N}^c}} - P_{S_n+h, X_{\mathbb{N}+n}} \|$$

$$+ E\| P_{S_n+h, X_{\mathbb{N}+n}} - P_{S_n, X_{\mathbb{N}+n}} \|$$

$$= EZ_n + E\| P_{S_n+h | X_{\mathbb{N}+n}} - P_{S_n | X_{\mathbb{N}+n}} \|$$

by the definition of Z_n and (4.1.5). The last term vanishes asymptotically by Theorem 5.3.2 and Lemma 5.2.5. Hence we have to prove $\lim_{n\to\infty} EZ_n = 0$.

Choose m so large that for some arbitrary $\varepsilon > 0$

$$\perp(X_{\mathbb{N}^c}, X_{\mathbb{N}+m}) < \varepsilon$$

and note that by Lemma 6.1.6, with $U := S_{-m}$,

(6.1.4) $$\| P_{S_n - S_m, X_{K_n^c}} - P_{S_n, X_{K_n^c}} \| = E\| P_{S_n - S_m | X_{K_n^c}} - P_{S_n | X_{K_n^c}} \| \to 0$$

for $n \to \infty$. Here the equality is justified by (4.1.5). Construct $\tilde{X}_{\mathbb{N}+n}$ independent of $X_{\mathbb{N}+n}$. This is possible by Corollary 4.2.5 in such a way that

$$P(\tilde{X}_{\mathbb{N}+m} \neq X_{\mathbb{N}+m}) = \perp(X_{\mathbb{N}^c}, X_{\mathbb{N}+m}) < \varepsilon.$$

Let $\tilde{X}$ and X be coupled over $\mathbb{N}^c+m$. Using (1.1.1) we obtain from (6.1.4) that for n sufficiently large

$$\| P_{\tilde{S}_n - \tilde{S}_m, \tilde{X}_{\mathbb{N}+m}, \tilde{X}_{\mathbb{N}^c}} - P_{\tilde{S}_n, \tilde{X}_{\mathbb{N}+n}, \tilde{X}_{\mathbb{N}^c}} \| < \varepsilon + 2\cdot 2P(X_{\mathbb{Z}} \neq \tilde{X}_{\mathbb{Z}}) < 5\varepsilon.$$

Because $\tilde{X}_{\mathbb{N}^c}$ and $\tilde{X}_{\mathbb{N}+m}$ are independent, the term on the left equals for $n \geq m$

$$P_{\tilde{S}_n - \tilde{S}_m, \tilde{X}_{\mathbb{N}+n}} \times P_{\tilde{X}_{\mathbb{N}^c}} .$$

Using (1.1.1) once again, we obtain for n sufficiently large,

$$\| P_{S_n - S_m, X_{\mathbb{N}+n}} \times P_{X_{\mathbb{N}^c}} - P_{S_n, X_{\mathbb{N}+n}, X_{\mathbb{N}^c}} \| < 5\varepsilon + 2\cdot 2P(X_{\mathbb{Z}} \neq \tilde{X}_{\mathbb{Z}}) < 9\varepsilon.$$

By Proposition 4.1.3 and the triangle inequality we have

$$EZ_n = \| P_{S_n, X_{\mathbb{N}+n}} \times P_{X_{\mathbb{N}^c}} - P_{S_n, X_{\mathbb{N}+n}, X_{\mathbb{N}^c}} \|$$

$$\leq \| P_{S_n, X_{\mathbb{N}+n}} \times P_{X_{\mathbb{N}^c}} - P_{S_n - S_m, X_{\mathbb{N}+n}} \times P_{X_{\mathbb{N}^c}} \|$$

$$+ \| P_{S_n - S_m, X_{\mathbb{N}+n}} \times P_{X_{\mathbb{N}^c}} - P_{S_n, X_{\mathbb{N}+n}, X_{\mathbb{N}^c}} \|$$

$$< 9\varepsilon + 9\varepsilon = 18\varepsilon$$

for n sufficiently large. This implies $\lim_{n\to\infty} EZ_n = 0$. □

PROOF of Theorem 6.1.2. The proof of this theorem is almost the same as the proof above. We give a sketch. To obtain the proof of (ii) → (i) we apply Theorem 5.2.3 instead of Theorem 5.3.2. With this change we can follow the proof above.

To prove the converse we have to adapt Corollary 6.1.6. By Theorem 5.2.3 we may suppose that for each absolutely continuous probability measure ν holds

$$\lim_{n\to\infty} E\| \nu * P_{S_n+h | X_{K_n^c}} - \nu * P_{S_n | X_{K_n^c}} \| = 0, \qquad h \text{ real}.$$

Using the note to Proposition 6.1.5 we prove that if U is X_K-measurable with $K \subset \mathbb{Z}$ finite, then

$$\lim_{n\to\infty} \| \nu * P_{S_n+U | X_{K_n^c}} - \nu * P_{S_n | X_{K_n^c}} \| = 0 \text{ a.s.}$$

If this limit relation is used instead of the limit relation of Corollary 6.1.6, then we can follow the argument in the proof above of (i) → (ii) to obtain the assertion. □

6.2. NONLATTICE AND STRONGLY NONLATTICE RANDOM WALKS

Let F be a probability distribution on the real line. In Chapter 1 Blackwell's theorem is formulated for random walks with independent, F-distributed increments under a nonlattice condition on F. Apart from a nonlattice condition a strongly nonlattice concept is used in Chapter 1. In Section 5.2 we generalized this strongly nonlattice concept for random walks with dependent increments. In this section we define a nonlattice concept for random walks controlled by a stationary sequence $X_{\mathbb{Z}}$. The main results, Theorem 6.2.2 and its corollary, study the gap between the nonlattice and the strongly nonlattice concepts, in case $X_{\mathbb{Z}}$ is weak Bernoulli. The proofs of these side-results are tedious and use techniques of Chapter 5.

A distribution F on the real line is called *weakly lattice* if F is concentrated on a displaced lattice $c + L_d$, $d > 0$, with L_d defined by (1.1.2). Equivalently we can say that F is weakly lattice if F^{n*} is concentrated on a displaced lattice $c_n + L_d$, $d > 0$, for all $n \geq 1$. Here the displacement c_n may be different for each $n \geq 1$, but note that we may choose $c_n = nc_1$, $n \geq 1$. The distribution F is said to be *lattice* if F is concentrated on a lattice L_d, $d > 0$, or equivalently if F^{n*} is concentrated on $c + L_d$ for all $n \geq 1$, where c does not depend on $n \geq 1$. Note that in the last formulation necessarily $c \in L_d$.

Let us assume in this section that $S_{\mathbb{Z}}$ is a random walk controlled by a stationary sequence $X_{\mathbb{Z}}$ of random variables with values in a Borel space Γ. In Section 5.2 we defined $S_{\mathbb{Z}}$ to be *weakly lattice with respect to* $X_{\mathbb{Z}}$ if for some $d > 0$ the random variable (S_n) mod d is, up to a null set, $(X_{\mathbb{N}^c}, X_{\mathbb{N}+n})$-measurable, $n \geq 1$, i.e. if there is a function

$$c_n : \Gamma^{\mathbb{N}^c} \times \Gamma^{\mathbb{N}} \to [0,d),$$

such that

$$(S_n) \bmod d = c_n(X_{\mathbb{N}^c}, X_{\mathbb{N}+n}) \quad \text{a.s.,} \qquad n \geq 1.$$

Define $S_{\mathbb{Z}}$ to be *lattice with respect to* $X_{\mathbb{Z}}$ if for some $d > 0$ there is a function

$$c: \Gamma^{\mathbb{N}^c} \times \Gamma^{\mathbb{N}} \to [0,d),$$

such that

$$(S_n) \bmod d = c(X_{\mathbb{N}^c}, X_{\mathbb{N}+n}) \quad \text{a.s.,} \qquad n \geq 1. \tag{6.2.1}$$

Here the function c does not depend on $n \geq 1$. We call $S_{\mathbb{Z}}$ *nonlattice* with respect to $X_{\mathbb{Z}}$ if there is no such $d > 0$. In case for a random walk $S_{\mathbb{Z}}$ no controlling process $X_{\mathbb{Z}}$ is specified, we take $X_{\mathbb{Z}}$ to be the process of increments.

EXAMPLE 6.2.1.

(i) Let F be a nonlattice distribution on the real line. A random walk $S_{\mathbb{Z}}$ with independent, F-distributed increments is nonlattice with respect to its process of increments $X_{\mathbb{Z}}$. To see this, note that we can replace the requirement (6.2.1) in the definition of a lattice random walk above by the condition

$$(S_n) \bmod d = c \quad \text{a.s.},$$

with c constant, because S_n and $(X_{\mathbb{N}^c}, X_{\mathbb{N}+n})$ are independent. Therefore, the random walk $S_{\mathbb{Z}}$ is lattice with respect to $X_{\mathbb{Z}}$ if and only if F^{n*} is concentrated on $c + L_d$, $n \geq 1$, for some $c \in [0,d)$, $d > 0$, so if and only if F is lattice. This proves the assertion above.

(ii) A strongly nonlattice random walk is nonlattice (see Example 5.2.1).

(iii) Let $S_{\mathbb{Z}}$ be controlled by a stationary, Markov dependent sequence $X_{\mathbb{Z}}$ with values in a Borel space Γ. Then $S_{\mathbb{Z}}$ is lattice if and only if for some $d > 0$ there exists a function $c: \Gamma\times\Gamma \to [0,d)$, such that

$$(S_n) \bmod d = c(X_0, X_{n+1}) \quad \text{a.s.}, \qquad n \geq 1.$$

To see this, note that S_n is $(X_1,\ldots,X_n)$-measurable (use (5.0.2)). Hence, by Markov dependence, S_n is independent of $(X_{\mathbb{N}^c}, X_{\mathbb{N}+n})$, given (X_0, X_{n+1}). The assertion is an easy consequence of this observation.

(iv) Let Γ be a finite set of positive numbers that are linearly independent over the rational numbers. Assume $X_{\mathbb{Z}}$ is a stationary, weak Bernoulli sequence with values in Γ such that $P(X_n = \gamma) > 0$ for each $\gamma \in \Gamma$. Let $S_{\mathbb{Z}}$ be the random walk with process of increments $X_{\mathbb{Z}}$. This random walk example is taken from GUREVIČ [1967]. The random walk $S_{\mathbb{Z}}$ is nonlattice with respect to $X_{\mathbb{Z}}$. This follows by comparing Theorem 4.2 in GUREVIČ [1967] and Theorem 6.5.9. A direct proof does not seem to be easy.

How can we justify the definition of the nonlattice concept above? In Chapter 1 we have shown that for random walks with stationary, independent increments the strongly nonlattice condition is equivalent with the limit relation

(6.2.2) $$\lim_{n\to\infty} \| \nu * P_{S_n} - \nu * P_{S_n + h} \| = 0, \qquad h \text{ real},$$

for any absolutely continuous probability distribution ν on the real line. In Section 5.2 we could justify our definition of strongly nonlattice by proving in Theorem 5.2.3 that the strongly nonlattice condition is equivalent to a limit relation similar as (6.2.2). Such a justification is also given for the nonlattice concept in Sections 6.4 and 6.5.

In the following theorem we compare the lattice and weakly lattice concepts. The proof of the theorem can be found at the end of the section. It needs a long preparation.

To explain the contents of the following theorem, consider a random walk $S_{\mathbb{Z}}$ that is weakly lattice with respect to $X_{\mathbb{Z}}$ with minimal weak lattice width $d = \infty$, i.e. there are measurable functions c_n such that

$$S_n = c_n(X_{\mathbb{N}^c}, X_{\mathbb{N}+n}) \text{ a.s.}, \quad n \geq 1.$$

Our aim is to investigate how the functions c_n depend on n. We prove that if $X_{\mathbb{Z}}$ is weak Bernoulli, there is a constant $\bar{c}$ such that

$$S_n = c(X_{\mathbb{N}^c}, X_{\mathbb{N}+n}) + n\bar{c} \text{ a.s.}, \quad n \geq 1.$$

In general we have the following result:

<u>THEOREM 6.2.2</u>. *Let* $S_{\mathbb{Z}}$ *be a random walk controlled by a stationary sequence* $X_{\mathbb{Z}}$ *with values in a Borel space* Γ. *Suppose* $X_{\mathbb{Z}}$ *is weak Bernoulli and satisfies condition* (5.1.3). *Let* $S_{\mathbb{Z}}$ *be weakly lattice with respect to* $X_{\mathbb{Z}}$, *with minimal weak lattice width* d. *Then there exists a real number* $\bar{c}$ *and a measurable function*

$$c: \Gamma^{\mathbb{N}^c} \times \Gamma^{\mathbb{N}} \to \mathbb{R}^1$$

such that

$$(S_n - n\bar{c}) \bmod d = c(X_{\mathbb{N}^c}, X_{\mathbb{N}+n}) \text{ a.s.}$$

<u>COROLLARY 6.2.3</u>. *Let the conditions of Theorem* 6.2.2 *hold. If* $S_{\mathbb{Z}}$ *is nonlattice with respect to* $X_{\mathbb{Z}}$ *then* d *is finite, and in case* $d > 0$, *the numbers* $\bar{c}$ *and* d *are mutually prime.*

<u>PROOF</u>. Use the definition of nonlattice. □

Assume that the conditions of Theorem 6.2.2 hold. By the definition of weakly lattice we have

$$(S_n) \bmod d = c_n(X_{\mathbb{N}^c}, X_{\mathbb{N}+n}) \text{ a.s.}, \quad n \geq 1,$$

for functions $c_n : \Gamma^{\mathbb{N}^c} \times \Gamma^{\mathbb{N}} \to \mathbb{R}^1$. Suppose that we succeed to select c_n, $n \geq 1$, such that on the domain of the functions c_n

(6.2.3) $\qquad (c_{n+1} - c_n) \bmod d \equiv \bar{c}, \quad n \geq 1,$

where $\bar{c}$ is a constant, not depending on n. Then we can define the function c by $c := (c_n - n\bar{c}) \bmod d$. Note that c does not depend on n. Then the assertion of the theorem follows because, using this definition of c, we have

$$(S_n) \bmod d = (c(X_{\mathbb{N}^c}, X_{\mathbb{N}+n}) + n\bar{c}) \bmod d \text{ a.s.}$$

Apparently we have to investigate the difference in (6.2.3).

Of great importance in the considerations below is the following property of weak Bernoulli processes. Define the measures P_n, $n \geq 1$, on $\Gamma^{\mathbb{N}^c} \times \Gamma^{\mathbb{N}}$ by

$$P_n := P_{X_{\mathbb{N}^c}, X_{\mathbb{N}+n}}, \quad n \geq 1.$$

Because $X_{\mathbb{Z}}$ is weak Bernoulli, we have

$$\lim_{n\to\infty} \| P_n - P_{X_{\mathbb{N}^c}} \times P_{X_{\mathbb{N}}} \| = 0.$$

Hence P_n, $n \geq 1$, is a Cauchy sequence and by (4.1.1) we have the property

(6.2.4) $\qquad \lim_{n\to\infty} \| P_n \wedge P_{n+1} \| = 1.$

This property will help us to investigate the difference $c_{n+1} - c_n$.

Let us consider the substochastic measures F_n, $n \geq 1$, on the real line, defined by

$$F_n(B) := (P_n \wedge P_{n+1})((c_{n+1} - c_n) \bmod d \in B), \quad B \in \mathcal{B}^1.$$

The measures F_n, $n \geq 1$, contain important information about $c_{n+1} - c_n$.

LEMMA 6.2.4. F_n, $n \geq 1$, *is a nondescending sequence of measures.*

PROOF. Note that for $k \geq 1$, using the notation (5.0.2),

$$c_{k+1}(X_{\mathbb{N}^c}, X_{\mathbb{N}+k+1}) = (S_{k+1}) \bmod d =$$

$$= (S_k + f(X_{k+1})) \bmod d$$

$$= (c_k(X_{\mathbb{N}^c}, X_{\mathbb{N}+k}) + f(X_{k+1})) \bmod d \text{ a.s.}$$

Apply this equality for $k = n$ and $k = n+1$ to get for any Borel set B on the real line

$$F_n(B) = \int \chi_B((c_{n+1}(x,y) - c_n(x,y)) \bmod d)\, dP_n \wedge P_{n+1}(x,y)$$

$$= \int \chi_B(([c_{n+2}(x,Ty) - f(y_1)] - [c_{n+1}(x,Ty) - f(y_1)] \bmod d)\, dP_n \wedge P_{n+1}(x,y).$$

Here we write $Ty := (y_{n+1})_{n\geq 1}$, if $y = (y_n)_{n\geq 1}$. Because for each measurable set $C \subset \Gamma^{\mathbb{N}^c} \times \Gamma^{\mathbb{N}}$ we have

$$P_n(\{(x,y): (x,Ty) \in C\}) = P_{n+1}(C),$$

it follows that

$$P_n \wedge P_{n+1}(\{(x,y): (x,Ty) \in C\}) \leq P_{n+1} \wedge P_{n+2}(C).$$

Therefore we have with $z = ty$

$$F_n(B) \leq \int \chi_B((c_{n+2}(x,z) - c_{n+1}(x,z))\, dP_{n+1} \wedge P_{n+2}(x,z)$$

$$= F_{n+1}(B), \quad B \in \mathcal{B}^1. \qquad \square$$

By (6.2.4) the total mass of F_n converges to 1 for $n \to \infty$. By the lemma above it follows that F_n converges in total variation to a probability measure F. Our aim is to prove that F is degenerate at $\{\bar{c}\}$. Once this is proved it is not very difficult to show that (6.2.3) holds. We already saw above that this proves the theorem.

The following proposition shows how we can study F.

<u>PROPOSITION 6.2.5.</u> *Let $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$ be processes on the same probability space, such that*

$$\delta(X_{\mathbb{Z}}, X'_{\mathbb{Z}}),\ \delta(X_{\mathbb{Z}}, X''_{\mathbb{Z}}) < \varepsilon,$$

while

(6.2.5) $\quad X'_j = X''_j \qquad j \le 0,$

$\qquad\quad = X''_{j+1} \qquad j > n.$

Then we have

$$\| P_{(S''_{n+1} - S'_n) \bmod d} - F \| < 5\varepsilon.$$

PROOF. Write C_n, C'_n and C''_{n+1} for $c_n(X_{\mathbb{N}^c}, X_{\mathbb{N}+n}), \dots$ and write P'_n and P''_{n+1} for $P_{X'_{\mathbb{N}^c}, X'_{\mathbb{N}+n}}, \dots$. Using (4.1.1) we obtain the inequality

$$P((S'_n) \bmod d \neq C'_n) \le \delta(X_{\mathbb{Z}}, X'_{\mathbb{Z}}) + P((S_n) \bmod d \neq C_n) < \varepsilon + 0 = \varepsilon.$$

Using a similar estimate for S''_{n+1} we obtain

(6.2.6) $\quad P((S''_{n+1} - S'_n) \bmod d \neq (C''_{n+1} - C'_n) \bmod d) < 2\varepsilon.$

By our assumption on $\delta(X_{\mathbb{Z}}, X'_{\mathbb{Z}})$ and $\delta(X_{\mathbb{Z}}, X''_{\mathbb{Z}})$ we have

$$\tfrac{1}{2}\| P_n - P'_n \| < \varepsilon, \quad \tfrac{1}{2}\| P_{n+1} - P''_{n+1} \| < \varepsilon$$

and because $P'_n = P''_{n+1}$ by (6.2.5), one easily derives that

$$R := P_n \wedge P'_n \wedge P_{n+1} \wedge P''_{n+1}$$

has at least mass $1 - 2\varepsilon$. The substochastic measure G on the real line, defined by

$$G(B) := \int \chi_B((c_{n+1} - c_n) \bmod d)\, dR$$

is dominated by F_n and hence

$$\| F_n - G \| \le 1 - \| G \| = 1 - \| R \| \le 2\varepsilon.$$

By the definitions of C'_n, C''_{n+1} and (6.2.5)

$$G \le P_{(C''_{n+1} - C'_n) \bmod d}.$$

Therefore, by the triangle inequality,

$$\|P_{(S''_{n+1}-S'_n) \bmod d} - F\|$$

$$\leq \|P_{(S''_{n+1}-S'_n) \bmod d} - P_{(C''_{n+1}-C'_n) \bmod d}\|$$

$$+ \|P_{(C''_{n+1}-C'_n) \bmod d} - G\| + \|G-F_n\| + \|F_n-F\|$$

$$< 2\cdot 2\varepsilon + (1- \|G\|) + 2\varepsilon + (1- \|G\|) < 10\varepsilon.$$

In this inequality we used in the estimation of the first term (6.2.6) and (1.1.1), and in the last term $F \geq F_n \geq G$. □

Using the proposition above it will be possible to obtain a proof that F is degenerate in the following way. Suppose that we succeed in constructing processes $X'_{\mathbb{Z}}$, $X''_{\mathbb{Z}}$ and $\tilde{X}_{\mathbb{Z}}$, approximately distributed as $X_{\mathbb{Z}}$, such that

(6.2.7) $$\begin{aligned} \tilde{X}_j &= X'_j \quad j \leq 0, & \tilde{X}_j &= X''_j \quad j \leq n,\\ &= X'_{j+1} \quad j > k, & &= X''_{j+1} \quad j > n+k,\end{aligned}$$

where $n \gg k$. Then we have

(6.2.8) $$\begin{aligned} \tilde{X}_j &= X'_j = X''_j \quad j \leq 0,\\ \tilde{X}_j &= X'_{j+1} = X''_{j+1} \quad j > n+k.\end{aligned}$$

As a consequence we have, with large probability,

$$(S'_{n+k+1}) \bmod d = c_{n+k+1}(X'_{\mathbb{N}^c}, X'_{\mathbb{N}+n+k+1}) = (S''_{n+k+1}) \bmod d.$$

Apart from this property we have, with large probability,

$$\begin{aligned}(S'_{n+k+1} - S''_{n+k+1}) \bmod d &= ((S'_{n+k+1} - \tilde{S}_{n+k}) + (\tilde{S}_{n+k} - S''_{n+k+1})) \bmod d\\ &= (Z_1 - Z_2) \bmod d,\end{aligned}$$

where

$$Z_1 = (S'_{k+1} - \tilde{S}_k) \bmod d$$

and

$$Z_2 = ((S''_{n+k+1} - S''_n) - (\tilde{S}_{n+k} - \tilde{S}_n)) \bmod d.$$

By (6.2.7) and Proposition 6.2.5 the random variables Z_1 and Z_2 are approximately distributed as F. Suppose that, using $n \gg k$, it is possible to show that Z_1 and Z_2 are independent. Then by means of

$$(Z_1 - Z_2) \bmod d = 0$$

it is easy to show that $(Z_1) \bmod d = \bar{c}$, with $\bar{c}$ constant. This will then prove that F is degenerate.

To be able to prove that $X_{\mathbb{Z}}$ can be approximated as mentioned above, we have to investigate the dependence structure of the weak Bernoulli process $X_{\mathbb{Z}}$. To this purpose we formulate below an approximation lemma. This lemma uses that $X_{\mathbb{Z}}$ is a weak Bernoulli sequence with values in a Borel space Γ, for which condition (5.1.3) holds.

Define the integer sets $K := \{1,\ldots,k\}$ and $L := \{-\ell_1,\ldots,\ell_2\} \supset K$ and write

$$L^- := \{-\ell_2,\ldots,0\}, \quad L^+ := \{k+1,\ldots,\ell_2\}.$$

Suppose $\tilde{X}_{\mathbb{Z}}$ is a sequence of random variables with values in Γ such that

(i) $\tilde{X}_{\mathbb{N}^c}$ and $\tilde{X}_{\mathbb{N}+k}$ are independent, while

$$\delta(X_{\mathbb{N}^c}, \tilde{X}_{\mathbb{N}^c}), \quad \delta(X_{\mathbb{N}+k}, \tilde{X}_{\mathbb{N}+k}) < \varepsilon;$$

(ii) $\tilde{X}_K$ is conditionally independent of $\tilde{X}_{L^c}$, given $\tilde{X}_{L\setminus K}=x$, with its conditional distribution given by

$$P_{X_K | X_{L\setminus K}=x}.$$

<u>LEMMA 6.2.6</u>. *Suppose* $X_{\mathbb{Z}}$ *is weak Bernoulli and satisfies* (5.1.3). *If* $\tilde{X}_{\mathbb{Z}}$ *satisfies* (i) *and* (ii) *then for each* $\varepsilon > 0$ *there is a number* $\tilde{k} = \tilde{k}(\varepsilon)$ *and for each* $k \geq \tilde{k}$ *there is a number* $\tilde{\ell} = \tilde{\ell}(\varepsilon,k)$ *such that, if* $\ell_1,\ell_2 \geq \tilde{\ell}$ *then*

$$\delta(X_{\mathbb{Z}}, \tilde{X}_{\mathbb{Z}}) < 10\varepsilon.$$

<u>PROOF</u>. Choose $\tilde{k} = \tilde{k}(\varepsilon)$ so large that

$$\perp(X_{\mathbb{N}^c}, X_{\mathbb{N}+\tilde{k}}) < \varepsilon$$

and take for any $k \geq \tilde{k}$ a number $\tilde{\ell} = \tilde{\ell}(k,\varepsilon)$ so large that if $\ell_1,\ell_2 \geq \tilde{\ell}$

$$E \perp_{X_{L\setminus K}}(X_K, X_{L^c}) < \varepsilon.$$

This is possible because $X_{\mathbb{Z}}$ satisfies (5.1.3). By (i) and the triangle inequality

$$\delta(X_{K^c},\tilde{X}_{K^c}) = \tfrac{1}{2}\| P_{X_{\mathbb{N}^c},X_{\mathbb{N}+k}} - P_{\tilde{X}_{\mathbb{N}^c}} \times P_{\tilde{X}_{\mathbb{N}+k}} \|$$

$$\leq \perp(X_{\mathbb{N}^c},X_{\mathbb{N}+k}) + \tfrac{1}{2}\| P_{X_{\mathbb{N}^c}} \times P_{X_{\mathbb{N}+k}} - P_{\tilde{X}_{\mathbb{N}^c}} \times P_{\tilde{X}_{\mathbb{N}+k}} \|$$

and hence, using in the first term Proposition 4.1.2 and in the second the triangle inequality,

$$\delta(X_{K^c},\tilde{X}_{K^c}) \leq \perp(X_{\mathbb{N}^c},X_{\mathbb{N}+\tilde{k}}) + \tfrac{1}{2}\| (P_{X_{\mathbb{N}^c}} - P_{\tilde{X}_{\mathbb{N}^c}}) \times P_{X_{\mathbb{N}+k}} \|$$

$$+ \tfrac{1}{2}\| P_{\tilde{X}_{\mathbb{N}^c}} \times (P_{X_{\mathbb{N}+k}} - P_{\tilde{X}_{\mathbb{N}+k}}) \| < \varepsilon+\varepsilon+\varepsilon = 3\varepsilon .$$

Because $K^c \supset L\backslash K$ we have by (4.1.2)

$$\delta(X_{L\backslash K},\tilde{X}_{L\backslash K}) \leq \delta(X_{K^c},\tilde{X}_{K^c}) < 3\varepsilon$$

and by (ii) and (4.1.6)

$$\delta(X_L,\tilde{X}_L) = \delta(X_{L\backslash K},\tilde{X}_{L\backslash K}) < 3\varepsilon .$$

Because $(X_K,X_{L\backslash K},X_{L^c})$ forms a Markov triple by (ii), we have by Proposition 4.1.4

$$\delta(X_{\mathbb{Z}},\tilde{X}_{\mathbb{Z}}) \leq E\perp_{X_{L\backslash K}}(X_K,X_{L^c}) + \delta(X_{L\backslash K},\tilde{X}_{L\backslash K})$$

$$+ \delta(X_L,X_{L^c}) + \delta(X_{K^c},\tilde{X}_{K^c})$$

$$< \varepsilon+3\varepsilon+3\varepsilon+3\varepsilon = 10\varepsilon .$$ □

<u>NOTE</u>. We can choose $\tilde{k}(\varepsilon)$ decreasing in ε and $\tilde{\ell}(k,\varepsilon)$ decreasing in ε and increasing in k.

With the approximation lemma above we can construct pairs of processes $(X'_{\mathbb{Z}},X''_{\mathbb{Z}})$ as were used in Proposition 6.2.4. The example below illustrates this. Denote for integer sets J

$$n+J := \{n + j\colon j \in J\}.$$

<u>EXAMPLE 6.2.7</u>. Consider a sequence $\tilde{X}_{\mathbb{Z}}$ of random variables with values in Γ. Suppose there are defined integer sets $K := n+\{1,\ldots,k\}$ and $L := n+\{-\ell_1,\ldots,\ell_2\} \supset K$. Let p be an integer, such that the sets

$$K' := n + \{1,\ldots,k+p\},$$

$$L' := n + \{-\ell_1,\ldots,\ell_2+p\}$$

are nonempty integer sets. Let $X'_{\mathbb{Z}}$ be a process, satisfying

$$X'_j = \tilde{X}_j \qquad j \le n,$$
$$= \tilde{X}_{j-p} \qquad j > n+k+p,$$

while $X'_{K'}$ is, given $X'_{L'\setminus K'}=x$, independent of $\tilde{X}_{\mathbb{Z}}$, with conditional distribution

$$P_{X_{K'}|X_{L'\setminus K'}=x}.$$

We call $X'_{\mathbb{Z}}$ a *lengthening* of $\tilde{X}_{\mathbb{Z}}$ *with length* p *at* L\K. Suppose that the process $\tilde{X}^{(n)}_{\mathbb{Z}}$, defined by

$$\tilde{X}^{(n)}_j := \tilde{X}_{n+j}, \qquad j \in \mathbb{Z},$$

satisfies (i) and (ii). By Lemma 6.2.6 we have, if $k+p \ge \tilde{k}(\varepsilon)$ and $\ell_1,\ell_2+p \ge \tilde{\ell}(k+p,\varepsilon)$, that

$$\delta(X_{\mathbb{Z}},X'_{\mathbb{Z}}) < \varepsilon.$$

We shall use lengthenings in applications of Proposition 6.2.4. In particular we construct the process $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$ in (6.2.7) as lengthenings of $\tilde{X}_{\mathbb{Z}}$.

The following result is the key to the proof of Theorem 6.2.2.

PROPOSITION 6.2.8. *Under the conditions of Theorem* 6.2.2, *the distribution* F *is degenerate.*

PROOF. The first part of the proof consists of a construction of a process $\tilde{X}_{\mathbb{Z}}$ such that (i) and (ii) hold for both $\tilde{X}_{\mathbb{Z}}$ and the process $\tilde{X}^{(n)}_{\mathbb{Z}}$, defined in the example above. The second part of the proof constructs lengthenings $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$ of the process $\tilde{X}_{\mathbb{Z}}$ such that (6.2.7) holds (see the example above for the definition of a lengthening).

Part 1. Let η be some positive number. We want to apply Lemma 6.2.6. Take an integer $k \ge k(\eta)$ and choose $\ell_1,\ell_2 \ge \tilde{\ell}(k+1,\eta)$. Define the integer sets

$$K_1 := \{1,\dots,k\}, \qquad L_1 := \{-\ell_1,\dots,\ell_2\},$$
$$K_2 := n+\{1,\dots,k\}, \qquad L_2 := n+\{-\ell_1,\dots,\ell_2\},$$

where n is chosen such that $\inf L_2 - \sup L_1 > m$, with m so large that

$$\perp(X_{\mathbb{N}^c}, X_{\mathbb{N}+m}) < \eta.$$

Apart from the conditions on $\tilde{X}_{\mathbb{Z}}$ and $\tilde{X}^{(n)}_{\mathbb{Z}}$ mentioned above, we shall also need that $\tilde{X}_{L_1}$ and $\tilde{X}_{L_2}$ are independent.

We can write $\mathbb{Z}$ as a disjoint union

$$\mathbb{Z} = M_1 \cup K_1 \cup M_2 \cup K_2 \cup M_3,$$

where $M_1 := \mathbb{N}^c$ and $M_3 := \mathbb{N}+n+k$. The set M_2 can be split up as a disjoint union

$$M_2 = L_1^+ \cup \{\ell_2+1,\dots,n-\ell_1-1\} \cup L_2^-,$$

where $L_1^+ \subset L_1$ and $L_2^- \subset L_2$. By Corollary 4.2.5 we can construct $X^*_{L_1^+}$, distributed as $X_{L_1^+}$ and independent of $X_{L_2^-}$, such that

$$P(X^*_{L_1^+} \neq X_{L_1^+}) = \perp(X_{L_1^+}, X_{L_2^-}).$$

By our choice of n and the stationarity of $X_{\mathbb{Z}}$

$$P(X^*_{L_1^+} \neq X_{L_1^+}) \le \perp(X_{\mathbb{N}^c}, X_{\mathbb{N}+m}) < \eta.$$

This defines X^*_j for $j \in L_1^+$. Let $X^*_j := X_j$ for all other $j \in M_2$. Thus we constructed $X^*_{M_2}$ such that $X^*_{L_1^+}$ and $X^*_{L_2^-}$ are independent, while

$$P(X^*_{M_2} \neq X_{M_2}) < \eta.$$

Construct $\tilde{X}_{M_1}$, $\tilde{X}_{M_2}$ and $\tilde{X}_{M_3}$ as independent random variables, distributed as X_{M_1}, $X^*_{M_2}$ and X_{M_3}. Construct $(\tilde{X}_{K_1},\tilde{X}_{K_2})$, given $(\tilde{X}_{L_1\setminus K_1},\tilde{X}_{L_2\setminus K_2}) = (x_1,x_2)$, independent of the random variables mentioned before, with its conditional distribution, given by

$$P_{X_{K_1}|X_{L_1\setminus K_1}=x_1} \times P_{X_{K_2}|X_{L_2\setminus K_2}=x_2}.$$

Because $\tilde{X}_{M_1 \cup L_1^+}$ and $\tilde{X}_{L_2^+ \cup M_3}$ are independent, also $\tilde{X}_{L_1\setminus K_1}$ and $\tilde{X}_{L_2\setminus K_2}$ are independent. Hence using the properties of $(\tilde{X}_{K_1},\tilde{X}_{K_2})$, also $\tilde{X}_{L_1}$ and $\tilde{X}_{L_2}$ are independent.

To derive (i) and (ii) for $\tilde{X}_{\mathbb{Z}}$ and $\tilde{X}^{(n)}_{\mathbb{Z}}$ we argue as follows. We have:

(j) $\tilde{X}_{M_2}$ and $\tilde{X}_{M_3}$ are independent and

$$\delta(X_{M_2},\tilde{X}_{M_2}) < \eta, \qquad \delta(X_{M_3},\tilde{X}_{M_3}) = 0.$$

Using the properties of $(\tilde{X}_{K_1},\tilde{X}_{K_2})$ and the independence of $\tilde{X}_{L_1\setminus K_1}$ and $\tilde{X}_{L_2\setminus K_2}$ we obtain

(jj) $\tilde{X}_{K_2}$ is, given $\tilde{X}_{L_2\setminus K_2}$, independent of $\tilde{X}_{M_2\cup M_3}$ and has the "right" conditional distribution.

By duplicating the proof of Lemma 6.2.6 we obtain from (j) and (jj), because $k \geq \tilde{k}(\eta)$ and $\ell_1,\ell_2 \geq \tilde{\ell}(k,\eta) \geq \tilde{\ell}(k+1,\eta)$, that

$$\delta(X_{M_2\cup K_2\cup M_3},\tilde{X}_{M_2\cup K_2\cup M_3}) < 10\eta.$$

Note that $\tilde{X}_{\mathbb{Z}}$ satisfies (i) and (ii) with $\varepsilon = 10\eta$. By Lemma 6.2.6

$$\delta(X_{\mathbb{Z}},\tilde{X}_{\mathbb{Z}}) < 10\varepsilon = 100\eta,$$

because $k \geq \tilde{k}(\eta) \geq \tilde{k}(10\eta)$ and $\ell_1,\ell_2 \geq \tilde{\ell}(k+1,\eta) \geq \tilde{\ell}(k,10\eta)$ by the note to Lemma 6.2.5. As a consequence also $\tilde{X}^{(n)}_{\mathbb{Z}}$ satisfies (i) and (ii) with $\varepsilon = 100\eta$.

<u>Part 2</u>. Consider $\tilde{X}_{\mathbb{Z}}$ (with $\tilde{X}_{L_1}$ and $\tilde{X}_{L_2}$ independent). Lengthen $\tilde{X}_{\mathbb{Z}}$ with 1 at $L_1\setminus K_1$ to get $X'_{\mathbb{Z}}$ and lengthen $\tilde{X}_{\mathbb{Z}}$ with 1 at $L_2\setminus K_2$ to get $X''_{\mathbb{Z}}$. Thus we constructed (6.2.7).

We still have to make one important remark. Let

$$K'_1 := \{1,\ldots,k+1\} \quad \text{and} \quad K''_2 := n + \{1,\ldots,k+1\}.$$

Note that the lengthenings can be given such that, given $(\tilde{X}_{L_1\setminus K_1},\tilde{X}_{L_2\setminus K_2})$, the random vectors $(X'_{K'_1},\tilde{X}_{K_1})$ and $(X''_{K''_2},\tilde{X}_{K_2})$ are independent. Therefore, also

$$Z_1 := S'_{k+1} - \tilde{S}_k \quad \text{and} \quad Z_2 := (S''_{n+k+1} - S''_n) - (\tilde{S}_{n+k} - \tilde{S}_n)$$

are independent. Because k, ℓ_1 and ℓ_2 are sufficiently large, we have by Example 6.2.7

$$\delta(X_{\mathbb{Z}}, X'_{\mathbb{Z}}),\delta(X_{\mathbb{Z}},X''_{\mathbb{Z}}) < 10.100\eta = 1000\eta.$$

We can now argue as we sketched in the paragraph following on Proposition

6.2.5. By this proposition we have, using (6.2.7),

$$(6.2.9) \qquad \|P_{Z_1} - F\|, \|P_{Z_2} - F\| < 5.1000\eta = 5000\eta.$$

Using (6.2.7) once more, we have

$$(6.2.10) \qquad (S''_{n+k+1} - S'_{n+k+1}) \bmod d = (Z_1 - Z_2) \bmod d.$$

For S'_{n+k+1} we have, using (4.1.1),

$$P(S'_{n+k+1} \neq c_{n+k+1}(X'_{\mathbb{N}^c}, X'_{\mathbb{N}+n+k+1}))$$

$$\leq \delta(X_{\mathbb{Z}}, X'_{\mathbb{Z}}) + P(S_{n+k+1} \neq c_{n+k+1}(X_{\mathbb{N}^c}, X_{\mathbb{N}+n+k+1}))$$

$$< 1000\eta + 0 = 1000\eta.$$

A similar inequality holds for S''_{n+k+1}. By the consequence (6.2.8) of (6.2.7)

$$c_{n+k+1}(X'_{\mathbb{N}^c}, X'_{\mathbb{N}+n+k+1}) - c_{n+k+1}(X''_{\mathbb{N}^c}, X''_{\mathbb{N}+n+k+1}) = 0$$

and therefore

$$P((S''_{n+k+1} - S'_{n+k+1}) \bmod d \neq 0) < 2.1000\eta = 2000\eta.$$

By (6.2.10) it follows that for the independent random variables Z_1 and Z_2

$$P(Z_1 - Z_2 \notin L_d) < 2000\eta.$$

Let ϕ_1 and ϕ_2 be independent, F-distributed random variables. By (6.2.9)

$$P(\phi_1 - \phi_2 \notin L_d) < 2.5000\eta + P(Z_1 - Z_2 \notin L_d) < 12000\eta.$$

Because $\eta > 0$ is arbitrary, it follows that $\phi_1 - \phi_2 \in L_d$ a.s. In case $d = \infty$ we have $L_d = \{0\}$. In that case the independent random variables ϕ_1 and ϕ_2 coincide up to a null set and hence are constants. It follows that F is degenerate at $\{\bar{c}\}$ for some real number $\bar{c}$. If $0 < d < \infty$, let f denote the characteristic function of F. Because $\phi_1 - \phi_2 \in L_d$ a.s. we have

$$|f(\omega)|^2 = f(\omega)\,\overline{f(\omega)} = 1, \qquad \omega \in L_{2\pi d},$$

and hence F is concentrated on a set of the form $c+L_d$. Because F is the limit in total variation of measures F_n, $n \geq 1$, concentrated on $[0,d)$, it follows that F is concentrated on $[0,d)$. Hence F is degenerate at $\{\bar{c}\}$ with $\bar{c} := (c) \bmod d$. □

The assertion of Theorem 6.2.2 follows without many problems from the proposition above. We still need one simple lemma.

LEMMA 6.2.9. *Let* e_n, $n \geq 1$, *be a sequence of functions and* P_n, $n \geq 1$, *be a sequence of measures on a measurable space* Γ_0. *Suppose that for* $n, m \geq 1$,

$$e_n = e_m, \quad P_n \wedge P_m\text{-a.s.}$$

Then there is a function e *such that for all* $n \geq 1$

$$e = e_n \quad P_n\text{-a.s.}$$

PROOF. We construct a sequence $e^{(1)}, e^{(2)}, \ldots$ such that

$$e^{(n)} = e_i, \quad P_i\text{-a.s.}, \quad 1 \leq i \leq n.$$

Take $e^{(1)} := e_1$. Let $e^{(n-1)}$ be given, $n > 1$. Observe

$$\begin{aligned} e^{(n-1)} &= e_i, \quad P_i\text{-a.s.}, \\ &= e_n, \quad P_n \wedge P_i\text{-a.s.}, \quad 1 \leq i \leq n-1. \end{aligned}$$

It is now easily seen that we can define $e^{(n)}$ such that

$$\begin{aligned} e^{(n)} &= e^{(n-1)}, \quad P_i\text{-a.s.}, \quad 1 \leq i \leq n-1, \\ &= e_n \quad , \quad P_n\text{-a.s.} \end{aligned}$$

This inductive definition yields a sequence $e^{(n)}$, $n \in \mathbb{N}$, such that $e^{(n)}$ coincides with e_i for $n \geq i$ apart from a P_i-null set. Define

$$e := \lim_{n\to\infty} e^{(n)}$$

on the set where this limit exists and $e := 0$ on the complement of this set. Clearly $e = e_i$ apart from a P_i-null set for all $i \geq 1$. □

PROOF of Theorem 6.2.2. By Proposition 6.2.8 there is a constant $\bar{c}$ such that F is degenerate at $\{\bar{c}\}$. Hence by Lemma 6.2.4 all F_n, $n \geq 1$, are degenerate at $\{\bar{c}\}$. Hence for all $n \geq 1$,

$$(c_{n+1} - c_n) \bmod d = \bar{c}, \quad P_n \wedge P_{n+1}\text{-a.s.}$$

and therefore we have for fixed $m \geq 1$,

$$(c_{n+m} - c_n) \bmod d = \Big(\sum_{k=1}^{m} (c_{n+k} - c_{n+k-1})\Big) \bmod d$$

$$= (m\,\bar{c}) \bmod d, \quad P_n \wedge \ldots \wedge P_{n+m}\text{-a.s.}$$

Define measures $F_{n,m}$ and $\tilde{F}_{n,m}$ on the real line by

$$F_{n,m}(B) = (P_n \wedge P_{n+m})((c_{n+m} - c_n) \bmod d \in B),$$

$$\tilde{F}_{n,m}(B) = (P_n \wedge \ldots \wedge P_{n+m})((c_{n+m} - c_n) \bmod d \in B), \quad B \in \mathcal{B}^1.$$

Clearly $F_{n,m} \geq \tilde{F}_{n,m}$. Note that

$$\|P_n \wedge \ldots \wedge P_{n+m}\| = \|\tilde{F}_{n,m}\| = \tilde{F}_{n,m}(\{(m\,\bar{c}) \bmod d\})$$

$$\leq F_{n,m}(\{(m\,\bar{c}) \bmod d\}).$$

By applying (6.2.4) it follows that the left-hand side converges to 1, so

$$\lim_{n\to\infty} F_{n,m}(\{(m\,\bar{c}) \bmod d\}) = 1.$$

Using the argument in the proof of Lemma 6.2.4, one shows that $F_{n,m}$, $n \geq 1$, forms a nondescending sequence of measures. Hence, using the limit relation above, it follows that for $n,m \geq 1$,

$$(c_{n+m} - c_n) \bmod d = (m\,\bar{c}) \bmod d, \quad P_n \wedge P_{n+m}\text{-a.s.}$$

So for $n,m \geq 1$,

$$(c_n - n\,\bar{c}) \bmod d = (c_m - m\,\bar{c}) \bmod d, \quad P_n \wedge P_m\text{-a.s.}$$

By Lemma 6.2.9 there exists a function c such that

$$c = (c_n - n\,\bar{c}) \bmod d, \quad P_n\text{-a.s.}$$

for all $n \geq 1$. Together with the definition of c_n this implies the assertion of the theorem. □

6.3. RENEWAL THEORY - THE GENERAL CASE

In this section we present renewal theorems for random walks, controlled by a weak Bernoulli sequence $X_{\mathbb{Z}}$. In Section 3.2 we already discussed renewal theoretic limit relations. These were of a much weaker type than the limit relations in this section.

Apart from the weak Bernoulli condition we assume that $X_{\mathbb{Z}}$ satisfies (5.1.3). This condition is valid, if $X_{\mathbb{Z}}$ is a Markov dependent sequence or if $X_{\mathbb{Z}}$ consists of countably valued random variables. In Sections 6.4 and 6.5 we study these two special cases closer and discuss the relevant literature.

The main results in this section are Theorems 6.3.1 and 6.3.2. We obtain a generalization of Blackwell's theorem as a corollary. The proofs consist of an application of the loss of memory results in Section 6.1 to the Palm theoretic formula (3.1.3). Theorem 6.3.1 considers spread out random walks and Theorem 6.3.2 nonlattice random walks. The proof of the last theorem causes some problems. For strongly nonlattice random walks the proof is a straightforward application of the loss of memory results in Section 6.1. To cover also the slightly larger class of nonlattice random walks, we need the long argument given in Section 6.2 and some additional propositions. As we shall see in Sections 6.4 and 6.5 the nonlattice random walks form the natural class of random walks for which Theorem 6.3.2 can be proved.

In the following two theorems we consider a random walk $S_{\mathbb{Z}}$, controlled by a stationary, weak Bernoulli sequence $X_{\mathbb{Z}}$, with values in a Borel space Γ. Define the marked point process N_t^+ on $(0,\infty) \times \Gamma$ by

$$\text{(6.3.1)} \qquad N_0^+(B) := \sum_{n\geq 0} \chi_B(S_n, X_n),$$

$$N_t^+(B) := T_t N_0^+(B),$$

where $B \subset (0,\infty) \times \Gamma$ is any measurable set and the translation T_t is defined as in Section 0.3. Let ES_1 be a finite and strictly positive number. In this section we do not impose the restriction that the increments of the random walk are positive.

Our aim is to prove convergence theorems for N_t^+, $t \to \infty$. The distribution Q^+ of the limit process can be described in the following way. Define the marked point process N_0 on the real line by

$$N_0(B) = \sum_{n \in \mathbb{Z}} \chi_B(S_n, X_n),$$

where $B \subset \mathbb{R}^1 \times \Gamma$ is any measurable set. Define the distribution Q of a marked point process N on the real line by

$$(6.3.2) \qquad Q(D) := \frac{1}{ES_1} E \int_0^{S_1} \chi_D(T_t N_0)\,dt, \qquad D \in \mathcal{D}.$$

The distribution Q is defined on a measurable space $(\mathcal{N},\mathcal{D})$, described in Section 0.3. With these definitions Q is the probability distribution with *Palm measure* $Q_0 := P_{N_0}$. The limit distribution Q^+ will be the distribution of the restriction N^+ to $(0,\infty) \times \Gamma$ of the marked point process N with distribution Q.

The proof of the first theorem below will be given, following on Corollary 6.3.3. The proof of the second theorem needs some preparation and is given at the end of this section.

<u>THEOREM 6.3.1</u>. *Let* $X_{\mathbb{Z}}$ *be stationary, weak Bernoulli and assume* (5.1.3). *Suppose* ES_1 *exists as a finite, strictly positive number. If the random walk* $S_{\mathbb{Z}}$ *is spread out with respect to* $X_{\mathbb{Z}}$, *then*

$$(6.3.3) \qquad \lim_{t\to\infty} \| P_{N_t^+ | X_{\mathbb{N}^c}} - Q^+ \| = 0 \text{ a.s.},$$

where Q^+ *is defined above.*

<u>THEOREM 6.3.2</u>. *Let* $X_{\mathbb{Z}}$ *be stationary, weak Bernoulli and assume* (5.1.3). *Suppose* ES_1 *exists as a finite, strictly positive number. If the random walk* $S_{\mathbb{Z}}$ *is nonlattice with respect to* $X_{\mathbb{Z}}$, *then for any absolutely continuous probability measure* ν *on* $(0,\infty)$

$$(6.3.4) \qquad \lim_{t\to\infty} \| \nu * P_{N_t^+ | X_{\mathbb{N}^c}} - Q^+ \| = 0 \text{ a.s.},$$

where Q^+ *is defined above.*

The behaviour of the renewal measure

$$H(B) := \sum_{n\geq 0} P(S_n \in B), \quad B \in \mathcal{B}^1,$$

can be obtained as a corollary to these theorems. Let ℓ be the Lebesgue measure and denote by ℓ^+ the measure on the real line that coincides with ℓ on $(0,\infty)$ and vanishes on $(-\infty,0]$. If m is a measure on the real line and I is an interval, denote by $\|m\|_I$ the total variation of the restriction of m to the interval I.

COROLLARY 6.3.3. *Let* $X_{\mathbb{Z}}$ *be stationary, weak Bernoulli and assume* (5.1.3). *Suppose* ES_1 *exists as a finite, strictly positive number. If the renewal measure* H *is finite on a neighbourhood of the origin, then we have:*

(i) *If the random walk* $S_{\mathbb{Z}}$ *is spread out with respect to* $X_{\mathbb{Z}}$*, then for any positive* h

$$\lim_{t\to\pm\infty} \|H - \frac{1}{ES_1}\ell^+\|_{(t,t+h]} = 0.$$

(ii) *If the random walk* $S_{\mathbb{Z}}$ *is nonlattice with respect to* $X_{\mathbb{Z}}$*, then for any positive* h

$$\lim_{t\to\pm\infty} H(t,t+h] - \frac{1}{ES_1}\ell^+(t,t+h] = 0.$$

PROOF of Corollary 6.3.3(i). We use the uniform integrability of N_0^+ (and N_t^+) following from Corollary 2.2.5, together with limit relation (6.3.3).

The process $X_{\mathbb{Z}}$ is ergodic by Propositions 4.4.1 and 4.4.2, so by note 3° to Proposition 3.1.1 the marked point process N with distribution Q is stationary and ergodic. Furthermore, its intensity, i.e. the intensity of the projection N^S of N on the real line, is $\frac{1}{ES_1}$. Let $(N^+)^S$ and $(N_t^+)^S$ be the projection onto $(0,\infty)$ of N^+ and N_t^+. Uniformly for Borel sets $B \subset (0,h]$ we can give for positive t the following estimate. Because the intensity of $(N^+)^S$ is $\frac{1}{ES_1}$,

$$|H(T_tB) - \frac{1}{ES_1}\ell(B)| \leq n \sum_{k=0}^{n} |P((N_t^+)^S(B)=k) - P((N^+)^S(B)=k)|$$

$$+ \sum_{k>n} k\, P((N_t^+)^S(0,h]=k) + \sum_{k>n} k\, P((N^+)^S(0,h]=k).$$

By choosing n large, the two last sums can be made arbitrarily small, because of the uniform integrability of N_0^+ (and N_t^+) and because $(N^+)^s$ has finite intensity respectively. Then by choosing t large, the first sum on the right can be made arbitrarily small because of (6.3.3). This proves (i) for $t \to \infty$.

Using that $\lim_{n\to\infty} S_n = \infty$ a.s., it follows that with probability 1 only finitely many S_n, $n \geq 0$, are negative. By the uniform integrability of N_0 one easily obtains (i) for $t \to -\infty$.

PROOF of (ii). Use the uniform integrability of N_0^+ and (6.3.4). A similar argument is already given in the proof of Corollary 3.2.4. □

NOTE to Corollary 6.3.3. For random walks with independent, stationary increments (i) and (ii) are known. In that case (ii) is Blackwell's theorem and (i) is known in a stronger form. BRETAGNOLLE and DACUNHA-CASTELLE [1966] presented the last mentioned result (see also STONE [1966] and REVUZ [chapter 5]).

PROOF of Theorem 6.3.1. By Theorem 6.1.1 we have the loss of memory property

$$(6.3.5) \qquad \lim_{n\to\infty} E\| P_{S_n+h,X_{\mathbb{N}+n}|X_{\mathbb{N}^c}} - P_{S_n,X_{\mathbb{N}+n}} \| = 0, \qquad h \text{ real.}$$

Because $X_{\mathbb{Z}}$ is weak Bernoulli, the process $X_{\mathbb{Z}}$ is also ergodic by Propositions 4.4.1 and 4.4.2. By the ergodic theorem $\lim_{n\to\infty} \frac{1}{n} S_n = ES_1$ a.s. and because $ES_1 > 0$, we have in particular $\lim_{n\to\infty} S_n = \infty$ a.s. In the first part of the proof we derive from the loss of memory property (6.3.5) above, the limit relation

$$(6.3.6) \qquad \lim_{t\to\infty} \| P_{N^+_{t+h}|X_{\mathbb{N}^c}} - P_{N^+_t} \| = 0 \text{ a.s.}, \qquad h \text{ real.}$$

In the second part of the proof we use Palm theory. By note 3° to Proposition 3.1.1 the distribution Q, defined by (6.3.2), is invariant under translations. In part 2 of the proof, we use (6.3.2) to prove

$$(6.3.7) \qquad \lim_{t\to\infty} \| P_{N^+_t} - Q^+ \| = 0.$$

The asserted limit relation follows from (6.3.6) and (6.3.7).

Part 1. Define the marked point process N^n on $(0,\infty) \times \Gamma$ by

$$N^n(B) := \sum_{k>n} \chi_B(S_k, X_k),$$

where $B \subset (0,\infty) \times \Gamma$ is any measurable set. This marked point process is $(S_n, X_{\mathbb{N}+n})$-measurable. We obtain

$$\lim_{n\to\infty} E\| P_{T_h N^n | X_{\mathbb{N}^c}} - P_{N^n} \| = 0,$$

by noting that the total variation expression above is dominated by the total variation expression in (6.3.5). Because $\lim_{n\to\infty} S_n = -\lim_{n\to-\infty} S_n = \infty$ a.s., the set

$$A_n(t) := \{S_k \le t \text{ for all } k \le n\}$$

is increasing in t, its probability tending to 1 for $t \to \infty$. On $A_n(t)$ the restriction of the point processes N_0 and N^n to $(t,\infty) \times \Gamma$ coincide and hence by (1.1.1)

$$E\| P_{N^+_{t+h} | X_{\mathbb{N}^c}} - P_{N^+_t} \| \le E\| P_{T_h N^n | X_{\mathbb{N}^c}} - P_{N^n} \|$$
$$+ 2\, P(A_n(t+h)^c) + 2\, P(A_n(t)^c).$$

By choosing n large, the first term on the right can be made arbitrarily small and then, by choosing t large, the other two terms on the right can be made arbitrarily small. This proves that

$$\| P_{N^+_{t+h} | X_{\mathbb{N}^c}} - P_{N^+_t} \|$$

converges to 0 in L_1-mean for $t \to \infty$. By (4.1.2) the expression above is nonascending. Hence we have also a.s.-convergence, i.e. (6.3.6) holds.

Part 2. Because Q is invariant under translations and $X_{\mathbb{Z}}$ is stationary, we have

$$Q(D) = \frac{1}{ES_1} E \int_{S_{-1}}^{0} \chi_D(T_{t+h} N_0)\, dh, \qquad D \in \mathcal{D}.$$

Hence Q is expressed in terms of N_0. On the set $A_0(t)$, defined above, we have $N^+_t = (T_t N_0)^+$, where $(\cdot)^+$ denotes the restriction to $(0,\infty) \times \Gamma$. Because

$A_0(t)^c \downarrow \phi$ a.s. for $t \to \infty$, it follows that

$$\lim_{t\to\infty} P(A_0(t)^c | X_{\mathbb{N}^c}) = 0 \text{ a.s.}$$

Hence we obtain from (6.3.6) that for $t \to \infty$

$$Y_t := \| P_{(T_{t+h}N_0)^+ | X_{\mathbb{N}^c}} - P_{N_t^+} \| \to 0 \text{ a.s.}$$

The random variable S_{-1} is $X_{\mathbb{N}^c}$-measurable. By the dominated convergence theorem for conditional expectations (given S_{-1}), we have for $t \to \infty$

$$\| P_{(T_{t+h}\, N_0)^+ | S_{-1}} - P_{N_t^+} \| \leq E(Y_t | S_{-1}) \to 0 \text{ a.s.}$$

for all real h. Hence by dominated convergence we have for $t \to \infty$

$$Z_t := \frac{1}{ES_1} \int_{S_{-1}}^{0} \| P_{(T_{t+h}\, N_0)^+ | S_{-1}} - P_{N_t^+} \| \, dh \to 0 \text{ a.s.}$$

Because $|Z_t| \leq 2|S_{-1}|$ is uniformly integrable, we obtain by dominated convergence for $t \to \infty$

$$\| Q^+ - P_{N_t^+} \| \leq E|Z_t| \to 0.$$

Hence (6.3.7) holds and together with (6.3.6) for h = 0 this yields (6.3.3). □

Theorem 6.3.1. is obtained above as an application of the loss of memory result in Theorem 6.1.1. Theorem 6.3.2 will be derived similarly by using the loss of memory result in Theorem 6.1.2. There is, however, a difference that causes some problems. Theorem 6.1.2 considers strongly nonlattice random walks, whereas Theorem 6.3.2 is involved with the slightly larger class of nonlattice random walks. Section 6.2 was meant to describe the gap between these two classes of random walks.

In Chapter 1, at the end of the proof of Theorem 1.1.3, we used a simple trick to construct out of a nonlattice random walk a strongly nonlattice random walk. We inserted in the distribution of the (independent) increments an atom at {0}. The required limit relation could be obtained easily from the corresponding limit relation for the new, strongly nonlattice random walk. The same idea will be used also in the present context. The following three propositions contain the necessary prerequisites.

Suppose $S_{\mathbb{Z}}$ is a random walk controlled by a stationary sequence $X_{\mathbb{Z}}$ with values in a Borel space Γ, such that

$$S_0 = 0, \quad S_n - S_{n-1} = f(X_n), \quad n \in \mathbb{Z},$$

where f is a measurable real function on Γ. Let $\zeta_{\mathbb{Z}}$ be a sequence of independent, $\{0,1\}$-valued random variables, with $p := P(\zeta_0 = 1)$ between 0 and 1, and suppose that $\zeta_{\mathbb{Z}}$ and $X_{\mathbb{Z}}$ are independent. Consider the random set

$$\{n \in \mathbb{Z}: \zeta_n = 1\}$$

and let

$$\ldots < \tau_{-1} < \tau_0 \leq 0 < \tau_1 < \ldots$$

be its elements. Suppose $\tilde{\gamma}$ is some element, not contained in Γ. Define the process $\tilde{X}_{\mathbb{Z}}$ by

$$\begin{aligned} \tilde{X}_n &:= X_k \quad \text{if } k = \tau_n \text{ for some integer } k, \\ &:= \tilde{\gamma} \quad \text{else,} \end{aligned}$$

for $n \in \mathbb{Z}$. Then $\tilde{X}_{\mathbb{Z}}$ is a stationary sequence of random variables with values in the Borel space $\tilde{\Gamma} := \{\tilde{\gamma}\} \cup \Gamma$. Extend f to $\tilde{\Gamma}$ by taking $f(\tilde{\gamma}) := 0$ and define the random walk $\tilde{S}_{\mathbb{Z}}$ by requiring

$$\tilde{S}_0 := 0, \quad \tilde{S}_n - \tilde{S}_{n-1} = f(\tilde{X}_n), \quad n \in \mathbb{Z}.$$

In the following propositions we investigate how the (Cesaro) weak Bernoulli property, the nonlattice concept and condition (5.1.3) behaves in the transition from $X_{\mathbb{Z}}$ to $\tilde{X}_{\mathbb{Z}}$.

A well known type of example of Cesaro weak Bernoulli processes are the periodic Markov chains in Example 4.4.11.

PROPOSITION 6.3.4. *If $X_{\mathbb{Z}}$ is Cesaro weak Bernoulli, then $\tilde{X}_{\mathbb{Z}}$ is weak Bernoulli.*

PROOF. By Theorem 4.4.9 the condition that $X_{\mathbb{Z}}$ is Cesaro weak Bernoulli is equivalent with the existence of a probability space with processes $X'_{\mathbb{Z}}$

and $X''_{\mathbb{Z}}$, both distributed as $X_{\mathbb{Z}}$, such that $X'_{\mathbb{N}^c}$ and $X''_{\mathbb{Z}}$ are independent and for certain finite random times σ_1 and σ_2

(6.3.8) $$X'_{n+\sigma_1} = X''_{n+\sigma_2}, \quad n \geq 1.$$

Assume that this probability space exists. We want to construct $\tilde{X}'_{\mathbb{Z}}$ and $\tilde{X}''_{\mathbb{Z}}$, both distributed as $\tilde{X}_{\mathbb{Z}}$, such that $\tilde{X}'_{\mathbb{N}^c}$ and $\tilde{X}''_{\mathbb{Z}}$ are independent and for some finite random time σ

$$\tilde{X}'_{n+\sigma} = \tilde{X}''_{n+\sigma}, \quad n \geq 1.$$

By Theorem 4.4.7 we shall then have proved that $\tilde{X}_{\mathbb{Z}}$ is weak Bernoulli.

Let $\bar{\tau}_{\mathbb{N}}$ and $\tau''_{\mathbb{N}}$ be mutually independent, independent of the random variables mentioned before and distributed as $\tau_{\mathbb{N}}$. Let $\bar{\sigma}$ be the meeting time of

$$\bar{\tau}_{n+\sigma_1} \quad \text{and} \quad \tau''_{n+\sigma_2}, \quad n \geq 1.$$

By the Chung Fuchs theorem this meeting time $\bar{\sigma}$ exists with probability 1 (compare the proof of Theorem 1.1.2). Define $\tau'_{\mathbb{N}}$ by

$$\begin{aligned} \tau'_{n+\sigma_1} &:= \bar{\tau}_{n+\sigma_1} \qquad n < \bar{\sigma}, \\ &:= \tau''_{n+\sigma_2} \qquad n \geq \bar{\sigma}. \end{aligned}$$

Given $Z := (\sigma_1, \sigma_2, X'_{\mathbb{Z}}, X''_{\mathbb{Z}})$, the processes $\bar{\tau}_{\mathbb{N}}$ and $\tau''_{\mathbb{N}}$ are independent and distributed as $\tau_{\mathbb{N}}$. Using the Markov property, one observes that, also given Z, $\tau'_{\mathbb{N}}$ is distributed as $\tau_{\mathbb{N}}$.

Let $\tau'_{\mathbb{N}^c}$ and $\tau''_{\mathbb{N}^c}$ be mutually independent, independent of the random variables mentioned before and distributed as $\tau_{\mathbb{N}^c}$. It follows that $\tau'_{\mathbb{Z}}$ and $X'_{\mathbb{Z}}$ are independent and hence $\tilde{X}'_{\mathbb{Z}}$ defined by

$$\begin{aligned} \tilde{X}'_n &:= X'_k \qquad \text{if } n = \tau_k \text{ for some } k, \\ &:= \tilde{\gamma} \qquad \text{else,} \end{aligned}$$

is distributed as $\tilde{X}_{\mathbb{Z}}$. A similar definition can be given for $\tilde{X}''_{\mathbb{Z}}$ and also this process is distributed as $X_{\mathbb{Z}}$. Because $(X'_{\mathbb{N}^c}, \tau'_{\mathbb{N}^c})$ and $(X''_{\mathbb{Z}}, \tau''_{\mathbb{Z}})$ are independent, also $\tilde{X}'_{\mathbb{N}^c}$ and $\tilde{X}''_{\mathbb{Z}}$ are independent.

If $k \geq \bar{\sigma}$ we have $\tau'_{k+\sigma_1} = \tau''_{k+\sigma_2}$. Note that for $n \geq \sigma := \tau'_{\bar{\sigma}+\sigma_1} = \tau''_{\bar{\sigma}+\sigma_2}$ we have two possibilities. If n does not have the form $\tau'_{k+\sigma_1} = \tau''_{k+\sigma_2}$ for some $k \geq \bar{\sigma}$, then

$$\tilde{X}'_n = \tilde{\gamma} = \tilde{X}''_n.$$

Otherwise $n = \tau'_{k+\sigma_1} = \tau''_{k+\sigma_2}$ for $k \geq \bar{\sigma}$, so

$$\tilde{X}'_n = X'_{k+\sigma_1} = X''_{k+\sigma_2} = \tilde{X}''_n.$$

Hence for all $n \geq \sigma$ holds

$$\tilde{X}'_n = \tilde{X}''_n.$$

Thus we verified in the last two paragraphs a necessary and sufficient condition for $\tilde{X}_{\mathbb{Z}}$ to be weak Bernoulli. □

PROPOSITION 6.3.5. *If $S_{\mathbb{Z}}$ is nonlattice with respect to $X_{\mathbb{Z}}$, then we have*

(i) *$\tilde{S}_{\mathbb{Z}}$ is nonlattice with respect to $\tilde{X}_{\mathbb{Z}}$;*

(ii) *if $X_{\mathbb{Z}}$ is weak Bernoulli, then $\tilde{S}_{\mathbb{Z}}$ is strongly nonlattice with respect to $\tilde{X}_{\mathbb{Z}}$;*

(iii) *if $X_{\mathbb{Z}}$ is Cesaro weak Bernoulli, then $\tilde{S}_{\mathbb{Z}}$ is strongly nonlattice with respect to $\tilde{X}_{\mathbb{Z}}$.*

PROOF of (i). We use the following easily verified property. Suppose (Y_1,Y_2) and Z are independent. If Y_1 is, up to a null set, (Y_2,Z)-measurable, then Y_1 is, up to a null set, Y_2-measurable.

We prove (i) by contradiction. Suppose $\tilde{S}_{\mathbb{Z}}$ is lattice with respect to $\tilde{X}_{\mathbb{Z}}$, i.e. for some $d > 0$ there is a measurable function $\tilde{c}$ such that

$$(\tilde{S}_n) \bmod d = \tilde{c}(\tilde{X}_{\mathbb{N}^c}, \tilde{X}_{\mathbb{N}+n}) \text{ a.s.}, \quad n \geq 1.$$

Let $\sigma_n := \zeta_1 + \ldots + \zeta_n$ and note that $(\tilde{X}_{\mathbb{N}^c}, \tilde{X}_{\mathbb{N}+n})$ and $(X_{\mathbb{N}^c}, X_{\mathbb{N}+\sigma_n}, \zeta_{\mathbb{N}^c}, \zeta_{\mathbb{N}+n})$ mutually determine each other. Hence there is a measurable function $\bar{c}$ such that

$$(S_{\sigma_n}) \bmod d = \bar{c}(X_{\mathbb{N}^c}, X_{\mathbb{N}+\sigma_n}, \zeta_{\mathbb{N}^c}, \zeta_{\mathbb{N}+n}) \text{ a.s.}, \quad n \geq 1.$$

Note that $(\sigma_n, X_{\mathbb{Z}})$ and $(\zeta_{\mathbb{N}^c}, \zeta_{\mathbb{N}+n})$ are independent. Using the first paragraph of the proof and this independence property, it is easily seen that there is a measurable function c such that

$$(S_{\sigma_n}) \bmod d = c(X_{\mathbb{N}^c}, X_{\mathbb{N}+\sigma_n}) \text{ a.s.}, \quad n \geq 1.$$

The process $X_{\mathbb{Z}}$ and the random variable σ_n are independent. Furthermore, $P(\sigma_n = n)$ is positive. It follows that

$$(S_n) \bmod d = c(X_{\mathbb{N}^c}, X_{\mathbb{N}+n}) \text{ a.s.}, \quad n \geq 1,$$

thus contradicting the assumption of the proposition.

PROOF of (ii). Suppose d_∞ is the width of the minimal weak lattice of $S_{\mathbb{Z}}$ with respect to $X_{\mathbb{Z}}$. Let d_n be the minimal lattice width of the distribution F_n on the real line, defined by

$$F_n(B) := E\, P_{S_n | X_{K_n^c}} * P_{-S_n | X_{K_n^c}}(B), \quad B \in \mathcal{B}^1,$$

where $K_n := \{1,\dots,n\}$, $n \geq 1$. By Lemma 5.2.6 and the definition of the minimal weak lattice width in Section 5.2, we have $d_n \downarrow d_\infty$ for $n \to \infty$. Because d_n divides d_{n+1} for each $n \geq 1$, the limit is attained if $d_\infty > 0$. Define $\tilde{d}_n, \tilde{d}_\infty$ and $\tilde{F}_n$ similarly for the random walk $\tilde{S}_{\mathbb{Z}}$ with respect to $\tilde{X}_{\mathbb{Z}}$. The property that $\tilde{S}_{\mathbb{Z}}$ is strongly nonlattice with respect to $\tilde{X}_{\mathbb{Z}}$ can be expressed by $\tilde{d}_\infty = 0$.

If for all $n \geq 1$

$$(\tilde{S}_n) \bmod d \text{ is a.s. } (\tilde{X}_{\mathbb{N}^c}, \tilde{X}_{\mathbb{N}+n})\text{-measurable},$$

or

$$(S_{\sigma_n}) \bmod d \text{ is a.s. } (X_{\mathbb{N}^c}, X_{\mathbb{N}+\sigma_n}, \zeta_{\mathbb{N}^c}, \zeta_{\mathbb{N}+n})\text{-measurable},$$

then, by the property mentioned in the first paragraph of the proof of (i), also

$$(S_{\sigma_n}) \bmod d \text{ is a.s. } (X_{\mathbb{N}^c}, X_{\mathbb{N}+\sigma_n})\text{-measurable},$$

and hence,

$$(S_n) \bmod d \text{ is a.s. } (X_{\mathbb{N}^c}, X_{\mathbb{N}+n})\text{-measurable}.$$

Therefore, $d_\infty \geq \tilde{d}_\infty$. Suppose $d_\infty > 0$. We have to prove that $\tilde{d}_\infty = 0$, under the assumption that $X_{\mathbb{Z}}$ is weak Bernoulli.

Because of the limit property (6.2.4) and the definition of d_∞, we can choose n so large that d_∞ is the lattice width of F_n and

(6.3.9) $\quad \mu := P_{X_{\mathbb{N}^c}, X_{\mathbb{N}+n}} \wedge P_{X_{\mathbb{N}^c}, X_{\mathbb{N}+n-1}} \neq 0.$

By Corollary 6.2.3 there exists a real number $\bar{c}$ and a real measurable function c, such that

(6.3.10) $\quad (S_n - n\,\bar{c}) \bmod d_\infty = c(X_{\mathbb{N}^c}, X_{\mathbb{N}+n})$ a.s.,

where $\bar{c}$ and d_∞ are mutually prime. Consider

$$\tilde{F}_n(B) := E\, P_{\tilde{S}_{\sigma_n} | \tilde{X}_{\mathbb{N}^c}, \tilde{X}_{\mathbb{N}+n}} * P_{-\tilde{S}_{\sigma_n} | \tilde{X}_{\mathbb{N}^c}, \tilde{X}_{\mathbb{N}+n}}(B), \qquad B \in \mathcal{B}^1.$$

Because $\{\sigma_n = n\} = \{\zeta_1 = \ldots = \zeta_n = 1\}$ has positive probability, it is easily seen that $\tilde{F}_n \gg F_n$ and hence $\tilde{d}_n$ divides d_∞. Similarly it can be seen that

$$\tilde{F}_n \gg G_n = \int P_{S_n | X_{\mathbb{N}^c}=x,\, X_{\mathbb{N}+n}=y} * P_{-S_{n-1} | X_{\mathbb{N}^c}=x,\, X_{\mathbb{N}+n-1}=y}\, d\mu(x,y),$$

where μ is defined by (6.3.9). By (6.3.10) the measure G_n is concentrated on $\bar{c} + L_{d_\infty}$. Because $\bar{c}$ and d_∞ are relatively prime and because the minimal lattice width $\tilde{d}_n$ divides d_∞, it follows that $\tilde{d}_n = 0$ and hence also $\tilde{d}_\infty \leq \tilde{d}_n$ vanishes. So $\tilde{S}_{\mathbb{Z}}$ is strongly nonlattice with respect to $\tilde{X}_{\mathbb{Z}}$.

<u>PROOF of (iii)</u>. The process $\tilde{X}_{\mathbb{Z}}$ is defined in terms of $X_{\mathbb{Z}}$ and $\zeta_{\mathbb{Z}}$, say as $\tilde{X}_{\mathbb{Z}} := g(X_{\mathbb{Z}}, \zeta_{\mathbb{Z}})$. Let $\zeta'_{\mathbb{Z}}$, $\zeta''_{\mathbb{Z}}$ and $X_{\mathbb{Z}}$ be independent, with $\zeta'_{\mathbb{Z}}$ and $\zeta''_{\mathbb{Z}}$ sequences of independent $\{0,1\}$-valued random variables, such that $\{\zeta'_n = 1\}$ and $\{\zeta''_n = 1\}$ both have probability $\sqrt{p}$, where $p := P(\zeta_n = 1)$. Then the process $\bar{\zeta}_{\mathbb{Z}}$ defined by $\bar{\zeta}_n := \zeta'_n \wedge \zeta''_n$, $n \in \mathbb{Z}$, is distributed as $\zeta_{\mathbb{Z}}$.

The process $X'_{\mathbb{Z}} := g(X_{\mathbb{Z}}, \zeta'_{\mathbb{Z}})$ is weak Bernoulli by Proposition 6.3.4 and the random walk $S'_{\mathbb{Z}}$ controlled by $X'_{\mathbb{Z}}$ is, by (i), nonlattice. The process $X''_{\mathbb{Z}} := g(X'_{\mathbb{Z}}, \zeta''_{\mathbb{Z}})$ is weak Bernoulli by Proposition 6.3.4 and the random walk $S''_{\mathbb{Z}}$ controlled by $X''_{\mathbb{Z}}$ is strongly nonlattice by (ii). Because $X''_{\mathbb{Z}} = g(X_{\mathbb{Z}}, \bar{\zeta}_{\mathbb{Z}})$, with $\bar{\zeta}_{\mathbb{Z}}$ distributed as $\zeta_{\mathbb{Z}}$ and independent of $X_{\mathbb{Z}}$, it follows that $\tilde{X}_{\mathbb{Z}}$ is distributed as $X''_{\mathbb{Z}}$. Hence we proved (iii). □

<u>PROPOSITION 6.3.6</u>. *If* $X_{\mathbb{Z}}$ *is weak Bernoulli and satisfies* (5.1.3), *then* $\tilde{X}_{\mathbb{Z}}$ *satisfies* (5.1.3).

<u>PROOF</u>. We have to show (by the note to Proposition 5.1.1) that for each set

$K := \{1,\ldots,k\}$

$$\lim_{\ell\to\infty} E \perp_{\tilde{X}_{L\setminus K}} (\tilde{X}_K,\tilde{X}_{L^c}) = 0,$$

where $L := \{-\ell,\ldots,\ell\}$. By the second inequality of Proposition 5.1.1, it follows that it is sufficient to prove for sets K and L of the form above

$$\lim_{k\to\infty} \limsup_{\ell\to\infty} E_{\tilde{X}_{L\setminus K}} (\tilde{X}_K,\tilde{X}_{L^c}) = 0.$$

To this purpose we construct a process $\tilde{X}'_{\mathbb{Z}}$, approximating $\tilde{X}_{\mathbb{Z}}$, such that $(\tilde{X}'_K,\tilde{X}'_{L\setminus K},\tilde{X}'_{L^c})$ is a Markov triple. The assertion can then be derived easily. The process $\tilde{X}'_{\mathbb{Z}}$ will be constructed such that for some process $X'_{\mathbb{Z}}$, approximating $X_{\mathbb{Z}}$, we have

$$\begin{aligned} \tilde{X}'_n &= X'_k && \text{if } n = \tau_k \text{ for some integer } k,\\ &= \tilde{\gamma} && \text{if } \zeta_n = 0. \end{aligned}$$

Construct $X^*_{\mathbb{N}^c}$ and $X^*_{\mathbb{N}}$ as independent processes, distributed as $X_{\mathbb{N}^c}$ and $X_{\mathbb{N}}$, respectively. Let τ^*_n, $n \geq 1$, be the n-th index for which $\zeta_j = 1$, $j > k$, and let $\tau^*_n := \tau_n$, $n \leq 0$. Define for $n \in K^c$

$$(6.3.11)\qquad \begin{aligned} \tilde{X}'_n &:= \tilde{\gamma} && \text{if } \zeta_n = 0,\\ &:= X^*_k && \text{if } n = \tau^*_k \text{ for some integer } k, \end{aligned}$$

and note that $\tilde{X}'_{K^c}$ is defined in terms of $(\zeta_{K^c},X^*_{\mathbb{N}^c},X^*_{\mathbb{N}})$. The random vector ζ_K is independent of this random vector. We define the process $X'_{\mathbb{Z}}$ as follows. Let $\sigma_k := \sum_{i=1}^k \zeta_i$ and take

$$\begin{aligned} X'_n &:= X^*_{n-\sigma_k} && n > \sigma_k,\\ &:= X^*_n && n \leq 0. \end{aligned}$$

This defines $X'_{K^c_\zeta}$ with $K_\zeta := \{1,\ldots,\sigma_k\}$. Let L_ζ be defined as

$$L_\zeta := \{-\sigma^-_\ell,\ldots,\sigma^+_\ell\}, \quad \text{with } \sigma^-_\ell := \sum_{i=-\ell}^{0} \zeta_i,$$
$$\sigma^+_\ell := \sum_{i=1}^{\ell} \zeta_i.$$

Define X'_{K_ζ}, independent of $(X'_{L^c_\zeta},\zeta_{\mathbb{Z}})$, given $(X'_{L_\zeta\setminus K_\zeta},K_\zeta,L_\zeta) = (x,K',L')$, with conditional distribution $P_{X_{K'}|X_{L'\setminus K'}=x}$. We can now define $\tilde{X}'_{\mathbb{Z}}$ by

$$\tilde{X}'_n := X'_k \quad \text{if } n = \tau_k \text{ for some integer } k,$$
$$:= \tilde{\gamma} \quad \text{if } \zeta_n = 0.$$

This definition is consistent with (6.3.11). Moreover, the independence of ζ_K and $(\zeta_{K^c}, X^*_{\mathbb{N}^c}, X^*_{\mathbb{N}})$ and the definition of X'_{K_ζ} imply that $\tilde{X}'_{L^c}$ and $\tilde{X}'_K$ are independent, given $\tilde{X}'_{L\setminus K}$.

To estimate $\delta(\tilde{X}'_{\mathbb{Z}}, \tilde{X}_{\mathbb{Z}})$ we use Lemma 6.2.6. Note that $X^*_{\mathbb{N}^c} = X'_{\mathbb{N}^c}$ and $X^*_{\mathbb{N}} = X'_{\mathbb{N}+\sigma_k}$ are independent and distributed as $X_{\mathbb{N}^c}$ and $X_{\mathbb{N}}$, respectively. Take some $\varepsilon > 0$ and let $\bar{k}$ be some integer, larger than $\tilde{k}(\varepsilon)$. By Lemma 6.2.6 and the note to this lemma

$$\tfrac{1}{2}\| P_{X'_{\mathbb{Z}}|\zeta_{\mathbb{Z}}} - P_{X_{\mathbb{Z}}} \| < 10\varepsilon$$

on the set

$$A := \{\tilde{k}(\varepsilon) \le \sigma_k \le \bar{k},\ \sigma^-_\ell, \sigma^+_\ell \ge \tilde{\ell}(\bar{k},\varepsilon)\}.$$

By the independence of $X_{\mathbb{Z}}$ and $\zeta_{\mathbb{Z}}$ we have $P_{X_{\mathbb{Z}}|\zeta_{\mathbb{Z}}} = P_{X_{\mathbb{Z}}}$ and hence on A

$$\tfrac{1}{2}\| P_{\tilde{X}'_{\mathbb{Z}}|\zeta_{\mathbb{Z}}} - P_{\tilde{X}_{\mathbb{Z}}|\zeta_{\mathbb{Z}}} \| = \tfrac{1}{2}\| P_{X'_{\mathbb{Z}}|\zeta_{\mathbb{Z}}} - P_{X_{\mathbb{Z}}|\zeta_{\mathbb{Z}}} \| < 10\varepsilon.$$

Using (4.1.5) we obtain

$$(6.3.12) \qquad \tfrac{1}{2}\| P_{\tilde{X}'_{\mathbb{Z}}} - P_{\tilde{X}_{\mathbb{Z}}} \| \le 10\varepsilon\, P(A) + P(A^c).$$

Using this estimate for $\tilde{X}'_{\mathbb{Z}}$ we can now prove the assertion. Let $\bar{X}_{\mathbb{Z}}$ be a process such that $(\bar{X}_K, \bar{X}_{L\setminus K}, \bar{X}_{L^c})$ forms a Markov triple and $\bar{X}_L$ and $\bar{X}_{K^c}$ are distributed as $\tilde{X}_L$ and $\tilde{X}_{K^c}$, respectively. We have to estimate

$$\delta(\tilde{X}_{\mathbb{Z}}, \bar{X}_{\mathbb{Z}}) = f_K(L) := E \perp_{\tilde{X}_{L\setminus K}} (\tilde{X}_K, \tilde{X}_{L^c}).$$

By the triangle inequality

$$\delta(\tilde{X}_{\mathbb{Z}}, \bar{X}_{\mathbb{Z}}) \le \delta(\bar{X}_{\mathbb{Z}}, \tilde{X}'_{\mathbb{Z}}) + \delta(\tilde{X}'_{\mathbb{Z}}, \tilde{X}_{\mathbb{Z}}).$$

To estimate the first term on the right, we use Proposition 4.1.4 and for the second term we use (6.3.12). We obtain with $K = \{1,\dots,k\}$ and

$L = \{-\ell,\ldots,\ell\}$

$$f_K(L) \le (0+\alpha+\alpha+\alpha)+\alpha = 4\alpha, \quad \alpha := 10\varepsilon\, P(A) + P(A^c).$$

Letting $\ell \to \infty$ we obtain

$$\limsup_{\ell\to\infty} f_K(L) \le 40\varepsilon\, P(\tilde{k}(\varepsilon) \le \sigma_k \le \bar{k}) + 4(1-P(\tilde{k}(\varepsilon) \le \sigma_k \le \bar{k})).$$

By choosing $\bar{k} \gg k$ and letting $k \to \infty$ we obtain for arbitrary positive ε

$$\lim_{k\to\infty} \limsup_{\ell\to\infty} f_K(L) \le 40\varepsilon,$$

which proves the assertion. □

NOTE. It is not clear whether the proposition is valid if the weak Bernoulli assumption on $X_{\mathbb{Z}}$ is weakened to the requirement that $X_{\mathbb{Z}}$ is Cesaro weak Bernoulli.

PROOF of Theorem 6.3.2. First we suppose that $\tilde{S}_{\mathbb{Z}}$ is strongly nonlattice with respect to $\tilde{X}_{\mathbb{Z}}$. The proof of (6.3.4) is obtained by some simple changes in the proof of Theorem 6.3.1. Instead of Theorem 6.1.1 we use Theorem 6.1.2. All probability distributions $P_{S_n, X_{\mathbb{N}+n}}$, $P_{N_t^+}$, Q^+, etc., have to be replaced by

$$\nu * P_{S_n, X_{\mathbb{N}+n}}, \ \nu * P_{N_t^+}, \ \nu * Q^+, \ \text{etc.},$$

where ν is an arbitrary, absolutely continuous probability measure on $(0,\infty)$. Note that because Q^+ is invariant under T_t, $t \ge 0$, we have $\nu * Q^+ = Q^+$. With these changes the proof of Theorem 6.3.1 can be followed to get (6.3.4).

In case the strongly nonlattice assumption is not satisfied, we consider the random walk $\tilde{S}_{\mathbb{Z}}$, controlled by $\tilde{X}_{\mathbb{Z}}$, as defined in the introduction of Proposition 6.3.6. By this proposition $\tilde{X}_{\mathbb{Z}}$ is weak Bernoulli and by Proposition 6.3.8 $\tilde{X}_{\mathbb{Z}}$ satisfies (5.1.3). By Proposition 6.3.5 (ii) the random walk $\tilde{S}_{\mathbb{Z}}$ is strongly nonlattice with respect to $\tilde{X}_{\mathbb{Z}}$ and hence by what we proved above

$$\text{(6.3.13)} \qquad \lim_{t\to\infty} \| \nu * P_{\tilde{N}_t^+ | \tilde{X}_{\mathbb{N}^c}} - \tilde{Q}^+ \| = 0 \text{ a.s.},$$

where $\tilde{N}_t^+$ and $\tilde{Q}^+$ are defined in the obvious way. Let $\tilde{N}_0$ be the point process, defined by

$$\tilde{N}_0(B) := \sum_{n\in\mathbb{Z}} \chi_B(\tilde{S}_n,\tilde{X}_n),$$

where $B \subset \mathbb{R}^1 \times \Gamma$ is any measurable set. We can obtain N_0 out of $\tilde{N}_0$ by removing all points $(\tilde{S}_n,\tilde{X}_n)$, $n \in \mathbb{Z}$, for which $\tilde{X}_n = \tilde{\gamma}$. Write $N_0 = \theta(\tilde{N}_0)$ and similarly $N_t^+ = \theta_+(\tilde{N}_t^+)$. Note that $Q^+ = \tilde{Q}^+\circ\theta_+^{-1}$. Using these remarks and the property that $X_{\mathbb{N}^c}$ is $\tilde{X}_{\mathbb{N}^c}$-measurable, one obtains from (6.3.13) that (6.3.4) holds. □

NOTE to Theorems 6.3.1 and 6.3.2.

1° In case the increments of $S_{\mathbb{Z}}$ are not strictly positive, we can strengthen both theorems above slightly. Define for positive t

$$\tau_t := \inf\{n \geq 0: S_n > t\}.$$

If $S_{\mathbb{Z}}$ has strictly positive increments, then N_t^+ and $(S_{\tau_t}, X_{\mathbb{N}+\tau_t})$ mutually determine each other. In case this condition is not fulfilled, it is possible that with positive probability for $n > \tau_t$ holds that $S_n < t$. So in that case N_t^+ does not determine $(S_{\tau_t}, X_{\mathbb{N}+\tau_t})$.

Define the marked point process $\bar{N}_t^+$ on $(0,\infty) \times \Gamma^{\mathbb{N}}$ by

$$\bar{N}_t^+(B) := \sum_{n\geq 0} \chi_B(S_n, X_{\mathbb{N}+n}),$$

where $B \subset (0,\infty) \times \Gamma^{\mathbb{N}}$ is any measurable set. Note that $\bar{N}_t^+$ and $(S_{\tau_t}, X_{\mathbb{N}+\tau_t})$ mutually determine each other. Both Theorems 6.3.1 and 6.3.2 remain valid if in (6.3.3) and (6.3.4) the marked point process N_t^+ is replaced by $\bar{N}_t^+$. The proofs of both theorems are easily adapted to cover also this assertion.

2° Note that $\nu * Q^+ = Q^+$. Hence the limit property (6.3.4) is equivalent with

$$\lim_{t\to\infty} d_\nu(P_{N_t^+|X_{\mathbb{N}^c}}, Q^+) = 0 \text{ a.s.},$$

where d_ν is the pseudo-metric defined by (0.3.3). This type of convergence is unusual. It is, however, strong enough to permit us in Section 6.4 to give converse results. The appendix discusses the topology corresponding to this convergence concept. Recently McDONALD [1978] used a similar convergence concept in renewal theory for semi-Markov chains.

Using the notations and assumptions given in the introduction to Theorems 6.3.1 and 6.3.2, we can prove the following result for a Cesaro weak Bernoulli sequence $X_{\mathbb{Z}}$ that consists of countably valued random variables.

THEOREM 6.3.7. *Let* $X_{\mathbb{Z}}$ *be a stationary, Cesaro weak Bernoulli sequence of countably valued random variables. Suppose* ES_1 *is finite and strictly positive. If the random walk* $S_{\mathbb{Z}}$ *is nonlattice with respect to* $X_{\mathbb{Z}}$, *then for each absolutely continuous probability measure* ν *on* $(0,\infty)$ *we have*

$$\lim_{t\to\infty} \| \nu * P_{N_t^+ | X_{\mathbb{N}^c}} - Q^+ \| = 0 \ a.s.$$

PROOF. We apply the idea given in the last paragraph of the proof of Theorem 6.3.2. Note that $\tilde{X}_{\mathbb{Z}}$ is countably valued, so satisfies (5.1.3) by Proposition 5.1.3. By Proposition 6.3.4 $\tilde{X}_{\mathbb{Z}}$ is weak Bernoulli and by Proposition 6.3.5 (iii) $\tilde{S}_{\mathbb{Z}}$ is nonlattice with respect to $\tilde{X}_{\mathbb{Z}}$. Hence by Theorem 6.3.2

$$\lim_{t\to\infty} \| \nu * P_{\tilde{N}_t^+ | \tilde{X}_{\mathbb{N}^c}} - \tilde{Q}^+ \| = 0 \text{ a.s.}$$

This implies the assertion. □

6.4. RENEWAL THEORY - THE COUNTABLE AND MARKOV CASE

In the preceding section we obtained renewal theorems for random walks, controlled by a stationary sequence $X_{\mathbb{Z}}$. We assumed that $X_{\mathbb{Z}}$ was weak Bernoulli and required that the random walk was spread out or nonlattice. The question that interests us in this section is whether these conditions are well chosen. To this purpose we study two special classes of processes that allow us to obtain more detailed results. In both cases condition 5.1.3 is satisfied.

In the first half of this section we study a countably valued sequence $X_{\mathbb{Z}}$. It appears to be natural to replace the weak Bernoulli condition on $X_{\mathbb{Z}}$ by the slightly weaker assumption that $X_{\mathbb{Z}}$ is Cesaro weak Bernoulli. We then obtain in Theorem 6.4.1 a necessary and sufficient condition for the validity of the limit relation (6.3.4) (or (6.4.2)) that was derived in the preceding section.

In the second half of Section 6.4 we apply our results to obtain renewal theorems for functionals on Markov chains. Also the renewal theory for semi-Markov chains fits into this context. Theorem 6.4.5 describes the main result. The literature in this direction is fairly extensive. Our method of proof differs considerably from the methods that are usually applied. However the result, Theorem 6.4.5, is close to other limit theorems in this direction. At the end of the section we give a survey of the literature.

Let $S_{\mathbb{Z}}$ be a random walk controlled by a stationary sequence $X_{\mathbb{Z}}$, with values in a Borel space Γ. Suppose $S_{\mathbb{Z}}$ has strictly positive increments with finite expectation. Let N_0 be the marked point process on the real line with marks in Γ, defined by

$$(6.4.1) \qquad N_0(B) := \sum_{n\in\mathbb{Z}} \chi_B(S_n,X_n),$$

where $B \subset \mathbb{R}^1 \times \Gamma$ is any measurable set. Define $N_t := T_t N_0$, t real, and let N_t^+ and N_t^- be the restrictions of N_t to $(0,\infty) \times \Gamma$ and $(-\infty,0] \times \Gamma$, respectively. Let Q^+ be defined as in the introduction to Theorem 6.3.1. With these notations we have the following theorem.

THEOREM 6.4.1. *Let $S_{\mathbb{Z}}$ be a random walk, controlled by a countably valued sequence $X_{\mathbb{Z}}$. Assume that $S_{\mathbb{Z}}$ has strictly positive increments with finite expectation. The following two statements are equivalent:*

(i) *$X_{\mathbb{Z}}$ is Cesaro weak Bernoulli and $S_{\mathbb{Z}}$ is nonlattice with respect to $X_{\mathbb{Z}}$.*

(ii) *For each absolutely continuous probability measure ν on $(0,\infty)$ we have*

$$(6.4.2) \qquad \lim_{t\to\infty} \| \nu * P_{N_t^+ | N_0^-} - Q^+ \| = 0 \ a.s.$$

PROOF. We may assume that Γ is countable. By Proposition 5.1.2, condition (5.1.3) is satisfied. By Theorem 6.3.7 we have (i) $\to$ (ii). The converse follows from the following two propositions. □

PROPOSITION 6.4.2. *Let $S_{\mathbb{Z}}$ be a random walk, controlled by a stationary sequence $X_{\mathbb{Z}}$, with values in a Borel space Γ. Suppose $S_{\mathbb{Z}}$ has strictly positive increments. If (6.4.2) holds, then $X_{\mathbb{Z}}$ is Cesaro weak Bernoulli.*

PROOF. By (6.4.2) we have

$$\lim_{t\to\infty} \| \nu * P_{N_t^+ | N_0^-} - \nu * P_{N_t^+} \| = 0 \text{ a.s.}$$

Let Y be a ν-distributed random variable, independent of N_0. The limit relation above can be written as

$$\lim_{t\to\infty} \| P_{N_{t+Y}^+ | N_0^- = m^-} - P_{N_{t+Y}^+} \| = 0,$$

up to a $P_{N_0^-}$-null set. Fix some m^- not contained in this null set. Consider the probability space on which N_0 is defined. For any positive t we can construct, using Proposition 4.2.1 and Lemma 4.2.4, a pair $(\bar{N}_0, \bar{Y})$ with distribution $P_{N_0 | N_0^- = m^-} \times \nu$, such that

(6.4.3) $$P^{m^-}(\bar{N}_{t+\bar{Y}}^+ \neq N_{t+Y}^+) = \tfrac{1}{2} \| P_{N_{t+Y}^+ | N_0^- = m^-} - P_{N_{t+Y}^+} \|.$$

Here the probability measure P^{m^-} depends on m^-. The marked point process $\bar{N}_0$ has the form

$$\bar{N}_0(B) := \sum_{n\in\mathbb{Z}} \chi_B(\bar{S}_n, \bar{X}_n),$$

where $B \subset \mathbb{R}^1 \times \Gamma$ is measurable, $\bar{S}_0 = 0$ and $\bar{S}_{\mathbb{Z}}$ has strictly positive increments. On the set $\{\bar{N}_{t+\bar{Y}}^+ = N_{t+Y}^+\}$ we have

$$S_{n+\sigma} = \bar{S}_{n+\bar{\sigma}}, \quad X_{n+\sigma} = \bar{X}_{n+\bar{\sigma}}, \quad n \geq 1,$$

where σ and $\bar{\sigma}$ are given by

$$\sigma := \inf\{n \geq 0 : S_n + Y > t\}, \quad \bar{\sigma} := \inf\{n \geq 0 : \bar{S}_n + \bar{Y} > t\}.$$

Because the right-hand side in (6.4.3) tends to 0 for $n \to \infty$, we have by Proposition 4.3.4 that

$$\lim_{n\to\infty} \| \frac{1}{n} \sum_{k=0}^{n-1} P_{X_{\mathbb{N}+k}} - \frac{1}{n} \sum_{k=0}^{n-1} P^{m^-}_{\bar{X}_{\mathbb{N}+k}} \| = 0.$$

The marked point process $\bar{N}_0$ is distributed as $P_{N_0 | N_0^- = m^-}$ and hence

$$P^{m^-}_{\bar{X}_{\mathbb{N}+k}} = P_{X_{\mathbb{N}+k} | N_0^- = m^-}.$$

The first term in the last limit relation equals $P_{X_{\mathbb{N}}}$ by stationarity. Because N_0^- and $X_{\mathbb{N}^c}$ mutually determine each other, it follows that, up to a null set,

$$\lim_{n\to\infty} \| P_{X_{\mathbb{N}}} - \frac{1}{n}\sum_{k=0}^{n-1} P_{X_{\mathbb{N}+k}|X_{\mathbb{N}^c}} \| = 0.$$

By Proposition 4.4.8 the process $X_{\mathbb{Z}}$ is Cesaro weak Bernoulli. □

<u>PROPOSITION 6.4.3</u>. *Let* $S_{\mathbb{Z}}$ *be a random walk, controlled by a stationary sequence* $X_{\mathbb{Z}}$, *with values in a Borel space* Γ. *Suppose* $S_{\mathbb{Z}}$ *has strictly positive increments. If* (6.4.2) *holds for any absolutely continuous probability measure* ν *on* $(0,\infty)$, *then* $S_{\mathbb{Z}}$ *is nonlattice with respect to* $X_{\mathbb{Z}}$.

<u>PROOF</u>. Suppose that $S_{\mathbb{Z}}$ is lattice with respect to $X_{\mathbb{Z}}$. For some $d > 0$ there is a real function c such that

$$(S_n) \bmod d = c(X_{\mathbb{N}^c}, X_{\mathbb{N}+n}) \quad \text{a.s.}, \qquad n \ge 1.$$

We derive a contradiction with (6.4.2).

Let $(N,\mathcal{D})$ be the measurable space, defined in Section 0.3 for marked point processes. Suppose m^- is the restriction to $(-\infty,0] \times \Gamma$ of some element in N. Define $D(m^-)$ to be the set of all $m \in N$, such that their restriction to $(-\infty,0] \times \Gamma$ is m^-, while m has the form

$$m(B) = \sum_{n\in\mathbb{Z}} \chi_B(s_n,x_n), \qquad B \subset \mathbb{R}^1 \times \Gamma,$$

$$\text{(6.4.4)} \qquad \ldots < s_{-1} < s_0 = 0 < s_1 < \ldots$$

$$(s_n) \bmod d = c(x_{\mathbb{N}^c}, x_{\mathbb{N}+n}), \qquad n \ge 1.$$

Let $t \ge 0$. Define $\phi_t(m)$ to be the element of N that vanishes on the set $(0,t) \times \Gamma$ and coincides with m on the complement of this set. Note that ϕ_t maps $D(m^-)$ onto $D(m^-)$.

Consider an element $m \in D(m^-)$ of the form (6.4.4), with $s_1 > t$. Let h be real, such that $h+t \ge 0$ and define $\tilde{m}$ by requiring

$$\tilde{s}_n = s_n, \quad n \le 0, \qquad \tilde{s}_n = s_n + h, \quad n \ge 1,$$

$$\tilde{m}(B) = \sum_{n\in\mathbb{Z}} \chi_B(\tilde{s}_n,x_n), \qquad B \subset \mathbb{R}^1 \times \Gamma.$$

Note that $\tilde{m} \in D(m^-)$ if and only if $h \in L_d$. In case $h \in L_d$, the mapping $m \to \tilde{m}$ is an invertible mapping from $\phi_t(D(m^-))$ onto $\phi_{t+h}(D(m^-))$.

Denote the restriction of $m \in N$ to $(0,\infty) \times \Gamma$ by m^+ and define for $t \geq 0$

$$D_t(m^-) := \{(T_t m)^+ : m \in D(m^-)\}$$

$$= \{(T_t m)^+ : m \in \phi_t(D(m^-))\}.$$

From the remarks in the preceding paragraphs, it follows that if $t+h \geq 0$, the sets $D_t(m^-)$ and $D_{t+h}(m^-)$ coincide if $h \in L_d$ and are disjoint if $h \notin L_d$. Define for arbitrary positive ε the set

$$D_t^\varepsilon(m^-) := \bigcup_{0 \leq h < \varepsilon} D_{t+h}(m^-).$$

Let ν be the homogeneous distribution on $[0,\varepsilon)$ with $0 < \varepsilon < \frac{1}{2}d$. Then the measures

$$(6.4.5) \qquad \nu * P_{N_t^+ | N_0^-} \qquad \text{and} \qquad \nu * P_{N_{t+\varepsilon}^+ | N_0^-}$$

are concentrated on

$$D_t^\varepsilon(N_0^-) \qquad \text{and} \qquad D_{t+\varepsilon}^\varepsilon(N_0^-),$$

respectively. Because these sets are disjoint, the two measures in (6.4.5) are mutually singular. This contradicts (6.4.2) because (6.4.2) implies that the total variation of the difference of these two measures vanishes asymptotically with probability 1. □

Suppose $S_{\mathbb{Z}}$ has strictly positive increments. The two propositions above imply that if the limit relation of Theorem 6.3.2 holds (i.e. (6.4.2)) then necessarily $X_{\mathbb{Z}}$ is Cesaro weak Bernoulli and $S_{\mathbb{Z}}$ is nonlattice with respect to $X_{\mathbb{Z}}$. The corresponding assertion holds also for the limit relation of Theorem 6.3.1. If this limit relation holds, then necessarily $X_{\mathbb{Z}}$ is Cesaro weak Bernoulli and, as we shall see below, $S_{\mathbb{Z}}$ is spread out with respect to $X_{\mathbb{Z}}$.

PROPOSITION 6.4.4. *Let* $S_{\mathbb{Z}}$ *be a random walk, controlled by a stationary sequence* $X_{\mathbb{Z}}$, *with values in a Borel space* Γ. *Suppose* $S_{\mathbb{Z}}$ *has strictly* *positive increments. If* (6.3.3) *holds, then* $S_{\mathbb{Z}}$ *is spread out with respect to* $X_{\mathbb{Z}}$.

PROOF. Suppose $S_{\mathbb{Z}}$ is nonlattice with respect to $X_{\mathbb{Z}}$. We derive a contradiction. By assumption $P_{S_n|X_{\mathbb{N}^c},X_{\mathbb{N}+n}}$ is singular with respect to the Lebesgue measure ℓ or, equivalently,

$$\mu_1 := P_{S_n,X_{\mathbb{N}^c},X_{\mathbb{N}+n}} \quad \text{and} \quad \mu_2 := \ell \times P_{X_{\mathbb{N}^c},X_{\mathbb{N}+n}}$$

are mutually singular. Let C^n and its complement form a Hahn-decomposition for $\mu_1 - \mu_2$. Define for $x \in \Gamma^{\mathbb{N}^c}$, $y \in \Gamma^{\mathbb{N}}$

$$C^n_{x,y} := \{s \in \mathbb{R}^1: \ (s,x,y) \in C^n\}.$$

We have

$$(6.4.6) \qquad P(S_n \in C^n_{X_{\mathbb{N}^c},X_{\mathbb{N}+n}}) = 1,$$

while

$$C^n_{X_{\mathbb{N}^c},X_{\mathbb{N}+n}}$$

is an ℓ-null set a.s.

Let the measurable space $(N,\mathcal{D})$ be defined as in Section 0.3. Define $D(m^-) \in \mathcal{D}$ to be the set of all $m \in N$ such that m^- is their restriction to $(-\infty,0] \times \Gamma$, while m has the form

$$m(B) = \sum_{n\in\mathbb{Z}} \chi_B(s_n,x_n)$$

$$\dots < s_{-1} < s_0 = 0 < s_1 < \dots$$

$$s_n \in C^n_{x_{\mathbb{N}^c},x_{\mathbb{N}+n}}, \qquad n \geq 1.$$

Let m^+ be the restriction of a measure $m \in N$ to $(0,\infty) \times \Gamma$ and define

$$D_t(m^-) := \{(T_t m)^+: \ m \in D(m^-)\}.$$

Remark that by (6.4.6) we have

$$(6.4.7) \qquad P_{N_t^+|N_0^-}(D_t(N_0^-)) = 1 \text{ a.s.}$$

Because (6.3.3) holds, we can choose t so large that

$$E\|(P_{N_t^+|N_0^-} - P_{N_{t+h}^+|N_0^-})^+\| < \tfrac{1}{2} \quad \text{for } h \geq 0,$$

and hence by (6.4.7)

$$(6.4.8) \qquad E\, P_{N_{t+h}^+|N_0^-}(D_t(N_0^-)) \geq \tfrac{1}{2} \qquad \text{for } h \geq 0.$$

By the definition of $D_t(N_0^-)$

$$\{N_{t+h}^+ \in D_t(N_0^-)\} \subset \bigcup_{n,m\geq 1} \{S_n \in C^m_{X_{\mathbb{N}^c}, X_{\mathbb{N}+n}} + h\},$$

and hence

$$\int P(N_{t+h}^+ \in D_t(N_0^-))\,dh \leq \int \sum_{n,m\geq 1} P(S_n \in C^m_{X_{\mathbb{N}^c}, X_{\mathbb{N}+n}} + h)\,dh = 0$$

by Fubini's theorem and because $C^m_{X_{\mathbb{N}^c}, X_{\mathbb{N}+n}}$ is a.s. an ℓ-null set. Because by (6.4.8) the left-hand side of the last inequality is infinite, we derived a contradiction. □

In Theorem 6.4.1 above we saw that if Γ is countable, there is a quite complete description of the conditions under which the limit relation (6.4.2) holds. It was possible to give necessary and sufficient conditions for the validity of (6.4.2). In case Γ is an arbitrary Borel space, it is more difficult to obtain such a complete theory. Below we discuss renewal theory for arbitrary Γ under a simplifying condition. We assume Markov dependence.

Suppose $(X_n)_{n\geq 0}$ is a Markov chain on a Borel space Γ, with transition probability P(x,C). If μ is the distribution of X_0, we call μ the *initial distribution* of the Markov chain and we denote the distribution of $(X_n)_{n\geq 0}$ by P_μ. In case μ is degenerate at {x}, we write P_x instead of P_μ.

The limit theory for the iterates $P^n(x,C)$ of the transition probability P(x,C) is thoroughly studied in case the state space Γ is countable. By using Doeblin's idea of looking at the excursions between successive visits of X_n to some fixed element of Γ, this limit theory is reduced to renewal theory for random walks with independent increments. In case Γ is not countable, it is possible that there are no recurrent points and so this idea cannot be used. However, for a special class of Markov chains on arbitrary state space Γ, the so-called Harris chains, there exists a fairly good analogy with the situation for countable Γ.

A Markov chain $(X_n)_{n\geq 0}$ is called *recurrent in the sense of Harris*, or shortly, a *Harris chain*, if there exists a positive, σ-finite, invariant measure π on Γ, such that for each initial distribution μ

$$P(X_n \in C \text{ infinitely often}) = 1$$

for all sets C with positive π-measure. The limit theory for this class of Markov chains can be found in OREY [1971] or REVUZ [1975]. The analogy between Harris chains and Markov chains on countable state space is fairly large. It is even possible to use Doeblin's idea, mentioned above, in a slightly revised form (see GRIFFEATH [1976], NUMMELIN [1978a], or ATHREYA, NEY [1978]).

Below we discuss renewal theory for functionals on a Harris chain. We assume that π is a probability measure and study the following random walk. Let f be a strictly positive measurable function on Γ and define

$$\text{(6.4.9)} \qquad S_0 := 0, \quad S_n := f(X_1) + \ldots + f(X_n), \quad n \geq 1.$$

Analogous to earlier definitions, we call the random walk *spread out* (with respect to the Markov chain) if for some $n \geq 1$ with positive P_π-probability $(P_\pi)_{S_{n-1}|X_0,X_n}$ is not singular with respect to the Lebesgue measure ℓ. The random walk is called *lattice* if for some $d > 0$ there is a real function c on Γ^2, such that

$$(S_{n-1}) \bmod d = c(X_0,X_n) \quad P_\pi\text{-a.s.}, \quad n > 1.$$

Otherwise, the random walk is called *nonlattice*. Define the marked point process N_t^+, t positive, on $(0,\infty) \times \Gamma$ by

$$N_t^+(B) := \sum_{n\geq 0} \chi_B(S_n - t, X_n),$$

where $B \subset (0,\infty) \times \Gamma$ is measurable. Let $(\mathcal{N}^+,\mathcal{D}^+)$ be the measurable space in which N_t^+ has its values. Assume that $E_\pi S_1$ is finite. Here E_π denotes the expectation with respect to P_π. Define Q^+ to be the probability distribution, given by

$$Q^+(D) := \frac{1}{E_\pi S_1} E_\pi \int_0^{S_1} \chi_D(N_t^+)\,dt, \quad D \in \mathcal{D}^+.$$

With these notations and assumptions we have the following result.

THEOREM 6.4.5. *For any arbitrary initial distribution* μ *on* Γ *we have*

(i) *if the random walk is spread out, then*

$$\lim_{t\to\infty} \| P_{N_t^+} - Q^+ \| = 0;$$

(ii) *if the random walk is nonlattice, then*

$$\lim_{t\to\infty} \| \nu * P_{N_t^+} - Q^+ \| = 0,$$

for all absolutely continuous probability measures ν *on* $(0,\infty)$.

PROOF. First we prove (i) for aperiodic Harris chains. By OREY [Theorem 1.7.1], or REVUZ [Theorem 6.2.8], we have, if the Harris chain has arbitrary initial distribution μ,

$$(6.4.10) \qquad \lim_{n\to\infty} \| (P_\mu)_{X_n} - \pi \| = 0.$$

Because π is invariant, the process $(X_n)_{n\geq 0}$ is stationary under P_π. Extend this process to a stationary sequence $X_{\mathbb{Z}}$ on a probability space $(\Omega,\mathcal{A},\mathcal{P})$. Let the random walk $S_{\mathbb{Z}}$ be given by (5.0.2). By (4.1.6) and the Markov property

$$\| P_{X_{\mathbb{N}+n} | X_{\mathbb{N}^c}} - P_{X_{\mathbb{N}+n}} \| = \| P_{X_{n+1} | X_0} - \pi \|.$$

By (6.4.10) this expression converges a.s. to 0 for $n \to \infty$ and so, by (4.4.2), the process $X_{\mathbb{Z}}$ is weak Bernoulli. Because $X_{\mathbb{Z}}$ is Markov dependent, it satisfies (5.1.3) by Proposition 5.1.2. Hence by the spread out condition in (i) and Theorem 6.3.1

$$(6.4.11) \qquad \lim_{t\to\infty} \| (P_x)_{N_t^+} - Q^+ \| = 0 \qquad \pi\text{-a.s.}$$

By the Markov property we can write for each $n \geq 0$ and $D \in \mathcal{D}^+$

$$P_\mu(N_t^+ \in D, S_n \leq t) = \int_\Gamma \int_{[0,t]} (P_x)_{N_{t-s}^+}(D) \; d(P_\mu)_{S_n | X_n = x}(s)\, d(P_\mu)_{X_n}(x).$$

Using the definition of total variation and the triangle inequality, we obtain

$$\|(P_\mu)_{N_t^+} - Q^+\|$$

$$\leq \int_{\Gamma[0,t]} \int \|(P_x)_{N_{t-s}^+} - Q^+\| d(P_\mu)_{S_n|X_n=x}(s)\, d\pi(x) + \|(P_\mu)_{X_n} - \pi\| + 2P_\mu(S_n > t).$$

By (6.4.10) and the bounded convergence theorem, together with (6.4.11), the limit relation of (i) follows from this inequality.

In case the Harris chain is periodic, we use an idea, similar to the argument at the end of the proof of Theorem 1.1.3. Let Γ' be a duplicate of Γ that is disjoint with Γ. We consider a new Markov chain $(\tilde{X}_n)_{n\geq 0}$ on $\tilde{\Gamma} := \Gamma \cup \Gamma'$. If $\tilde{x} \in \tilde{\Gamma}$, we denote by x and x' the elements of Γ and Γ' respectively that correspond to $\tilde{x}$. Define the transition probability of the new chain, by requiring

$$\tilde{P}(\tilde{x},C) := p\,\chi_C(x') \qquad C \subset \Gamma',$$

$$:= (1-p)P(x,C) \qquad C \subset \Gamma,$$

where $0 < p < 1$ for some p. Extend f to $\tilde{\Gamma}$, by defining $f(x') := 0$ for $x' \in \Gamma'$. Consider the n-th transition of the new chain. With probability p this transition is to Γ', and with probability (1-p) to Γ. In the first case nothing changes: $\tilde{S}_n = \tilde{S}_{n-1}$ and if Γ and Γ' would be identified, $\tilde{X}_n = \tilde{X}_{n-1}$. In the second case the chain makes, conditionally, a transition that is described by the transition probability of the original chain.

The new Markov chain is easily seen to be an aperiodic Harris chain and the random walk $(\tilde{S}_n)_{n\geq 0}$ is spread out with respect to $(\tilde{X}_n)_{n\geq 0}$. The argument above yields the limit relation of (i) for the new chain. Clearly, this implies the limit relation for the original chain.

To prove (ii) one uses Theorem 6.3.2 instead of Theorem 6.3.1. The proof is similar as above and will be left to the reader. □

NOTE. If the limit relation of (i) holds for any initial distribution μ, then necessarily the random walk is spread out. This follows from Proposition 6.4.4. If the limit relation of (ii) holds for any initial distribution μ, then necessarily the random walk is nonlattice. This follows from Proposition 6.4.3.

The theorem above holds also for semi-Markov chains. Let $(X_n)_{n\geq 0}$ be a Harris chain on a Borel space Γ with an invariant probability measure π. Let $(\xi_n)_{n\geq 1}$ be a sequence of strictly positive random variables, such that ξ_n, given $((X_j)_{j\geq 0},(\xi_j)_{j\geq 0, j\neq n})$, only depends on (X_{n-1},X_n), with its conditional distribution

$$F_{x,y} = P_{\xi_n | X_{n-1}=x, X_n=y}$$

not depending on $n \geq 1$. Renewal theory for the random walk

$$S_0 := 0, \quad S_n := \xi_1 + \ldots + \xi_n, \quad n \geq 1,$$

is known as limit theory for *semi-Markov chains*. We assume that $E_\pi S_1$ is finite. Here E_π denotes expectation under the assumption that the initial distribution of $(X_n)_{n\geq 0}$ is π. Define the spread out and nonlattice condition as above.

PROPOSITION 6.4.6. *With these definitions, Theorem* 6.4.5 *holds again.*

PROOF. Define $\tilde{\Gamma} := \Gamma \times (0,\infty) \times \Gamma$ and consider the Markov chain $(\tilde{X}_n)_{n\geq 0}$ on $\tilde{\Gamma}$, given by

$$\tilde{X}_n := (X_n, \xi_{n+1}, X_{n+1}), \quad n \geq 0.$$

Note that $(\tilde{X}_n)_{n\geq 0}$ is a Markov chain with invariant probability measure

$$\tilde{\pi}(\tilde{C}) := E_\pi \chi_{\tilde{C}}(\tilde{X}_n).$$

To prove that $(\tilde{X}_n)_{n\geq 0}$ is a Harris chain, select a set $\tilde{C}$ with positive $\tilde{\pi}$-measure and take $\varepsilon > 0$ so small, that

$$C := \{x \in \Gamma: E_x \chi_{\tilde{C}}(x,\xi_1,X_1) > \varepsilon\}$$

has positive π-measure. By OREY [Proposition 1.5.1(i)] and our choice of C,

$$\{X_n \in C \text{ infinitely often}\} \subset \{\tilde{X}_n \in \tilde{C} \text{ infinitely often}\} \text{ a.s.}$$

Because $(X_n)_{n\geq 0}$ is a Harris chain, it follows that for each initial distribution of $(X_n)_{n\geq 0}$ on Γ, the set on the left has probability 1. Hence also

$(\tilde{X}_n)_{n\geq 0}$ is distributed as a Harris chain. Define f on $\tilde{\Gamma}$ to be the projection on the second coordinate of $\tilde{\Gamma}$. We apply Theorem 6.4.5 to the new Markov chain $(\tilde{X}_n)_{n\geq 0}$. By using the Markov property, it follows that if $(S_n)_{n\geq 0}$ is spread out (nonlattice) with respect to $(X_n)_{n\geq 0}$, this holds also for $(S_n)_{n\geq 0}$ with respect to $(\tilde{X}_n)_{n\geq 0}$. Theorem 6.4.5 applied to the new Markov chain yields the required assertion. □

There is a much simpler approach to the results discussed in Theorem 6.4.5, if the transition probability P(x,C) of the Harris chain $(X_n)_{n\geq 0}$ satisfies the following domination property. There is a nonvanishing measure ϕ on a subset $C \subset \Gamma$, such that for all $x \in C$

$$P(x,\cdot) \geq \phi(\cdot).$$

Using this assumption, it is possible to consider the Harris chain as a regenerative phenomenon. Then one can obtain renewal theoretic theorems, using classical methods as described in SMITH [1958] or FELLER [1971,XI.8]. This approach is given in NUMMELIN [1978b] and ATHREYA, McDONALD, NEY [1978]. At present it does not seem possible to obtain Theorem 6.4.5 and Proposition 6.4.6 in full generality with this approach (see Section 3 of the last mentioned paper).

The literature concerning renewal theorems for semi-Markov chains is quite large. In case Γ is countable, the times of visit to some fixed recurrent point form a sequence of regeneration epochs for the process and then one can apply the theory of regenerative phenomena. A well known survey is SMITH [1958]. Also Markov chains with noncountable state space are investigated, and in particular Harris chains. KESTEN [1974] contains a good list of references. We can mention OREY [1961] who uses operator theoretic theorems. Also RUNNENBURG [1960] discusses a class of Harris chains, satisfying a Doeblin condition. JACOD [1971,1974] has a result that is close to Proposition 6.4.6(ii). His method of proof makes use of space-time harmonic functions, a technical tool known in the theory of Harris chains as presented in OREY [1971]. McDONALD [1978] also discussed this topic for Markov chains on a general state space. He uses space-time harmonic functions too. Some of his results come close to both Proposition 6.4.6(i) and (ii). However, his main interest is transient chains. KESTEN [1974] considers Markov chains on a separable metric space and is quite different from the last mentioned papers. It is possible to apply the results in this paper also to processes that are not described by means of a Markov chain.

EXAMPLE 6.4.7 (Kesten). Let $X_{\mathbb{Z}}$ be a stationary sequence of random variables with values in a finite space Γ. Let f be a function on Γ and define $(S_n)_{n\geq 0}$ by (6.4.9). The process $Y_n := X_{\mathbb{N}^c+n}$, $n \in \mathbb{Z}$, is a Markov chain on $\tilde{\Gamma} := \Gamma^{\mathbb{N}^c}$ with stationary transition probabilities. Let d_1 be the metric on Γ, defined by

$$d_1(\gamma_1,\gamma_2) := 0 \quad \text{if } \gamma_1 = \gamma_2,$$
$$:= 1 \quad \text{else,}$$

and define d on $\tilde{\Gamma}$ by

$$d(\tilde{\gamma}_1,\tilde{\gamma}_2) := \sum_{n\geq 0} 2^{-n} d_1((\tilde{\gamma}_1)_{-n},(\tilde{\gamma}_2)_{-n}).$$

This makes $\tilde{\Gamma}$ a separable metric space. Under some conditions, under which the requirement that if $d(\tilde{\gamma}_1,\tilde{\gamma}_2)$ is small, then in some particular sense

$$P_{X_{\mathbb{N}},S_{\mathbb{N}}|X_{\mathbb{N}^c}=\tilde{\gamma}_1} \quad \text{is close to} \quad P_{X_{\mathbb{N}},S_{\mathbb{N}}|X_{\mathbb{N}^c}=\tilde{\gamma}_2},$$

it is possible to derive renewal theorems of a different type than we did in Chapter 6 (see KESTEN [1974, Conditions I]). The model given above, is related to the distance diminishing models in IOSIFESCU and THEODORESCU [1969].

6.5. MIXING AND REMIXING FOR FLOWS

The main result in Section 3.1, Theorem 3.1.4, establishes a 1-1 correspondence between a class $\mathcal{Q}$ of distributions of stationary point processes N on the real line and a class $\mathcal{Q}_0$ of distributions of point processes N_0, given in terms of a random walk $S_{\mathbb{Z}}$ with stationary increments. One of the interesting properties of this 1-1 correspondence is that it behaves nicely with respect to ergodicity: N is ergodic if and only if the process of increments of $S_{\mathbb{Z}}$ is ergodic. So a mixing property of N, ergodicity, induces and is induced by a mixing property of N_0. This section considers a similar "mixing and remixing" property. It investigates how the weak Bernoulli condition behaves under the 1-1 correspondence.

The first two theorems are the main results. We assume that N_0 is given in terms of a random walk $S_{\mathbb{Z}}$ controlled by a stationary sequence $X_{\mathbb{Z}}$. Especially the second of these theorems, where we require that $X_{\mathbb{Z}}$ is countably valued, has a quite satisfying form. Both theorems are obtained as simple applications of results in the preceding two sections.

In the second half of this section we deduce from the results mentioned above a property of flows. Flows are studied in ergodic theory. In particular the theory of mixing properties of special flows is narrowly related to renewal theory. Theorem 6.5.9 forms the main result in this direction. We discuss the related literature and show that this theorem improves on results in GUREVIČ [1967] and TOTOKI [1971].

Let N be a marked point process on $\mathbb{R}^1 \times \Gamma$, where Γ is a Borel space. We denote the restriction of N to $(0,\infty) \times \Gamma$ and $(-\infty,0] \times \Gamma$ by N^+ and N^- respectively. Suppose N is stationary. N is called *weak Bernoulli* if

$$\lim_{t\to\infty} \perp(N^-,(T_tN)^+) = 0$$

or equivalently (compare (4.4.2))

$$\lim_{t\to\infty} \|P_{(T_tN)^+|N^-} - P_{N^+}\| = 0 \text{ a.s.}$$

To describe the most interesting of our results in this section, the weak Bernoulli condition is slightly too strong. Say N is *smoothed weak Bernoulli* if for each absolutely continuous probability measure ν on $(0,\infty)$ holds

$$\lim_{t\to\infty} \|\nu * P_{(T_tN)^+|N^-} - P_{N^+}\| = 0 \text{ a.s.}$$

In the results below we assume that $X_{\mathbb{Z}}$ is a stationary sequence with values in the Borel space Γ. Let $S_{\mathbb{Z}}$ be a random walk, controlled by $X_{\mathbb{Z}}$, with strictly positive increments having finite expectation. Define the marked point process N_0 by

$$(6.5.1) \qquad N_0(B) := \sum_{n\in\mathbb{Z}} \chi_B(S_n,X_n)$$

for any measurable set $B \subset \mathbb{R}^1 \times \Gamma$. Let N be a marked point process with distribution Q given by

$$(6.5.2) \qquad Q(D) := \frac{1}{ES_1} E \int_0^{S_1} \chi_D(T_sN_0)\,ds, \qquad D \in \mathcal{D}.$$

Here $(\mathcal{N},\mathcal{D})$ is the measurable space, defined at the end of Section 0.3. The marked point process N is stationary. This is proved by using notes 2° and 3° to Proposition 3.1.1 (or see MATTHES [1963]).

With these notations and assumptions we have the following two theorems.

THEOREM 6.5.1. *If* $X_{\mathbb{Z}}$ *is weak Bernoulli and satisfies* (5.1.3), *then*

(i) *if* $S_{\mathbb{Z}}$ *is spread out with respect to* $X_{\mathbb{Z}}$ *then* N *is weak Bernoulli;*

(ii) *if* $S_{\mathbb{Z}}$ *is nonlattice with respect to* $X_{\mathbb{Z}}$ *then* N *is smoothed weak Bernoulli.*

A more complete result can be obtained if $X_{\mathbb{Z}}$ is countably valued.

THEOREM 6.5.2. *Suppose* $X_{\mathbb{Z}}$ *is countably valued.* N *is smoothed weak Bernoulli if and only if* $X_{\mathbb{Z}}$ *is Cesaro weak Bernoulli and* $S_{\mathbb{Z}}$ *is nonlattice with respect to* $X_{\mathbb{Z}}$.

The first theorem, case (i), is a consequence of Proposition 6.5.3 and Theorem 6.3.1. Case (ii) of the first theorem follows from Proposition 6.5.4 and Theorem 6.3.2. The last theorem follows from Proposition 6.5.4 and Theorem 6.4.1.

With the notations and assumptions in the introduction to both theorems above, we have the following two propositions.

PROPOSITION 6.5.3. N *is weak Bernoulli if and only if*

$$(6.5.3) \qquad \lim_{t\to\infty} \| P_{(T_tN_0)^+|N_0^-} - P_{N^+} \| = 0 \text{ a.s.}$$

PROOF. The proposition compares a limit property of N with a limit property of N_0.

First we prove the if-part. By the stationarity of $X_{\mathbb{Z}}$ and (6.5.2)

$$P_N(D') = \frac{1}{ES_1} E \int_{S_{-1}}^{0} \chi_{D'}(T_sN_0)\,ds$$

$$= \frac{1}{ES_1} E \int_{S_{-1}}^{0} P_{T_sN_0|N_0^-}(D')\,ds, \qquad D' \in \mathcal{D}.$$

The last equality follows because S_{-1} is N_0^--measurable. Let $D \in \mathcal{D}^+ \times \mathcal{D}^-$ and take $t \geq 0$. With the choice $D' := \{m \in N\colon ((T_tm)^+, m^-) \in D\}$ we have

$$P_{T_sN_0|N_0^-}(D') = P_{(T_{s+t}N_0)^+,(T_sN_0)^-|N_0^-}(D).$$

Let δ_x be the probability measure degenerate at x. Note that

$$P_{(T_tN)^+,N^-}(D) = \frac{1}{ES_1} E \int_{S_{-1}}^{0} P_{(T_{t+s}N_0)^+|N_0^-}$$

$$\times \delta_{(T_sN_0)^-}(D)\,ds, \qquad D \in \mathcal{D}^+ \times \mathcal{D}^-.$$

Here we assume that the distributions Q^+ and Q^- of N^+ and N^- are defined on the measurable spaces $(\mathcal{N}^+,\mathcal{D}^+)$ and $(\mathcal{N}^-,\mathcal{D}^-)$. We also have

$$Q^+ \times Q^-(D) = \frac{1}{ES_1} E \int_{S_{-1}}^{0} Q^+ \times \delta_{(T_sN_0)^-}(D)\,ds, \qquad D \in \mathcal{D}^+ \times \mathcal{D}^-.$$

Using the last two equalities we obtain

$$(6.5.4) \qquad \perp((T_tN)^+,N^-) = \tfrac{1}{2}\| P_{(T_tN)^+,N^-} - Q^+ \times Q^- \|$$
$$\leq \frac{1}{ES_1} E \int_{S_{-1}}^{0} \tfrac{1}{2}\| P_{(T_{t+s}N_0)^+|N_0^-} - Q^+ \|\,ds.$$

By (6.5.3) and the bounded convergence theorem, the last expression vanishes for $t \to \infty$.

To prove the only if-part we use the analogue of Proposition 3.1.4 for marked point processes. Let N^s be the projection of N on the real line. Suppose first that U is the smallest nonnegative point of N^s. We have (see MATTHES [1963])

$$(6.5.5) \qquad \lim_{\delta \downarrow 0} \| P_{T_UN|\,|U|<\delta} - P_{N_0} \| = 0.$$

By using a reflection around the origin it follows that we may also take U to be the largest nonpositive point of N^s. Write $N_\delta := T_UN$. Assume that N is weak Bernoulli, so with $Q^+ = P_{N^+}$

$$\lim_{t\to\infty} \| P_{(T_tN)^+|N^-} - Q^+ \| = 0 \text{ a.s.}$$

The latter random variable U is N^--measurable. Hence we may replace t by $t+U$ in the limit relation above and therefore $(T_tN)^+$ by $(T_tN_\delta)^-$. Note that N^- and (N_δ^-,U) mutually determine each other. By (4.1.5) we have the equality

$$\| P_{(T_tN_\delta)^+,N_\delta^-,U} - Q^+ \times P_{N_\delta^-,U} \| = E\| P_{(T_tN_\delta)^+|N_\delta^-,U} - Q^+ \|.$$

By the observations we made above, the right-hand side vanishes for $t \to \infty$. Hence for all $\delta > 0$

$$\lim_{t\to\infty} \| P_{(T_t N_\delta)^+, N_\delta^- \,|\, |U|<\delta} - Q^+ \times P_{N_\delta^- \,|\, |U|<\delta} \| = 0$$

and by (6.5.5)

$$\| P_{(T_t N_0)^+, N_0^-} - Q^+ \times P_{N_0^-} \| = E \| P_{(T_t N_0)^+ | N_0^-} - Q^+ \| \to 0$$

for $t \to \infty$. Using $Q^+ = P_{N^+} = P_{(T_t N)^+}$ and (4.1.2), one proves that the expression in (6.5.3) is nonincreasing. Hence the limit relation above implies (6.5.3). □

PROPOSITION 6.5.4. N *is smoothed weak Bernoulli if and only if for each absolutely continuous probability measure* ν *on* $(0,\infty)$

$$\text{(6.5.6)} \qquad \lim_{t\to\infty} \| \nu * P_{(T_t N_0)^+ | N_0^-} - P_{N^+} \| = 0 \text{ a.s.}$$

PROOF. The proof is parallel to the proof of Proposition 6.5.3. We only give a sketch.

In the if-part one derives instead of (6.5.4)

$$\begin{aligned} \perp^\nu((T_t N)^+, N^-) &:= E\, \tfrac{1}{2} \| \nu * P_{(T_t N)^+ | N^-} - Q^+ \| \\ &\le \frac{1}{ES_1} E \int_{S_{-1}}^{0} \tfrac{1}{2} \| \nu * P_{(T_{t+s} N_0)^+ | N_0^-} - Q^+ \| ds. \end{aligned}$$

If (6.5.6) holds, then this expression converges to 0 for $t \to \infty$. Hence

$$\| \nu * P_{(T_t N)^+ | N^-} - Q^+ \| \to 0$$

for $t \to \infty$ in L_1-mean and because this expression is nonincreasing, we also have a.s.-convergence.

To prove the only if-part, one derives, following the proof of Proposition 6.5.3, from the smoothed weak Bernoulli property, that

$$\lim_{t\to\infty} E \| \nu * P_{(T_t N_\delta)^+ | N_\delta^-, U} - Q^+ \| = 0.$$

Also as before, one proves, by means of (6.5.5), that this implies

$$\lim_{t\to\infty} E\| \nu * P_{(T_tN_0)^+|N_0^-} - Q^+\| = 0.$$

Hence one obtains (6.5.6). □

In Proposition 6.5.6 we compare the smoothed weak Bernoulli property with other, better known concepts of asymptotic independence. First we prove a lemma.

LEMMA 6.5.5. *If Q is the distribution of a stationary marked point process on $\mathbb{R}^1 \times \Gamma$, with Γ a Borel space, then*

$$\lim_{t\to 0} Q(D \,\Delta (T_tD)) = 0, \quad D \in \mathcal{D}.$$

PROOF. By the definition of a Borel space, we may assume that $\Gamma \subset [0,1]$. The mapping f defined by

$$f(m,t) := \chi_D(T_tm), \quad m \in N, \ t \text{ real},$$

is measurable for $D \in \mathcal{D}$. To see this, let $\mathcal{D}_1$ be the class of all $D \in \mathcal{D}$ for which f is measurable. Then $\mathcal{D}_1$ contains all sets of the form

$$D := \{m \in N\colon m(I\times J) \le k\},$$

where I and J are intervals of the form (a,b] and $k \in \mathbb{N}$. Hence by KALLENBERG [1.4] the set $\mathcal{D}_1$ generates $\mathcal{D}$. Because $\mathcal{D}_1$ is a monotone field, it follows by HALMOS [1950,I.6] that $\mathcal{D} = \mathcal{D}_1$.

By stationarity we have for each real s

$$Q(D\,\Delta\, (T_tD)) = \int |f(m,s)-f(m,s+t)|\,dQ(m)$$

and hence for any $h > 0$

$$Q(D\,\Delta\, (T_tD)) = \frac{1}{h}\int_0^h \int |f(m,s)-f(m,s+t)|\,dQ(m)\,ds.$$

By Fubini's theorem we may exchange the integrals and because $f(m,\cdot)$ is measurable.

$$g(t) := \frac{1}{h}\int_0^h |f(m,s)-f(m,s+t)|\,ds \to 0$$

for $t \to 0$. The assertion follows by the dominated convergence theorem. □

Define $\mathcal{D}_s^t \subset \mathcal{D}$, $-\infty \le s < t \le \infty$, to be the σ-field, induced by the mappings

$$m \to m(B),$$

with $B \subset (s,t) \times \Gamma$ measurable. A marked point process N has *trivial right* (or *left*) *tail σ-field*, if the sets

$$\{N \in D\}, \quad D \in \bigcap_{t \in \mathbb{R}^1} \mathcal{D}_t^\infty \quad (\text{or } \bigcap_{t \in \mathbb{R}^1} \mathcal{D}_{-\infty}^t)$$

have probability 0 or 1.

PROPOSITION 6.5.6. *If a stationary marked point process* N *is smoothed weak Bernoulli, then it has trivial right (and left) tail σ-field.*

PROOF. Lemma 3.2.1 holds also for marked point processes, i.e. N has trivial right tail σ-field, if its distribution Q satisfies

$$(6.5.7) \qquad \lim_{t\to\infty} \sup_{D \in \mathcal{D}_t^\infty} |Q(D \cap D_0) - Q(D)Q(D_0)| = 0$$

for all $D_0 \in \mathcal{D}$. This is proved similarly as in Lemma 3.2.1. Because each set $D_0 \in \mathcal{D}$ can be approximated arbitrarily close by sets $D_0 \in \mathcal{D}_{-\infty}^t$, t real, it suffices to prove (6.5.7) for all $D \in \mathcal{D}_{-\infty}^t$, t real. By stationarity we only have to prove (6.5.7) for all $D_0 \in \mathcal{D}_{-\infty}^0$.

Also by stationarity we have $Q((T_s^{-1}D) \cap D_0) = Q(D \cap (T_s D_0))$ for any $s \ge 0$, and so

$$|Q(D \cap D_0) - Q((T_s^{-1}D) \cap D_0)| \le Q(D_0 \Delta (T_s D_0)).$$

Suppose $D_0 \in \mathcal{D}_{-\infty}^0$ and let ν be the homogeneous distribution on (0,ε). Uniformly for $D \in \mathcal{D}_t^\infty$ we have, using Lemma 6.5.5,

$$|Q(D \cap D_0) - Q(D)Q(D_0)|$$

$$\le \int Q(D_0 \Delta (T_s D_0))\,d\nu(s) + \int |Q((T_s^{-1}D) \cap D_0) - Q(D)Q(D_0)|\,d\nu(s)$$

$$\le o(1) + E\|\nu * P_{(T_t N)^+ | N^-} - P_{N^+}\|, \quad \varepsilon \to 0,$$

and because N is smoothed weak Bernoulli the assertion follows. □

The two theorems in the beginning of this section, are results on the theory of mixing properties for flows. An introduction in the theory of flows and their relation with point processes can be found in DE SAM LAZARO and MEYER [1975] and NEVEU [1977].

A *measurable flow* on a measurable space $(\Omega,\mathcal{A})$ is a family $(\theta_t,\ t\in\mathbb{R}^1)$ of mappings $\theta_t\colon \Omega\to\Omega$ such that

(i) $\theta_s\circ\theta_t = \theta_{s+t}$ on Ω for s and t real, with θ_0 the identity;

(ii) the mapping $(t,\omega)\to\theta_t\omega$ from $\mathbb{R}^1\times\Omega\to\Omega$ is measurable.

The flow is called *measure preserving* if for all real t

$$P(\theta_t^{-1}A) = P(A), \quad A\in\mathcal{A}$$

A set A is called *invariant* under the flow if $\theta_t^{-1}A = A$ for all real t. The *invariant σ-field* J is the set of invariant events $A\in\mathcal{A}$. A flow on a probability space is called *ergodic,* if each invariant set has probability 0 or 1.

EXAMPLE 6.5.7. Let Q be the distribution on $(\mathcal{N},\mathcal{D})$ of a stationary marked point process. The family $(T_t,\ t\in\mathbb{R}^1)$ on $(\mathcal{N},\mathcal{D})$ is a measure preserving flow on $(\mathcal{N},\mathcal{D},Q)$.

Let $(\theta_t,\ t\in\mathbb{R}^1)$ be a measure preserving flow on a probability space $(\Omega,\mathcal{A},P)$. The flow is called a *K-flow* if there is a σ-field $\mathcal{B}\subset\mathcal{A}$ such that

(i) $\mathcal{B}$ is *increasing,* i.e. $\theta_s^{-1}\mathcal{B}\subset\theta_t^{-1}\mathcal{B}$ for all real $s\le t$;

(ii) $\mathcal{B}$ is *generating,* i.e. $\bigcup_{t\in\mathbb{R}^1}\theta_t^{-1}\mathcal{B}$ generates $\mathcal{A}$;

(iii) the σ-field $\bigcap_{t\in\mathbb{R}^1}\theta_t^{-1}\mathcal{B}$ is *trivial,* i.e. contains only sets with probability 0 and 1.

As in SMORODINSKY [1971, Theorem 7.5] one proves that a K-flow is *mixing,* i.e.

$$\lim_{t\to\infty} P(A_1\cap\theta_t^{-1}A_2) = P(A_1)P(A_2), \quad A_1,A_2\in\mathcal{A}.$$

It follows that a K-flow is ergodic.

EXAMPLE 6.5.8. Let Q be the distribution of a stationary, marked point process N with trivial left tail σ-field. Using the definitions of Section 0.3, the flow $(T_t\colon t\in\mathbb{R}^1)$ on $(\mathcal{N},\mathcal{D},Q)$ is a K-flow. To see this, let $D_s^t\subset\mathcal{D}$ be defined as in the introduction to Proposition 6.5.6 and take $\mathcal{B} := \mathcal{D}_{-\infty}^0$.

Suppose that $S_{\mathbb{Z}}$ is a random walk, controlled by a stationary sequence $X_{\mathbb{Z}}$. Let $S_{\mathbb{Z}}$ have strictly positive increments with finite expectation. Let N be a marked point process with distribution Q, given by (6.5.1) and (6.5.2).

THEOREM 6.5.9. *Let* $X_{\mathbb{Z}}$ *be weak Bernoulli and satisfy* (5.1.3). *In case* $S_{\mathbb{Z}}$ *is nonlattice with respect to* $X_{\mathbb{Z}}$, *the flow of translations on* $(N,\mathcal{D},Q)$ *is a* K*-flow. Otherwise there is some* $t \neq 0$, *such that* T_t *is not ergodic on* $(N,\mathcal{D},Q)$.

PROOF. The first assertion forms a weakening of Theorem 6.5.1. This follows from Proposition 6.5.6 and Example 6.5.8. To obtain the second assertion, assume that $S_{\mathbb{Z}}$ is lattice with respect to $X_{\mathbb{Z}}$, i.e. for some $d > 0$ there exists a measurable function $c\colon \Gamma^{\mathbb{N}^c} \times \Gamma^{\mathbb{N}} \to [0,d)$, such that

$$(S_n) \bmod d = c(X_{\mathbb{N}^c}, X_{\mathbb{N}+n}) \text{ a.s.}, \quad n \geq 1.$$

Let $D_0 \subset N$ be the set of all $m \in N$ of the form

$$m(B) = \sum_{n\in\mathbb{Z}} \chi_B(s_n,x_n), \quad B \subset \mathbb{R}^1 \times \Gamma,$$

$$\dots < s_{-1} < s_0 = 0 < s_1 < \dots$$

for $x_{\mathbb{Z}} \in \Gamma^{\mathbb{Z}}$, such that

$$(s_n - s_m) \bmod d = c(x_{\mathbb{N}^c}, x_{\mathbb{N}+n}), \quad -\infty < m < n < \infty.$$

The point process N_0 defined by (6.5.1) has its values in D_0 and the measure Q is concentrated on $D := \bigcup_{t\in\mathbb{R}^1} D_t$, with $D_t := T_t D_0$.

The sets D_t and D_u coincide if $(t-u) \bmod d = 0$, and otherwise D_t and D_u are disjoint. Define a function ψ on D (so Q-a.s. on N) by

$$\psi(m) := \inf\{t \geq 0\colon m \in D_t\}$$

and note that $T_t\psi(m)$ is, as a function in t, a saw-teeth function with period d. Hence ψ is Q-a.s. invariant under T^d. Furthermore, the sets $\{0 \leq \psi < \frac{1}{2}d\}$ and $\{\frac{1}{2}d \leq \psi < d\}$ have probability $\frac{1}{2}$. Because these sets are Q-a.s. invariant, it follows that T^d is not ergodic on $(N,\mathcal{D},Q)$. □

NOTE. For a countably valued sequence $X_{\mathbb{Z}}$, the theorem holds also under the weaker condition that $X_{\mathbb{Z}}$ is Cesaro weak Bernoulli. This follows by using Theorem 6.5.2 instead of Theorem 6.5.1.

The theorem above is related to the theory of mixing properties of special flows.

EXAMPLE 6.5.10 (special flow under a function). Let T be a bimeasurable, measure preserving bijection on a probability space $(\Omega,\mathcal{A},P)$. Suppose F is a strictly positive, measurable function on Ω. Define a random walk $S_{\mathbb{Z}}$ by

$$S_0(\omega) := 0, \quad S_n(\omega) - S_{n-1}(\omega) = F(T^{n-1}\omega), \quad n \in \mathbb{Z}, \tag{6.5.8}$$

and assume that S_1 has finite expectation. Define the probability space $(\tilde{\Omega},\tilde{\mathcal{A}},\tilde{P})$ to be the restriction to

$$\tilde{\Omega} := \{(u,\omega): 0 \le u < F(\omega)\}$$

of the product

$$(\mathbb{R}^1, \mathcal{B}^1, \frac{1}{ES_1}\ell) \times (\Omega,\mathcal{A},P),$$

where ℓ is the Lebesgue measure. The family $(\theta_t,\ t \in \mathbb{R}^1)$ defined by

$$\theta_t(u,\omega) = (u+t - S_n(\omega), T^n\omega), \quad \text{if} \quad S_n(\omega) \le u+t < S_{n+1}(\omega),$$

is a measurable flow on $(\tilde{\Omega},\tilde{\mathcal{A}})$, called a *special flow under a function*.

The concept special flow under a function, described in the example above, is quite old. It is introduced by AMBROSE [1941] and it appears that also Von Neumann was interested in this concept. AMBROSE [1941] proves that every ergodic flow can be represented as a special flow. This result was extended by AMBROSE and KAKUTANI [1942] to a very general class of flows. In these results the following isomorphy concept is used. A flow $(\theta_t, t \in \mathbb{R}^1)$ on $(\Omega,\mathcal{A},P)$ is *isomorphic* to a flow $(\theta'_t,\ t \in \mathbb{R}^1)$ on $(\Omega',\mathcal{A}',P')$ if there is an a.s. bimeasurable bijection $\phi: \Omega \to \Omega'$ that commutes with the flow and is measure preserving.

For the flow of translations on $(N,\mathcal{D},Q)$ in Theorem 6.5.9 it is quite simple to construct the isomorphy with a special flow.

EXAMPLE 6.5.11. Assume $X_{\mathbb{Z}}$ is the coordinate process on $\Omega := \Gamma^{\mathbb{Z}}$. The random walk $S_{\mathbb{Z}}$ is controlled by $X_{\mathbb{Z}}$ and so, is given by (5.0.2). Hence $S_{\mathbb{Z}}$ is given by (6.5.8) with

$$F(\omega) = f(x_1), \quad \omega = x_{\mathbb{Z}},$$

where f is a strictly positive, measurable function on Γ. Define P on $(\Omega,\mathcal{A}) = \prod_{n\in\mathbb{Z}} (\Gamma,\mathcal{T})$ to be the distribution of $X_{\mathbb{Z}}$. We show that the special flow $(\theta_t, t \in \mathbb{R}^1)$ on $(\tilde{\Omega},\tilde{\mathcal{A}},\tilde{P})$ is isomorphic to the flow $(T_t, t \in \mathbb{R}^1)$ on $(N,\mathcal{D},Q)$, with Q defined by (6.5.1) and (6.5.2).

Define for $\tilde{\omega} = (u,\omega) \in \tilde{\Omega}$ the measure $m = \phi(\tilde{\omega})$ on $\mathbb{R}^1 \times \Gamma$ by

$$m(B) := T_u \sum_{n\in\mathbb{Z}} \chi_B(S_n,X_n),$$

with $B \subset \mathbb{R}^1 \times \Gamma$ measurable. Because P is concentrated on

$$\{\lim_{n\to\infty} S_n = -\lim_{n\to-\infty} S_n = \infty\}$$

the mapping $\phi\colon \tilde{\Omega} \to N$ is defined with probability 1. From the definition of Q it follows that $\phi(\tilde{\Omega})$ has Q-measure 1. Note that $X_{\mathbb{Z}}(\omega) = \omega$. It is easily proved that ϕ is an a.s. bimeasurable bijection that commutes with the flow. The mapping ϕ is measure preserving because

$$Q(D) = \frac{1}{ES_1} E \int_0^{S_1} \chi_D(T_s N_0)\,ds$$

$$= \int \int_0^{F(\omega)} \chi_D(\phi(u,\omega)) \frac{1}{ES_1}\, d\ell(u)\,dP(\omega).$$

In ergodic theory most results are formulated for the special flow described above, rather than the flow of translations on $(N,\mathcal{D},Q)$. Several papers give conditions under which the flow above is a K-flow. Important is GUREVIČ [1967], that discusses several sets of conditions under which the K-property can be proved. The technique that he uses is quite different from our approach.

Close to Theorem 6.5.9 is the following result in GUREVIČ [1967]. Suppose $S_{\mathbb{Z}}$ is a random walk with stationary increments $X_{\mathbb{Z}}$ having values in a finite set $\Gamma \subset (0,\infty)$. If $X_{\mathbb{Z}}$ is weak Bernoulli and the elements of $\{\gamma \in \Gamma\colon P(X_0=\gamma) > 0\}$ are independent over the rational numbers, then the special flow mentioned above is a K-flow.

Throughout his paper GUREVIČ [1967] assumes that the "ceiling function" F admits a finite or countable number of values. To remove the finiteness condition on F, TOTOKI [1970] discusses the K-property for the special case where $S_{\mathbb{Z}}$ has i.i.d. increments that are not necessarily countably valued. Note that instead of a finiteness condition, our result in this direction, Theorem 6.5.9, uses condition (5.1.3).

BLANCHARD [1976] discusses the nonfinite case too. His result is in the spirit of one of the theorems in GUREVIČ [1967] for the finite case. Specialized to a probabilistic context his result can be described in the following way. Let F_n^+ and F_n^- be the completion of the σ-fields generated by $X_{\mathbb{N}+n}$ and $X_{\mathbb{N}^c+n}$ respectively, $n \in \mathbb{Z}$. Suppose that

$$F_0^+ \cap F_0^-$$

contains only events with probability 0 or 1. If the distribution of S_1 is nonlattice, then the special flow discussed above is a K-flow.

Other results for special flows are given by RATNER [1974,1978]. In these papers it is assumed that $X_{\mathbb{Z}}$ is i.i.d. but there is used another choice for F, than in Example 6.5.11. Both papers discuss the K-property but consider also stronger mixing properties.

APPENDIX A

A. A TOPOLOGY ON A SET OF DISTRIBUTIONS OF POINT PROCESSES

Let $\mathcal{P}$ be the set of distributions of (marked) point processes. This appendix investigates d, defined by (0.3.4). First we show that d is a metric on $\mathcal{P}$ and later we compare the topology induced by d with better known topologies on $\mathcal{P}$. Our results apply both to point processes on the real line $R := \mathbb{R}^1$ and to marked point processes on $R := \mathbb{R}^1 \times \Gamma$. We assume that the Borel space Γ is the unit interval $\Gamma := [0,1]$. Only in the second proposition this assumption forms a restriction. The measures $P \in \mathcal{P}$ are defined on the measurable space $(\mathcal{N},\mathcal{D})$ defined in Section 0.3. If f is a measurable function on R, write for $m \in \mathcal{N}$

$$mf := \int_R f dm,$$

if this integral exists.

<u>PROPOSITION A.1</u>. *d is a metric on* $\mathcal{P}$.

<u>PROOF</u>. The symmetry of d and the triangle inequality are easily checked. We have to prove that d separates. Let N_1 and N_2 be point processes on the same probability space with distributions P_1 and P_2, for which $d(P_1,P_2) = 0$. Suppose Y_n, $n \geq 1$, are random variables, homogeneously distributed on $(0,\frac{1}{n})$, and independent of N_1 and N_2. Let f be an arbitrary continuous function on R with compact support C. By KALLENBERG [3.1] it is sufficient to prove $N_1 f \overset{d}{=} N_2 f$ to get $P_1 = P_2$. We let $n \to \infty$ in

$$N_1(T_{Y_n} f) = (T_{Y_n} N_1) f \overset{d}{=} (T_{Y_n} N_2) f = N_2(T_{Y_n} f).$$

Take a compact set $C_1 \supset C$ such that $T_y C \subset C_1$ for all $0 \leq y \leq 1$. Because f is uniformly continuous on C_1 and N_1 and N_2 are finite on C_1 we have

$\lim_{n\to\infty} N_i(T_{Y_n} f) = N_i f$ a.s., $i = 1,2$, and hence it follows that $N_1 f \stackrel{d}{=} N_2 f$. □

Let μ and ν be absolutely continuous probability measures on the real line. The pseudometric d_ν, defined by (0.3.3), is invariant under translations of ν. Furthermore, because $\|\nu * P - \mu * P\| \le \|\nu - \mu\|$, we have

$$d_\nu(P_1,P_2) \le d_\mu(P_1,P_2) + 2\|\nu-\mu\|.$$

The measure ν can be approximated in the total variation metric by a probability measure with a continuous density with respect to the Lebesgue measure ℓ (see HALMOS [1950, Section 55]) and also by a probability measure μ of the form

$$\mu = \frac{1}{k} \sum_{j=1}^{k} T_{t_j} \nu_\varepsilon,$$

where ν_ε, $\varepsilon > 0$, is the homogeneous distribution on $(0,\varepsilon)$. We may even suppose $\varepsilon = \frac{1}{n}$, $n \ge 1$. Therefore we can obtain that

$$\text{(A.1)} \qquad d_\nu(P_1,P_2) \le d_{\nu_{1/n}}(P_1,P_2) + 2\|\nu-\mu\|,$$

where $\|\nu-\mu\|$ can be made arbitrarily small. Hence to prove that $\lim_{n\to\infty} d_\nu(P_n,P'_n) = 0$ it is sufficient to show $\lim_{n\to\infty} d(P_n,P'_n) = 0$.

PROPOSITION A.2. *The topology introduced by* d *on* $\mathcal{P}$ *is stronger than the weak topology on* $\mathcal{P}$ *with respect to the vague topology on* $\mathcal{N}$.

PROOF. The space of Radon measures $\mathcal{N}$ on R is Polish in the vague topology (see KALLENBERG [A7.7]). By PARTHASARATY [II.6.2] it follows that the space of probability measures $\mathcal{P}$ on $(\mathcal{N},\mathcal{D})$, provided with the weak topology, is a separable metric space. Hence to prove that the weak topology on $\mathcal{P}$ is weaker than the topology introduced by the metric d it suffices to show that sequential convergence in d-metric implies weak convergence (see DUGUNDJI [Chapter X]).

Let N_n, $n \ge 0$, be (marked) point processes on a probability space with distributions P_n, $n \ge 0$, such that $\lim_{n\to\infty} d(P_n,P_0) = 0$. To prove weak convergence KALLENBERG [4.2] shows that it is enough to prove for $n \to \infty$

$$\text{(A.2)} \qquad N_n f \xrightarrow{d} N_0 f$$

for all $f \in F$, the set of continuous real functions with compact support on R. Let Y_k, $k \geq 1$, be random variables that are homogeneously distributed on $(0,1/k)$ and independent of N_n, $n \geq 0$. Because $\lim_{n\to\infty} d(P_n,P_0) = 0$, we have

$$\text{(A.3)} \qquad N_n(T_{Y_k} f) \xrightarrow{d} N_0(T_{Y_k} f)$$

if $n \to \infty$, for any $k \geq 1$ and $f \in F$.

Take a compact set $C_1 \supset C$ so large that $T_y C \subset C_1$ for all $0 \leq y \leq 1$ and let $g \in F$ be a function on R such that $T_y g \geq 1$ on C_1 for all $0 \leq y \leq 1$. Hence

$$(T_{Y_1} N_n) g \geq N_n(C_1).$$

Because the left-hand side converges in distribution, the sequence $N_n(C_1)$, $n \geq 0$, is uniformly tight. Observe that

$$\begin{aligned}\{|N_n f - N_0 f| > \varepsilon\} \subset \{N_n(C_1) > m\} &\cup \{N_0(C_1) > m\} \\ &\cup \{|N_n(T_{Y_k} f) - N_0(T_{Y_k} f)| > \tfrac{1}{2}\varepsilon\} \\ &\cup \{\sup|f - T_{Y_k} f| > \tfrac{\varepsilon}{2m}\}\end{aligned}$$

for any $m > 0$. Because of (A.3) and because f is uniformly continuous, we obtain (A.2). □

<u>NOTE</u>. To obtain weak convergence of P_n to P_0, $n \to \infty$, we used the assumption $\lim_{n\to\infty} d(P_n,P_0) = 0$. However, this assumption can be replaced by

$$\lim_{n\to\infty} (\nu * P_n(D) - \nu * P_0(D)) = 0, \quad D \in \mathcal{D}.$$

To see this, observe that (A.3) holds for simple functions f. This can be proved by using the limit relation above with $D := \{m \in N\colon m(E_i) = k_i,\ 1 \leq i \leq j\}$, where E_i are bounded measurable sets and k_i, $1 \leq i \leq j$, are integers. By an enclosure argument one obtains (A.3) for all $f \in F$. Then weak convergence of P_n to P_0, $n \to \infty$, can be proved as before.

There is another metric on P: the total variation metric. This metric is stronger than d, because

$$\|\nu*P_1 - \nu*P_2\| \le 2 \sup_{D\in\mathcal{D}}|\nu*P_1(D) - \nu*P_2(D)|$$

$$\le 2 \sup_{D\in\mathcal{D}}|P_1(D) - P_2(D)| = \|P_1-P_2\|.$$

In many results we prove convergence with respect to d_ν, with ν absolutely continuous, or better with respect to the metric d. There are two reasons that force us to use the "smoothing" $\nu*P$ of the elements $P \in \mathcal{P}$. On the one hand the total variation metric is quite often too strong (see Example 3.2.6). On the other hand the weak topology mentioned in the proposition above, is too weak to allow us to describe limit results like Theorem 6.4.2.

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