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ASYMPTOTIC OPTIMALITY THEORY FOR TESTING PROBLEMS WITH RESTRICTED ALTERNATIVES

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CONTENTS

PREFACE AND SUMMARY

The monograph lying before you contains a theory of asymptotic optimality for tests for a class of testing problems for exponential families (special attention is paid to testing problems for contingency tables), at a fixed level of significance and with an emphasis on restricted alternatives. Two sources may be mentioned here from which I derived inspiration for this study. Firstly there are several experimental scientists who have consulted me for advice over their statistical problems. They made it clear to me that testing problems with restricted alternatives arise rather frequently in practice, and are worth being investigated. On the other hand Willem Schaafsma introduced me to these problems. He had treated them in SCHAAFSMA (1966), but his approach was mathematically not completely satisfactory.

Statistically inclined experimental scientists may be interested in Chapters 1 and 9, and in parts of Sections 3.2, 3.3, 3.4, 5.6, 5.7, 6.3 and 6.4. Chapter 1 is an introduction and summary, using a minimum of mathematical language. Chapter 9 considers several testing problems with restricted alternatives for contingency tables, and gives tests which are asymptotically optimal in the sense of Chapters 7 and 8.

Chapter 2 reviews some basic concepts from the theory of hypothesis testing, and serves as a reference for later chapters. Readers familiar with this subject can omit it. Chapters 3 to 8 constitute the body of this thesis. In Section 3.1 the class of testing problems to be studied is formulated; in order to pursue an asymptotic approach, such problems are considered as members of a sequence of similar testing problems (Section 3.5), where the sample sizes tend to infinity. Chapter 4 is devoted to the development of a technical tool: the limit of a sequence of testing problems with a fixed outcome space. In Chapter 5, the concepts "asymptotically of level α" and "asymptotically uniformly most powerful" are extensively discussed and asymptotically uniformly most powerful - level α tests are given for certain testing problems.

The asymptotic optimality theory of Chapters 6 to 8 contains the main results of this study. The approach is based on minimizing the maximum shortcoming. For testing problems with fixed sample sizes this leads to the most stringent test (Section 2.6). For testing problems with unrestricted and a few with restricted alternatives, the asymptotically most stringent test is asymptotically unique; this test is derived in Chapter 6. For many testing problems with restricted alternatives, however, the asymptotically most stringent test is not asymptotically unique. This phenomenon had been noticed by Willem Schaafsma, and was one of the problems leading to this research. In Chapter 7 this problem is treated, and in order to resolve it a new optimum property is proposed: "everywhere asymptotically most stringent", abbreviated to "EAMS".

The EAMS - level α test can be determined explicitly, when the most stringent level α tests for certain "limiting" problems for normal distributions are known. Unfortunately the most stringent - level α test is unknown for many of these limiting problems. However, it is possible to construct tests which are EAMS in certain subclasses of the class of all asymptotically level α tests. This leads to the tests of Chapter 9, which can be regarded as versions of the Wilcoxon-Mann-Whitney test, the Kruskal-Wallis test, etcetera, with an "optimal" treatment of ties.

I thank the Mathematical Centre for the opportunity to publish this monograph in their series Mathematical Centre Tracts and all those at the Mathematical Centre who have contributed to its technical realization.

SOME SYMBOLS AND TERMINOLOGY

In most instances, the meaning of the symbols used will be clear.

$\mathbb{R}^m$ is used to denote the m-dimensional Euclidean space, and to denote the measurable space $(\mathbb{R}^m, \mathcal{B}_m)$; $\mathcal{B}_m$ is the σ-field of the Borel subsets of $\mathbb{R}^m$.

A subset F of the topological space X is said to be relatively compact, if it has compact closure in X.

$\{x_\nu\}$ is used both to denote the sequence $x_1, x_2, \ldots$ and to denote the set $\{x_1, x_2, \ldots\}$.

A subsequential limit of $\{x_\nu\}$ is a limit of a subsequence of $\{x_\nu\}$.

$x_\nu = \mathcal{O}(y_\nu)$ means that $\lim\sup_\nu y_\nu^{-1} x_\nu < \infty$.

' denotes transposition.

I is the identity matrix.

I_A is the indicator function of the set A.

$\mathcal{L}(X)$ is the probability distribution of the random variable X.

$\otimes$ denotes taking a product measure.

P^n is the n-fold product measure of P.

$\|P - Q\|$ is the variation distance between the probability distributions P and Q.

$\mathcal{N}_m(\mu,\Sigma)$ is the m-variate normal distribution with mean μ and covariance matrix Σ(if m is omitted, then the dimension will be clear).

$\mathcal{B}(n,p)$	is the binomial distribution with parameters n and p.
$\mathcal{M}_m(n,p)$	is the m-nomial distribution with parameters n and p.
χ^2_m	is the chi square distribution with m degrees of freedom.
$\chi^2_{m;\delta^2}$	is the non-central chi square distribution with m degrees of freedom and non-centrality δ^2.
u_α	is the real number with $(\mathcal{N}(0,1))\ [u_\alpha,\infty) = \alpha$.
$\chi^2_{m;\alpha}$	is the real number with $\chi^2_m\ [\chi_{m;\alpha},\infty) = \alpha$.
*	indicates sections, examples and proofs which can be omitted at first reading.
□	indicates the end of a proof or example.
$\mathcal{C}$	see page 237.
$\mathcal{K}$	see page 83.
$\mathcal{M}_1(X)$	see page 222.
$\mathcal{S}, \mathcal{S}_H, \mathcal{S}_A$	see page 116.

CHAPTER 1

INTRODUCTION

In this study, testing problems with restricted alternatives are investigated; much attention is devoted to testing problems with restricted alternatives for contingency tables. A testing problem in ethological research, which was one of the starting points of this study, is presented as an example in Section 1.1. Sections 1.2 and 1.3 give a "non-mathematical" introduction to some of the central ideas of this study.

1.1. AN EXAMPLE OF A TESTING PROBLEM WITH A RESTRICTED ALTERNATIVE

The following experiment was carried out by the biologist ms. (now dr.) Nance Vodegel and was discussed with the author in 1974. It is one of the experiments studied in VODEGEL (1978), and it is presented here in a slightly simplified way. A fish was isolated in a tank and one of three fish dummies (a small dummy, a medium-sized one and a large one) was presented to the fish. The dummies were presented in a random order; dummy i (1 = small, 2 = medium, 3 = large) was presented n_i times. The numbers n_1, n_2 and n_3 will be regarded as predetermined constants (this may be justified by a conditioning argument). The experiment was designed in such a way that it was reasonable to assume that the trials were independent and identical (except for the dummy sizes). The behavioural activities of the fish were classified into a number of mutually exclusive behavioural categories, in such a way that exactly one behavioural category was displayed at any given moment. The dummy-associated behavioural categories were ranked according to decreasing aggresiveness as follows: 1 = butting, 2 = frontal display, 3 = lateral display or vibrating, 4 = turning around or leaving. After each presentation of the dummy, the first dummy-associated activity of the fish was recorded:

x_{ij} = number of behavioural category, displayed as the first dummy-associated activity after the j'th presentation of dummy i,

where $1 \le j \le n_i$ and $1 \le i \le 3$. As it was assumed that the trials were independent and identical, the data were condensed into a contingency table with entries n_{ih} $(1 \le i \le 3,\ 1 \le h \le 4)$, where n_{ih} denotes the number of trials j with $x_{ij} = h$.

The assumption of independent and identical trials is expressed in the following probabilistic model. It is postulated that x_{ij} is the outcome of the random variable X_{ij}, and that the X_{ij} are independent and have probability distributions given by

$$P_p\{X_{ij} = h\} = p_{ih} \qquad 1 \le j \le n_i,\ 1 \le i \le 3,\ 1 \le h \le 4,$$

for a parameter $p = (p_1, p_2, p_3)$ with $p_i = (p_{i1}, p_{i2}, p_{i3}, p_{i4})$ and

$$p_{ih} > 0, \qquad \sum_{h=1}^{4} p_{ih} = 1.$$

The value of p is unknown and depends on the individual fish. This probabilistic model implies that n_{ih} is the outcome of a random variable N_{ih} and that N_1, N_2 and N_3 defined by

$$N_i = (N_{i1}, N_{i2}, N_{i3}, N_{i4})$$

are independent random variables, N_i having the multinomial distribution with parameters n_i and $(p_{i1}, p_{i2}, p_{i3}, p_{i4})$.

For each of a number of individual fish, ms. Vodegel was interested in the question whether this fish exhibited a different degree of aggressiveness towards the different dummies. It was felt that the statement "more aggressiveness is directed towards dummy i_1 than towards dummy i_2" can be expressed by "for every h $(1 \le h \le 4)$, the probability that a behavioural category at least as aggressive as category h is displayed, is larger for dummy i_1 than for dummy i_2", or equivalently

$$(1.1.1) \qquad \sum_{g=1}^{h} p_{i_1 g} \ge \sum_{g=1}^{h} p_{i_2 g} \qquad 1 \le h \le 4.$$

Ms. Vodegel believed that, if there should be any difference at all between the probability distributions of the behavioural categories displayed towards the different dummies, then it would be possible to rank the dummies in the sense of (1.1.1), but she did not want to tie herself to any particular rank order. The larger dummy might elicit both a stronger tendency to attack and a stronger tendency to escape; it was not clear beforehand which tendency would be the stronger. So the question, focusing on one single fish, is formulated as a testing problem with null hypothesis

$$H:\ p_1 = p_2 = p_3$$

and alternative hypothesis

A: a permutation (i_1,i_2,i_3) of $(1,2,3)$ exists such that

$$\sum_{g=1}^{h} p_{i_1 g} \geq \sum_{g=1}^{h} p_{i_2 g} \geq \sum_{g=1}^{h} p_{i_3 g}$$

holds for $1 \leq h \leq 4$, with at least one of the inequalities strict. We shall return to this testing problem in Section 9.3.

1.2. CHOOSING A TEST

A general formulation of a testing problem can be given as follows. The observed data are denoted by x. A probabilistic model is postulated, stating that x is the outcome of a random variable X with probability distribution P_θ. Here θ is a parameter, the true value of which is unknown. In Section 1.1, $(x_{11},\ldots,x_{1n_1}, x_{21},\ldots,x_{3n_3})$ plays the role of x and $p = (p_1,p_2,p_3)$ that of θ; the contingency table $(n_{ih})_{1\leq i\leq 3;1\leq h\leq 4}$ may also play the role of x. The null hypothesis and the alternative hypothesis are two mutually exclusive propositions concerning the true value of θ. They will be expressed as H and A,

H: θ is an element of Θ_H

A: θ is an element of Θ_A,

for certain disjoint sets Θ_H and Θ_A.

The statistician has to propose a test: a procedure to decide on the basis of the outcome x of X, whether "the null hypothesis is rejected" or "the null hypothesis is not rejected". For theoretical reasons it is convenient to allow randomized tests: procedures where a random mechanism may be used in order to decide whether or not the null hypothesis is rejected. Randomized tests are unattractive for practical purposes, and they will play a minor role in this study. Tests will be indicated by test functions ϕ, which assume values $\phi(x)$ with $0 \leq \phi(x) \leq 1$, and where $\phi(x)$ [or $1-\phi(x)$] is the probability with which the null hypothesis is to be [or not to be] rejected, conditionally given the outcome x of X. In practice, tests are often given in the form of a test statistic $t(x)$ and a critical value c: if $t(x) > c$ then the null hypothesis is to be rejected ($\phi(x)=1$), if $t(x) < c$ then the null hypothesis is not to be rejected ($\phi(x)=0$), while for $t(x) = c$ a further specification of the value of $\phi(x)$ is necessary ($0 \leq \phi(x) \leq 1$).

The probability of rejecting the null hypothesis if X has probability distribution P_θ and the test ϕ is used, is the expectation of ϕ, and will be denoted by

$$E_\theta \phi(X).$$

Rejection of the null hypothesis, when the null hypothesis is true, is called an error of the first kind; not rejecting the null hypothesis, when the alternative hypothesis is true, is called an error of the second kind. In the Neyman-Pearson approach to hypothesis testing, attention is restricted to tests of level α: the probability of making an error of the first kind must not exceed a preassigned value α, or equivalently,

$$(1.2.1) \qquad E_\theta \phi(X) \leq \alpha \qquad \text{for all } \theta \text{ belonging to } \Theta_H.$$

The number α is called "level of significance"; common values of α range from .001 to .05. Different level α tests are compared by their power functions $\beta_\phi(\theta)$ defined by

$$\beta_\phi(\theta) = E_\theta \phi(X),$$

for θ belonging to Θ_A.

If ϕ_0 is a level α test which has, among all level α tests, the highest power attainable at θ, simultaneously for all θ belonging to Θ_A:

$$\beta_{\phi_0}(\theta) \geq \beta_\phi(\theta) \qquad \text{for all } \theta \text{ belonging to } \Theta_A \text{ and all level } \alpha \text{ tests } \phi,$$

then ϕ_0 is called a uniformly most powerful (UMP)-level α test. (See Section 2.4.) For most testing problems, such as that of Section 1.1, a UMP-level α test does not exist. Several considerations can then be used as a guide in the selection of a test, such as

(1) considerations about the "over-all" power properties of the test; more specifically, one can try to construct a test which is optimal in some compromising sense, or which is uniformly most powerful in a subclass of the class of all level α tests (Sections 2.5 and 3.2);

(2) considerations pertaining to the nature of the question investigated (e.g., in the problem of Section 1.1, one might be interested especially in differences between the probabilities of butting p_{11}, p_{21} and p_{31} and only in the second place in differences between the probabilities p_{1h}, p_{2h} and p_{3h} for $h \neq 1$);

(3) practical considerations, such as computational feasibility or interpretability of the test statistic.

This study focuses on asymptotic optimality considerations in the spirit of (1). These are chiefly based on the optimum property "most stringent" and the new asymptotic optimum property "everywhere asymptotically most stringent". The word "asymptotic" means that certain approximations are made, which are satisfactory for large sample sizes (approximations of probability distributions by multivariate normal distributions will play a central role).

1.3. MOST STRINGENT AND EVERYWHERE ASYMPTOTICALLY MOST STRINGENT TESTS

The concept of a most stringent test was introduced by ABRAHAM WALD (1942). (See Section 2.6.) The class of all level α tests will be denoted by Φ_α. The envelope power at θ, for θ belonging to Θ_A, is defined as the highest power at θ attainable by a level α test, and denoted by $\beta^*(\theta)$:

$$\beta^*(\theta) = \sup_{\phi \in \Phi_\alpha} \beta_\phi(\theta).$$

The shortcoming of the test ϕ at θ is denoted by $\gamma(\phi,\theta)$ and defined by

$$\gamma(\phi,\theta) = \beta^*(\theta) - \beta_\phi(\theta).$$

This definition implies that $\gamma(\phi,\theta) \geq 0$ for all θ belonging to Θ_A and for all level α tests ϕ. The maximum shortcoming of the test ϕ when θ ranges over Θ_A is denoted by $\gamma^*(\phi)$:

$$\gamma^*(\phi) = \sup_{\theta\in\Theta_A} \gamma(\phi,\theta).$$

The test ϕ_0 is most stringent-level α if it is a level α test for which the maximum shortcoming is as small as possible:

$$\gamma^*(\phi_0) = \min_{\phi\in\Phi_\alpha} \gamma^*(\phi)$$

or, equivalently,

$$\sup_{\theta\in\Theta_A} \gamma(\phi_0,\theta) = \min_{\phi\in\Phi_\alpha} \sup_{\theta\in\Theta_A} \gamma(\phi,\theta).$$

The right hand side of the latter equation is called the minimax shortcoming. Note that if the minimax shortcoming equals 0 then a test is most stringent-level α if and only if it is uniformly most powerful-level α; we are interested especially in testing problems where the minimax shortcoming is strictly positive.

The concept "most stringent" can be called conservative in the sense that it takes into account only "the worst which can happen". There are testing problems where a test ϕ_1 exists with a slightly larger maximum shortcoming than the most stringent test ϕ_0, but on the other hand the power of ϕ_1 is considerably larger than the power of ϕ_0 in large regions of Θ_A. This is the case, e.g., for the problem of combining two independent test statistics (see part 1 of Section 3.3): computations of OOSTERHOFF and VAN ZWET (1967) show that for this testing problem, Fisher's combination method may be considered to be preferable to the most stringent test, for $\alpha = .05$. (This point is discussed also in Section 1.6 of LEHMANN (1959); see also the second testing problem of Example 2.8.1.) So the most stringent test is not always to be preferred. For many testing problems, however, the most stringent test has quite satisfactory power properties.

A most stringent test exists for all testing problems which satisfy certain regularity conditions (Theorem 2.6.1); but the explicit determination of this test is often exceedingly difficult. In the following discussion it is assumed that the variable X originates from one or more independent random samples, with "large" sample sizes. For large sample sizes the testing problem can often be approximated by a "simpler" testing problem for multivariate normal distributions with a known covariance matrix. The latter problem will be called the "limiting problem". In a few cases there exists a uniformly most powerful test for the limiting problem; this leads to an "asymptotically uniformly most powerful test" for the actual testing problem. Instances of this situation will be found in Examples 5.6.1 and 5.6.2 and in Section 5.7. In some other cases a uniformly most powerful test does not exist but the most stringent test can be constructed explicitly for the limiting problem. This leads to an "asymptotically most stringent test" for the actual testing problem.

In a number of these cases, notably when the alternative hypothesis is unrestricted, the asymptotically most stringent test is "asymptotically unique" (see Definition 5.4.2), and has (for large sample sizes) quite satisfactory power properties. An example is provided by the well known χ^2 test for testing homogeneity or independence in a contingency table (Section 6.3).

In other cases, notably when the alternative hypothesis is restricted (as in the testing of Section 1.1), the asymptotically most stringent test is not asymptotically unique, and the concept "asymptotically most stringent" is not a satisfactory asymptotic optimum property (see Sections 7.1 and 7.2). Therefore another asymptotic optimum property, stronger than the property "asymptotically most stringent" and called "everywhere asymptotically most stringent", is introduced (Section 7.4).

The following is a heuristic explanation of this concept. The parameter spaces Θ_H and Θ_A will be embedded in a natural way in some Euclidean space $\mathbb{R}^m$, and their common boundary will be denoted by Θ_B. Denote the total sample size by n and let $\hat{\theta}$ be a good estimate for θ. For large n, $\hat{\theta}$ will be very close to θ, with probability almost equal to 1. Under regularity conditions, the distance between $\hat{\theta}$ and θ will be of the order of magnitude of $n^{-\frac{1}{2}}$. For every θ_0 belonging to Θ_B, let $\bar{\gamma}(\theta_0)$ denote the minimax shortcoming when the parameter θ is restricted to a neighbourhood of θ_0 with diameter tending to zero more slowly than $n^{-\frac{1}{2}}$; this quantity will be called the "local minimax shortcoming at θ_0". The test ϕ_0 is everywhere asymptotically most stringent-level α, or EAMS-level α, if it satisfies the following three conditions:

(i) ϕ_0 is asymptotically of level α (Sections 5.2, 5.3);

(ii) the maximum shortcoming of ϕ_0 in any neighbourhood of θ_0 with diameter of the order of magnitude of $n^{-\frac{1}{2}}$, does not exceed $\bar{\gamma}(\theta_0)$ by more than a vanishingly small amount; this holds for all θ_0 belonging to Θ_B;

(iii) the power of ϕ_0 at θ tends to 1, for all θ belonging to Θ_A and at a distance from Θ_H of an order of magnitude larger than $n^{-\frac{1}{2}}$

(Definition 6.1.2).

The minimax shortcoming for the whole testing problem is the maximum of $\bar{\gamma}(\theta_0)$ when θ_0 ranges over Θ_B, and will be denoted by $\bar{\gamma}$. For some testing problems, $\bar{\gamma}(\theta_0)$ is asymptotically constant and equal to $\bar{\gamma}$. Then the asymptotically most stringent test will be asymptotically unique, and the property "EAMS" contains nothing more than the property "asymptotically most stringent". (Sections 6.3, 6.4.) For other testing problems, where $\bar{\gamma}(\theta_0)$ varies with θ_0, the asymptotically most stringent test will not be asymptotically unique and the property "EAMS" is proposed as a relevant asymptotic optimum property, stronger than the property "asymptotically most stringent" and needed in order to obtain asymptotic uniqueness for the asymptotically optimal test.

In Sections 3.1 and 3.5 a large class of testing problems is described, containing many testing problems for contingency tables, for which an EAMS test exists. In Chapter 8 the EAMS test for such problems is expressed in terms of the most stringent test for the limiting problems (the limiting problem depends on θ_0; the minimax shortcoming for the limiting problem will be equal to $\bar{\gamma}(\theta_0)$).

Unfortunately, for many testing problems with restricted alternatives the most stringent-level α test has not been constructed explicitly, and it seems very hard to do so. (Section 3.3 contains some testing problems with restricted alternatives where the most stringent-level α test has been constructed explicitly.) For this reason, it is proposed in Sections 8.2 and 8.4 to focus attention on suitable subclasses of the class of all level α tests, for which the EAMS test in this subclass can be constructed explicitly and has attractive power properties. For many testing problems with restricted alternatives for contingency tables, the concept of an "EAMS-asymptotically-Ψ test", developed in Section 8.2, seems to be promising. In Chapter 9, EAMS-asymptotically linear tests, constructed in Section 8.3, are given for some testing problems from practice. It seems to be worthwhile to develop EAMS-asymptotically-Ψ or EAMS-conditionally-Ψ tests for other classes Ψ than the class of linear tests; see Section 9.1.

CHAPTER 2

TESTING STATISTICAL HYPOTHESES: SOME BASIC CONCEPTS

This chapter is devoted to the introduction of some basic concepts and results from the theory of testing statistical hypotheses. We shall follow the Neyman-Pearson approach to hypothesis testing (section 2.1). The emphasis is on results which will be used in later chapters. Except for those of Section 2.8, all concepts and results to be presented are well known; an excellent reference for most of them is LEHMANN (1959).

2.1. THE NEYMAN-PEARSON APPROACH TO HYPOTHESIS TESTING

Before a researcher formulates an inference problem as a problem of hypothesis testing, he usually has to go through a process where his questions and assumptions are made more explicit. In order to set the stage for a formulation of the Neyman-Pearson approach to hypothesis testing, some elements of this process will be briefly outlined.

A researcher R wants to gain knowledge about what can be called "the state of nature" or "the state of the world". R will express his view of "the part of the world which is relevant for his research" in a more or less formalized model M. The present knowledge of R about "the state of the world" is incomplete; this is reflected by a certain indeterminateness of the model. This indeterminateness will be expressed by including a parameter θ, with a set Θ of possible values, in the model. The true value of θ is not completely specified. Sometimes the parameter θ is regarded as an unknown constant, in some other cases θ is regarded as the outcome of a random variable with a given probability distribution. The aim of R will be to gain knowledge about the value of θ.

It is often possible for R to collect empirical data which he believes to have a bearing on the value of θ. The process of collecting data will be called "performing an experiment"; R has to choose an experimental design. The totality of data collected, which R wishes to include in his inference about the value of θ, will be denoted by x.

R has to formulate a model concering the relation between θ and x. In order to make on the basis of x an inference about the value of θ (or, more generally: to take an action, of which the merits depend on θ), it is often necessary to consider the experiment performed not as a completely unique one, but as a representative of a hypothetical class of "similar" experiments. In such cases, the data x is often considered to be the outcome of a random variable X, with outcome space $\mathcal{X}$ and an unknown probability distribution. It can be convenient to let $\mathcal{X}$ be a mathematical idealization of the set of possible values for x; e.g., x may be a vector of m numbers measured with a finite precision while one takes $\mathcal{X} = \mathbb{R}^m$. In order to be able to apply measure theory, $\mathcal{X}$ will often be equipped with a σ-field $\mathcal{F}$ of measurable subsets. The term "outcome space" will be used both for the set $\mathcal{X}$ and for the measurable space $(\mathcal{X}, \mathcal{F})$. The relation between θ and x will be expressed in the probability distribution of X.

The model M involving the parameter θ has to be sufficiently rich, so that the probability distribution of X is completely determined by θ; it will be denoted by P_θ. The experiment is represented by the probabilistic model $((\mathcal{X},\mathcal{F}), \Theta, P)$, P being the function which gives the correspondence between θ and P_θ. Parsimonious model building will often lead to models M and $((\mathcal{X},\mathcal{F}), \Theta, P)$ where the correspondence between θ and P_θ is one-to-one: the parameter θ is then said to be identifiable. R may have his reasons for using models where the parameter is not identifiable, however. Working within the confines of the experiment modelled by $((\mathcal{X}, \mathcal{F}),\Theta, P)$, we shall only consider questions to which a "good" answer can be given if the true probability distribution P_θ is known. Of course the data x may suggest that X has none of the probability distributions P_θ; but this can be perceived only by transcending the confines of the model $((\mathcal{X}, \mathcal{F}), \Theta, P)$. As the model is never a final one, one should always keep this possibility in mind.

In the following, θ will denote the true parameter value. The theory of hypothesis testing is designed for inference problems which can be formulated as the question whether the proposition H (the null hypothesis),

$$H : \theta \in \Theta_H ,$$

holds true; Θ_H is a subset of Θ . In many practical applications, Θ_H is the set of all those parameter values which imply a kind of "standard situation":

independence, homogeneity, etcetera. The inference problem will be formalized by requiring that on the basis of x, exactly one of the statements d_0 and d_1,

d_0: "H is not rejected"
d_1: "H is rejected" ,

be made. If $\theta \in \Theta_H$, then it is desired to make statement d_0.

R has to indicate a subset Θ_A of Θ, disjoint from Θ_H, so that especially if $\theta \in \Theta_A$ it is desired to make statement d_1, while if $\theta \in \Theta \setminus (\Theta_H \cup \Theta_A)$ it is less clear which statement is to be preferred. For example, R may be confident that $\theta \in \Theta_H \cup \Theta_A$; or R may be indifferent with respect to the statement made, if $\theta \in \Theta \setminus (\Theta_H \cup \Theta_A)$; or R may wish another statement than d_0 or d_1 to be made if $\theta \in \Theta \setminus (\Theta_H \cup \Theta_A)$. In the present study we shall only consider the situation (which often amounts to a simplification) that for $\theta \in \Theta \setminus (\Theta_H \cup \Theta_A)$, R is indifferent with respect to the statement made. The proposition A,

$$A : \theta \in \Theta_A ,$$

will be called the alternative hypothesis. One says that R wishes to test the null hypothesis H against the alternative hypothesis (or: against the alternative) A. It will be assumed that the sets of probability distributions

$$\mathcal{P}_H = \{P_{\theta'} \mid \theta' \in \Theta_H\}, \ \mathcal{P}_A = \{P_{\theta'} \mid \theta' \in \Theta_A\}$$

are disjoint. In the sequel, the parameter plays a role only via the sets of probability distributions $\mathcal{P}_H$ and $\mathcal{P}_A$; the set

$$\{P_{\theta'} \mid \theta' \in \Theta \setminus (\Theta_H \cup \Theta_A)\}$$

will not play a role at all in our discussion. So the formulation of the testing problem in this study will involve the outcome space $(\mathcal{X}, \mathcal{F})$, the classes of probability distributions $\mathcal{P}_H$ and $\mathcal{P}_A$, and the statements d_0 and d_1.

In the construction of the model M, the choice of the experimental design, the construction of the model $((\mathcal{X}, \mathcal{F}), \Theta, P)$ relating M and the data to be observed, and the determination of null hypothesis and alternative hypothesis, many scientific results as well as R's insights and interests can play a role. If R consults a statistician, an important task for the statistician can lie in joining R in this process of model construction and making choices. The outline of this process given above is an idealized one; in practice, the researcher often proceeds along somewhat different lines and in another chronological order.

The hypothesis testing problem developed above can be summarized in the triple $((\mathcal{X}, \mathcal{F}), \mathcal{P}_H, \mathcal{P}_A)$, where $\mathcal{P}_H$ and $\mathcal{P}_A$ are disjoint classes of probability distributions on the outcome space $(\mathcal{X}, \mathcal{F})$. The formulation of the statements d_0 and d_1 reflects that this theory of hypothesis testing is intended to be used in situations where it is desired to treat null hypothesis and alternative hypothesis asymmetrically. Making statement d_1 if H is true is called an error of the first kind, while making statement d_0 if A is true is called an error of the second kind. An error of the first kind is considered to be more serious than an error of the second kind.

In the Neyman-Pearson approach to hypothesis testing (NEYMAN and PEARSON (1933)), it is required that the probability of making an error of the first kind does not exceed a pre-assigned number, called level of significance and denoted by α. Common values of α are in the range from .001 to .05. Subject to this restriction, the quality of a procedure for determining which of the statements d_0 and d_1 will be made is inversely related to the probability of making an error of the second kind, which is a function of $P \in \mathcal{P}_A$.

The class of permitted decision rules will be the class of all measurable functions

$$\phi : \mathcal{X} \to [0,1].$$

For every $x \in \mathcal{X}$, $\phi(x)$ is to be interpreted as the probability with which decision d_1 is to be made: the decision made is a random variable Y with values in $\{d_0, d_1\}$ and the function ϕ determines the conditional probability that $Y = d_1$, given X. The joint probability distribution $\bar{P}$ of (X, Y) if $\mathcal{L}(X) = P$ is completely determined by P and ϕ. The probability of making statement d_1, when X has probability distribution P, which is the unconditional probability that $Y = d_1$, is given by

$$\bar{P}\{Y = d_1\} = E_P[\bar{P}\{Y = d_1 \mid X\}] = E_P\,\phi(X) ,$$

also denoted by $E_P\phi$. The random variable Y is not needed in the expression $E_P\,\phi(X)$ and will be omitted from the considerations. Measurable functions $\phi : \mathcal{X} \to [0,1]$ will be called tests or test functions.

When using the test ϕ, the probability of making an error of the first kind is given by $E_P\phi$ for $P \in \mathcal{P}_H$, and the probability of an error of the second kind is $1 - E_P\phi$ for $P \in \mathcal{P}_A$. A test is said to be of level α if

$$E_P\phi \leq \alpha \quad \text{for all } P \in \mathcal{P}_H .$$

In the exact Neyman-Pearson approach attention is restricted to the class of all level α tests; sometimes a subclass of this class is considered. The quality of a test is judged by its power function, which is the function $P \mapsto E_P\phi$, defined on $\mathcal{P}_A$.

Admitting all measurable functions $\phi : \mathcal{X} \to [0,1]$ may raise the following two objections . In the first place, the practical user of statistical tests dislikes the additional randomness in his decision, prescribed by randomized tests. In the second place, even if the test ϕ is non-randomized (i.e., $\phi(\mathcal{X}) \subset \{0,1\}$), it may be impossible in practice to evaluate $\phi(x)$: for example suppose that x is a real number observed with a finite precision, while ϕ is the indicator function of the rational numbers. In spite of these two objections, all measurable functions $\phi : \mathcal{X} \to [0,1]$ are admitted for the purpose of elegance of the mathematical theory. The reader may be comforted by knowing that for most testing problems occurring in practice, the theory leads to optimal test functions which are almost everywhere continuous (with respect to a suitable topology on $\mathcal{X}$, such as the Euclidean topology on $\mathbb{R}^m$) and hence can be well evaluated, and where randomization plays no, or a minor, role. (See, e.g., Theorem 2.7.2.)

In this study, testing problems will be denoted by

$$((\mathcal{X}, \mathcal{F}), \mathcal{P}_H, \mathcal{P}_A)$$

or by

$$((\mathcal{X}, \mathcal{F}), \mathcal{P}_H, \mathcal{P}_A, \Phi)$$

where Φ is the class of tests to which attention is restricted; often Φ is the class of level α tests. When the probability distributions are indexed by a parameter θ, the testing problem can also be specified by giving the probability distributions, together with null hypothesis and alternative hypothesis:

$$L_\theta(X) = P_\theta$$

$$H : \theta \in \Theta_H \quad , \quad A : \theta \in \Theta_A \ .$$

It is often more convenient to state H and the disjunction $H \vee A$ instead of H and A. In the remainder of this study, all aspects of the process leading to the formulation of the testing problem $((X, F), P_H, P_A)$ will be taken for granted; we shall only study the choice of a test function ϕ for a given testing problem.

DEFINITION 2.1.1. A test ϕ for the testing problem $((X, F), P_H, P_A)$ is of level α if

$$E_P\phi \leq \alpha \qquad \text{for all } P \in P_H.$$

The size of a test ϕ is

$$\sup_{P \in P_H} E_P\phi \ .$$

In this chapter, the class of all level α tests will be denoted by Φ_α. In particular, the class of all tests will be denoted by Φ_1.
If P_H has exactly one element, then H is called a simple hypothesis; otherwise, H is called a composite hypothesis.

2.2. SUFFICIENT STATISTICS

When P is a class of probability distributions on a measurable space (X, F), the pair $((X, F), P)$ will be referred to as an experiment. A statistic for an experiment $((X, F), P)$ is a measurable function t from (X, F) to a measurable space (T, G). The statistic t is often identified with the associated random variable $T = t(X)$. A basic concept in mathematical statistics is the concept of a "sufficient statistic", introduced by FISHER (1922b). Fisher formulated the "criterion of sufficiency" by requiring "that the statistic chosen should summarize the whole of the information supplied by the sample". A statistic $T = t(X)$ is called sufficient, loosely speaking, if the conditional distribution Q_τ of X given $T = \tau$ is the same for all $P \in \mathcal{P}$.

Suppose that T is a sufficient statistic, and that only the outcome of T and not the outcome of X is communicated to the statistician. Then, without knowing the probability distribution P of X, the statistician can use a "randomization mechanism" which yields a random variable X' which conditionally on $T = \tau$ has probability distribution Q_τ. Proceeding in this fashion, the statistician has at his disposal a random variable X' which has, irrespective of τ, the same unconditional probability distribution P as the original random variable X, whatever be P (provided that $P \in \mathcal{P}$). This argument is used in a more formal way in Theorem 2.2.1 in order to show that if t is a sufficient statistic and ϕ a test function, then a test function $\tilde{\phi}(t(.))$ exists which is "just as good" as ϕ in the sense that $E_P\tilde{\phi}(t(X)) = E_P\phi(X)$ for all $P \in \mathcal{P}$.

DEFINITION 2.2.1. Let $((\mathcal{X}, \mathcal{F}), \mathcal{P})$ be an experiment, $(\mathcal{T}, \mathcal{G})$ a measurable space, and $t : (\mathcal{X}, \mathcal{F}) \to (\mathcal{T}, \mathcal{G})$ a measurable function. Then $T = t(X)$ is a sufficient statistic for $((\mathcal{X}, \mathcal{F}), \mathcal{P})$ if for every bounded measurable function $f : \mathcal{X} \to \mathbb{R}$ a version $g(T)$ of $E_P \{f(X) | T\}$ exists, which is the same for all $P \in \mathcal{P}$.

THEOREM 2.2.1. *Suppose that* $T = t(X)$ *is a sufficient statistic for* $((\mathcal{X}, \mathcal{F}), \mathcal{P})$. *Then for every test function* $\phi : \mathcal{X} \to [0,1]$ *there exists a test function* $\tilde{\phi} : \mathcal{T} \to [0,1]$ *with*

$$E_P \tilde{\phi} (t(X)) = E_P \phi (X) \quad \text{for all } P \in \mathcal{P} .$$

PROOF. Let $\tilde{\phi}(\tau)$ be a version of $E_P \{\phi(X) | T = \tau\}$, which does not depend on P. Such a version exists, since $t(X)$ is sufficient. Then

$$E_P \tilde{\phi}(t(X)) = E_P [E \{\phi(X) | T = \tau\}] = E_P \phi(X) ,$$

for all $P \in \mathcal{P}$. □

As a consequence: when the statistician restricts himself to tests ϕ for which a measurable function $\tilde{\phi} : \mathcal{T} \to [0,1]$ exists with $\phi(x) = \tilde{\phi}(t(x))$ for all $x \in \mathcal{X}$, where t is a sufficient statistic, then the same power functions are available to him at a fixed level of significance, as when he refrains from this restriction.

The following Factorization Theorem (originating with FISHER (1925) and NEYMAN (1935)) yields, for many experiments, a particularly simple criterion for determining whether a statistic is sufficient.

THEOREM 2.2.2. (Factorization Theorem). *Let* $((X, F), P)$ *be an experiment for which a* σ*-finite measure* λ *on* (X, F) *exists such that* $P << \lambda$ *for all* $P \in P$. *A statistic* $t : (X, F) \to (T, G)$ *is sufficient iff there exist an* F*-measurable function* $h : X \to [0, \infty)$ *and for every* $P \in P$ *a* G*-measurable function* $f_P : T \to [0, \infty)$ *such that*

$$h(.)\ f_P(t(.))$$

is a version of $d P / d \lambda$.

PROOF. See LEHMANN(1959), Section 2.6. □

2.3. EXPONENTIAL FAMILIES

In this section certain classes of probability distributions (experiments, in other words), called exponential families, are defined. In the main part of this study attention will be restricted to testing problems for random samples from exponential families. This is motivated by the facts that (i) the mathematical techniques needed for treating these testing problems are simpler than the techniques required for treating more general testing problems; (ii) this kind of testing problem occurs in practice rather frequently (for example, the testing problems in Chapters 1 and 9).

An exponential family is an experiment $((X, F), P)$ for which a σ-finite measure λ on (X, F) exists such that the densities of P with respect to λ, for $P \in P$, can be expressed by

$$d P / d \lambda (x) = h(x) \exp \{[\theta(P)]'\, t(x) - \psi(\theta(P))\}, \tag{2.3.1}$$

for certain measurable functions $t : X \to \mathbb{R}^m$ and $h : X \to [0, \infty)$ and certain functions $\theta : P \to \mathbb{R}^m$ and $\psi : \theta(P) \to \mathbb{R}$. The function ψ is a normalizing function, and it is determined by λ, t, h according to

$$\psi(\theta) = \log \int h(x) \exp \{\theta'\, t(x)\}\, d \lambda(x)$$

for all $\theta \in \Theta(\mathcal{P})$. It can be shown that many well-known families of distributions (univariate and multivariate normal families, gamma families, binomial and multinomial families, Poisson families, etcetera) are exponential families.

The Factorization Theorem shows that $t(X)$ is a sufficient statistic. So it suffices to know the outcome of $T = t(X)$. The experiment of observing T, induced by the experiment $((\mathcal{X}, \mathcal{F}), \mathcal{P})$ with densities (2.3.1), is

$$(\mathbb{R}^m, \{P_\theta^T \mid \theta \in \Theta(\mathcal{P})\})$$

with

$$d\, P_\theta^T / d\, \lambda^T(t) = \exp\{\theta' t - \psi(\theta)\} ,$$

where λ^T is the σ-finite measure on $\mathbb{R}^m$ given by

$$\lambda^T(A) = \int_{t^{-1}(A)} h(x)\, d\, \lambda(x) \qquad \text{for all } A \in \mathcal{B}_m .$$

Note that the probability distributions of T are parametrized by the m-dimensional parameter θ. It is possible that different parameters θ_1, $\theta_2 \in \mathbb{R}^m$ are associated with the same probability distribution $P_{\theta_1}^T = P_{\theta_2}^T$; see Theorem 2.3.2. The parameter θ is called the natural parameter, the statistic $t(X)$ the canonical sufficient statistic. In the sequel, exponential families are considered where the reduction to the canonical sufficient statistic has been made.
Denoting the canonical sufficient statistic by X, P_θ^T by P_θ, $\Theta(\mathcal{P})$ by $\tilde{\Theta}$, and λ^T by λ, we obtain the experiment

$$(2.3.2) \qquad (\mathbb{R}^m, \{P_\theta \mid \theta \in \tilde{\Theta}\})$$

with

$$(2.3.3) \qquad d\, P_\theta / d\, \lambda\, (x) = \exp\{\theta' x - \psi(\theta)\} .$$

The set of all $\theta \in \mathbb{R}^m$ for which

$$\int \exp(\theta' x)\, d\, \lambda(x) < \infty$$

is the largest set of θ's for which (2.3.3) defines a probability distribution P_θ. This set will be called the natural parameter space. If θ_0 is an element of the natural parameter space, then

$$d\,P_\theta \;/\; d\,P_{\theta_0}(x) = \exp\{(\theta - \theta_0)'x - \psi(\theta) + \psi(\theta_0)\} \quad .$$

This shows that P_{θ_0} can be used as dominating measure instead of λ. When the attention is directed to one particular θ_0, it is not a restriction to assume that $\theta_0 = 0$, $\psi(0) = 0$, and $E_0 X = 0$, as the transformation $Y = X - E_{\theta_0} X$, $\eta = \theta - \theta_0$ yields an exponential family of distributions Q_η with

$$d\,Q_\eta \;/\; d\,Q_0(y) = \exp\{\eta' y - \tilde{\psi}(\eta)\} \quad ,$$

where

$$\tilde{\psi}(\eta) = \psi(\eta + \theta_0) - \psi(\theta_0) - \eta' E_{\theta_0} X \ ,$$

and it satisfies $E_0 Y = E_{\theta_0}(X - E_{\theta_0} X) = 0$ and $\tilde{\psi}(0) = 0$.

If $X_1, \ldots, X_n$ is a random sample from the distribution with density (2.3.3), the density of the sample is

$$d\,P_\theta^n \;/\; d\,\lambda^n\,(x_1, \ldots, x_n) = \exp\{\theta' \sum_{i=1}^{n} x_i - n\,\psi(\theta)\} \ .$$

This is the form (2.3.1). The statistic $\sum_{i=1}^{n} X_i$ is sufficient,and has a probability distribution from an exponential family with outcome space $\mathbb{R}^m$. For families of absolutely continuous probability distributions, the property that a sufficient statistic of fixed dimension m exists for samples of arbitrary size n, characterizes exponential families, provided that certain regularity conditions are satisfied. See, e.g., HIPP (1974) and the references cited there.

The random variables for the testing problems in Chapters 1 and 9 are random samples from probability distributions on finite outcome spaces. The following example shows that families of probability distributions on finite outcome spaces are exponential families.

<u>EXAMPLE 2.3.1</u>. *Experiments with finite outcome spaces are exponential*

families. Let $\mathcal{X}$ be a finite outcome space $\{x_1, \ldots, x_{m+1}\}$ with the σ-field of all its subsets, and let $\mathcal{P}$ be a class of probability distributions on $\mathcal{X}$ with $P\{x\} > 0$ for all $P \in \mathcal{P}$ and all $x \in \mathcal{X}$. Let λ be counting measure on $\mathcal{X}$, and let $a_1, \ldots, a_m$ be a basis for $\mathbb{R}^m$. Define $t : \mathcal{X} \to \mathbb{R}^m$ by

$$t(x_i) = a_i \qquad (1 \le i \le m), \qquad t(x_{m+1}) = 0 ,$$

let A be the m x m matrix

$$A = (a_1, \ldots, a_m) \quad ,$$

and define

$$q_i(P) = \log P\{x_i\} - \log P\{x_{m+1}\} , \quad q(P) = (q_1(P), \ldots, q_m(P))'$$
$$\theta(P) = A^{-1} q(P)$$
$$\psi(\theta(P)) = - \log P\{x_{m+1}\} \quad .$$

Then

$$[\theta(P)]' \, t(x_i) = [\theta(P)]' \, a_i = q_i(P) \qquad (1 \le i \le m) ,$$

and for all $x \in \mathcal{X}$ one has

$$d P / d \lambda(x) = \exp \{[\theta(P)]' \, t(x) - \psi(\theta(P))\} .$$

This is of the form (2.3.1). Hence $(\mathcal{X}, \mathcal{P})$ is an exponential family. If $X_1, \ldots, X_n$ is a random sample from a distribution P with $P \in \mathcal{P}$, then the sufficient statistic $\sum_{j=1}^{n} t(X_j)$ is equivalent to the statistic $(S_1, \ldots S_{m+1})$ where S_i is the number of X_j with $X_j = x_i$. □

It will be useful to have available same well-known properties of exponential families of distributions.

<u>THEOREM 2.3.1</u>. *Let λ be a σ-finite measure on $\mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}$ a measurable function. Let $U \subset \mathbb{R}^m$ be an open set with*

$$\int | f(x) | \exp (\theta' x) \, d \lambda(x) < \infty$$

for all $\theta \in U$. *Then the function* $g : U \to \mathbb{R}$, *defined by*

$$g(\theta) = \int f(x) \exp(\theta' x) \, d\lambda(x)$$

is continuously differentiable on U, *and*

$$\partial g(\theta) / \partial\theta_i = \int x_i f(x) \exp(\theta' x) \, d\lambda(x) .$$

The right hand side of this expression is an absolutely convergent integral for all $\theta \in U$.

PROOF. This theorem can be regarded as a property of the Laplace transformation. A proof is contained in LEHMANN (1959), Section 2.7. □

Corollary 2.3.1. *Let* λ *be a* σ*-finite measure on* $\mathbb{R}^m$ *and*

$$\psi(\theta) = \log \int \exp(\theta' x) \, d\lambda(x)$$
$$\Theta = \text{int} \{\theta \in \mathbb{R}^m \mid \psi(\theta) < \infty\} .$$

Let $X = (X_1, \ldots, X_m)'$ *be a random variable with* $L_\theta(X) = P_\theta$ *defined by* (2.3.3). *Then* ψ *is infinitely often differentiable on* Θ, $E_\theta \|X\|^r < \infty$ *for all* $r > 0$, *and*

$$E_\theta X_i = \partial \psi(\theta) / \partial\theta_i$$
$$\text{cov}_\theta (X_i, X_j) = \partial^2 \psi(\theta) / \partial\theta_i \partial\theta_j .$$

PROOF. An application of Theorem 2.3.1 with $f \equiv 1$ shows that ψ is differentiable on Θ; and that $E_\theta X_i = \partial\psi(\theta) / \partial\theta_i$. An application of Theorem 2.3.1 with $f(x) = x_j$ shows that $\text{cov}_\theta (X_i, X_j) = \partial^2\psi(\theta) / \partial\theta_i\partial\theta_j$. An induction argument shows that ψ is infinitely often differentiable, and that all moments of X exist. □

As defined in Appendix 1, $M_1(\mathbb{R}^m)$ is the set of all probability distributions on $\mathbb{R}^m$, and the sequence $\{P_n\}$ in $M_1(\mathbb{R}^m)$ converges weakly to $P \in M_1(\mathbb{R}^m)$ iff

$$\int f \, dP_n \to \int f \, dP$$

for every bounded continuous function $f : \mathbb{R}^m \to \mathbb{R}$. The corresponding weak topology is used in the following theorem. In this theorem, results of BERK (1972, Section 2) and BARNDORFF-NIELSEN (1969) are combined.

THEOREM 2.3.2. *Let λ, ψ, Θ, X and P_θ be as in the corollary above. Suppose $\Theta \neq \emptyset$. Define $\mu(\theta) = E_\theta X$ and $\Sigma_\theta = \mathrm{cov}_\theta\, X$.*

Then Θ is a convex set and ψ a convex function, and the following statements are equivalent.

(i) *λ is not concentrated in a hyperplane*

(ii) *$\psi : \Theta \to \mathbb{R}$ is strictly convex*

(iii) *$\theta \mapsto P_\theta$ is 1 : 1 on Θ*

(iv) *$\theta \mapsto P_\theta$ is a homeomorphism from Θ to $\{P_\theta \mid \theta \in \Theta\}$ with the relative weak topology as a subset of $M_1(\mathbb{R}^m)$*

(v) *$\theta \mapsto \mu(\theta)$ is 1 : 1 on Θ*

(vi) *$\theta \mapsto \mu(\theta)$ is a homeomorphism from Θ to $\mu(\Theta) \subset \mathbb{R}^m$*

(vii) *Σ_θ is a positive definite matrix for some $\theta \in \Theta$*

(viii) *Σ_θ is a positive definite matrix for all $\theta \in \Theta$.*

PROOF. The convexity of Θ and of ψ follows straightforwardly from Hölder's inequality.

(i) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii). As $d\,P_\theta \,/\, d\,\lambda\,(x) > 0$ a.e. $[\lambda]$ for all $\theta \in \Theta$, the null sets of λ coincide with those of P_θ, for all $\theta \in \Theta$. For every $\theta \in \Theta$, Σ_θ is positive definite iff P_θ is not concentrated in a hyperplane.

(viii) $\Rightarrow$ (ii). Corollary 2.3.1 states that Σ_θ is the matrix with elements $\partial^2\psi(\theta) \,/\, \partial\theta_i\, \partial\theta_j$. If this matrix is positive definite for all θ, then ψ is strictly convex.

(ii) $\Rightarrow$ (vi). Corollary 2.3.1 states that $\mu(\theta) = \mathrm{grad}\, \psi(\theta)$, and that this is a continuous function. As ψ is strictly convex, μ is 1 : 1. It remains to be proved that μ^{-1} is continuous. Suppose that $\{\theta_\nu\} \subset \Theta$ and $\mu(\theta_\nu) \to \mu(\theta_0)$, for some $\theta_0 \in \Theta$. It must be proved that $\theta_\nu \to \theta_0$. It is not a restriction to assume that $\theta_0 = 0$, $\psi(0) = 0$ and $\mu(0) = 0$. The strict convexity of ψ and $\mu(0) = 0$ imply that $\psi(\theta) > \psi(0) = 0$ for all $\theta \neq 0$. For $\varepsilon > 0$, define

$$\delta(\varepsilon) = \inf\{\psi(\theta) \mid \theta \in \Theta,\ \|\theta\| = \varepsilon\}\ .$$

Then $\delta(\varepsilon) > 0$, and $\mu(\theta) \geq \delta(\varepsilon)$ for all $\|\theta\| \geq \varepsilon$. As $\mu(\theta_\nu) \to 0$, this implies that $\theta_\nu \to 0$.

(vi) $\Rightarrow$ (v) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). If λ is concentrated in a hyperplane, then there exist $y \in \mathbb{R}^m$

and $c \in \mathbb{R}$ with $y \neq 0$ and $x'y = c$ a.e. $[\lambda]$. This implies that

$$\exp\{\theta'x - \psi(\theta)\} = \exp\{(\theta + y)'x - \psi(\theta) - c\} \quad \text{a.e. } [\lambda]$$

for all $\theta \in \Theta$. Therefore if $\theta \in \Theta$, then $\theta + y \in \Theta$ and $P_\theta = P_{\theta+y}$.
(i) $\Rightarrow$ (iv). The implication (i) $\Rightarrow$ (iii) proved above shows that $\theta \mapsto P_\theta$ is 1 : 1. The continuity of the functions

$$\theta \mapsto \int f \, d\, P_\theta \quad ,$$

for every bounded continuous function $f : \mathbb{R}^m \to \mathbb{R}$, follows from Theorem 2.3.1. It remains to be proved that if $\{\theta_\nu\} \subset \Theta$ and

$$\int f \, d\, P_{\theta_\nu} \to \int f \, d\, P_{\theta_0}$$

for every bounded continuous $f : \mathbb{R}^m \to \mathbb{R}$, and some $\theta_0 \in \Theta$, then $\theta_\nu \to \theta_0$. It is not a restriction to assume that $\theta_0 = 0$, $\psi(0) = 0$, $\lambda = P_0$ and $\mu(0) = 0$. Then $\psi(\theta) \geq 0$ for all $\theta \in \Theta$. A subsequence $\{\theta_\xi\}$ of $\{\theta_\nu\}$ exists which satisfies one of the following four conditions:

(a) $\theta_\xi \to \theta$ for some $\theta \in \Theta$
(b) $\|\theta_\xi\| \to \infty$ and $\|\theta_\xi\|^{-1}\, \theta_\xi \to y$ for some y
(c) $\theta_\xi \to \theta$ for some $\theta \in \partial\Theta$ and $\psi(\theta_\xi) \to p$ for some $p \geq 0$
(d) $\theta_\xi \to \theta$ for some $\theta \in \partial\Theta$ and $\psi(\theta_\xi) \to \infty$.

It must be shown that (a) implies $\theta = 0$, while (b), (c) and (d) all lead to contradictions.
(a) The continuity of the function $\theta \mapsto P_\theta$ demonstrated above implies $P_\theta = P_0$. As $\theta \mapsto P_\theta$ is 1 : 1, this shows that $\theta = 0$.
(b) It follows from (i) and $E_0 X = \mu(0) = 0$, that $P_0\{y'X > 0\}$ and $P_0\{y'X < 0\}$ both are positive. Assumption (b) implies

$$\{x \mid y\, x < 0\} \subset \bigcup_{\xi_0} \bigcap_{\xi \geq \xi_0} \{x \mid \theta_\xi'\, x < 0\} .$$

Hence a ξ_0 exists with $P_0(A) > 0$, where

$$A = \bigcap_{\xi \geq \xi_0} \{x \mid \theta_\xi'\, x < 0\} .$$

A bounded continuous function $f : \mathbb{R}^m \to \mathbb{R}$ exists with $0 \leq f \leq 1$, $f(x) = 0$

for $x \notin A$ and $\int f \, d P_0 > 0$. For all $\xi \geq \xi_0$ one has

(1) $$\int f(x) \exp(\theta_\xi' x) \, d P_0(x) \leq P_0(A) < 1 .$$

It follows from $P_0 \{y'X > 0\} > 0$ that

(2) $$\psi(\theta_\xi) = \log \int \exp(\theta_\xi' x) \, d P_0(x) \to \infty .$$

From (1) and (2) it can be concluded that

$$\int f \, d P_{\theta_\xi} = \int f(x) \exp\{\theta_\xi' x - \psi(\theta_\xi)\} \, d P_0(x) \to 0 .$$

But $\int f \, d P_0 > 0$ and it was assumed that $\int f \, d P_{\theta_\nu} \to \int f \, d P_0$; this is a contradiction.

(c) It follows from (i) that $P_0 \{\theta'X = p\} < 1$. Hence a function $f : \mathbb{R}^m \to \mathbb{R}$ with compact support exists for which

$$\int f \, d P_0 \neq \int f(x) \exp\{\theta' x - p\} \, d P_0(x) .$$

Assumption (c) and Lebesgue's dominated convergence theorem imply

$$\int f \, d P_{\theta_\xi} \to \int f(x) \exp\{\theta' x - p\} \, d P_0(x) .$$

This is a contradiction with $\int f \, d P_{\theta_\nu} \to \int f \, d P_0$.

(d) Apply the method of (c) with $p = \infty$ and $\exp\{\theta' x - p\} = 0$ for all x.

(iv) ⇒ (iii). Trivial. □

When the equivalent conditions (i) - (viii) of Theorem 2.3.2 are not satisfied, the outcome space can be transformed to a lower-dimensional space in order to obtain an experiment which does satisfy conditions (i) - (viii). For example if $X = (X_1, \ldots, X_m)$ has the multinomial $\mathcal{M}(n;p)$ distribution for some $p \in S_m$ where

$$S_m = \{p \in \mathbb{R}^m \mid p_i > 0, \sum_i p_i = 1\} ,$$

then $\sum_{i=1}^{m} X_i = n$ with probability 1, contradicting (i). The statistic $\tilde{X} = (X_1, \ldots, X_{m-1})$ is in 1 :1 correspondence with X. Hence $\tilde{X}$ is a sufficient statistic and has a distribution from an exponential family; the distribution of $\tilde{X}$ is not concentrated in a hyperplane. We shall always work with exponential families with densities (2.3.3), where the conditions of Theorem 2.3.2 are satisfied.

DEFINITION 2.3.1. A canonical exponential family is a class $\{P_\theta \mid \theta \in \Theta\}$ of probability distributions on $\mathbb{R}^m$ for which a σ-finite measure λ on $\mathbb{R}^m$ exists which is not concentrated in a hyperplane, such that

$$\Theta = \text{int } \{\theta \in \mathbb{R}^m \mid \int \exp(\theta' x) \, d\lambda(x) < \infty\} \neq \emptyset$$

$$d P_\theta / d\lambda(x) = \exp(\theta' x - \psi(\theta))$$

$$\psi(\theta) = \log \int \exp(\theta' x) \, d\lambda(x) \ .$$

Note that many different measures λ can be used for the same exponential family; e.g., any P_θ can be used. In this definition Θ has been taken as the interior of the natural parameter space because that will be convenient later.

2.4. UNIFORMLY MOST POWERFUL TESTS

In sections 2.4 - 2.7 the testing problem

$$((X, F), P_H, P_A, \Phi)$$

is studied; attention is restricted to tests in the class Φ. In many applications, Φ will be the class Φ_α of all level α tests. The Neyman-Pearson approach to hypothesis testing, discussed in Section 2.1, leads to the following definition.

DEFINITION 2.4.1. The power function of a test φ is the restriction of the function $P \mapsto E_P\phi$ to the domain P_A. For $P \in P_A$, a test φ is most powerful -Φ against P, or MP -Φ against P, if

(i) $\phi \in \Phi$

(ii) $E_P\phi \geq E_P\phi'$ for all $\phi' \in \Phi$.

A test φ is uniformly most powerful -Φ, or UMP-Φ, if it is most powerful -Φ against all $P \in P_A$.

The theory of most powerful - level α tests is based on the Fundamental Lemma of Neyman and Pearson (1933), which gives the most powerful - level α test for testing problems with a simple null hypothesis and a simple alternative. Note that for every pair P_0, P_1 of probability distributions on (X, F) there exist σ-finite measures λ on (X, F) with $P_i << \lambda$ $(i = 0,1)$: e.g., $\lambda = P_0 + P_1$.

THEOREM 2.4.1 (Neyman-Pearson Fundamental Lemma). *Suppose* $P_H = \{P_0\}$ *and* $P_A = \{P_1\}$. *Let* λ *be a* σ*-finite measure on* (X, F) *with* $P_i << \lambda$ $(i = 0,1)$ *and define* $p_i = d P_i / d \lambda$ $(i = 0,1)$.

Let Φ^* *be the class of tests* ϕ *which are of size* α, *and for which a* $k \in [0, \infty]$ *exists such that* ϕ *satisfies, for almost all* $[P_0 + P_1]$ x,

$$\phi(x) = \begin{cases} 1 & p_1(x) > k\, p_0(x) \\ 0 & p_1(x) < k\, p_0(x) . \end{cases} \tag{2.4.1}$$

Then Φ^* *is not empty, and it is the class of all MP-size* α *tests. It is also the class of all MP-level* α *tests, unless a test exists of size less than* α *and with power 1.*
(If in (2.4.1) $k = \infty$ *and* $p_0(x) = 0$, *then* $k\, p_0(x)$ *is to be interpreted as* 0.*)*

PROOF. See LEHMANN (1959), page 65. □

A most powerful - level α test for testing a composite null hypothesis against a simple alternative exists if there is a σ-finite measure which dominates all $P \in P_H$. This test can in general not be indicated as explicity as in the Neyman-Pearson Fundamental Lemma. The weak*-topology on the class of all test functions plays a part in the existence proof.

DEFINITION 2.4.2. Let (X, F) be a measurable space, λ a σ-finite measure on (X, F) and Φ_1 the class of all test functions on (X, F). The weak* topology on Φ_1 is the weakest topology for which the functions

$$\phi \mapsto \int \phi\, f\, d\lambda ,$$

are continuous, for all λ - integrable functions $f : X \to \mathbb{R}$.

The weak* topology is defined in a more general setting in many textbooks on functional analysis, e.g. in ASH (1972) Section 3.5. In order to use the weak* topology on a class of test functions, it will have to be clear which measure λ is used. Note that if λ and λ' are σ-finite measures with

$\lambda << \lambda' << \lambda$, then λ and λ' induce the same weak* topology. In the sequel it will often be assumed that a σ-finite measure λ exists which dominates all $P \in \mathcal{P}_H$ or all $P \in \mathcal{P}_H \cup \mathcal{P}_A$; the weak* topology will then be taken with respect to this measure λ.

THEOREM 2.4.2. *If λ is a σ-finite measure on $(X, \mathcal{F})$, then the weak* topology with respect to λ on Φ_1 is compact; if moreover $(X, \mathcal{F}) = \mathbb{R}^m$, then the weak* topology on Φ_1 is pseudo-metrizable. If $P << \lambda$ for all $P \in \mathcal{P}_H$ and $0 \leq \alpha \leq 1$, then Φ_α is a weakly* closed subset of Φ_1, and hence weakly* compact.*

PROOF. The weak*compactness of Φ_1 is a direct consequence of the Banach-Alaoglu Theorem on the weak* compactness of the unit ball in a normed linear space (ASH (1972) Theorem 3.5.16), applied to the linear space $L_\infty(X, \mathcal{F}, \lambda)$ with the sup-norm. For every $P \in \mathcal{P}_H$, $dP / d\lambda$ is λ-integrable; hence

$$\Phi_\alpha = \bigcap_{P \in \mathcal{P}_H} \{\phi \in \Phi_1 \mid \int \phi (dP / d\lambda)\, d\lambda \leq \alpha\}$$

is an intersection of weakly* closed subsets of Φ_1, so that Φ_α is weakly* closed. The Borel σ-field on $\mathbb{R}^m$ is countably generated. Hence $L_1(\mathbb{R}^m, \lambda)$ is separable for every σ-finite measure λ. Let $\{p_n \mid n \in \mathbb{N}\}$ be a set of functions with $\int |p_n| d\lambda \leq 1$ for all n, so that the linear hull of $\{p_n \mid n \in \mathbb{N}\}$ is dense in $L_1(\mathbb{R}^m, \lambda)$. Then $\rho(\phi, \psi) = \sum_n 2^{-n} |\int p_n(\phi - \psi) d\lambda|$ is a pseudometric on Φ_1 which generates the weak* topology. □

Theorem 2.4.2 implies that Φ_1 is sequentially compact if $(X, \mathcal{F}) = \mathbb{R}^m$ and a σ-finite dominating measure λ exists. This is also proved by LEHMANN (1959), Appendix 4. A more general theorem about the weak* sequential compactness of Φ_1 is given by NÖLLE and PLACHKY (1967) .

COROLLARY 2.4.1. *Suppose that a σ-finite measure λ exists with $P << \lambda$ for all $P \in \mathcal{P}_H$. Then for all $P_1 \in \mathcal{P}_A$, there exists a most powerful - level α test against P_1.*

PROOF. (Also in LEHMANN (1959), Section 3.8.) Let $P_1 \in \mathcal{P}_A$. It may be assumed that $P_1 << \lambda$ (if necessary, replace λ by $\lambda + P_1$). The function

$$\phi \mapsto E_{P_1}\phi = \int \phi\, (dP_1 / d\lambda)\, d\lambda$$

is weakly* continuous. As Φ_α is weakly* compact, the supremum of this function on Φ_α is assumed in some $\phi_1 \in \Phi_\alpha$. This ϕ_1 is MP - level α against P_1. □

It can be proved in the same way that if the assumption of this corollary is satisfied and Φ is any weakly* closed class of test functions, then a MP- Φ test against P_1 exists.

If the alternative hypothesis is composite, a UMP - level α test often does not exist. Such a test does exist for one-sided testing problems for experiments with monotone likelihood ratio.

DEFINITION 2.4.3. Consider an experiment $((X, F), \{P_\theta \mid \theta \in \Theta\})$ with $\Theta \subset \mathbb{R}$, and a statistic $t : X \to \mathbb{R}$. This experiment has monotone likelihood ratio in $t(x)$ if for all $\theta, \theta' \in \Theta$ with $\theta < \theta'$, one has that $P_\theta \neq P_{\theta'}$ and the ratio

$$p_{\theta'}(x) \,/\, p_\theta(x) ,$$

where p_θ and $p_{\theta'}$ are versions of the densities of P_θ and $P_{\theta'}$ with respect to some dominating measure λ, is a.e. $[P_\theta + P_{\theta'}]$ a non-decreasing function of $t(x)$.

One-dimensional exponential families with densities (2.3.1), where $m = 1$, are examples of experiments with monotone likelihood ratio.

THEOREM 2.4.3. *Suppose that the experiment* $((X, F), \{P_\theta \mid \theta \in \Theta\})$, *with* $\Theta \subset \mathbb{R}$, *has monotone likelihood ratio in* $t(x)$, *and consider the testing problem*

$$H : \theta = \theta_0, \qquad A : \theta > \theta_0.$$

Then $k \in \mathbb{R}$ *and* $\gamma \in [0,1]$ *can be chosen in such a way that the test* ϕ_0 *given by*

$$\text{(2.4.2)} \qquad \phi_0(x) = \begin{cases} 1 & t(x) > k \\ \gamma & t(x) = k \\ 0 & t(x) < k \end{cases}$$

is of size α; *for these* k *and* γ, ϕ_0 *is a UMP - level* α *test. The function* $\theta \mapsto E_\theta\phi_0$ *is strictly increasing on* $\{\theta \in \Theta \mid 0 < E_\theta\phi_0 < 1\}$.

PROOF. See LEHMANN (1959), Theorem 3.2. □

2.5. OTHER OPTIMUM PROPERTIES; BAYES TESTS

Many testing problems do not admit a UMP - level α test. In such situations one can (1) restrict the attention to a smaller class of tests, (2) employ an other optimum property than "UMP", or combine these two approaches.

(1) *Restriction to a smaller class of tests.* Sometimes, a UMP-Φ test exists for a "reasonable" class Φ. E.g., attention can be restricted to the unbiased level α tests, which are those level α tests which satisfy

$$E_P \phi \geq \alpha \qquad \text{for all } P \in \mathcal{P}_A .$$

See LEHMANN (1959), Chapters 4,5. For certain testing problems one can apply invariance considerations and restrict attention to the class of all invariant level α tests. See LEHMANN (1959), Chapter 6. Other intuitive or practical reasons may exist for restricting attention to a particular class of tests, or even for selecting one particular test statistic.

(2) *Other optimum properties.* Another approach is the construction of optimum properties which are weaker (and consequently less compelling) than the property "uniformly most powerful". As examples, we shall consider the optimum properties "maximin" ,"most stringent" and "Bayes".

A test ϕ_0 is defined to be maximin -Φ, if $\phi_0 \in \Phi$ and

$$\inf_{P\in\mathcal{P}_A} E_P\phi_0 = \sup_{\phi\in\Phi} \inf_{P\in\mathcal{P}_A} E_P\phi .$$

For many testing problems one has that

$$\inf \{ \| P - Q \| \; | P \in \mathcal{P}_H, \; Q \in \mathcal{P}_A \} = 0, \tag{2.5.1}$$

$\| P - Q \|$ denoting the variation distance between P and Q. If (2.5.1) is satisfied, then

$$\sup_{\phi\in\Phi_\alpha} \inf_{P\in\mathcal{P}_A} E_P\phi = \alpha .$$

In such cases, a test is maximin-level α iff it is unbiased level α, and the property "maximin" is not very useful. This difficulty can be circumvented by the introduction of an indifference zone $\mathcal{P}_I \subset \mathcal{P}_A$ of probability distributions P which are so "close" to $\mathcal{P}_H$ that a large probability of making an error of the second kind is not deemed to be very important for these P, so that (2.5.1) is not valid when $\mathcal{P}_A$ is replaced by $\mathcal{P}_{A'}$, where

$$A' : P \in \mathcal{P}_A \setminus \mathcal{P}_I, \qquad \mathcal{P}_{A'} = \mathcal{P}_A \setminus \mathcal{P}_I .$$

One can apply the maximin property to the testing problem with alternative A'. See, e.g., LEHMANN (1959) Sections 8.1, 8.2. This approach can lead to meaningful results. The maximin test depends on the indifference zone $\mathcal{P}_I$, however, and in most applications there is ample room for disagreement concerning the choice of $\mathcal{P}_I$. It may also be noted that in many instances where the maximin test has been constructed explicitly, the least favourable distribution (see Proposition 2.6.1) is concentrated in one point, and the maximin test is most powerful against some simple alternative. (E.g., see part (3) of Section 3.2.) In such cases, use of the maximin property entails that one implicitly restricts attention to the class of all tests which are "somewhere most powerful".

For technical reasons, the introduction of the property "most stringent" is deferred to the next section. We shall first treat the interesting optimum property "Bayes". This property is studied here mainly because Bayes tests are important for the construction of most stringent tests (Section 2.6) and complete classes (Section 2.7).

DEFINITION 2.5.1. Let $\mathcal{P}_A$ be equipped with a σ-field with respect to which the power functions of all tests are measurable, and let τ be a probability distribution on $\mathcal{P}_A$. The test ϕ_0 is Bayes - level α against τ if ϕ_0 is of level α, and

$$\int E_P \phi_0 \, d\,\tau(P) = \sup_{\phi \in \Phi_\alpha} \int E_P \phi \, d\,\tau(P) .$$

If τ is discrete, then it is not necessary to specify a σ-field on $\mathcal{P}_A$. We shall short-circuit a lot of theory and dispute, and not discuss the various interpretations of the distribution τ. Of course the choice of τ leaves room for disagreement. It can be desirable to use "improper" distributions τ: σ-finite distributions on $\mathcal{P}_A$ with $\tau(\mathcal{P}_A) = \infty$ (think of Lebesgue measure on $\mathbb{R}^m$). For such a τ, it will often happen that

$$\int E_P \, \phi \, d\, \tau(P) = \infty$$

for all size α tests ϕ. The test ϕ_0 can be defined to be comparative generalized Bayes - level α with respect to τ (STONE (1967)) if ϕ_0 is of level α and

$$\int E_P(\phi_0 - \phi) \, d\, \tau(P) \geq 0$$

for every level α test ϕ.

In the following proposition, the Neyman-Pearson Fundamental Lemma is used to derive the form of the Bayes tests for testing a simple null hypothesis. The assumption that P_A is parametrized by a Borel subset of $\mathbb{R}^m \setminus \{0\}$ is not essential.

PROPOSITION 2.5.1. *Let* H *be a simple null hypothesis and let* P_A *be parametrized by a Borel subset* $\Theta_A \subset \mathbb{R}^m \setminus \{0\}$:

$$P_H = \{P_0\}, \qquad P_A = \{P_\theta \mid \theta \in \Theta_A\} .$$

Define $\Theta = \Theta_A \cup \{0\}$. *Suppose that a* σ*-finite measure* λ *on* (X, F) *exists with*

$$P_\theta << \lambda \qquad \text{for all } \theta \in \Theta ,$$

and that versions $p_\theta(x)$ *of the densities*

$$p_\theta = d\, P_\theta \,/\, d\, \lambda$$

exist such that $p_\theta(x)$ *is a jointly measurable function of* θ *and* x. *Let* τ *be a probability distribution on* Θ_A, *and let* Φ^τ *be the class of tests* ϕ *which are of size* α, *and for which a* $k \in [0,\infty]$ *exists such that* ϕ *satisfies a.e.* $[\lambda]$

$$(2.5.2) \qquad \phi(x) = \begin{cases} 1 & \int p_\theta(x) \, d\, \tau(\theta) > k\, p_0(x) \\ 0 & \int p_\theta(x) \, d\, \tau(\theta) < k\, p_0(x) \end{cases} .$$

Then Φ^τ *is not empty, and every* $\phi \in \Phi^\tau$ *is Bayes - level* α *against* τ. *If*

$\lambda << P_\theta$ *for all* $\theta \in \Theta$, *then* Φ^τ *is the class of all Bayes - level* α *tests. (If in (2.5.2)* $k = \infty$ *and* $p_0(x) = 0$, *then* $k\, p_0(x)$ *is to be interpreted as* 0.)

PROOF. Fubini's theorem and the joint measurability of $p_\theta(x)$ imply that for every test ϕ, one has

$$\int E_\theta\, \phi\, d\,\tau(\theta) = \int \{\int \phi(x) p_\theta(x)\, d\,\lambda(x)\}\, d\,\tau(\theta)$$

$$= \int \phi(x)\, \{\int p_\theta(x)\, d\,\tau(\theta)\}\, d\,\lambda(x) = \int \phi(x)\, d\, P_\tau(x)\ ,$$

where P_τ is the probability distribution on $(X,\, F)$ with

$$d\, P_\tau \,/\, d\,\lambda(x) = \int p_\theta(x)\, d\,\tau(\theta)\ .$$

Hence a test ϕ is Bayes - level α against τ iff it is most powerful - level α for testing $H_0 : P = P_0$ against $A_\tau : P = P_\tau$. The proposition now follows immediately from the Neyman-Pearson Fundamental Lemma. (Note that if $P_\theta << \lambda$ and $\lambda << P_\theta$ for all $\theta \in \Theta$, then P_0 and P_τ are equivalent to λ, and no test of size less than 1 and with power against P_τ equal to 1 exists.) □

2.6. MOST STRINGENT TESTS

The concept of a most stringent test was introduced by WALD (1942). Define the envelope power function with respect to the class Φ_α of all level α tests, for $P \in \mathcal{P}_A$, by

$$\beta^*_\alpha(P) = \sup_{\phi\in\Phi_\alpha} E_P\phi\ ;$$

and the shortcoming of the test ϕ with respect to β^*_α by

$$\gamma_\alpha\,(\phi,\, P) = \beta^*_\alpha(P) - E_P\phi\ .$$

It is clear that $\gamma_\alpha\,(\phi,\, P) \geq 0$ for all $\phi \in \Phi_\alpha$ and $P \in \mathcal{P}_A$; and that $\gamma_\alpha\,(\phi,\, P) = 0$ for $\phi \in \Phi_\alpha$ iff ϕ is MP - level α against P. Wald defined the test ϕ_0 to be most stringent - level α if $\phi_0 \in \Phi_\alpha$ and

$$\sup_{P\in\mathcal{P}_A} \gamma_\alpha\,(\phi_0, P) = \inf_{\phi\in\Phi_\alpha}\ \sup_{P\in\mathcal{P}_A} \gamma_\alpha\,(\phi,\, P).$$

WALD(1942) introduced most stringent tests in the context of asymptotic testing theory. In those cases where an asymptotically uniformly most powerful - level α sequence of tests exists, there will exist many different sequences of tests which are asymptotically uniformly most powerful - level α. Wald argued that in such a case it seems to be desirable to use a sequence of most stringent tests, as the power function of such a sequence of tests "will approach the envelope function, in a certain sense, faster than any other power function". In his 1943 paper, Wald studies (asymptotically) most stringent tests in their own right. Some references to early papers concerned with (exact) most stringent tests can be found in LEHMANN (1959), Section 8.7.

The concept of a most stringent test can be extended by including the possibility of defining the shortcoming with respect to other functions β^*. For example if $\beta^* \equiv 1$ then the modified property "most stringent" is identical to the property "maximin". The function β^* can also be the envelope power function with respect to a class Ψ of test functions which one likes to use as a reference class. The dependence of the shortcoming γ on the function β^* will be suppressed in the notation.

DEFINITION 2.6.1. Let Φ be a class of tests and $\beta^* : \mathcal{P}_A \to [0,1]$ a function. The shortcoming of a test ϕ (with respect to β^*) is

$$\gamma\ (\phi,\ P) = \beta^*\ (P) - E_P\phi\ .$$

The test ϕ_0 is most stringent - $(\Phi,\ \beta^*)$, or MS - $(\Phi,\ \beta^*)$, if

(i) $\phi_0 \in \Phi$

(ii) $\sup_{P\in\mathcal{P}_A} \gamma(\phi_0,\ P) = \inf_{\phi\in\Phi} \sup_{P\in\mathcal{P}_A} \gamma(\phi,\ P)\ .$

The number

$$\inf_{\phi\in\Phi} \sup_{P\in\mathcal{P}_A} \gamma(\phi,\ P)$$

is called the minimax shortcoming - $(\Phi,\ \beta^*)$, or MXS - $(\Phi,\ \beta^*)$. If

$$\beta^*(P) = \sup_{\phi\in\Phi} E_P\phi \qquad \text{for all } P \in \mathcal{P}_A\ ,$$

then β^* is called the envelope power function with respect to Φ, and the suffix " - (Φ, β^*)" can be replaced by " - Φ".

Many optimality considerations in this study will be based on the property "most stringent". Although one may consider this optimum property to be attractive, it cannot be regarded as compelling. For testing problems which do not admit a UMP - level α test, optimum properties other than UMP, such as MS, can be a useful guide for obtaining tests; but the merits of a test satisfying such an optimum property have to be rated according to its "over-all power properties".

The following theorem is well known; see, e.g., LEHMANN (1959) Exercise 8.12.

THEOREM 2.6.1. *Suppose that a σ-finite measure λ on (X, F) exists with $P << \lambda$ for all $P \in P_H \cup P_A$, and let Φ be a weakly* closed class of tests and $\beta^* : P_A \to [0,1]$ a function. Then a most stringent - (Φ, β^*) test exists. In particular, a most stringent - level α test exists.*

PROOF. The function $\phi \mapsto \sup_{P \in P_A} \gamma(\phi, P)$ is weakly* lower semicontinuous and Φ is weakly* compact (Theorem 2.4.2). Hence this function achieves its infimum for some $\phi \in \Phi$. The compactness of the class of all level α tests follows from Theorem 2.4.2. □

For testing problems which are invariant under a group of transformations, there are conditions which guarantee that an invariant most stringent test exists. See, e.g., LEHMANN (1959) Section 8.5 and PLACHKY (1970). Hence under these conditions, if a UMP - invariant level α test exists, this test is MS - level α.

EXAMPLE 2.6.1. For the testing problem

$$L_\theta(X) = N_m(\theta, I)$$
$$H : \theta = 0, \qquad A : \theta \neq 0 ,$$

the test ϕ_0 which rejects for

$$||x||^2 \geq \chi^2_{m;\,\alpha} ,$$

is most stringent - level α. LEHMANN (1959, Section 8.5) proves this by invariance considerations; ϕ_0 is UMP - invariant level α. Another proof, essentially that of WALD (1943), is obtained by noting that

$$\gamma(\phi_0, \theta) = g(||\theta||) ,$$

for a continuous function $g : (0, \infty) \to (0,1)$ with

$$\lim_{t\to 0} g(t) = \lim_{t\to\infty} g(t) = 0 .$$

Let $t_0 \in (0, \infty)$ satisfy

$$g(t_0) = \sup_t g(t) ,$$

and let τ be the prior distribution which is uniform on $\{\theta \mid ||\theta|| = t_0\}$. It can be concluded from Proposition 2.5.1 that ϕ_0 is Bayes - level α against τ. The definitions of t_0 and τ yield that

$$\gamma(\phi_0, \theta) \leq g(t_0) = \int \gamma(\phi_0, \theta')\, d\,\tau(\theta') ,$$

for every $\theta \neq 0$. Theorem 2.11.2 of FERGUSON (1967) yields that ϕ_0 is most stringent - level α. □

In this example, the most stringent test was constructed as a Bayes test. The following proposition shows that for many testing problems the most stringent test is Bayes. It is a special case of general results stating that minimax procedures are Bayes with respect to least favourable prior distributions, such as Theorem 3.9 of WALD (1950).

PROPOSITION 2.6.1. *Let the conditions of Theorem 2.6.1 be satisfied. Consider only prior distributions which are defined on σ-fields of P_A, with respect to which β^* is measurable. For prior distributions τ, define*

$$\gamma(\tau) = \inf_{\phi\in\Phi_\alpha} \int \gamma(\phi, P)\, d\,\tau(P) .$$

Let T be the class of all prior distributions with finite support and γ_0 the minimax shortcoming. Then

$$\gamma_0 = \sup_{\tau\in T} \gamma(\tau) .$$

If a distribution τ_0 exists with $\gamma(\tau_0) = \gamma_0$, then every most stringent - (Φ_α, β^*) test is Bayes - level α against τ_0. (Such a distribution is called least favourable.)

PROOF. Note that Φ_α is weakly* compact (Theorem 2.4.2) and $\gamma(\phi, P)$ is a weakly* continuous function of ϕ. Application of a minimax theorem, e.g. that of KY FAN (1953) (for a review of many minimax theorems see PARTHASARATHY and RAGHAVAN (1971), Chapter 5) yields that $\gamma_0 = \sup_\tau \gamma(\tau)$. If ϕ_0 is MS - (Φ_α, β^*) then

$$(1) \qquad \int E_P\phi_0 \, d\,\tau_0(P) = \int \beta^*(P)\, d\,\tau_0(P) - \int \gamma(\phi_0, P)\, d\,\tau_0(P) \geq \int \beta^*(P)\, d\,\tau_0(P) - \gamma_0 .$$

As

$$\gamma_0 = \gamma(\tau_0) = \int \beta^*(P)\, d\,\tau_0(P) - \sup_{\phi \in \Phi_\alpha} \int E_P\phi \, d\,\tau_0(P) ,$$

(1) implies that ϕ_0 is Bayes - level α against τ_0. □

Not every testing problem admits a least favourable distribution. It is rather easy to show that under some regularity conditions, and if P_A is a weakly compact subset of $M_1(\mathbb{R})$, then a least favourable distribution exists; see WALD (1950), Theorem 3.14. The following proposition gives a sufficient condition for the existence of a least favourable distribution without this compactness requirement on Θ_A. It is an extension of a result of LEHMANN (1952) about the existence of least favourable distributions for the property "maximin".

PROPOSITION 2.6.2. *Let Θ be a closed subset of $\mathbb{R}^s$ with $0 \in \Theta$, $\Theta \neq \{0\}$ and let $\{P_\theta \mid \theta \in \Theta\}$ be a family of probability distributions on $\mathbb{R}^m$ with the properties*

(i) *there exists a σ-finite measure λ on $\mathbb{R}^m$ with $P_\theta << \lambda$ for all $\theta \in \Theta$*

(ii) *the densities $p_\theta = d\,P_\theta / d\,\lambda$ can be determined so that if $\theta_n \to \theta$ then $p_{\theta_n}(x) \to p_\theta(x)$ a.e. $[\lambda]$*

(iii) $\lim_{\|\theta\| \to \infty} P_\theta\{x \mid \|x\| \leq r\} = 0$ *for all r.*

Let $\beta^* : \Theta \to [0,1]$ be a continuous function with $\beta^*(0) = \alpha$. If the minimax shortcoming - (Φ_α, β^*) for the testing problem

$$(\mathbb{R}^m, \{P_0\}, \{P_\theta \mid \theta \in \Theta \setminus \{0\}\})$$

is positive, then there exists a least favourable distribution on $\Theta \setminus \{0\}$.

PROOF. Note that (ii) with Scheffé's Lemma implies that if $\theta_n \to \theta$, then

$$\sup_{B \in \mathcal{B}} | P_{\theta_n}(B) - P_\theta(B) | \to 0,$$

where $\mathcal{B}$ is the Borel σ-field on $\mathbb{R}^m$. This implies that $\gamma(\phi, \theta)$ is a continuous function of θ for every ϕ, and that $P_\theta \{x \mid \|x\| < r\}$ is a continuous function of θ.

Theorem 2.6.1 establishes the existence of a MS - (Φ_α, β^*) test ϕ_0. It may be assumed that $E_0 \phi_0 = \alpha$. Denote the minimax shortcoming by γ.

Proposition 2.6.1 yields a sequence $\{\tau_n\}$ of distributions on $\Theta \setminus \{0\}$ with

$$\inf_{\phi \in \Phi_\alpha} \int \gamma(\phi, \theta)\, d\tau_n(\theta) \to \gamma .$$

If a probability distribution τ on $\Theta \setminus \{0\}$ exists which is a weak subsequential limit of $\{\tau_n\}$ in $M_1(\Theta)$, then

$$\inf_{\phi \in \Phi_\alpha} \int \gamma(\phi, \theta)\, d\tau(\theta) \geq \gamma ,$$

so that τ is least favourable. Hence it is sufficient to show that such a τ exists.

Let $0 < \varepsilon < \frac{1}{2}$, and define $S_r = \{x \in \mathbb{R}^m \mid \|x\| \leq r\}$, $B_t = \{\theta \in \Theta \mid \|\theta\| \leq t\}$. Passing to a subsequence if necessary, it may be assumed (see Lemma A.5.2) that

$$\sup_t \liminf_n \tau_n(B_t) = \sup_t \limsup_n \tau_n(B_t) .$$

This implies the existence of a t_0 such that for all $t > t_0$ one has

$$\limsup_n \tau_n(B_t \setminus B_{t_0}) < \varepsilon .$$

Let r_0 be such that $P_\theta(S_{r_0}) \geq 1 - \varepsilon\alpha$ for all $\theta \in B_{t_0}$, and denote S_{r_0} by S. Let $t > t_0$ be such that $P_\theta(S) \leq \varepsilon$ for all $\theta \in \Theta \setminus B_t$, and denote B_{t_0} by B, $B_t \setminus B_{t_0}$ by C and $\Theta \setminus B_{t_0}$ by D.
The test $\phi_1 = (1 - \varepsilon)\phi_0 I_S + I_{\mathbb{R}^m \setminus S}$ is of level α and

$$\int_D \gamma(\phi_0, \theta)\, d\,\tau_n(\theta) = \int \{\gamma(\phi_0, \theta) - \gamma(\phi_1, \theta)\}\, d\,\tau_n(\theta)$$

$$+ \int_D \gamma(\phi_1, \theta)\, d\,\tau_n(\theta) + \int_B E_\theta\,(\phi_0 - \phi_1)\, d\,\tau_n(\theta).$$

Consider the right hand side. As

$$\int \gamma(\phi_0, \theta)\, d\,\tau_n(\theta) \leq \gamma \leq \liminf_n \int \gamma(\phi_1, \theta)\, d\,\tau_n(\theta) \;,$$

the first term is asymptotically non-positive. As $D \subset C \cup \{\theta \mid P_\theta(S) < \varepsilon\}$ and

$$\gamma(\phi_1, \theta) \leq 1 - E_\theta I_{\mathbb{R}^m \setminus S} = P_\theta(S) \;,$$

the second term is not greater than $\tau_n(C) + \varepsilon$. As $\phi_0 - \phi_1 \leq \varepsilon\phi_0$, the third term is not greater than ε. With $\limsup_n \tau_n(C) < \varepsilon$ this shows that

$$\limsup_n \int_D \gamma(\phi_0,\theta)\, d\,\tau_n(\theta) < 3\varepsilon \;.$$

Hence

$$\gamma = \lim_n \int \gamma(\phi_0, \theta)\, d\,\tau_n(\theta) \leq 3\varepsilon + \gamma \liminf_n \tau_n(B) \;.$$

Summarizing this discussion, we see that for all $\varepsilon > 0$ there exists a compact set $B \subset \Theta$ with $\liminf_n \tau_n(B) \geq (\gamma - 3\varepsilon) / \gamma$, so that $\{\tau_n\}$ is tight. According to Theorem A.1.1, $\{\tau_n\}$ has a subsequence which converges weakly to some probability distribution τ on Θ. As $\beta^*(0) - E_0\phi_0 = \alpha - \alpha = 0$, τ is concentrated on $\Theta \setminus \{0\}$. □

If β^* is the envelope power function with respect to a class of test functions, then condition (ii) of Proposition 2.6.2 implies the continuity of β^*. If β^* is the envelope power function with respect to Φ_α, and the minimax shortcoming is 0, then the conclusion of Proposition 2.6.2 holds; every distribution on $\Theta \setminus \{0\}$ is least favourable.

The following corollary plays a role in Chapter 8.

COROLLARY 2.6.1. *Let* Θ *be a closed subset of* $\mathbb{R}^m$ *with* $0 \in \Theta$ *and* $\Theta \neq \{0\}$, *let* Σ *be a positive definite symmetric matrix and let* $\beta^* : \Theta \to [0,1]$ *be a continuous function with* $\beta^*(0) = \alpha$. *If the minimax shortcoming* $-(\Phi_\alpha,\beta^*)$ *is positive, then the most stringent* $-(\Phi_\alpha,\beta^*)$ *test for the testing problem*

$$(\mathbb{R}^m, \{N_m(0, \Sigma)\}, \{N_m(\theta, \Sigma) \mid \theta \in \Theta \setminus \{0\}\})$$

is unique up to equivalence a.e. with respect to Lebesgue measure.

PROOF. Theorem 2.6.1 establishes the existence of a MS - (Φ_α, β^*) test. Proposition 2.6.1 and Proposition 2.6.2 establish the existence of a least favourable distribution τ, against which every MS - (Φ_α, β^*) test is Bayes - level α. Proposition 2.5.1, with the density function of the normal distribution, shows that the Bayes - level α test is uniquely determined up to equivalence a.e.. □

2.7. COMPLETE CLASSES

In this section the concept of a complete class, introduced by LEHMANN (1947), is studied.

DEFINITION 2.7.1. A class C of test functions is a complete class if for every $\phi \in \Phi_1 \setminus C$ a $\phi' \in C$ exists with

$$E_P\, \phi' \leq E_P\, \phi \qquad \text{for all } P \in \mathcal{P}_H$$

$$E_P\, \phi' \geq E_P\, \phi \qquad \text{for all } P \in \mathcal{P}_A\ ,$$

with strict inequality for some $P \in \mathcal{P}_H \cup \mathcal{P}_A$.
A class C of test functions is an essentially complete class if for every $\phi \in \Phi_1$ a $\phi' \in C$ exists with

$$E_P\, \phi' \leq E_P\, \phi \qquad \text{for all } P \in \mathcal{P}_H$$

$$E_P\, \phi' \geq E_P\, \phi \qquad \text{for all } P \in \mathcal{P}_A\ .$$

In the Neyman-Pearson approach, one would like to restrict attention to $\Phi_\alpha \cap C$, if C is an essentially complete class. Theorem 2.2.1 demonstrates that the class of all tests depending on x through the sufficient statistic t(x), is essentially complete. The following theorem is a special case of Wald's results on the essential completeness of the closure of the class of Bayes decision rules; see WALD (1950), Section 3.6 and LE CAM (1955), Section 5.

THEOREM 2.7.1. *Suppose that* H *is a simple hypothesis, and that there exists a* σ*-finite measure* λ *on* (X, F) *dominating all* $P \in \mathcal{P}_H \cup \mathcal{P}_A$. *Let* B *be the class of all tests which, for some* $\alpha \in [0,1]$ *and some distribution* τ *on* $\mathcal{P}_A$ *with finite support, are Bayes - level* α *against* τ. *Then the weak* closure* $\bar{B}$ *of* B *is an essentially complete class.*

PROOF. Let $\mathcal{P}_H = \{P_0\}$, let $\phi_0 \in \Phi_1$ and define $\alpha = E_{P_0}\phi_0$, $\beta^*(P) = E_P\phi_0$ and $\gamma(\phi, P) = \beta^*(P) - E_P\phi$. Denote the class of all probability distributions on the finite subset F of $\mathcal{P}_A$ by T_F. Define $\gamma(\tau)$ and γ_0 as in Proposition 2.6.1 and define

$$\gamma_F = \sup_{\tau \in T_F} \gamma(\tau) = \sup_{\tau \in T_F} \inf_{\phi \in \Phi_\alpha} \sum_{P \in F} \gamma(\phi, P)\, \tau\{P\} .$$

Note that T_F has a natural topology in which T_F is a compact set and γ a continuous function. Hence for every finite $F \subset \mathcal{P}_A$ there exists a least favourable $\tau_F \in T_F$ with $\gamma(\tau_F) = \gamma_F$, and a most stringent $-(\Phi_\alpha, \beta^*)$ test ϕ_F for testing H against the alternative "$P \in F$" (Theorem 2.6.1); ϕ_F is Bayes - level α against τ_F and the minimax shortcoming $-(\Phi_\alpha, \beta^*)$ for testing against F is γ_F (Proposition 2.6.1).

The class of finite subsets of $\mathcal{P}_A$ is directed by inclusion. Hence $\{\phi_F\}$ can be regarded as a net in Φ_α. The weak* compactness of Φ_α implies that $\{\phi_F\}$ has a weakly* convergent subnet; denote its limit by ϕ_1. As it may be assumed that $E_{P_0}\phi_F = \alpha$ for all F, it can be concluded that $E_{P_0}\phi_1 = \alpha$. For all $P \in \mathcal{P}_A$, $\gamma(\phi, P)$ is a continuous function of ϕ. Hence for all $P \in \mathcal{P}_A$ and $\varepsilon > 0$, there exists a finite $F \subset \mathcal{P}_A$ with $P \in F$ and $|\gamma(\phi_F, P) - \gamma(\phi_1, P)| < \varepsilon$. This implies

$$\gamma(\phi_1, P) < \gamma_F + \varepsilon \leq \gamma_0 + \varepsilon \quad .$$

Hence ϕ_1 is MS $-(\Phi_\alpha, \beta^*)$, and

$$\sup_{P \in \mathcal{P}_A} \gamma(\phi_1, P) \leq \sup_{P \in \mathcal{P}_A} \gamma(\phi_0, P) = 0 .$$

This shows that $E_P\phi_1 \geq E_P\phi_0$ for all $P \in \mathcal{P}_A$. As $E_{P_0}\phi_1 = E_{P_0}\phi_0$ and $\phi_1 \in \bar{B}$, the proof is complete. □

Theorem 2.7.1 can be very rewardingly applied to testing problems with a simple null hypothesis for exponential families. Let $\Theta \subset \mathbb{R}^m$ and let $\{P_\theta \mid \theta \in \Theta\}$ be an exponential family of distributions on $\mathbb{R}^m$ with

$$d\, P_\theta \,/\, d\, \lambda(x) = \exp\,(\theta' x - \psi(\theta))$$

for a certain σ-finite measure λ and a function $\psi : \Theta \to \mathbb{R}$. Consider the testing problem

$$H : \theta = \theta_0 , \qquad A : \theta \in \Theta_A$$

for some $\theta_0 \in \Theta$, $\Theta_A \subset \Theta \setminus \{\theta_0\}$. Proposition 2.5.1 yields that ϕ is a Bayes - level α test, for some α, against the prior distribution τ on Θ_A with $\tau\{\theta_i\} = t_i$ $(1 \leq i \leq n)$, $\sum_i t_i = 1$, iff a k exists such that ϕ satisfies for almost all $[\lambda]$ x

$$\phi(x) = \begin{cases} 1 & s(x) > k \\ 0 & s(x) < k \end{cases} ,$$

where the statistic s is defined by

$$s(x) = \sum_{i=1}^{n} t_i \exp\,[(\theta_i - \theta_0)'\, x - \psi(\theta_i) + \psi(\theta_0)] .$$

Note that s is a convex function with $\{x \mid s(x) < k\} = \text{int}\,\{x \mid s(x) \leq k\}$. Hence for every Bayes test ϕ against a distribution with finite support on Θ_A, there exists a closed convex set $C \subset \mathbb{R}^m$ such that a.e. $[\lambda]$, ϕ satisfies

$$\phi(x) = \begin{cases} 1 & x \notin C \\ 0 & x \in \text{int}\, C . \end{cases}$$

This property motivates the following definition.

DEFINITION 2.7.2. A test function $\phi : \mathbb{R}^m \to [0,1]$ has acceptance region C if C is a closed subset of $\mathbb{R}^m$ with

$$\phi(x) = \begin{cases} 1 & x \notin C \\ 0 & x \in \text{int } C \ . \end{cases}$$

The class of test functions with convex acceptance region is denoted by Φ_C. Not every test has an acceptance region. If the test ϕ has acceptance region C, then ϕ is continuous on $\mathbb{R}^m \setminus \partial C$. If also C is convex, then ϕ is continuous almost everywhere with respect to Lebesgue measure.

The following theorem was given by BIRNBAUM (1955) with an incorrect proof. A correct proof was given by Sacks and published by MATTHES and TRUAX (1967).

THEOREM 2.7.2. *Let $\Theta \subset \mathbb{R}^m$ and let $\{P_\theta \mid \theta \in \Theta\}$ be a family of probability distributions on $\mathbb{R}^m$ with*

$$d\, P_\theta \;/\; d\, \lambda(x) = \exp\, (\theta' x - \psi(\theta)),$$

for a σ-finite measure λ on $\mathbb{R}^m$ and a function $\psi : \Theta \to \mathbb{R}$. Let $\theta_0 \in \Theta$, $\Theta_A \subset \Theta \setminus \{\theta_0\}$ and consider the testing problem

$$H : \theta = \theta_0 \qquad\qquad A : \theta \in \Theta_A \ .$$

Then Φ_C is an essentially complete class .

PROOF. Theorem 2.7.1 and the discussion preceding Definition 2.7.2 yield that the weak* closure $\bar{\Phi}_C$ of Φ_C is essentially complete. Theorem A.4.1 (i) states that $\bar{\Phi}_C$ is weakly* compact. This implies that

$$\bar{\Phi}_C = \{\phi \mid \text{a } \phi' \in \Phi_C \text{ exists with } \phi'(x) = \phi(x) \text{ a.e. } [\lambda]\}.$$

(Note that the class of all test functions is not Hausdorff.) Hence for every $\phi \in \bar{\Phi}_C$ there exists a $\phi' \in \Phi_C$ with

$$E_\theta \phi' = E_\theta \phi \qquad\qquad \text{for all } \theta \in \Theta \ .$$

This shows that Φ_C inherits the essential completeness from $\bar{\Phi}_C$. □

2.8. THE POWER OF THE MOST STRINGENT TEST FAR FROM THE NULL HYPOTHESIS

This section is concerned with the testing problem

$$(2.8.1) \qquad (\mathbb{R}^m, \{N_m(0, \Sigma)\}, \{N_m(\theta, \Sigma) \mid \theta \in \Theta_A\})$$

where Σ is a known positive definite matrix and $\Theta_A \cup \{0\}$ is a closed cone in $\mathbb{R}^m$. Corollary 2.6.1 shows that the most stringent $-(\Phi_\alpha, \beta^*)$ test ϕ_0 is, under some conditions on β^*, uniquely determined. Some results are derived which play a role in proofs of the sharp consistency of sequences of tests in Chapter 6 and 8. These results are related to the question, whether

$$(2.8.2) \qquad \lim_{||\theta|| \to \infty, \theta \in \Theta_A} \gamma(\phi_0, \theta) = 0 .$$

This clearly is a desirable property for the test ϕ_0. The following example shows that if Θ_A is not a cone, the most stringent - level α test does not necessarily satisfy (2.8.2).

EXAMPLE 2.8.1. Consider the testing problem (2.8.1) with $m = 2$, $\Sigma = I$ and

$$\Theta_A = \{\theta \mid \theta_2 = 0, \theta_1 \neq 0\} .$$

Denote the observed random variable by $X = (X_1, X_2)$. Then X_1 is a sufficient statistic, and example 2.6.1 shows that the test ϕ_0 which rejects for

$$|X_1| \geq u_{\frac{1}{2}\alpha}$$

is most stringent - level α. (Note that $\chi^2_{1;\alpha} = (u_{\frac{1}{2}\alpha})^2$.) Denote the minimax shortcoming by γ_0, and the envelope power function with respect to the class of all level α tests by β^*. Let θ_{10} be the positive number with

$$E_{(\theta_{10}, 0)}\phi_0 = 1 - \gamma_0 .$$

Then for every θ_2,

$$\gamma(\phi_0, (\theta_{10}, \theta_2)) = \beta^*(\theta_{10}, \theta_2) - E_{(\theta_{10}, \theta_2)}\phi_0 \leq 1 - (1-\gamma_0) = \gamma_0;$$

furthermore

$$\lim_{\theta_2 \to \infty} \gamma(\phi_0, (\theta_{10}, \theta_2)) = \gamma_0 .$$

Hence if Θ_A is replaced by

$$\Theta_A = \{\theta \mid \theta_2 = 0 \text{ or } \theta_1 = \theta_{10};\ \theta_1 \neq 0\} ,$$

then ϕ_0 is still the most stringent - level α test; but (2.8.2) fails to hold. □

In order to show that the most stringent test ϕ_0 satisfies (2.8.2) when Θ_A is a cone, the concept of the recession cone of a convex set will be used. For a detailed study of recession cones, see ROCKAFELLAR (1970), Section 8.

<u>DEFINITION 2.8.1</u>. Let C be a non-empty convex subset of $\mathbb{R}^m$. The recession cone of C is

$$0^+C = \{y \in \mathbb{R}^m \mid x + ty \in C \text{ for every } x \in C \text{ and } t \geq 0\} .$$

ROCKAFELLAR (1970) motivates the notation 0^+C by the property that for non-empty closed convex C, 0^+C consists exactly of all possible limits of sequences $\{t_n x_n\}$ with $t_n \to 0^+$ and $x_n \in C$. In terms of Definition 4.2.1, 0^+C is the topological limit of $t_n C$, for $t_n \to 0$, $t_n > 0$. The intuitive understanding of the concept of the recession cone may also be enhanced by noting that if C is a closed convex set, and if y is such that $x_0 + ty \in C$ for every $t \geq 0$, for some $x_0 \in C$, then $x + ty \in C$ for every $t \geq 0$ and every $x \in C$, so that $y \in 0^+C$.

<u>PROPOSITION 2.8.1</u>. *Let* C *be a non-empty closed convex subset of* $\mathbb{R}^m$. *Then*

(i) 0^+C *is a closed convex cone containing the origin*

(ii) *for every* $y \in 0^+C$, $C + y \subset C$ *and* $\text{int}\, C + y \subset \text{int}\, C$.

(iii) *Let* K *be a closed cone in* $\mathbb{R}^m$ *with* $K \cap 0^+C = \{0\}$. *Then for every sequence* $\{y_n\} \subset K$ *with* $\|y_n\| \to \infty$,

$$\inf_{x \in C} \|x - y_n\| \to \infty .$$

<u>PROOF</u>. (i) and (ii) follow from Theorems 8.1 and 8.2 of ROCKAFELLAR (1970).

(iii) Argue by contradiction and suppose that sequences $\{x_n\} \subset C$ and $\{y_n\} \subset K$ exist with $||y_n|| \to \infty$ and $\liminf ||x_n - y_n|| < \infty$. It may be assumed, after passing to a subsequence if necessary, that $x_n - y_n \to z$ and that $y_n / ||y_n|| \to y$. Then $x_n / ||y_n|| \to y$. As K is a closed cone, one has $y \in K$ and therefore $y \notin 0^+C$. Hence there exist $x_0 \in C$, $t > 0$ with $x_0 + ty \notin C$. Also,

$$z_n = (1 - t / ||y_n||)\, x_0 + (t / ||y_n||)\, x_n \to x_0 + ty \; .$$

The convexity of C implies that $z_n \in C$; the closedness of C implies that $x_0 + ty \in C$. This is in contradiction with $x_0 + ty \notin C$. □

<u>PROPOSITION 2.8.2</u>. *Consider testing problem (2.8.1), where* $\Theta_A \cup \{0\}$ *is a closed cone. Let* ϕ *be a test with convex acceptance region* C. *If* $\Theta_A \cap 0^+C \neq \emptyset$, *then*

$$\liminf_{||\theta|| \to \infty, \theta \in \Theta_A} E_\theta \phi \leq E_0 \phi \; .$$

If $\Theta_A \cap 0^+C = \emptyset$, *then*

$$\lim_{||\theta|| \to \infty, \theta \in \Theta_A} E_\theta \phi = 1 \; .$$

<u>PROOF</u>. (i) Suppose that $\theta \in \Theta_A \cap 0^+C$ and let $t > 0$. Then $\text{int } C + t\theta \subset \text{int } C$ according to Proposition 2.8.1, so that $\phi(x + t\theta) \leq \phi(x)$ for all $x \in \mathbb{R}^m \setminus \partial C$. Hence for all $t > 0$,

$$E_{t\theta}\, \phi = E_0\, (X + t\theta) \leq E_0 \phi(X) \; .$$

(ii) For every n, let $\theta_n \in \Theta_A$ be such that $||\theta_n|| \geq n$ and

$$E_{\theta_n} \phi \leq \inf_{||\theta|| \geq n, \theta \in \Theta_A} E_\theta \phi + n^{-1} \; .$$

Proposition 2.8.1 (iii) implies that

$$\inf_{x \in C} ||x - \theta_n|| \to \infty \; ,$$

so that

$$P_0 \{X \in C - \theta_n\} \to 0.$$

Therefore

$$\inf_{||\theta|| > n, \theta \in \Theta_A} E_\theta \phi \geq E_{\theta_n} \phi - n^{-1} =$$

$$= 1 - P_{\theta_n} \{X \in C\} - n^{-1} = 1 - P_0 \{X \in C - \theta_n\} - n^{-1} \to 1 .$$

□

COROLLARY 2.8.1. *Consider testing problem (2.8.1) where* $\Theta_A \cup \{0\}$ *is a closed cone. Let* $\beta^* : \Theta_A \to [0,1]$ *be a continuous function with*

$$\lim_{||\theta|| \to \infty} \beta^*(\theta) = 1 \text{ , } \limsup_{\theta \to 0} \beta^*(\theta) < 1 .$$

Let Φ *be a class of level* α *tests.*

(i) *If* Φ *contains at least one test* ϕ *with convex acceptance region* C *for which* $\Theta_A \cap 0^+C = \emptyset$ *and* $E_\theta \phi \geq \alpha$ *for all* $\theta \in \Theta_A$*, then the minimax shortcoming* $-(\Phi, \beta^*)$ *is strictly less than* $1 - \alpha$.

(ii) *If the minimax shortcoming* $-(\Phi, \beta^*)$ *is strictly less than* $1 - \alpha$ *while the test* ϕ_0 *has convex acceptance region and is most stringent* $-(\Phi, \beta^*)$, *then*

$$\lim_{||\theta|| \to \infty} E_\theta \phi_0 = 1 .$$

PROOF. (i) It is sufficient to show that

(1) $$\sup_{\theta \in \Theta_A} \gamma(\phi, \theta) < 1 - \alpha.$$

But $\gamma(\phi, \theta)$ is a continuous function of θ with $\gamma(\phi, \theta) < 1 - \alpha$ for all $\theta \in \Theta_A$ and

$$\lim_{\|\theta\|\to\infty} \gamma(\phi,\ \theta) = 0 \qquad \text{(Proposition 2.8.2)}$$

$$\limsup_{\theta\to 0} \gamma(\phi,\ \theta) < 1 - \alpha.$$

This implies (1).

(ii) This follows immediately from Proposition 2.8.2. □

Note that the MS - level α test against the unrestricted alternative, which is the test rejecting for

$$X'\ \Sigma^{-1}\ X \geq \chi^2_{m;\alpha}$$

(see Example 2.6.1), satisfies the requirements for the test ϕ in (i) of Corollary 2.8.1. Hence the conclusions of Corollary 2.8.1 are valid for $\Phi = \Phi_\alpha$.

CHAPTER 3

TESTING PROBLEMS FOR EXPONENTIAL FAMILIES WITH RESTRICTED ALTERNATIVES

The class of testing problems which is the subject of this study will be described in Section 3.1. It is a fairly large class of testing problems for exponential families of distributions, including both testing problems with restricted and with unrestricted alternative hypotheses. The emphasis will be on testing problems with restricted alternatives. There are many methods to produce tests for these testing problems. A few of these methods will be reviewed in Sections 3.2 to 3.4. In Section 3.5 the testing problem of Section 3.1 will be embedded in a sequence of similar testing problems, where the sample sizes tend to infinity. This sets the stage for the asymptotic approach to be developed in the following chapters.

3.1. THE CLASS OF TESTING PROBLEMS TO BE STUDIED

This study is chiefly concerned with testing problems of the following form. The experiment is constituted by k ($k \geq 1$) random samples

$$X_{i1}, \ldots, X_{in_i} \qquad 1 \leq i \leq k .$$

The variables X_{ij} ($1 \leq j \leq n_i$, $1 \leq i \leq k$) are independent. The variables $X_{i1}, \ldots, X_{in_i}$ are identically distributed m_i-dimensional random variables with probability distribution $P_{i\theta_i}$, for some $\theta_i \in \Theta_i$. For every i,

$$\{P_{i\theta_i}, \theta_i \in \Theta_i\}$$

is a canonical m_i-dimensional exponential family of distributions. So Θ_i is an open subset of $\mathbb{R}^{m_i}$, and there exist σ-finite measures λ_i on $\mathbb{R}^{m_i}$ and functions $\psi_i : \Theta_i \to \mathbb{R}$ such that

$$d\,P_{i\theta_i} \,/\, d\,\lambda_i(x) = \exp\,\{x'\theta_i - \psi_i(\theta_i)\}\;;$$

none of the λ_i is concentrated on a hyperplane.

Denote

$$\theta = \begin{pmatrix} \theta_1 \\ \cdot \\ \cdot \\ \cdot \\ \theta_k \end{pmatrix} \qquad \Theta = \prod_{i=1}^{k} \Theta_i \qquad m = \sum_{i=1}^{k} m_i$$

$$\mu_i(\theta_i) = E_{\theta_i} X_{i1} \qquad \mu(\theta) = \begin{pmatrix} \mu_1(\theta_1) \\ \cdot \\ \cdot \\ \cdot \\ \mu_k(\theta_k) \end{pmatrix} .$$

According to Theorem 2.3.2, the correspondence between θ and $\mu(\theta)$ is 1 : 1. So the probability distributions can be parametrized by $\mu = \mu(\theta)$ as well as by θ. The parameter μ assumes values in $F = \mu(\Theta)$.

Null hypothesis and alternative hypothesis for the testing problem are given by

$$H \quad : f(\mu) \in V$$
$$H \vee A : f(\mu) \in V + K ,$$

where

$f : F \to f(F) \subset \mathbb{R}^m$ is a twice continuously differentiable 1 : 1 function, of which the matrix $(\partial f \,/\, \partial\mu)$ of first-order partial derivatives is non-singular for all $\mu \in F$, and which has a continuous inverse f^{-1} ,

V is a linear subspace of $\mathbb{R}^m$ with $V \cap f(F) \neq \emptyset$,

K is a closed cone in $\mathbb{R}^m$ with $K \setminus V \neq \emptyset$.

Many testing problems for exponential families, both with restricted and with unrestricted alternatives, have this form. In most cases f can be taken to be the identity function.

EXAMPLE 3.1.1. *The testing problem of Section 1.1.* Let

$$\tilde{X}_{i1}, \ldots, \tilde{X}_{in_i} \qquad 1 \leq i \leq k$$

be independent identically distributed random variables with

$$P_{p_i}\{\tilde{X}_{i1} = h\} = p_{ih} \qquad 1 \le h \le m_0 + 1$$

for

$$p_i = (p_{i1}, \ldots, p_{i,m_0+1})'$$

with $p_{ih} > 0$, $\Sigma_h p_{ih} = 1$. The null hypothesis of homogeneity can be tested against one of several alternative hypotheses: e.g., the alternative of decreasing stochastic ordering (A_1), the alternative of stochastic ordering in an unspecified direction (A_2), or the unrestricted alternative (A_3). These are defined by

$$H \quad : p_1 = p_2 = \ldots = p_k$$

$$H \vee A_1 : \sum_{h=1}^{g} p_{i+1,h} \ge \sum_{h=1}^{g} p_{ih} \qquad 1 \le g \le m_0,\ 1 \le i \le k-1$$

$H \vee A_2$: a permutation $(i_1, \ldots, i_k)$ of $(1, \ldots, k)$ exists such that

$$\sum_{h=1}^{g} p_{i_{t+1}h} \ge \sum_{h=1}^{g} p_{i_t h} \qquad 1 \le g \le m_0,\ 1 \le t \le k-1$$

$$A_3 \quad : p_i \ne p_1 \text{ for some } i.$$

In Section 1.1, this testing problem was considered with $k = 3$, $m_0 = 3$ and alternative A_2.

The $\tilde{X}_{ij}$ can be transformed to random variables X_{ij} having a distribution from a canonical m_0-dimensional exponential family (see Example 2.3.1):

$$X_{ij} = (X_{ij1}, \ldots, X_{ijm_0})'$$

$$X_{ijh} = \begin{cases} 1 & \tilde{X}_{ij} = h \\ 0 & \tilde{X}_{ij} \ne h \, . \end{cases}$$

Then

$$\mu_i = E_{p_i} X_{ij} = (p_{i1}, \ldots, p_{im_0})' \, .$$

This problem is of the form mentioned above, for all three alternatives. The function f can be taken to be the identity function. □

The following example demonstrates that f cannot always be taken to be the identity function.

<u>EXAMPLE 3.1.2</u>. *Independence problems for bivariate categorical data.* Let

$$(\tilde{X}_{11}, \tilde{X}_{12}), (\tilde{X}_{21}, \tilde{X}_{22}), \ldots, (\tilde{X}_{n1}, \tilde{X}_{n2})$$

be independent identically distributed random variables with

$$P_p\{(\tilde{X}_{11}, \tilde{X}_{12}) = (h_1, h_2)\} = p(h_1, h_2)$$

where p is a $m_{01} \times m_{02}$ matrix with positive entries $p(h_1, h_2)$ and

$$\sum_{1 \le h_1 \le m_{01},\ 1 \le h_2 \le m_{02}} p(h_1, h_2) = 1 .$$

Consider the testing problem of independence against the alternative of positive regression dependence of $\tilde{X}_{12}$ on $\tilde{X}_{11}$ (A_1) or the unrestricted alternative of dependence (A_2) :

H : $p(h_1, h_2) = p(h_1,+)p(+, h_2)$ for all (h_1, h_2)

$H \vee A_1$: $P_p\{\tilde{X}_{12} \ge h_2 \mid \tilde{X}_{11} = h_1\}$ is, for all h_2, non-decreasing in h_1

A_2 : $p(h_1, h_2) \ne p(h_1,+)p(+, h_2)$ for some (h_1, h_2) ,

where

$$p(h_1,+) = \sum_{1 \le h_2 \le m_{02}} p(h_1, h_2)$$

$$p(+, h_2) = \sum_{1 \le h_1 \le m_{01}} p(h_1, h_2) .$$

Note that $H \vee A_1$ can be expressed alternatively by

$$H \vee A_1 : \sum_{g \ge h_2} p(h_1+1, g) / p(h_1+1, +) \ge$$

$$\ge \sum_{g \ge h_2} p(h_1, g) / p(h_1,+) \quad \text{for all } (h_1, h_2) .$$

The $(\tilde{X}_{j1}, \tilde{X}_{j2})$ can be transformed to random variables X_j having a distribution from a canonical m-dimensional exponential family (see Example 2.3.1) with $m = m_{01}m_{02} - 1$. It is convenient to represent X_j by an $m_{01} \times m_{02}$ matrix where the (m_{01}, m_{02}) element is deleted, and where the (h_1, h_2) element is

$$X_{jh_1h_2} = \begin{cases} 1 & (\tilde{X}_{j1}, \tilde{X}_{j2}) = (h_1, h_2) \\ 0 & (\tilde{X}_{j1}, \tilde{X}_{j2}) \neq (h_1, h_2) . \end{cases}$$

Then

$$\mu = E_p X_j = p^*,$$

where p^* is obtained from p by deleting $p(m_{01}, m_{02})$. Define $f(p^*)$ as the $m_{01} \times m_{02}$ matrix where the (m_{01}, m_{02}) element is deleted and where

$$[f(p^*)](h_1, h_2) = \sum_{g \geq h_2} p(h_1, g) \,/\, p(h_1, +)$$

$$\text{for } 1 \leq h_1 \leq m_{01},\ 1 \leq h_2 \leq m_{02} - 1$$

$$[f(p^*)]\,(h_1, m_{02}) = p(h_1, +) \quad \text{for } 1 \leq h_1 \leq m_{01} - 1 .$$

Then it is seen that these testing problems are of the form mentioned above, with

$$V = \{x \mid x(1, h_2) = \ldots = x(m_{01}, h_2) \text{ for } 1 \leq h_2 \leq m_{02} - 1\}$$

$$K_1 = \{x \mid x(1, h_2) \leq \ldots \leq x(m_{01}, h_2) \text{ for } 1 \leq h_2 \leq m_{02} - 1\}$$

$$K_2 = \{x \mid \text{no restrictions on } x\} .$$

f cannot be taken to be the identity function, because the set of all μ for which the null hypothesis is satisfied, is "curved". □

3.2. SOME METHODS FOR OBTAINING TESTS

A concise and far from exhaustive review will be given of some methods for obtaining tests for testing problems of the kind of Section 3.1.

Attention is directed in particular to testing problems with a composite null hypothesis ($V \neq \{0\}$) and a restricted alternative (K is a cone, but not a linear subspace). At the end of this section , the formulation of the alternative hypothesis for testing problems from practice is briefly discussed.

Five approaches will be mentioned which can be used in order to construct tests: (1) a Bayesian approach, (2) the construction of a UMP test in a suitable class of tests, (3) determining an optimal test for an optimum property weaker than UMP, (4) using standard construction methods which do not automatically produce a test with a specified optimum property, (5) ad hoc methods.

(1) *Bayes tests*. When there are reasons to postulate a certain prior distribution, a Bayes test may be advisable. See, for example, LINDLEY (1965). We shall not pursue this approach.

(2) *UMP tests*. For some testing problems, a UMP test exists in the class of all level α tests, or in the class of unbiased level α tests; or in the class of invariant level α tests, if invariance considerations are applicable. If a UMP test in such a class exists, it can be found by methods employing the Neyman-Pearson Fundamental Lemma. Let

$$(\mathbb{R}^m, \{P_\theta \mid \theta \in \Theta\})$$

be a canonical m-dimensional exponential family of distributions. Then $\theta = (\theta_1, \ldots, \theta_m)$ is the canonical parameter, and $\Theta \subset \mathbb{R}^m$. If the testing problem can be given the form

$$H : \theta = 0, \qquad A : \theta_1 > 0, \ \theta_i = 0 \quad \text{for all } i \geq 2,$$

then a UMP - level α test exists (Theorem 2.4.3). If the testing problem can be given one of the forms

$$H : \theta_1 = 0, \qquad A : \theta_1 \neq 0$$

$$H : \theta_1 = 0, \qquad A : \theta_1 > 0 \ ,$$

then a UMP-unbiased level α test exists (LEHMANN (1959), Section 4.4). In some cases, e.g. if the testing problem of the latter form is a testing problem for the means of normal distributions with known variances, this

UMP-unbiased level α test is even UMP - level α. (For examples of UMP - level α tests for exponential families with more than one unknown parameter see LEHMANN (1959) Section 3.9 and problems 3.27, 3.32, 3.33.)

Few testing problems of the kind of Section 3.1 admit a UMP - invariant level α test . The general linear hypothesis problem is an important problem for which a UMP - invariant level α test exists (LEHMANN (1959), Chapter 7). Another instance is provided by the testing problem where X_i $(1 \leq i \leq m)$ are independent random variables with Bernoulli distributions and success probabilities $P_p\{X_i = 1\} = p_i$, and where null hypothesis and alternative hypothesis are

$$H : p_i = \tfrac{1}{2} \quad (1 \leq i \leq m), \quad H \vee A : p_i \geq \tfrac{1}{2} \quad (1 \leq i \leq m) .$$

The test which rejects for large values of $\Sigma_i X_i$ is UMP - invariant level α (LEHMANN (1959), Example 6.7). This test is also most stringent - level α (LEHMANN (1959), Section 8.5; compare our remark following Theorem 2.6.1). As invariance considerations seem to be of little use for most testing problems with restricted alternatives, they will not play an important part in this study.

(3) *Other optimum properties*. For most testing problems of the kind of Section 3.1 there exists no UMP - level α or UMP - unbiased level α test. One can try to construct a test with a weaker optimum property. Of the many optimum properties which can be considered, we shall only pay attention to "maximin" and "most stringent".

An example of an application of the maximin property is given by LEE (1977). He considers the testing problem where $(X_1, \ldots, X_m)$ has the multinomial distribution with parameters n and $p = (p_1, \ldots, p_m)$, and where the null hypothesis and alternative hypothesis are given by

$$H : p_1 = \ldots = p_m = m^{-1}, \; H \vee A : p_1 \leq \ldots \leq p_m .$$

After the removal of an indifference zone, Lee considers the alternatives

$$A'_\delta \;:\; p_{i+1} - p_i \geq \delta_i \qquad 1 \leq i \leq m-1$$

$$A''_\lambda \;:\; p_{i+1} \geq \lambda_i p_i \qquad 1 \leq i \leq m-1 ,$$

where $\delta = (\delta_1, \ldots, \delta_{m-1})$ is a vector with $\delta_i \geq 0$ (all i), $\delta \neq 0$ and

$\lambda = (\lambda_1, \ldots, \lambda_{m-1})$ is a vector with $\lambda_i \geq 1$ (all i), $\lambda \neq (1, \ldots, 1)$. It appears that the parameter value $p^* = (p_1^*, \ldots, p_m^*)$ determined by

$$p_{i+1}^* - p_i^* = \delta_i$$

for A'_δ and by

$$p_{i+1}^* = \lambda_i p_i^*$$

for A''_λ, is a least favourable parameter value: the distribution giving probability 1 to p^* is least favourable. Hence the test which is most powerful - level α against p^*, which is the test rejecting for large values of

$$\sum_{i=1}^{m} X_i \log p_i^* ,$$

is maximin - level α. Note that this test is a function of δ or λ, respectively. (The assertion of LEE (1977) Section 1, that if $\min \lambda_i > 1$, then the maximin test against A''_λ rejects for large values of $\Sigma_i iX_i$ is a mistake.)

In practical situations, there will rarely be compelling reasons for choosing a certain indifference region. In the case discussed here, all maximin level α tests are "somewhere most powerful" in the sense that they are most powerful against certain simple alternatives. In cases where the choice of an indifference zone cannot be clearly motivated, the use of an optimal or asymptotically optimal (e.g., most stringent) test in the class of all somewhere most powerful tests seem to be preferable to the use of the maximin test after the removal of an indifference zone. It is also sensible to consider tests which are not somewhere most powerful; CHACKO (1966) and ROBERTSON (1978) study the likelihood-ratio test for this testing problem. See also Example 4.3.2. Maximin tests will not be considered in the sequel.

(4) *Standard construction methods*. Several methods for constructing tests exist which are motivated primarily by intuitive arguments and only indirectly by optimality considerations. The most well-known method is the (generalized) likelihood ratio principle and will be outlined below.

Consider the testing problem

$$((X, F), \{P_\theta \mid \theta \in \Theta_H\}, \{P_\theta \mid \theta \in \Theta_A\}) ,$$

where a σ-finite measure λ on $(\mathcal{X}, \mathcal{F})$ exists with

$$P_\theta << \lambda \qquad \text{for all } \theta \in \Theta_H \cup \Theta_A .$$

Let $\{p_\theta \mid \theta \in \Theta_H \cup \Theta_A\}$ be a "smooth" family of versions of the densities

$$p_\theta = d\,P_\theta \,/\, d\lambda .$$

NEYMAN and PEARSON (1928) proposed the likelihood ratio test, which rejects the null hypothesis for large values of the test statistic

$$\ell(X) = \sup_{\theta\in\Theta_A} p_\theta(X) \,/\, \sup_{\theta\in\Theta_H} p_\theta(X) .$$

In their paper of 1933, they proved that if the null hypothesis and the alternative hypothesis are simple, then the likelihood ratio test is most powerful: the Neyman-Pearson Fundamental Lemma.

WILKS (1938) studied the testing problem where $X_1, \ldots, X_n$ is a random sample from a probability distribution P_θ for some $\theta \in \Theta \subset \mathbb{R}^m$, and where Θ_H is the intersection of Θ with a linear subspace of dimension m', and $\Theta_A = \Theta \setminus \Theta_H$. He showed that if $\theta \in \Theta_H$ is an interior point of Θ then, under certain regularity conditions, the asymptotic distribution of $2 \log \ell(X)$ for $n \to \infty$ is χ^2 with $m - m'$ degrees of freedom. This result permits an approximation of the critical value for the test statistic $2 \log \ell(X)$, for testing problems with unrestricted alternatives and large sample sizes.

WALD (1943) proved that for the testing problem considered by Wilks, the likelihood ratio test is "asymptotically most stringent". For asymptotic optimum properties of the likelihood ratio test when the sample size tends to infinity and the significance level tends to 0 see, e.g., KALLENBERG (1978) and the references cited by him in his Chapter 1. For many testing problems with restricted alternatives, the power properties of the likelihood ratio test are quite good. For such problems, however, the computation of test statistic and critical value is often rather cumbersome.

(5) *Ad hoc methods*. It may happen that a statistician with a good insight proposes a test which appears to have good power properties. An excellent example is provided by Fisher's method of combining independent tests. For m independent test statistics yielding one-sided tail probabilities $Q_1, \ldots, Q_m$ which are uniformly distributed on [0,1] under the null hypo-

thesis, FISHER (1932) proposes as a combined test the test which rejects for

$$-2 \sum_{i=1}^{m} \log Q_i \geq \chi^2_{2m;\alpha} .$$

Numerical computations for small m have demonstrated that this test has very good power properties; see VAN ZWET and OOSTERHOFF (1967). Another example of an "ad hoc" test is the Wilcoxon test with mid-rank scores for the treatment of ties, to be mentioned in Section 3.4.

These five approaches are, of course, not really distinct. For example, K. PEARSON (1900) introduced the χ^2 test for goodness of fit as the fruit of ad hoc methods (5); NEYMAN and PEARSON (1928) showed that the χ^2 test is approximately equal to the likelihood ratio test (4); WALD (1943) demonstrated the asymptotic optimality of the likelihood ratio test (3). (See also Section 6.3.)

Before a method for obtaining a test can be applied, the practical problem must be translated into a testing problem $((X,F), P_H, P_A)$. The determination of the class P_A of probability distributions, for which one likes to have a high probability of rejecting the null hypothesis, is often more or less a matter of taste. Sometimes a researcher devotes no attention to the formulation of the alternative hypothesis against which his null hypothesis is to be tested, and he uses a test which is good for testing against an unrestricted alternative. MOLENAAR (1978) is an example of a paper which criticizes the use of a standard analysis of variance F-test in an experimental psychological study, where a test against a restricted alternative might be more appropiate. In many other cases, the researcher or statistician selects a subalternative A_0 such that the corresponding class P_{A_0} is a lower-dimensional subclass of the (sometimes vaguely described) class P_A against which a high rejection probability is desirable; and he tries to construct a test which is good for testing H against A_0. See NEYMAN (1969) Section 3 and MOLENAAR (1978) for examples of this.

The adoption of an unrestricted alternative in cases where a restricted alternative might be more appropiate, or of a lower-dimensional subalternative A_0, can be caused by, e.g.,

(i) the difficulty of precisely defining a more satisfactory alternative hypothesis A;

(ii) the difficulty of obtaining a test which is good for testing against A;

(iii) the availability of tests, or of methods for constructing a test, for testing H against the unrestricted alternative or against the subalternative A_0.

NEYMAN (1969) makes an appeal to the experimenter's intuition in order to justify the lower-dimensional subalternative against which he constructs an optimal test. Although the experimenter's intuition can be very valuable in model building, it sometimes can be misleading: the experimenter may adopt a particular subalternative without fully grasping the statistical consequences. Nevertheless it may be sensible to propose a test which is good for testing against an "important" subalternative A_0, provided that one makes sure that this test is "not too bad" for probability distributions belonging to $\mathcal{P}_A \setminus \mathcal{P}_{A_0}$.

It seems to be preferable, although often difficult to achieve, to determine an alternative hypothesis A which corresponds to the class of all probability distributions against which a high power is desirable, and to use a test with "satisfactory power properties against A". In the sequel we shall study the testing problem of Section 3.1, assuming that the alternative hypothesis A has been determined in this way.

3.3. SOME TESTING PROBLEMS WITH RESTRICTED ALTERNATIVES FOR NORMAL DISTRIBUTIONS

Testing problems for normal distributions are of interest not only because they arise in practice, but also because they can play the role of "limiting testing problems" for (sequences of) testing problems involving other than normal distributions; see the following chapters of this study. In this section, a few results and references to the literature will be given concerning a number of testing problems for normal distributions with restricted alternatives.

1. *Positive quadrant.* The testing problem

$$\mathcal{L}_\theta(X_1, X_2) = \mathcal{N}_2(\theta, I)$$
$$H : \theta = 0, \; H \vee A : \theta_1 \geq 0, \; \theta_2 \geq 0$$

arises in the problem of combining two independent test statistics. VAN ZWET and OOSTERHOFF (1967) prove that for certain values of the level of significance α, the test which rejects for

$$\exp(r_\alpha X_1) + \exp(r_\alpha X_2) \geq d_\alpha ,$$

for certain r_α and d_α , is MS-level α. They have strong evidence suggesting that this test is MS - level α for all $\alpha \geq \alpha_0 \approx .043$, and prove that this is indeed the case for $\alpha = .10$ and $\alpha = .05$, with

$$r_{.10} = 1.635 \qquad d_{.10} = 16.52$$

$$r_{.05} = 1.900 \qquad d_{.05} = 44.47 .$$

They also study the likelihood ratio test (test statistic $(\max(X_1, 0))^2 + (\max(X_2, 0))^2$), Fisher's combination procedure (test statistic $- 2 \log Q_1 - 2 \log Q_2$, where Q_i is the tail probability under H of X_i) and the linear test (test statistic $X_1 + X_2$). The likelihood ratio test and Fisher's combination procedure appear to have very good power properties, both for $\alpha = .05$ and asymptotically for $\alpha \to 0$. See also OOSTERHOFF (1969), which includes a review of many other combination procedures for independent test statistics.

2. *Convex cone in* $\mathbb{R}^2$. The testing problem

$$L_\theta(X_1, X_2) = N_2(\theta, \Sigma)$$

$$H : \theta = 0, H \vee A : \theta_1 \geq 0, \theta_2 \geq 0,$$

with a known covariance matrix Σ, can be transformed into the problem

$$L_\theta(X_1, X_2) = N_2(\theta, I)$$

$$H : \theta = 0, H \vee A : |\theta_1| \leq \theta_2 \operatorname{tg} \omega ,$$

where ω is an angle with $0 < \omega < \frac{1}{2}\pi$. For $\omega = \frac{1}{4}\pi$, the testing problem discussed above ("positive quadrant") is obtained. SCHAAFSMA (1968) demonstrates, that critical angles $\omega_{cr}(\alpha)$ exist such that for $0 < \omega < \omega_{cr}(\alpha)$, the MS - level α test is the test $\phi_{1\alpha}$ which rejects for

$$X_2 \geq (b \cos \omega)^{-1} \{c - \log (\exp(bX_1 \sin \omega) + \exp(-bX_1 \sin \omega))\} ,$$

where $b = b_\alpha(\omega)$ and $c = c_\alpha(\omega)$ are given by the following figures.

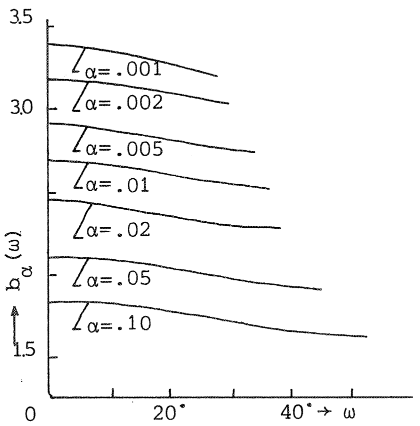

Fig. 1.

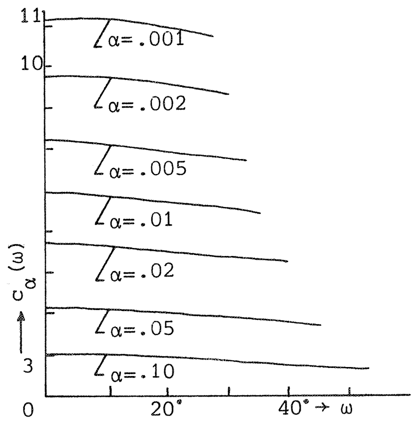

Fig. 2.

The MS - somewhere most powerful level α test (see (4) below) $\phi_{2\alpha}$ rejects for

$$X_2 \geq u_\alpha \ .$$

Power comparisons in SCHAAFSMA (1968) show that if α = .05 or .01 and $\omega \leq \pi/3$ then $\phi_{2\alpha}$ is a good competitor of $\phi_{1\alpha}$, from an over-all power point of view. The test $\phi_{2\alpha}$ has the advantage of greater computational simplicity. Computations of VAN ZWET and OOSTERHOFF (1967) show, however, that for the case $\omega = \pi/4$ the likelihood ratio test and Fisher's test seem to be more attractive than both $\phi_{1\alpha}$ and $\phi_{2\alpha}$. SCHAAFSMA (1971) considers this testing problem and other testing problems for two independent normally distributed random variables with known variances, and reviews many tests which can be used for these problems.

3. *Positive orthant.* The testing problem for a random sample $X_1, \ldots, X_n$ from $N_m(\theta, \Sigma)$ with

$$H : \theta = 0, \ H \vee A : \theta_i \geq 0 \qquad (1 \leq i \leq m),$$

has been studied by several authors in three cases: for known covariance matrix Σ, for $\Sigma = \sigma^2\Lambda$ with σ^2 unknown and Λ known, and for completely unknown Σ. Many results and references can be found in BARLOW, BARTHOLOMEW, BREMNER and BRUNK (1972). The testing problem (2) discussed above is a special case of this problem, namely with m = 2 and known Σ.

For $m = 3,4$, $\alpha = .1$ and $\Sigma = I$, the MS -level α test was obtained by SCHAAFSMA (1968) and VAN LINDE, SCHAAFSMA and VELVIS (1967) (m = 3, 4) and by OOSTERHOFF (1969) (m = 3). These tests reject, respectively, for

$$\sum_{i=1}^{3} \exp\,(1.706\; n^{-\frac{1}{2}} \sum_{j=1}^{n} X_{ij}) \geq 27.36$$

$$\sum_{i=1}^{4} \exp\,(1.755\; n^{-\frac{1}{2}} \sum_{j=1}^{n} X_{ij}) \geq 39.05\ .$$

KUDÔ (1963) derived for the case of known Σ the probability distribution under H of the (transformed) likelihood ratio statistic

$$n\; \hat{\theta}'\; \Sigma^{-1}\; \hat{\theta}\ ,$$

where $\hat{\theta}$ is the maximum likelihood estimator of θ under H $\vee$ A. It can be shown that $\hat{\theta}$ is the projection of $n^{-1}\Sigma_i X_i$ on the convex cone $\{\theta \in \mathbb{R}^m \,|\, \theta_i \geq 0$ for all i$\}$ with respect to the inner product $(x, y) = x'\,\Sigma^{-1}\,y$. In most cases this projection is rather difficult to obtain. The probability distribution of the test statistic under H is a weighted average of χ^2 distributions; for $m \geq 4$ the weights must be determined numerically.

The likelihood ratio test for the case that $\Sigma = \sigma^2\Lambda$ with known Λ was also studied by KUDÔ (1963). PERLMAN (1969) derived the likelihood ratio test for the case that Σ is completely unknown. BARLOW et al. (1972) contains many details and further references . For another approach to these testing problems, see (4) below.

4. *Polyhedral cone*. (A cone is polyhedral if it is the intersection of finitely many half-spaces, each of which has the origin as a boundary point.) The testing problem of (3) with Σ known and $\Sigma = \sigma^2\Lambda$ (Λ known) can be transformed into special cases of the testing problem

$$X_1, X_2, \ldots, X_n \text{ independent}$$

$$L_{\theta,\eta}(X_i) = \begin{cases} N(0, \sigma^2) & 1 \leq i \leq n-s \\ N(\theta_{i-n+s}, \sigma^2) & n-s+1 \leq i \leq n-s+r \\ N(\eta_{i-n+s-r}, \sigma^2) & n-s+r+1 \leq i \leq n \end{cases}$$

$$H : \theta = 0$$

$$H \vee A : d_j'\theta \geq 0 \qquad (1 \leq j \leq t) ;$$

where $\theta = (\theta_1, \ldots, \theta_r)'$, $\eta = (\eta_1, \ldots, \eta_{s-r})'$, $r \leq s \leq n$, and $d_1, \ldots, d_t$ are vectors in $\mathbb{R}^r$ which span $\mathbb{R}^r$. In many applications one has $t = r$. The vector $(X_1, \ldots, X_n)'$ will be partitioned into

$$X^{(1)} = (X_1, \ldots, X_{n-s})' , \; X^{(2)} = (X_{n-s+1}, \ldots, X_{n-s+r})' ,$$

$$X^{(3)} = (X_{n-s+r+1}, \ldots, X_n)' .$$

This testing problem can be considered either with known or with unknown σ^2.

In this general formulation it was studied by SCHAAFSMA (1966) and by SCHAAFSMA and SMID (1966). In the case $\sigma^2 = 1$ they restricted attention to the class of somewhere most powerful - level α , or SMP level α, tests. A test is SMP level α if it is MP - level α for testing H against some simple subalternative of A. The Neyman-Pearson Fundamental Lemma can be used to show that for testing H against the simple alternative that $\theta = \theta_0$, $\eta = \eta_0$, the MP - level α test rejects H for

$$\theta_0' X^{(2)} \geq u_\alpha \|\theta_0\| .$$

Let the vector $a \in \mathbb{R}^r$ of length 1 correspond to the half-line $\{pa \mid p > 0\}$, and for two such vectors a and b let $\omega(a, b) = \arccos a'b$ be the smaller angle between a and b. It is proved in SCHAAFSMA (1966) and SCHAAFSMA and SMID (1966) that if $a_0 \in K$ satisfies

$$\sup_{b \in K} \omega(a_0, b) = \inf_{a \in K} \sup_{b \in K} \omega(a, b) < \tfrac{1}{2}\pi$$

where

$$K = \{x \in \mathbb{R}^r \mid d_j' x \geq 0 \text{ for } 1 \leq j \leq t; \|x\| = 1\} ,$$

then the test which rejects for

$$a_0' X^{(2)} \geq u_\alpha$$

is most stringent - SMP level α. The half-line a_0 can be determined by applying a method of ABELSON and TUKEY (1963). These authors introduced the concept of a "maximin r^2 linear contrast", which is closely related to the concept of a MS - SMP test. Their method can be summarized as follows. Let $e_1, \ldots, e_q$ be the elements of K corresponding to the q edges of the cone spanned by K. (If t = r, then q = r and the edges are determined by $d_j' e_j > 0$, $d_j' e_h = 0$ for $j \neq h$.) If t = r and an $a_0 \in K$ exists with

$$\omega(a_0, e_1) = \omega(a_0, e_2) = \ldots = \omega(a_0, e_r) ,$$

then this is the desired half-line. If t > r or if such an a_0 does not exist then there exists a rearrangement of the q edges, a number $q' \leq q$ and a vector $a_0 \in K$ spanned by $e_1, \ldots, e_{q'}$ with

$$\omega(a_0, e_1) = \omega(a_0, e_2) = \ldots = \omega(a_0, e_{q'}) \geq \omega(a_0, e_j) \quad (q' < j \leq q)$$

and this a_0 is the desired half-line. The minimax angle is

$$\omega_0 = \inf_{a \in K} \sup_{b \in K} \omega(a, b) = \sup_{b \in K} \omega(a_0, b)$$

(to be denoted by ω elsewhere). SCHAAFSMA (1966) contains many examples where a_0 has been determined explicitly.

In the case where σ^2 is unknown, SCHAAFSMA (1968) and SCHAAFSMA and SMID (1966) restrict attention to the class of somewhere most powerful-similar size α tests. (The test ϕ is said to be similar size α if $E_P \phi = \alpha$ for all $P \in \mathcal{P}_H$.) They prove that the test which rejects for

$$a_0' X^{(2)} \geq t_{n-s+r-1;\alpha} \; (n-s+r-1)^{-\frac{1}{2}} \left[\sum_{i=1}^{n-s+r} X_i^2 - (a_0' X^{(2)})^2 \right]^{\frac{1}{2}}$$

is most stringent - SMP similar size α, if a_0 is determined as above, and provided that $\omega_0 < \omega_0^*$ for a certain upper bound ω_0^* depending on α and n-s+r.

It is an advantage of these tests that they can be carried out easily. The maximum shortcoming of the MS - SMP level α test (and of the MS - SMP similar size α test) on the half-line $\{p\theta \mid p > 0\}$, for $\theta \in K$, is an increasing function of $\omega(a_0, \theta)$. For small values of ω_0 the MS - SMP test is very satisfactory; for large values of ω_0 the shortcoming on the half-line corresponding to a_0 is 0, but the maximum shortcoming on the edges $e_1, \ldots, e_q$ is high, especially for small values of α. Hence the MS - SMP test is unsatisfactory for large values of ω_0. Power calculations, nearly always restricted to $r = 2$, of SCHAAFSMA (1966) Section 2.13, VAN ZWET and OOSTERHOFF (1967) and SCHAAFSMA (1968) point to the conclusion that for $\omega_0 \leq \pi/6$ the MS - SMP test has very good power properties; for $\pi/6 < \omega_0 \leq \pi/3$ its power properties are reasonable (very good in "the middle" of the alternative, rather poor "near the boundary" of the alternative); for $\pi/3 < \omega_0 < \pi/2$ its power properties are rather poor, and its main qualities are the high power in "the middle" of the alternative, and the fact that it is an easy test for a (for $r \geq 3$) very complicated testing problem. Note that as r increases, it becomes intrinsically more difficult to obtain a satisfactory test; e.g., the minimax - level α shortcoming tends to $1 - \alpha$ (provided that some conditions on the d_j are satisfied).

It seems worthwile, at least for $\omega_0 \geq \pi/4$, to develop tests which have better over-all power properties than the MS - SMP test, and which are not much more difficult to carry out. For some work in this direction, see (5) and (6) below, and AKKERBOOM and STEERNEMAN (1979).

5. *Circular cone*. PINCUS (1975) studied the testing problem where the experiment is as in (4) above, but null hypothesis and alternative are given by

$$H : \theta = 0 \quad , \quad H \vee A : \theta' a \geq \|\theta\| \cos \omega ,$$

where a is a vector in $\mathbb{R}^r$ of unit length, and ω an angle with $0 \leq \omega \leq \pi/2$. See also HUMAK (1977). PINCUS (1975) derives the likelihood ratio test for this testing problem. He suggests that for testing against a polyhedral cone (as in 4 above), the likelihood ratio test against a suitable circular cone might be used. This test is considerably easier to carry out than the likelihood ratio test against the polyhedral cone alternative. Pincus's procedure may lead to tests which are preferable to the MS-SMP test, especially for large ω. Hans Akkerboom is working on the production of tables for the null distribution of the likelihood ratio statistic.

6. *Trend.* A very interesting special case of the testing problem with a polyhedral cone alternative is the k-sample trend problem where $X_{i1}, \ldots, X_{in_i}$ $(1 \leq i \leq k)$ are independent random variables, $\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}^k$ and

$$L_\theta\ (X_{ij}) = N(\theta_i, \sigma^2) \qquad 1 \leq j \leq n_i \ ; \ 1 \leq i \leq k$$

$$H : \theta_1 = \theta_2 = \ldots = \theta_k$$

$$H \vee A : \theta_1 \leq \theta_2 \leq \ldots \leq \theta_k \ .$$

This testing problem can be considered either with known σ^2 or with unknown σ^2. It was studied by many authors, among whom are Van Eeden, Bartholomew, Chacko, Shorack (focusing on the likelihood ratio test); Abelson and Tukey, Schaafsma and Smid (focusing on "best linear tests"). See BARLOW et al. (1972) for an extensive review of this and related testing problems, and for further references.

3.4. SOME TESTING PROBLEMS WITH RESTRICTED ALTERNATIVES FOR NON-NORMAL DISTRIBUTIONS

For many testing problems with restricted alternatives for non-normal distributions, it is rather difficult, or laborious, to produce satisfactory tests wich are exactly of level α. In practice, one is often satisfied when a test is "approximately" of level α. For these testing problems it is often particularly difficult to produce tests satisfying some exact optimum property. Many authors follow an asymptotic approach, relating these testing problems to testing problems for normal distributions. This will also be done in the present study.

Many authors studied likelihood - ratio tests. CHERNOFF (1954) derived a general result about the asymptotic distribution under the null hypothesis of the likelihood ratio statistic. CHACKO (1966) studied the likelihood ratio ratio test for the testing problem

$$L_p(X) = M_m(n; (p_1, \ldots, p_m))$$

$$H : p_1 = p_2 = \ldots = p_m = m^{-1}$$

$$H \vee A : p_1 \leq p_2 \leq \ldots \leq p_m \ .$$

His results were extended by ROBERTSON (1978) to more general restricted alternatives. In Section 3.2 (3), LEE's (1977) treatment of this problem was discussed. BOSWELL and BRUNK (1969) studied the likelihood ratio test for testing homogeneity against an upward trend, for k independent random samples from a one-dimensional exponential family: if the value of the parameter for the i'th sample is θ_i, then null hypothesis and alternative hypothesis are given by

$$H : \theta_1 = \theta_2 = \dots = \theta_k$$

$$H \vee A : \theta_1 \leq \theta_2 \dots \leq \theta_k \ .$$

The reader may also consult Section 4.3 of BARLOW et al (1972).

Other tests have also been proposed; much attention has been devoted to testing problems where the data can be summarized in a two-way contingency table. VAN EEDEN and HEMELRIJK (1955) studied the problem of testing homogeneity against an upward trend for success probabilities. Their testing problem is given by

$$N_1, N_2, \dots, N_k \text{ independent; } \mathcal{L}_{p_i}(N_i) = \mathcal{B}(n_i, p_i)$$

$$H : p_1 = \dots = p_k \quad , A : \sum_{i<j} (p_j - p_i) > 0 \ .$$

They restricted attention to "linear" test statistics and proposed to use a design-free test: a test where the region of consistency does not depend on the asymptotic ratios of the sample sizes. However, it seems that a more relevant formulation of the alternative hypothesis of an upward trend is

$$A : p_1 \leq p_2 \leq \dots \leq p_k \quad , p_1 < p_k \quad .$$

In our approach, the "power" of the test for parameter values satisfying neither H nor A is not taken into consideration; so we do not advocate design-free tests.

Another testing problem which arises rather frequently in practice is the two-sample problem for discrete random variables, obtained by taking k = 2 and alternative hypothesis A_1 in Example 3.1.1. After a reduction by sufficiency, this problem is given by

$$N_i = (N_{i1}, \ldots, N_{im})' , \quad i = 1,2; \quad N_1 \text{ and } N_2 \text{ independent}$$

$$p_i = (p_{i1}, \ldots, p_{im})' , \quad i = 1,2; \quad p_{ih} > 0, \ \sum_h p_{ih} = 1$$

$$\mathcal{L}_{(p_1,p_2)} (N_i) = \mathcal{M}(n_i, p_i) \quad i = 1,2$$

$$H : p_1 = p_2$$

$$H \vee A : \sum_{h=1}^{g} p_{1h} \leq \sum_{h=1}^{g} p_{2h} \quad 1 \leq g \leq m .$$

Three approaches to this testing problem will be mentioned.

(i) Do not bother about the one-sided formulation of the alternative hypothesis. Use the χ^2 test for the 2 x m table, which is a good test against the unrestricted alternative.

(ii) Assign increasing scores $a_1, \ldots, a_m$ to the m ordered categories (e.g. $a_h = h$). The scores should not depend on the outcomes of N_1 and N_2. Apply Student's test; or a test based on the difference between the sample means

$$T_a = \sum_{h=1}^{m} a_h (N_{1h} / n_1 - N_{2h} / n_2)$$

such that the test is of size α, conditionally given the marginal totals $(N_{+1}, \ldots, N_{+m})$ where $N_{+h} = N_{1h} + N_{2h}$. Note that, under H,

$$E \{T_a \mid N_{+1}, \ldots, N_{+m}\} = 0$$

$$\text{var} \{T_a \mid N_{+1}, \ldots, N_{+m}\} = n_+\{n_1 n_2(n_+ - 1)\}^{-1} \sum_{h=1}^{m} N_{+h}(a_h - M^{(a)})^2$$

where $n_+ = n_1 + n_2$ and

$$M^{(a)} = n_+^{-1} \sum_{h=1}^{m} a_h N_{+h} .$$

These formulas and the conditional asymptotic normality under H of T_a are derived, e.g.,in the appendix of LEHMANN (1975). An asymptotic approximation for the critical value of the conditional test based on T_a yields the test which rejects for

$$[n_+ \{n_1 n_2 (n_+ - 1)\}^{-1} \sum_{h=1}^{m} N_{+h}(a_h - M^{(a)})^2]^{-\frac{1}{2}} T_a > u_\alpha .$$

(iii) Assign increasing scores $a_1, \ldots, a_m$ to the categories such that the

a_h depend only on the marginal totals $(N_{+1}, \ldots, N_{+m})$. Apply the conditional test given $(N_{+1}, \ldots, N_{+m})$. This gives rise to the same formulas as in (ii). The most frequently used scores are the mid-ranks $R_1, \ldots, R_m$

$$R_h = \sum_{g=1}^{h-1} N_{+g} + \frac{1}{2}(N_{+h} + 1) .$$

The conditional test with mid-rank scores is usually regarded as the version of the Wilcoxon-Mann-Whitney test with the mid-rank treatment of ties. SEN (1967) showed that this test is asymptotically optimal if N_1 and N_2 count the number of outcomes in the two samples, falling in m fixed intervals, if the underlying random variables have shifted logistic distributions. In most applications this very particular probabilistic model cannot be supported by arguments, and mid-rank scores are not necessarily preferable to other scores. The frequent use of mid-rank scores seems to spring mainly from tradition and computational facility (the conditional variance of T_a under H can be expressed simply).

SCHAAFSMA (1966) proposed the scores $W_1, \ldots, W_m$

$$W_1 = 0,\ W_{h+1} = W_h + (n_+ / N_{+h} + n_+ / N_{+,h+1})^{\frac{1}{2}}$$

and stated that the corresponding test is "approximately" most stringent - somewhere most powerful level α . This statement, however, was not supported by a precisely formulated limit theorem. The scores $W_1, \ldots, W_m$ can be regarded as an attempt to provide an "approximately optimal treatment of ties" for Wilcoxon's test.

For many other "one-sided" testing problem for contingency tables, a similar picture can be sketched. More testing problems with restricted alternatives for non-normal distributions can be found in Section 4.4 of BARLOW et al. (1972).

3.5. AN ASYMPTOTIC APPROACH TO THE TESTING PROBLEM OF SECTION 3.1

For many problems of the kind of Section 3.1, such as the two-sample problem for discrete random variables discussed in the preceding section for $m \geq 3$, there does not exist a UMP - level α or UMP-unbiased level α test, while the construction of a MS - level α test seems to be very difficult. Therefore an asymptotic theory will be developed, relating the problems of Section 3.1 to testing problems for normal distributions with known covariance matrices. The latter kind of testing problem yields itself somewhat more easily to the construction of optimal tests.

The testing problem of Section 3.1 will be "embedded" in a sequence of similar testing problems, indexed by the variable ν. The sample sizes will tend to infinity. In order to define this sequence $\{T_\nu\}$ of testing problems let k, Θ_i, Θ, λ_i, $\{P_{i\theta_i} \mid \theta_i \in \Theta_i\}$, θ, μ, F, f, V and K be as in Section 3.1, not depending on ν. The experiment for testing problem T_ν is constituted by k independent random samples

$$X_{i1}^{(\nu)}, \ldots, X_{in_i(\nu)}^{(\nu)}$$

with

$$L_\theta(X_{ij}^{(\nu)}) = P_{i\theta_i} \qquad 1 \le j \le n_i(\nu) \ , \ 1 \le i \le k \ .$$

Null hypothesis and alternative hypothesis for T_ν are given by

$$H : f(\mu) \in V$$

$$H \vee A : f(\mu) \in V + K \ ,$$

independently of ν. The sequence $\{T_\nu\}$ will be called an "asymptotic testing problem". It will be assumed that the level of significance α is independent of ν, with $0 < \alpha < 1$ (see the discussion later in this section); and that

$$\lim_\nu n_i(\nu) = \infty \qquad 1 \le i \le k$$

$$\liminf_\nu n_i(\nu) \,/\, n_j(\nu) > 0 \qquad 1 \le i, j \le k \ .$$

Some notation will be introduced for use in the following chapters. Define

$$\Theta_H = \{\theta \mid f(\mu(\theta)) \in V\} \qquad \Theta_A = \{\theta \mid f(\mu(\theta)) \in (V + K) \setminus V\} \ ,$$

$$n(\nu) = \sum_{i=1}^{k} n_i(\nu) \ , \qquad \rho_i(\nu) = n_i(\nu) \,/\, n(\nu) \ .$$

The assumptions concerning the sample sizes are equivalent to

$$\lim_{\nu} n(\nu) = \infty \qquad \liminf_{\nu} \rho_i(\nu) > 0 \qquad 1 \leq i \leq k .$$

Let R_ν be the m x m diagonal matrix with, successively, m_1 entries $\rho_1(\nu)$, m_2 entries $\rho_2(\nu)$, ..., m_k entries $\rho_k(\nu)$. This matrix will be called the proportion matrix.

The vector of sample means

$$X_{\cdot}^{(\nu)} = \begin{pmatrix} X_{1.}^{(\nu)} \\ \cdot \\ \cdot \\ \cdot \\ X_{k.}^{(\nu)} \end{pmatrix} ,$$

where

$$X_{i.}^{(\nu)} = [n_i(\nu)]^{-1} \sum_{j=1}^{n_i(\nu)} X_{ij}^{(\nu)} ,$$

is a sufficient statistic. Its expectation and covariance matrix are

$$E_\mu X_{\cdot}^{(\nu)} = \mu$$

$$\operatorname{cov}_\mu X_{\cdot}^{(\nu)} = \begin{pmatrix} [n_1(\nu)]^{-1} \Sigma_{1\mu_1} & & O \\ & \ddots & \\ O & & [n_k(\nu)]^{-1} \Sigma_{k\mu_k} \end{pmatrix}$$

$$= [n(\nu)]^{-1} R_\nu^{-1} \Sigma_\mu ,$$

where

$$\Sigma_{i\mu_i} = \operatorname{cov}_{\mu_i} X_{i1}^{(\nu)}$$

$$\Sigma_\mu = \begin{pmatrix} \Sigma_{1\mu_1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \Sigma_{k\mu_k} \end{pmatrix} .$$

Note that Σ_μ and R_ν commute: $\Sigma_\mu R_\nu^{-1} = R_\nu^{-1} \Sigma_\mu = R_\nu^{-\frac{1}{2}} \Sigma_\mu R_\nu^{-\frac{1}{2}}$. Define

$$Y_\nu = \begin{cases} [n(\nu)]^{\frac{1}{2}} f(X_{\cdot}^{(\nu)}) & \text{if } X_{\cdot}^{(\nu)} \in F \\ 0 & \text{if } X_{\cdot}^{(\nu)} \notin F . \end{cases}$$

It is possible that $P_\mu \{X_{\cdot}^{(\nu)} \notin F\} > 0$ for all μ (consider the binomial distribution). In later chapters, we shall only study sequences $\{\mu_\nu\}$ which are relatively compact in F (see Section 5.3). For such sequences one has $P_{\mu_\nu}\{X_{\cdot}^{(\nu)} \notin F\} \to 0$. Therefore the possibility that $X_{\cdot}^{(\nu)} \notin F$ may be ignored. In the sequel we shall proceed as if Y_ν is always defined by

$$Y_\nu = [n(\nu)]^{\frac{1}{2}} f(X_{\cdot}^{(\nu)}) .$$

Since f is one-to one, Y_ν is a sufficient statistic just like $X_{\cdot}^{(\nu)}$. Let

$$D_\mu = (\partial f / \partial \mu)$$

be the matrix of first-order partial derivatives of f (coordinates of f in the rows, coordinates of μ in the columns). Recall that D_μ is assumed to be non-singular for all $\mu \in F$ (Section 3.1). The matrix

$$\Lambda_{\nu\mu} = D_\mu R_\nu^{-1} \Sigma_\mu D_\mu'$$

is the asymptotic covariance matrix of Y_ν : if $R_\nu \to R$, then

$$\Lambda_{\nu\mu} \to \Lambda_\mu = D_\mu R^{-1} \Sigma_\mu D_\mu'$$

and

$$\mathcal{L}_\mu (Y_\nu - [n(\nu)]^{\frac{1}{2}} f(\mu)) \to \mathcal{N}(0, \Lambda_\mu)$$

(see, e.g., RAO (1973) Section 6a.2). Sometimes the notation

$$\Lambda(R, \mu) = D_\mu R^{-1} \Sigma_\mu D'_\mu$$

will be used; note that $\Lambda_{\nu\mu} = \Lambda(R_\nu, \mu)$.

A test $\{\phi_\nu\}$ is a sequence of tests ϕ_ν for T_ν. Unless stated otherwise, it will be assumed that for every ν, ϕ_ν is a measurable function

$$\phi_\nu : \mathbb{R}^m \to [0,1] ,$$

which is understood to be applied to Y_ν. Thus the notation

$$E_\mu \phi_\nu = E_\mu \phi_\nu (Y_\nu)$$

will be used without causing confusion. For $\{\mu_\nu\}$ with $f(\mu_\nu) \in (V + K) \setminus V$ for every ν, $\{E_{\mu_\nu} \phi_\nu\}$ will be called the power sequence of $\{\phi_\nu\}$.

There exist extensive studies about asymptotic testing problems where the level of significance α_ν depends on ν and approaches 0. Important pioneering work in this field has been done by CHERNOFF and by BAHADUR; see e.g., the survey paper of GROENEBOOM and OOSTERHOFF (1977). KALLENBERG (1978) proves that for many testing problems for exponential families with an unrestricted alternative (i.e., $\Theta_H \cup \Theta_A$ is the natural parameter space) the shortcoming of the likelihood ratio test tends to zero uniformly on compact subsets of Θ_A, sometimes even uniformly on Θ_A, when $\alpha_\nu \to 0$. For these testing problems, the likelihood ratio test can be said to be "asymptotically UMP" for $\alpha_\nu \to 0$; this seems to lessen the need for considering a restricted alternative on which to "concentrate" the power of a test. (The only advantage of considering a restricted alternative might lie in the possibility that the shortcoming should approach 0 at a faster rate.) Two objections will be made against theories where $\alpha_\nu \to 0$. In the first place, when the sample sizes are so large that extremely small significance levels could be considered, then testing theory loses much of its relevance: instead of testing a null hypothesis, it will often be more relevant to construct confidence regions for certain parameters, or to try and construct more refined models. In the second place, the approximations provided by this asymptotic approach seem to be close, in many cases, only for values of α which are so small as hardly to arise in practice. This is demonstrated for a very simple case by the example below. So it seems that besides the asymptotic theory for $\alpha_\nu \to 0$, an asymptotic theory for fixed α remains relevant.

<u>EXAMPLE 3.5.1</u>. *Testing against a one-sided or two-sided alternative.* Consider the testing problems for a random variable X having probability distribution $N(\mu, 1)$, with null hypothesis and alternative hypotheses

$$H : \mu = 0 \ , \quad A_1 : \mu \neq 0 \ , \quad A_2 : \mu > 0 \ .$$

The likelihood ratio test $\phi_{(\alpha)}$ against A_1 rejects for $|X| \geq u_{\frac{1}{2}\alpha}$. This test is also most stringent - level α, UMP - unbiased level α and UMP - invariant level α for testing against A_1. Its shortcoming with respect to the class of all level α tests is, for $\mu > 0$,

$$\gamma_\alpha \ (\phi_{(\alpha)}, \mu) = \Phi(u_{\frac{1}{2}\alpha} - \mu) - \Phi(u_\alpha - \mu) - \Phi(-u_{\frac{1}{2}\alpha} - \mu),$$

where Φ is the distribution function of the standard normal distribution. Its maximum shortcoming satisfies

$$\gamma^*_\alpha \ (\phi_{(\alpha)}) = \sup_{\mu>0} \gamma_\alpha \ (\phi_{(\alpha)}, \mu) \leq \sup_{\mu>0} \ [\Phi(u_{\frac{1}{2}\alpha} - \mu) - \Phi(u_\alpha - \mu)].$$

The supremum is reached for $\mu = \frac{1}{2}(u_\alpha + u_{\frac{1}{2}\alpha})$. Hence

$$\Phi(\tfrac{1}{2}(u_{\frac{1}{2}\alpha} - u_\alpha)) - \Phi(\tfrac{1}{2}(u_\alpha - u_{\frac{1}{2}\alpha})) - \Phi(-\tfrac{1}{2}(3u_{\frac{1}{2}\alpha} + u_\alpha)) \leq$$

$$\leq \gamma^*_\alpha \ (\phi_{(\alpha)}) \leq \Phi(\tfrac{1}{2}(u_{\frac{1}{2}\alpha} - u_\alpha)) - \Phi(\tfrac{1}{2}(u_\alpha - u_{\frac{1}{2}\alpha})) \leq$$

$$\leq (2\pi)^{-\frac{1}{2}} \ (u_{\frac{1}{2}\alpha} - u_\alpha).$$

It follows from $\lim_{\alpha\to 0} (u_{\frac{1}{2}\alpha} - u_\alpha) = 0$ that $\lim_{\alpha\to 0} \gamma^*_\alpha \ (\phi_{(\alpha)}) = 0$. This is in accordance with the general results of KALLENBERG (1978) mentioned above. In particular, the test $\phi_{(\alpha)}$ is, for $\alpha \to 0$, "asymptotically UMP" for testing against A_2. The upper and lower bounds for $\gamma^*_\alpha \ (\phi_{(\alpha)})$ are hardly different and yield the following table.

α	.05	.01	.001
$\gamma^*_\alpha(\phi_{(\alpha)})$	.13	.10	.08

It is seen that, even for $\alpha = .001$, the maximum shortcoming of $\phi_{(\alpha)}$ is not negligible. □

CHAPTER 4

ASYMPTOTIC TESTING PROBLEMS WITH A SEQUENCE OF SIMPLE NULL HYPOTHESES

In Section 4.1 the well-known concept of contiguity is introduced. In the other sections of this chapter, some techniques are developed for use in later chapters; these sections can be skipped by readers who are not interested in the proofs of the results obtained in this study. (Example 4.3.2 may be interesting to them, however.) Of central importance in Sections 4.2 to 4.4 is the concept of the limit of a sequence of testing problems with simple null hypotheses and a fixed outcome space (Definition 4.2.2). Technical details of some proofs in this chapter have been relegated to Appendices A.2 to A.4.

4.1. CONTIGUITY

Any reasonable test for the problem of Section 3.5 will satisfy $\lim_\nu E_\mu \phi_\nu = 1$ for every parameter value μ satisfying the alternative hypothesis. This shows that in order to distinguish between "reasonable" tests, it is not sufficient to consider the limiting power for fixed parameter values. One will have to study the rate of convergence of the power, or to consider sequences of parameter values. We shall follow the latter approach.

Especially important are sequences $\{\mu_\nu\}$ which "approach the null hypothesis fast enough but not too fast" in order that the envelope power at μ_ν, with respect to the class of all level α tests, is bounded away from 1 and from α. The concept of contiguity is useful for the study of such sequences. It was introduced by LE CAM (1960), who generalized earlier work such as NEYMAN's (1937) consideration of parameter sequences approaching the null hypothesis at a speed of order $n^{-\frac{1}{2}}$, n being the sample size. See also, e.g., HAJEK and SIDAK (1967), LE CAM (1969), WITTING and NÖLLE (1970), ROUSSAS (1972), HALL and LOYNES (1977). In this section and in Appendix A.2 some well-known results about contiguity are presented.

DEFINITION 4.1.1. For every $\nu \in \mathbb{N}$, let P_ν and Q_ν be probability measures on the measurable space (X_ν, F_ν). $\{Q_\nu\}$ is contiguous to $\{P_\nu\}$, denoted by $\{Q_\nu\} \triangleleft \{P_\nu\}$, if for every sequence $\{B_\nu\}$ of measurable sets $B_\nu \in F_\nu$,

$$P_\nu(B_\nu) \to 0 \text{ implies that } Q_\nu(B_\nu) \to 0 .$$

$\{Q_\nu\}$ and $\{P_\nu\}$ are mutually contiguous, denoted by $\{Q_\nu\} \triangleleft\triangleright \{P_\nu\}$, if $\{Q_\nu\} \triangleleft \{P_\nu\}$ and $\{P_\nu\} \triangleleft \{Q_\nu\}$.

The following useful proposition gives alternative characterizations by means of test functions.

PROPOSITION 4.1.1. *For every ν, let P_ν and Q_ν be probability measures on (X_ν, F_ν) and let ϕ_ν be a test function $\phi_\nu : X_\nu \to [0,1]$. The following three statements are equivalent.*

(i) $\{Q_\nu\} \triangleleft \{P_\nu\}$

(ii) $\phi_\nu \to 0$ in $\{P_\nu\}$ - prob. implies that $\phi_\nu \to 0$ in $\{Q_\nu\}$ - prob.

(iii) $E_{P_\nu} \phi_\nu \to 0$ implies that $E_{Q_\nu} \phi_\nu \to 0$.

PROOF. Immediate when taking $B_\nu = \{x \mid \phi_\nu(x) \geq \varepsilon\}$. □

Characterization (iii) can be interpreted by regarding ϕ_ν as a test for the null hypothesis P_ν against the alternative hypothesis Q_ν : if the level of significance tends to 0, then the power (against a contiguous sequence of alternatives) also tends to 0.

Our asymptotic testing problems will usually be such that for any pair $\{P_\nu\}$, $\{Q_\nu\}$ of sequences of probability measures, either $\{P_\nu\} \triangleleft\triangleright \{Q_\nu\}$ holds, or a sequence $\{B_\nu\}$ of measurable sets exists and a subsequence $\{\xi\}$ of $\{\nu\}$ with $P_\xi(B_\xi) \to 0$, $Q_\xi(B_\xi) \to 1$ (see Theorem 4.1.1). The following example shows that for other testing problems there may exist intermediate possibilities.

EXAMPLE 4.1.1. Let $X_\nu = [0,1]^\nu$, let P_ν be the ν-fold product measure of the uniform distribution on $[0,1]$ and let Q_ν be the ν-fold product measure of the uniform distribution on $[0,1 - p / \nu]$, for some fixed $p > 0$. Define

$$C_\nu = \{(x_1, \ldots, x_\nu) \in X_\nu \mid \max_i x_i \leq 1 - p / \nu\} .$$

Then for every measurable set $B \subset \mathcal{X}_\nu$

$$Q_\nu(B) = (1 - p / \nu)^{-\nu} P_\nu(B \cap C_\nu) .$$

It follows from Definition 4.1.1 that $\{Q_\nu\} \lhd \{P_\nu\}$. Furthermore,

$$Q_\nu(\mathcal{X}_\nu \setminus C_\nu) = 0$$
$$P_\nu(\mathcal{X}_\nu \setminus C_\nu) = 1 - (1 - p / \nu)^\nu \to 1 - e^{-p} .$$

This shows that $\{P_\nu\}$ is not contiguous to $\{Q_\nu\}$. For testing the null hypothesis Q_ν against the alternative hypothesis P_ν, the test function $1 - I_{C_\nu}$ has size 0 and asymptotic power $1 - e^{-p} > 0$. □

For every ν, let t_ν be a statistic defined on $(\mathcal{X}_\nu, \mathcal{F}_\nu)$, and let $T_\nu = t_\nu(X_\nu)$. It follows immediately from the definition that $\{Q_\nu\} \lhd \{P_\nu\}$ implies $\{\mathcal{L}_{Q_\nu}(T_\nu)\} \lhd \{\mathcal{L}_{P_\nu}(T_\nu)\}$. If for every ν, T_ν is a sufficient statistic for the experiment $((\mathcal{X}_\nu, \mathcal{F}_\nu), \{P_\nu, Q_\nu\})$, then the reverse implication is also true. To see this, suppose that $\{\mathcal{L}_{Q_\nu}(T_\nu)\} \lhd \{\mathcal{L}_{P_\nu}(T_\nu)\}$ and that $P_\nu(B_\nu) \to 0$. The sufficiency of T_ν implies that

$$P_\nu \{B_\nu \mid T_\nu\} = Q_\nu \{ B_\nu \mid T_\nu\} .$$

Let $\phi_\nu(T_\nu) = P_\nu\{B_\nu \mid T_\nu\}$ a.e.; then

$$E_{P_\nu} \phi_\nu(T_\nu) = P_\nu(B_\nu) \to 0,$$

so that, by Proposition 4.1.1,

$$Q_\nu(B_\nu) = E_{Q_\nu} \phi_\nu(T_\nu) \to 0 .$$

It can be concluded that $\{Q_\nu\} \lhd \{P_\nu\}$.

The following theorem characterizes contiguity for sequences of probability distributions of the form

$$P_\mu^{(\nu)} = \mathcal{L}_\mu (X_{11}^{(\nu)}, \ldots, X_{1n_1(\nu)}^{(\nu)}, X_{21}^{(\nu)}, \ldots, X_{kn_k(\nu)}^{(\nu)}) ,$$

for the testing problem of Section 3.5. In view of the remark above, instead of $P_\mu^{(\nu)}$ one can as well consider the probability distributions of the sample means $X_{\cdot}^{(\nu)}$, or Y_ν; these are both sufficient statistics.

<u>THEOREM 4.1.1</u>. *Let* $\{\mu_\nu\}$ *and* $\{\mu'_\nu\}$ *be sequences in* F *and let at least one of these sequences be relatively compact in* F. *Then*

$$\lim_\nu \sup \; [n(\nu)]^{\frac{1}{2}} \; \| \mu_\nu - \mu'_\nu \| < \infty$$

if and only if

$$\{P^{(\nu)}_{\mu_\nu}\} \Leftrightarrow \{P^{(\nu)}_{\mu'_\nu}\} \; .$$

If $[n(\nu)]^{\frac{1}{2}} \; \| \mu_\nu - \mu'_\nu \| \; \to \infty$ *then a sequence of measurable sets* B_ν *exists with the property that*

$$P^{(\nu)}_{\mu_\nu}(B_\nu) \to 0 \qquad , \qquad P^{(\nu)}_{\mu'_\nu}(B_\nu) \to 1 \; .$$

<u>PROOF</u>. Let θ_ν and θ'_ν be the natural parameter values with $\mu_\nu = \mu(\theta_\nu)$, $\mu'_\nu = \mu(\theta'_\nu)$. Theorems 2.3.1,2 and Corollary 2.3.1 show that $[n(\nu)]^{\frac{1}{2}} \; \| \mu_\nu - \mu'_\nu \| \; \to \infty$ iff $[n(\nu)]^{\frac{1}{2}} \; \| \theta_\nu - \theta'_\nu \| \; \to \infty$.

Corollary A.2.1 and Lemma A.2.1 show that $\{P^{(\nu)}_{\mu_\nu}\} \Leftrightarrow \{P^{(\nu)}_{\mu'_\nu}\}$ iff

$$\{[n_i(\nu)]^{\frac{1}{2}} \; \| \theta_{\nu i} - \theta'_{\nu i} \| \; \} \text{ is bounded for every } i;$$

and that if

$$[n_i(\nu)]^{\frac{1}{2}} \; \| \theta_{\nu i} - \theta'_{\nu i} \| \to \infty \quad \text{for some } i \; ,$$

then a sequence $\{B_\nu\}$ of measurable sets exists with

$$P^{(\nu)}_{\mu_\nu}(B_\nu) \to 0 \quad , \quad P^{(\nu)}_{\mu'_\nu}(B_\nu) \to 1.$$

As it was assumed in Section 3.5 that $\lim \inf_\nu n_i(\nu) \; / \; n(\nu) > 0$ for every i, this yields the desired conclusions. □

We shall often use the notation "$\{\mu_\nu\} \Leftrightarrow \{\mu'_\nu\}$" as a shorthand notation for

$$\{P^{(\nu)}_{\mu_\nu}\} \Leftrightarrow \{P^{(\nu)}_{\mu'_\nu}\} \quad ,$$

or equivalently for the boundedness of $\{[n(\nu)]^{\frac{1}{2}} \| \mu_\nu - \mu'_\nu \| \}$. Attention will be restricted in this study to sequences $\{\mu_\nu\}$ which are relatively compact in F.

4.2. LIMIT THEOREMS FOR ASYMPTOTIC TESTING PROBLEMS WITH A SEQUENCE OF SIMPLE NULL HYPOTHESES AND A FIXED OUTCOME SPACE

The testing problem $\mathcal{T}_\nu$ of Section 3.5 has been reduced to a testing problem with outcome space $\mathbb{R}^m$, independently of ν, by the reduction to the sufficient statistic $X^{(\nu)}_{.}$ or, rather, Y_ν. A fundamental tool for the treatment of the asymptotic testing problem $\{\mathcal{T}_\nu\}$, to be used in Chapters 5, 6 and 8, is the consideration of auxiliary asymptotic testing problems with sequences of simple null hypotheses

$$H_\nu : \mu = \mu_\nu,$$

where $\{\mu_\nu\} \subset \mu(\Theta_H)$. In this section, a theory for asymptotic testing problems with a sequence of simple null hypotheses is developed in which the feature of a fixed outcome space is exploited. For the general orientation of this section I am indebted to WALD (1950) and LE CAM (1972), but the particular approach followed below seems to be new.

In this section we consider the outcome space $\mathbb{R}^m$ with a fixed class Φ of test functions on $\mathbb{R}^m$ and a sequence

$$\{(P_{H\nu}, \mathcal{P}_{A\nu}, \Phi_\nu)\}$$

of testing problems: $P_{H\nu}$ is a probability measure on $\mathbb{R}^m$ (null hypothesis), $\mathcal{P}_{A\nu}$ is a non-empty class of probability measures on $\mathbb{R}^m$ with $P_{H\nu} \notin \mathcal{P}_{A\nu}$ (alternative hypothesis), and Φ_ν is a non-empty subset of Φ (class of tests under consideration). $\{P_{H\nu}\} \cup \mathcal{P}_{A\nu}$ will be denoted by $\mathcal{P}_\nu$.
The theory of this section will be applied e.g. with $\Phi = \Phi_C$ (the class of all tests with convex acceptance region),

$$P_{H\nu} = \mathcal{L}_{\mu_\nu}(\tilde{X}_\nu)$$

$$\Phi_\nu = \{\phi \in \Phi \mid E_{\mu_\nu} \phi(\tilde{X}_\nu) \leq \alpha \}$$

where $\{\mu_\nu\} \subset \mu(\Theta_H)$ and $\tilde{X}_\nu = [n(\nu)]^{\frac{1}{2}}(X^{(\nu)}_{.} - \mu_\nu)$, and with $\mathcal{P}_{A\nu}$ a suitable subset of

$$\{L_\mu(\tilde{X}_\nu) \mid \mu \in \mu(\Theta_A)\} ;$$

it will be assumed that $\mu_\nu \to \mu$ and $\rho_i(\nu) \to \rho_i$ for certain $\mu \in \mu(\Theta_H)$ and $\rho_i \in (0,1)$ $(1 \leq i \leq k)$.

The limit of a sequence $\{(P_{H\nu}, \mathcal{P}_{A\nu}, \Phi_\nu)\}$ will be defined, using the concept of the topological limit of a sequence of subsets of a pseudo-metrizable space.

DEFINITION 4.2.1. Let X be a pseudo-metrizable topological space and $\{B_\nu\}$ a sequence of subsets of X. The lower and upper limits of $\{B_\nu\}$ are defined, respectively, by

$$\mathrm{Li}_\nu B_\nu = \{x \in X \mid \text{a sequence } \{b_\nu\} \text{ exists with } b_\nu \in B_\nu \text{ for all } \nu, \text{ and } b_\nu \to x\}$$

$$\mathrm{Ls}_\nu B_\nu = \{x \in X \mid \text{a subsequence } \{\xi\} \text{ of } \{\nu\} \text{ and a sequence } \{b_\xi\} \text{ exist with } b_\xi \in B_\xi \text{ for all } \xi, \text{ and } b_\xi \to x\} .$$

If $\mathrm{Li}_\nu B_\nu = \mathrm{Ls}_\nu B_\nu$, then this set is called the topological limit of $\{B_\nu\}$ and denoted by $\mathrm{Lt}_\nu B_\nu$.

LEMMA 4.2.1. $\mathrm{Li}_\nu B_\nu$ *and* $\mathrm{Ls}_\nu B_\nu$ *are closed sets.*

PROOF. Suppose that $x \notin \mathrm{Ls}_\nu B_\nu$, and denote the pseudo-metric on X by d. Then an $\varepsilon > 0$ exists such that

$$\{y \in X \mid d(x,y) < \varepsilon\} \cap B_\nu = \emptyset$$

for ν sufficiently large. This implies that

$$\{y \in X \mid d(x,y) < \tfrac{1}{2}\varepsilon\} \cap \mathrm{Ls}_\nu B_\nu = \emptyset .$$

Hence $\mathrm{Ls}_\nu B_\nu$ is closed. The closedness of $\mathrm{Li}_\nu B_\nu$ is proved similarly. □

More about topological limits can be found in ALEXANDROV and HOPF (1935, II §5), KURATOWSKI (1966, §29) and Appendix A.3. Note that always $\mathrm{Li}_\nu B_\nu \subset \mathrm{Ls}_\nu B_\nu$. Some examples for $X = \mathbb{R}$ may be helpful for the intuitive understanding of these concepts:

$\mathrm{Lt}_\nu\ (\nu^{-1},\ 1-\nu^{-1}) = [0,1]$; $\mathrm{Lt}_\nu\ [\nu,\infty) = \emptyset$;
$\mathrm{Lt}_\nu\ \{1/\nu,\ 2/\nu,\ \ldots,\ (\nu-1)/\nu\} = [0,1]$; $\mathrm{Lt}_\nu(0,1) = [0,1]$;
$\mathrm{Li}_\nu\ [(-1)^\nu,\ (-1)^\nu + 2] = \{1\}$; $\mathrm{Ls}_\nu\ [(-1)^\nu,\ (-1)^\nu + 2] = [-1,\ 3]$.

The following is a summary of a non-rigourous method which is sometimes used by statisticians when dealing with testing problems for large sample sizes:

"Invoke the central limit theorem and replace the probability distributions by normal distributions with the same means, plugging in estimated values for the variances and covariances. This yields a testing problem for multivariate normal distributions with unknown means and known covariance matrices; apply standard optimality theory to this testing problem. This method will produce approximately optimal tests."

The following definition will be used to incorporate this approach in a rigourous theory. Regarding the topologies used in this definition, note that the weak topology on $M_1(\mathbb{R}^m)$ is metrizable (Theorem A.1 (i)), and that, for any σ-finite measure λ on $\mathbb{R}^m$, the weak* topology with respect to λ on the class of all test functions (of which Φ is a subclass) is pseudo-metrizable (Theorem 2.4.2).

<u>DEFINITION 4.2.2</u>. Let λ be a σ-finite measure on $\mathbb{R}^m$. Consider $M_1(\mathbb{R}^m)$ with the weak topology and Φ with the weak* topology with respect to λ. Let Φ be compact.

The sequence $\{(P_{H\nu},\ \mathcal{P}_{A\nu},\ \Phi_\nu)\}$ converges to $(P_{H\infty},\ \mathcal{P}_{A\infty},\ \Phi_\infty)$, denoted by $(P_{H\nu},\ \mathcal{P}_{A\nu},\ \Phi_\nu) \to (P_{H\infty},\ \mathcal{P}_{A\infty},\ \Phi_\infty)$, if

L(i) $P_{H\infty} = \lim_\nu P_{H\nu}$

L(ii) $\mathcal{P}_\infty = \mathrm{Lt}_\nu \mathcal{P}_{A\nu}$ as subsets of $M_1(\mathbb{R}^m)$

L(iii) $\mathcal{P}_{A\infty} = \mathcal{P}_\infty \setminus \{P_{H\infty}\} \neq \emptyset$;
if $P_{H\infty} \in \mathcal{P}_\infty$, then $P_{H\infty} \in \mathrm{cl}\ \mathcal{P}_{A\infty}$

L(iv) $\Phi_\infty = \mathrm{Lt}_\nu \Phi_\nu$ as subsets of Φ

L(v) $P << \lambda$ for all $P \in \{P_{H\infty}\} \cup \mathcal{P}_{A\infty}$

L(vi) every $\phi \in \Phi_\infty$ is continuous a.e. $[\lambda]$

L(vii) if $\phi_\nu \in \Phi_\nu$ and $P_\nu \in \mathcal{P}_\nu$ for all ν, while $\phi_\nu \to \phi$ and $P_\nu \to P$, then
$E_{P_\nu} \phi_\nu \to E_P \phi$.

The conditions L(i - vii) are intended to deal with the "local part" of the asymptotic testing problem $\{(P_{H\nu},\ \mathcal{P}_{A\nu},\ \Phi_\nu)\}$. They are relevant especially in combination with the following condition; a sequence $\{P_\xi\}$ will be called divergent if $\{\xi\}$ is a subsequence of $\{\nu\}$ and $\{P_\xi\}$ has no subsequences which converge in $M_1(\mathbb{R}^m)$.

L(viii) For every divergent sequence $\{P_\xi\}$ with $P_\xi \in \mathcal{P}_{A_\xi}$ for all ξ, one has

$$\sup_{\phi \in \Phi_\xi} E_{P_\xi} \phi \to 1 \ .$$

This condition will play a role only in Section 4.3.

Some remarks may elucidate the roles of some of the conditions in Definition 4.2.2. The second part of condition L(iii) plays an essential role in part (i) of the proof of Proposition 4.2.3. Condition L(iv) implies, with the compactness of Φ and Lemma 4.2.1, that Φ_∞ is compact and non-empty.

If $\Phi_\nu = \Phi_\infty$ for every ν and L(vi) is satisfied, then condition L(vii) is equivalent to the condition that if $P_\nu \in \mathcal{P}_\nu$ for all ν, then $P_\nu \to P$ weakly implies that $d_w(P_\nu, P) \to 0$, where d_w is the intrinsic metric of WALD (1950), pages 85, 89:

$$d_w(P, Q) = \sup_{\phi \in \Phi_\infty} | E_P\phi - E_Q\phi | \ .$$

The following very simple example is included to improve the understanding of Definition 4.2.2.

<u>EXAMPLE 4.2.1.</u> Consider the testing problem

$$L_p(X_\nu) = \mathcal{B}(\nu, p)$$
$$H : p = p_0, \qquad A : p \neq p_0 \ .$$

Define

$$Y_\nu = [\nu p_0(1 - p_0)]^{-\frac{1}{2}} (X_\nu - \nu p_0)$$
$$Q_{\nu p} = L_p(Y_\nu) \ .$$

According to Theorem 2.7.2, the class of all tests with convex acceptance region is essentially complete. In the one-dimensional case considered here, ϕ has convex acceptance region iff ϕ is a two-sided or one-sided test, randomization at the boundary of the acceptance region being allowed, or $\phi \equiv 0$ or $\phi \equiv 1$.

Definition 4.2.2 will be applied with $m = 1$, λ = Lebesgue measure (the dominating measure for the limiting problem), and Φ the class of all tests with convex acceptance region. It will be proved that

$$(Q_{\nu p_0}, \{Q_{\nu p} \mid p \neq p_0\}, \Phi_\nu) \to (N(0,1), \{N(\eta, 1) \mid \eta \neq 0\}, \Phi_\infty)$$

where

$$\Phi_\nu = \{\phi \in \Phi \mid E_{p_0} \phi(Y_\nu) \leq \alpha\}, \quad \Phi_\infty = \{\phi \in \Phi \mid E_{N(0,1)} \phi \leq \alpha\} .$$

L(i) The Central Limit Theorem implies that $Q_{\nu p_0} \to N(0,1)$ weakly.

L(ii) Note that it can be proved, e.g. using the Central Limit Theorem of Lindeberg-Feller, that $\nu^{\frac{1}{2}}(p_\nu - p_0) \to \eta$ implies that $Q_{\nu p_\nu} \to N([p_0(1 - p_0)]^{-\frac{1}{2}} \eta, 1)$. This shows that

$$\{N(\eta, 1) \mid \eta \in \mathbb{R}\} \subset \mathrm{Li}_\nu \{Q_{\nu p} \mid p \neq p_0\} .$$

If $\{\xi\}$ is a subsequence of $\{\nu\}$ and $|\xi^{\frac{1}{2}} (p_\xi - p_0)| \to \infty$, then $\{Q_{\xi p_\xi}\}$ has no convergent subsequence in $M_1(\mathbb{R})$. This shows that

$$\{N(\eta, 1) \mid \eta \in \mathbb{R}\} \supset \mathrm{Ls}_\nu \{Q_{\nu p} \mid p \neq p_0\} .$$

With $\mathrm{Li}_\nu \{Q_{\nu p} \mid p \neq p_0\} \subset \mathrm{Ls}_\nu \{Q_{\nu p} \mid p \neq p_0\}$, this establishes L(ii) with

$$P_\infty = \{N(\eta, 1) \mid \eta \in \mathbb{R}\} .$$

L(iii) Trivial.

L(iv) This follows from Proposition 4.2.1 below and L(vii).

L(v) Trivial.

L(vi) Every $\phi \in \Phi$ has at most two points of discontinuity.

L(vii) It is well-known and easy to prove that if $\{P_\nu\}$ is a sequence in $M_1(\mathbb{R})$ with $P_\nu \to P$ weakly for a P which is absolutely continuous with respect to Lebesgue measure, then $P_\nu(-\infty, x) \to P(-\infty, x)$ and $P_\nu\{x\} \to 0$, both limits being uniform in $x \in \mathbb{R}$. This shows that $\nu^{\frac{1}{2}}(p_\nu - p_0) \to \eta$ implies

$$\sup_{\phi \in \Phi} \left| E_{p_\nu} \phi(Y_\nu) - E_{N(\eta,1)} \phi \right| \to 0 .$$

With L(vi), this establishes L(vii).

The two following modifications may elucidate condition L(iii). First consider the sequence

$$\{(Q_{\nu p_0}, \{Q_{\nu p} \mid 0 < |p - p_0| \leq \nu^{-1}\}, \Phi_\nu)\} .$$

The change of alternative hypothesis affects only L(ii, iii): condition L(ii) holds with $\mathcal{P}_\infty = \{N(0,1)\}$ and L(iii) does not hold, as $\mathcal{P}_\infty \setminus \{P_{H\infty}\} = \emptyset$. The alternative hypothesis "shrinks too fast".

Secondly consider the sequence

$$(1) \qquad \{(Q_{\nu p_0}, \{Q_{\nu p} \mid 0 < |p - p_0| \leq \nu^{-1} \text{ or } |p - p_0| \geq \nu^{-\frac{1}{2}}\} , \Phi_\nu)\} .$$

Again the change of alternative hypothesis affects only L(ii, iii). Now L(ii) holds with

$$\mathcal{P}_\infty = \{N(\eta, 1) \mid \eta = 0 \text{ or } |\eta| \geq [p_0(1 - p_0)]^{-\frac{1}{2}} \} .$$

The first part of L(iii) holds with

$$\mathcal{P}_{A\infty} = \{N(\eta, 1) \mid |\eta| > [p_0(1 - p_0)]^{-\frac{1}{2}}\},$$

but the second part of L(iii) does not hold: $P_{H\infty} \in \mathcal{P}_\infty$, but $P_{H\infty} \notin \partial\mathcal{P}_{A\infty}$. Cases like this are excluded in Definition 4.2.2, because they are rather irrelevant and the correspondence between asymptotically optimal tests for (1) and optimal tests for the potential limit problem

$$(2) \qquad (N(0,1), \mathcal{P}_{A\infty}, \Phi_\infty)$$

is too poor. E.g., the maximin test for (2) is "uniquely" determined (testing problem (2) has an "indifference zone") and the maximin power for (2) is larger than α, whereas for every testing problem in the sequence (1) the maximin power is α and every unbiased - level α test is maximin. □

Condition L(iii) can often be verified with the following proposition.

<u>PROPOSITION 4.2.1</u>. *Suppose that* L(i) *is satisfied and that* $\{\phi_\nu\} \subset \Phi$, $\phi_\nu \overset{*}{\to} \phi$ *implies*

$$E_{P_{H\nu}} \phi_\nu \to E_{P_{H\infty}} \phi .$$

Define

$$\Phi_\nu = \{\phi \in \Phi \mid E_{P_{H\nu}} \phi \le \alpha\} \qquad \nu \in \mathbb{N} \cup \{\infty\} ,$$

and suppose that for every $\phi \in \Phi_\infty$ *a sequence* $\{\phi_h\} \subset \Phi$ *exists with* $\phi_h \to \phi$ *and* $E_{P_{H\infty}} \phi_h < \alpha$ *for all* h.
Then $\Phi_\infty = \mathrm{Lt}_\nu \Phi_\nu$.

PROOF. As $\mathrm{Li}_\nu \Phi_\nu \subset \mathrm{Ls}_\nu \Phi_\nu$, it suffices to prove that $\mathrm{Ls}_\nu \Phi_\nu \subset \Phi_\infty \subset \mathrm{Li}_\nu \Phi_\nu$.

(i) Let $\phi \in \mathrm{Ls}_\nu \Phi_\nu$. Then a subsequence $\{\xi\}$ of $\{\nu\}$ and a sequence $\{\phi_\xi\}$ with $\phi_\xi \in \Phi_\xi$ exist with $\phi_\xi \overset{*}{\to} \phi$. As $E_{P_{H\xi}} \phi_\xi \le \alpha$ and $E_{P_{H\xi}} \phi_\xi \to E_{P_{H\infty}} \phi$, it can be concluded that $\phi \in \Phi_\infty$.

(ii) Let $\phi \in \Phi_\infty$. First suppose that $E_{P_{H\infty}} \phi < \alpha$. Then there exists a ν_0 with $E_{P_{H\nu}} \phi < \alpha$ for all $\nu \ge \nu_0$. Hence a sequence $\{\phi_\nu\}$ with $\phi_\nu \in \Phi_\nu$ exists such that $\phi_\nu = \phi$ for all $\nu > \nu_0$. One has $\phi_\nu \overset{*}{\to} \phi$, so that $\phi \in \mathrm{Li}_\nu \Phi_\nu$.
Secondly suppose that $E_{P_{H\infty}} \phi = \alpha$ and let $\{\phi_h\}$ be the sequence of which the existence is assumed. Then $\phi_h \in \mathrm{Li}_\nu \Phi_\nu$ because of the result above. Lemma 4.2.1 implies that $\phi \in \mathrm{Li}_\nu \Phi_\nu$. □

In order to see that we cannot dispense with the last condition of Proposition 4.2.1, let $P_{H\nu} = N(\nu^{-1}, 1)$, $P_{H\infty} = N(0,1)$, λ = Lebesgue measure and

$$\Phi = \{\phi \mid \phi(x) = 1 \text{ for all } x \ge u_\alpha\} .$$

Then $E_{P_{H\nu}} \phi > \alpha$ for all ν, so that $\Phi_\nu = \emptyset$ and $\mathrm{Lt}_\nu \Phi_\nu = \emptyset$. But Φ_∞ is not empty.

In the remainder of this section, it will be assumed that Φ is weakly* compact and that $(P_{H\nu}, \mathcal{P}_{A\nu}, \Phi_\nu) \to (P_{H\infty}, \mathcal{P}_{A\infty}, \Phi_\infty)$.

In the following proposition, the function β^*_∞ is also defined for $P_{H\infty}$, if $P_{H\infty} \in \mathcal{P}_\infty$. This will be convenient, although it makes the name "envelope power function" somewhat abusive. In Sections 4.2 and 4.3, the class of all sequences $\{P_\nu\}$ with $P_\nu \in \mathcal{P}_{A\nu}$ for all ν, which are relatively compact in $M_1(\mathbb{R}^m)$ (in other words: which have no divergent subsequences), will be denoted by K.

PROPOSITION 4.2.2. *Let* $\beta^*_\nu : \mathcal{P}_{A\nu} \to [0,1]$ *and* $\beta^*_\infty : \mathcal{P}_\infty \to [0,1]$ *be the envelope power functions with respect to* Φ_ν *and* Φ_∞*, respectively:*

$$\beta^*_\nu(P) = \sup_{\phi\in\Phi_\nu} E_P\phi \qquad\qquad \nu \in \mathbb{N} \cup \{\infty\} .$$

Suppose that $\{P_\nu\} \in K$ *and* $P_\nu \to P$. *Then* $\beta^*_\nu(P_\nu) \to \beta^*_\infty(P)$.

PROOF. Let $\phi \in \Phi_\infty$. Since $\Phi_\infty = \mathrm{Lt}_\nu\Phi_\nu$, a sequence $\{\phi_\nu\}$ exists with $\phi_\nu \in \Phi_\nu$ and $\phi_\nu \to \phi$. Hence with L(vii),

$$E_P\phi = \lim_\nu E_{P_\nu}\phi_\nu \le \liminf_\nu \beta^*_\nu(P_\nu) .$$

This implies that $\beta^*_\infty(P) \le \liminf_\nu \beta^*_\nu(P_\nu)$.
Now let $\{\xi\}$ be a subsequence of $\{\nu\}$ and $\phi_\xi \in \Phi_\xi$ such that

$$E_{P_\xi}\phi_\xi \to \limsup_\nu \beta^*_\nu(P_\nu) .$$

As Φ is pseudo-metrizable and compact, it may be assumed (if necessary replace $\{\xi\}$ by a further subsequence) that $\phi_\xi \to \phi$ for some $\phi \in \Phi$. From $\Phi_\infty = \mathrm{Lt}_\nu\Phi_\nu$ it follows that $\phi \in \Phi_\infty$. Hence with L(vii),

$$\limsup_\nu \beta^*_\nu(P_\nu) = \lim_\xi E_{P_\xi}\phi_\xi = E_P\,\phi \le \beta^*_\infty(P). \quad \square$$

In accordance with Section 2.6, the functions β^*_ν and β^*_∞ will not always be the envelope power functions of Proposition 4.2.2. It will be assumed that $\beta^*_\nu : P_{A\nu} \to [0,1]$ and $\beta^*_\infty : P_\infty \to [0,1]$ are functions with the property

(4.2.1) if $\{P_\nu\} \in K$, $P_\nu \to P$ then $\beta^*_\nu(P_\nu) \to \beta^*_\infty(P)$.

In other words, the conclusion of Proposition 4.2.2 must be satisfied. The shortcomings with respect to β^*_ν and β^*_∞ will be denoted by

(4.2.2) $\gamma_\nu\,(\phi, P) = \beta^*_\nu(P) - E_P\phi$.

LEMMA 4.2.2. *Assumption* (4.2.1) *implies that* $\beta^*_\infty : P_\infty \to [0,1]$ *is a continuous function.*

PROOF. Suppose that $\{P_n\} \subset P_\infty$ and $P_n \to P \in P_\infty$. It is not a restriction to assume that $\beta^*_\infty(P_n) \to p$ for some p; it must be proved that $p = \beta^*_\infty(P)$. Condition L(ii) implies that there exists a $\{Q_\nu\} \in K$ with $Q_\nu \to P$, and for every n there exists a $\{P_{n\nu}\} \in K$ with $\lim_\nu P_{n\nu} = P_n$. Let d be a metric generating the weak topology on $M_1(\mathbb{R}^m)$. For every n, there exists a $\bar{\nu}(n) \geq n$ with

$$d(P_{n,\bar{\nu}(n)}, P_n) \leq n^{-1}, \quad |\beta^*_{\bar{\nu}(n)}(P_{n,\bar{\nu}(n)}) - \beta^*_\infty(P_n)| \leq n^{-1}.$$

Define

$$\bar{n}(\nu) = \min\{n \mid \bar{\nu}(n) = \nu\},$$

using the convention that $\min \emptyset = \infty$. Define

$$Q'_\nu = \begin{cases} P_{\bar{n}(\nu),\nu} & \bar{n}(\nu) < \infty \\ Q_\nu & \bar{n}(\nu) = \infty \end{cases}$$

Then $Q'_\nu \in P_{A\nu}$ and $Q'_\nu \to P$. There are infinitely many ν with $\bar{n}(\nu) < \infty$; hence p is a subsequential limit of the sequence $\{\beta^*_\nu(Q'_\nu)\}$. As (4.2.1) implies that $\beta^*_\nu(Q'_\nu) \to \beta^*_\infty(P)$, this establishes $p = \beta^*_\infty(P)$. □

The following proposition and its two corollaries will be used as tools for proving asymptotic optimality of certain sequences of tests.

<u>PROPOSITION 4.2.3</u>. *Let $\phi_\nu \in \Phi_\nu$ for all ν, and denote the class of all subsequential limits of $\{\phi_\nu\}$ by Ψ. Then*

$$\inf_{\phi\in\Psi} \sup_{P\in P_{A\infty}} \gamma_\infty(\phi, P) = \sup_{\{P_\nu\}\in K} \liminf_\nu \gamma_\nu(\phi_\nu, P_\nu)$$

$$\sup_{\phi\in\Psi} \sup_{P\in P_{A\infty}} \gamma_\infty(\phi, P) = \sup_{\{P_\nu\}\in K} \limsup_\nu \gamma_\nu(\phi_\nu, P_\nu).$$

<u>PROOF</u>. (i) First it is proved that for every $\phi \in \Psi$

$$(1) \qquad \sup_{P\in P_{A\infty}} \gamma_\infty(\phi, P) = \sup_{P\in P_\infty} \gamma_\infty(\phi, P).$$

According to L(iii), one has $P_{A\infty} = P_\infty \setminus \{P_{H\infty}\}$. Hence it is sufficient to prove that if $P_{H\infty} \in P_\infty$, then

$$\gamma_\infty(\phi, P_{H\infty}) \leq \sup_{P\in P_{A\infty}} \gamma_\infty(\phi, P).$$

Suppose that $P_{H\infty} \in \mathcal{P}_\infty$ and $\phi \in \Psi$. Condition L(iv) implies that $\Psi \subset \Phi_\infty$. With L(v, vi) this implies that $E_P\phi$ is a continuous function of $P \in \mathcal{P}_\infty$. Lemma 4.2.2 implies that $\gamma_\infty(\phi, P)$ is a continuous function of $P \in \mathcal{P}_\infty$. As $P_{H\infty} \in \partial\mathcal{P}_{A\infty}$ according to L(iii), this establishes (1).
(ii) Let $\{P_\nu\} \in K$ and $\phi \in \Psi$. Let $\{\phi_\xi\}$ be a subsequence of $\{\phi_\nu\}$ with $\phi_\xi \to \phi$. Then, with L(ii),

$$\liminf_\nu \gamma_\nu (\phi_\nu, P_\nu) \leq \limsup_\xi \gamma_\xi (\phi_\xi, P_\xi) \leq \sup_{P\in\mathcal{P}_\infty} \gamma_\infty (\phi, P) .$$

This holds for arbitrary $\{P_\nu\} \in K$ and $\phi \in \Psi$; hence

$$\sup_{\{P_\nu\}\in K} \liminf_\nu \gamma_\nu(\phi_\nu, P_\nu) \leq \inf_{\phi\in\Psi} \sup_{P\in\mathcal{P}_\infty} \gamma_\infty (\phi, P).$$

(iii) Let $\varepsilon > 0$ and define

$$\gamma = \inf_{\phi\in\Psi} \sup_{P\in\mathcal{P}_\infty} \gamma_\infty (\phi, P) .$$

For every $\phi \in \Psi$ there is a $P_\phi \in \mathcal{P}_\infty$ with $\gamma_\infty (\phi, P_\phi) > \gamma - \varepsilon$. It was proved in (i) that for every $\phi \in \Psi$, $\gamma_\infty (\phi, P)$ is a continuous function of $P \in \mathcal{P}_\infty$. Hence for every $\phi \in \Psi$, the set

$$U(\phi) = \{\psi \in \Psi \mid \gamma_\infty (\psi, P_\phi) > \gamma - \varepsilon\}$$

is a neighbourhood of ϕ in Ψ. As Ψ is a closed subset of the compact set Φ, Ψ is compact. Hence there exist finitely many $\phi_1, \ldots, \phi_n \in \Psi$ with

$$\Psi = \bigcup_{h=1}^{n} U(\phi_h) .$$

Define $P_h = P_{\phi_h}$; then

(2) $$\inf_{\phi\in\Psi} \max_h \gamma_\infty (\phi, P_h) \geq \gamma - \varepsilon .$$

Condition L(ii) implies that for every h there exists a $\{P_{h\nu}\} \in K$ with $P_{h\nu} \to P_h$. Let $\{\xi\}$ be a subsequence of $\{\nu\}$ with

$$\lim_\xi \max_h \gamma_\xi (\phi_\xi, P_{h\xi}) = \liminf_\nu \max_h \gamma_\nu (\phi_\nu, P_{h\nu})$$

and with $\phi_\xi \to \psi$ for some $\psi \in \Psi$. Then

$$(3) \qquad \max_h \gamma_\infty (\psi, P_h) = \lim_\xi \max_h \gamma_\xi (\phi_\xi, P_{h\xi}) =$$

$$= \liminf_\nu \max_h \gamma_\nu (\phi_\nu, P_{h\nu}).$$

Define $h(\nu)$ by

$$\gamma_\nu (\phi_\nu, P_{h(\nu),\nu}) = \max_h \gamma_\nu (\phi_\nu, P_{h\nu}).$$

Then (3) and the fact that $\{P_{h(\nu),\nu}\} \in K$ imply that

$$\inf_{\phi\in\Psi} \max_h \gamma_\infty (\phi, P_h) \le \liminf_\nu \gamma_\nu (\phi_\nu, P_{h(\nu),\nu}) \le$$

$$\le \sup_{\{P_\nu\}\in K} \liminf_\nu \gamma_\nu (\phi_\nu, P_\nu) .$$

The definition of γ and (2) show that it has been proved that

$$\inf_{\phi\in\Psi} \sup_{P\in P_\infty} \gamma_\infty (\phi, P) \le \sup_{\{P_\nu\}\in K} \liminf_\nu \gamma_\nu (\phi_\nu, P_\nu).$$

(iv) The first equality to be proved follows from (1), (ii) and (iii).
(v) The proof of the second equality is straightforward (use (1), and prove both inequalities which together yield the equality). □

COROLLARY 4.2.1. *Let* β^*_ν *and* β^*_∞ *be as in Proposition* 4.2.2, *and let* $\phi_\nu \in \Phi_\nu$ *for every* ν.
The two following statements are equivalent.

(1) $\gamma_\nu (\phi_\nu, P_\nu) \to 0$ *for all* $\{P_\nu\} \in K$

(2) *every subsequential limit of* $\{\phi_\nu\}$ *is* UMP - Φ_∞ *for the testing problem* $(\mathbb{R}^m, \{P_{H\infty}\}, P_{A\infty})$.

PROOF. Denote the class of all subsequential limits of $\{\phi_\nu\}$ by Ψ. Then (1) can be written as

$$\sup_{\{P_\nu\}\in K} \limsup_\nu \gamma_\nu (\phi_\nu, P_\nu) = 0,$$

and (2) as

$$\sup_{\phi\in\Psi}\ \sup_{P\in\mathcal{P}_{A\infty}} \gamma_\infty(\phi, P) = 0 .$$

The equivalence of (1) and (2) follows immediately from the second part of Proposition 4.2.3. □

COROLLARY 4.2.2. *Denote by* $\tilde{\Phi}$ *the class of all sequences* $\{\phi_\nu\}$ *with* $\phi_\nu \in \Phi_\nu$ *for all* ν.

(i)
$$\inf_{\{\phi_\nu\}\in\tilde{\Phi}}\ \sup_{\{P_\nu\}\in K}\ \liminf_{\nu}\ \gamma_\nu(\phi_\nu, P_\nu) =$$

$$= \inf_{\{\phi_\nu\}\in\tilde{\Phi}}\ \sup_{\{P_\nu\}\in K}\ \limsup_{\nu}\ \gamma_\nu(\phi_\nu, P_\nu) = \inf_{\phi\in\Phi_\infty}\ \sup_{P\in\mathcal{P}_{A\infty}} \gamma_\infty(\phi, P) .$$

(ii) *For* $\{\psi_\nu\} \in \tilde{\Phi}$, *the following two statements are equivalent.*

(1)
$$\sup_{\{P_\nu\}\in K}\ \limsup_{\nu}\ \gamma_\nu(\psi_\nu, P_\nu) =$$

$$= \inf_{\{\phi_\nu\}\in\tilde{\Phi}}\ \sup_{\{P_\nu\}\in K}\ \liminf_{\nu}\ \gamma_\nu(\phi_\nu, P_\nu)$$

(2) *every subsequential limit of* $\{\psi_\nu\}$ *is* MS - $(\Phi_\infty,\beta_\infty)$ *for the testing problem* $(\mathbb{R}^m,\{P_{H\infty}\},\mathcal{P}_{A\infty})$.

PROOF. (i) This follows from L(iv) and Proposition 4.2.3.
(ii) Denote by Ψ the class of all subsequential limits of $\{\psi_\nu\}$, and

$$\gamma = \inf_{\phi\in\Phi_\infty}\ \sup_{P\in\mathcal{P}_{A\infty}} \gamma_\infty(\phi, P) .$$

The second part of Proposition 4.2.3 and (i) above imply that (1) is equivalent to

$$\sup_{\phi\in\Psi}\ \sup_{P\in\mathcal{P}_{A\infty}} \gamma_\infty(\phi, P) = \gamma;$$

according to the definition of "most stringent", this is also equivalent to (2). □

*4.3. ASYMPTOTICALLY OPTIMAL TESTS FOR THE ASYMPTOTIC TESTING PROBLEM OF SECTION 4.2.

In this section again, the assumptions of Section 4.2 will be made: Φ is weakly* compact, $(P_{H\nu}, \mathcal{P}_{A\nu}, \Phi_\nu) \to (P_{H\infty}, \mathcal{P}_{A\infty}, \Phi_\infty)$ in the sense of Definition 4.2.2, and the functions $\beta^*_\nu : \mathcal{P}_{A\nu} \to [0,1]$ and $\beta^*_\infty : \mathcal{P}_\infty \to [0,1]$ satisfy (4.2.1). Moreover, it will be assumed that condition L(viii) is satisfied. The shortcomings γ_ν and γ_∞ are defined by (4.2.2). The sequence $\{\beta^*_\nu\}$ is denoted by $\tilde{\beta}^*$ and the class of all sequences $\{\phi_\nu\}$ with $\phi_\nu \in \Phi_\nu$, for every ν , is denoted by $\tilde{\Phi}$.

Correspondences will be established between asymptotically optimal tests for the asymptotic testing problem $\{(\mathbb{R}^m, \{P_{H\nu}\}, \mathcal{P}_{A\nu}, \Phi_\nu)\}$ and optimal tests for the limiting problem $(\mathbb{R}^m, \{P_{H\infty}\}, \mathcal{P}_{A\infty}, \Phi_\infty)$. The definitions of "asymptotically UMP" and "asymptotically MS" are direct asymptotic analogues of the definitions "UMP" and "MS" given in Chapter 2. More general definitions of asymptotic optimum properties are given in Chapters 5, 6 and 7.

DEFINITION 4.3.1. Consider the asymptotic testing problem

$$\{(\mathbb{R}^m, \{P_{H\nu}\}, \mathcal{P}_{A\nu}, \Phi_\nu)\} .$$

The test $\{\phi_\nu\}$ is asymptotically uniformly most powerful - $\tilde{\Phi}$, or AUMP - $\tilde{\Phi}$, if
(i) $\{\phi_\nu\} \in \tilde{\Phi}$
(ii) for every $\{\psi_\nu\} \in \tilde{\Phi}$ one has

$$\lim_\nu \sup \sup_{P \in \mathcal{P}_{A\nu}} E_P(\psi_\nu - \phi_\nu) \leq 0 .$$

The test $\{\phi_\nu\}$ is asymptotically most stringent - $(\tilde{\Phi}, \tilde{\beta}^*)$, or AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$, if
(i) $\{\phi_\nu\} \in \tilde{\Phi}$
(ii) $\sup_{P \in \mathcal{P}_{A\nu}} \gamma_\nu (\phi_\nu, P) - \inf_{\phi \in \Phi_\nu} \sup_{P \in \mathcal{P}_{A\nu}} \gamma_\nu (\phi, P) \to 0.$

If

$$\lim_\nu \inf_{\phi \in \Phi_\nu} \sup_{P \in \mathcal{P}_{A\nu}} \gamma_\nu (\phi, P)$$

exists, this number is called the asymptotic minimax shortcoming - $(\tilde{\Phi}, \tilde{\beta}^*)$, or AMXS - $(\tilde{\Phi}, \tilde{\beta}^*)$.

A test $\{\phi_\nu\}$ is sharply consistent, if

$$E_{P_\xi}\phi_\xi \to 1$$

for all divergent sequences $\{P_\xi\}$ with $P_\xi \in \mathcal{P}_{A\xi}$ for all ξ.

It follows immediately from the definitions, that a sequence of UMP-Φ_ν tests is AUMP - $\tilde{\Phi}$ and that a sequence of MS - (Φ_ν, β^*_ν) tests is AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$. It is also clear that for $\beta^*_\nu(P) = \sup_{\phi\in\Phi_\nu} E_P\phi$, a test $\{\phi_\nu\}$ is AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$ with asymptotic minimax shortcoming equal to 0 iff $\{\phi_\nu\}$ is AUMP - $\tilde{\Phi}$.

THEOREM 4.3.1. *The test* $\{\phi_\nu\}$ *is* AUMP - $\tilde{\Phi}$ *iff*

(i) $\{\phi_\nu\} \in \tilde{\Phi}$

(ii) *every subsequential limit of* $\{\phi_\nu\}$ *is* UMP - Φ_∞ *for the testing problem* $(\mathbb{R}^m, \{P_{H\infty}\}, \mathcal{P}_{A\infty})$

(iii) $\{\phi_\nu\}$ *is sharply consistent.*

PROOF. Let $\beta^*_\nu(P) = \sup_{\phi\in\Phi_\nu} E_P\phi$. Then $\{\phi_\nu\}$ is UMP - Φ_∞ iff $\{\phi_\nu\} \in \tilde{\Phi}$ and

(1) $$\sup_{P\in\mathcal{P}_{A\nu}} \gamma_\nu(\phi_\nu, P) \to 0.$$

Condition (1) is satisfied iff $\{\phi_\nu\}$ is sharply consistent and

(2) $$\gamma_\nu(\phi_\nu, P_\nu) \to 0 \text{ for all } \{P_\nu\} \in K.$$

Corollary 4.2.1 shows that (2) is equivalent to (ii). □

THEOREM 4.3.2. *Suppose that* $\beta^*_\xi(P_\xi) \to 1$ *for all divergent sequences* $\{P_\xi\}$ *with* $P_\xi \in \mathcal{P}_{A\xi}$. *Let*

$$\gamma = \inf_{\phi\in\Phi_\infty} \sup_{P\in\mathcal{P}_{A\infty}} \gamma_\infty(\phi, P) .$$

If there exists at least one test $\{\phi_\nu\} \in \tilde{\Phi}$ *such that*

(i) *every subsequential limit of* $\{\phi_\nu\}$ *is* MS - $(\Phi_\infty, \beta^*_\infty)$ *for the testing problem* $(\mathbb{R}^m, \{P_{H\infty}\}, \mathcal{P}_{A\infty})$

(ii) $\liminf_\xi E_{P_\xi}\phi_\xi \geq 1 - \gamma$ *for every divergent sequence* $\{P_\xi\}$ *with* $P_\xi \in \mathcal{P}_{A\xi}$, *then any test* $\{\phi_\nu\}$ *is* AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$ *iff* $\{\phi_\nu\} \in \tilde{\Phi}$ *and* $\{\phi_\nu\}$ *satisfies* (i) *and* (ii); *in that case, moreover, the* AMXS - $(\tilde{\Phi}, \tilde{\beta}^*)$ *exists and is equal to* γ.

PROOF. "if" Suppose that $\{\phi_\nu\} \in \tilde{\Phi}$ satisfies (i) and (ii). Then Corollary 4.2.2 yields that

$$(1) \qquad \sup_{\{P_\nu\} \in K} \limsup_{\nu} \gamma_\nu(\phi_\nu, P_\nu) = \sup_{\{P_\nu\} \in K} \liminf_{\nu} \gamma_\nu(\phi_\nu, P_\nu) = \gamma$$

$$(2) \qquad \inf_{\{\psi_\nu\} \in \tilde{\Phi}} \sup_{\{P_\nu\} \in K} \liminf_{\nu} \gamma_\nu(\psi_\nu, P_\nu) = \gamma.$$

It follows from (1) and (ii) that

$$(3) \qquad \sup_{P \in \mathcal{P}_{A\nu}} \gamma_\nu(\phi_\nu, P) \to \gamma.$$

Result (2) yields that

$$(4) \qquad \liminf_{\nu} \inf_{\phi \in \Phi_\nu} \sup_{P \in \mathcal{P}_{A\nu}} \gamma_\nu(\phi, P) \geq \gamma.$$

Since $\{\phi_\nu\} \in \tilde{\Phi}$, (3) and (4) yield that $\{\phi_\nu\}$ is AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$ and that the AMXS - $(\tilde{\Phi}, \tilde{\beta}^*)$ is equal to γ.

"only if". Suppose that $\{\phi_{0\nu}\} \in \tilde{\Phi}$ satisfies (i) and (ii), and that $\{\phi_\nu\}$ is AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$. From the "if" part it follows that the AMXS - $(\tilde{\Phi}, \tilde{\beta}^*)$ is equal to γ. Hence

$$(5) \qquad \sup_{P \in \mathcal{P}_{A\nu}} \gamma_\nu(\phi_\nu, P) \to \gamma.$$

Together with (2) and Corollary 4.2.2, this implies that $\{\phi_\nu\}$ satisfies (i). If $\{P_\xi\}$ is a divergent sequence with $P_\xi \in \mathcal{P}_{A\xi}$, then $\beta^*_\xi(P_\xi) \to 1$, while (5) implies that $\limsup_\xi \gamma_\xi(\phi_\xi, P_\xi) \leq \gamma$. Hence $\{\phi_\nu\}$ satisfies (ii). □

The assumptions of this section do not exclude the possibility that no test $\{\phi_\nu\} \in \tilde{\Phi}$ exists which satisfies conditions (i) and (ii) of Theorem 4.3.2. In such cases, the classes of test functions Φ_ν are "not large enough", as the following example demonstrates.

*EXAMPLE 4.3.1. Let $m = 2$, and let λ be Lebesgue measure on $\mathbb{R}^2$. Let

$$P_{H\nu} = N_2(0,\ I)$$
$$\mathcal{P}_{A\nu} = \{N_2(\mu,\ I) \mid (\mu_1 > 0,\ \mu_2 = 0) \text{ or } ||\mu|| \geq \nu^{\frac{1}{2}}\}$$
$$\Phi_\nu = \Phi = \{I_{(u_\alpha,\infty)}(d'x) \mid d \in \mathbb{R}^2,\ ||d|| = 1\}.$$

For every ν, Φ_ν is a class of level α tests for the testing problem $(\mathbb{R}^2, \{P_{H\nu}\}, \mathcal{P}_{A\nu})$. It can easily be verified that $(P_{H\nu}, \mathcal{P}_{A\nu}, \Phi_\nu) \to (P_{H\infty}, \mathcal{P}_{A\infty}, \Phi)$ with

$$P_{H\infty} = N_2(0,\ I)$$
$$\mathcal{P}_{A\infty} = \{N_2(\mu,\ I) \mid \mu_1 > 0,\ \mu_2 = 0\}\ .$$

Condition L(viii) is also satisfied.

For every ν, let β^*_ν be the envelope power function with respect to Φ; then β^*_ν does not depend on ν. There is a unique MS - Φ test for the testing problem $(\mathbb{R}^2, \{P_{H\infty}\}, \mathcal{P}_{A\infty})$; this is of course the UMP - Φ test $I_{(u_\alpha,\infty)}(X_1)$. The minimax shortcoming for the limit problem is 0. The class $\tilde{\Phi}$ does not contain a sharply consistent test, however. Hence there is no test $\{\phi_\nu\} \in \tilde{\Phi}$ which satisfies conditions (i) and (ii) of Theorem 2.

The asymptotic minimax shortcoming for this testing problem is 1, and every $\{\phi_\nu\} \in \tilde{\Phi}$ is asymptically most stringent - $(\tilde{\Phi}, \tilde{\beta}^*)$. □

For testing problems in practice, the classes Φ_ν will always be chosen "large enough" so that AMS tests exist which do not only satisfy (ii) of Theorem 4.3.2 but which are even sharply consistent.

<u>THEOREM 4.3.3</u>. *A test* $\{\phi_\nu\}$ *is sharply consistent and* AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$ *iff*

(i) $\{\phi_\nu\} \in \tilde{\Phi}$

(ii) $\{\phi_\nu\}$ *is sharply consistent*

(iii) *every subsequential limit of* $\{\phi_\nu\}$ *is* MS - $(\Phi_\infty, \beta^*_\infty)$ *for the testing problem* $(\mathbb{R}^2, \{P_{H\infty}\}, \mathcal{P}_{A\infty})$.

<u>PROOF</u>. This follows immediately from Theorem 4.3.2. The condition that $\beta^*_\xi(P_\xi) \to 1$ for all divergent subsequences $\{P_\xi\}$ with $P_\xi \in \mathcal{P}_{A\xi}$, is superfluous in this case: it was used only for proving that all AMS tests necessarily satisfy condition (ii) of Theorem 4.3.2. That part of the proof of Theorem 4.3.2 is not needed for the proof of this theorem. □

EXAMPLE 4.3.2. Consider the testing problem

$$L_p(X_\nu) = M_3(\nu, p)$$
$$H : p_1 = p_2 = p_3 = 1/3, \ H \vee A : p_1 \le p_2 \le p_3 .$$

This is the case m = 3 of the testing problem considered in Section 3.2 part (3) and in Section 3.4. Transform $X_\nu = (X_{\nu 1}, X_{\nu 2}, X_{\nu 3})'$ to $Y_\nu = (Y_{\nu 1}, Y_{\nu 2})'$ by

$$Y_\nu = (2\nu)^{-\frac{1}{2}} \begin{pmatrix} 3 X_{\nu 2} - \nu \\ 3^{\frac{1}{2}}(X_{\nu 3} - X_{\nu 1}) \end{pmatrix} .$$

This is a 1 : 1 transformation. It has been chosen so that under H one has cov $Y_\nu = I$. The probability distributions of Y_ν will be parametrized by $\mu = \nu^{-\frac{1}{2}} E_p Y_\nu$:

$$\mu = \begin{pmatrix} 2^{-\frac{1}{2}} (3p_2 - 1) \\ (3/2)^{\frac{1}{2}} (p_3 - p_1) \end{pmatrix} .$$

In terms of μ, H and A are given by

$$H : \mu = 0, \ H \vee A : |\mu_1| \le 3^{-\frac{1}{2}} \mu_2 .$$

Let Φ be the class of all tests with convex acceptance region and define

$$P_{H\nu} = L_0(Y_\nu)$$
$$P_{A\nu} = \{L_\mu(Y_\nu) \mid \mu \in M_A\}$$
$$\Phi_\nu = \{\phi \in \Phi \mid E_0\phi(Y_\nu) \le \alpha\}$$

where

$$M_A = \{\mu \in \mathbb{R}^2 \mid \mu \ne 0, |\mu_1| \le 3^{-\frac{1}{2}} \mu_2\} .$$

Definition 4.2.2 will be applied with m = 2 and λ = Lebesgue measure. It will be proved that

$$(P_{H\nu}, P_{A\nu}, \Phi_\nu) \to (N_2(0, I), \{N_2(\mu, I) \mid \mu \in M_A\}, \Phi_\infty)$$

where

$$\Phi_\infty = \{\phi \in \Phi \mid E_{N(0,I)}\ \phi \leq \alpha\}.$$

Conditions L(i-v) can be verified in a similar way as in Example 4.2.1. Every test function with convex acceptance region is continuous a.e. [λ], so that L(vi) holds. The remark following Theorem A.4.2 implies that L(vii) holds. The sharp consistency of the test $\{\phi_\nu\}$ below implies that L(viii) holds.

Let

$$\beta^*_\nu(P) = \sup_{\phi\in\Phi_\nu} E_P\phi \qquad \nu \in \mathbb{N} \cup \{\infty\}\ .$$

A most stringent - $(\Phi_\infty, \beta^*_\infty)$ test for the limiting problem is given in part 2 of Section 3.3. According to Corollary 2.6.1, this test is unique up to equivalence a.e. [λ]. An asymptotically most stringent - $(\tilde{\Phi}, \tilde{\beta}^*)$ test can be given with the aid of Theorem 4.3.3. Let ϕ_ν be the test rejecting for

$$Y_{\nu 2} \geq 2\cdot 3^{-\frac{1}{2}}\, b^{-1}_{\nu\alpha}\, [c_{\nu\alpha} - \log\{\exp(\tfrac{1}{2}b_{\nu\alpha} Y_{\nu 1}) + \exp(-\tfrac{1}{2}b_{\nu\alpha} Y_{\nu 1})\}]\ ,$$

where $b_{\nu\alpha}$ and $c_{\nu\alpha}$ are chosen so that ϕ_ν is of size α and $b_{\nu\alpha} \to b_\nu(\pi/6)$, $c_{\nu\alpha} \to c_\alpha(\pi/6)$, b_α and c_α being given in the figures reproduced in part 2 of Section 3.3. More accurate values for b_α and c_α can be found in tables 3 and 4 of VAN LINDE, SCHAAFSMA and VELVIS (1967). Some values are

α	.05	.02	.01	.005
$b_\alpha(\pi/6)$	1.979	2.308	2.539	2.757
$c_\alpha(\pi/6)$	3.980	5.445	6.600	7.788 .

The test $\{\phi_\nu\}$ satisfies conditions (i) and (iii) of Theorem 4.3.3. In order to show that it is also sharply consistent, define ψ_ν as the test rejecting for

$$Y_{\nu 2} \geq 2\cdot 3^{-\frac{1}{2}}\, b^{-1}_{\nu\alpha}\, (c_{\nu\alpha} - \log 2)\ .$$

The inequality $t + t^{-1} \geq 2$ (for $t > 0$) implies that $\psi_\nu \leq \phi_\nu$. Hence it is sufficient to show that $\{\psi_\nu\}$ is sharply consistent. Let $\{L_{\mu_\xi}(Y_\xi)\}$ be a divergent sequence; then $\mu_{2\xi} \to \infty$. As $\mathrm{var}_\mu\, Y_{2\nu}$ is bounded uniformly in μ, it follows from $\mu_{2\xi} \to \infty$ that $Y_{2\xi} \to \infty$ in $\{\mu_\xi\}$ - prob.. Hence $E_{\mu_\xi}\, \psi_\xi \to 1$, establishing the sharp consistency of $\{\phi_\nu\}$.

Theorem 4.3.3 implies that $\{\phi_\nu\}$ is AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$. In practice one will not be concerned about the size of the test being slightly different from α, and one will use $b_{\nu\alpha} = b_\alpha(\pi/6)$, $c_{\nu\alpha} = c_\alpha(\pi/6)$.

It will be shown now that $\{\phi_\nu\}$ is also AMS in the following sense (stronger than Definition 4.3.1): if $\{\phi'_\nu\}$ is any test with

$$\limsup_{\nu} \; E_0\, \phi'_\nu\, (Y_\nu) \le \alpha,$$

then

$$\text{(1)} \qquad \limsup_{\nu} \; \{ \sup_{P \in \mathcal{P}_{A\nu}} \gamma_\nu\, (\phi_\nu, P) - \sup_{P \in \mathcal{P}_{A\nu}} \gamma_\nu\, (\phi'_\nu, P)\} \le 0 \; .$$

Define

$$c_\nu = \min\, \{\alpha \,/\, E_0 \phi'_\nu(Y_\nu),\; 1\}$$

$$\phi''_\nu = c_\nu \phi'_\nu \; .$$

Then $E_0\phi''_\nu\, (Y_\nu) \le \alpha$ for every ν and $E_{\mu_\nu}\, \mid \phi'_\nu(Y_\nu) - \phi''(Y_\nu) \mid \to 0$ for every sequence $\{\mu_\nu\}$. Hence it suffices to prove (1) for $\{\phi''_\nu\}$. According to Theorem 2.7.2, for every ϕ''_ν there exists a $\phi'''_\nu \in \Phi_\nu$ with

$$E_\mu \phi'''_\nu \ge E_\mu \phi''_\nu \quad \text{for all } \mu \in M_A \; .$$

This implies that

$$\sup_{P \in \mathcal{P}_{A\nu}} \gamma_\nu\, (\phi''_\nu, P) \ge \inf_{\phi \in \Phi_\nu} \; \sup_{P \in \mathcal{P}_{A\nu}} \gamma_\nu\, (\phi, P) \; .$$

Hence the result that $\{\phi_\nu\}$ is AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$ implies that $\{\phi_\nu\}$ is AMS in the stronger sense described above. (Compare Corollary 4.4.1.)

It follows from the proof of Theorem 4.3.2 that the AMXS - $(\tilde{\Phi}, \tilde{\beta}^*)$ is equal to the MXS - level α of the limit problem. Tables in VAN LINDE, SCHAAFSMA and VELVIS (1967) yield that the MXS-level .05 is .069 and the MXS-level .01 is .073. The test for the limit problem which rejects for $Y_2 \ge u_\alpha$

has also good power properties; its shortcoming is 0 for certain parameter values satisfying the alternative hypothesis, while its maximum shortcoming is .108 for α = .05 and .184 for α = .01. (A more detailed comparison between this test and the MS - level α test is made in SCHAAFSMA (1968).) It can be inferred that the test $\{\phi'_\nu\}$ where ϕ'_ν rejects for $Y_{\nu 2} \geq u_\alpha$, is a good test for the asymptotic testing problem. This test is AMS - $(\tilde{\Psi}, \tilde{\beta}^*)$ when Ψ_ν is defined by

$$\Psi_\nu = \{I_{(u_\alpha, \infty)} (d'Y_\nu) \mid d \in \mathbb{R}^2 , \| d \| = 1\} .$$

□

4.4. APPLICATION OF SECTION 4.2 TO ASYMPTOTIC TESTING PROBLEMS FOR EXPONENTIAL FAMILIES

In the remainder of this chapter we consider the asymptotic testing problem $\{\mathcal{T}_\nu\}$ of Section 3.5 of which Examples 4.2.2 and 4.3.2 are special cases, and use the notation introduced in Section 3.5. Recall that both the vector of sample means $X_{.}^{(\nu)}$ and $Y_\nu = [n(\nu)]^{\frac{1}{2}} f(X_{.}^{(\nu)})$ are sufficient statistics for $\mathcal{T}_\nu$, that the probability distributions are parametrized by $\mu \in F \subset \mathbb{R}^m$,

$$E_\mu X_{.}^{(\nu)} = \mu$$

$$\text{cov}_\mu X_{.}^{(\nu)} = [n(\nu)]^{-1} R_\nu^{-1} \Sigma_\mu \quad ,$$

and that the "asymptotic covariance matrix"of Y_ν is denoted by

$$\Lambda_{\nu\mu} = \Lambda(R_\nu, \mu) = D_\mu R_\nu^{-1} \Sigma_\mu D'_\mu \quad .$$

Null hypothesis and alternative hypothesis are given by

$$H : f(\mu) \in V$$

$$H \vee A : f(\mu) \in V + K ,$$

where V is a linear subspace of $\mathbb{R}^m$ and K a cone in $\mathbb{R}^m$. Consult Sections 3.1 and 3.5 for other assumptions and definitions.

In the present section, we consider sequences of simple null hypotheses and alternative hypotheses given by

$$H_\nu : \mu = \mu_\nu$$
$$A_\nu : \mu \in M_\nu .$$

Theorem 2.7.2 shows that the class of tests of the form $\phi(X_.^{(\nu)})$, with $\phi \in \Phi_C$, constitutes an essentially complete class. For the proofs of sharp consistency, it will appear that Y_ν is a more convenient random variable to work with than $X_.^{(\nu)}$. In this section, the basic random variable is

$$\tilde{Y}_\nu = Y_\nu - [n(\nu)]^{\frac{1}{2}} f(\mu_\nu) = [n(\nu)]^{\frac{1}{2}} (f(X_.^{(\nu)}) - f(\mu_\nu)) .$$

The class of tests of the form $\phi(X_.^{(\nu)})$ with $\phi \in \Phi_C$, corresponds to the class of tests $\phi(\tilde{Y}_\nu)$ with $\phi \in \Phi_{1\nu}$, where

$$\Phi_{1\nu} = \{\phi \mid \text{a } \psi \in \Phi_C \text{ exists with } \psi(x) = \phi([n(\nu)]^{\frac{1}{2}}(f(x) - f(\mu_\nu)))$$
$$\text{for all } x \in F;\ \phi(x) = 1 \text{ for } x \notin [n(\nu)]^{\frac{1}{2}}(f(F) - f(\mu_\nu))\} .$$

Section 4.2 will be applied with λ = Lebesgue measure. The class of test functions Φ must be weakly* compact with $\Phi_{1\nu} \subset \Phi$ for all ν. A convenient choice is

(4.4.1) $$\Phi = (\cup_\nu \Phi_{1\nu}) \cup \Phi_C .$$

<u>LEMMA 4.4.1</u>. *The class* Φ *defined by* (4.4.1) *is weakly* compact.*

<u>PROOF</u>. It is sufficient to prove that (i) $\Phi_{1\nu}$ is compact for every ν; and (ii) if $\phi_\nu \in \Phi_{1\nu}$ for all ν, then there exist a subsequence $\{\phi_\xi\}$ of $\{\phi_\nu\}$ and a $\phi \in \Phi_C$ with $\phi_\xi \overset{*}{\to} \phi$. Assertion (i) follows from the compactness of Φ_C (Theorem A.4.1 (i)) and the assumption that f, and hence also the function $x \mapsto [n(\nu)]^{\frac{1}{2}} (f(x) - f(\mu_\nu))$, has a continuous inverse (Section 3.1). Assertion (ii) follows from the weak* compactness of the class of all test functions (Theorem 2.4.2) and Lemma A.4.4 (i). □

It may be noted that Φ is not weakly* closed as a subset of the class of all test functions: if $\phi \in \Phi$ and $\psi = \phi$ a.e. $[\lambda]$ then it is possible that $\psi \notin \Phi$. (The class of all test functions with the weak* toplogy is not Hausdorff.)

THEOREM 4.4.1. *Suppose that*

(i) $\rho_i(\nu) \to \rho_i > 0 \qquad (1 \le i \le k)$

(ii) $\mu_\nu \to \mu_0 \in F$

(iii) $\mathrm{Lt}_\nu [n(\nu)]^{\frac{1}{2}} (f(M_\nu) - f(\mu_\nu)) = M$ *as subsets of* $\mathbb{R}^m$

(iv) $M_A = M \setminus \{0\} \neq \emptyset$, $M = \mathrm{cl}\, M_A$

(v) $\Phi_\nu \subset \Phi_{1\nu} \cup \Phi_C$ *for every* ν, $\Phi_\nu \neq \emptyset$

(vi) $\mathrm{Lt}_\nu\, \Phi_\nu = \Phi_\infty$ *as subsets of* Φ.

Then

$$(L_{\mu_\nu}(\tilde{Y}_\nu),\ \{L_\mu(\tilde{Y}_\nu) \mid \mu \in M_\nu\},\ \Phi_\nu) \to$$

$$\to (N(0, \Lambda),\ \{N(\eta, \Lambda) \mid \eta \in M_A\},\ \Phi_\infty)$$

where $\Lambda = \Lambda(R, \mu_0)$, R *being the proportion matrix corresponding to proportions* $\rho_1, \ldots, \rho_k$. *Moreover* $\Phi_\infty \subset \Phi_C$, *and* $\{L_{\mu_{1\nu}}(\tilde{Y}_\nu)\}$ *is relatively compact in* $M_1(\mathbb{R}^m)$ *iff* $\{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}$.

PROOF. The conditions of Definition 4.2.2 will be verified. For L(i,ii,iii) it is sufficient to prove that

(a) $[n(\nu)]^{\frac{1}{2}}(f(\mu_{1\nu}) - f(\mu_\nu)) \to \eta$ implies that $L_{\mu_{1\nu}}(\tilde{Y}_\nu) \to N(\eta, \Lambda)$

(b) $[n(\nu)]^{\frac{1}{2}}\|f(\mu_{1\nu}) - f(\mu_\nu)\| \to \infty$ implies that $\{L_{\mu_{1\nu}}(\tilde{Y}_\nu)\}$ is divergent.

(a) Suppose that $[n(\nu)]^{\frac{1}{2}}(f(\mu_{1\nu}) - f(\mu_\nu)) \to \eta$. Then $\mu_{1\nu} \to \mu_0$ by (ii) and the continuity of the inverse of f. It can be concluded from Theorem 2.3.1 that the third moments of $X_{ij}^{(\nu)}$ are uniformly bounded for $\mu \in K$, when K is a compact neighbourhood of μ_0 with $K \subset F$. Hence Liapounov's Theorem can be applied, so that

$$L_{\mu_{1\nu}}([n(\nu)]^{\frac{1}{2}} (X_{\cdot}^{(\nu)} - \mu_{1\nu})) \to N(0, R^{-1} \Sigma_{\mu_0}) .$$

Since f is continously differentiable this implies (see, e.g., RAO (1973) Section 6a.2) that

$$L_{\mu_{1\nu}}([n(\nu)]^{\frac{1}{2}} (f(X_{\cdot}^{(\nu)}) - f(\mu_{1\nu}))) \to N(0, \Lambda) .$$

With $[n(\nu)]^{\frac{1}{2}}(f(\mu_{1\nu}) - f(\mu_\nu)) \to \eta$, this shows that $L_{\mu_{1\nu}}(\tilde{Y}_\nu) \to N(\eta, \Lambda)$.

(b) Suppose that $[n(\nu)]^{\frac{1}{2}}\| f(\mu_{1\nu}) - f(\mu_\nu) \| \to \infty$. Then also

$[n(\nu)]^{\frac{1}{2}} \, || \mu_{1\nu} - \mu_\nu || \to \infty$. Proposition A.2.1 yields the existence of a sequence $\{y_\nu\} \subset \mathbb{R}^m$, $||y_\nu|| = 1$ with $[n(\nu)]^{\frac{1}{2}} y'_\nu (X^{(\nu)}_{\cdot} - \mu_\nu) \to \infty$ in $\{\mu_{1\nu}\}$ - prob.. Hence

$$P_{\mu_{1\nu}} \{[n(\nu)]^{\frac{1}{2}} \, || X^{(\nu)}_{\cdot} - \mu_\nu || \le r\} \to 0$$

for every r. For every r, there exists an r' such that

$$\{x \mid [n(\nu)]^{\frac{1}{2}} \, || f(x) - f(\mu_\nu) || \le r\} \subset \{x \mid [n(\nu)]^{\frac{1}{2}} \, || x - \mu_\nu || \le r'\}.$$

This shows that

$$P_{\mu_{1\nu}} \{ || \tilde{Y}_\nu || \le r\} \to 0$$

for every r, so that $\{L_{\mu_{1\nu}}(\tilde{Y}_\nu)\}$ is divergent.

L(iv) follows immediately from condition (vi).

L(v) follows from $|\Lambda| \neq 0$.

L(vi) According to Lemma A.4.4 (i), every $\phi \in \Phi_\infty$ is a.e. $[\lambda]$ equal to a test with convex acceptance region. The definition of Φ implies that every $\phi \in \Phi_\infty$ has itself a convex acceptance region. In particular, every $\phi \in \Phi_\infty$ is a.e. $[\lambda]$ continuous.

L(vii) follows from Theorem A.4.2.

That $\Phi_\infty \subset \Phi_C$ has been demonstrated in the proof of L(vi). Note that $\{\mu_{1\nu}\} \triangleleft\triangleright \{\mu_\nu\}$ is equivalent to $\lim\sup_\nu [n(\nu)]^{\frac{1}{2}} \, || \mu_{1\nu} - \mu_\nu || < \infty$ (see Section 4.1), which again is equivalent to $\lim\sup_\nu [n(\nu)]^{\frac{1}{2}} || f(\mu_{1\nu}) - f(\mu_\nu) || < \infty$. Hence (a) and (b) show that $\{L_{\mu_{1\nu}}(\tilde{Y}_\nu)\}$ is relatively compact iff $\{\mu_{1\nu}\} \triangleleft\triangleright \{\mu_\nu\}$. □

<u>THEOREM 4.4.2</u>. *Let* $\Phi_\nu = \{\phi \in \Phi_{1\nu} \mid E_{\mu_\nu} \phi(\tilde{Y}_\nu) \le \alpha\}$. *Then*

$$Lt_\nu \, \Phi_\nu = \{\phi \in \Phi_C \mid E_{N(0,\Lambda)} \phi \le \alpha\} .$$

<u>PROOF</u>. Denote $\{\phi \in \Phi_C \mid E_{N(0,\Lambda)} \phi \le \alpha\}$ by Φ_∞. It must be proved that $Ls_\nu \, \Phi_\nu \subset \Phi_\infty \subset Li_\nu \, \Phi_\nu$. As $L_{\mu_\nu}(\tilde{Y}_\nu) \to N(0,\Lambda)$ weakly and $|\Lambda| \neq 0$, it follows from Lemma A.4.4 (i) and Theorem A.4.2 that $Ls_\nu \, \Phi_\nu \subset \Phi_\infty$. Now let $\phi \in \Phi_\infty$. According to Lemma A.4.4 (iv), a sequence $\{\phi_{1\nu}\}$ exists with $\phi_{1\nu} \in \Phi_{1\nu}$ and $\phi_{1\nu} \overset{*}{\to} \phi$. Theorem A.4.2 shows that $E_{\mu_\nu} \phi_{1\nu}(\tilde{Y}_\nu) \to E_{N(0,\Lambda)} \phi \le \alpha$. Hence there

exist $\phi_\nu \in \Phi_{1\nu}$ with $E_{\mu_\nu}\phi_\nu(\tilde{Y}_\nu) \leq \alpha$ for all ν and $\phi_\nu \overset{*}{\to} \phi$. Hence $\phi \in Li_\nu \Phi_\nu$. □

THEOREM 4.4.3. *Let Φ_0 be a weakly* compact subset of Φ_C such that for every $\phi \in \Phi_0$ a sequence $\{\phi_h\} \subset \Phi_0$ exists with $\phi_h(x) \leq \phi(x)$ for all x, $\lambda\{\phi_h(x) < \phi(x)\} > 0$ for all h, and $\phi_h \overset{*}{\to} \phi$. Let $\Phi_\nu = \{\phi \in \Phi_0 \mid E_{\mu_\nu}\phi(\tilde{Y}_\nu) \leq \alpha\}$. Then*

$$Lt_\nu \Phi_\nu = \{\phi \in \Phi_0 \mid E_{N(0,\Lambda)}\phi \leq \alpha\} .$$

PROOF. This follows immediately from Proposition 4.2.1 and Theorem A.4.2. □

These theorems will be applied in Chapters 5, 6 and 8. The following two corollaries will play a role in these applications.

COROLLARY 4.4.1. *Suppose that assumptions (i), (ii), (iii) and (iv) of Theorem 4.4.1 are satisfied. Let $\phi \in \Phi_C$ be the MS - level α test and let γ be the minimax - level α shortcoming for the testing problem*

$$(\mathbb{R}^m, \{N(0, \Lambda)\}, \{N(\eta, \Lambda) \mid \eta \in M_A\})$$

and let the shortcoming at stage ν be defined by

$$\gamma_\nu(\phi',\mu) = \sup \{E_\mu \psi(\tilde{Y}_\nu) \mid E_{\mu_\nu} \psi(\tilde{Y}_\nu) \leq \alpha\} - E_\mu\phi'(\tilde{Y}_\nu) .$$

Then

$$\gamma = \sup_{\{\mu_{1\nu}\}\in K} \limsup_\nu \gamma_\nu (\phi, \mu_{1\nu})$$

and for every test $\{\phi_\nu\}$ with $\limsup_\nu E_{\mu_\nu}\phi_\nu(\tilde{Y}_\nu) \leq \alpha$ one has that

$$\gamma \leq \sup_{\{\mu_{1\nu}\}\in K} \liminf_\nu \gamma_\nu (\phi_\nu, \mu_{1\nu})$$

where

$$K = \{\{\mu_{1\nu}\} \mid \mu_{1\nu} \in M_\nu, \ \{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}\} .$$

PROOF. Let $\Phi_\nu = \{\phi' \in \Phi_{1\nu} \mid E_{\mu_\nu}\phi'(\tilde{Y}_\nu) \leq \alpha\} \cup \{\phi\}$. It can be concluded from Theorem 4.4.2 that

$$\mathrm{Lt}_\nu \Phi_\nu = \{\phi' \in \Phi_C \mid E_{N(0,\Lambda)} \phi' \leq \alpha\} .$$

Let $\{\phi_\nu\}$ be a test with $\lim\sup_\nu E_{\mu_\nu} \phi_\nu(Y_\nu) \leq \alpha$ and let

$$c_\nu = \min \{\alpha / E_{\mu_\nu} \phi_\nu(\tilde{Y}_\nu), 1\} .$$

Then $E_{\mu_\nu} c_\nu \phi_\nu(\tilde{Y}_\nu) \leq \alpha$ for every ν.
As Φ_C is a complete class for the testing problem in terms of $X^{(\nu)}_{\cdot}$ (Theorem 2.7.2), there exists for every ν a $\phi'_\nu \in \Phi_\nu$ with $E_\mu c_\nu \phi_\nu(\tilde{Y}_\nu) \leq E_\mu \phi'_\nu(\tilde{Y}_\nu)$ for every $\mu \in M_\nu$. With $c_\nu \to 1$, this shows that for every $\{\mu_{1\nu}\} \in K$,

$$\liminf_\nu \gamma_\nu (\phi'_\nu, \mu_{1\nu}) \leq \liminf_\nu \gamma_\nu (\phi_\nu, \mu_{1\nu}).$$

The assertions to be proved now follow from Theorem 4.4.1 and Corollary 4.2.3. □

<u>COROLLARY 4.4.2</u>. *Suppose that* $\{\mu_\nu\}$ *is a relatively compact sequence in* F *and that* $\{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}$. *Let the test* ϕ_ν *reject for*

$$[n(\nu)]^{\frac{1}{2}}(\mu_{1\nu} - \mu_\nu)' R_\nu \Sigma^{-1}_{\mu_\nu}(X^{(\nu)}_{\cdot} - \mu_\nu) \geq u_\alpha [(\mu_{1\nu} - \mu_\nu)' R_\nu \Sigma^{-1}_{\mu_\nu}(\mu_{1\nu} - \mu_\nu)]^{\frac{1}{2}}.$$

Then $E_{\mu_\nu} \phi_\nu \to \alpha$; *and for every test* $\{\psi_\nu\}$ *with* $\lim\sup_\nu E_{\mu_\nu} \psi_\nu \leq \alpha$ *one has that*

$$\liminf_\nu \; E_{\mu_{1\nu}} (\phi_\nu - \psi_\nu) \geq 0 .$$

<u>PROOF</u>. It is not a restriction to assume that $[n(\nu)]^{\frac{1}{2}} (\mu_{1\nu} - \mu_\nu) \to \eta \neq 0$, that $\rho_i(\nu) \to \rho_i > 0$ for all i and that $\mu_\nu \to \mu_0 \in$ F. Corollary 4.4.1 will be applied, f being the identity function so that $\tilde{Y}_\nu = [n(\nu)]^{\frac{1}{2}} (X^{(\nu)}_{\cdot} - \mu_\nu)$, and M_ν consisting only of $\mu_{1\nu}$.
Conditions (i) - (iv) of Theorem 4.4.1 are satisfied with $M_A = \{\eta\}$. The MS - level α test ϕ for the limiting problem is even UMP - level α (in other words, $\gamma = 0$) and it is given by

$$\phi(x) = \begin{cases} 1 & \eta' \Lambda^{-1} x > u_\alpha [\eta' \Lambda^{-1} \eta]^{\frac{1}{2}} \\ 0 & \eta' \Lambda^{-1} x \leq u_\alpha [\eta' \Lambda^{-1} \eta]^{\frac{1}{2}} \end{cases} .$$

As $E_{\mu_\nu} \mid \phi(\tilde{Y}_\nu) - \phi_\nu(\tilde{Y}_\nu) \mid \to 0$ and $L_{\mu_\nu}(\tilde{Y}_\nu) \to N(0, \Lambda)$,one has that $E_{\mu_\nu}\phi_\nu \to \alpha$. Corollary 4.4.1 and Proposition 4.1.1 yield that

$$\lim_\nu \gamma_\nu (\phi_\nu, \mu_{1\nu}) = 0 \leq \liminf_\nu \gamma_\nu (\psi_\nu, \mu_{1\nu}) .$$

This implies that $\liminf_\nu E_{\mu_{1\nu}} (\phi_\nu - \psi_\nu) \geq 0$. □

Of course, Corollary 4.4.2 can also be proved more directly by using the Neyman-Pearson Fundamental Lemma to produce the exact MP - level α test for the testing problem

$$(\mathbb{R}^m, \{L_{\mu_\nu}(\tilde{Y}_\nu)\}, \{L_{\mu_{1\nu}}(\tilde{Y}_\nu)\}) .$$

4.5. ASYMPTOTIC UNIQUENESS OF THE ASYMPTOTICALLY MOST STRINGENT TEST

In this section the asymptotic testing problem of Section 4.4 will be specialized to the case where the sequences of null hypotheses and of alternative hypotheses are given by

$$H_\nu : \mu = \mu_\nu$$

$$A_\nu : f(\mu) - f(\mu_\nu) \in M_A$$

where $\mu_\nu \to \mu_0$ for some $\mu_0 \in F$, and where $M_A \cup \{0\}$ is a closed cone in $\mathbb{R}^m$. This sequence of testing problems will play a central role in the proofs of several theorems in Chapters 6 and 8. Furthermore it will be supposed that $\rho_i(\nu) \to \rho_i$ for all i, and that Φ_ν is as in Theorem 4.4.2. Then all conditions of Theorem 4.4.1 are satisfied, and it can be concluded that

$$(L_{\mu_\nu}(\tilde{Y}_\nu), \{L_\mu(\tilde{Y}_\nu) \mid \mu \in M_\nu\}, \Phi_\nu) \to$$

$$\to (N(0, \Lambda), \{N(\eta, \Lambda) \mid \eta \in M_A\}, \Phi_\infty)$$

where $\Lambda = \Lambda(R, \mu_0)$ and

$$\Phi_\infty = \{\phi \in \Phi_C \mid E_{N(0, \Lambda)}\phi \leq \alpha\} .$$

The functions β^*_ν and β^*_∞ will be the envelope power functions

$$\beta_\nu^*(\mu) = \sup_{\phi \in \Phi_\nu} E_\mu \phi_\nu(\tilde{Y}_\nu)$$

$$\beta_\infty^*(\eta) = \sup_{\phi \in \Phi_\infty} E_{N(\eta,\Lambda)} \phi .$$

Corollary 2.6.1 shows that the most stringent - $(\Phi_\infty, \beta_\infty^*)$ test for this testing problem is unique up to equivalence a.e.. Let C be the acceptance region of this most stringent test, and $\phi = 1 - I_C$ the most stringent test; denote the minimax shortcoming by γ. Corollary 4.2.2 shows that if $\phi_\nu \in \Phi_\nu$ for every ν, then the condition that $\limsup_\nu \gamma_\nu(\phi_\nu, \mu_{1\nu}) \leq \gamma$ for every $\{\mu_{1\nu}\} \in K$ where

$$K = \{\{\mu_{1\nu}\} | \ f(\mu_{1\nu}) - f(\mu_\nu) \in M_A, \{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}\}$$

is equivalent to $\phi_\nu \overset{*}{\to} \phi$. The following theorem can be interpreted as establishing the asymptotic uniqueness of the asymptotically most stringent test without the restriction that $\phi_\nu \in \Phi_\nu$ for every ν.

<u>THEOREM 4.5.1</u>. *If* $\{\phi_\nu\}$ *is a test which satisfies*

$$\limsup_\nu E_{\mu_\nu} \phi_\nu(\tilde{Y}_\nu) \leq \alpha$$

$$\limsup_\nu \gamma_\nu(\phi_\nu, \mu_{1\nu}) \leq \gamma \quad \text{for all } \{\mu_{1\nu}\} \in K ,$$

then $E_{\mu_\nu} | \phi_\nu(\tilde{Y}_\nu) - \phi(\tilde{Y}_\nu)| \to 0$.

<u>PROOF</u>. For notational simplicity, the proof will be given only for $k = 1$ and $n(\nu) = \nu$.

(i) It will be indicated first, that it is sufficient to consider only the case where f is the identity function. Define

$$K' = \{\{\mu_{1\nu}\} \mid D_{\mu_0}(\mu_{1\nu} - \mu_\nu) \in M_A, \{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}\} .$$

If $\nu^{\frac{1}{2}} \| \mu_{1\nu} - \mu_{2\nu} \| \to 0$, then $\| L_{\mu_{1\nu}}(\tilde{Y}_\nu) - L_{\mu_{2\nu}}(\tilde{Y}_\nu) \| \to 0$. Hence the condition that $\limsup_\nu \gamma_\nu(\phi_\nu, \mu_{1\nu}) \leq \gamma$ for all $\{\mu_{1\nu}\} \in K$ is equivalent to the condition that this holds for all $\{\mu_{1\nu}\} \in K'$. Therefore it is sufficient to consider the case where $f(x) = D_{\mu_0} x$. But then A_ν can be expressed by $A_\nu : \mu - \mu_\nu \in D_{\mu_0} M_A$. As $D_{\mu_0}^{-1} M_A \cup \{0\}$ is a closed cone, this shows that it is sufficient to consider the case where $f(x) \equiv x$. It will be assumed in the remainder of this proof that $f(x) \equiv x$. Note that this implies that $\tilde{Y}_\nu = \nu^{\frac{1}{2}}(X^{(\nu)}_{\cdot} - \mu_\nu)$ and $\Lambda = \Sigma_{\mu_0}$.

(ii) Proposition 2.6.2 shows that a least favourable distribution τ exists for the limiting testing problem

$$\tilde{T}_\infty = (\mathbb{R}^m, \{N(0, \Lambda)\}, \{N(\eta, \Lambda) \mid \eta \in M_A\}, \Phi_\infty) .$$

Corollary 2.8.1 (ii) implies that τ has a bounded support on M_A. Define

$$p_\eta(y) = \exp (\eta' \Lambda^{-1} y - \tfrac{1}{2}\eta' \Lambda^{-1} \eta) = d\,N(\eta, \Lambda) \,/\, d\,N(0, \Lambda)$$

$$\ell_\tau(y) = \int p_\eta(y)\, d\tau(\eta).$$

As ϕ is Bayes - level α against τ, Proposition 2.5.1 shows that a constant c exists such that the acceptance region of ϕ is the set $C = \{y \mid \ell_\tau(y) \leq c\}$. Then $\phi = 1 - I_C$. Define

$$d_\nu = E_{\mu_\nu}\, \ell_\tau\, (\tilde{Y}_\nu)(\phi_\nu(\tilde{Y}_\nu) - \phi(\tilde{Y}_\nu)) .$$

It will be shown that $\lim\inf_\nu d_\nu \geq 0$.

Let θ be the natural parameter for the exponential family and let $Q_\theta^{(\nu)} = L_\mu(\tilde{Y}_\nu)$, where $\mu = \mu(\theta)$. Let $\mu_\nu = \mu(\theta_\nu)$. Then

$$d\, Q^{(\nu)}_{\theta_\nu + \nu^{-\frac{1}{2}}\eta} \,/\, d\, Q^{(\nu)}_{\theta_\nu}(y) =$$

$$= \exp\, [\eta' y - \nu\{\psi(\theta_\nu + \nu^{-\frac{1}{2}}\eta) - \psi(\theta_\nu) - \nu^{-\frac{1}{2}}\, \eta'\mu_\nu\}]$$

$$= p_{\Lambda\eta}\, (y)\, c_\nu(\eta)$$

where

$$\log c_\nu(\eta) = \tfrac{1}{2}\eta'\Lambda\eta - \nu\{\psi(\theta_\nu + \nu^{-\frac{1}{2}}\eta) - \psi(\theta_\nu) - \nu^{-\frac{1}{2}}\eta'\mu_\nu\} .$$

Define

$$\ell_\tau^{(\nu)}(y) = \int p_\eta(y)\, c_\nu(\Lambda^{-1}\eta)\, d\tau(\eta).$$

Corollary 2.3.1 implies that $c_\nu(\xi) \to 1$ uniformly on compacts. Hence

(1) $\ell_\tau^{(\nu)}(y) \,/\, \ell_\tau(y) \to 1$ uniformly in y.

Define $\mu_\nu(\eta) = \mu(\theta_\nu + \nu^{-\frac{1}{2}}\Lambda^{-1}\eta)$. Fubini's Theorem implies

(2) $$E_{\mu_\nu}\phi(\tilde{Y}_\nu)\ell_\tau^{(\nu)}(\tilde{Y}_\nu) = \int E_{\mu_\nu(\eta)}\phi(\tilde{Y}_\nu)\, d\tau(\eta) \; .$$

As ϕ is a.e. continuous and

$$Q^{(\nu)}_{\theta_\nu+\nu^{-\frac{1}{2}}\Lambda^{-1}\eta} \to N(\eta,\, \Lambda) \text{ weakly}$$

for all η, Lebesgue's Theorem of dominated convergence implies

(3) $$\int E_{\mu_\nu(\eta)}\phi(\tilde{Y}_\nu)\, d\tau(\eta) \to \int E_{N(\eta,\Lambda)}\phi(Y)\, d\tau(\eta) \; .$$

Define

$$\beta^*(\tau) = \int \beta^*(\eta)\, d\tau(\eta) \; ;$$

then (2) and (3) yield

(4) $$E_{\mu_\nu}\phi(\tilde{Y}_\nu)\, \ell_\tau^{(\nu)}(\tilde{Y}_\nu) \to \beta^*(\tau) - \gamma \; .$$

For every $\eta \in M_A$ there is a sequence $\{\mu_{1\nu}\} \in K$ with $\nu^{\frac{1}{2}}\, \|\mu_{1\nu} - \mu_\nu(\eta)\| \to 0$. Therefore the second assumption about $\{\phi_\nu\}$ implies that for every $\eta \in M_A$ one has

$$\limsup_\nu \{\beta^*(\eta) - E_{\mu_\nu(\eta)}\, \phi_\nu(\tilde{Y}_\nu)\} \le \gamma \; .$$

With Fatou's Lemma this implies

$$\limsup_\nu \int \{\beta^*(\eta) - E_{\mu_\nu(\eta)}\phi_\nu(\tilde{Y}_\nu)\}\, d\tau(\eta) \le \gamma.$$

Fubini's Theorem applied to the left hand side yields

(5) $$\liminf_\nu E_{\mu_\nu}\phi_\nu(\tilde{Y}_\nu)\; \ell_\tau^{(\nu)}(\tilde{Y}_\nu) \ge \beta^*(\tau) - \gamma \; .$$

(1), (4) and (5) imply that $\liminf_\nu d_\nu \geq 0$.

(iii) Define

$$b_\nu = -2d_\nu + 2c\, E_{\mu_\nu}(\phi_\nu(\tilde{Y}_\nu) - \phi(\tilde{Y}_\nu)).$$

It follows from Lemma 4.5.1 that for every ν and for $c < h_\nu \leq 2c$,

$$E_{\mu_\nu}|\phi_\nu(\tilde{Y}_\nu) - \phi(\tilde{Y}_\nu)| \leq b_\nu/(h_\nu - c) + 2P_{\mu_\nu}\{c \leq \ell_\tau(\tilde{Y}_\nu) < h_\nu\}.$$

The result that $\liminf_\nu d_\nu \geq 0$ and the first assumption made about $\{\phi_\nu\}$ imply that $\limsup_\nu b_\nu \leq 0$. Hence a sequence $\{h_\nu\}$ can be chosen with $h_\nu > c$, $h_\nu \to c$ and $\limsup_\nu b_\nu/(h_\nu - c) \leq 0$. Then also $P_{\mu_\nu}\{c \leq \ell_\tau(\tilde{Y}_\nu) < h_\nu\} \to 0$, so that one has $E_{\mu_\nu}|\phi_\nu(\tilde{Y}_\nu) - \phi(\tilde{Y}_\nu)| \to 0$. □

The following lemma is used in the proof above, but it is also of some interest in itself. Theorem 2.4.1 and Proposition 2.5.1 show that most powerful and Bayes tests have the form

$$\phi(x) = \begin{cases} 1 & \ell(x) > c \\ 0 & \ell(x) < c \end{cases},$$

where ℓ is the likelihood ratio and c a constant, determined by the size of the test. How different can a test be from such a most powerful test, and still be almost as powerful as the most powerful test? A special case of the lemma yields the following answer: if $\ell = dP_A / dP_H$, ϕ is as above, $E_{P_H}\phi' \leq E_{P_H}\phi$ and $E_{P_A}\phi' \geq E_{P_A}\phi - \varepsilon$, then

$$\tfrac{1}{2}E_{P_H}|\phi - \phi'| \leq \inf_{h>c}[\varepsilon / (h-c) + P_H\{c \leq \ell(X) < h\}].$$

<u>LEMMA 4.5.1</u>. *Let* P *be a probability measure on* (X, F), $\ell : X \to [0, \infty)$ *an integrable function and* $\phi : X \to [0,1]$ *a test function with*

$$\phi(x) = \begin{cases} 1 & \ell(x) > c \\ 0 & \ell(x) < c \end{cases},$$

for some c. *Then for every test function* ϕ' *and for every* $h > c$,

$$E_P|\phi - \phi'| \leq (h-c)^{-1} \{2\, E_P\, \ell(\phi - \phi') - (h+c) E_P(\phi - \phi')\} +$$

$$+ 2\, P\{c \leq \ell(X) < h\}\ .$$

PROOF. Let $A = \ell^{-1}(c, \infty)$, $B = \ell^{-1}[0,c]$, $C = \ell^{-1}[h, \infty)$ and $D = \ell^{-1}(c, h)$. Note that $A = C \cup D$. First an upper bound will be obtained for $E_P|\phi_0 - \phi'|$, where $\phi_0 = I_A$. For $d_0 = E_P(\phi_0 - \phi')$, $d_1 = E_P\, \ell(\phi_0 - \phi')$ we have that

(1) $$\int_B \phi'\, dP = \int_A (1-\phi')\, dP - d_0\ ,$$

(2) $$\int_A \ell(1-\phi')\, dP = \int_B \ell\phi'\, dP + d_1 \leq c \int_B \phi'\, dP + d_1 \overset{(1)}{=}$$

$$= c \int_A (1-\phi')\, dP - c\, d_0 + d_1 \leq$$

$$\leq c\, h^{-1} \int_C \ell(1-\phi')\, dP + \int_D \ell(1-\phi')\, dP - c\, d_0 + d_1\ ,$$

(3) $$(h-c) \int_C (1-\phi')\, dP \leq (1 - ch^{-1}) \int_C \ell(1-\phi')\, dP \overset{(2)}{\leq} d_1 - c\, d_0 .$$

Hence

$$E_P|\phi_0 - \phi'| = \int_A (1-\phi')\, dP + \int_B \phi'\, dP \overset{(1)}{=} 2 \int_A (1-\phi')\, dP - d_0 \overset{(3)}{\leq}$$

$$\leq (h-c)^{-1} \{2d_1 - (h+c)\, d_0\} + 2P(D).$$

Let $d_2 = E_P|\phi - \phi_0| = \int_{\{\ell(x)=c\}} \phi\, dP$. Then $d_2 \leq P\{\ell(X) = c\}$ and

$$E_P\ (\phi - \phi') = d_0 + d_2$$

$$E_P\ell(\phi - \phi') = d_1 + c d_2\ .$$

Hence

$$E_P|\phi - \phi'| \leq E_P|\phi - \phi_0| + E_P|\phi_0 - \phi'|$$

$$\leq d_2 + (h-c)^{-1} \{2E_P\ell(\phi - \phi') - (h+c)\ E_P(\phi - \phi') +$$

$$+ (h-c)d_2\} + 2P(D)$$

$$\leq (h-c)^{-1} \{2E_P\, \ell(\phi - \phi') - (h+c)\, E_P(\phi - \phi')\} +$$

$$+ 2P\{c \leq \ell(X) < h\} .$$

□

Of course it is possible to consider tests of the form $\phi_\nu(X_{11}^{(\nu)}, \ldots, X_{kn_k(\nu)}^{(\nu)})$ instead of only those tests which depend on $\tilde{Y}_\nu$. For such tests, the conclusion of Theorem 4.5.1 remains valid. To see this, let $\{\phi_\nu\}$ be a sequence of tests of the form $\phi_\nu(X_{11}^{(\nu)}, \ldots, X_{kn_k(\nu)}^{(\nu)})$ with

$$\limsup_\nu E_{\mu_\nu} \phi_\nu\, (X_{11}^{(\nu)}, \ldots, X_{kn_k(\nu)}^{(\nu)}) \leq \alpha$$

$$\limsup_\nu \gamma_\nu(\phi_\nu, \mu_{1\nu}) \leq \gamma \qquad \text{for all } \{\mu_{1\nu}\} \in K .$$

Define

$$\psi_\nu\, (\tilde{Y}_\nu) = E\,\{\phi_\nu(X_{11}^{(\nu)}, \ldots, X_{kn_k}^{(\nu)}) \mid \tilde{Y}_\nu\}.$$

As $\tilde{Y}_\nu$ is a sufficient statistic, this conditional expectation is independent of μ. The test $\{\psi_\nu\}$ satisfies the assumptions of Theorem 4.5.1. As the test function ϕ assumes only the values 0 and 1, the conclusion of Theorem 4.5.1 can be written as

$$E_{\mu_\nu}\, \{\psi_\nu(\tilde{Y}_\nu)\, I_{\{0\}}(\phi(\tilde{Y}_\nu)) + (1 - \psi_\nu(\tilde{Y}_\nu))\, I_{\{1\}}(\phi(\tilde{Y}_\nu))\} \to 0.$$

This implies that

$$E_{\mu_\nu} |\phi_\nu(X_{11}^{(\nu)}, \ldots, X_{kn_k(\nu)}^{(\nu)}) - \phi(\tilde{Y}_\nu)| =$$

$$= E_{\mu_\nu}\, [E\,\{\phi_\nu(X_{11}^{(\nu)}, \ldots, X_{kn_k(\nu)}^{(\nu)})\, I_{\{0\}}(\phi(\tilde{Y}_\nu)) +$$

$$+ (1-\phi_\nu(X_{11}^{(\nu)}, \ldots, X_{kn_k(\nu)}^{(\nu)}))\, I_{\{1\}}(\phi(\tilde{Y}_\nu)) \mid \tilde{Y}_\nu\}] =$$

$$= E_{\mu_\nu}\, [\psi_\nu(\tilde{Y}_\nu)\, I_{\{0\}}(\phi(\tilde{Y}_\nu)) + (1-\psi_\nu(\tilde{Y}_\nu))\, I_{\{1\}}(\phi(\tilde{Y}_\nu))] \to 0 .$$

CHAPTER 5

ASYMPTOTICALLY LEVEL α AND ASYMPTOTICALLY UNIFORMLY MOST POWERFUL TESTS FOR PROBLEMS WITH COMPOSITE NULL HYPOTHESES

In this chapter, asymptotic versions of the concepts "level α" and "uniformly most powerful" are studied for a class of testing problems which is somewhat more general than the class defined in Section 3.5. Adapting these concepts to asymptotic testing problems is less straightforward when the null hypothesis is composite than when it is simple. It will be seen that it can be advantageous to take error probabilities of tests into account only for parameter sequences $\{\theta_\nu\}$ for which a compact subset K of the parameter space exists with $\{\theta_\nu\} \subset K$. Definitions of the concepts "asymptotically of level α" and "asymptotically uniformly most powerful" are given in Sections 5.2 and 5.5, respectively. In Section 5.6, asymptotically uniformly most powerful tests are derived for certain testing problems for exponential families.

5.1. THE SEQUENCE OF TESTING PROBLEMS TO BE CONSIDERED

In Section 3.5, attention was focused on a class of testing problems for exponential families. Since the relevance of the asymptotic properties to be defined is determined by other matters than whether or not the classes of probability distributions from which the samples are drawn constitute exponential families, the assumption of exponential families will be dropped in Chapters 5,6 and 7 (unless indicated otherwise).

A sequence $T_1, T_2, \ldots$ of testing problems is considered such that for every ν, the experiment for T_ν is constituted by $k (k \geq 1)$ independent random samples

$$X_{i1}^{(\nu)}, \ldots, X_{in_i(\nu)}^{(\nu)} \qquad\qquad 1 \leq i \leq k ,$$

$X_{ij}^{(\nu)}$ being an m_i - dimensional random variable with probability distribution

$P_{i\theta_i}$, where $\theta_i \in \Theta_i$. Denote $\theta = (\theta_1', \ldots, \theta_k')'$ and $\Theta = \Pi_{i=1}^k \Theta_i$. Disjoint subsets Θ_H and Θ_A of Θ are given such that for every ν, null hypothesis and alternative hypothesis for T_ν are

$$H : \theta \in \Theta_H \qquad\qquad A : \theta \in \Theta_A .$$

Denote $\Theta_T = \Theta_H \cup \Theta_A$; values of θ outside Θ_T will be left out of consideration. Thus, T_ν can be represented by

$$T_\nu = (\mathbb{R}^{m(\nu)}, \{P_\theta^{(\nu)} \mid \theta \in \Theta_H\}, \{P_\theta^{(\nu)} \mid \theta \in \Theta_A\})$$

where

$$m(\nu) = \Sigma_{i=1}^k m_i n_i(\nu)$$

$$P_\theta^{(\nu)} = \otimes_{i=1}^k (P_{i\theta_i})^{n_i(\nu)} .$$

It is assumed that $\min_i n_i(\nu) \to \infty$. A test for the asymptotic testing problem $\{T_\nu\}$ is a sequence $\{\phi_\nu\}$ where, for every ν, ϕ_ν is a test function for T_ν. The notation $E_\theta \phi_\nu$ is used for $E_\theta \phi_\nu (X_{11}^{(\nu)}, \ldots, X_{kn_k(\nu)}^{(\nu)})$.

The parametrization is assumed to be identifiable: if $\theta, \theta' \in \Theta_T$ and $\theta \neq \theta'$, then $P_{i\theta_i} \neq P_{i\theta_i'}$ for some i. It is useful to endow Θ_T with the topology which is the coarsest topology such that for every i, the function $\theta \mapsto P_{i\theta_i}$ from Θ_T to $M_1(\mathbb{R}^{m_i})$ is a continuous function, when $M_1(\mathbb{R}^{m_i})$ is considered with the weak topology (see Appendix A.1). Define

$$m = \Sigma_{i=1}^k m_i$$

$$P_\theta = \otimes_{i=1}^k P_{i\theta_i} .$$

This topology on Θ_T can be characterized by the property that $\theta_n \to \theta$ iff

$$\int f \, dP_{\theta_n} \to \int f \, dP_\theta$$

for all bounded continuous functions $f : \mathbb{R}^m \to \mathbb{R}$. It follows from the metrizability and separability of $M_1(\mathbb{R}^m)$, that Θ_T with this topology is a metrizable separable space.

The topological assumptions are made, that Θ_T is locally compact and that Θ_H is a closed subset of Θ_T. Usually, Θ_T can be taken to be a subset of a

Euclidean space, and the topology defined above coincides with the relative Euclidean topology. In most cases, the variation topology on $M_1(\mathbb{R}^m)$ induces the same relative topology on $\{P_\theta \mid \theta \in \Theta_T\}$ as the weak topology does.

If the $\{P_{i\theta_i} \mid \theta_i \in \Theta_i\}$ are canonical exponential families, where θ_i is the natural parameter and Θ_i the interior of the natural parameter space, then Theorem 2.3.2 says that the relative Euclidean topology of Θ_i corresponds to the relative weak topology of $\{P_{i\theta_i} \mid \theta_i \in \Theta_i\}$ as a subset of $M_1(\mathbb{R}^{m_i})$. If, moreover, Θ_T is a closed subset of Θ and Θ_H a closed subset of Θ_T, then the topological assumptions are satisfied. (Recall that closed subsets and open subsets of locally compact spaces are locally compact.) Hence the testing problem of Section 3.5 satisfies the assumptions made here.

The local compactness assumption will be directly used only in the following lemma, which is essential to several theorems later in Chapters 5 and 6.

<u>LEMMA 5.1.1</u>. *A sequence* $\{K_h\}$ *of compact subsets of* Θ_T *exists with*

$$K_h \subset \operatorname{int} K_{h+1}\ , \quad \cup_h K_h = \Theta_T\ .$$

This statement remains true if Θ_T *is replaced by* Θ_H.

*<u>PROOF</u>. The proof needs to be given only for Θ_T. Let d be a metric for the topology of Θ_T, and let $\{\theta^{(r)}\}$ be a sequence which is dense in Θ_T. Define for $\theta \in \Theta_T$

$$S(\theta;\ \varepsilon) = \{\theta' \in \Theta_T \mid d(\theta', \theta) \leq \varepsilon\}$$

$$\varepsilon(\theta) = \sup\ \{\varepsilon \leq 1 \mid S(\theta;\ \varepsilon) \text{ is compact}\}\ .$$

The local compactness of Θ_T implies that $\varepsilon(\theta) > 0$ for all $\theta \in \Theta_T$. If $d(\theta', \theta) = \delta < \varepsilon < \varepsilon(\theta)$, then $S(\theta';\ \varepsilon - \delta) \subset S(\theta;\ \varepsilon)$, so that $S(\theta'; \varepsilon - \delta)$ is compact. This shows that $|\varepsilon(\theta') - \varepsilon(\theta)| \leq d(\theta', \theta)$, implying that ε is a continuous function.

For $r \in \mathbb{N}$, define $S_r = S(\theta^{(r)};\ \frac{1}{2}\varepsilon(\theta^{(r)}))$. Then S_r is a compact subset of Θ_T. As ε is continuous and positive and $\{\theta^{(r)}\}$ is dense, one has

$$(1) \qquad \Theta_T = \cup_{r \in \mathbb{N}} \operatorname{int} S_r\ .$$

Define $\{K_h\}$ inductively in the following way. Let $K_1 = S_1$. Suppose that

$K_h \subset \Theta_T$ has been defined and is compact. Because of (1), a finite set $H_h \subset \mathbb{N}$ exists with

$$K_h \subset \bigcup_{r \in H_h} \text{int } S_r .$$

Define $H_{+h} = H_h \cup \{h + 1\}$ and

$$K_{h+1} = \bigcup_{r \in H_{+h}} S_r .$$

Then K_{h+1} is a compact subset of Θ_T with $K_h \subset \text{int } K_{h+1}$ and $S_{h+1} \subset K_{h+1}$. With (1), this implies $\Theta_T = \bigcup_h K_h$. □

5.2. DEFINITIONS OF "ASYMPTOTICALLY OF LEVEL α"

When one tries to give mathematical formulations for concepts such as "asymptotically of level α" and "asymptotically UMP" for the testing problem of Section 5.1, two obstacles are encountered. The first one arises from the inadequacy of considering only only fixed parameter values in the approach followed here. The other obstacle lies in the possible "degeneration" at the boundary of the parameter space. In this section these obstacles are indicated for the formulation of "asymptotically of level α".

A very strong asymptotic level α requirement for the test $\{\phi_\nu\}$ is that for every ν, ϕ_ν be of level :

$$(5.2.1) \qquad \sup_{\theta \in \Theta_H} E_\theta \, \phi_\nu \leq \alpha \qquad \text{for all } \nu.$$

This requirement, with the inequality replaced by an equality sign, is the one used by WALD (1943). There are cases where this requirement can be met, e.g., when one uses exact similar-size α tests. In many cases, however, this requirement is too strong for practical purposes. The critical value for a test statistic is often computed by means of a normal approximation, and the user only knows that the size of his test is "approximately" equal to α. For (moderately) large sample sizes, determining a test which is exactly of size α is often a troublesome and irrelevant affair: one does not care too much whether the size of the test used is .054 or .050. Instead of (5.2.1), therefore, one might impose the requirement

$$\text{(5.2.2)} \qquad \limsup_{\nu} E_\theta \, \phi_\nu \le \alpha \qquad \text{for all } \theta \in \Theta_H \, .$$

This requirement is used, for instance, by NEYMAN (1959). The following example, due to Willem Schaafsma, demonstrates that (5.2.2) does not rule out "super-power". It is somewhat similar to the examples of super-efficient estimators, first given by Hodges and reported by LE CAM (1953).

<u>EXAMPLE 5.2.1</u>. *A test with super-power*. For two random samples of size ν from normal distributions $N(\theta_1, 1)$ and $N(\theta_2, 1)$, respectively, consider the testing problem

$$H : 0 \le \theta_1 \le 1, \qquad \theta_2 = 0$$

$$A : 0 \le \theta_1 \le 1, \qquad 0 < \theta_2 \le 1 \, .$$

(The first sample is redundant from a conceptual point of view.)
The associated parameter spaces are

$$\Theta_H = [0,1] \times \{0\}$$

$$\Theta_A = [0,1] \times (0,1] \, .$$

The pair $(X_{1.}^{(\nu)}, X_{2.}^{(\nu)})$ of sample means is a sufficient statistic. The test ϕ_ν, rejecting for

$$X_{2.}^{(\nu)} \ge u_\alpha \, \nu^{-\frac{1}{2}} \quad ,$$

is UMP - level α. Let ψ_ν be the test which rejects for

$$X_{2.}^{(\nu)} \ge u_\alpha \nu^{-\frac{1}{2}} \qquad \text{or} \qquad \nu^{-\frac{1}{4}} \le X_{1.}^{(\nu)} \le 3\nu^{-\frac{1}{4}} \, .$$

The sequence of test functions $\{\psi_\nu\}$ has super-power among the asymptotically level α tests in the sense that it satisfies (5.2.2), while

$$E_\theta \, \phi_\nu \le E_\theta \, \psi_\nu \qquad \text{for all } \theta \in \Theta_A$$

and

$$\sup_{\theta \in \Theta_A} (E_\theta \psi_\nu - E_\theta \phi_\nu) \to 1 - \alpha \, .$$

The first two assertions are obvious, the last one is proved by considering the sequence $\{\theta_\nu\}$ with $\theta_\nu = (2\nu^{-\frac{1}{4}}, \nu^{-1})$. □

In NEYMAN (1959), the adoption of (5.2.2) as the definition of "asymptotically of level α" creates no problems, because there the possibility of "super-power" is excluded by restricting the attention to the class of "all $C(\alpha)$ tests".

A requirement which is intermediate between (5.2.1) and (5.2.2) is given by

$$(5.2.3) \qquad \limsup_{\nu} \sup_{\theta\in\Theta_H} E_\theta \, \phi_\nu \leq \alpha \; .$$

It can be proved that this requirement excludes "super-power": if $\{\theta_\nu\} \subset \Theta_A$ and ϕ_ν is the most powerful - level α test for testing H against the simple alternative $\theta = \theta_\nu$, and if $\{\psi_\nu\}$ satisfies (5.2.3), then

$$\limsup_{\nu} E_{\theta_\nu}(\psi_\nu - \phi_\nu) \leq 0 \; .$$

For many testing problems, (5.2.3) can be used as a relevant and easily applicable asymptotic level α -requirement. Many other testing problems, however, exhibit a "degeneration" at the "boundary" of the parameter space which makes (5.2.3) less suitable. (In this context, a sequence $\{\theta_\nu\}$ will be said to tend to the boundary of the parameter space, if every compact subset of Θ_T contains only finitely many θ_ν's.)
This "degeneration" poses the problem with respect to the property "asymptotically of level α", that for parameter sequences which approach the boundary of the parameter space sufficiently rapidly, other limiting distributions occur than for parameter sequences which stay away from the boundary. E.g., the multinomial distribution tends asymptotically to a normal distribution if the probabilities are bounded away from 0, but if at least one probability tends to 0 sufficiently rapidly, then the multinomial distribution tends asymptotically to a distribution with at least one Poisson marginal. This implies that the verification of the condition

$$\limsup_{\nu} E_{\theta_\nu} \phi_\nu \leq \alpha$$

requires other methods for sequences $\{\theta_\nu\}$ tending to the boundary of the parameter space, than for the sequences $\{\theta_\nu\}$ which stay away from the boundary. However, the behaviour at the boundary of the parameter space is usually not important: the statistician is confident that the true, but unknown, value of θ lies "well inside" Θ_T. The theory to be developed in this chapter is designed for such situations. A corresponding asymptotic level α - requirement is

$$(5.2.4) \qquad \limsup_{\nu} \sup_{\theta \in K} E_\theta\, \phi_\nu \leq \alpha \qquad \text{for all compact } K \subset \Theta_H .$$

A study where the compact subsets of the parameter space occupy a similar position is LE CAM (1956), Section 8. Requirement (5.2.4) is adopted in Definition 5.3.1 as the asymptotic level α requirement to be used in this study.

The implications between the four considered asymptotic level α - restrictions can be summarized by

$$(5.2.1) \Rightarrow (5.2.3) \Rightarrow (5.2.4) \Rightarrow (5.2.2).$$

The example of super-power shows that (5.2.2) is essentially weaker than the other three requirements. It will be demonstrated in Section 5.4 that the class of tests satisfying (5.2.4) does not contain essentially more tests, in a certain sense, than the class of tests satisfying (5.2.1).

5.3. THE CHOICE OF "INTERIOR SEQUENCE" AS A FUNDAMENTAL CONCEPT

The asymptotic level α -restrictions (5.2.3) and (5.2.4) are defined in terms of (suprema over) sets of parameter values. They can alternatively be formulated in terms of sequences of parameter values. This will turn out to be advantageous in later sections. So the following classes of sequences are defined.

$$S_1 = \{\{\theta_\nu\} \subset \Theta_H \mid \theta_\nu \equiv \theta \quad \text{for some } \theta \in \Theta_H\}$$
$$S_2 = \{\{\theta_\nu\} \subset \Theta_H \mid \theta_\nu \to \theta \quad \text{for some } \theta \in \Theta_H\}$$
$$S_3 = \{\{\theta_\nu\} \subset \Theta_H \mid \{\theta_\nu\} \subset K \text{ for some compact } K \subset \Theta_H\}$$
$$S_4 = \{\{\theta_\nu\} \subset \Theta_H\}.$$

Note that $S_1 \subset S_2 \subset S_3 \subset S_4$. The class S_3 is "not much larger" than S_2, since every $\{\theta_\nu\} \in S_3$ has a subsequence converging to a limit in Θ_H. When for i = 1, 2, 3, 4 the asymptotic level α - requirement $A_i(\alpha)$ is defined by

$$\limsup_{\nu} E_{\theta_\nu} \phi_\nu \leq \alpha \quad \text{for all } \{\theta_\nu\} \in S_i \ ,$$

then (5.2.2) is equivalent to $A_1(\alpha)$, (5.2.4) is equivalent both to $A_2(\alpha)$ and $A_3(\alpha)$, and (5.2.3) is equivalent to $A_4(\alpha)$. WITTING and NÖLLE (1970, Section 2.2) treat asymptotic testing problems in a similar fashion: they always relate the concepts "asymptotically of level α" and "asymptotically UMP" to classes of sequences of probability distributions. For testing problems with composite null hypotheses where a "degeneration" at the boundary of the parameter space occurs as discussed in Sections 5.2 and 5.5, the use of sequences of parameter values seems to lead to a more elegant formulation of asymptotic optimality considerations than the use of sets of parameter values. Therefore, all asymptotic properties in this study will be formulated in terms of sequences of parameter values.

The discussion in the preceding section shows that S_2 and S_3 are very relevant for the formulation of "asymptotically of level α". In following sections it will be argued that for asymptotic optimum properties, too, it is often relevant to consider only those parameter sequences which stay away from the boundary of the parameter space. One could consider the class of all sequences with compact closure in Θ_T or, more restrictively, the class of all sequences which are convergent in Θ_T. Choosing between these two is a rather academic affair. The class of sequences with compact closure, because of its greater generality, gives a little bit more room for theoretical developments in Chapter 7. These sequences will be called "interior sequences".

DEFINITION 5.3.1. The class of interior sequences is

$$S = \{ \{\theta_\nu\} \subset \Theta_T \mid \{\theta_\nu\} \subset K \text{ for some compact } K \subset \Theta_T \} \ .$$

The classes of interior sequences corresponding to null hypothesis and alternative hypothesis, respectively, are denoted by

$$S_H = \{ \{\theta_\nu\} \in S \mid \{\theta_\nu\} \subset \Theta_H \}$$
$$S_A = \{ \{\theta_\nu\} \in S \mid \{\theta_\nu\} \subset \Theta_A \} \ .$$

A test $\{\phi_\nu\}$ is asymptotically of level α, if

$$\limsup_{\nu} E_{\theta_\nu} \phi_\nu \leq \alpha \qquad \text{for all } \{\theta_\nu\} \in S_H .$$

Asymptotically level α tests will be compared by means of the power sequences $\{E_{\theta_\nu}\phi_\nu\}$ for $\{\theta_\nu\} \in S_A$. Many sequences of tests are "not essentially different" from our asymptotic point of view. Hence a concept of asymptotic equivalence is needed. This is given in the following definition. Like the definition above, this one is suitable especially for situations where the behaviour of tests at the boundary of the parameter space is felt to be not very important.

DEFINITION 5.3.2. Two tests $\{\phi_\nu\}$ and $\{\psi_\nu\}$ are said to be asymptotically equivalent if

$$E_{\theta_\nu}(\phi_\nu - \psi_\nu) \to 0 \quad \text{for all } \{\theta_\nu\} \in S_A .$$

They are said to be strongly asymptotically equivalent if

$$E_{\theta_\nu} |\phi_\nu - \psi_\nu| \to 0 \quad \text{for all } \{\theta_\nu\} \in S_A .$$

For two asymptotically equivalent tests, the "asymptotic powers" against all interior sequences corresponding to the alternative are the same for both tests. For two strongly asymptotically equivalent tests (the randomization, if necessary, can be carried out so that) the probability that the two tests lead to different decisions tends to 0. It may be sensible to try to differentiate between tests which are (strongly) asymptotically equivalent. A large amount of literature exists about this subject, but it will not be considered in this study.

5.4. RELATIONS BETWEEN SEVERAL DEFINITIONS OF "ASYMPTOTICALLY OF LEVEL α"

In Section 5.2, four different definitions are given for the concept "asymptotically of level α", and the implication between them are summarized as

$$(5.2.1) \Rightarrow (5.2.3) \Rightarrow (5.2.4) \Rightarrow (5.2.2) .$$

Requirement (5.2.4) is adopted to be used in the rest of this study (Definition 5.3.1). It is demonstrated in this section, that (5.2.4) does not admit essentially more tests than (5.2.1) does: for every test $\{\phi_\nu\}$ satisfying (5.2.4), there exists a test $\{\psi_\nu\}$ which satisfies (5.2.1) and is strongly asymptotically equivalent to $\{\phi_\nu\}$.

First suppose that $\{\phi_\nu\}$ satisfies (5.2.3), and define

$$c_\nu = \min \{\alpha \,/\, \sup_{\theta\in\Theta_H} E_\theta \phi_\nu,\ 1\} .$$

Then $c_\nu \to 1$. The test $\{\psi_\nu\}$, defined by $\psi_\nu = c_\nu \phi_\nu$, satisfies (5.2.1) and is strongly asymptotically equivalent to $\{\phi_\nu\}$, as

$$E_{\theta_\nu} \mid \phi_\nu - \psi_\nu \mid = E_{\theta_\nu} (1 - c_\nu)\phi_\nu \leq 1 - c_\nu \to 0$$

for all $\{\theta_\nu\} \subset \Theta_T$.

It remains to be shown that for $\{\phi_\nu\}$ satisfying (5.2.4), there exists a $\{\psi_\nu\}$ which satisfies (5.2.3) and is strongly asymptotically equivalent to $\{\phi_\nu\}$. The following lemma is used to prove this result. It is a version of Lemma 4 of LE CAM (1956).

LEMMA 5.4.1. *Let* K *be a compact subset and* G *an open subset of* Θ_T *with* $\emptyset \neq K \subset G \neq \Theta_T$. *Then there exists a test* $\{\chi_\nu\}$ *with*

$$\inf_{\theta\in K} E_\theta \chi_\nu \to 1$$

$$\sup_{\theta\in\Theta_T\setminus G} E_\theta \chi_\nu \to 0 .$$

*PROOF. Define m and P_θ as in Section 5.2, and let $\mathcal{F}$ be the class of finite sets of pairs (f, t), where $f : \mathbb{R}^m \to \mathbb{R}$ is a bounded continuous function and t a real number. Since the topology of Θ_T corresponds to the relative weak topology of $\{P_\theta | \theta \in \Theta_T\}$ as a subset of $\mathcal{M}_1(\mathbb{R}^m)$, the class of subsets

$$U(F) = \{\theta \in \Theta_T | \ |\int f\, dP_\theta - t| < 1 \text{ for all } (f, t) \in F\}$$

for $F \in \mathcal{F}$, constitutes a base for the topology of Θ_T. As K is compact and G open, finitely many $F_1, \ldots, F_N \in \mathcal{F}$ exist with

$$K \subset \bigcup_{h=1}^{N} U(F_h) \subset G \ .$$

Define

$$d(\theta) = \min_h \max_{(f,t)\in F_h} \left| \int f \, dP_\theta - t \right| .$$

Then d is a continuous function and $d(\theta) < 1$ is equivalent to $\theta \in \bigcup_h U(F_h)$. Hence

$$K \subset \{\theta \in \Theta_T | \ d(\theta) < 1\} \subset G \ .$$

With the continuity of d and the compactness of K, this implies

$$\sup_{\theta\in K} d(\theta) < 1 \leq \inf_{\theta\in\Theta_T \setminus G} d(\theta) \ .$$

Define

$$a = \tfrac{1}{2} (\sup_{\theta\in K} d(\theta) + 1) \quad , \qquad \varepsilon = 1 - a$$

$$n_o(\nu) = \min_i n_i(\nu)$$

$$T_\nu f = [n_o(\nu)]^{-1} \sum_{j=1}^{n_0(\nu)} f(X_{1j}^{(\nu)}, \ \ldots \ , X_{kj}^{(\nu)}) \ .$$

($T_\nu f$ is not the most sophisticated estimator for $\int f \, dP_\theta$, but it works easily.) Let $|f(x)| \leq M_1$ for all $(f, t) \in \bigcup_h F_h$ and $x \in \mathbb{R}^m$, and let M_2 be the number of elements of $\bigcup_h F_h$. Chebychev's and Boole's inequalities yield

$$P_\theta \ \{ \ |T_\nu f - \int f \, dP_\theta| \geq \varepsilon \text{ for some } (f, t) \in \bigcup_h F_h \} \leq$$

$$\leq M_1^2 \ M_2 \ / \ \varepsilon^2 n_0(\nu) \ .$$

Let χ_ν be the indicator function of the event

$$\{\min_h \max_{(f,t)\in F_h} |T_\nu f - t| < a\} \ .$$

Then for all $\theta \in K$ one has

$$P_\theta\ \{\chi_\nu = 0\} \le M_1^2\ M_2\ /\ \varepsilon^2 n_0(\nu)\ ,$$

and for all $\theta \in \Theta_T \setminus G$

$$P_\theta\ \{\chi_\nu = 1\} \le M_1^2\ M_2\ /\ \varepsilon^2 n_0(\nu)\ .$$

As $n_0(\nu) \to \infty$, this shows that $\{\chi_\nu\}$ satisfies the requirements. □

THEOREM 5.4.1. *Let* $\{\phi_\nu\}$ *be a test satisfying* (5.2.4). *Then a test* $\{\psi_\nu\}$ *exists which satisfies* (5.2.3), *and which is strongly asymptotically equivalent to* $\{\phi_\nu\}$.

*PROOF. If Θ_T is compact, the theorem is trivial. Therefore assume that Θ_T is not compact. According to Lemma 5.1.1, there exists a sequence $\{K_h\}$ of compact subsets of Θ_T with $K_1 \neq \emptyset$ and

$$K_h \subset \text{int}\ K_{h+1}\quad ,\quad \cup_h K_h = \Theta_T\ .$$

Since Θ_T is not compact, $K_h \neq \Theta_T$. Note that $\sup_{\theta \in K_h} E_\theta \phi_\nu$ is non-decreasing in h, for every ν, and that (5.2.4) implies

$$\limsup_\nu \sup_{\theta \in K_h} E_\theta \phi_\nu \le \alpha \quad \text{for every } h\ .$$

Lemma A.5.1 implies the existence of a sequence $\{h_1(\nu)\}$ with $h_1(\nu) \to \infty$ and

$$\limsup_\nu \sup\ \{E_\theta \phi_\nu \mid \theta \in K_{h_1(\nu)}\} \le \alpha\ .$$

For every h, Lemma 5.4.1 yields a test $\{\chi_{h\nu}\}$ with

$$\inf_{\theta \in K_h} E_\theta\ \chi_{h\nu} \to 1 \qquad , \qquad \sup_{\theta \in \Theta_T \setminus K_h} E_\theta\ \chi_{h\nu} \to 0\ .$$

It is not a restriction to assume that $\chi_{h\nu} \le \chi_{h+1,\nu}$. (If necessary, replace $\chi_{h\nu}$ by $\max_{n \le h} \chi_{n\nu}$.) Define

$$a(h,\ \nu) = \sup\ \{E_\theta \chi_{h\nu} \mid \theta \notin K_{h_1(\nu)}\}\ .$$

Then $a(h, \nu) \leq a(h+1, \nu)$ and for every h one has $a(h, \nu) \to 0$. Lemma A.5.1 now yields a sequence $\{h_2(\nu)\}$ with $h_2(\nu) \to \infty$ and $a(h_2(\nu), \nu) \to 0$. Define $h_3(\nu) = \min\{h_1(\nu), h_2(\nu)\}$, $\chi_\nu = \chi_{h_3(\nu),\nu}$ and $\psi_\nu = \chi_\nu \phi_\nu$. It will be proved that $\{\psi_\nu\}$ has the required properties.

In order to prove the strong asymptotic equivalence with $\{\phi_\nu\}$, let $\{\theta_\nu\} \subset K$ for a compact $K \subset \Theta_T$. As $K \subset \bigcup_h \operatorname{int} K_h$ and K is compact, there exists an h with $K \subset K_h$, and hence

$$\limsup_\nu E_{\theta_\nu} |\phi_\nu - \psi_\nu| \leq \limsup_\nu E_{\theta_\nu}(1-\chi_\nu) \leq 1 - \liminf_\nu E_{\theta_\nu} \chi_{h\nu}$$

$$\leq 1 - \liminf_\nu \inf_{\theta \in K_h} E_\theta \chi_{h\nu} = 0 .$$

In order to show that $\{\psi_\nu\}$ satisfies (5.2.3), note that a subsequence $\{\xi\}$ of $\{\nu\}$ and a $\{\theta_\xi\} \subset \Theta_H$ exist such that

$$\lim_\xi E_{\theta_\xi} \psi_\xi = \limsup_\nu \sup_{\theta \in \Theta_H} E_\theta \psi_\nu ,$$

and such that either $\theta_\xi \in K_{h_1(\xi)}$ for all ξ, or $\theta_\xi \notin K_{h_1(\xi)}$ for all ξ. In the first case,

$$\lim_\xi E_{\theta_\xi} \psi_\xi \leq \limsup_\xi E_{\theta_\xi} \phi_\xi \leq \limsup_\xi \sup \{E_\theta \phi_\xi \mid \theta \in K_{h_1(\xi)}\} \leq \alpha .$$

In the second case,

$$\lim_\xi E_{\theta_\xi} \psi_\xi \leq \limsup_\xi E_{\theta_\xi} \chi_\xi \leq \limsup_\xi a(h_3(\xi), \xi) \leq$$

$$\leq \limsup_\xi a(h_2(\xi), \xi) = 0 .$$

This establishes that $\{\psi_\nu\}$ satisfies (5.2.3). □

5.5. DEFINITIONS OF "ASYMPTOTICALLY UNIFORMLY MOST POWERFUL"

In this section, we try to give a relevant formulation for the concept "asymptotically uniformly most powerful", abbreviated to "AUMP". In accordance with Section 5.3, the concept "AUMP" will be formulated in terms of sequences of parameter values. The attention will always be restricted to a certain class $\tilde{\Phi}$ of tests. For example, $\tilde{\Phi}$ can be the class of all asymptotically level α tests. In Chapters 8 and 9, smaller classes $\tilde{\Phi}$ will also be considered.

DEFINITION 5.5.1. Let $\{\theta_\nu\} \subset \Theta_A$. The test $\{\phi_\nu\}$ is said to be asymptotically most powerful - $\tilde{\Phi}$, or AMP- $\tilde{\Phi}$, against $\{\theta_\nu\}$ if

(i) $\{\phi_\nu\} \in \tilde{\Phi}$

(ii) $\liminf_\nu E_{\theta_\nu}(\phi_\nu - \psi_\nu) \geq 0$ for all $\{\psi_\nu\} \in \tilde{\Phi}$.

For a class S_0 of sequences in Θ_A, a test $\{\phi_\nu\}$ can be said to be AUMP-$\tilde{\Phi}$ against S_0 if it is AMP- $\tilde{\Phi}$ against $\{\theta_\nu\}$ for all $\{\theta_\nu\} \in S_0$. Possibilities for S_0 are

$$S_1 = \{ \{\theta_\nu\} \subset \Theta_A \mid \theta_\nu \equiv \theta \text{ for some } \theta \in \Theta_A\}$$
$$S_2 = S_A = \{ \{\theta_\nu\} \subset \Theta \mid \{\theta_\nu\} \subset K \text{ for some compact } K \subset \Theta_A\}$$
$$S_3 = \{ \{\theta_\nu\} \subset \Theta_A\} .$$

Usually, $\tilde{\Phi}$ is large enough so that there exists a $\{\phi_\nu\} \in \tilde{\Phi}$ with $E_\theta\phi_\nu \to 1$ for every $\theta \in \Theta_A$. In such cases, $\{\phi_\nu\} \in \tilde{\Phi}$ is AUMP-$\tilde{\Phi}$ against S_1 iff $\{\phi_\nu\}$ is consistent, and the optimum property "AUMP-$\tilde{\Phi}$ against S_1" is rather weak.

The optimum property "AUMP-$\tilde{\Phi}$ against S_3" leads to meaningful results only if $\tilde{\Phi}$ is a class of tests satisfying (5.2.1) or (5.2.3) as asymptotic level α - restriction. This is the approach usually followed; see WALD (1941) or JOHNSON and ROUSSAS (1969). The following example shows that for some testing problems this approach is too restrictive: there may exist an AUMP - $\tilde{\Phi}$ test against S_2, while no test is AUMP - $\tilde{\Phi}$ against S_3. This can happen when a "degeneration at the boundary of the parameter space" occurs (see Section 5.2).

EXAMPLE 5.5.1. *Comparison of two succes probabilities.* Let $X_{ij}^{(\nu)}$, for i = 1,2 and $1 \leq j \leq n_i(\nu)$, be independent random variables, $X_{ij}^{(\nu)}$ having the Bernoulli distribution with succes probability θ_i:

$$P_\theta \{X_{ij}^{(\nu)} = 1\} = \theta_i \quad , P_\theta \{X_{ij}^{(\nu)} = 0\} = 1 - \theta_i \quad ,$$

where $\theta = (\theta_1, \theta_2)$. Suppose that $n_1(\nu) / n_2(\nu) \to \rho \in (0,1)$ and consider the testing problem

$$H : \theta_1 = \theta_2 \qquad\qquad A : \theta_1 > \theta_2 .$$

Denote by $\tilde{\Phi}$ the class of all tests which are asymptotically of level α in the sense (5.2.3). A sufficient statistic is $(X_{1+}^{(\nu)}, X_{2+}^{(\nu)})$ with

$$X_{i+}^{(\nu)} = \sum_{j=1}^{n_i(\nu)} X_{ij}^{(\nu)} .$$

It will be shown in Example 5.6.1 that Fisher's exact test is AUMP- $\tilde{\Phi}$ against S_2. The purpose of the present example is to show that no AUMP- $\tilde{\Phi}$ test against S_3 exists.

Argue by contradiction, and assume that $\{\phi_\nu\}$ is AUMP- $\tilde{\Phi}$ against S_3. It may be assumed that test functions $\phi_{0\nu} : \mathbb{N}_0^2 \to [0,1]$ exist with $\phi_\nu = \phi_{0\nu}(X_{1+}^{(\nu)}, X_{2+}^{(\nu)})$. A diagonal sequence argument establishes the existence of a subsequence $\{\xi\}$ of $\{\nu\}$ for which

$$\phi_{0\xi}(x_1, x_2) \to \phi(x_1, x_2) \text{ for all } (x_1, x_2) \in \mathbb{N}_0^2$$

for a certain test function ϕ. Note that if $n_i(\nu)\,\theta_{\nu i} \to p_i$, then $L_{\theta_{\nu i}}(X_{i+}^{(\nu)})$ tends in variation distance to the Poisson distribution with parameter p_i. Let Y_1 and Y_2 be independent random variables having Poisson distributions with parameters p_1 and p_2, and let $\theta_\xi(p_1, p_2) = (p_1 / n_1(\xi), p_2 / n_2(\xi))$; then

$$\text{(1)} \qquad E_{\theta_\xi(p)}\, \phi_\xi(X_{1+}^{(\xi)}, X_{2+}^{(\xi)}) \to E_p \phi(Y_1, Y_2)$$

for all $p = (p_1, p_2)$. With (5.2.3), this implies that ϕ is of level α for the "limiting" testing problem

$$H' : p_1 = \rho p_2 \quad , \qquad A' : p_1 > \rho p_2$$

for Poisson parameters p_1 and p_2. This testing problem admits no UMP-level α test (see below), so there exist a test ψ and parameter values q_1, q_2 with $q_1 > \rho q_2$ and

$$E_{(\rho p, p)}\, \psi(Y_1, Y_2) \leq \alpha \text{ for all } p > 0$$

$$E_q\, \psi(Y_1, Y_2) > E_q\, \phi(Y_1, Y_2)$$

where $q = (q_1, q_2)$. Define

$$\psi^{(h)}(x_1, x_2) = \psi(x_1, x_2)\, I_{[0,h]}(\max\{x_1, x_2\});$$

there exists an h with

(2) $$E_q\, \psi^{(h)}(Y_1, Y_2) > E_q \phi(Y_1, Y_2).$$

The test $\{\psi^{(h)}(X_{1+}^{(\nu)}, X_{2+}^{(\nu)})\}$ satisfies (5.2.3) and

(3) $$E_{\theta_\xi(q)}\, \psi^{(h)}(X_{1+}^{(\xi)}, X_{2+}^{(\xi)}) \to E_q\, \psi^{(h)}(Y_1, Y_2).$$

Relations (1), (2) and (3) contradict the assumption that $\{\phi_\nu\}$ is AUMP-$\tilde{\Phi}$ against S_3.

(In order to prove that no UMP-level α test exists for the "limiting" testing problem, it is sufficient to prove that a level α test ϕ_0 exists which is not dominated by the UMP-unbiased level α test ϕ_u. Define ϕ_0 by

$$\phi_0(y_1, y_2) = \begin{cases} 1 & y_1 > c, \quad y_2 = 0 \\ t & y_1 = c, \quad y_2 = 0 \\ 0 & \text{elsewhere}, \end{cases}$$

where $c \in \mathbb{N}$ and $t \in [0,1)$ are determined so that

(1) $$\sup_{p>0} E_{(\rho p, p)}\, \phi_0(Y_1, Y_2) = \alpha\ .$$

Since $E_{(\rho p,p)}\phi_0$ is a continuous function of p which tends to 0 for $p \to \infty$, a p_0 exists which is the largest nonnegative real number with

$$E_{(\rho p_0, p_0)}\, \phi_0 = \alpha\ .$$

The test ϕ_0 is the unique most powerful - level α test for testing $(p_1, p_2) = (\rho p_0, p_0)$ against $(p_1, p_2) = (p, 0)$, for all $p > \rho p_0$. Since $\phi_u \neq \phi_0$, it can be concluded that

$$E_{(p,0)}\phi_u(Y_1, Y_2) < E_{(p,0)}\phi_0(Y_1, Y_2)$$

for all $p > \rho p_0$. Since the expectations of ϕ_0 and ϕ_u are continuous functions of (p_1, p_2) on $[0, \infty)^2$, there exists a (p_1, p_2) with $p_1 > \rho p_2 > 0$, $p_1 > \rho p_0$ such that

$$(2) \qquad E_{(p_1,p_2)} \phi_u(Y_1, Y_2) < E_{(p_1,p_2)} \phi_0(Y_1,Y_2) .$$

It follows from (1) and (2) that ϕ_0 is a level α test which is not dominated by ϕ_u.) □

As mentioned in Section 5.2, the theory to be developed in this chapter is designed for situations, where the statistician is not so concerned about the error probabilities of his test for parameter values which are very close to the boundary of Θ_T. In such situations the property "AUMP-$\tilde{\Phi}$ against $S_2 = S_A$" is satisfactory. If a test is AUMP - $\tilde{\Phi}$ against S_2, the question whether it is also AUMP-$\tilde{\Phi}$ against S_3 (or the question to determine the largest class of interior sequences, against which the test is AUMP-$\tilde{\Phi}$) is interesting, but often of minor importance. For certain testing problems (e.g., in the case of normal distributions), invariance considerations may be used to prove that the test is indeed AUMP-$\tilde{\Phi}$ against S_3; for many testing problems for contingency tables where an AUMP-$\tilde{\Phi}$ test against S_2 exists, such as that of Example 5.5.1, an AUMP-$\tilde{\Phi}$ test against S_3 will not exist.

<u>DEFINITION 5.5.2</u>. The test $\{\phi_\nu\}$ is asymptotically uniformly most powerful -$\tilde{\Phi}$, or AUMP-$\tilde{\Phi}$, if it is AMP-$\tilde{\Phi}$ against all $\{\theta_\nu\} \in S_A$. If $\tilde{\Phi}$ is the class of all asymptotically level α tests, "AUMP-$\tilde{\Phi}$" can be replaced by "AUMP - level α".

5.6. AUMP TESTS FOR EXPONENTIAL FAMILIES

The literature contains many results concerning asymptotically most powerful tests. These results are often valid for testing problems involving families of distributions which are more general than exponential families. Many articles on AUMP test, such as WALD (1941) and JOHNSON and ROUSSAS (1969), consider testing problems with simple null hypotheses. NEYMAN (1959) considers testing problems with composite null hypotheses, but he proves asymptotic optimality of the proposed test only within the class "C(α)" of tests, which does not contain all asymptotically level α tests. Theorem 5.6.1 gives an AUMP - level α test for certain testing problems with composite null hypotheses for exponential families. This test will be asymptotically equivalent to "optimal" tests derived by other authors; the conclusion that this test is AUMP - level α, in the sense of Definitions 5.5.2 and 5.3.1, is new. Because of the restriction to exponential families, the test statistic can be expressed in a relatively simple way.

The asymptotic testing problem of Section 3.5 is considered again, with the notation introduced there; it is assumed that the cone K used to define the alternative hypothesis can be chosen to be of the form

$$(5.6.1) \qquad K = \{t\,a \mid t \geq 0\} ,$$

for some $a \in \mathbb{R}^m$. (Note that if dim $V \geq 1$, then there exist many different cones K' with $V + K' = V + K$; the cone (5.6.1) is a choice with minimal dimension.) So, null hypothesis and alternative hypothesis are

$$H : f(\mu) \in V$$

$$A : f(\mu) \in V + \{t\,a \mid t > 0\} ;$$

the parametrization is done with $\mu = E_\mu X^{(\nu)}_{\cdot}$. A sequence $\{\mu_\nu\} \subset F$ is an interior sequence iff it is relatively compact in F.

In part (2) of Section 3.2 one can find conditions ensuring that a UMP - level α or a UMP - unbiased level α test exists for this testing problem. If a UMP - level α test is known, then the construction of an AUMP - level α test is superfluous. If there is no UMP - level α test, then it is relevant to construct an AUMP - level α test, or to determine whether a given test is AUMP - level α. All AUMP - level α tests will be strongly asymptotically equivalent in the sense of Definition 5.3.2 (this can be proved using Theorem 4.5.1). For intermediate sample sizes, however, there may be different "good" level α tests with sizeable differences in their power functions; in spite of the existence of an AUMP - level α test, selecting a test can still be a non-trivial affair.

Corollary 4.4.2 states that if $\{\mu_{0\nu}\}$ and $\{\mu_{1\nu}\}$ are interior sequences with $\lim\sup_\nu [n(\nu)]^{\frac{1}{2}} \, || \mu_{1\nu} - \mu_{0\nu} || < \infty$, then the sequence of tests ϕ_ν rejecting for

$$(5.6.2) \qquad [n(\nu)]^{\frac{1}{2}}(\mu_{1\nu} - \mu_{0\nu})' R_\nu \Sigma^{-1}_{\mu_\nu}(X^{(\nu)}_{\cdot} - \mu_{0\nu}) \geq u_\alpha \, [(\mu_{1\nu} - \mu_{0\nu})' R_\nu \Sigma^{-1}_{\mu_\nu}(\mu_{1\nu} - \mu_{0\nu})]^{\frac{1}{2}}$$

is AMP - level α for the sequence of testing problems with null hypotheses and alternative hypotheses

$$H_\nu : \mu = \mu_{0\nu} \qquad , \qquad A_\nu : \mu = \mu_{1\nu} .$$

For arbitrary $\{\mu_{0\nu}\} \in S_H$ and $\{\mu_{1\nu}\} \in S_A$, there does not always exists a test which has "asymptotically the same power" against $\{\mu_{1\nu}\}$ as the sequence of tests with rejection regions (5.6.2), and which is asymptotically of level α for the whole null hypothesis. Suppose for simplicity that f is the identity function; if $\mu_{0\nu}$ is the projection of $\mu_{1\nu}$ on V with respect to the inner product $x'R_\nu \Sigma^{-1}_{\mu_{0\nu}} Y$, then the left hand side of (5.6.2) equals

$$[n(\nu)]^{\frac{1}{2}} (\mu_{1\nu} - \mu_{0\nu})' \quad R_\nu \Sigma^{-1}_{\mu_{0\nu}} X^{(\nu)}_{\cdot} \quad ;$$

this statistic has expectation 0 under H, while its variance can be consistently estimated. In this case, an asymptotically level α test $\{\phi_\nu\}$ exists which has "asymptotically the same power" against $\{\mu_{1\nu}\}$ as the sequence of tests with rejection regions (5.6.2). This test is AMP - level α against $\{\mu_{1\nu}\}$. In the testing problem of this section, it is possible to choose $\{\phi_\nu\}$ independently of $\{\mu_{1\nu}\}$, which results in an AUMP - level α test. This is the essence of Theorem 5.6.1. This theorem uses the concept of a "uniformly consistent estimator".

DEFINITION 5.6.1. A uniformly consistent estimator for μ under H is a sequence of statistics $\hat{\mu}_\nu$ with values in $\mu(\Theta_H)$, such that

$$\hat{\mu}_\nu - \mu_{0\nu} \to 0 \text{ in } \{\mu_{0\nu}\} \text{ - prob.}$$

for all $\{\mu_{0\nu}\} \in S_H$, while for every $\{\mu_\nu\} \in S_A$ and every subsequence of $\{\nu\}$ there exist a further subsequence $\{\xi\}$ and a $\mu_0 \in \mu(\Theta_H)$ with

$$\hat{\mu}_\xi \to \mu_0 \text{ in } \{\mu_\xi\} \text{ - prob..}$$

The first property required in this definition is essential; the second one is technical in character, and it is postulated only because it is handy for proofs of sharp consistency (see Definition 6.1.2 and part (iii) of the proof of Theorem 5.6.1). For the testing problem of Section 3.5, there always exists a uniformly consistent estimator for μ under H. For example, let L_ν be the projection of $\mathbb{R}^m$ onto V with respect to the inner product $x'R_\nu y$, and suppose for simplicity that f is the identity function and $L_\nu X_{\cdot}^{(\nu)} \in \mu(\Theta_H)$ with probability 1. Then $\hat{\mu}_\nu = L_\nu X_{\cdot}^{(\nu)}$ is a uniformly consistent estimator for μ under H : for every $\{\mu_\nu\} \in S$ one has

$$||L_\nu X_{\cdot}^{(\nu)} - L_\nu \mu_\nu|| \leq ||X_{\cdot}^{(\nu)} - \mu_\nu|| \to 0 \text{ in } \{\mu_\nu\} \text{ - prob..}$$

If the k families $\{P_{i\theta_i} | \theta_i \in \Theta_i\}$ are identical and $V = \{(\mu_1', \ldots, \mu_k')' | \mu_1 = \ldots = \mu_k\}$, then $L_\nu^1 X_{\cdot}^{(\nu)}$ is the UMVU estimator for μ under H.

<u>THEOREM 5.6.1</u>. *For the positive definite symmetric matrix* Λ, *denote by* L_Λ *the projection on* V *with respect to the inner product* $x'\Lambda^{-1}y$ *and let* w_Λ *be the vector of weights*

$$w_\Lambda = \frac{\Lambda^{-1}(a - L_\Lambda a)}{\{(a - L_\Lambda a)'\Lambda^{-1}(a - L_\Lambda a)\}^{\frac{1}{2}}} .$$

Let $\{\hat{\mu}_\nu\}$ *be a uniformly consistent estimator for* μ *under* H *and define*

$$T_\nu = w'_{\Lambda(R_\nu, \hat{\mu}_\nu)} Y_\nu$$

$$\phi_\nu = \begin{cases} 1 & \text{if } T_\nu > u_\alpha \\ 0 & \text{if } T_\nu \leq u_\alpha \end{cases} .$$

Then $\{\phi_\nu\}$ *is AUMP - level* α.

<u>PROOF</u>. Define

$$W_\nu = w_{\Lambda(R_\nu, \hat{\mu}_\nu)} .$$

As w_Λ is a continuous function of Λ, the uniform consistency of $\{\hat{\mu}_\nu\}$ implies that

(1) $$W_\nu - w_{\Lambda(R_\nu, \mu_\nu)} \to 0 \quad \text{in } \{\mu_\nu\} \text{ - prob. ,}$$

for every $\{\mu_\nu\} \in S_H$.

(i) First it will be proved that $\{\phi_\nu\}$ is asymptotically of level α. Let $\{\mu_\nu\} \in S_H$. Liapounov's Central Limit Theorem implies (as in part (a) of the proof of Theorem 4.4.1) that

$$(2) \qquad L_{\mu_\nu}\,(w'_{\Lambda(R_\nu,\mu_\nu)}(Y_\nu - [n(\nu)]^{\frac{1}{2}} f(\mu_\nu))) \to N(0,1)\ .$$

From $\{\mu_\nu\} \in S_H$ it follows that $f(\mu_\nu) \in V$, and hence $w'_\Lambda\, f(\mu_\nu) = 0$ for every Λ. With (1) and (2), this implies

$$L_{\mu_\nu}(T_\nu) \to N(0,1)\ .$$

Hence $E_{\mu_\nu}\phi_\nu \to \alpha$.

(ii) Let $\{\mu_{1\nu}\} \in S_A$ and let $\{\psi_\nu\}$ be an asymptotically level α test. Suppose that there exists a $\mu \in \mu\ (\Theta_H)$ with $\mu_{1\nu} \to \mu$ and suppose that

$$(3) \qquad \limsup_{\nu}\ [n(\nu)]^{\frac{1}{2}} ||\,\mu_{1\nu} - \mu_{0\nu}|| < \infty\ ,$$

where $\mu_{0\nu}$ is defined by

$$f(\mu_{0\nu}) = L_{\Lambda(R_\nu,\mu)}\ f(\mu_{1\nu}).$$

Let $\phi_{1\nu}$ be the test given by Corollary 4.4.2, with rejection region (5.6.2). Then

$$(4) \qquad \liminf_{\nu}\ E_{\mu_{1\nu}}(\phi_{1\nu} - \psi_\nu) \geq 0\ .$$

Note that

$$\Lambda^{-1}(R_\nu,\ \mu) = (D_\mu^{-1})'\ \Sigma_\mu^{-1}\ R_\nu D_\mu^{-1}\ ,$$

where $D_\mu = (\partial f\ /\ \partial\mu)$. The assumption that f is twice continuously differentiable and a first-order Taylor expansion yield that if $\phi_{2\nu}$ has rejection region

$$d'_\nu\ \Lambda^{-1}(R_\nu,\ \mu)\ \tilde{Y}_\nu \geq u_\alpha[d'_\nu\ \Lambda^{-1}(R_\nu,\ \mu)d_\nu]^{\frac{1}{2}}$$

where

$$d_\nu = f(\mu_{1\nu}) - f(\mu_{0\nu}) \ , \quad \tilde{Y}_\nu = Y_\nu - [n(\nu)]^{\frac{1}{2}} f(\mu_{0\nu}) ,$$

then

$$(5) \qquad E_{\mu_{0\nu}} |\phi_{2\nu} - \phi_{1\nu}| \to 0 \ .$$

The definition of $\mu_{0\nu}$ and the fact that $f(\mu_{1\nu}) \in V + K$ imply that $\phi_{2\nu}$ rejects iff

$$w'_{\Lambda(R_\nu,\mu)} \tilde{Y}_\nu > u_\alpha \ ;$$

the property that $w'_\Lambda f(\mu_{0\nu}) = 0$ for every Λ implies that ϕ_ν rejects iff $W'_\nu Y_\nu > u_\alpha$. With (1), this shows that

$$(6) \qquad E_{\mu_{0\nu}} |\phi_{2\nu} - \phi_\nu| \to 0 \ .$$

Together with (3) and Proposition 4.1.1, (5) and (6) imply that $E_{\mu_{1\nu}} |\phi_{1\nu} - \phi_\nu| \to 0$. With (4), this yields

$$\liminf_\nu E_{\mu_{1\nu}} (\phi_\nu - \psi_\nu) \geq 0 \ .$$

(iii) Let again $\{\mu_{1\nu}\} \in S_A$, and let $\{\psi_\nu\}$ be an asymptotically level α test. Suppose now that

$$(7) \qquad [n(\nu)]^{\frac{1}{2}} || \mu_{1\nu} - \mu_{0\nu} || \to \infty$$

for every $\{\mu_{0\nu}\} \in S_H$. The second assumption in Definition 5.6.1 implies (passing to a subsequence, if necessary) the existence of a $\mu_0 \in \mu(\Theta_H)$ with

$$(8) \qquad W_\nu - w_{\Lambda(R_\nu, \mu_0)} \to 0 \quad \text{in } \{\mu_{1\nu}\}\text{- prob..}$$

Let

$$d_\nu = (I - L_{\Lambda(R_\nu,\mu_0)}) f(\mu_{1\nu}) \ .$$

It follows from (7), (8) and $f(\mu_{1\nu}) \in V + K$ that

$$[n(\nu)]^{\frac{1}{2}} W'_\nu d_\nu \to \infty \quad \text{in } \{\mu_{1\nu}\}\text{- prob.}.$$

Note that

$$T_\nu = W'_\nu (Y_\nu - [n(\nu)]^{\frac{1}{2}} f(\mu_{1\nu})) + [n(\nu)]^{\frac{1}{2}} W'_\nu d_\nu ;$$

the first term of the right hand side is bounded in $\{\mu_{1\nu}\}$- prob. and the second term goes to ∞ . This yields

$$E_{\mu_{1\nu}} \phi_\nu \to 1 .$$

(iv) Let $\{\mu_{1\nu}\} \in S_A$, and let $\{\psi_\nu\}$ be an asymptotically level α test. There exists a subsequence $\{\xi\}$ of $\{\nu\}$ for which

$$\lim_\xi E_{\mu_{1\xi}} (\phi_\xi - \psi_\xi) = \liminf_\nu E_{\mu_{1\nu}} (\phi_\nu - \psi_\nu),$$

and to which either (ii) or (iii) can be applied. In both cases it can be concluded that

$$\liminf_\nu E_{\mu_{1\nu}} (\phi_\nu - \psi_\nu) \geq 0 .$$

This establishes, together with (i), that $\{\phi_\nu\}$ is AUMP - level α. □

<u>REMARK</u>. It may be rather tedious to determine the projection $L_{\Lambda(R_\nu,\hat{\mu}_\nu)}a$. In many practical applications, f is the identity function while (possibly after a linear transformation)

(5.6.3) $\quad \Sigma_\mu V = V \qquad$ for all $\mu \in \mu\ (\Theta_H)$.

This implies that $x'\ \Lambda^{-1}(R_\nu, \mu)\ v = 0$ for all $v \in V$ iff $x'\ R_\nu v = 0$ for all $v \in V$. So under condition (5.6.3), $L_{\Lambda(R_\nu,\mu)}a$ is the projection of a on V with respect to the inner product $x'\ R_\nu y$, and independent of μ.

For a result which corresponds to the theorem above but which is valid for more general families of probability distributions, NEYMAN (1959) needs "root n - consistent estimators". These are estimators for which, roughly said,

$$\{L_{\mu_\nu}([n(\nu)]^{\frac{1}{2}}(\hat{\mu}_\nu - \mu_\nu))\} \text{ is tight}$$

for all $\{\mu_\nu\} \in S_H$. We need to assume of the estimator $\{\hat{\mu}_\nu\}$ only that it is uniformly consistent, because of our restriction to exponential families.

EXAMPLE 5.6.1. *Comparison of two success probabilities (continued).* This is a continuation of Example 5.5.1. In order to use the same symbols as in the present section, the success probabilities will be denoted by μ_1 and μ_2. This testing problem provides an instance of the testing problem of this section, with $k = 2$, $m_1 = m_2 = 1$, $m = 2$, f = the identity function, and

$$V = \{(\mu_1, \mu_2)' \in \mathbb{R}^2 \mid \mu_1 = \mu_2\} \quad , \quad a = (1, -1)' \ .$$

The average number of sucesses in the combined sample

$$\bar{X}_{\cdot}^{(\nu)} = (X_{1+}^{(\nu)} + X_{2+}^{(\nu)})/n(\nu) = \rho_1(\nu)\, X_{1.}^{(\nu)} + \rho_2(\nu)\, X_{2.}^{(\nu)}$$

is a uniformly consistent estimator for the success probability under H. For $\mu = (\mu_0, \mu_0)' \in V$, one has

$$\Sigma_\mu = \mu_0(1 - \mu_0)\, I \ .$$

It is clear that (5.6.3) is satisfied and

$$a - L_{\Lambda(R_\nu,\mu)}a = 2(\rho_2(\nu), -\rho_1(\nu))'$$

$$w_{\Lambda(R_\nu,\mu)} = \{\rho_1(\nu)\, \rho_2(\nu) / \mu_0(1 - \mu_0)\}^{\frac{1}{2}}(1, -1)'$$

for $\mu = (\mu_0, \mu_0)'$. This yields the test statistic

$$T_\nu = \{\rho_1(\nu)\, \rho_2(\nu)\, n(\nu) / \bar{X}_{\cdot}^{(\nu)}(1 - \bar{X}_{\cdot}^{(\nu)})\}^{\frac{1}{2}} (X_{1.}^{(\nu)} - X_{2.}^{(\nu)}) \ .$$

The test which rejects for $T_\nu > u_\alpha$ is AUMP - level α. It is asymptotically equivalent to Fisher's exact test $\phi_{F\nu}$. The latter test is UMP-unbiased level α (LEHMANN (1959), Section 4.5) and it is given by $\phi_{F\nu}(X_{1+}^{(\nu)}, X_{2+}^{(\nu)})$ with

$$\phi_{F\nu}(x_1, x_2) = \begin{cases} 1 & > \\ t_\alpha(n, n_1, x_1 + x_2) & x_1 = h_\alpha(n, n_1, x_1 + x_2), \\ 0 & < \end{cases}$$

where $n = n(\nu)$, $n_1 = n_1(\nu)$, and t_α and h_α are determined by the requirement

$$E_{(\mu_0,\mu_0)}\{\phi_{F\nu}(X_{1+}^{(\nu)}, X_{2+}^{(\nu)}) \mid X_{1+}^{(\nu)} + X_{2+}^{(\nu)} = x\} = \alpha \text{ for all } x.$$

The test which rejects for $T_\nu > u_\alpha$ can also be regarded as a one-sided version of the χ^2 test for the 2 x 2 table, which rejects for $T_\nu^2 > u_{\frac{1}{2}\alpha}^2 = \chi^2_{1;\alpha}$.
□

EXAMPLE 5.6.2. *The Behrens-Fisher problem.* Let $Z_{11}^{(\nu)}, \ldots, Z_{1n_1(\nu)}^{(\nu)}$ and $Z_{21}^{(\nu)}, \ldots, Z_{2n(\nu)}^{(\nu)}$ be independent random samples from normal distributions with expectation η_1 and η_2 and variance σ_1^2 and σ_2^2, respectively. Consider the testing problem

$$H : \eta_1 \leq \eta_2 \quad , \; A : \eta_1 > \eta_2 \; .$$

This is called the Behrens-Fisher problem. It has received much attention in the literature (see, e.g., LINNIK (1968) and references cited there) because of its practical and theoretical importance. The subset Θ_H of the natural parameter space corresponding to the null hypothesis is "curved", and the standard methods for obtaining UMP-unbiased level α tests cannot be applied.

With the definition

$$X_{ij}^{(\nu)} = \begin{pmatrix} Z_{ij}^{(\nu)} \\ (Z_{ij}^{(\nu)})^2 \end{pmatrix} \qquad i = 1, 2; \; 1 \leq j \leq n_i(\nu) \; ,$$

this problem is brought into the form of the testing problem of this section. Expectation and covariance matrix of $X_{ij}^{(\nu)}$ are given by

$$\mu_i = \begin{pmatrix} \eta_i \\ \eta_i^2 + \sigma_i^2 \end{pmatrix}$$

$$\Sigma_{i\mu_i} = \begin{pmatrix} \sigma_i^2 & 2\eta_i\sigma_i^2 \\ 2\eta_i\sigma_i^2 & 2\sigma_i^4 + 4\eta_i^2\sigma_i^2 \end{pmatrix} .$$

Some computations show that for $\mu = (\eta,\ \eta^2 + \sigma_1^2,\ \eta,\ \eta^2 + \sigma_2^2)'$,

$$w_{\Lambda(R_\nu,\mu)} = \{\sigma_1^2 \,/\, \rho_1(\nu) + \sigma_2^2 \,/\, \rho_2(\nu)\}^{-\frac{1}{2}} (1,\ 0,\ -1,\ 0)' .$$

The variances σ_i^2 can be estimated by $S_{\nu i}^2 \,/\, n_i(\nu)$, where

$$S_{\nu i}^2 = \sum_{j=1}^{n_i(\nu)} (Z_{ij}^{(\nu)} - Z_{i.}^{(\nu)})^2$$

$$Z_{i.}^{(\nu)} = [n_i(\nu)]^{-1} \sum_{j=1}^{n_i(\nu)} Z_{ij}^{(\nu)} .$$

Theorem 5.6.1 yields that the test which rejects for

$$n(\nu)\ [S_{\nu 1}^2 \,/\, \rho_1^2(\nu) + S_{\nu 2}^2 \,/\, \rho_2^2(\nu)]^{-\frac{1}{2}}\ (Z_{1.}^{(\nu)} - Z_{2.}^{(\nu)}) > u_\alpha$$

is AUMP - level α. Most tests which are proposed for this problem, such as the test of WELCH (1947), are asymptotically equivalent to this test. As a matter of fact, the reasons why the Behrens-Fisher problem is found to be interesting are related to the difficulties inherent in the size α - restriction

$$\sup_{\eta_1=\eta_2} E_{(\eta_1,\eta_2,\sigma_1^2,\sigma_2^2)}\ \phi = \alpha$$

and in the similarity restriction

$$E_{(\eta_1,\eta_2,\sigma_1^2,\sigma_2^2)}\ \phi = \alpha \qquad \text{for all } (\eta_1,\ \eta_2,\ \sigma_1^2,\sigma_2^2) \text{ with } \eta_1 = \eta_2.$$

These difficulties disappear in the asymptotic approach followed here. The "flavour" of the Behrens-Fisher problem is lost in the present treatment.
□

5.7. AN ELABORATE EXAMPLE: PAIRED COMPARISONS WITH ORDER EFFECTS

For the comparison of two treatments, an "efficient design" can be obtained by letting each individual serve as his own control: the two treatments are administered to each of ν individuals, and each individual produces a score indicating the preference for the one over the other treatment. There will in general be an order effect, as one of the treatments is administered first and the other treatment last. The methodological and statistical procedures will have to take this order effect into account.

The problem of testing whether a treatment effect exists, is discussed in SCHEFFÉ (1952), GART (1969), SCHAAFSMA (1973) and RAY (1976), under different assumptions concerning the experimental design. We study the same design as Schaafsma: a fair coin is tossed for each individual separately in order to decide which treatment is administered first. Each individual is requested to state whether he prefers the treatment tried first or the one tried last. Indicate the two treatments by the numbers 0 and 1, and define, for $1 \leq j \leq \nu$,

z_{j1} = number of treatment administered first to individual j
z_{j2} = 0 (or 1) if individual j preferred the treatment administered first (or last).

The ν pairs (z_{j1}, z_{j2}) are regarded as the outcomes of a random sample $(Z_{11}, Z_{12}), \ldots, (Z_{\nu 1}, Z_{\nu 2})$. Denote the conditional probability that a randomly selected individual prefers the treatment administered last, if treatment g was administered first, by θ_g:

$$\theta_g = P_\theta\{Z_{j2} = 1 \mid Z_{j1} = g\}$$

$$\theta = (\theta_0, \theta_1) \in (0,1)^2 .$$

Since the coin is supposed to be fair, we have

$$P_\theta\{(Z_{j1}, Z_{j2}) = (g, h)\} = \begin{cases} \tfrac{1}{2}(1-\theta_g) & \text{if } h = 0 \\ \tfrac{1}{2}\,\theta_g & \text{if } h = 1 \end{cases} .$$

The null hypothesis of no treatment effect

$$H : \theta_0 = \theta_1$$

is to be tested against the alternative that the second treatment is preferred,

$$A : \theta_0 > \theta_1 .$$

(Note that the order effect is absent iff $\theta_0 + \theta_1 = 1$.)
This is a testing problem for an exponential family of distributions where the part of the natural parameter space corresponding to the null hypothesis is "curved"; there does not exist a UMP - unbiased level α test.

A sufficient statistic is $N = (N_{00}, N_{01}, N_{10}, N_{11})$ where

$$N_{gh} = \text{number of } j \text{ with } (Z_{j1}, Z_{j2}) = (g, h).$$

In practice, the order effect is often ignored, and the data are reduced to $(N_{00} + N_{11}, N_{01} + N_{10})$, being the numbers of times that treatment 0 or 1, respectively, is preferred. The sign test, rejecting for large values of $N_{01} + N_{10}$, is UMP among the level α tests based on $N_{01} + N_{10}$. It is also most powerful - level α against all (θ_0, θ_1) with $\theta_0 > \theta_1$ and $\theta_0 + \theta_1 = 1$. The test which rejects for large values of $N_{01} + N_{10}$, conditionally given $(N_{0+}, N_{1+}, N_{+0}, N_{+1})$ where

$$N_{g+} = N_{g0} + N_{g1} \quad , \quad N_{+h} = N_{0h} + N_{1h} \quad ,$$

is formally equivalent to Fisher's test for the 2 x 2 table; it is studied by GART (1969) and SCHAAFSMA (1973). The latter author computes power functions of the sign test and of Fisher's test, leading him to the conclusion that "as ν increases, Fisher's test becomes more and more attractive whereas the sign test becomes less attractive". We shall prove that Fisher's test is AUMP - level α.

An asymptotic version of Fisher's test can be given by

$$\phi_{0\nu} = \begin{cases} 1 & T_{0\nu} > u_\alpha \\ 0 & T_{0\nu} \leq u_\alpha \end{cases} ,$$

where the test statistic is

$$T_{0\nu} = \{N_{0+}^{(\nu)} N_{1+}^{(\nu)} N_{+0}^{(\nu)} N_{+1}^{(\nu)} / \nu\}^{-\frac{1}{2}} (N_{10}^{(\nu)} N_{01}^{(\nu)} - N_{00}^{(\nu)} N_{11}^{(\nu)}) .$$

In order to apply the theory of Section 5.6, the testing problem must be given the form of Section 3.5. In view of the remark following Theorem 5.6.1, a convenient transformation of the $(Z_{j1}^{(\nu)}, Z_{j2}^{(\nu)})$ to random variables having a distribution from a canonical exponential family (compare Example 2.3.1) is

$$X_j^{(\nu)} = \begin{cases} (-1,0,0)' & \\ (0,1,1)' & \text{if } (Z_{j1}^{(\nu)}, Z_{j2}^{(\nu)}) = \\ (1,0,0)' & \\ (0,-1,1)' & \end{cases} \begin{matrix} (0,0) \\ (0,1) \\ (1,0) \\ (1,1) . \end{matrix}$$

This transformation transforms $N^{(\nu)}$ to

$$X_+^{(\nu)} = \sum_{j=1}^{\nu} X_j^{(\nu)} = (N_{10}^{(\nu)} - N_{00}^{(\nu)}, N_{01}^{(\nu)} - N_{11}^{(\nu)}, N_{01}^{(\nu)} + N_{11}^{(\nu)})' .$$

Note that

$$\mu = \mu(\theta) = E_\theta X_j^{(\nu)} = \tfrac{1}{2}(\theta_0 - \theta_1, \theta_0 - \theta_1, \theta_0 + \theta_1)' .$$

(Unlike in Section 3.5, θ is here not the natural parameter.) This transformation is chosen because in terms of μ, the testing problem obtains the simple form

$$H : \mu_1 = \mu_2 = 0, \qquad A : \mu_1 = \mu_2 > 0 ,$$

while the covariance matrix under H is diagonal:

$$\Sigma_\mu = \text{cov}_\mu\, X_j^{(\nu)} = \begin{pmatrix} 1-\mu_3 & 0 & 0 \\ 0 & \mu_3 & 0 \\ 0 & 0 & \mu_3(1-\mu_3) \end{pmatrix}$$

for $\mu = (0,0, \mu_3)$. This testing problem is of the form of Section 5.6, with $k = 1$, $m = m_1 = 3$, f = the identity function and

$$V = \{(0,0,\mu_3)' | \mu_3 \in \mathbb{R}\} \ , \ a = (1,\ 1,\ 0)' \ .$$

Note that $\Lambda(R_\nu, \mu) = \Sigma_\mu$ and that condition (5.6.3) is satisfied; we have $L_{\Sigma_\mu} a = 0$ and

$$w_{\Sigma_\mu} = \{\mu_3(1 - \mu_3)\}^{-\frac{1}{2}} (\mu_3,\ 1 - \mu_3,\ 0)' \ .$$

A uniformly consistent estimator for μ_3 under H is $X_{3.}^{(\nu)} = \nu^{-1} N_{+1}^{(\nu)}$. Theorem 5.6.1 yields that the test ϕ_ν which rejects for $T_\nu > u_\alpha$, where

$$T_\nu = \nu^{\frac{1}{2}} \{X_{3.}^{(\nu)}(1 - X_{3.}^{(\nu)})\}^{-\frac{1}{2}} \{X_{3.}^{(\nu)} X_{1.}^{(\nu)} + (1 - X_{3.}^{(\nu)}) X_{2.}^{(\nu)}\}$$

$$= 2(\nu N_{+0}^{(\nu)} N_{+1}^{(\nu)})^{-\frac{1}{2}} (N_{01}^{(\nu)} N_{10}^{(\nu)} - N_{00}^{(\nu)} N_{11}^{(\nu)}) \ ,$$

is AUMP - level α. For all μ one has $\mathcal{L}_\mu(N_{0+}^{(\nu)}) = \mathcal{B}(\nu, \frac{1}{2})$, so that $\nu^{-2} N_{0+}^{(\nu)} N_{1+}^{(\nu)} \to \frac{1}{4}$ in prob.. Hence

$$T_\nu \ / \ T_{0\nu} \to 1 \text{ in prob.},$$

implying that $\{\phi_\nu\}$ and $\{\phi_{0\nu}\}$ are strongly asymptotically equivalent. Hence Fisher's test $\{\phi_{0\nu}\}$ is AUMP - level α.

The considerations and power computations in SCHAAFSMA (1973) show that, for very small ν, Fisher's exact test is inadmissible; and that for ν reasonably large, say, $\nu \geq 30$, the sign test is only slightly better than Fisher's test, for $\theta_0 + \theta_1$ close to 1, whereas Fisher's test is much better than the sign test in a large region of the alternative. Although no comparisons with other tests have been made, it seems that for this testing problem the statement "$\{\phi_{0\nu}\}$ is AUMP - level α" can be interpreted roughly as "for $\nu \geq 40$, no level α test exists which has, for some (θ_0, θ_1) with $\theta_0 > \theta_1$ and $\theta_0(\theta_1)$ not too close to 0(1), a worthwile power advantage over $\phi_{0\nu}$".

CHAPTER 6

ASYMPTOTICALLY MOST STRINGENT TESTS FOR PROBLEMS WITH COMPOSITE NULL HYPOTHESES

In this chapter, asymptotic versions of the concept "most stringent" are treated for the testing problem of Section 5.1. A definition of the concept "asymptotically most stringent" is given in Section 6.1. In Section 6.3, the asymptotically most stringent test for testing problems with unrestricted alternatives for exponential families is derived. This result is closely related to the results of WALD (1943), and includes as a special case that the familiar χ^2 test for contingency tables is asymptotically most stringent. In Section 6.4, the asymptotically most stringent test for certain testing problems with restricted alternatives for exponential families is derived.

6.1. DEFINITIONS OF "ASYMPTOTICALLY MOST STRINGENT"

For many asymptotic testing problems there does not exist an AUMP - $\tilde{\Phi}$ test in "attractive" classes $\tilde{\Phi}$. Therefore it is relevant to define asymptotic optimum properties which can (almost) always be satisfied, and which are equivalent to "AUMP" if there exists an AUMP test. The property "asymptotically most stringent" (abbreviated to "AMS"), to be defined in this section, is such an optimum property. It is an asymptotic version of the property "most stringent", defined in Section 2.6. Recall that a test ϕ is most stringent -(Φ, β^*) if $\phi \in \Phi$ and

$$\sup_{\theta \in \Theta_A} \gamma(\phi, \theta) = \inf_{\phi' \in \Phi} \sup_{\theta \in \Theta_A} \gamma(\phi', \theta), \tag{6.1.1}$$

where γ is the shortcoming with respect to β^*, defined by

$$\gamma(\phi, \theta) = \beta^*(\theta) - E_\theta \phi .$$

A useful alternative formulation of condition (6.1.1) is the requirement that for every $\phi' \in \Phi$, one has

$$(6.1.2) \qquad \inf_{\theta \in \Theta_A} \sup_{\theta' \in \Theta_A} \{\gamma(\phi', \theta') - \gamma(\phi, \theta)\} \geq 0 .$$

In Sections 6.1 and 6.2, the sequence of testing problems $T_1, T_2, \ldots$ of Section 5.1 is considered again. The property "asymptotically most stringent" will be defined with respect to a sequence

$$\tilde{\beta}^* = \{\beta^*_\nu\}$$

of functions $\beta^*_\nu : \Theta_A \to [0,1]$. For every ν, the shortcoming with respect to β^*_ν is

$$\gamma_\nu(\phi, \theta) = \beta^*_\nu(\theta) - E_\theta \phi ,$$

where ϕ is a test for T_ν. In most applications, β^*_ν is the envelope power function with respect to the class of all level α tests for T_ν.

The concept "asymptotically most stringent" was introduced by WALD (1943). He defines a test $\{\phi_\nu\}$ to be asymptotically most stringent if

(i) $$\sup_{\theta \in \Theta_H} E_\theta \phi_\nu = \alpha \qquad \text{for all } \nu$$

(ii) for every test $\{\phi'_\nu\}$ which satisfies (i),

$$\liminf_\nu \, [\sup_{\theta \in \Theta_A} \gamma_\nu(\phi'_\nu, \theta) - \sup_{\theta \in \Theta_A} \gamma_\nu(\phi_\nu, \theta)] \geq 0 .$$

(γ_ν is the shortcoming with respect to the class of all level α tests for T_ν.) An alternative formulation for (ii), resembling (6.1.2), is the requirement that for every test $\{\phi'_\nu\}$ satisfying (i), one has

$$\inf_{\{\theta_\nu\} \subset \Theta_A} \sup_{\{\theta'_\nu\} \subset \Theta_A} \liminf_\nu \, [\gamma_\nu(\phi'_\nu, \theta'_\nu) - \gamma_\nu(\phi_\nu, \theta_\nu)] \geq 0.$$

This requirement involves not only interior sequences, but also sequences $\{\theta_\nu\}$ and $\{\theta'_\nu\}$ which "tend to the boundary of Θ_T". In Sections 5.2 and 5.5 it is demonstrated that many testing problems "degenerate at the boundary of Θ_T", and that it can be relevant to define asymptotic optimum properties of tests which involve only interior sequences. For a class S_0 of sequences in Θ_A, a test $\{\phi_\nu\} \in \tilde{\Phi}$ will be called asymptotically most stringent -$(\tilde{\Phi}, \tilde{\beta}^*)$ against S_0 if for every $\{\phi'_\nu\} \in \tilde{\Phi}$, one has

$$(6.1.3) \qquad \inf_{\{\theta_\nu\}\in S_0} \sup_{\{\theta'_\nu\}\in S_0} \liminf_\nu [\gamma_\nu(\phi'_\nu,\theta'_\nu) - \gamma_\nu(\phi_\nu,\theta_\nu)] \geq 0.$$

This is Wald's definition if $\tilde{\Phi}$ is the class of all sequences of size α test functions, β^*_ν is the envelope power function with respect to the class of all level α tests for T_ν, and S_0 is the class of all sequences in Θ_A. We shall call a test $\{\phi_\nu\} \in \tilde{\Phi}$ asymptotically most stringent -$(\tilde{\Phi}, \tilde{\beta}^*)$ if $\{\phi_\nu\}$ is asymptotically most stringent -$(\tilde{\Phi}, \tilde{\beta}^*)$ against S_A, the class of all interior sequences in Θ_A. In Chapter 7, the property "AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$ against S_0" will be considered for $S_0 \subset S_A$.

The optimum property "AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$" is especially relevant if $\tilde{\beta}^*$ corresponds to $\tilde{\Phi}$ in the sense that for every $\{\theta_\nu\} \in S_A$ a test $\{\phi_\nu\}$ exists which is AMP - $\tilde{\Phi}$ against $\{\theta_\nu\}$, and

$$(6.1.4) \qquad \gamma_\nu(\phi_\nu, \theta_\nu) = \beta^*_\nu(\theta_\nu) - E_{\theta_\nu}\phi_\nu \to 0 \ .$$

In this case, $\{\phi_\nu\} \in \tilde{\Phi}$ is AMP - $\tilde{\Phi}$ against $\{\theta_\nu\}$ iff (6.1.4) holds. Theorem 5.4.1 shows that if $\tilde{\Phi}$ is the class of all asymptotically level α tests and β^*_ν is the envelope power function with respect to the class of all level α tests for T_ν, then β^*_ν corresponds to $\tilde{\Phi}$ in this sense. If $\tilde{\beta}^*_1$ and $\tilde{\beta}^*_2$ are sequences of envelope power functions which both correspond to the class of tests $\tilde{\Phi}$, then

$$\beta^*_{1\nu}(\theta_\nu) - \beta^*_{2\nu}(\theta_\nu) \to 0 \quad \text{for all } \{\theta_\nu\} \in S_A \ .$$

This implies that for $S_0 \subset S_A$, a test is AMS - $(\tilde{\Phi}, \tilde{\beta}^*_1)$ against S_0 iff it is AMS - $(\tilde{\Phi}, \tilde{\beta}^*_2)$ against S_0. Hence it makes sense to replace the name "AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$ against S_0" by "AMS - $\tilde{\Phi}$ against S_0" if $\tilde{\beta}^*$ corresponds to $\tilde{\Phi}$, and $S_0 \subset S_A$.

An asymptotic analogue of the minimax shortcoming does not always exist. This should be a number γ_0 such that $\{\phi_\nu\} \in \tilde{\Phi}$ is AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$ iff

$$\sup_{\{\theta_\nu\}\in S_A} \limsup_\nu \gamma_\nu(\phi_\nu,\theta_\nu) \le \gamma_0 .$$

The latter condition is equivalent to the condition that (6.1.3) holds for every $\{\phi'_\nu\} \in \tilde{\Phi}$, with $S_0 = S_A$, iff

$$\gamma_0 = \inf_{\{\phi'_\nu\}\in\tilde{\Phi}} \sup_{\{\theta_\nu\}\in S_A} \liminf_\nu \gamma_\nu (\phi'_\nu,\theta_\nu)$$

$$= \inf_{\{\phi'_\nu\}\in\tilde{\Phi}} \sup_{\{\theta_\nu\}\in S_A} \limsup_\nu \gamma_\nu (\phi'_\nu,\theta_\nu) .$$

If this equality holds, then we say that the asymptotic minimax shortcoming - $(\tilde{\Phi}, \tilde{\beta}^*)$ exists and is equal to γ_0. We shall see in Section 6.3 that the asymptotic minimax shortcoming for the class of asymptotically level α tests exists, e.g, for a large class of testing problems for exponential families with unrestricted alternatives. For the testing problem of Section 3.5, the asymptotic minimax shortcoming exists if the sequence of proportion matrices R_ν is convergent.

After WALD (1943) started the investigation of asymptotically most stringent tests with his monumental work, several other authors have studied similar asymptotic optimum concepts. JOHNSON and ROUSSAS (1972) consider particular classes Ψ_ν of tests such that if $\phi_\nu \in \Psi_\nu$ for all ν, then $\{\phi_\nu\}$ is asymptotically of level α; they take $\tilde{\Phi}$ as the class of all such tests $\{\phi_\nu\}$, and β^*_ν as the envelope power function with respect to Ψ_ν. Presumably, it can be proved that $\tilde{\Phi}$ contains "all good asymptotically level α tests" in some sense. They define certain sequences $\{B_\nu\}$ of subsets of Θ_A such that $\beta^*_\nu(\theta)$ is approximately constant for $\theta \in B_\nu$, and consider the optimum property "AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$ against S_0", where

$$S_0 = \{\{\theta_\nu\} \mid \theta_\nu \in B_\nu \text{ for all } \nu\} .$$

This optimum property is so heavily loaded with technical details, that it seems less appealing than the optimum property "AMS - level α" defined below. On the other hand, Johnson and Roussas derive asymptotically optimal tests for a class of testing problems which is much more general than the testing problem for exponential families for which an AMS-level α test is derived in Section 6.3.

BHAT and NAGNUR (1965) consider the shortcoming with respect to the class of all level α tests, and take $\tilde{\Phi}$ to be the class of $C(\alpha)$ tests introduced by NEYMAN (1959); they define the test $\{\phi_\nu\} \in \tilde{\Phi}$ to be "locally asymptotically most stringent" if for all $\{\phi'_\nu\} \in \tilde{\Phi}$, one has

$$\liminf_\nu \Big[\sup_{\{\theta_\nu\} \in S_0} \gamma_\nu(\phi'_\nu, \theta_\nu) - \sup_{\{\theta_\nu\} \in S_0} \gamma_\nu(\phi_\nu, \theta_\nu) \Big] \geq 0,$$

where S_0 is a certain subclass of S_A. This requirement is equivalent to Wald's requirement (ii) mentioned above, since

$$\sup_{\{\theta_\nu\} \in S_0} \gamma_\nu(\phi, \theta_\nu) = \sup_{\theta \in \Theta_A} \gamma_\nu(\phi, \theta).$$

However, the remainder of their paper seems to imply that they intend to use definition (6.1.3).

The preceding discussion leads to the following definitions.

<u>DEFINITION 6.1.1</u>. Let $\tilde{\Phi}$ be a class of tests and $\tilde{\beta}^*$ a sequence $\beta_1^*, \beta_2^*, \ldots$ of functions $\beta_\nu^* : \Theta_A \to [0,1]$. For a test ϕ for T_ν , the shortcoming γ_ν with respect to β_ν^* is

$$\gamma_\nu(\phi, \theta) = \beta_\nu^*(\theta) - E_\theta \phi.$$

Let S_0 be a class of sequences in Θ_A.

A test $\{\phi_\nu\}$ is asymptotically most stringent - $(\tilde{\Phi}, \tilde{\beta}^*)$ against S_0, or AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$ against S_0, if

(i) $\{\phi_\nu\} \in \tilde{\Phi}$

(ii) for every $\{\phi'_\nu\} \in \tilde{\Phi}$, one has

$$\inf_{\{\theta_\nu\} \in S_0} \sup_{\{\theta'_\nu\} \in S_0} \liminf_\nu [\gamma_\nu(\phi'_\nu, \theta'_\nu) - \gamma_\nu(\phi_\nu, \theta_\nu)] \geq 0.$$

A test is asymptotically most stringent - $(\tilde{\Phi}, \tilde{\beta}^*)$ if it is asymptotically most stringent - $(\tilde{\Phi}, \tilde{\beta}^*)$ against S_A.

If $\tilde{\Phi}$ and $\tilde{\beta}^*$ are such that for every $\{\theta_\nu\} \in S_A$, a $\{\phi_\nu\} \in \tilde{\Phi}$ exists which is asymptotically most powerful - $\tilde{\Phi}$ against $\{\theta_\nu\}$ and satisfies

$$\gamma_\nu(\phi_\nu, \theta_\nu) \to 0 ,$$

then $\tilde{\beta}^*$ is said to agree with $\tilde{\Phi}$, and for $S_0 \subset S_A$, "AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$ against S_0" can be replaced by "AMS - $\tilde{\Phi}$ against S_0".

If $\tilde{\Phi}$ is the class of all asymptotically level α tests and $S_0 \subset S_A$, then "AMS - $\tilde{\Phi}$ against S_0" can be replaced by "AMS - level α against S_0".

If

$$\inf_{\{\phi_\nu\} \in \tilde{\Phi}} \; \sup_{\{\theta_\nu\} \in S_0} \; \liminf_{\nu} \gamma_\nu(\phi_\nu, \theta_\nu) =$$

$$= \inf_{\{\phi_\nu\} \in \tilde{\Phi}} \; \sup_{\{\theta_\nu\} \in S_0} \; \limsup_{\nu} \gamma_\nu(\phi_\nu, \theta_\nu) \quad ,$$

then this number is called the asymptotic minimax shortcoming - $(\tilde{\Phi}, \tilde{\beta}^*)$ against S_0, or the AMXS - $(\tilde{\Phi}, \tilde{\beta}^*)$ against S_0. The suffix "$(\tilde{\Phi}, \tilde{\beta}^*)$" can be replaced by "$\tilde{\Phi}$" or "level α" as above. If $S_0 = S_A$ the indication "against S_0" can be omitted.

For asymptotic testing problems where no AUMP test is available, the property "AMS" does not ensure that $E_{\theta_\nu}\phi_\nu \to 1$ for all sequences $\{\theta_\nu\} \in S_A$ for which an asymptotically level α test $\{\phi'_\nu\}$ exists with $E_{\theta_\nu}\phi'_\nu \to 1$, and for all AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$ tests $\{\phi_\nu\}$ where $\tilde{\Phi}$ and $\tilde{\beta}^*$ are "reasonable". So an additional requirement is made, expressed in the following definition.

<u>DEFINITION 6.1.2</u>. A sequence $\{\theta_\nu\}$ is remote if $\{\theta_\nu\} \in S_A$ and a test $\{\phi_\nu\}$ exists which is asymptotically of level 0 and satisfies

$$E_{\theta_\nu}\phi_\nu \to 1 \; .$$

A test $\{\phi_\nu\}$ is sharply consistent if

$$E_{\theta_\nu}\phi_\nu \to 1$$

for all remote sequences $\{\theta_\nu\}$.

Note that for the testing problem of Section 3.5, a sequence $\{\mu_\nu\} \in S_A$ is remote iff there exists no subsequence $\{\xi\}$ of $\{\nu\}$ and no $\{\mu_{0\nu}\} \in S_H$ with $\{\mu_\nu\} \Leftrightarrow \{\mu_{0\nu}\}$. Example 4.1.1 shows that this property does not hold in general for the testing problem of Section 5.1. Note also that if β^*_ν is the envelope power function with respect to the class of all level α tests for T_ν, then $\beta^*_\nu(\theta_\nu) \to 1$ for every remote sequence $\{\theta_\nu\}$.

6.2. CHARACTERIZATIONS OF AMS TESTS

This section contains two propositions which relate AMS tests to the AMXS. The first proposition shows that if the AMXS exists, then a test is AMS iff its asymptotic shortcoming nowhere exceeds the AMXS.

PROPOSITION 6.2.1. *For* $\{\phi_\nu\} \in \tilde{\Phi}$ *and* $\gamma \in [0,1]$ *the assertions (i) and (ii) are equivalent.*

(i) *the* AMXS - $(\tilde{\Phi}, \tilde{\beta}^*)$ *against* S_0 *exists and is equal to* γ*;*
$\{\phi_\nu\}$ *is* AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$ *against* S_0 .

(ii) $\gamma = \sup_{\{\theta_\nu\}\in S_0} \limsup_\nu \gamma_\nu(\phi_\nu,\theta_\nu) \le \inf_{\{\phi'_\nu\}\in\tilde{\Phi}} \sup_{\{\theta_\nu\}\in S_0} \liminf_\nu \gamma_\nu(\phi'_\nu,\theta_\nu)$.

PROOF. For $\{\phi'_\nu\} \in \tilde{\Phi}$, let

$$\gamma_-\{\phi'_\nu\} = \sup_{\{\theta_\nu\}\in S_0} \liminf_\nu \gamma_\nu(\phi'_\nu,\theta_\nu)$$

$$\gamma_+\{\phi'_\nu\} = \sup_{\{\theta_\nu\}\in S_0} \limsup_\nu \gamma_\nu(\phi'_\nu,\theta_\nu) \ .$$

(i) ⇒ (ii). Note that

$$\liminf_\nu [\gamma_\nu(\phi'_\nu,\theta'_\nu) - \gamma_\nu(\phi_\nu,\theta_\nu)] \le \liminf_\nu \gamma_\nu(\phi'_\nu,\theta'_\nu) + - \liminf_\nu \gamma_\nu(\phi_\nu,\theta_\nu)$$

$$\liminf_\nu [\gamma_\nu(\phi'_\nu,\theta'_\nu) - \gamma_\nu(\phi_\nu,\theta_\nu)] \le \limsup_\nu \gamma_\nu(\phi'_\nu,\theta'_\nu) + - \limsup_\nu \gamma_\nu(\phi_\nu,\theta_\nu) \ .$$

These inequalities permit to conclude from the assumption that $\{\phi_\nu\}$ is AMS, that

$$0 \le \inf_{\{\theta_\nu\}\in S_0} \sup_{\{\theta'_\nu\}\in S_0} \liminf_\nu [\gamma_\nu(\phi'_\nu,\theta'_\nu) - \gamma_\nu(\phi_\nu,\theta_\nu)] \le$$

$$\le \min [\gamma_-\{\phi'_\nu\} - \gamma_-\{\phi_\nu\},\ \gamma_+\{\phi'_\nu\} - \gamma_+\{\phi_\nu\}]$$

for all $\{\phi'_\nu\} \in \tilde{\Phi}$. Hence

$$\gamma_- \{\phi_\nu\} = \inf_{\{\phi'_\nu\} \in \tilde{\Phi}} \gamma_- \{\phi'_\nu\}$$

$$\gamma_+ \{\phi_\nu\} = \inf_{\{\phi'_\nu\} \in \tilde{\Phi}} \gamma_+ \{\phi'_\nu\} .$$

Both right hand sides are equal to γ, so that (ii) follows from

$$\gamma = \gamma_+\{\phi_\nu\} = \inf_{\{\phi'_\nu\} \in \tilde{\Phi}} \gamma_- \{\phi'_\nu\} .$$

(ii) $\Rightarrow$ (i). It follows from (ii), that

$$\inf_{\{\phi'_\nu\} \in \tilde{\Phi}} \gamma_- \{\phi'_\nu\} \leq \inf_{\{\phi'_\nu\} \in \tilde{\Phi}} \gamma_+ \{\phi'_\nu\} \leq \gamma_+\{\phi_\nu\} = \gamma \leq \inf_{\{\phi'_\nu\} \in \tilde{\Phi}} \gamma_-\{\phi'_\nu\}.$$

In this chain of inequalities, equality must hold. Therefore the AMXS exists and is equal to γ. For every $\{\phi'_\nu\} \in \tilde{\Phi}$, (ii) immediately implies the second inequality in

$$\inf_{\{\theta_\nu\} \in S_0} \sup_{\{\theta'_\nu\} \in S_0} \liminf_\nu [\gamma_\nu(\phi'_\nu,\theta'_\nu) - \gamma_\nu(\phi_\nu,\theta_\nu)] \geq$$

$$\geq \gamma_- \{\phi'_\nu\} - \gamma_+\{\phi_\nu\} \geq 0.$$

This establishes that $\{\phi_\nu\}$ is AMS. □

<u>COROLLARY 6.2.1</u>. *Suppose that* $\tilde{\beta}^*$ *agrees with* $\tilde{\Phi}$. *Then* $\{\phi_\nu\}$ *is AUMP* $- \tilde{\Phi}$ *iff* $\{\phi_\nu\}$ *is AMS* $- \tilde{\Phi}$ *and the AMXS* $- \tilde{\Phi}$ *is equal to* 0.

<u>PROOF</u>. The assumption that $\tilde{\beta}^*$ agrees with $\tilde{\Phi}$ implies that for all $\{\theta_\nu\} \in S_A$ and $\{\phi'_\nu\} \in \tilde{\Phi}$, one has

$$\liminf_\nu \gamma_\nu(\phi'_\nu,\theta_\nu) \geq 0;$$

and that $\{\phi'_\nu\}$ is AMP $- \tilde{\Phi}$ against $\{\theta_\nu\}$ iff $\gamma_\nu(\phi'_\nu,\theta_\nu) \to 0$. So $\{\phi_\nu\}$ is AUMP $- \tilde{\Phi}$ iff condition (ii) of Proposition 6.2.1 holds with $\gamma = 0$ and $S_0 = S_A$. The corollary follows from this observation and Proposition 6.2.1. □

The following proposition states, loosely said, that for certain S_0 a test is AMS against S_0 iff it is AMS against S_0 for all subsequences of $\{\nu\}$ for which the AMXS against S_0 exists. (Compare the inequality in the statement of Proposition 6.2.2 with condition (ii) of Proposition 6.2.1.) The proof permits the conclusion that the "only if" statement is valid for all classes S_0 of sequences in Θ_A. It is possible to give examples of classes S_0 for which the "if" statement fails to hold.

PROPOSITION 6.2.2. *Let* S_0 *be a class of sequences for which countably many subsets* $B_{h\nu}$ *of* Θ_A *exist with* $B_{h\nu} \subset B_{h+1,\nu}$ *and*

$$S_0 = \bigcup_h \{ \{\theta_\nu\} \mid \theta_\nu \in B_{h\nu} \text{ for every } \nu\} .$$

A test $\{\phi_\nu\} \in \tilde{\Phi}$ *is* AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$ *against* S_0, *iff every subsequence of* $\{\nu\}$ *has a further subsequence* $\{\xi\}$ *such that*

$$\sup_{\{\theta_\nu\}\in S_0} \limsup_{\xi} \gamma_\xi(\phi_\xi, \theta_\xi)$$

$$\leq \inf_{\{\phi'_\nu\}\in\tilde{\Phi}} \sup_{\{\theta_\nu\}\in S_0} \liminf_{\xi} \gamma_\xi(\phi'_\xi, \theta_\xi) .$$

PROOF. "if". Suppose that $\{\phi_\nu\} \in \tilde{\Phi}$ is not AMS. Then a $\{\phi'_\nu\} \in \tilde{\Phi}$, $\{\theta_\nu\} \in S_0$ and an $\varepsilon > 0$ exist for which

$$\sup_{\{\theta'_\nu\}\in S_0} \liminf_{\nu} [\gamma_\nu(\phi'_\nu, \theta'_\nu) - \gamma_\nu(\phi_\nu, \theta_\nu)] < -\varepsilon .$$

This can be expressed as

$$\sup_h \liminf_\nu a_{h\nu} < -\varepsilon,$$

where

$$a_{h\nu} = \sup_{\theta\in B_{h\nu}} \gamma_\nu(\phi'_\nu, \theta) - \gamma_\nu(\phi_\nu, \theta_\nu).$$

Since $a_{h\nu} \leq a_{h+1,\nu}$ for every h and ν, Lemma A.5.2 yields a subsequence $\{\xi\}$ of $\{\nu\}$ with

$$\sup_{h} \limsup_{\xi} a_{h\xi} < -\varepsilon.$$

It may be assumed, after taking a further subsequence if necessary, that $\gamma_\xi(\phi_\xi, \theta_\xi)$ converges. This implies

$$\sup_{\{\theta'_\nu\}\in S_0} \limsup_{\xi} \gamma_\xi(\phi'_\xi,\theta'_\xi) = \sup_{h} \limsup_{\xi} a_{h\xi} + \lim_{\xi} \gamma_\xi(\phi_\xi,\theta_\xi) <$$

$$< \lim_{\xi} \gamma_\xi(\phi_\xi, \theta_\xi) - \varepsilon.$$

So $\{\phi_\nu\}$ does not have the second property mentioned in the proposition.
"only if". Suppose that $\{\phi_\nu\}$ is AMS. Lemma A.5.3 demonstrates that for every subsequence of $\{\nu\}$, a further subsequence $\{\xi\}$ exists with

(1) $$\sup_{\{\theta_\nu\}\in S_0} \liminf_{\xi} \gamma_\xi(\phi_\xi,\theta_\xi) = \sup_{\{\theta_\nu\}\in S_0} \limsup_{\xi} \gamma_\xi(\phi_\xi,\theta_\xi) .$$

Since $\{\phi_\nu\}$ is AMS, for every $\{\phi'_\nu\}$ one has

$$0 \leq \inf_{\{\theta_\nu\}} \sup_{\{\theta'_\nu\}} \liminf_{\nu} [\gamma_\nu(\phi'_\nu,\theta'_\nu) - \gamma_\nu(\phi_\nu,\theta_\nu)]$$

(2) $$\leq \inf_{\{\theta_\nu\}} \sup_{\{\theta'_\nu\}} \liminf_{\xi} [\gamma_\xi(\phi'_\xi,\theta'_\xi) - \gamma_\xi(\phi_\xi,\theta_\xi)] \leq$$

$$\leq \sup_{\{\theta'_\nu\}} \liminf_{\xi} \gamma_\xi(\phi'_\xi,\theta'_\xi) - \sup_{\{\theta_\nu\}} \liminf_{\xi} \gamma_\xi(\phi_\xi,\theta_\xi) .$$

(1) and (2) show that $\{\phi_\nu\}$ has the desired property. □

COROLLARY 6.2.2. *A test* $\{\phi_\nu\} \in \tilde{\Phi}$ *is* AMS - $(\tilde{\Phi}, \tilde{\beta}^*)$ *iff every subsequence of* $\{\nu\}$ *has a further subsequence* $\{\xi\}$ *such that*

$$\sup_{\{\theta_\nu\}\in S_A} \limsup_{\xi} \gamma_\xi(\phi_\xi,\theta_\xi) \leq$$

$$\leq \inf_{\{\phi'_\nu\} \in \tilde{\Phi}} \sup_{\{\theta_\nu\} \in S_A} \liminf_{\xi} \gamma_\xi(\phi'_\xi, \theta_\xi) .$$

PROOF. Let $\{K_h\}$ be the sequence of compact sets given by Lemma 5.2.1 with $K_h \subset \text{int } K_{h+1}$ and $\Theta_T = \bigcup_h K_h$. For every compact $K \subset \Theta_T = \bigcup_h \text{int } K_h$, there exists an h with $K \subset K_h$. Hence a sequence $\{\theta_\nu\} \subset \Theta_T$ is relatively compact in Θ_T iff it is contained in some K_h. Hence

$$S_A = \bigcup_h \{ \{\theta_\nu\} \mid \theta_\nu \in B_h \text{ for all } \nu\}$$

with $B_h = K_h \cap \Theta_A$, and Proposition 6.2.2 can be applied. □

6.3. AMS TESTS AGAINST UNRESTRICTED ALTERNATIVES FOR EXPONENTIAL FAMILIES

This section gives the AMS - level α test for the testing problem for exponential families of Section 3.5, in the case that the cone K used to define the alternative is a linear subspace of $\mathbb{R}^m$. If $V + K = \mathbb{R}^m$, the alternative is unrestricted. If $V + K$ is a linear subspace of $\mathbb{R}^m$ but not $\mathbb{R}^m$ itself then the alternative is not really "unrestricted", but the testing problem will be treated here in the same way as when $V + K = \mathbb{R}^m$. Hence the title of this section.

Important practical examples of this testing problem are the k-sample problem and the independence problem for random samples from probability distributions with finitely many possible outcomes, with the unrestricted alternatives (Examples 3.1.1 and 3.1.2 with alternatives A_3 and A_2, respectively). These testing problems are commonly formulated as testing problems for contingency tables. KARL PEARSON (1900) proposed the now familiar χ^2 test for contingency tables; however, the number of degrees of freedom used by him is incorrect for cases with composite null hypotheses. Many of the "fathers of mathematical statistics" made important contributions to the theory of the χ^2 test. FISHER (1922a) showed the correct way to compute the number of degrees of freedom. NEYMAN and E. PEARSON (1928) showed that the χ^2 test can be derived as an asymptotic approximation to the likelihood ratio test. WILKS (1935) derived the exact likelihood ratio test for these testing problems for contingency tables, and argued that there is no theoretical reason why the χ^2 test should be preferred to the likelihood ratio test. Much work has been done on the χ^2 test; we refrain from a more extensive historical impression.

WALD (1943) proved that, for testing problems for very general families of distributions, the likelihood ratio test is asymptotically most stringent for testing against an unrestricted alternative. Wald's assumptions III and V, however, contain uniformity conditions which are not satisfied by full multinomial families. He does not take the possibility into account that the testing problem "degenerates at the boundary of Θ_T" (see Sections 5.2 and 5.5). So Wald's result cannot be directly applied to show that χ^2 tests are AMS for the corresponding testing problems for contingency tables with composite null hypotheses and unrestricted alternatives. The question, whether χ^2 tests for these testing problems are indeed AMS in the sense of Wald's original definition (see Section 6.1), remains open. (See the remark at the end of this section.)

The result that the test $\{\phi_\nu\}$ of Theorem 6.3.1 is AMS - level α can also be proved by applying WALD's (1943) theorem to arbitrary large compact subsets of Θ_T. We give another proof, which is similar to the proofs of Theorems 6.4.1 and 8.1.1 concerning testing problems with restricted alternatives (Wald's proofs cannot be extended to the latter kind of testing problem). Wald's theorem is more general because he does not make the restriction to exponential families. BHAT and NAGNUR (1965) and JOHNSON and ROUSSAS (1972) also derive AMS tests for testing problems with unrestricted alternatives for very general families of probability distributions. We argued in Section 6.1 that their definitions of "asymptotically most stringent" are not completely satisfactory. The asymptotic uniqueness of the AMS - level α test proved in Theorem 6.3.1 seems to be new.

In this section the asymptotic testing problem of Section 3.5 is considered again, with the notation introduced there; it is assumed that the cone K used to define the alternative can be chosen to be a linear subspace of $\mathbb{R}^m$. For the positive definite symmetric matrix Λ, define

$$[x, y]_\Lambda = x' \Lambda^{-1} y$$

$$||x||_\Lambda = \{[x, x]_\Lambda\}^{\frac{1}{2}}$$

$L_{H\Lambda} : \mathbb{R}^m \to V$ the projection on V with respect to $[.,.]_\Lambda$

$L_{A\Lambda} : \mathbb{R}^m \to V + K$ the projection on V + K with respect to $[.,.]_\Lambda$.

THEOREM 6.3.1. *Let* $\{\hat{\mu}_\nu\}$ *be a uniformly consistent estimator for* μ *under* H *and let*

$$\hat{\Lambda}_\nu = \Lambda\ (R_\nu, \hat{\mu}_\nu)\ .$$

Let ϕ_ν *be the test which rejects for*

$$T_\nu = \|\ (L_{A\hat{\Lambda}_\nu} - L_{H\hat{\Lambda}_\nu})\ Y_\nu\ \|^2_{\hat{\Lambda}_\nu} \geq \chi^2_{r;\alpha}\ ,$$

where $r = \dim\ (V + K) - \dim V$.

The test $\{\phi_\nu\}$ *is AMS - level* α *and sharply consistent. Every sharply consistent AMS - level* α *test is asymptotically equivalent to* $\{\phi_\nu\}$*. The AMXS- level* α *exists and is equal to the minimax shortcoming for the testing problem*

$$L_\eta(U) = N_r(\eta,\ I)$$
$$H : \eta = 0,\ A : \eta \neq 0\ ,$$

and given by the formula

$$\text{(6.3.1)} \qquad \gamma = \sup_{\delta>0}\ [P_\delta\ \{Z_1 \geq u_\alpha\} - P_\delta\ \{Z_2 \geq \chi^2_{r;\alpha}\}]\ ,$$

where $L_\delta(Z_1) = N_1(\delta,\ 1)$ *and* $L_\delta(Z_2) = \chi^2_{r,\delta^2}$.

PROOF. Because of Corollary 6.2.2, it is not a restriction to assume that $R_\nu \to R$ for a non-singular diagonal matrix R. Let Ψ_ν be the class of all level α tests for T_ν, and $\tilde{\Phi}$ the class of all asymptotically level α tests.
(i) Let $\{\mu_{0\nu}\} \in S_H$, $\mu_{0\nu} \to \mu_0$. Define

$$\tilde{Y}_\nu = Y_\nu - [n(\nu)]^{\frac{1}{2}}\ f(\mu_{0\nu})$$
$$w(\Lambda,\ y) = \|\ (L_{A\Lambda} - L_{H\Lambda})\ y\ \|_\Lambda\ .$$

Then $w(\Lambda,\ y)$ is continuous in $(\Lambda,\ y)$. It follows from $(L_{A\Lambda} - L_{H\Lambda})\ f(\mu_{0\nu}) = 0$ that

$$T_\nu = w^2(\hat{\Lambda}_\nu,\ \tilde{Y}_\nu)\ .$$

It can be proved as in Theorem 4.4.1 that $L_{\mu_{0\nu}}(\tilde{Y}_\nu) \to N(0,\Lambda_0)$ with $\Lambda_0 = \Lambda(R,\mu_0)$. With $\hat{\Lambda}_\nu \to \Lambda_0$ this implies

(1) $$T_\nu - w^2(\Lambda_0, \tilde{Y}_\nu) \to 0 \qquad \text{in } \{\mu_{0\nu}\}\text{-prob.} ,$$

and hence $L_{\mu_{0\nu}}(T_\nu) \to \chi^2_r$. This establishes that $\{\phi_\nu\} \in \tilde{\Phi}$.

(ii) Let again $\{\mu_{0\nu}\} \in S_H$, $\mu_{0\nu} \to \mu_0$ and define $\tilde{Y}_\nu$ and Λ_0 as in (i), and

$$M_\nu = \{\mu \mid f(\mu) \in V + K,\ L_{H\Lambda_0} f(\mu) = f(\mu_{0\nu}),\ \mu \neq \mu_{0\nu}\}.$$

Then conditions (i) - (iv) of Theorem 4.4.1 are satisfied with

$$M = \{\eta \in V + K \mid L_{H\Lambda_0}\eta = 0\},\ M_A = M \setminus \{0\} .$$

Corollary 4.4.1 can be applied. The limiting problem is

$$L_\eta(Y) = N_m(\eta, \Lambda_0)$$
$$H : \eta = 0, \qquad A : \eta \in M_A .$$

For this testing problem, $(L_{A\Lambda_0} - L_{H\Lambda_0})Y$ is a sufficient statistic; it can be concluded from Example 2.6.1 that the test which rejects for $w^2(\Lambda_0, Y) \geq \chi^2_{r;\alpha}$ is MS - level α and that the minimax shortcoming - level α is given by (6.3.1).

Corollary 4.4.1, Proposition 4.1.1 and (1) together yield the following conclusion: if we define

$$\Phi_\nu = \{\phi \mid E_{\mu_{0\nu}} \phi(Y_\nu) \leq \alpha \}$$
$$\tilde{\gamma}_\nu(\phi,\mu) = \sup_{\psi\in\Phi_\nu} E_\mu \psi(Y_\nu) - E_\mu \phi(Y_\nu)$$
$$K = \{ \{\mu_\nu\} \mid \mu_\nu \in M_\nu,\ \{\mu_\nu\} \mathrel{\diamond} \{\mu_{0\nu}\} \} ,$$

then

(2) $$\gamma = \sup_{\{\mu_\nu\}\in K} \limsup_\nu \tilde{\gamma}_\nu(\phi_\nu,\mu_\nu) ,$$

and for every test $\{\phi'_\nu\} \in \tilde{\Phi}$ we have

(3) $$\gamma \le \sup_{\{\mu_\nu\}\in K} \liminf_\nu \tilde{\gamma}_\nu(\phi'_\nu,\mu_\nu) \ .$$

Lemma 6.3.1 below shows that in these conclusions, $\tilde{\gamma}_\nu$ can be replaced by γ_ν where

$$\gamma_\nu(\phi,\mu) = \sup_{\psi\in\Psi_\nu} E_\mu\psi(Y_\nu) - E_\mu\phi(Y_\nu) \ .$$

(iii) Before (ii) is used to prove that $\{\phi_\nu\}$ is AMS - level α, it will first be demonstrated that $\{\phi_\nu\}$ is sharply consistent. Let $\{\mu_\nu\}$ be a remote sequence and denote $[n(\nu)]^{\frac{1}{2}}f(\mu_\nu)$ by η_ν. Then

$$w(\hat{\Lambda}_\nu,\eta_\nu) \to \infty \qquad \text{in } \{\mu_\nu\}\text{-prob.}$$

$$\{L_{\mu_\nu}(Y_\nu - \eta_\nu)\} \quad \text{is tight.}$$

Hence

$$w(\hat{\Lambda}_\nu, Y_\nu) \ge w(\hat{\Lambda}_\nu,\eta_\nu) - w(\hat{\Lambda}_\nu, Y_\nu - \eta_\nu) \to \infty$$

in $\{\mu_\nu\}$-prob..This implies $E_{\mu_\nu}\phi_\nu \to 1$.

(iv) It will be proved that for every $\{\mu_\nu\} \in S_A$, one has

(4) $$\limsup_\nu \gamma_\nu(\phi_\nu,\mu_\nu) \le \gamma \ .$$

Because of (iii), and passing to a subsequence if necessary, it may be assumed that $\{\mu_\nu\}$ is contiguous to some sequence in S_H, and that $\mu_\nu \to \mu_0 \in \mu(\Theta_H)$. Let $\Lambda_0 = \Lambda(R,\mu_0)$ and $\mu_{0\nu} = f^{-1}(L_{H\Lambda_0} f(\mu_\nu))$. Then $\{\mu_\nu\} \Leftrightarrow \{\mu_{0\nu}\}$ and (ii) can be applied; (4) follows immediately from (2).

(v) It follows from (3) that

$$\gamma \le \inf_{\{\phi'_\nu\}\in\tilde{\Phi}} \ \sup_{\{\mu_\nu\}\in S_A} \liminf_\nu \gamma_\nu(\phi'_\nu,\mu_\nu) \ .$$

With (iv) and Corollary 6.2.2, this establishes that $\{\phi_\nu\}$ is AMS - level α, and that the AMXS - level α is γ.

(vi) Finally, the asymptotic uniqueness of $\{\phi_\nu\}$ will be proved. Let $\{\phi'_\nu\}$ be any other sharply consistent AMS - level α test. For remote sequences $\{\mu_\nu\}$, the sharp consistency implies

$$E_{\mu_\nu} \mid \phi_\nu - \phi'_\nu \mid \to 0.$$

Now let $\{\mu_\nu\} \in S_A$ be contiguous to some sequence in S_H. It may be assumed that $\mu_\nu \to \mu_0$; let Λ_0 and $\{\mu_{0\nu}\}$ be as in (iv), and K as in (ii). Since $\{\phi'_\nu\}$ is AMS - level α and the AMXS is γ, we have

$$\sup_{\{\mu'_\nu\} \in K} \limsup_\nu \gamma_\nu(\phi'_\nu, \mu'_\nu) \leq \gamma .$$

Theorem 4.5.1 implies that

$$E_{\mu_{0\nu}} \mid \phi_\nu - \phi'_\nu \mid \to 0;$$

with Proposition 4.1.1, this yields

$$E_{\mu_\nu} \mid \phi_\nu - \phi'_\nu \mid \to 0 .$$

□

LEMMA 6.3.1. *Suppose that* $R_\nu \to R$, $\{\mu_{0\nu}\} \in S_H$ *and* $\mu_{0\nu} \to \mu_0$; *let* $\Lambda_0 = \Lambda(R, \mu_0)$. *Define*

$$\Psi_\nu = \{\phi \mid E_\mu \phi\ (Y_\nu) \leq \alpha \quad \text{for all } \mu \in \mu\ (\Theta_H)\}$$
$$\Phi_\nu = \{\phi \mid E_{\mu_{0\nu}} \phi\ (Y_\nu) \leq \alpha \ \} .$$

If $\{\mu_\nu\} \Leftrightarrow \{\mu_{0\nu}\}$ *and* $L_{H\Lambda_0} f(\mu_\nu) = f(\mu_{0\nu})$ *for all* ν, *then*

$$\sup_{\phi \in \Phi_\nu} E_{\mu_\nu} \phi - \sup_{\phi \in \Psi_\nu} E_{\mu_\nu} \phi \to 0 .$$

PROOF. As $\Psi_\nu \subset \Phi_\nu$, it is sufficient to show that for the test $\{\phi_{1\nu}\}$ presented in Corollary 4.4.2 as the asymptotically most powerful - level α test for testing " $\mu = \mu_{0\nu}$" against " $\mu = \mu_\nu$ ", there exists a test $\{\psi_\nu\}$ which is asymptotically equivalent to $\{\phi_{1\nu}\}$ and with $\psi_\nu \in \Psi_\nu$ for every ν. Part (ii) of the proof of Theorem 5.6.1 yields an asymptotically level α test $\{\phi_\nu\}$ which is asymptotically equivalent to $\{\phi_{1\nu}\}$. It follows from Section 5.4 that there exists a test $\{\psi_\nu\}$ with $\psi_\nu \in \Psi_\nu$, which is asymptotically equivalent to $\{\phi_{1\nu}\}$. □

In many applications of Theorem 6.3.1, f can be taken to be the identity function. If this is not the case (as in the independence problem for a contingency table), the indicated procedure is rather roundabout: it involves f (in Y_ν) and its "local inverse" $D^{-1}_{\hat{\mu}_\nu}$ (in $\hat{\Lambda}^{-1}_\nu$). The following corollary yields a test statistic $\bar{T}_\nu$ which is identical to T_ν if f is the identity function, and which may be easier to compute than T_ν in other cases.

<u>COROLLARY 6.3.1</u>. *Suppose that* $V + K = \mathbb{R}^m$. *Define*

$$\bar{\Sigma}_\nu = R_\nu^{-1} \Sigma_{\hat{\mu}_\nu}$$

$$\bar{T}_\nu = [n(\nu)] \inf_{\mu \in \mu(\Theta_H)} \| X^{(\nu)}_{\cdot} - \mu \|^2_{\bar{\Sigma}_\nu} .$$

The test $\{\bar{\phi}_\nu\}$ *which rejects for* $\bar{T}_\nu > \chi^2_{r;\alpha}$ *is a sharply consistent AMS - level* α *test.*

<u>PROOF</u>. The sharp consistency can be proved as in the proof of Theorem 6.3.1. With Proposition 4.1.1, this implies that it is sufficient to prove that $\bar{T}_\nu - T_\nu \to 0$ in $\{\mu_\nu\}$-prob., for every $\{\mu_\nu\} \in S_H$ with $\mu_\nu \to \mu$ for some $\mu \in \mu(\Theta_H)$. Define

$$Z_\nu(\mu) = [n(\nu)]^{\frac{1}{2}} D^{-1}_{\hat{\mu}_\nu} \{f(X^{(\nu)}_{\cdot}) - f(\mu)\}$$

$$\bar{Z}_\nu(\mu) = [n(\nu)]^{\frac{1}{2}} (X^{(\nu)}_{\cdot} - \mu)$$

and let $\{\bar{\mu}_\nu\}$ be a sequence of statistics with

$$\bar{T}_\nu - \| \bar{Z}_\nu(\bar{\mu}_\nu) \|^2 \to 0 \quad \text{in } \{\mu_\nu\}\text{-prob.} .$$

The tightness of $\{L_{\mu_\nu}(\bar{Z}_\nu(\mu))\}$ implies that $\{L_{\mu_\nu}(\bar{Z}_\nu(\bar{\mu}_\nu))\}$ is tight. With a Taylor expansion for f, and $\hat{\mu}_\nu \to \mu$ in $\{\mu_\nu\}$ - prob., this yields that

$$D_{\hat{\mu}_\nu}(Z_\nu(\bar{\mu}_\nu) - \bar{Z}_\nu(\bar{\mu}_\nu)) =$$
$$= [n(\nu)]^{\frac{1}{2}} \{f(X^{(\nu)}_{\cdot}) - f(\bar{\mu}_\nu) - D_{\hat{\mu}_\nu}(X^{(\nu)}_{\cdot} - \bar{\mu}_\nu)\} \to 0$$

in $\{\mu_\nu\}$ - prob., and hence that $Z_\nu(\bar{\mu}_\nu) - \bar{Z}_\nu(\bar{\mu}_\nu) \to 0$ in $\{\mu_\nu\}$ - prob..

As

$$T_\nu = \inf_{\mu \in \mu(\Theta_H)} \| Z_\nu(\mu) \|^2_{\bar{\Sigma}_\nu} \leq \| Z_\nu(\bar{\mu}_\nu) \|^2_{\bar{\Sigma}_\nu} ,$$

this implies

$$P_{\mu_\nu}\{T_\nu - \bar{T}_\nu > \varepsilon\} \to 0$$

for every $\varepsilon > 0$. It can be proved similarly that

$$P_{\mu_\nu}\{T_\nu - \bar{T}_\nu < \varepsilon\} \to 0$$

for every $\varepsilon > 0$. Hence $\bar{T}_\nu - T_\nu \to 0$ in $\{\mu_\nu\}$ - prob. . □

If $X_1^{(\nu)}, \ldots, X_\nu^{(\nu)}$ is a sample from the multinomial $M(1;\mu)$ distribution, then

$$\| X_{\cdot}^{(\nu)} - \mu \|^2_{\bar{\Sigma}_\nu} = \sum_j (X_{j\cdot}^{(\nu)} - \mu_j)^2 / \hat{\mu}_j .$$

So the familiar χ^2 tests for the testing problems of homogeneity and of independence for contingency tables are instances of the AMS tests provided by the theorem of this section and its corollary. Are these tests also asymptotically most stringent in the sense of Wald's definition? In the terms of Section 6.1, this can be expressed as the question whether the χ^2 test is AMS - $(\tilde{\Phi},\tilde{\beta}^*)$ against S_0, where $\tilde{\Phi}$ is the class of all asymptotically level α tests in the sense of (5.2.3), β_ν^* the envelope power function with respect to the class of all level α tests for T_ν, and S_0 the class of all sequences in $\mu(\Theta_A)$. Let γ_0 be the AMXS - $(\tilde{\Phi},\tilde{\beta}^*)$ against S_0.

It follows from $S_A \subset S_0$ that $\gamma_0 \geq \gamma$, where γ is as in Theorem 6.3.1. According to Proposition 6.2.1, the test $\{\phi_\nu'\}$ is AMS - $(\tilde{\Phi},\tilde{\beta}^*)$ against S_0 iff $\{\phi_\nu'\} \in \tilde{\Phi}$ and

$$\limsup_\nu \gamma_\nu(\phi_\nu',\mu_\nu) \leq \gamma \qquad \text{for all } \{\mu_\nu\} \in S_0 .$$

The χ^2 test $\{\phi_\nu\}$ satisfies

$$\limsup_\nu \gamma_\nu(\phi_\nu,\mu_\nu) \leq \gamma \leq \gamma_0 \qquad \text{for all } \{\mu_\nu\} \in S_A ;$$

so $\{\phi_\nu\}$ is AMS - $(\tilde{\Phi},\tilde{\beta}^*)$ against S_0 iff $\{\phi_\nu\} \in \tilde{\Phi}$ and

$$\limsup_\nu \gamma_\nu(\phi_\nu,\mu_\nu) \leq \gamma_0 \quad \text{for all } \{\mu_\nu\} \in S_0 \smallsetminus S_A \ .$$

It is unknown whether the χ^2 test satisfies this condition. If $\gamma_0 = \gamma$, then the AMXS - $(\tilde{\Phi},\tilde{\beta}^*)$ against S_0 is determined by the class S_A of interior sequences, and every test which is AMS - $(\tilde{\Phi},\tilde{\beta}^*)$ against S_0 must be asymptotically equivalent to the χ^2 test. (Recall that asymptotic equivalence was defined in Definition 5.3.2 with respect to only the class S_A of interior sequences in Θ_A.) If $\gamma_0 > \gamma$, then the AMXS - $(\tilde{\Phi},\tilde{\beta}^*)$ against S_0 is determined by $S_0 \smallsetminus S_A$, and there will exist AMS - $(\tilde{\Phi},\tilde{\beta}^*)$ tests against S_0 which are not asymptotically equivalent to the χ^2 test. So if $\gamma_0 > \gamma$, the optimum property "AMS - $(\tilde{\Phi},\tilde{\beta}^*)$ against S_0" is not a very relevant optimum property. A closely related phenomenon is studied in Chapter 7.

6.4. AMS TESTS FOR CERTAIN TESTING PROBLEMS WITH RESTRICTED ALTERNATIVES FOR EXPONENTIAL FAMILIES

The asymptotic testing problem of Section 3.5 is considered again. In this section, testing problems with restricted alternatives are studied, for which the sharply consistent AMS - level α test is asymptotically unique.

The latter qualification is not superfluous, because for most testing problems for exponential families with restricted alternatives there are many sharply consistent AMS - level α tests, which are not asymptotically equivalent to each other. This phenomenon will be studied in the next chapter. In the proof of Theorem 6.3.1, limiting problems T^*_Λ of the form

$$L_\eta(Y) = N_m(\eta,\Lambda)$$
$$H : \eta = 0, \quad H \vee A : \eta \in V + K,\ L_{H\Lambda}\eta = 0$$

play a role, where $\Lambda = \Lambda(R,\mu)$ for $\mu \in \mu\ (\Theta_H)$. It is essential in the proof of the asymptotic uniqueness of the sharply consistent AMS - level α test, that the minimax shortcoming for $T^*_{\Lambda(R,\mu)}$ be independent of μ. The assumptions made below are sufficient, but not necessary, conditions for the minimax shortcoming of the limiting problem to be independent of $\mu \in \mu\ (\Theta_H)$. The theorem could be stated in greater generality, but the present form seems to suffice for practical purposes. For the assumptions made, compare the remark following Theorem 5.6.1.

<u>THEOREM 6.4.1</u>. *Suppose that there exists a continuous function* $\sigma^2 : \mu(\Theta_H) \to (0,\infty)$ *such that for all* $\mu \in \mu(\Theta_H)$ *and for all proportion matrices* R, *one has*

$$(6.4.1)\qquad \begin{aligned} &\Lambda(R,\mu)\ R\ V = V \\ &\Lambda(R,\mu)\ R\ x = \sigma^2(\mu)\ x \qquad \text{for all } x \in K. \end{aligned}$$

Let L_ν *be the projection on* V *with respect to the inner product* $x'R_\nu y$, *and let* $\phi_{0\nu}$ *be the MS - level* α *test with convex acceptance region for the testing problem*

$$L_\eta(Y) = N_m(\eta, R_\nu^{-1})$$
$$H : \eta = 0, \quad H \vee A : \eta \in (I - L_\nu)K .$$

Let $\{\hat{\mu}_\nu\}$ *be a uniformly consistent estimator for* μ *under* H *and define*

$$\phi_\nu(Y_\nu) = \phi_{0\nu}(\sigma^{-1}(\hat{\mu}_\nu)Y_\nu) .$$

The test $\{\phi_\nu\}$ *is AMS - level* α *and sharply consistent. Every sharply consistent AMS - level* α *test is strongly asymptotically equivalent to* $\{\phi_\nu\}$.

<u>PROOF</u>. It may be assumed that $R_\nu \to R$. Denote the linear hull of K by W. Then $\Lambda(R,\mu)\ Rx = \sigma^2(\mu)x$ for all $\mu \in \mu(\Theta_H)$, $x \in W$. Let L be the projection on V with respect to the inner product $x'Ry$, and let B_ν [B] be the projection on $(I - L_\nu)W$ [$(I - L)W$] with respect to $x'R_\nu y$ [$x'Ry$]. Some linear algebra shows that for all $\Lambda = \Lambda(R,\mu)$ with $\mu \in \mu(\Theta_H)$, one has

$$(1)\qquad \begin{aligned} &BR^{-1} = R^{-1}B' \quad , \Lambda RL = L\Lambda R , \\ &\Lambda R(I - L)W = (I - L)W,\ \Lambda RB = \sigma^2(\mu)B . \end{aligned}$$

(i) Consider the testing problems

$$T_0^* = (\mathbb{R}^m, \{N(0,R^{-1})\}, \{N(\eta,R^{-1}) \mid \eta \neq 0,\ \eta \in (I-L)K\})$$
$$T_1^* = (\mathbb{R}^m, \{N(0,\Lambda)\}, \{N(\eta,\Lambda) \mid \eta \neq 0,\ \eta \in (I - L)K\})$$

where $\Lambda = \Lambda(R, \mu_0)$ for some $\mu_0 \in \mu(\Theta_H)$; let ϕ_0^* and ϕ_1^* be the MS - level α tests for T_0^* and T_1^*, respectively. It will be demonstrated that

(2) $\qquad \phi_1^*(y) = \phi_0^*(\sigma^{-1}(\mu)y) \qquad$ a.e. .

If $\eta \in (I - L)K$ then $R^{-1}\Lambda^{-1}\eta \in (I - L)W$ by (1), implying that

$$\eta'\Lambda^{-1}y = (R^{-1}\Lambda^{-1}\eta)'Ry = (R^{-1}\Lambda^{-1}\eta)'RBy = \eta'\Lambda^{-1}By$$

$$\eta'Ry = \eta'RBy$$

for all $y \in \mathbb{R}^m$. Hence BY is a sufficient statistic for T_1^* and T_0^*. The covariance matrix of BY in T_0^* is $BR^{-1}B'$; in T_1^* it is (use (1))

$$B\Lambda B' = B\Lambda RR^{-1}B' = B\Lambda RBR^{-1} = \sigma^2(\mu)\,B^2R^{-1} = \sigma^2(\mu)BR^{-1}B' \ .$$

With Corollary 2.6.1, this implies (2) .

(ii) Let $\{\mu_{0\nu}\} \in S_H$, $\mu_{0\nu} \to \mu_0$; define

$$\tilde{Y}_\nu = Y_\nu - [n(\nu)]^{\frac{1}{2}} f(\mu_{0\nu}) \ .$$

$B_\nu Y$ is a sufficient statistic for the testing problem mentioned in the statement of the theorem, and the definitions of L_ν and B_ν imply that $B_\nu f(\mu_{0\nu}) = 0$. Hence

$$\phi_\nu(Y_\nu) = \phi_{0\nu}(\sigma^{-1}(\hat{\mu}_\nu)Y_\nu) = \phi_{0\nu}(\sigma^{-1}(\hat{\mu}_\nu)\tilde{Y}_\nu) \ .$$

Lemma 8.1.1 implies that $\phi_{0\nu} \overset{*}{\to} \phi_0^*$. It follows from (i), the uniform consistency of $\{\hat{\mu}_\nu\}$ and Lemma's A.4.5, 6 that

$$\phi_\nu(Y_\nu) - \phi_1^*(\tilde{Y}_\nu) \to 0 \qquad \text{in } \{\mu_{0\nu}\}\text{-prob. .}$$

(iii) Let again $\{\mu_{0\nu}\} \in S_H$, $\mu_{0\nu} \to \mu_0$ and define $\tilde{Y}_\nu$ as in (ii), $\Lambda = \Lambda(R,\mu_0)$ and

$$M_\nu = \{\mu \mid f(\mu) \in V + K,\ L_\nu f(\mu) = f(\mu_{0\nu}),\ \mu \neq \mu_{0\nu}\}.$$

Then conditions (i) - (iv) of Theorem 4.4.1 are satisfied with

$$M = \{\eta \in V + K \mid L\eta = 0\} = (I - L)K \ , \ M_A = M \setminus \{0\} \ .$$

Hence $L_{\mu_{0\nu}}(\tilde{Y}_\nu) \to N(0,\Lambda)$. With (ii), it can be concluded that $\{\phi_\nu\}$ is asymptotically of level α. Furthermore, Corollary 4.4.1 can be applied. The limiting problem is T_1^*. Denote the minimax shortcoming for T_1^* by γ.

For the remainder of the proof, see the second half of part (ii), and parts (iv), (v), (vi), of the proof of Theorem 6.3.1. Only the sharp consistency of $\{\phi_\nu\}$ remains to be proved.

(iv) Let $\{\mu_\nu\}$ be a remote sequence. Define

$$x_\nu = [n(\nu)]^{\frac{1}{2}} (I - L_\nu) f(\mu_\nu) .$$

Then $\| x_\nu \| \to \infty$; it is not a restriction to assume that $x_\nu / \| x_\nu \| \to x \in (I-L)K$. It can be concluded from Proposition 2.8.2 and the fact that $\gamma < 1 - \alpha$, that $x \notin 0^+$ (acc ϕ_1^*). Define

$$\tilde{Y}_\nu = Y_\nu - [n(\nu)]^{\frac{1}{2}} f(\mu_\nu);$$

it can be proved as in (ii) that

$$\phi_\nu(Y_\nu) = \phi_{0\nu}(\sigma^{-1}(\hat{\mu}_\nu)\tilde{Y}_\nu + \sigma^{-1}(\hat{\mu}_\nu)x_\nu).$$

Corollary A.4.3 can be applied with $X_\nu = \tilde{Y}_\nu$, $T_\nu = \sigma^{-1}(\hat{\mu}_\nu)$ and $\phi = \phi_1^*$. With (i) and the uniform consistency of $\{\hat{\mu}_\nu\}$, it yields that $E_{\mu_\nu} \phi_\nu \to 1$. □

EXAMPLE 6.4.1. *Testing homogeneity against an upward trend.* The following k-sample problems are instances of the testing problem of this section. The index ν is deleted.

(i) For $1 \leq i \leq k$,

$$X_{i1}, X_{12}, \ldots, X_{in_i}$$

are independent random samples from Bernoulli distributions with success probabilities μ_i. Null hypothesis and alternative hypothesis are

$$H : \mu_1 = \mu_2 = \ldots = \mu_k \qquad H \vee A : \mu_1 \leq \mu_2 \leq \ldots \leq \mu_k.$$

This testing problem is mentioned in Section 3.4. Take f = the identity function, $m = k$, $\mu(\theta) = (0,1)^k$,

$$V = \{x \in \mathbb{R}^k \mid x_1 = \dots = x_k\}, \quad K = \{x \in \mathbb{R}^k \mid x_1 \leq \dots \leq x_k\}.$$

For $\mu = (p, \dots, p) \in \mu(\Theta_H)$ we have $\Lambda(R,\mu) = p(1 - p)R^{-1}$. Assumptions (6.4.1) are satisfied with $\sigma^2(p, \dots, p) = p(1 - p)$.

(ii) For $1 \leq i \leq k$,

$$X_{i1}, \dots, X_{in_i}$$

are independent random samples from Poisson distributions with parameters μ_i. Null hypothesis and alternative hypothesis are

$$H : \mu_1 = \mu_2 = \dots = \mu_k \quad , \quad H \vee A : \mu_1 \leq \mu_2 \leq \dots \leq \mu_k \ .$$

Take f = the identity function, $m = k$, $\mu(\Theta) = (0, \infty)^k$, V and K as in (i). For $\mu = (p, \dots, p) \in \mu(\Theta_H)$ we have $\Lambda(R, \mu) = pR^{-1}$. Assumptions (6.4.1) are satisfied with $\sigma^2(p, \dots, p) = p$.

(iii) For $1 \leq i \leq k$,

$$Z_{i1}, \dots, Z_{in_i}$$

are independent random samples from normal distributions with means η_i and common (unknown) variance τ^2. Null hypothesis and alternative hypothesis are

$$H : \eta_1 = \eta_2 = \dots = \eta_k \quad , \quad H \vee A : \eta_1 \leq \eta_2 \leq \dots \leq \eta_k \ .$$

To obtain random variables with a distribution in a canonical exponential family, let

$$X_{ij} = (Z_{ij}, Z_{ij}^2)' \ .$$

Then $m = 2k$,

$$\mu_i = E_{(\eta_i,\tau_i^2)} X_{ij} = (\eta_i, \eta_i^2 + \tau_i^2)' \ ,$$

$$\mu(\Theta) = \{x \in \mathbb{R}^m \mid x_{2h} > x_{2h-1}^2 \text{ for } 1 \leq h \leq k\} \ .$$

Take $f_0(x_1, x_2)' = (x_1, x_2 - x_1^2)'$ and

$$f(\mu) = \begin{pmatrix} f_0(\mu_1) \\ \cdot \\ \cdot \\ \cdot \\ f_0(\mu_k) \end{pmatrix}$$

$$V = \{x \in \mathbb{R}^m \mid x_h = x_{h-2} \text{ for } 3 \leq h \leq m\}$$

$$K = \{x \in \mathbb{R}^m \mid x_1 \leq x_3 \leq \ldots \leq x_{m-1} , x_2 = x_4 = \ldots = x_m = 0\} .$$

For $\mu = (\eta, \eta^2 + \tau^2, \eta, \ldots, \eta^2 + \tau^2) \in \mu(\Theta_H)$ we have

$$\Lambda(R,\mu) = R^{-1} \begin{pmatrix} \tau^2 & & & & 0 \\ & 2\tau^4 & & & \\ & & \tau 2 & & \\ & & & \ddots & \\ 0 & & & & 2\tau^4 \end{pmatrix}$$

Assumptions (6.4.1) are satisfied with $\sigma^2(\eta, \eta^2 + \tau^2, \eta, \ldots, \eta^2 + \tau^2) = \tau^2$.
(iv) For $1 \leq i \leq k$,

$$Z_{i1}, \ldots, Z_{in_i}$$

are independent random samples from normal distributions with means η_i and variances τ_i^2 . Null hypothesis and alternative hypothesis are

$$H : \tau_1^2 = \ldots = \tau_k^2$$

$$H \vee A : \tau_1^2 \leq \ldots \leq \tau_k^2 \quad .$$

Let m, X_{ij}, μ and f be as in (iii). Take

$$V = \{x \in \mathbb{R}^m \mid x_2 = x_4 = .. = x_m\}$$

$$K = \{x \in \mathbb{R}^m \mid x_1 = x_3 = .. = x_{m-1} = 0, \ x_2 \leq x_4 \leq \ldots \leq x_m\} .$$

For $\mu \in \mu(\Theta_H)$, $\Lambda(R,\mu)$ is as in (iii). Assumptions (6.4.1) are satisfied with $\sigma^2(\eta_1, \eta_1^2 + \tau^2, \eta_2, \ldots, \eta_k^2 + \tau^2) = 2\tau^4$.

These four examples (as well as the related testing problems mentioned in Section 4.3 of BARLOW et al. (1972)) have the following form (for (iii) and (iv) the coordinates must be rearranged):

$$V = \{x \in \mathbb{R}^m \mid x_1 = \dots = x_k\}$$
$$K = \{x \in \mathbb{R}^m \mid x_1 \leq \dots \leq x_k,\ x_{k+1} = \dots = x_m = 0\},$$

with $k \leq m$. The projection L_ν is given by $L_\nu x = y$, where

$$y_1 = \dots = y_k = \sum_{i=1}^{k} \rho_i(\nu)\, x_i$$
$$y_h = x_h \qquad k+1 \leq h \leq m\ ,$$

and $L_\nu X^{(\nu)}_{\cdot}$ is the UMVU estimator for μ under H (see the remark following Definition 5.6.1), and uniformly consistent for μ under H.

For $k = 2$, AUMP - level α tests exist for these testing problems; see Section 5.6. Now consider the case $k = 3$. The AMS test involves the MS test for the testing problem

$$\mathcal{L}_\eta(Y) = \mathcal{N}_3(\eta, R^{-1})$$
$$H : \eta = 0, \qquad H \vee A : \eta \in K_R$$

where

$$R = \begin{pmatrix} \rho_1 & & 0 \\ & \rho_2 & \\ 0 & & \rho_3 \end{pmatrix}$$

$$K_R = \{x \in \mathbb{R}^3 \mid x_1 \leq x_2 \leq x_3,\ \sum_i \rho_i x_i = 0\}$$

and $\rho_i > 0$, $\rho_1 + \rho_2 + \rho_3 = 1$. A sufficient statistic is $(Y_2 - Y_1, Y_3 - Y_2)$. A more convenient form of the sufficient statistic is (Z_1, Z_2) with

(6.4.2)
$$Z_1 = \{\rho_2 / 2(\rho_2 + \rho_1\rho_3 + \kappa_1\kappa_3)\}^{\frac{1}{2}} \{\kappa_1(Y_2 - Y_1) - \kappa_3(Y_3 - Y_2)\}$$
$$Z_2 = \{\rho_2 / 2(\rho_2 + \rho_1\rho_3 - \kappa_1\kappa_3)\}^{\frac{1}{2}} \{\kappa_1(Y_2 - Y_1) + \kappa_3(Y_3 - Y_2)\}$$

where

(6.4.3)
$$\kappa_i = \{\rho_i(1 - \rho_i)\}^{\frac{1}{2}} \qquad i = 1, 3\ .$$

In terms of (Z_1, Z_2) the testing problem is

$$L_\zeta(Z) = N_2(\zeta, I)$$

$$H : \zeta = 0, \qquad H \vee A : d\,|\zeta_1| \leq \zeta_2$$

where

$$d = \{(\rho_2 + \rho_1\rho_3 + \kappa_1\kappa_3) / (\rho_2 + \rho_1\rho_3 - \kappa_1\kappa_3)\}^{\frac{1}{2}} =$$

$$= \{(1 - \kappa_0) / (1 - \kappa_0)\}^{\frac{1}{2}} \quad ,$$

(6.4.4) $$\kappa_0 = \{\rho_1\rho_3 / (\rho_2 + \rho_1\rho_3)\}^{\frac{1}{2}} \quad .$$

The angle ω between the lines $\zeta_2 = 0$ and $\zeta_2 = d\zeta_1$ has $\operatorname{cotg}\omega = d$ and hence

$$\sin\omega = \{\tfrac{1}{2}(1 - \kappa_0)\}^{\frac{1}{2}} \quad , \cos\omega = \{\tfrac{1}{2}(1 + \kappa_0)\}^{\frac{1}{2}} \; .$$

Since $0 < \kappa_0 < 1$, w satisfies $0 < \omega < \frac{1}{4}\pi$. The MS - level α test for this problem is, for certain values of α and ω, given in part 2 of Section 3.3.

Theorem 6.4.1 now yields the following conclusion. The test which rejects for

$$Z_2 \geq 2^{\frac{1}{2}}(1 + \kappa_0)^{-\frac{1}{2}}\, b^{-1} \, *$$

$$* [\, c - \log\{\exp(2^{-\frac{1}{2}}(1 - \kappa_0)^{\frac{1}{2}}\, b\, Z_1) + \exp\,(-2^{-\frac{1}{2}}(1 - \kappa_0)^{\frac{1}{2}}\, b\, Z_1)] \quad ,$$

where b and c are determined as in Section 3.3 part 2, where κ_0 is given by (6.4.4), (Z_1, Z_2) by (6.4.2,3) and (Y_1, Y_2, Y_3) for the four testing problems, respectively, by

(i) $$Y_i = n^{\frac{1}{2}}\{\hat{p}(1 - \hat{p})\}^{-\frac{1}{2}}\, X_{i.}$$

$$\hat{p} = \sum_{i=1}^{3} \rho_i\, X_{i.}$$

(ii) $$Y_i = n^{\frac{1}{2}}\, \hat{p}^{-\frac{1}{2}}\, X_{i.}$$

$$\hat{p} = \sum_{i=1}^{3} \rho_i\, X_{i.}$$

(iii) $Y_i = n^{\frac{1}{2}}\,\hat{\tau}^{-1}\,Z_{i.}$

$$\hat{\tau}^2 = n^{-1}\sum_{i=1}^{3}\sum_{j=1}^{n_i}(Z_{ij} - Z_{i.})^2$$

(iv) $Y_i = 2^{-\frac{1}{2}}\,n^{\frac{1}{2}}\,\hat{\tau}^{-2}\,n_i^{-1}\sum_{j=1}^{n_i(\nu)}(Z_{ij} - Z_{i.})^2$

$$\hat{\tau}^2 = n^{-1}\sum_{i=1}^{3}\sum_{j=1}^{n_i}(Z_{ij} - Z_{i.})^2$$

and where $n = n_1 + n_2 + n_3$ and $\rho_i = n_i / n$, is AMS - level α.

The examples (i), (ii) and (iv) can also be approached by means of variance - stabilizing transformations (see e.g. RAO (1973) Section 6g). In terms of Theorem 6.4.1: the function f can be chosen in such a way that $\sigma^2(\mu) = 1$ for all $\mu \in \mu(\Theta_H)$. This leads to the following alternative expressions for (Y_1, Y_2, Y_3) :

(i) $Y_i = 2n^{\frac{1}{2}}\arcsin X_{i.}^{\frac{1}{2}}$

(ii) $Y_i = 2n^{\frac{1}{2}}\,X_{i.}^{\frac{1}{2}}$

(iv) $Y_i = 2^{-\frac{1}{2}}\,n^{\frac{1}{2}}\log\{n_i^{-1}\sum_{j=1}^{n_i}(Z_{ij} - Z_{i.})^2\}.$

Using these expressions for Y_i in the procedures indicated above leads also to an AMS - level α test.

The discussion in part 2 of Section 3.3 implies that the test which rejects for $Z_2 \geq u_\alpha$ is also a good test for the limiting problem. The corresponding test for the testing problem (i) is presented in Section 9.2 as the AMS - asymptotically linear test.

For $k \geq 4$, the AMS - level α test for the limiting problem is unknown. The AMS - asymptotically linear test for testing problem (i), for general k, can be found in Section 9.2. □

CHAPTER 7

TOWARDS AN ASYMPTOTIC OPTIMUM PROPERTY WHICH IS STRONGER THAN "ASYMPTOTICALLY MOST STRINGENT"

For many testing problems with restricted alternatives, the asymptotically most stringent test is not asymptotically unique. An example is provided in Section 7.1. A stronger optimum property is necessary in order to obtain an asymptotically unique optimal test. This chapter is devoted to the introduction of the new optimum property "everywhere asymptotically most stringent", which is proposed as the proper asymptotic analogue of the property "most stringent" for testing problems where the asymptotically most stringent test is not asymptotically unique.

The approach in this chapter is complementary to the (less rigourous) description of the property "everywhere asymptotically most stringent" in Section 1.3. In Sections 7.2 and 7.3, the possibility is investigated of partitioning the asymptotic testing problem into subproblems, and defining a test to be optimal if it is asymptotically most stringent for every subproblem. It appears in Sections 7.3 and 7.4, that the equivalence relation of mutual contiguity can be used for an attractive partition into subproblems. This leads to the definition of "everywhere asymptotically most stringent".

7.1. AMS TESTS AGAINST RESTRICTED ALTERNATIVES ARE OFTEN NOT ASYMPTOTICALLY UNIQUE

In the proofs of Theorems 6.3.1 and 6.4.1, a central role is played by the limiting testing problems

$$T^*_\Lambda = (\mathbb{R}^m, \{N(0,\Lambda)\}, \{N(\eta,\Lambda) \mid \eta \in M_A\})$$

for $\Lambda = \Lambda(R,\mu)$, $\mu \in \mu(\Theta_H)$. The relation between these limiting problems and the asymptotic testing problem is established by Theorem 4.4.1. The proof of the asymptotic uniqueness of the AMS test in Theorems 6.3.1 and 6.4.1 is based on the property that the minimax shortcoming for $T^*_{\Lambda(R,\mu)}$ is the same for all $\mu \in \mu(\Theta_H)$. In many cases, the minimax shortcoming for $T^*_{\Lambda(R,\mu)}$ does depend on μ; this implies that there exist many AMS tests, which are not mutually asymptotically equivalent. This is demonstrated in the following example. The index ν is deleted.

Consider the testing problem for two independent random samples from probability distributions with three possible outcomes, where the null hypothesis of homogeneity is tested against the alternative hypothesis of an ordering with increasing likelihood ratio. After a reduction by sufficiency, one obtains the testing problem where (N_{11}, N_{12}, N_{13}) and (N_{21}, N_{22}, N_{23}) are independent, with

$$L_p(N_{i1}, N_{i2}, N_{i3}) = M(n_i, (p_{i1}, p_{i2}, p_{i3})) \qquad i = 1,2$$

where $p = (p_{11}, p_{12}, p_{13}, p_{21}, p_{22}, p_{23})$, $p_{ih} > 0$ and $\Sigma_h p_{ih} = 1$, and where null hypothesis and alternative hypothesis are given by

$$H : (p_{11}, p_{12}, p_{13}) = (p_{21}, p_{22}, p_{23})$$

$$H \vee A : p_{11} / p_{21} \leq p_{12} / p_{22} \leq p_{13} / p_{23} .$$

In accordance with the notation introduced in Section 3.5 let $n = n_1 + n_2$, $\rho_i = n_i / n$, $X_{i.h} = N_{ih} / n_i$. Note that $(N_{11}, N_{13}, N_{21}, N_{23})$ is a sufficient statistic and let

$$\mu = E_p(X_{1.1}, X_{1.3}, X_{2.1}, X_{2.3})' = (p_{11}, p_{13}, p_{21}, p_{23})' .$$

As $p_{i2} = 1 - p_{i1} - p_{i3}$, an alternative expression for $H \vee A$ is

$$H \vee A : p_{11}(1 - p_{23}) \leq p_{21}(1 - p_{13}), \; p_{23}(1 - p_{11}) \leq p_{13}(1 - p_{21}).$$

This testing problem is of the form of Section 3.1 with

$$(7.1.1) \qquad f\begin{pmatrix} p_{11} \\ p_{13} \\ p_{21} \\ p_{23} \end{pmatrix} = \begin{pmatrix} p_{21}(1-p_{13}) - p_{11}(1-p_{23}) \\ p_{13}(1-p_{21}) - p_{23}(1-p_{11}) \\ \rho_1 p_{11} + \rho_2 p_{21} \\ \rho_1 p_{13} + \rho_2 p_{23} \end{pmatrix},$$

$$V = \{x \in \mathbb{R}^4 \mid x_1 = x_2 = 0\},$$
$$K = \{x \in \mathbb{R}^4 \mid x_1 \geq 0,\ x_2 \geq 0,\ x_3 = x_4 = 0\}.$$

(In contravention of Section 3.5, the function f depends on the deleted index ν through $\rho_i = \rho_i(\nu)$. This has no consequences for the application of the theory of Chapter 4; the present choice of f leads to a simple form of $\Lambda(R,\mu)$ below.)

As it was done in the proofs of Theorems 6.3.1 and 6.4.1, we can use Theorem 4.4.1 to relate limiting testing problems

$$T^*_\Lambda = (\mathbb{R}^4, \{N(0,\Lambda)\}, \{N(\eta,\Lambda) \mid \eta \neq 0,\ \eta_1 \geq 0,\ \eta_2 \geq 0,\ \eta_3 = \eta_4 = 0\})$$

to this (asymptotic) testing problem; $\Lambda = \Lambda(R,\mu)$ is the asymptotic covariance matrix of

$$(7.1.2) \qquad Y = n^{\frac{1}{2}} f(X_{1.1}, X_{1.3}, X_{2.1}, X_{2.3})'.$$

An expression for $\Lambda(R,\mu)$ is given in Section 3.5. Some computations show that for $\mu = (p_1, p_3, p_1, p_3)$ satisfying the null hypothesis,

$$\Lambda = \begin{pmatrix} \Sigma(p_1,p_3) & & 0 & \\ & & p_1(1-p_1) & -p_1p_3 \\ 0 & & -p_1p_3 & p_3(1-p_3) \end{pmatrix}$$

where

$$\Sigma(p_1,p_3) = (\rho_1^{-1} + \rho_2^{-1})\, p_2 \begin{pmatrix} p_1(1-p_3) & -p_1p_3 \\ -p_1p_3 & p_3(1-p_1) \end{pmatrix}.$$

If the random variable for T^*_Λ is denoted by (Y_1, Y_2, Y_3, Y_4), then (Y_1, Y_2) is a sufficient statistic. For fixed (p_1, p_3), it is convenient to apply a transformation to $(Z_1(p_1, p_3), Z_2(p_1, p_3))$ where

$$Z_1(p_1,p_3) = \{2(1+\kappa(p_1,p_3))\}^{-\frac{1}{2}}\{p_1(1-p_3))^{-\frac{1}{2}} Y_1 - (p_3(1-p_1))^{-\frac{1}{2}} Y_2\}$$

(7.1.3) $$Z_2(p_1,p_3) = \{2(1-\kappa(p_1,p_3))\}^{-\frac{1}{2}}\{p_1(1-p_3))^{-\frac{1}{2}} Y_1 + (p_3(1-p_1))^{-\frac{1}{2}} Y_2\}$$

$$\kappa(p_1,p_3) = \{p_1 p_3 \; / \; (1-p_1)(1-p_3)\}^{\frac{1}{2}} .$$

The transformed testing problem is $\mathcal{T}^t_{\omega(p_1,p_3)}$, where

$$\mathcal{T}^t_\omega = (\mathbb{R}^2 , \{N_2(0,I)\}, \{N(\eta,I) \mid \eta_2 > 0, \; |\eta_1| \leq \eta_2 \text{ tg } \omega\})$$

$$\text{tg } \omega(p_1,p_3) = \{(1-\kappa(p_1,p_3)) \; / \; (1+\kappa(p_1,p_3))\}^{\frac{1}{2}}$$

(7.1.4) $$\sin \omega(p_1,p_3) = \{\tfrac{1}{2}(1-\kappa(p_1,p_3))\}^{\frac{1}{2}}$$

$$\cos \omega(p_1,p_3) = \{\tfrac{1}{2}(1+\kappa(p_1,p_3))\}^{\frac{1}{2}} .$$

The minimax shortcoming - level α for $\mathcal{T}^t_\omega$ is denoted by $\gamma(\omega)$, and the MS - level α test by ϕ_ω (deleting the index α). It follows from SCHAAFSMA (1968) that $\gamma(\omega)$ is a strictly increasing function for $0 < \omega < \frac{1}{4}\pi$ and certain (presumably all) values of α, including $\alpha = .05$.
As in part (ii) of the proof of Theorem 6.3.1, it can be concluded from Corollary 4.4.1 that for every asymptotically level α test for the original testing problem, the "asymptotic maximum shortcoming" is at least

$$\sup_{p_1,p_3} \gamma(\omega(p_1,p_3)) = \gamma(\tfrac{1}{4}\pi) .$$

The asymptotic minimax shortcoming - level α is indeed $\gamma(\frac{1}{4}\pi)$; one can prove that it is attained by the test

(7.1.5) $$\phi_{\frac{1}{4}\pi} \; (Z_1(\hat{p}_1,\hat{p}_3), \; Z_2(\hat{p}_1,\hat{p}_3))$$

where

(7.1.6) $$\hat{p}_h = \rho_1 \, X_{1.h} + \rho_2 \, X_{2.h} = (N_{1h} + N_{2h}) \; / \; n \qquad h = 1, 3 ,$$

and where Z_1 and Z_2 are given by (7.1.1,2,3). So this test is AMS - level

α. But (using the index ν for a moment) for an asymptotically level α test $\{\phi_\nu\}$ and a $\{\mu_\nu\} \in S_A$ with $\mu_\nu \to (p_1, p_3, p_1, p_3)$, the requirement

$$\limsup_\nu \gamma_\nu(\phi_\nu, \mu_\nu) \leq \gamma(\tfrac{1}{4}\pi)$$

is a sharp requirement only for $\omega(p_1, p_3)$ close to $\frac{1}{4}\pi$. This implies that one has a lot of freedom in the choice of an AMS -level α test. E.g., it can be proved that if $\{\psi_\omega \mid 0 < \omega < \frac{1}{4}\pi\}$ is a family of test functions on $\mathbb{R}^2$ with convex acceptance regions, with the properties that $\omega \mapsto \psi_\omega$ is a weakly* continuous function while for every ω, ψ_ω is of level α for $\mathcal{T}^t_\omega$ and its maximum shortcoming - level α for $\mathcal{T}^t_\omega$ does not exceed $\gamma(\frac{1}{4}\pi)$, then

$$\psi_{\omega(\hat{p}_1,\hat{p}_3)}\ (Z_1(\hat{p}_1,\hat{p}_3),\ Z_2(\hat{p}_1,\hat{p}_3))$$

is an AMS - level α test. The test (7.1.5) is of this form, with $\psi_\omega = \phi_{\frac{1}{4}\pi}$; one can also take $\psi_\omega = \phi_\omega$, etcetera.
Other AMS - level α tests exist, which do not depend continuously on $(\hat{p}_1, \hat{p}_3)$. E.g., let ψ be any level α test for $\mathcal{T}^t_\omega$ which is a.e. continuous, let (p_1, p_3) satisfy $p_1 > 0$, $p_3 > 0$, $p_1 + p_3 < 1$ and define

$$\psi_t = \begin{cases} \phi_{\frac{1}{4}\pi}(Z_1, Z_2) & (\hat{p}_1 - p_1)^2 + (\hat{p}_3 - p_3)^2 \geq t/n \\ \psi(Z_1, Z_2) & (\hat{p}_1 - p_1)^2 + (\hat{p}_3 - p_3)^2 < t/n \end{cases}$$

where $Z_h = Z_h(\hat{p}_1, \hat{p}_3)$; there exists a $t_0 > 0$ such that ψ_t is AMS - level α for $0 \leq t \leq t_0$. One can continue at will, and propose many strange AMS - level α tests.

7.2. A TENTATIVE OPTIMUM PROPERTY: EVERYWHERE LOCALLY AMS WITH RESPECT TO A PARTITION OF S_A

For many testing problems with restricted alternatives, the AMS - level α test is not asymptotically unique. A typical example is exhibited in the preceding section. As we desire that an asymptotic optimum property recommends a unique test (up to asymptotic equivalence) as "the optimal test", this means that the property "AMS" is unsatisfactory for many testing problems with restricted alternatives. This section starts the construction of an optimum property which selects one "optimal" test from the class of all AMS tests. The testing problem of Section 5.1 is considered again.

The construction is based on the following idea. For testing problems where the AMS test is not asymptotically unique, the asymptotic minimax shortcoming is determined by a "small part" of the parameter space (in the testing problem of Section 7.1: by those parameter values $(p_{11}, p_{13}, p_{21}, p_{23})$ satisfying the alternative hypothesis, for which $p_{i1}\, p_{i3}$ is small $(i = 1,2)$ while of course $p_{1h} - p_{2h}$ is of the order of magnitude of $n^{-\frac{1}{2}}(h = 1,3)$). The property "AMS" is a strong requirement on the asymptotic power function of a test only near that "small part". In other "parts" of the parameter space, a "local" asymptotic minimax shortcoming might be defined, being not larger and in many places smaller than the asymptotic minimax shortcoming for the whole problem. One would like a test to attain the "local" asymptotic minimax shortcoming in all "parts" of the parameter space. But which precise meaning will be given to the words "local" and "part"?

Note that a similar problem was encountered earlier in this study. We saw in Sections 6.3 and 6.4 that an AMS test is not necessarily sharply consistent. The property "AMS" is a strong requirement only for the class of contiguous sequences in S_A. One could say that for the class of remote sequences the "local" asymptotic minimax shortcoming is 0, and that a test attains this "local" asymptotic minimax shortcoming iff it is sharply consistent. For the testing problems treated in Chapter 6, the sharply consistent AMS tests are asymptotically uniformly most powerful in the class of all AMS tests. In the present case where the non-uniqueness is related to the property that the minimax shortcoming for the limiting testing problems varies with μ, for $\mu \in \mu\ (\Theta_H)$, an AUMP - AMS test does not exist. The new optimum property will have to provide a compromise between different AMS tests.

In accordance with Section 5.3, the vague notion "part of the parameter space" used above, will be given the meaning "subset of the class S_A of interior sequences". Any equivalence relation $\sim$ on S_A can be considered as a device to partition the asymptotic testing problem into subproblems, each subproblem corresponding to a $\sim$ - equivalence class of S_A. A test is "everywhere locally asymptotically most stringent" if it is AMS against every $\sim$ - equivalence class S_0. This approach yields a great flexibility, because any equivalence relation may be considered.

<u>DEFINITION 7.2.1.</u> Let $\tilde{\Phi}$ be a class of tests, let $\tilde{\beta}^*$ be a sequence of envelope power functions with respect to which the shortcomings γ_ν are defined, and let $\sim$ be an equivalence relation on S_A.

The test $\{\phi_\nu\}$ is everywhere locally asymptotically most stringent $-(\tilde{\Phi},\tilde{\beta}^*)$ with respect to $\sim$, or ELAMS $-(\tilde{\Phi},\tilde{\beta}^*)$ with respect to $\sim$, if $\{\phi_\nu\}$ is asymptotically most stringent $-(\tilde{\Phi},\tilde{\beta}^*)$ against S_0, for every $\sim$ - equivalence class S_0.

The suffix "$-(\tilde{\Phi},\tilde{\beta}^*)$" can be replaced by "$-\tilde{\Phi}$" or "-level α" subject to the same conditions as in Definition 6.1.1.

This definition can be paraphrased by stating that $\{\phi_\nu\}$ is ELAMS - $(\tilde{\Phi},\tilde{\beta}^*)$ with respect to $\sim$ iff

(i) $\{\phi_\nu\} \in \tilde{\Phi}$

(ii) for every $\{\phi'_\nu\} \in \tilde{\Phi}$,

$$\text{(7.2.1)} \qquad \inf_{\{\theta_\nu\}\in S_A} \; \sup_{\{\theta'_\nu\}\sim\{\theta_\nu\}} \; \liminf_{\nu} \, [\gamma_\nu(\phi'_\nu,\theta'_\nu) - \gamma_\nu(\phi_\nu,\theta_\nu)] \geq 0 \; .$$

It follows immediately from this formulation that if $\{\phi_\nu\}$ is ELAMS - $(\tilde{\Phi},\tilde{\beta}^*)$ with respect to $\approx$, and $\sim$ is a coarser equivalence relation than $\approx$ (i.e., $\{\theta_\nu\} \approx \{\theta'_\nu\}$ implies $\{\theta_\nu\} \sim \{\theta'_\nu\}$), then $\{\phi_\nu\}$ is also ELAMS - $(\tilde{\Phi},\tilde{\beta}^*)$ with respect to $\sim$. There are two extremes. The finest equivalence relation is equality, = . If $\tilde{\beta}^*$ agrees with $\tilde{\Phi}$, then a test is ELAMS $-\tilde{\Phi}$ with respect to = iff it is AUMP - $\tilde{\Phi}$. The coarsest equivalence relation declares all $\{\theta_\nu\} \in S_A$ to be equivalent. A test is ELAMS - $(\tilde{\Phi},\tilde{\beta}^*)$ with respect to this relation iff it is AMS - $(\tilde{\Phi},\tilde{\beta}^*)$. Thus, if a test is AUMP - $\tilde{\Phi}$, it is ELAMS $-\tilde{\Phi}$ with respect to every equivalence relation; if a test is ELAMS - $(\tilde{\Phi},\tilde{\beta}^*)$ with respect to some equivalence relation, then it is AMS - $(\tilde{\Phi},\tilde{\beta}^*)$. For the equivalence relation defined by

$$\{\theta_\nu\} \sim \{\theta'_\nu\} \text{ iff both are remote, or neither one is ,}$$

a test is ELAMS - $(\tilde{\Phi},\tilde{\beta}^*)$ with respect to $\sim$ iff it is sharply consistent and AMS - $(\tilde{\Phi},\tilde{\beta}^*)$; provided that $\tilde{\Phi}$ contains at least one sharply consistent test.

The approach of this section is reminiscent of COGBURN's (1967) approach to stringency. Cogburn considers a decision problem with parameter space Θ, space of decision functions Δ, and risk function $R : \Theta \times \Delta \to [0,\infty)$, together with an equivalence relation $\sim$ on Θ. The envelope risk function and the excess relative to $\sim$ are

$$r_{\sim}(\theta) = \inf_{\delta\in\Delta} \sup_{\theta'\sim\theta} R(\theta'\delta)$$

$$e_{\sim} = \inf_{\delta\in\Delta} \sup_{\theta\in\Theta} \{R(\theta,\delta) - r_{\sim}(\theta)\} .$$

Cogburn defines a decision function $\delta_{\sim}$ to be stringent relative to $\sim$ if

$$R(\theta, \delta_{\sim}) \leq r_{\sim}(\theta) + e_{\sim} \quad \text{for all } \theta \in \Theta.$$

In other words, stringency is the minimax property with respect to the new loss function $R(\theta,\delta) - r_{\sim}(\theta)$. In my opinion, for non-trivial relations $\sim$ this definition is relevant only if the excess $e_{\sim}$ is small; then $R(\theta,\delta_{\sim})$ hardly exceeds $r_{\sim}(\theta)$. If $e_{\sim}$ is large, the equivalence relation $\sim$ does not seem to be a natural one for the testing problem at hand.
The definition of "ELAMS" is an asymptotic analogue of Cogburn's definition of stringency, with $e_{\sim} = 0$.

7.3. LOCALLY UNIQUE ELAMS TESTS

The property "ELAMS" has been defined with respect to an equivalence relation on S_A, which can be chosen freely. The two trivial equivalence relations considered in Section 7.2 show that for many testing problems with composite null hypotheses and restricted alternatives, some equivalence relations are so fine that they do not admit an ELAMS test at all, while other equivalence relations are so coarse that with respect to them there is a large class of ELAMS tests, containing tests which are not asymptotically equivalent. It seems natural to ask, whether there exists a finest partition among all partitions admitting an ELAMS test. The answer is negative: for those testing problems of Section 3.5 where the AMS - level α test is not asymptotically unique, it is possible to construct equivalence relations $\sim$ and $\approx$ such that

(i) there exist ELAMS - level α tests $\{\phi_\nu\}$ and $\{\phi'_\nu\}$ with respect to $\sim$ and $\approx$, respectively;

(ii) if $\{\psi_\nu\}$ is ELAMS - level α with respect to $\sim$ ($\approx$), then $\{\psi_\nu\}$ is asymptotically equivalent to $\{\phi_\nu\}$ ($\{\phi'_\nu\}$) (i.e., the ELAMS tests are asymptotically unique);

(iii) $\{\phi_\nu\}$ and $\{\phi'_\nu\}$ are not asymptotically equivalent.

These three properties imply that the equivalence relations $\sim$ and $\approx$ are not comparable. The following is a typical example for the construction of such relations $\sim$ and $\approx$.

<u>EXAMPLE 7.3.1</u>. *Non-equivalent ELAMS tests.* Consider the testing problem of Section 7.1, with the notation introduced there. Denote the maximum shortcoming - level α of $\phi_{\omega'}$ for $\mathcal{T}^t_\omega$ by $\gamma(\omega,\omega')$. Then $\gamma(\omega,\omega) = \gamma(\omega)$ and $\gamma(\omega,\omega')$ is a continuous function of (ω,ω'). There exist $\omega_0 \in (0,\frac{1}{4}\pi)$ and $\chi : (0,\frac{1}{4}\pi) \to$ $\to [0,\frac{1}{4}\pi)$ with

$$\chi(\omega) < \omega \qquad \text{for } 0 < \omega < \omega_0$$

$$\chi(\omega) = \omega \qquad \text{for } \omega_0 \le \omega < \tfrac{1}{4}\pi$$

$$\gamma(\omega, \chi(\omega)) < \gamma(\tfrac{1}{4}\pi) \qquad \text{for } 0 < \omega < \omega_0 \quad .$$

Let

$$\gamma_0 = \sup_{0 < \omega < \omega_0} \gamma(\omega,\chi(\omega));$$

then $\gamma_0 \ge \gamma(\omega_0)$. Use the index ν again and define

$$\phi_\nu = \phi_{\omega(\hat{p}_{\nu1},\hat{p}_{\nu3})}\,(Z_1^{(\nu)}(\hat{p}_{\nu1},\hat{p}_{\nu3}),\ Z_2^{(\nu)}(\hat{p}_{\nu1},\hat{p}_{\nu3}))$$

$$\phi'_\nu = \phi_{\chi(\omega(\hat{p}_{\nu1},\hat{p}_{\nu3}))}\,(Z_1^{(\nu)}(\hat{p}_{\nu1},\hat{p}_{\nu3}),\ Z_2^{(\nu)}(\hat{p}_{\nu1},\hat{p}_{\nu3})) \quad .$$

Then $\{\phi_\nu\}$ and $\{\phi'_\nu\}$ are not asymptotically equivalent. Equivalence relations $\sim$ and $\approx$ on S_A will be indicated, such that $\{\phi_\nu\}$ ($\{\phi'_\nu\}$) is the asymptotically unique ELAMS - level α test with respect to $\sim$ ($\approx$).

It will be convenient to use parameters which can be easily identified in terms of the limiting problems $\mathcal{T}^t_\omega$. To that end, define $\theta = \theta(p)$ by

$$\theta_3 = (p_{11} + p_{21}) \ / \ 2$$

$$\theta_4 = (p_{13} + p_{23}) \ / \ 2$$

$$\theta_1 = \{2(1+\kappa(\theta_3,\theta_4))\}^{-\frac{1}{2}}\{(\theta_3(1-\theta_4))^{-\frac{1}{2}}\,\eta_1 - (\theta_4(1-\theta_3))^{-\frac{1}{2}}\,\eta_2\}$$

$$\theta_2 = \{2(1-\kappa(\theta_3,\theta_4))\}^{-\frac{1}{2}}\{(\theta_3(1-\theta_4))^{-\frac{1}{2}}\,\eta_1 + (\theta_4(1-\theta_3))^{-\frac{1}{2}}\,\eta_2\}$$

where

$$\eta_1 = p_{21}(1-p_{13}) - p_{11}(1-p_{23})$$

$$\eta_2 = p_{13}(1-p_{21}) - p_{23}(1-p_{11})$$

(compare(7.1.1,3)). Note that

$$\Theta_A = \{\theta \in \mathbb{R}^4 \mid \theta_2 > 0, |\theta_1| \leq \theta_2 \text{ tg } \omega(\theta_3,\theta_4)\}$$

and that $\{\theta_\nu\} \in S_A$ is remote iff

$$n(\nu)\ (\theta_{\nu 1}^2 + \theta_{\nu 2}^2) \to \infty \quad .$$

Define $\{\theta_\nu\} \sim \{\theta'_\nu\}$ iff

$$\theta_{\nu 3} = \theta'_{\nu 3},\quad \theta_{\nu 4} = \theta'_{\nu 4} \qquad \text{for all } \nu$$
$$\limsup_\nu\ n(\nu)\ \{(\theta_{\nu 1} - \theta'_{\nu 1})^2 + (\theta_{\nu 2} - \theta'_{\nu 2})^2\} < \infty .$$

Choose $(\theta_{03}, \theta_{04})$ so that $\gamma(\omega(\theta_{03},\theta_{04})) = \gamma_0$ and partition Θ_A into the sets

$$B(\theta_3, \theta_4) = \{(\theta_1,\theta_2,\theta_3,\theta_4) \mid \theta_2 > 0, |\theta_1| \leq \text{tg } \chi(\omega(\theta_3,\theta_4))\}$$
$$\text{for } (\theta_3,\theta_4) \neq (\theta_{03},\theta_{04})$$

$$B(\theta_{03},\theta_{04}) = \{\theta \in \Theta_A \mid (\theta_3,\theta_4) = (\theta_{03},\theta_{04})$$
$$\text{or } |\theta_1| > \text{tg } \chi(\omega(\theta_3,\theta_4))\} \quad .$$

Denote the equivalence relation corresponding to this partition by $\sim\sim$. Define $\{\theta_\nu\} \approx \{\theta'_\nu\}$ iff

$$\theta_\nu \sim\sim \theta'_\nu \qquad \text{for all } \nu$$
$$\limsup_\nu\ n(\nu)\ \{(\theta_{\nu 1} - \theta'_{\nu 1})^2 + (\theta_{\nu 2} - \theta'_{\nu 2})^2\} < \infty .$$

It can be proved (in a way similar to the proof of Theorem 8.1.1) that $\{\phi_\nu\}$ is the asymptotically unique ELAMS - level α test with respect to $\sim$, and also that $\{\phi'_\nu\}$ is the asymptotically unique ELAMS - level α test with respect to $\sim$, when the alternative hypothesis A is replaced by A' defined by

$$A' : \theta_2 > 0, |\theta_1| \leq \text{tg } \chi(\omega(\theta_3,\theta_4)) \quad .$$

The asymptotic maximum shortcoming of $\{\phi'_\nu\}$ for sequences $\{\theta_\nu\}$ contained in

$$\{\theta \in \Theta_A \mid |\theta_1| > \mathrm{tg}\ \chi\ \omega(\theta_3,\theta_4))\}$$

does not exceed γ_0, which is the asymptotic minimax shortcoming for the class of sequences contained in

$$\{\theta \in \Theta_A \mid (\theta_3,\theta_4) = (\theta_{03},\theta_{04})\} .$$

It can be concluded, using Proposition 6.2.2, that $\{\phi'_\nu\}$ is the asymptotically unique ELAMS - level α test with respect to $\approx$. □

This example demonstrates that even if attention is restricted to those equivalence relations on S_A which are coarse enough for the existence of an ELAMS test and fine enough for the asymptotic uniqueness of the ELAMS test, one still is faced with many, essentially different, ELAMS tests. So a further restriction is necessary: a stronger uniqueness condition will be imposed.

Recall that the basic principle for the concept of an ELAMS test is that one wishes to minimalize the maximum shortcoming, for each "subproblem" simultaneously: if it were told that attention can be restricted to one particular subproblem, one still could use the same ELAMS test. Now suppose that $\{\phi_\nu\}$ is ELAMS with respect to $\approx$, and that S_0 is a $\approx$ - equivalence class with the property that another test $\{\psi_\nu\}$ exists which is also AMS against S_0, and that a $\{\theta_\nu\} \in S_0$ exists with

$$E_{\theta_\nu}(\phi_\nu - \psi_\nu) \not\to 0 :$$

"the AMS test against S_0 is not asymptotically unique on S_0". If attention can be restricted to S_0, then one is faced with a choice between different AMS tests against S_0; in the spirit of the basic principle expounded above, one can consider partitions of S_0 and try to minimalize the maximum shortcoming, simultaneously for each of the corresponding subclasses of S_0. If this yields a test which is not asymptotically equivalent to $\{\phi_\nu\}$, then restricting attention to S_0 leads to a test which is different from the ELAMS test. So the basic principle for the concept of an ELAMS test can lead to inconsistencies, unless for every equivalence class S_0, the AMS test against S_0 is "asymptotically unique on S_0". The latter requirement is expressed in the following definition.

DEFINITION 7.3.1. The ELAMS - $(\tilde{\Phi},\tilde{\beta}^*)$ test $\{\phi_\nu\}$ with respect to $\sim$ is locally unique if for every $\sim$ - equivalence class S_0 and every $\{\phi'_\nu\}$ which is AMS - $(\tilde{\Phi},\tilde{\beta}^*)$ against S_0, one has

$$E_{\theta_\nu}(\phi_\nu - \phi'_\nu) \to 0 \qquad \text{for all } \{\theta_\nu\} \in S_0.$$

We now try to find equivalence relations on S_A which lead to locally unique ELAMS tests. The surprising result will be proved that, under certain conditions, this approach leads to a unique test for the testing problem of Section 5.1: all locally unique ELAMS tests, with respect to different equivalence relations, are asymptotically equivalent. In other words, the locally unique ELAMS tests do not depend on the equivalence relation. This result will be proved for equivalence relations which are regular in the following sense.

DEFINITION 7.3.2. The equivalence relation $\sim$ on S_A is regular, if the conditions

$$\{\theta_\nu\} \sim \{\theta'_\nu\}$$

$$\text{for every } \nu, \text{ it holds that } \theta''_\nu = \theta_\nu \text{ or } \theta''_\nu = \theta'_\nu$$

imply that $\{\theta_\nu\} \sim \{\theta''_\nu\}$.

Most "reasonable" equivalence relations are regular. This concept of regularity is introduced, because I do not know whether the theorem below is true, let alone how it can be proved, for non-regular equivalence relations.

THEOREM 7.3.1. *Let* d *be a metric on* Θ, $\{s_\nu\}$ *a sequence in* $[0,\infty)$ *and* $\{\hat{\theta}_\nu\}$ *a sequence of estimators with values in* Θ, *such that*

$$(7.3.1) \qquad \{L_{\theta_\nu}(s_\nu d(\hat{\theta}_\nu,\theta_\nu))\} \text{ is tight for all } \{\theta_\nu\} \in S.$$

Define the equivalence relation $\approx$ *on* S_A *by*

$$\{\theta_\nu\} \approx \{\theta'_\nu\} \text{ iff } \limsup_\nu \; s_\nu d(\theta_\nu,\theta'_\nu) < \infty .$$

If $\sim$ *is a regular equivalence relation with respect to which there exists a locally unique ELAMS - level* α *test, then this test is also ELAMS - level*

α *with respect to* $\approx$.

(*Remark*. This theorem will be applied in cases where $\Theta \subset \mathbb{R}^m$, d is the Euclidean metric and $s_\nu = [n(\nu)]^{\frac{1}{2}}$.)

PROOF. Denote by $\tilde{\Phi}$ the class of all asymptotically level α tests. Let $\{\phi_{0\nu}\}$ be a strongly unique ELAMS - level α test with respect to $\sim$. Assume that a $\approx$ - equivalence class S_0 exists against which $\{\phi_{0\nu}\}$ is not AMS - level α. It is sufficient to prove that this assumption leads to a contradiction.

(0) First, the argument will be sketched for the case that AMXS's exists against all subclasses of S_A to be encountered in the proof, and that all infima and suprema are attained.

(i) Let $\{\psi_\nu\}$ be AMS against S_0, with AMXS γ_0. Let γ_{00} be the asymptotic maximum shortcoming of $\{\phi_{0\nu}\}$ against S_0, attained for the sequence $\{\theta_{0\nu}\} \in S_0$. Then $\gamma_{00} > \gamma_0$. Consider the equivalence class

$$S_1 = \{ \{\theta_\nu\} \in S_A \mid \{\theta_\nu\} \sim \{\theta_{0\nu}\} \} .$$

(ii) Test functions χ_ν will constructed with properties which ensure that the "convex combinations"

$$\phi_{1\nu} = \chi_\nu \phi_{0\nu} + (1-\chi_\nu)\, \psi_\nu$$

satisfy

(iii) $\{\phi_{1\nu}\}$ is asymptotically of level α,

(iv) $\{\phi_{1\nu}\}$ is AMS against S_1,

(v) $\limsup_\nu \; E_{\theta_{0\nu}} (\phi_{1\nu} - \phi_{0\nu}) > 0$.

This contradicts the strong uniqueness of $\{\phi_{0\nu}\}$. As a matter of fact, $\chi_\nu \to 0$ in $\{\theta_{0\nu}\}$ - prob.; in some way, high values of χ_ν indicate that "θ is far from $\theta_{0\nu}$" .

Now the real proof starts.

(i) Since $\{\phi_{0\nu}\}$ is asymptotically of level α but not AMS - level α against S_0 , a $\{\psi_\nu\} \in \tilde{\Phi}$, a $\{\theta_{0\nu}\} \in S_0$ and an $\varepsilon > 0$ exist with

$$(1) \qquad \sup_{\{\theta_\nu\} \in S_0} \; \liminf_\nu [\gamma_\nu(\psi_\nu,\theta_\nu) - \gamma_\nu(\phi_{0\nu},\theta_{0\nu})] \leq -\varepsilon.$$

Let $\{K_h\}$ be the non-decreasing sequence of compact sets with $\cup_h K_h = \Theta_T$ which was given by Lemma 5.1.1 and define $H_h = K_h \cap \Theta_A$. Then

$$S_0 = \{ \{\theta_\nu\} \subset \Theta_A \mid \limsup_\nu \; s_\nu \, d(\theta_\nu, \theta_{0\nu}) < \infty \text{ and } \{\theta_\nu\} \subset K_h \text{ for some } h\}$$

$$= \cup_h \{ \{\theta_\nu\} \subset H_h \mid s_\nu \, d(\theta_\nu, \theta_{0\nu}) \leq h \text{ for all } \nu\} .$$

So (1) can be written as

$$\sup_h \; \liminf_\nu \; a(h,\nu) \leq -\varepsilon ,$$

where

(2) $$a(h,\nu) = \sup \{\gamma_\nu(\psi_\nu,\theta) - \gamma_\nu(\phi_{0\nu},\theta_\nu) \mid \theta \in H_h, \; s_\nu \, d(\theta,\theta_{0\nu}) \leq h\}.$$

Lemma A.5.2 demonstrates the existence of a subsequence $\{\xi\}$ of $\{\nu\}$ with

(3) $$\sup_h \; \liminf_\xi \; a(h,\xi) = \sup_h \; \limsup_\xi \; a(h,\xi) \leq - \varepsilon.$$

Passing to a further subsequence if necessary, it may be assumed that

$$\gamma_{00} = \lim_\xi \; \gamma_\xi(\phi_{0\xi},\theta_{0\xi})$$

exists. It follows from (2) and (3) that

$$\gamma_0 = \sup_{\{\theta_\nu\} \in S_0} \; \limsup_\xi \; \gamma_\xi(\psi_\xi,\theta_\xi) \leq \gamma_{00} - \varepsilon .$$

Define the $\sim$ - equivalence class

$$S_1 = \{ \{\theta_\nu\} \in S_A \mid \{\theta_\nu\} \sim \{\theta_{0\nu}\} \} .$$

According to Lemma A.5.3 it may be assumed, passing to a further subsequence of $\{\xi\}$ if necessary, that

(4) $$\sup_{\{\theta_\nu\} \in S_1} \; \liminf_\xi \; \gamma_\xi(\phi_{0\xi},\theta_\xi) = \sup_{\{\theta_\nu\} \in S_1} \; \limsup_\xi \; \gamma_\xi(\phi_{0\xi},\theta_\xi).$$

Denote this number by γ_1. It follows from $\{\theta_{0\nu}\} \in S_1$ that $\gamma_{00} \leq \gamma_1$ .Since

$\{\phi_{0\nu}\}$ is ELAMS with respect to $\sim$, $\{\phi_{0\nu}\}$ is AMS against S_1. With (4) and the "only if" part of Proposition 6.2.2 (see also the remark preceding that proposition) it can be concluded that

(5) $$\gamma_1 = \inf_{\{\phi_\nu\}\in\tilde{\Phi}} \sup_{\{\theta_\nu\}\in S_1} \liminf_{\xi} \gamma_\xi(\phi_\xi,\theta_\xi) \quad .$$

(ii) Test functions χ_ν will be constructed so that $E_{\theta_{0\xi}}\chi_\xi \to 0$, $E_{\theta_\xi}\chi_\xi \to 1$ if $\gamma_\xi(\psi_\xi,\theta_\xi)$ is "too large", and so that for every $\{\theta_\xi\}$, $\{\chi_\xi\}$ is "asymptotically constant in $\{\theta_\xi\}$-prob.". Define

$$b_\xi = \sup\ \{\ b \mid \gamma_\xi(\psi_\xi,\theta) \le \gamma_0 + \varepsilon \text{ for all } \theta \in \Theta_A \text{ with } d(\theta,\theta_{0\xi}) \le b\}.$$

It will be proved that $s_\xi b_\xi \to \infty$. Argue by contradiction, and suppose that for some subsequence $\{\zeta\}$ of $\{\xi\}$, $\{s_\zeta b_\zeta\}$ is bounded. Then a sequence $\{\theta_{1\zeta}\}$ exists for which $\limsup_\zeta s_\zeta d(\theta_{1\zeta}, \theta_{0\zeta}) < \infty$ and

$$\liminf_{\zeta} \gamma_\zeta(\psi_\zeta,\theta_{1\zeta}) \ge \gamma_0 + \varepsilon .$$

For $\nu \notin \{\zeta\}$ define $\theta_{1\nu} = \theta_{0\nu}$. Then $\{\theta_{1\nu}\} \in S_0$ and

$$\limsup_{\nu} \gamma_\nu(\psi_\nu,\theta_{1\nu}) \ge \gamma_0 + \varepsilon .$$

This contradicts the definition of γ_0. Hence $s_\xi b_\xi \to \infty$.
Define

$$t_\xi(\theta) = \min\ \{1,\ b_\xi^{-1} d(\theta,\ \theta_{0\xi})\}$$
$$t_\nu(\theta) = 1 \qquad \text{for all } \nu \notin \{\xi\} \text{ and all } \theta$$
$$\chi_\nu = t_\nu(\hat\theta_\nu) \ .$$

The assumption about $\{\hat\theta_\nu\}$ and $s_\xi b_\xi \to \infty$ imply that for all $\{\theta_\nu\} \in S$

$$|t_\xi(\theta_\xi) - t_\xi(\hat\theta_\xi)| \le b_\xi^{-1} d(\theta_\xi,\hat\theta_\xi) \to 0 \qquad \text{in } \{\theta_\xi\}\text{-prob.},$$

and hence

(6) $$\chi_\nu - t_\nu(\theta_\nu) \to 0 \qquad \text{in } \{\theta_\nu\}\text{-prob., for all } \{\theta_\nu\} \in S \ .$$

With $t_\xi(\theta_{0\xi}) = 0$, this yields in particular that

$$\chi_\xi \to 0 \qquad \text{in } \{\theta_{0\xi}\}\text{-prob. .}$$

Define

$$\phi_{1\nu} = \chi_\nu \, \phi_{0\nu} + (1-\chi_\nu)\, \psi_\nu \, .$$

Then $\phi_{1\nu} = \phi_{0\nu}$ for all $\nu \notin \{\xi\}$, and $\phi_{1\xi} - \psi_\xi \to 0$ in $\{\theta_{0\xi}\}$-prob. .
(iii) For every ν, $\phi_{1\nu}$ is a test function for T_ν. For every $\{\theta_\nu\} \in S$,

$$\phi_{1\nu} - [t_\nu(\theta_\nu)\, \phi_{0\nu} + (1-t_\nu(\theta_\nu))\psi_\nu] \to 0 \qquad \text{in } \{\theta_\nu\}\text{-prob. .}$$

Since $\{\phi_{0\nu}\}$ and $\{\psi_\nu\}$ are asymptotically of level α, this implies that $\{\phi_{1\nu}\}$ is asymptotically of level α.
(iv) Let $\{\phi_\nu\} \in \tilde{\Phi}$ and $\{\theta_{2\nu}\} \in S_1$ be arbitrary. In order to prove that $\{\phi_{1\nu}\}$ is AMS - level α against S_1, it suffices to show that

$$\sup_{\{\theta_\nu\} \in S_1} \liminf_\nu \, [\gamma_\nu(\phi_\nu,\theta_\nu) - \gamma_\nu(\phi_{1\nu},\theta_{2\nu})] \geq 0 \, .$$

It follows from (6) that

$$\liminf_\nu \, [\gamma_\nu(\phi_\nu,\theta_\nu) - \gamma_\nu(\phi_{1\nu},\theta_{2\nu})] =$$
$$= \liminf_\nu \, [\gamma_\nu(\phi_\nu,\theta_\nu) - t_\nu(\theta_{2\nu})\gamma_\nu(\phi_{0\nu},\theta_{2\nu}) - (1-t_\nu(\theta_{2\nu}))\gamma_\nu(\psi_\nu,\theta_{2\nu})].$$

Let $\delta > 0$. From (5) follows the existence of a $\{\theta_{3\nu}\} \in S_1$ with

$$\liminf_\xi \;\; \gamma_\xi(\phi_\xi,\theta_{3\xi}) \geq \gamma_1 - \delta \, .$$

Since $\{\phi_{0\nu}\}$ is AMS against S_1, a $\{\theta_{4\nu}\} \in S_1$ exists with

$$\liminf_\nu \, [\gamma_\nu(\phi_\nu,\theta_{4\nu}) - \gamma_\nu(\phi_{0\nu},\theta_{2\nu})] > -\delta.$$

Define $\{\theta_{5\nu}\}$ by

$$\theta_{5\nu} = \begin{cases} \theta_{3\nu} & t_\nu(\theta_{2\nu}) < 1 \\ \theta_{4\nu} & t_\nu(\theta_{2\nu}) = 1 \ . \end{cases}$$

Since $\sim$ is regular, $\{\theta_{5\nu}\} \in S_1$. It will be proved that

$$\liminf_{\nu} [\gamma_\nu(\phi_\nu,\theta_{5\nu}) - \gamma_\nu(\phi_{1\nu}, \theta_{2\nu})] > -\delta \ .$$

For the subsequence $\{\zeta\}$ of all ν with $t_\nu(\theta_{2\nu}) = 1$ (if infinitely many such ν exist),

$$\liminf_{\zeta} [\gamma_\zeta(\phi_\zeta,\theta_{5\zeta}) - \gamma_\zeta(\phi_{1\zeta},\theta_{2\zeta})] =$$

$$= \liminf_{\zeta} [\gamma_\zeta(\phi_\zeta,\theta_{4\zeta}) - \gamma_\zeta(\phi_{0\zeta},\theta_{2\zeta})] > -\delta \ .$$

For the subsequence $\{\zeta\}$ of all ν with $t_\nu(\theta_{2\nu}) < 1$ (if infinitely many such ν exist),

$$\liminf_{\zeta} [\gamma_\zeta(\phi_\zeta,\theta_{5\zeta}) - \gamma_\zeta(\phi_{1\zeta},\theta_{2\zeta})] \geq$$

$$\geq \liminf_{\zeta} \gamma_\zeta(\phi_\zeta,\theta_{3\zeta}) - \limsup_{\zeta} [t_\zeta(\theta_{2\zeta})\gamma_\zeta(\phi_{0\zeta},\theta_{2\zeta}) +$$

$$+ (1-t_\zeta(\theta_{2\zeta}))\gamma_\zeta(\psi_\zeta,\theta_{2\zeta})]$$

$$\geq \gamma_1 - \delta - \limsup_{\zeta} [\gamma_1 t_\zeta(\theta_{2\zeta}) + (1-t_\zeta(\theta_{2\zeta}))\gamma_\zeta(\psi_\zeta,\theta_{2\zeta})] \ .$$

Therefore, it is sufficient to prove that for this subsequence $\{\zeta\}$,

$$\limsup_{\zeta} \gamma_\zeta(\psi_\zeta,\theta_{2\zeta}) \leq \gamma_1 \ .$$

This inequality follows from $t_\zeta(\theta_{2\zeta}) < 1$, which implies $d(\theta_{2\zeta},\theta_{0\zeta}) < b_\zeta$, which implies $\gamma_\zeta(\psi_\zeta,\theta_{2\zeta}) \leq \gamma_0 + \varepsilon \leq \gamma_{00} \leq \gamma_1$.

(v) Since

$$\limsup_{\nu} E_{\theta_{0\nu}}(\phi_{1\nu} - \phi_{0\nu}) \geq \liminf_{\xi} E_{\theta_{0\xi}}(\psi_\xi - \phi_{0\xi}) =$$

$$= \liminf_{\xi} [\gamma_\xi(\phi_{0\xi},\theta_{0\xi}) - \gamma_\xi(\psi_\xi,\theta_{0\xi})] \geq \gamma_{00} - \gamma_0 \geq \varepsilon ,$$

$\{\phi_{0\nu}\}$ and $\{\phi_{1\nu}\}$ are not asymptotically equivalent on S_1.

The conclusion that $\{\phi_{1\nu}\}$ is AMS - level α against S_1, but not asymptotically equivalent to $\{\phi_{0\nu}\}$ on S_1, contradicts the local uniqueness of $\{\phi_{0\nu}\}$. □

It may be noted from the proof, that Theorem 7.3.1 remains valid if "ELAMS - level α" is replaced by "ELAMS - $(\tilde{\Phi},\tilde{\beta}^*)$", where $\tilde{\Phi}$ is the class of all asymptotically level α tests and $\tilde{\beta}^*$ is arbitrary.

The following proposition gives a natural upper bound for the rapidity, with which the sequence $\{s_\nu\}$ can tend to infinity.

PROPOSITION 7.3.1. *Let* d, $\{s_\nu\}$ *and* $\{\hat{\theta}_\nu\}$ *satisfy* (7.3.1).
Then $\{\theta_\nu\} \triangleleft \{\theta'_\nu\}$ *implies that* $\limsup_\nu s_\nu d(\theta_\nu,\theta'_\nu) < \infty$.

PROOF. Suppose, for a subsequence $\{\xi\}$ of $\{\nu\}$, that $s_\xi d(\theta_\xi,\theta'_\xi) \to \infty$. Define $z_\xi = \tfrac{1}{2} s_\xi d(\theta_\xi,\theta'_\xi)$. Then

$$P_{\theta_\xi} \{s_\xi d(\hat{\theta}_\xi,\theta_\xi) \leq z_\xi\} \to 1 .$$

Since $s_\xi d(\hat{\theta}_\xi,\theta_\xi) \leq z_\xi$ implies

$$s_\xi d(\hat{\theta}_\xi,\theta'_\xi) \geq s_\xi d(\theta_\xi,\theta'_\xi) - s_\xi d(\theta_\xi,\hat{\theta}_\xi) \geq z_\xi ,$$

we have also

$$P_{\theta'_\xi} \{s_\xi d(\hat{\theta}_\xi,\theta_\xi) \leq z_\xi\} \leq P_{\theta'_\xi} \{s_\xi d(\hat{\theta}_\xi,\theta'_\xi) \geq z_\xi\} \to 0 .$$

Hence, neither $\{\theta_\nu\} \triangleleft \{\theta'_\nu\}$ nor $\{\theta'_\nu\} \triangleleft \{\theta_\nu\}$ can hold. □

COROLLARY 7.3.1. *Suppose that* d, $\{s_\nu\}$ *and* $\{\hat{\theta}_\nu\}$ *satisfy* (7.3.1) *and that*

$$\limsup_\nu \; s_\nu d(\theta_\nu,\theta'_\nu) < \infty \qquad \textit{implies} \;\; \{\theta_\nu\} \triangleleft\triangleright \{\theta'_\nu\}.$$

Suppose that $\sim$ *is a regular equivalence relation on* S_A *and that* $\{\phi_\nu\}$ *is the locally unique ELAMS - level* α *test with respect to* $\sim$. *Then* $\{\phi_\nu\}$ *is ELAMS - level* α *with respect to* $\triangleleft\triangleright$.

PROOF. It follows from the first assumption and from Proposition 7.3.1 that $\Leftrightarrow$ and $\approx$ are identical. This makes the corollary to a restatement of Theorem 7.3.1. □

In most applications of Theorem 7.3.1, Θ can be taken to be a subset of $\mathbb{R}^m$, for some m, and d the Euclidean metric. In the case of exponential families parametrized by the expectation μ, $\hat{\mu}_\nu = X^{(\nu)}_{\cdot}$ and $s_\nu = [n(\nu)]^{\frac{1}{2}}$ satisfy the assumptions of the theorem. More generally, under regularity conditions including the finiteness of the Fisher information matrix, one can take $s_\nu = [n(\nu)]^{\frac{1}{2}}$ and $\hat{\theta}_\nu$, e.g., the maximum likelihood estimator. See LE CAM (1955), Lemma 5. In the case of uniform distributions on intervals in $\mathbb{R}$, one can take $s_\nu = n(\nu)$. LE CAM (1969) and DACUNHA - CASTELLE (1978) give examples of families of densities with other "speeds of distinguishability" $\{s_\nu\}$.

7.4. THE DEFINITION OF "EVERYWHERE ASYMPTOTICALLY MOST STRINGENT"

Mutual contiguity can be regarded as an outstanding relation for the construction of ELAMS tests. It is clear that mutual contiguity (or contiguity, for short) is an equivalence relation. If two sequences in S_A are contiguous, say $\{\theta_\nu\} \Leftrightarrow \{\theta'_\nu\}$, then the power sequences $\{E_{\theta_\nu}\phi_\nu\}$ and $\{E_{\theta'_\nu}\phi_\nu\}$ are related. If two sequences in S_A have no contiguous subsequences then, under regularity conditions, the corresponding probability distributions are asymptotically concentrated on disjoint sets (see Theorem 4.1.1). Hence they can be "treated" separately by a single test. This shows that contiguity is a suitable equivalence relation for partitioning the asymptotic testing problem into subproblems.

For testing problems with $\Theta \subset \mathbb{R}^m$, under regularity conditions satisfied by the testing problems of Section 3.5, the first assumption of Corollary 7.3.1 is satisfied when d is the Euclidean metric, $s_\nu = [n(\nu)]^{\frac{1}{2}}$ and $\hat{\theta}_\nu$ is the maximum likelihood estimator. So Corollary 7.3.1 demonstrates that if a locally unique ELAMS test with respect to contiguity exists, then the argumentation of Section 7.3 leads to the selection of this test from the class of all AMS tests.

Section 1.3 contains another, more intuitive, introduction of the optimum property "EAMS".

DEFINITION 7.4.1. The test $\{\phi_\nu\}$ is everywhere asymptotically most stringent - $(\tilde{\Phi},\tilde{\beta}^*)$, or EAMS - $(\tilde{\Phi},\tilde{\beta}^*)$, if $\{\phi_\nu\}$ is locally unique ELAMS - $(\tilde{\Phi},\tilde{\beta}^*)$ with respect to mutual contiguity.

The indication "- $(\widetilde{\Phi},\widetilde{\beta}^*)$" can be replaced by "-$\widetilde{\Phi}$" or " - level α" as in Definition 6.1.1.

For testing problems where the AMS test is not asymptotically unique and where an EAMS test exists, this optimum property is proposed as the most natural method for selecting one test from the class of AMS tests. The property "ELAMS with respect to mutual contiguity", without the local uniqueness, is also an attractive asymptotic optimum property. It will appear in Chapter 8 that for the testing problems treated in this study, the local uniqueness can always be established.

The following proposition characterizes EAMS tests. In conditions (ii) and (iii) it is tacitly assumed that all sequences $\{\theta_\nu\}$ considered are elements of S_A.

<u>PROPOSITION 7.4.1</u>. *Suppose that a metric* d *on* Θ_T *and a sequence* $\{s_\nu\}$ *of nonnegative numbers exist such that for* $\{\theta_\nu\}$, $\{\theta'_\nu\} \in S_A$,

$$\{\theta_\nu\} \Leftrightarrow \{\theta'_\nu\} \text{ iff } \limsup_\nu s_\nu d(\theta_\nu,\theta'_\nu) < \infty .$$

Then $\{\phi_\nu\}$ *is EAMS* $-(\widetilde{\Phi},\widetilde{\beta}^*)$ *iff*

(i) $\{\phi_\nu\} \in \widetilde{\Phi}$

(ii) *for every* $\{\theta_{0\nu}\} \in S_A$ *and every subsequence of* $\{\nu\}$ *a further subsequence* $\{\xi\}$ *exists for which*

$$(7.4.1) \qquad \sup_{\{\theta_\nu\} \Leftrightarrow \{\theta_{0\nu}\}} \limsup_\xi \gamma_\xi(\phi_\xi,\theta_\xi) \leq$$

$$\leq \inf_{\{\psi_\nu\}\in\widetilde{\Phi}} \sup_{\{\theta_\nu\} \Leftrightarrow \{\theta_{0\nu}\}} \liminf_\xi \gamma_\xi(\phi_\xi,\theta_\xi)$$

(iii) *if* $\{\phi'_\nu\} \in \widetilde{\Phi}$ *and* $\{\theta_{0\nu}\} \in S_A$ *are such that for every subsequence of* $\{\nu\}$ *a further subsequence* $\{\xi\}$ *exists with*

$$\sup_{\{\theta_\nu\} \Leftrightarrow \{\theta_{0\nu}\}} \limsup_\xi \gamma_\xi(\phi'_\xi,\theta_\xi) \leq \sup_{\{\theta_\nu\} \Leftrightarrow \{\theta_{0\nu}\}} \liminf_\xi \gamma_\xi(\phi_\xi,\theta_\xi),$$

then $E_{\theta_{0\nu}}(\phi_\nu - \phi'_\nu) \to 0$.

PROOF. In Proposition 6.2.2 take

$$S_0 = \{ \{\theta_\nu\} \in S_A \mid \{\theta_\nu\} \triangleleft\triangleright \{\theta_{0\nu}\} \}$$
$$B_{h\nu} = \{\theta \in \Theta_A \cap K_h \mid s_\nu d(\theta,\theta_{0\nu}) \leq h\} ,$$

where $\{K_h\}$ is the sequence of compact subsets of Θ_T given by Lemma 5.1.1. Proposition 6.2.2 then shows that $\{\phi_\nu\}$ is ELAMS - level α with respect to contiguity iff (i) and (ii) hold. Condition (iii) is the local uniqueness of $\{\phi_\nu\}$. □

CHAPTER 8

EVERYWHERE ASYMPTOTICALLY MOST STRINGENT TESTS

This chapter concludes the theoretical part of this study: the optimum property "everywhere asymptotically most stringent" is applied to the asymptotic testing problem of Section 3.5. It is demonstrated in Section 8.1 that the EAMS - level α test can be obtained by using the test which is MS - level α for the "limiting problem": a corresponding testing problem for multivariate normal distributions, where the unknown covariance matrix is replaced by an estimate. Unfortunately, for many of these limiting problems the MS - level α tests are not known explicitly. One can focus attention on subclasses Ψ of the class of all level α tests for the limiting problem, containing tests with good power properties and for which the MS - Ψ test can be constructed. In Sections 8.2, 3, 4 asymptotic versions of this approach are developed, leading to the concepts of "EAMS - asymptotically - Ψ" and "EAMS - conditionally - Ψ" tests.

Throughout this chapter, the testing problem studied is that of Sections 3.1 and 3.5.

8.1. EAMS - LEVEL α TESTS

It will be demonstrated that the EAMS - level α test can be obtained in the following way. Replace the considered testing problem (that of Section 3.1) by a corresponding testing problem for multivariate normal distributions

$$(8.1.1) \qquad (\mathbb{R}^m, \{N(\eta,\Lambda) \mid \eta \in V\}, \{N(\eta,\Lambda) \mid \eta \in (V + K) \setminus V\}) ,$$

and use for the unknown covariance matrix $\Lambda = \Lambda(R,\mu)$ an estimate provided by a uniformly consistent estimator $\{\hat{\Lambda}_\nu\}$. Employ the test which is most stringent - level α for this "estimated limiting problem".

It will be convenient to replace testing problem (8.1.1) by the corresponding (and, in a sense, equivalent) testing problem with a simple null hypothesis

$$(8.1.2) \qquad T_\Lambda^* = (\mathbb{R}^m,\ \{N(0,\Lambda)\},\ \{N(\eta,\Lambda)\ |\ \eta \in (V+K) \setminus V,\ L_\Lambda \eta = 0\}),$$

where

$$L_\Lambda : \mathbb{R}^m \to V$$

is the projection onto V with respect to the inner product

$$[x,\ y]_\Lambda = x'\Lambda^{-1}\ y.$$

There exists a MS - level α test for T_Λ^* (Theorem 2.6.1) and this test is unique up to equivalence a.e. (Corollary 2.6.1). Let ϕ_Λ be the version of the MS - level α test which has a convex acceptance region (Theorem 2.7.2), denoted by acc ϕ_Λ, and with $\phi_\Lambda(x) = 0$ for all $x \in \partial$ acc ϕ_Λ. The recession cone of acc ϕ_Λ (Section 2.8) is denoted by 0^+ (acc ϕ_Λ).

The test $\{\phi_{\hat{\Lambda}_\nu}(Y_\nu)\}$ is EAMS - level α, as it will be demonstrated in Theorem 8.1.1. The following lemma is used in the proof.

<u>LEMMA 8.1.1</u>. *The most stringent test* ϕ_Λ, *specified as above, satisfies*

(i) $\phi_\Lambda(y+v) = \phi_\Lambda(y)$ *for all* $y \in \mathbb{R}^m$, $v \in V$, Λ

(ii) $(V+K) \cap 0^+(\text{acc } \phi_\Lambda) = V$ *for all* Λ

(iii) ϕ_Λ *is a weakly* continuous function of* Λ.

<u>PROOF</u>. (i) Denote the random variable for the testing problem T_Λ^* by Y. For every $y \in \mathbb{R}^m$ and $\eta \in L_\Lambda^{-1}\{0\}$,

$$y'\Lambda^{-1}\eta = [y,\eta]_\Lambda = [y - L_\Lambda y, \eta]_\Lambda = (y - L_\Lambda y)'\Lambda^{-1}\eta\ .$$

Hence $Y - L_\Lambda Y$ is a sufficient statistic. The determination of ϕ_Λ ensures that ϕ_Λ depends on Y through $Y - L_\Lambda Y$. Since $(y+v) - L_\Lambda(y+v) = y - L_\Lambda y$ for all y, this demonstrates (i).

(ii) Proposition 2.8.2 and Corollary 2.8.1 show that $((V+K) \cap L_\Lambda^{-1}\{0\}) \cap 0^+(\text{acc } \phi_\Lambda) = \{0\}$. With (i), this implies (ii).

(iii) The test $\psi_\Lambda(y) = \phi_\Lambda(\Lambda^{\frac{1}{2}}y)$ is MS - level α for the testing problem

$$L_\eta(Y) = N_m(\eta, I)$$
$$H : \eta = 0$$
$$H \vee A_\Lambda : \eta \in M_\Lambda = \Lambda^{-\frac{1}{2}}((V+K) \cap L_\Lambda^{-1}\{0\}) \ .$$

It is sufficient to prove that ψ_Λ is a weakly* continuous function of Λ. Let

$$\beta^*(\eta) = \sup \{E_\eta \psi(Y) \mid E_0\psi(Y) \leq \alpha\}, \quad \gamma(\psi,\eta) = \beta^*(\eta) - E_\eta\psi$$
$$\gamma^*(M_\Lambda) = \sup_{\eta \in M_\Lambda} \gamma(\psi_\Lambda,\eta).$$

Then $\gamma^*(M_\Lambda)$ is the minimax - level α shortcoming for testing against A_Λ, and

$$\gamma^*(M_\Lambda) \leq \gamma^* < 1-\alpha \qquad \text{for all } \Lambda,$$

where γ^* is the minimax - level α shortcoming for the testing problem with the unrestricted alternative (see Corollary 2.8.1 and the subsequent remark for the inequality $\gamma^* < 1-\alpha$). For every Λ, ψ_Λ is the a.e. unique level α test ψ satisfying

$$(1) \qquad \sup_{\eta \in M_\Lambda} \gamma(\psi,\eta) = \gamma^*(M_\Lambda) \ .$$

Suppose that $\Lambda_\nu \to \Lambda$ and let $\psi_\nu = \psi_{\Lambda_\nu}$. The compactness of Φ_C (Theorem A.4.1) implies that every subsequence of $\{\psi_\nu\}$ has a further subsequence $\{\psi_\xi\}$ with $\psi_\xi \overset{*}{\to} \psi$ for some $\psi \in \Phi_C$. It is sufficient to show that (1) holds for every subsequential limit (it is immediately seen that ψ is of level α, so that (1) will imply that $\psi = \psi_\Lambda$ a.e.). For every $\eta \in M_\Lambda$ there exists a sequence $\{\eta_\xi\}$ with $\eta_\xi \in M_{\Lambda_\xi}$, $\eta_\xi \to \eta$. With the joint continuity of γ, this shows that

$$(2) \qquad \sup_{\eta \in M_\Lambda} \gamma(\psi,\eta) \leq \liminf_\xi \ \gamma^*(M_{\Lambda_\xi}) \leq \gamma^* \ .$$

Since $\gamma^* < 1-\alpha$, Proposition 2.8.2 shows that $M_\Lambda \cap 0^+(\text{acc } \psi) = \{0\}$. Let $\{\eta_\xi\}$ be a sequence with $\eta_\xi \in M_{\Lambda_\xi}$ and

$$\gamma^*(M_{\Lambda_\xi}) - \gamma(\psi_\xi,\eta_\xi) \to 0 \ .$$

It may be assumed that either $\|\eta_\xi\| \to \infty$ or $\eta_\xi \to \eta$. If $\|\eta_\xi\| \to \infty$, then $\gamma(\psi_\xi,\eta_\xi) \to 0$ (Corollary A.4.2). If $\eta_\xi \to \eta$, then $\eta \in M_\Lambda$ and $\gamma(\psi_\xi,\eta_\xi) \to \gamma(\psi,\eta)$. Hence

$$\text{(3)} \qquad \limsup_\xi \ \gamma^*(M_{\Lambda_\xi}) \le \sup_{\eta \in M_\Lambda} \ \gamma(\psi,\eta) \ .$$

It follows from (2) and (3) that ψ satisfies (1). □

<u>THEOREM 8.1.1</u>. *Let* $\{\hat\mu_\nu\}$ *be a uniformly consistent estimator for* μ *under* H *and define*

$$\hat\Lambda_\nu = \Lambda(R_\nu,\hat\mu_\nu) \ .$$

Let ϕ_Λ *be the MS - level* α *test for* T^*_Λ*, as specified above Lemma 8.1.1.The test* $\{\phi_\nu\}$ *with*

$$\phi_\nu(Y_\nu) = \phi_{\hat\Lambda_\nu}(Y_\nu)$$

is a sharply consistent EAMS - level α *test.*

<u>PROOF</u>. In (i) the sharp consistency (which is necessary for $\{\phi_\nu\}$ to be EAMS) is proved; in (ii) and (iii), some preparations are made; in (iv), (v) and (vi) the three conditions of Proposition 7.4.1, which are equivalent to $\{\phi_\nu\}$ being EAMS - level α, are verified. It is not a restriction to assume that $R_\nu \to R$ for some proportion matrix R. The class of all asymptotically level α tests is denoted by $\tilde\Phi$. γ_ν is the shortcoming with respect to the class of all level α tests for T_ν.

(i) Let $\{\mu_\nu\}$ be a remote sequence. Because of the uniform consistency of $\{\hat\mu_\nu\}$, every subsequence of $\{\nu\}$ has a further subsequence $\{\xi\}$ such that $\hat\mu_\xi \to \mu_0$ in $\{\mu_\xi\}$ - prob., for some $\mu_0 \in \mu(\Theta_H)$. Let $\Lambda = \Lambda(R,\mu_0)$ and $x_\xi = [n(\xi)]^{\frac12}(I - L_\Lambda)f(\mu_\xi)$. Then $\|x_\xi\| \to \infty$ and for every subsequential limit x of $x_\xi / \|x_\xi\|$ one has $x \in (V+K) \setminus V$. Lemma 8.1.1 yields that every such x satisfies $x \notin 0^+(\text{acc}\ \phi_\Lambda)$; that $\phi_\xi(Y_\xi) = \phi_{\hat\Lambda_\xi}(\tilde Y_\xi + x_\xi)$ with $\tilde Y_\xi = Y_\xi - [n(\xi)]^{\frac12} f(\mu_\xi)$; and that the assumptions of Lemma A.4.7 are satisfied with $T_\xi = \hat\Lambda_\xi$, $X_\xi = \tilde Y_\xi$, $\phi_\xi(X_\xi + x_\xi, T_\xi) = \phi_{\hat\Lambda_\xi}(\tilde Y_\xi + x_\xi)$ and $\phi = \phi_\Lambda$. Hence it can be concluded from Corollary A.4.3 that $E_{\mu_\xi}\phi_\xi \to 1$.

(ii) Suppose that $\{\mu_{0\nu}\} \in S_H$ and that $\{\xi\}$ is a subsequence of $\{\nu\}$ with $\mu_{0\xi} \to \mu_0$; let $\Lambda = \Lambda(R,\mu_0)$ and $\tilde{Y}_\xi = Y_\xi - [n(\xi)]^{\frac{1}{2}} f(\mu_{0\xi})$. Lemma 8.1.1 (i) implies that $\phi_\xi(Y_\xi) = \phi_{\hat{\Lambda}_\xi}(\tilde{Y}_\xi)$. An application of Lemma A.4.7 as in (i) above yields

$$E_{\mu_{0\xi}} |\phi_\xi(Y_\xi) - \phi_\Lambda(\tilde{Y}_\xi)| \to 0.$$

(iii) Let $\{\mu_{0\nu}\}$, $\{\xi\}$, μ_0, Λ and $\tilde{Y}_\xi$ be as in (ii). Define

$$M_\xi = \{\mu \mid f(\mu) \in V + K,\ L_\Lambda f(\mu) = f(\mu_{0\xi}),\ \mu \neq \mu_{0\xi}\}$$

$$K = \{ \{\mu_\xi\} \mid \mu_\xi \in M_\xi \quad \{\mu_\xi\} \Leftrightarrow \{\mu_{0\xi}\} \},$$

and let γ_Λ be the minimax shortcoming for T^*_Λ. Corollary 4.4.1 and Lemma 6.3.1 yield that

$$\sup_{\{\mu_\xi\} \in K} \limsup_\xi \gamma_\xi(\phi_\Lambda, \mu_\xi) \leq \gamma_\Lambda \leq$$

$$\leq \inf_{\{\phi'_\nu\} \in \tilde{\Phi}} \sup_{\{\mu_\xi\} \in K} \liminf_\xi \gamma_\xi(\phi'_\xi, \mu_\xi) .$$

(iv) Let $\{\mu_{0\nu}\} \in S_H$. Every subsequence of $\{\mu_{0\nu}\}$ has a further subsequence $\{\mu_{0\xi}\}$ which converges to some $\mu_0 \in \mu(\Theta_H)$. Since

$$L_{\mu_{0\xi}}(\tilde{Y}_\xi) \to N(0,\Lambda)$$

(see Theorem 4.4.1), where $\tilde{Y}_\xi$ and Λ are as in (ii), while $E_{N(0,\Lambda)}\phi_\Lambda = \alpha$, it follows from (ii) that $E_{\mu_{0\xi}} \phi_\xi \to \alpha$. Hence $\{\phi_\nu\}$ is asymptotically of level α.

(v) Let $\{\mu_\nu\} \in S_A$. Every subsequence of $\{\nu\}$ has a further subsequence $\{\xi\}$ such that either $\{\mu_\xi\}$ is remote, or $\{\mu_\xi\} \Leftrightarrow \{\mu_{0\xi}\}$ for some $\{\mu_{0\nu}\} \in S_H$ with $\mu_{0\nu} \to \mu_0$ for some μ_0. In the first case it follows from (i) and Proposition 4.1.1 that

$$\text{(1)} \qquad \sup_{\{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}} \limsup_\xi \gamma_\xi(\phi_\xi, \mu_\xi) = 0 .$$

In the second case let Λ be as in (ii). It follows from (iii) that

$$\text{(2)} \qquad \gamma_\Lambda \leq \inf_{\{\phi'_\nu\}\in\tilde{\Phi}} \sup_{\{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}} \liminf_\xi \gamma_\xi(\phi'_\xi,\mu_{1\xi})$$

(note that every $\{\mu_{1\xi}\} \in K$ can be extended to a sequence $\{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}$). For every $\{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}$, (iii) can be applied with $\{f^{-1}(L_\Lambda f(\mu_{1\nu}))\}$ instead of $\{\mu_{0\nu}\}$; then $\{\mu_{1\xi}\} \in K$ and $f^{-1}(L_\Lambda f(\mu_{1\xi})) \to \mu_0$, so that

$$\text{(3)} \qquad \limsup_\xi \gamma_\xi(\phi_\Lambda,\mu_{1\xi}) \leq \gamma_\Lambda \ .$$

Since (3) holds for all $\{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}$, it shows together with (1) and (2) that condition (ii) of Proposition 7.4.1 is satisfied.

(vi) Let $\{\mu_\nu\} \in S_A$, and let $\{\phi'_\nu\} \in \tilde{\Phi}$ be a test such that every subsequence of $\{\nu\}$ has a further subsequence $\{\xi\}$ with

$$\text{(4)} \qquad \begin{aligned} &\sup_{\{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}} \limsup_\xi \gamma_\xi(\phi'_\xi,\mu_{1\xi}) \leq \\ &\leq \sup_{\{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}} \liminf_\xi \gamma_\xi(\phi_\xi,\mu_{1\xi}) \ . \end{aligned}$$

Distinguish again the two cases of (v). In the first case the right hand side of (4) is 0, implying that

$$\limsup_\xi E_{\mu_\xi} \mid \phi'_\xi - \phi_\xi \mid \leq \limsup_\xi E_{\mu_\xi}((1-\phi_\xi) + (1-\phi'_\xi)) = 0 \ .$$

In the second case the right hand side of (4) is γ_Λ, and Theorem 4.5.1 yields that $E_{\mu_{0\xi}} \mid \phi_\xi - \phi'_\xi \mid \to 0$,so that also $E_{\mu_\xi} \mid \phi_\xi - \phi'_\xi \mid \to 0$ (Proposition 4.1.1). Hence $E_{\mu_\nu} \mid \phi_\nu - \phi'_\nu \mid \to 0$. □

We saw in Section 3.3 that the MS - level α test ϕ_Λ has been determined only in a few cases and that it seems to be very difficult to obtain the MS - level α test in considerably more cases. Moreover, it may be expected that in general the MS - level α test is computationally rather unattractive. For these reasons, Theorem 8.1.1 seems to be of a limited practical interest.

EXAMPLE 8.1.1. *Testing homogeneity against increasing likelihood ratio for a 2 × 3 table.* Consider the testing problem of Section 7.1. The limiting problem T^*_Λ is obtained in that section, and transformed to the more convenient form T^t_ω. The MS - level α test for T^t_ω is given in part 2 of Section 3.3. So Theorem 8.1.1 yields that the test which rejects for

$$Z_2(\hat{p}_1, \hat{p}_3) \geq \{b \cos \omega(\hat{p}_1,\hat{p}_3)\}^{-1} *$$

$$* [c - \log \{\exp(b Z_1(\hat{p}_1,\hat{p}_3) \sin \omega(\hat{p}_1,\hat{p}_3)) +$$

$$+ \exp(- b Z_1(\hat{p}_1,\hat{p}_3) \sin \omega(\hat{p}_1,\hat{p}_3))\}] ,$$

where Z_1 and Z_2 are given by (7.1.1, 2, 3), ω by (7.1.4), $\hat{p}_1,\hat{p}_3$ by (7.1.6) and $b = b_\alpha(\omega(\hat{p}_1,\hat{p}_3))$ and $c = c_\alpha(\omega(\hat{p}_1,\hat{p}_3))$ by the figures in part 2 of Section 3.3, is EAMS - level α. □

8.2. ASYMPTOTICALLY - Ψ TESTS

For many testing problems from practice, the explicit construction of the EAMS - level α test seems to be a forbidding task. It may be sensible to focus attention on a judiciously chosen subclass $\tilde{\Phi}$ of the class of all asymptotically level α tests, for which the EAMS - $\tilde{\Phi}$ test can be constructed explicitly and has satisfactory power properties. In this section, it will be assumed that a class Ψ of tests is available for suitably transformed versions of the limiting testing problems T^*_Λ. The class $\tilde{\Psi}$ of asymptotically - Ψ tests will be defined in such a way, that the EAMS - $\tilde{\Psi}$ test can be obtained by a method similar to that of Section 8.1.

Note that a family of linear transformations $B_\Lambda : \mathbb{R}^m \to \mathbb{R}^r$ can be chosen such that

$$B_\Lambda V = \{0\}, \quad B_\Lambda \Lambda B_\Lambda' = I \quad \text{for all } \Lambda, \tag{8.2.1}$$
$$\Lambda \mapsto B_\Lambda \text{ is a continuous function.}$$

(An example is given by the transformations $Y \mapsto (Z_1(p_1,p_3), Z_2(p_1,p_3))$ in Section 7.1.) The limiting testing problem T^*_Λ is transformed by B_Λ to the testing problem

$$(\mathbb{R}^r, \{N(0,I)\}, \{N(\eta,I) \,|\, \eta \neq 0, \eta \in B_\Lambda K\}).$$

The class Ψ will be a class of tests for the latter testing problem, satisfying the following assumptions.

ASSUMPTIONS CONCERNING Ψ.

(i) Ψ is a weakly* (with respect to Lebesgue measure) compact class of test functions on $\mathbb{R}^r$,

(ii) every $\psi \in \Psi$ has a convex acceptance region and satisfies $E_{N(0,I)}\psi \leq \alpha$.

Interesting classes Ψ of tests are, e.g., the class of linear tests

$$\Psi = \{I_{(u_\alpha,\infty)}(a'x) \mid a \in \mathbb{R}^r, \ ||a|| = 1\}$$

(see part 4 of Section 3.3); and the class of all tests $\phi_{a,\omega}$, where $\phi_{a,\omega}$ is the likelihood ratio tests at level α for the testing problem with a "circular cone alternative"

$$L_\eta(X) = N_r(\eta, I)$$

$$H : \eta = 0, \quad H \vee A : \eta'a \geq ||\eta|| \cos\omega$$

where $a \in \mathbb{R}^r$, $||a|| = 1$ and $0 \leq \omega \leq \pi/2$ (see part 5 of Section 3.3). (It may be noted that linear test are likelihood ratio tests $\phi_{a,\omega}$ with $\omega = 0$.)

DEFINITION 8.2.1. The class $\tilde{\Psi}$ of asymptotically - Ψ tests is the class of all tests $\{\phi_\nu\}$ such that for every $\{\mu_\nu\} \in S_H$ there exists a sequence $\{\psi_\nu\} \subset \Psi$ with

$$(8.2.2) \qquad E_{\mu_\nu} \mid \phi_\nu(Y_\nu) - \psi_\nu(B_{\Lambda(R_\nu,\mu_\nu)}Y_\nu) \mid \to 0 .$$

In most applications, the class Ψ will be invariant under orthogonal transformations (if $\psi \in \Psi$ and U is an $r \times r$ matrix with $UU' = I$ then $\psi(U.) \in \Psi$). Then the class Ψ does not depend on the particular choice of the family of transformations B_Λ.

PROPOSITION 8.2.1. *Let $\{\phi_\nu\}$ be an asymptotically - Ψ test. For every $\{\mu_\nu\} \in S_H$ and every subsequence of $\{\nu\}$, there exists a further subsequence $\{\xi\}$ such that $\Lambda(R_\xi,\mu_\xi) \to \Lambda$ for some Λ and*

$$E_{\mu_\xi} \mid \phi_\xi(Y_\xi) - \psi(B_\Lambda Y_\xi) \mid \to 0$$

for some $\psi \in \Psi$.

PROOF. Let $\{\mu_\nu\} \in S_H$ and define $\Lambda_\nu = \Lambda(R_\nu,\mu_\nu)$; let $\{\psi_\nu\} \subset \Psi$ be a sequence satisfying (8.2.2). For every subsequence of $\{\nu\}$ there exists a further subsequence $\{\xi\}$ with $\Lambda_\xi \to \Lambda$ and (because of the compactness of Ψ) $\psi_\xi \overset{*}{\to} \psi$ for some $\psi \in \Psi$. Assumption (iv) implies that

$$\psi_\xi(B_{\Lambda_\xi}(y - [n(\xi)]^{\frac{1}{2}} f(\mu_\xi))) = \psi_\xi(B_{\Lambda_\xi} y) \quad ,$$

and similarly for $\psi(B_\Lambda\cdot)$. It follows from Lemma A.4.6 that

$$E_{\mu_\xi} \mid \psi_\xi(B_{\Lambda_\xi} Y_\xi) - \psi(B_\Lambda(Y_\xi)) \mid \to 0;$$

with (8.2.2), this yields the desired conclusion. □

COROLLARY 8.2.1. *Every asymptotically - Ψ test is asymptotically of level α.*

PROOF. This follows from Proposition 8.2.1, assumption (ii) and (8.2.1.), wich imply that

$$E_{\mu_\xi} \psi(B_\Lambda Y_\xi) = E_{\mu_\xi} \psi(B_\Lambda(Y_\xi - [n(\xi)]^{\frac{1}{2}} f(\mu_\xi))) \to E_{N(0,I)} \psi \leq \alpha.$$

□

In order to "replace" Lemma 6.3.1, certain assumptions are needed about the sequence $\tilde{\beta}^*$ of functions with respect to which the shortcoming functions γ_ν are defined.

ASSUMPTIONS CONCERNING $\tilde{\beta}^*$.

(i) There exists a function

$$\beta^0 : \cup_\Lambda B_\Lambda K \to [0,1]$$

such that for every $\{\mu_\nu\} \in S_H$ and $\{\mu_{1\nu}\} \in S_A$ with $\{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}$, one has

$$\beta^*_\nu(\mu_{1\nu}) - \beta^0(B_{\Lambda(R_\nu,\mu_\nu)} [n(\nu)]^{\frac{1}{2}} f(\mu_{1\nu})) \to 0 \; .$$

(ii) For every remote sequence $\{\mu_{1\nu}\}$, one has

$$\beta^*_\nu(\mu_{1\nu}) \to 1 \; .$$

It may be noted that assumption (i) implies that β^0 is a continuous function; see Lemma 4.2.2.

LEMMA 8.2.1. *The assumptions concerning* $\tilde{\beta}^*$ *imply that if* $\Lambda = \Lambda(R,\mu_0)$ *for some subsequential limit* R *of* $\{R_\nu\}$ *and some* $\mu_0 \in \mu(\Theta_H)$, *then*

$$\lim_{r\to\infty} \inf \{\beta^0(\eta) \mid \eta \in B_\Lambda K, \|\eta\| \geq r\} = 1 \quad .$$

PROOF. Let $L_\Lambda : \mathbb{R}^m \to V$ be the projection on V with respect to the inner product $x'\Lambda^{-1}Y$ and define

$$a(r,\nu) = \sup \{\beta^*_\nu(\mu) - \beta^0(B_\Lambda[n(\nu)]^{\frac{1}{2}} f(\mu)) \mid$$
$$\mu \in V + K,\ [n(\nu)]^{\frac{1}{2}} \|f(\mu) - f(\mu_0)\| \leq r,\ L_\Lambda f(\mu) = f(\mu_0)\}.$$

Assumption (i) implies that $\lim_\nu a(r,\nu) = 0$, for all r. Lemma A.5.1 implies the existence of a sequence $\{r_\nu\}$ with $r_\nu \to \infty$ and $a(r_\nu,\nu) \to 0$. Let $\{\eta_\nu\} \subset B_\Lambda K$ be a sequence with $\|\eta_\nu\| \leq r_\nu$, $\|\eta_\nu\| \to \infty$ and

$$\beta^0(\eta_\nu) \to \lim_{r\to\infty} \inf \{\beta^0(\eta) \mid \eta \in B_\Lambda K,\ \|\eta\| \geq r\} \ ;$$

let $L_\Lambda f(\mu_{1\nu}) = f(\mu_0)$ and $B_\Lambda[n(\nu)]^{\frac{1}{2}} f(\mu_{1\nu}) = \eta_\nu$. Then $\{\mu_{1\nu}\}$ is a remote sequence; assumption (ii) implies that $\beta^*_\nu(\mu_{1\nu}) \to 1$. Hence

$$\beta^0(\eta_\nu) \geq \beta^*_\nu(\mu_{1\nu}) - a(r_\nu,\nu) \to 1 \ ,$$

which proves the assertion. □

It can be attractive to use a sequence $\tilde{\beta}^*$ which agrees which $\tilde{\Psi}$ in the sense of Definition 6.1.1: for every $\{\mu_\nu\} \in S_A$ there exists an asymptotically most powerful - $\tilde{\Psi}$ test $\{\phi_\nu\}$ against $\{\mu_\nu\}$, and this test satisfies

$$\gamma_\nu(\phi_\nu,\mu_\nu) \to 0 \ ,$$

where $\gamma_\nu(\phi,\mu) = \beta^*_\nu(\mu) - E_\mu\phi$. In such a case, the sequence $\tilde{\beta}^*$ can be regarded as a sequence of envelope power functions with respect to $\tilde{\Psi}$. The following proposition shows that the assumptions concerning Ψ_Λ and the definition of $\tilde{\Psi}$ imply the existence of such a sequence $\tilde{\beta}^*$.

PROPOSITION 8.2.2. *Suppose that for every* Λ, *there exists a* $\psi \in \Psi$ *with* $B_\Lambda K \cap 0^+(\text{acc}\ \psi) = \{0\}$. *Then there exists a sequence* $\tilde{\beta}^*$ *of functions*

$\beta^*_\nu : \mu(\Theta_A) \to [0,1]$ *which agrees with* $\tilde{\Psi}$ *and which satisfies assumptions (i), (ii) with*

$$\beta^0(\eta) = \sup_{\psi \in \Psi} E_{N(\eta,I)}\, \psi \; .$$

PROOF. Let $\{\hat{\mu}_\nu\}$ be a uniformly consistent estimator for μ under H and define

$$\hat{\Lambda}_\nu = \Lambda(R_\nu, \hat{\mu}_\nu)$$

$$\Phi_\nu = \{\psi(B_{\hat{\Lambda}_\nu}\,.) \mid \psi \in \Psi\}$$

$$\beta^*_\nu(\mu) = \sup_{\phi \in \Phi_\nu} E_\mu \phi(Y_\nu) \; .$$

It will be demonstrated that the sequence $\tilde{\beta}^* = \{\beta^*_\nu\}$ satisfies the requirements. First the assumption concerning $\tilde{\beta}^*$ are verified.

(i) Let $\{\mu_\nu\} \in S_H$ and define $\Lambda_\nu = \Lambda(R_\nu, \mu_\nu)$. It follows from Corollary A.4.1 that for all $\{\psi_\nu\} \subset \Psi$,

$$E_{\mu_\nu} \mid \psi_\nu(B_{\hat{\Lambda}_\nu} Y_\nu) - \psi_\nu(B_{\Lambda_\nu} Y_\nu) \mid \to 0.$$

If $\{\mu_{1\nu}\} \lhd\rhd \{\mu_\nu\}$ and $B_{\Lambda_\nu} [n(\nu)]^{\frac{1}{2}} f(\mu_{1\nu}) \to \eta$, then

$$L_{\mu_{1\nu}}(B_{\Lambda_\nu} Y_\nu) \to N_r(\eta, I) \; .$$

It can be demonstrated similarly to the proof of Proposition 4.2.2, that the first assumption concerning $\tilde{\beta}^*$ is satisfied.

(ii) Let $\{\mu_{1\nu}\}$ be a remote sequence. The uniform consistency of $\{\hat{\mu}_\nu\}$ implies that every subsequence of $\{\nu\}$ has a further subsequence $\{\xi\}$ with $\hat{\mu}_\xi \to \mu$ in $\{\mu_{1\xi}\}$ - prob., for some $\mu \in \mu(\Theta_H)$, and with $R_\xi \to R$. Let $\Lambda = \Lambda(R,\mu)$ and let $\psi \in \Psi$ be a test with $B_\Lambda K \cap 0^+(\text{acc}\ \psi) = \{0\}$. Corollary A.4.3 can be applied with $x_\xi = [n(\xi)]^{\frac{1}{2}} f(\mu_{1\xi})$ and $T_\xi = \hat{\Lambda}_\xi$; it yields that

$$\beta^*_\xi(\mu_{1\xi}) \geq E_{\mu_{1\xi}} \psi(B_{\hat{\Lambda}_\xi} Y_\xi) \to 1 \; .$$

(iii) It remains to be proved that $\tilde{\beta}^*$ agrees with $\tilde{\Psi}$. Note that if $\phi_\nu \in \Phi_\nu$ for every ν, then $\{\phi_\nu\} \in \tilde{\Psi}$. Hence it is sufficient to prove that for every $\{\mu_{1\nu}\} \in S_A$, there exists a test $\{\phi_\nu\}$ which is asymptotically most powerful $-\tilde{\Psi}$ against $\{\mu_{1\nu}\}$, with $\phi_\nu \in \Phi_\nu$ for every ν.

For remote $\{\mu_{1\nu}\}$, the desired conclusion follows from (ii) above. If $\{\mu_{1\nu}\}$ is not remote, it may be assumed that $\{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}$ for some $\{\mu_\nu\} \in S_H$. Let ϕ_ν the most powerful $-\Phi_\nu$ test against $\mu_{1\nu}$. It is sufficient to prove that for every $\{\phi'_\nu\} \in \tilde{\Psi}$, we have

$$\liminf_{\nu} E_{\mu_{1\nu}}(\phi_\nu - \phi'_\nu) \geq 0 \ .$$

It follows from Proposition 8.2.1 that for every subsequence of $\{\nu\}$, there exists a further subsequence $\{\xi\}$ with $\Lambda(R_\xi,\mu_\xi) \to \Lambda$ for some Λ and

$$E_{\mu_\xi} \left| \phi'_\xi (Y_\xi) - \psi(B_\Lambda Y_\xi) \right| \to 0,$$

for some $\psi \in \Psi$. With Proposition 4.1.1, this implies

$$E_{\mu_{1\xi}} \left| \phi'_\xi (Y_\xi) - \psi(B_\Lambda Y_\xi) \right| \to 0.$$

Hence

$$\liminf_{\xi} E_{\mu_{1\xi}}(\phi_\xi - \phi'_\xi) \geq \liminf_{\xi} E_{\mu_{1\xi}}(\psi(B_{\hat{\Lambda}_\xi} Y_\xi) - \phi'_\xi(Y_\xi)) = 0 \ .$$

□

Other approaches, closely related to the one of this section, are also possible. Note that if Ψ contains tests ψ with $(B_\Lambda K) \cap 0^+(\mathrm{acc}\ \psi) \neq \{0\}$, then there will exist asymptotically $-\Psi$ tests which are not sharply consistent. This is the case, e.g., for the two classes Ψ mentioned before Definition 8.2.1. One can object against the use of a sequence $\tilde{\beta}^*$ which agrees with $\tilde{\Psi}$ and wish, instead, to use a sequence $\tilde{\beta}^*$ which agrees with the class of all sharply consistent tests in $\tilde{\Psi}$. Such an approach will lead to slightly different (and more complicated) assumptions concerning Ψ_Λ and $\tilde{\beta}^*$, and to a theory which can be developed along the lines of Sections 8.2 and 8.3.

8.3. EAMS - ASYMPTOTICALLY $-\Psi$ TESTS

From the point of view of practical applicability, the theorem of this section is one of the main results of this study. It shows that the EAMS -

asymptotically $-\Psi$ test can be obtained from the MS $-\Psi$ tests for the testing problems

(8.3.1) $\quad (\mathbb{R}^r, \{N(0,I)\}, \{N(\eta,I) \mid \eta \in B_\Lambda K, \eta \neq 0\})$

in a way which is similar to the method of Section 8.1.

The introduction of the optimum property "EAMS - $(\tilde{\Phi},\tilde{\beta}^*)$" has been motivated in Chapter 7 especially by Theorem 7.3.1, where $\tilde{\Phi}$ is the class of all asymptotically level α tests. A similar theorem can be proved for the case that $\tilde{\Phi}$ is the class of all asymptotically $-\Psi$ tests, under certain assumptions on Ψ. We shall not tire the reader with such a theorem, but only demonstrate how the EAMS - asymptotically $-\Psi$ test can be obtained.

Theorem 8.3.1. *Suppose that the assumptions of Section 8.2 concerning* Ψ *and* $\tilde{\beta}^*$ *are satisfied. Let* $\{\hat{\mu}_\nu\}$ *be a uniformly consistent estimator for* μ *under* H *and define*

$$\hat{\Lambda}_\nu = \Lambda(R_\nu,\hat{\mu}_\nu) .$$

Suppose that for every Λ *there exists a unique MS -* (Ψ,β^0) *test* ψ_Λ *for the testing problem* (8.3.1), *with maximum shortcoming with respect to* β^0 *less than* $1 - \alpha$. *Then the test* $\{\phi_\nu\}$ *with*

$$\phi_\nu(Y_\nu) = \psi_{\hat{\Lambda}_\nu}(B_{\hat{\Lambda}_\nu} Y_\nu)$$

is a sharply consistent EAMS - $(\tilde{\Psi},\tilde{\beta}^*)$ *test.*

PROOF. The assumption that the maximum shortcoming of ψ_Λ is less than $1 - \alpha$ implies that $(B_\Lambda K) \cap 0^+(\mathrm{acc}\ \psi_\Lambda) = \{0\}$; use Proposition 2.8.2, Corollary 2.8.1 (ii) and Lemma 8.2.1 to prove this. The uniqueness of the MS tests ψ_Λ implies that ψ_Λ is a weakly* continuous function of Λ; this is proved just like Lemma 8.1.1 (iii).

The proof of this theorem proceeds like that of Theorem 8.1.1, with some minor modifications. Let $\phi_\Lambda(x) = \psi_\Lambda(B_\Lambda x)$ and replace $\tilde{\Phi}$ by $\tilde{\Psi}$.

(i,ii) See the proof of Theorem 8.1.1.

(iii) Let $\{\mu_{0\nu}\}$, $\{\xi\}$, μ_0, Λ and $\tilde{Y}_\xi$ be as in (ii). Define M_ξ and K as in the proof of Theorem 8.1.1. Let $\Phi_\xi = \Psi_\Lambda$ for all ξ, and let γ_Λ be the minimax shortcoming - (Ψ,β^0) for testing problem (8.3.1). Then the conditions of Theorem 4.4.1 are satisfied for the subsequence $\{\xi\}$. Corollary 4.2.2 (i), assumption (i) concerning $\tilde{\beta}^*$, and Proposition 8.2.1 show that

$$\sup_{\{\mu_\xi\}\in K} \limsup_{\xi} \gamma_\xi(\phi_\Lambda,\mu_\xi) \le \gamma_\Lambda \le$$

$$\le \inf_{\{\phi'_\xi\}\in\tilde{\Psi}} \sup_{\{\mu_\xi\}\in K} \liminf_{\xi} \gamma_\xi(\phi'_\xi,\mu_\xi) .$$

(iv) It follows from (ii) that (8.2.1) holds with $\psi_\nu = \psi_{\Lambda(R_\nu,\mu_\nu)}$. So $\{\phi_\nu\} \in \tilde{\Psi}$.

(v,vi) See the proof of Theorem 8.1.1. The sentence before the last one must be replaced by the following two sentences. In the second case the uniqueness of the MS test ψ_Λ for testing problem (8.3.1) implies, with (iii) above, Proposition 8.2.2 and Corollary 4.2.2 (ii), that $E_{\mu_{0\xi}} \mid \phi'_\xi - \phi_\Lambda \mid \to 0$. It follows from Proposition 4.1.1 that $E_{\mu_\xi} \mid \phi'_\xi - \phi_\xi \mid \to 0$. □

Corollary 8.3.1. *Suppose that the assumptions of Section 8.2 concerning* Ψ *are satisfied. Let* $\{\hat{\mu}_\nu\}$ *be a uniformly consistent estimator for* μ *under* H *and define*

$$\hat{\Lambda}_\nu = \Lambda(R_\nu,\hat{\mu}_\nu) .$$

Suppose that for every Λ *there exists a unique MS* $-\Psi$ *test* ψ_Λ *for the testing problem* (8.3.1)*, with maximum shortcoming with respect to* Ψ *less than* $1-\alpha$ *and with* $B_\Lambda K \cap 0^+(\text{acc } \psi_\Lambda) = \{0\}$. *Then the test* $\{\phi_\nu\}$ *with*

$$\phi_\nu(Y_\nu) = \psi_{\hat{\Lambda}_\nu}(B_{\hat{\Lambda}_\nu} Y_\nu)$$

is a sharply consistent EAMS - asymptotically -Ψ *test.*

PROOF. This follows from Theorem 8.3.1 and Proposition 8.2.2. □

Corollary 8.3.1 demonstrates, e.g., that the tests which are derived as "approximately most stringent somewhere most powerful" in SCHAAFSMA (1966), are everywhere asymptotically most stringent - asymptotically - linear. Instead of "asymptotically-linear", one can say "asymptotically - somewhere most powerful". The term "asymptotically-linear" is shorter and indicates the computational simplicity of this class of tests; the term "asymptotically-somewhere most powerful" indicates a property of the (asymptotic) power functions. EAMS - asymptotically-linear tests for some testing problems from practice are given in Chapter 9.

Essential for a sensible application of Corollary 8.3.1 (or Theorem 8.3.1) is the judicious choice of Ψ (and β^0). This choice will have to ensure that the MS - Ψ test for testing problem (8.3.1) (i) can be constructed explicitly, (ii) has attractive power properties and (iii) is computationally not too demanding. The class of linear tests satisfies (i) and (iii), and (ii) to a certain but not altogether satisfactory extent; see the discussion in part 4 of Section 3.3. The class of likelihood ratio tests against circular cone alternatives seems to be more promising from the point of view of power properties.

8.4. CONDITIONALLY -Ψ TESTS

The class $\tilde{\Psi}$ of all asymptotically - Ψ tests is but one instance of a class of tests for the asymptotic testing problem, related to the class Ψ of tests for the limiting problems (8.3.1). The present section is devoted to an alternative approach, using the concept of a conditionally -Ψ test. At first sight this approach looks attractive, but we shall see that it is doubtful whether it is of much practical use.

For the testing problems of Chapter 9, the subset Θ_H of the natural parameter space corresponding to the null hypothesis is the intersection of the natural parameter space Θ with a linear subspace. This implies that the experiment

$$(8.4.1) \qquad (\mathbb{R}^m,\ \{L_\theta(\underset{\cdot}{X}^{(\nu)}) \mid \theta \in \Theta_H\})$$

admits a complete sufficient statistic (see LEHMANN (1959), Section 4.3, for the concept of completeness of a class of probability distributions). It can be convenient to perform a test conditionally on the outcome of a statistic which is complete sufficient for the experiment (8.4.1): the resulting conditional testing problem has a simple null hypothesis.

The assumptions of Section 8.2 concerning Ψ and $\tilde{\beta}^*$ will be supplied with the following assumptions concerning $\{\hat{\mu}_\nu\}$. For all testing problems of Chapter 9, these assumptions are satisfied if one takes $\hat{\mu}_\nu$ as the uniformly minimum variance unbiased estimator for μ under H. (See FERGUSON (1967), Section 3.6 for this concept; compare our remark following Theorem 5.6.1.)

<u>ASSUMPTIONS CONCERNING $\{\hat{\mu}_\nu\}$</u>.

(i) For every ν, $\hat{\mu}_\nu$ is a complete sufficient statistic for the experiment (8.4.1).

(ii) The sequence $\{\hat{\mu}_\nu\}$ is a uniformly consistent estimator for μ under H.

(iii) There exist versions of the conditional distributions of Y_ν given $\hat{\mu}_\nu$ under H, such that for all $\{\mu_\nu\} \in S_H$ with $\Lambda(R_\nu,\mu_\nu) \to \Lambda$, one has

$$L(B_\Lambda Y_\nu \mid \hat{\mu}_\nu = \mu_\nu) \to N_r(0,I) \ .$$

It may be possible to deduce assumption (iii) from assumption (i); the author has not succeeded in finding a proof of this implication.

<u>DEFINITION 8.4.1</u>. The class $\bar{\Psi}$ of conditionally $-\Psi$ tests is the class of all tests $\{\phi_\nu\}$ such that for every ν, ϕ_ν has the form

$$\phi_\nu(Y_\nu) = \Psi_\nu(B_{\hat{\Lambda}_\nu} Y_\nu;\ \hat{\mu}_\nu)$$

where $\psi_\nu(\cdot\,;\ \mu) \in \Psi$ for every μ, and where $\hat{\Lambda}_\nu = \Lambda(R_\nu,\hat{\mu}_\nu)$.

The class of conditionally $-\Psi$ tests is defined not in terms of asymptotic properties of the test $\{\phi_\nu\}$ (like the class of asymptotically $-\Psi$ tests), but in terms of the form of ϕ_ν; this may be a reason for preferring conditionally $-\Psi$ tests.

The EAMS - $(\tilde{\Psi},\tilde{\beta}^*)$ test of Theorem 8.3.1 is a conditionally $-\Psi$ test; one might expect (the author did) that this test is also EAMS - $(\bar{\Psi},\tilde{\beta}^*)$. However, it turns out that this is the case if and only if for every Λ we have

$$(8.4.2) \qquad \sup_{\tau \in T_\Lambda} \ \inf_{\psi \in \Psi} \ \gamma(\psi,\tau) = \inf_{\psi \in \Psi} \ \sup_{\tau \in T_\Lambda} \ \gamma(\psi,\tau) \ ,$$

where T_Λ is the class of probability distributions with finite support on $B_\Lambda K$, and

$$\gamma(\psi,\eta) = \beta^0(\eta) - E_{N(\eta,I)}\psi \qquad \eta \in B_\Lambda K$$

$$\gamma(\psi,\tau) = \int \gamma(\psi,\eta)\, d\,\tau(\eta) \qquad \tau \in T_\Lambda \ .$$

For many "manageable" classes Ψ, condition (8.4.2) is not satisfied. E.g., it can be shown by examples that if Ψ is the class of linear tests, then

(8.4.2) will be violated (try r = 2). If (8.4.2) does not hold, then it can be proved by means of a minimax theorem that

$$\inf_{\psi \in \Psi'} \sup_{\tau \in T_\Lambda} \gamma(\psi,\tau) = \sup_{\tau \in T_\Lambda} \inf_{\psi \in \Psi} \gamma(\psi,\tau) <$$

$$< \inf_{\psi \in \Psi} \sup_{\tau \in T_\Lambda} \gamma(\psi,\tau),$$

where Ψ' denotes the convex hull of Ψ. This implies that in order to minimize the maximum shortcoming, it is advantageous to randomize the choice of $\psi \in \Psi$. This randomization can be effected by a conditionally -Ψ test but not by an asymptotically -Ψ test. To see how this can be done, suppose that $\psi' = \Sigma_{j=1}^{h} t_j \psi_j$ is a convex combination of $\psi_j \in \Psi$ with

$$\text{(8.4.3)} \qquad \sup_{\tau \in T_\Lambda} \gamma(\psi',\tau) < \inf_{\psi \in \Psi} \sup_{\tau \in T_\Lambda} \gamma(\psi,\tau).$$

Then $\mu(\Theta_H)$ can be partioned into sets $C_{\nu 1}, \ldots, C_{\nu h}$ in such a way that $P_{\mu_\nu}\{\hat{\mu}_\nu \in C_{\nu j}\} \to t_j$ for all j and all $\{\mu_\nu\} \in \mathcal{S}_H$. (E.g., let μ_1 be a coordinate of $\mu \in \mu(\Theta_H)$ such that the variance of $\hat{\mu}_{\nu 1}$ is of the order of magnitude of $[n(\nu)]^{-1}$, let $0 < \varepsilon < \frac{1}{2}$ and define

$$C_{\nu j 1} = [n(\nu)]^{-\frac{1}{2}-\varepsilon} \left(\left[\sum_{g=1}^{j-1} t_g, \sum_{g=1}^{j} t_g\right) + \mathbb{Z}\right)$$

$$C_{\nu j} = \{\mu \in \mu(\Theta_H) \mid \mu_1 \in C_{\nu j 1}\} .$$

These sets $C_{\nu j}$ have the required property.)

The test $\{\phi'_\nu\}$ with

$$\phi'_\nu(Y_\nu) = \sum_{j=1}^{h} \psi_j(B_{\hat{\Lambda}_\nu} Y_\nu) \, I_{C_{\nu j}}(\hat{\mu}_\nu)$$

is a conditionally -Ψ test with the property that for all $\{\mu_\nu\} \in \mathcal{S}_H$ with $\Lambda(R_\nu,\mu_\nu) \to \Lambda$ and for all $\{\mu_{1\nu}\} \in \mathcal{S}_A$ with $\{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}$ one has

$$\limsup_{\nu} \gamma_\nu(\phi'_\nu,\mu_{1\nu}) \leq \sup_{\tau \in T_\Lambda} \gamma(\psi',\tau);$$

whereas for the EAMS - $(\tilde{\Phi},\tilde{\beta}^*)$ test $\{\phi_\nu\}$ there exists a $\{\mu_{1\nu}\} \in \mathcal{S}_A$ with

$\{\mu_{1\nu}\} \Leftrightarrow \{\mu_\nu\}$ and

$$\liminf_{\nu} \gamma_\nu(\phi_\nu,\mu_{1\nu}) = \inf_{\psi\in\Psi} \sup_{\tau\in T_\Lambda} \gamma(\psi,\tau).$$

With (8.4.3), this implies that $\{\phi_\nu\}$ is not EAMS - $(\bar{\Phi},\tilde{\beta}^*)$.

The EAMS - $(\bar{\Phi},\tilde{\beta}^*)$ test, for problems where (8.4.2) does not hold, will be a test where the estimator $\hat{\mu}_\nu$ is not only used to estimate the value of μ under H, but also to randomize between tests in Ψ. This is a very "unnatural" use of $\hat{\mu}_\nu$; the concept of an EAMS - conditionally -Ψ test is relevant only if (8.4.2) is satisfied. It seems to be typical for classes of tests Ψ which are "manageable" and continuously parametrized by a subset of $\mathbb{R}^s$, that they do not satisfy (8.4.2). We have no practical examples of EAMS - conditionally -Ψ tests.

For this reason we refrain from giving a proof of the result, that if the assumptions of Theorem 8.3.1, the assumptions concerning $\{\hat{\mu}_\nu\}$, and equation (8.4.2) are satisfied, then the EAMS - $(\tilde{\Psi},\tilde{\beta}^*)$ of Theorem 8.3.1 is also EAMS - $(\bar{\Psi},\tilde{\beta}^*)$.

CHAPTER 9

EAMS - ASYMPTOTICALLY - Ψ TESTS FOR SOME TESTING PROBLEMS FOR CONTINGENCY TABLES

EAMS - asymptotically -Ψ tests are presented for several testing problems for contingency tables, for suitable classes Ψ. These tests can be considered as versions of well-known tests such as the Wilcoxon-Mann-Whitney two sample test, the Kruskal-Wallis k sample test, etcetera, with an EAMS treatment of ties.

9.1. INTRODUCTION

In this chapter, some testing problems for two-dimensional (in the case of univariate symmetry: one-dimensional) contingency tables

$$(n_{ij})_{1 \leq i \leq k; 1 \leq j \leq m}$$

are studied. The contingency table (n_{ij}) will be considered to be the outcome of a matrix of random variables

$$(N_{ij})_{1 \leq i \leq k; 1 \leq j \leq m} \quad .$$

The probabilistic assumptions concerning (N_{ij}) will be specified in each of the following sections.

Many examples of testing problems where the data can be arranged in contingency tables have been presented in this study: see Sections 1.1, 3.1, 3.4 and 7.1. Optimal tests for some testing problems of this kind are derived in Examples 4.3.2, 5.6.1, 6.4.1(i) and 8.1.1, and in Section 5.7. In Section 3.4 a few approaches by other authors to these problems are briefly reviewed. All testing problems of this chapter are instances of the testing problem of Section 3.1, after the reduction by sufficiency.

It is demonstrated in Section 8.1 how the everywhere asymptotically most stringent - level α (EAMS - level α) test for the testing problems of this chapter can be constructed, if the most stringent - level α test for the "limiting problem" is known. Unfortunately the MS - level α tests for the "limiting problems" for the testing problems of this chapter are not explicitly known, except for some very special cases. We content ourselves with the approach of Sections 8.2 and 8.3, and present EAMS tests in certain subclasses of the class of all asymptotically level α tests.

In Sections 9.2, 9.4 and 9.5 attention is focused on asymptotically - linear tests; these are related to tests based on linear test statistics for the limiting testing problems. The tests presented are EAMS among the asymptotically linear tests. They are also conditionally - linear tests: the test statistics are linear combinations of the N_{ij}'s, conditionally given the outcomes of the marginal totals $(N_{1+}, \ldots, N_{k+})$ and $(N_{+1}, \ldots, N_{+m})$, for Sections 9.2 and 9.3. (In Section 9.5, other marginal statistics are used for the conditioning.) Many tests which are used in practice for these testing problems are asymptotically - linear and/or conditionally - linear; see Section 3.4.

Asymptotically - linear tests are not suitable for the testing problem of Section 9.3. Attention is focused there on asymptotically - squared means tests; these are related to tests for the limiting testing problems which are based on sums of squared standardized sample means.

In part 4 of Section 3.3 some of the pros and cons of linear tests for testing problems with restricted alternatives for normal distributions are discussed. The same kind of arguments can be used in a discussion about the merits of asymptotically - linear tests for the testing problems of this chapter. For these testing problems, the region in the parameter space corresponding to the alternative hypothesis (if the parametrization is done, as usual, by the probabilities of the different outcomes; in Section 9.4, conditional probabilities have to be used) is a cone. The power properties of the EAMS - asymptotically - linear test are excellent for "directions in the center of the cone" and rather poor for "directions near the edges of the cone". The maximum shortcoming of this test at the edges of the cone increases with k and m. (The testing problem itself becomes intrinsically more difficult for larger values of k and m; the minimax shortcoming tends to $1 - \alpha$ as $km \to \infty$.)

It seems desirable to look for other classes Ψ of tests for the limiting problem, with the properties that the EAMS - asymptotically - Ψ tests are "near the edges" more attractive than the EAMS - asymptotically - linear test and "near the center" not too much less attractive. The class of likelihood - ratio tests against circular cone alternatives (see part 5 of Section 3.3) seems to be promising; this class of tests is currently being studied by Mr. Hans Akkerboom. Among the asymptotically - linear tests, however, the EAMS - asymptotically - linear tests may be preferred from the point of view of power properties; unless there are reasons to give particular attention to a certain subalternative, and to use a test which is good for testing against this subalternative (compare the discussion at the end of Section 3.2).

The tests presented in Sections 9.2, 9.4 and 9.5 are the tests derived in Chapters 8,9 and 10 of SCHAAFSMA (1966) as "approximately" most stringent - somewhere most powerful tests; see also part 4 of Section 3.3, and Section 3.4. These tests are derived by Schaafsma as the most stringent - somewhere most powerful tests for the limiting problems (the somewhere most powerful tests for the limiting problem are based on linear test statistics), but without a theoretical justification of this method. It follows immediately from Corollary 8.3.1 that they are EAMS - asymptotically - linear. A similar method is used in Appendix A.6 in order to derive the test presented in Section 9.3.

9.2. TESTING HOMOGENEITY AGAINST TREND IN A SPECIFIED DIRECTION

Consider the problem of testing homogeneity against trend for k independent random samples from probability distributions with m possible outcomes. The experimental data can be condensed into a k × m contingency table

		outcome						sample size
		1	2	.	.	.	m	
s	1	n_{11}	n_{12}	.	.	.	n_{1m}	n_1
a	2	n_{21}	n_{22}	.	.	.	n_{2m}	n_2
m	.	.	.	.	.	.	.	.
p	.	.	.	.	.	.	.	.
l	.	.	.	.	.	.	.	.
e	k	n_{k1}	n_{k2}	.	.	.	n_{km}	n_k
total		n_{+1}	n_{+2}	.	.	.	n_{+m}	n

where n_i is the size of the i'th sample, and n_{ij} is the frequency count of outcome j for the i'th sample. The probability model states that for every i, $(n_{i1}, \dots, n_{im})$ is the outcome of a random variable $N_i = (N_{i1}, \dots, N_{im})$ which has the multinomial distribution with parameters n_i and $p_i =$ $= (p_{i1}, \dots, p_{im})$, while $N_1, N_2, \dots, N_k$ are independent. The parameters p_i satisfy

$$p_{ij} > 0, \qquad \sum_{j=1}^{m} p_{ij} = 1.$$

The null hypothesis of homogeneity

$$H : p_1 = p_2 = \dots = p_k$$

is to be tested against the one-sided alternative of an upward trend

$$A_1 : \sum_{j=h}^{m} p_{i+1,j} \geq \sum_{j=h}^{m} p_{ij} \qquad 2 \leq h \leq m, \quad 1 \leq i \leq k-1 ,$$

with at least one inequality strict;

or against the two-sided alternative of an upward or a downward trend

A_2 : either A_1 holds, or

$$\sum_{j=h}^{m} p_{i+1,j} \leq \sum_{j=h}^{m} p_{ij} \qquad 2 \leq h \leq m, \quad 1 \leq i \leq k-1 ,$$

with at least one inequality strict.

We restrict attention to one-sided (for A_1) or two-sided (for A_2) tests, based on test statistics of the form

$$\Sigma_{i,j}\, a_{ij}\, N_{ij}$$

where the weights a_{ij} are allowed to depend on the outcome $(n_{+1}, \dots n_{+m})$ of the combined sample and on the sample sizes:

$$a_{ij} = a_{ij}\,(n_{+1}, \dots, n_{+m};\, n_1, \dots, n_k),$$

and where the weights satisfy, asymptotically for $n \to \infty$, certain continuity conditions. (More precisely: we restrict attention to tests which are asymptotically - linear and conditionally - linear, as defined in Sections 8.2 and 8.4.) The weights a_{ij} must be determined in an "optimal" way; we use the optimum property "everywhere asymptotically most stringent".

The EAMS - asymptotically - linear level α test for testing H against A_1 rejects the null hypothesis if

$$(9.2.1) \qquad (S^2)^{-\frac{1}{2}} \sum_{i=1}^{k} b_i \sum_{j=1}^{m} a_j \frac{N_{ij}}{n_i} > u_\alpha$$

where

$$a_j = \sum_{h=1}^{j-1} \left\{ \frac{n}{N_{+h}} + \frac{n}{N_{+,h+1}} \right\}^{\frac{1}{2}}, \quad a_1 = 0$$

$$b_i = -\{c_i\,(n-c_i)\}^{\frac{1}{2}} + \{c_{i-1}\,(n-c_{i-1})\}^{\frac{1}{2}}$$

$$c_i = \sum_{g=1}^{i} n_g, \quad c_0 = 0$$

$$S^2 = \frac{1}{n-1} \{\sum_{i=1}^{k} n_i^{-1} b_i^2\}\{\sum_{j=1}^{m} N_{+j}(a_j - a.)^2\}$$

$$a. = \sum_{j=1}^{m} a_j \frac{N_{+j}}{n} .$$

Note the $a_1 < a_2 < \ldots < a_m$ and that a small value of n_{+h} leads to wide spacings between the weights a_{h-1}, a_h and a_{h+1}. For the frequently used midrank weights, a small value of n_{+h} leads to small spacings between the weights for the indices $h-1$, h and $h+1$. The concavity of the function $t \mapsto \{t(1-t)\}^{\frac{1}{2}}$ for $0 \le t \le 1$ implies that $b_i/n_i \le b_{i+1}/n_{i+1}$. Furthermore, $\Sigma_i\, b_i = 0$; this implies that the test statistic (9.2.1) remains unaltered if a_j is replaced, for all j, by $a_j - a_*$ where a_* is arbitrary but independent of j. For numerical reasons it can be recommendable to replace (for the computation) a_j by $a_j - a_*$ where a_* is equal or close to a. .

The test statistic has under the null hypothesis, conditionally given $(n_{+1}, \ldots, n_{+m})$, mean 0 and variance 1. The test statistic can be regarded as a linear combination of sample means

$$\sum_{j=1}^{m} a_j \frac{N_{ij}}{n_i} \;;$$

this sample mean can be interpreted as a measure for the tendency of the i'th sample towards high outcomes.

The EAMS - asymptotically - linear level α test for testing H against A_2 is the natural two-sided analogue of (9.2.1), and rejects the null hypothesis if

$$(9.2.2) \qquad \left| (S^2)^{-\frac{1}{2}} \sum_{i=1}^{k} b_i \sum_{j=1}^{m} a_j \frac{N_{ij}}{n_i} \right| > u_{\frac{1}{2}\alpha} \; ,$$

where a_j, b_i and S^2 are as above.

SCHAAFSMA (1966, Section 8.2) treats also the problem of the combination of r independent problems of testing homogeneity against an upward trend in $k \times m$ tables. He proposes a test which is (in terms of the present study) EAMS - asymptotically - linear level α. The maximum shortcoming of this test will be rather large, however, unless r, k and m are very small. If the r sample sizes are not too different, and if there is no reason to desire that the combination test is powerful especially for a certain subalternative of the alternative hypothesis "there is an upward trend for at least one of the r testing problems", it may be preferable to combine several independent test statistics (9.2.1) and/or (9.2.2) by Fisher's method of combining tests (see Section 3.2, part 5 and Section 3.3, part 1) rather than by the procedure proposed by Schaafsma.

In the case $k = 2$, the tests (9.2.1) and (9.2.2) can be regarded as versions of the Wilcoxon-Mann-Whitney two-sample test, with an EAMS treatment of ties. In that case, the rejection regions (9.2.1) and (9.2.2) can be written in simplified form as

$$(9.2.3) \qquad (S^2)^{-\frac{1}{2}} \sum_{j=1}^{m} a_j \left(\frac{N_{2j}}{n_2} - \frac{N_{1j}}{n_1} \right) > u_\alpha$$

$$(9.2.4) \qquad \left| (S^2)^{-\frac{1}{2}} \sum_{j=1}^{m} a_j \left(\frac{N_{2j}}{n_2} - \frac{N_{1j}}{n_1} \right) \right| > u_{\frac{1}{2}\alpha}$$

where a_j and $a.$ are as above and

$$S^2 = \frac{n}{n_1 n_2 (n-1)} \sum_{j=1}^{m} N_{+j} \, (a_j - a.)^2 \; .$$

In the case $k \geq 3$, the tests (9.2.1) and (9.2.2) can be regarded, respectively, as one-sided and two-sided trend analogues of the Kruskal-Wallis test, with an EAMS treatment of ties. They are related to the Jonckheere-Terpstra test which is not conditionally-linear (it is linear conditionally given a set of statistics different from $(N_{+1}, \ldots, N_{+m})$), but which is an asymptotically-linear test. So the result that the tests (9.2.1) and (9.2.2) are EAMS - asymptotically - linear demonstrates that, in the sense of the optimum property "EAMS", these tests can be preferred over the Wilcoxon-Mann-Whitney test (for $k = 2$) and the Jonckheere-Terpstra test (for $k \geq 3$).

In the case $m = 2$, the testing problem considered is the problem of testing homogeneity against an upward trend for k probabilities. This problem is treated also in Example 6.4.1 (i), where the asymptotically most stringent - level α test is derived. This is one of the few testing problems with restricted alternatives for contingency table, where the "local minimax shortcoming" (see Section 1.3) is constant, and where the AMS - level α test is asymptotically unique. The AMS - asymptotically - linear level α test is also asymptotically unique and given by (9.2.1) for the one-sided and by (9.2.2) for the two-sided case. For $m = 2$ these rejection regions can be written in simplified form as

$$(9.2.5) \qquad (S^2)^{-\frac{1}{2}} \sum_{i=1}^{k} b_i \frac{N_{i2}}{n_i} > u_\alpha$$

$$(9.2.6) \qquad \left| (S^2)^{-\frac{1}{2}} \sum_{i=1}^{k} b_i \frac{N_{i2}}{n_i} \right| > u_{\frac{1}{2}\alpha} \quad ,$$

where b_i is as above and

$$S^2 = \frac{N_{+1} \, N_{+2}}{n(n-1)} \sum_{i=1}^{k} \frac{b_i^2}{n_i} \quad .$$

For this case $(m = 2)$ one can also apply the variance - stabilizing - arc-sine - transformation (see RAO (1973), Section 6 g.3): the tests (9.2.5) and (9.2.6) can be demonstrated to be asymptotically equivalent to the tests which reject for

$$(9.2.7) \qquad 2(S^2)^{-\frac{1}{2}} \sum_{i=1}^{k} b_i \arcsin \left(\frac{N_{i2}}{n_i}\right)^{\frac{1}{2}} > u_\alpha$$

$$(9.2.8) \qquad \left| 2(S^2)^{-\frac{1}{2}} \sum_{i=1}^{k} \arcsin \left(\frac{N_{i2}}{n_i} \right)^{\frac{1}{2}} \right| > u_{\frac{1}{2}\alpha} \quad ,$$

where b_i is as above and

$$S^2 = \sum_{i=1}^{k} \frac{b_i^2}{n_i} \quad .$$

In the case $m = k = 2$, the testing problem for the 2×2 table is obtained of which the one-sided case was treated in Example 5.6.1, where Fisher's test was demonstrated to be asymptotically uniformly most powerful - level α. This test is for $m = k = 2$ asymptotically equivalent to the tests (9.2.1), (9.2.3), (9.2.5), (9.2.7). For the two-sided case with $m = k = 2$, the alternative A_2 is the unrestricted alternative, and the tests (9.2.2), (9.2.4), (9.2.6), (9.2.8) are all asymptotically equivalent to the χ^2 test for the 2×2 table which is AMS - level α according to the remark following Corollary 6.3.1.

9.3. TESTING HOMOGENEITY AGAINST TREND WITHOUT A SPECIFIED DIRECTION

Consider the testing problem where the data are of the kind of Section 9.2, with the same probabilistic assumptions. The null hypothesis of homogeneity

$$H : p_1 = p_2 = \ldots = p_k$$

is to be tested against the alternative of a trend in an unspecified direction

A : a permutation $(i_1, i_2, \ldots, i_k)$ of $(1, 2, \ldots, k)$ exists such that $\Sigma_{j=h}^{m} p_{i_{r+1}j} \geq \Sigma_{j=h}^{m} p_{i_r j}$, for $2 \leq h \leq m$, $1 \leq r \leq k-1$; with at least one inequality strict.

For $k = 2$ this alternative hypothesis is identical to alternative hypothesis A_2 of Section 9.2. The testing problem of Section 1.1 is an instance of this testing problem, with $k = 3$ and $m = 4$.

Linear test statistics are not suitable for this testing problem. In this section, we restrict attention to test statistics of the form

$$(9.3.1) \qquad \sum_{i=1}^{k} n_i^{-1} \{ \sum_{j=1}^{m} a_j (N_{ij} - n_i n^{-1} N_{+j}) \}^2 ,$$

where the weights a_j are allowed to depend on the outcome $(n_{+1}, \ldots, n_{+m})$ of the combined sample and on the sample sizes $n_1, \ldots, n_k$. The statistic (9.3.1) can be interpreted as a sum of squared "standardized" sample means, where outcome j has received score a_j. The corresponding class of tests for the limiting problem will be called the class of "squared means" tests.

The optimal weights, resulting in an EAMS - asymptotically - squared means level α test, are derived in Appendix A.6. This test rejects the null hypothesis H if

$$(9.3.2) \qquad S^{-2} \sum_{i=1}^{k} n_i^{-1} \{ \sum_{j=1}^{m} a_j (N_{ij} - n_i n^{-1} N_{+j}) \}^2 > \chi^2_{k-1;\alpha} ,$$

where

$$a_j = \sum_{h=1}^{j-1} \{ \frac{n}{N_{+h}} + \frac{n}{N_{+,h+1}} \}^{\frac{1}{2}} , \quad a_1 = 0$$

$$S^2 = \frac{1}{n-1} \sum_{j=1}^{m} N_{+j} (a_j - a_.)^2$$

$$a_. = \sum_{j=1}^{m} a_j \frac{N_{+j}}{n} .$$

Note that the weights a_j are identical to those of Section 9.2. Again, the test statistic (9.3.2) remains unaltered if a_j is replaced by $a_j - a_*$ for all j; it can be recommendable for numerical reasons to replace, for the computation, a_j by $a_j - a_*$ where a_* is equal or close to $a_.$.

The test (9.3.2) can be regarded as a version of the Kruskal-Wallis test, with an EAMS treatment of ties. For $m = 2$ the alternative hypothesis A is the unrestricted alternative, and the test (9.3.2) is the usual χ^2 test for the $k \times 2$ table, except for a factor $1 - n^{-1}$ in the test statistic.

As an example, the test (9.3.2) will be applied to the data for the fish indicated by P6 and studied by VODEGEL (1978) in the experiment described in Section 1.1. These data are given by the following table. The weights $a_j - a_.$ are given in the last row.

		behavioural category				sample size
		1	2	3	4	
d	small	80	55	18	6	159
u						
m	medium	113	81	26	9	229
m						
y	large	20	149	29	4	202
total		213	285	73	19	590
$a_j - a.$		-2.10	.10	3.28	9.54	

The outcome of the test statistic (9.3.2) is 14.3. The null hypothesis is rejected at the level of significance $\alpha = .001$. It may be interesting to know that VODEGEL (1978) proceeds from the rejection of the null hypothesis H to the construction of more refined models.

9.4. TESTING INDEPENDENCE AGAINST "POSITIVE DEPENDENCE"

Consider the problem of testing independence against "positive dependence" for a random sample from a bivariate probability distribution with k x m possible outcomes. The experimental data can be condensed into a k x m contingency table

			outcome variable 2							total
			1	2					m	
o	v	1	n_{11}	n_{12}	.	.	.		n_{1m}	n_{1+}
u	a	2	n_{21}	n_{22}	.	.	.		n_{2m}	n_{2+}
t	r	.	.	.	.	.	.		.	.
c	.	.	.	.	.	.	.		.	.
o		.	.	.	.	.	.		.	.
m	1	.	.	.	.	.	.		.	.
e		k	n_{k1}	n_{k2}	.	.	.		n_{km}	n_{k+}
total			n_{+1}	n_{+2}	.	.	.		n_{+m}	n

where n_{ij} is the frequency count of outcome (i, j) and n is the sample size. The probability model states that $(n_{11}, n_{12}, \ldots, n_{1m}, n_{21}, \ldots, n_{km})$ is the outcome of a random variable $(N_{11}, N_{12}, \ldots, N_{km})$ which has the multinomial distribution with parameters n and $p = (p_{11}, p_{12}, \ldots, p_{1m}, p_{21}, \ldots, p_{km})$. The parameter p satisfies

$$p_{ij} > 0, \qquad \sum_{i=1}^{k} \sum_{j=1}^{m} p_{ij} = 1 \ .$$

The null hypothesis of independence

$$H : p_{ij} = p_{i+}p_{+j} \qquad \text{for all } (i,j) \ ,$$

where $p_{i+} = \Sigma_j \, p_{ij}$ and $p_{+j} = \Sigma_i \, p_{ij}$, is to be tested against an alternative hypothesis which specifies the concept of "positive dependence". Two such alternatives are considered.

The first one is the alternative of positive regression dependence of variable 2 on variable 1,

$$A_1 \ : p_{i+}^{-1} \sum_{j=h}^{m} p_{ij} \leq p_{i+1,+}^{-1} \sum_{j=h}^{m} p_{i+1,j}$$

for $1 \leq i \leq k-1$, $2 \leq h \leq m$; with at least one inequality strict.

This concept of dependence is discussed in LEHMANN (1959, 1966). It is not symmetric in two variables, and it can be relevant especially if, from a methodological point of view, variable 1 is regarded as the "independent variable" and variable 2 as the "dependent variable ".

The second alternative is that of positive quadrant dependence

$$A_2 \ : \sum_{i=1}^{g} \sum_{j=1}^{h} p_{ij} \geq \sum_{i=1}^{g} p_{i+} \sum_{j=1}^{h} p_{+j}$$

for $1 \leq g \leq k-1$, $1 \leq h \leq m-1$; with at least one inequality strict.

This concept of dependence was introduced by LEHMANN (1966) and SCHAAAFSMA (1966). It is symmetric in the two variables.

We restrict attention to one-sided tests, based on test statistics of the form

$$\Sigma_{i,j}\, a_{ij}\, N_{ij}$$

where the weights a_{ij} are allowed to depend on the marginal outcomes $(n_{+1}, \ldots, n_{+m})$ and $(n_{1+}, \ldots, n_{k+})$, in such a way that the resulting tests are asymptotically - linear and conditionally linear, as defined in Sections 8.2 and 8.4. The weights a_{ij} must be determined in an "optimal" way; we use the optimum property "everywhere asymptotically most stringent".

The EAMS - asymptotically - linear level α test for testing H against A_1 is given by the same formulae as the test (9.2.1), if n_i is replaced by n_{i+} $(1 \leq i \leq k)$. It may be noted that this testing problem is "transformed" into the testing problem of Section 9.2, with the alternative hypothesis A_1 defined in that section, by conditioning on the outcomes $(n_{1+}, \ldots, n_{k+})$ of the frequency counts $(N_{1+}, \ldots, N_{k+})$ for the first variable.

The EAMS - asymptotically - linear level α test for testing H against A_2 rejects the null hypothesis if

$$(S^2)^{-\frac{1}{2}} \sum_{i=1}^{k} \sum_{j=1}^{m} N_{ij}(a_i - a_.)(b_j - b_.) > u_\alpha \quad , \tag{9.4.1}$$

where

$$a_i = \sum_{g=1}^{i-1} \left\{ \frac{n}{N_{g+}} + \frac{n}{N_{g+1,+}} \right\}^{\frac{1}{2}} \quad , \; a_1 = 0$$

$$a_. = \sum_{i=1}^{k} a_i \frac{N_{i+}}{n}$$

$$b_j = \sum_{h=1}^{j-1} \left\{ \frac{n}{N_{+h}} + \frac{n}{N_{+,h+1}} \right\}^{\frac{1}{2}} \quad , \; b_1 = 0$$

$$b_. = \sum_{j=1}^{m} b_j \frac{N_{+j}}{n}$$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{k} N_{i+}(a_i - a_.)^2 \sum_{j=1}^{m} N_{+j}(b_j - b_.)^2$$

The test statistic (9.4.1), divided by $(n-1)^{\frac{1}{2}}$, can be regarded as a version of Spearman's rank correlation coefficient with an EAMS treatment of ties. The test statistic has under the null hypothesis, conditionally given $(n_{1+}, \ldots, n_{k+})$ and $(n_{+1}, \ldots, n_{+m})$, mean 0 and variance 1.

One can also wish to test the null hypothesis H against one of the two-sided alternative hypotheses which are the natural two-sided analogues of A_1 and A_2. The EAMS - asymptotically linear level α tests for these testing problems are the natural two-sided analogues of the one-sided tests mentioned above (compare Section 9.2.).

9.5. BIVARIATE AND UNIVARIATE SYMMETRY PROBLEMS

Bivariate symmetry problems can occur when samples are drawn from a bivariate probability distribution, where the two variables are similar in a certain sense: e.g., when they represent measurements for the left eye and the right eye, respectively; or the reactions to two treatments when both treatments are administered to the same persons. Consider the problem of testing symmetry against "asymmetry towards high values on the first variable" for a random sample from a bivariate probability distribution with $m \times m$ possible outcomes. The experimental data can be condensed into an $m \times m$ contingency table

			outcome variable 2					total
			1	2	.	.	.	m	
o	v	1	n_{11}	n_{12}	.	.	.	n_{1m}	n_{1+}
u	a	2	n_{21}	n_{22}	.	.	.	n_{2m}	n_{2+}
t	r	.	.	.	.	.	.	.	.
c	.	.	.	.	.	.	.	.	.
o		.	.	.	.	.	.	.	.
m	1	.	.	.	.	.	.	.	.
e		.	.	.	.	.	.	.	.
		m	n_{m1}	n_{m2}	.	.	.	n_{mm}	n_{m+}
total			n_{+1}	n_{+2}	.	.	.	n_{+m}	n

where n_{ij} is the frequency count for outcome (i,j) and n is the sample size. The probability model states that $(n_{11}, n_{12}, \ldots, n_{1m}, n_{21}, \ldots, n_{mm})$ is the outcome of a random variable $(N_{11}, N_{12}, \ldots, N_{mm})$ which has the multinomial distribution with parameters n and $p = (p_{11}, p_{12}, \ldots , p_{1m}, p_{21}, \ldots, p_{mm})$.

The parameter p satisfies

$$p_{ij} > 0, \quad \sum_{i=1}^{m} \sum_{j=1}^{m} p_{ij} = 1 .$$

The null hypothesis of symmetry

$$H : p_{ij} = p_{ji} \qquad \text{for all } (i,j)$$

states that the two variables are exchangeable. It is to be tested against the alternative hypothesis of "asymmetry towards high values on the first variable"

$$A : \sum_{(i,j) \in B} p_{ij} \geq \sum_{(i,j) \in B} p_{ji} \qquad \text{for all } B \in \mathcal{A}$$

with at least one inequality strict,

where $\mathcal{A}$ is the class of all subsets of $\{1, \ldots, m\}^2$ which are increasing in the first coordinate and decreasing in the second coordinate:

$$\mathcal{A} = \{B \subset \{1, \ldots, m\}^2 \mid \text{if } (i,j) \in B \text{ and } i \leq g,\ j \geq h \text{ then } (g,h) \in B\} .$$

This alternative hypothesis was introduced by SCHAAFSMA (1966). It is discussed in SCHAAFSMA and SNIJDERS (1979).

We restrict attention to one-sided tests, based on test statistics of the form

$$\Sigma_{i,j}\, a_{ij}\, N_{ij}$$

where the weights a_{ij} are allowed to depend on the "marginal" outcomes $(n_{ij} + n_{ji})_{i \leq j}$ in such a way, that the resulting tests are asymptotically - linear and conditionally - linear, as defined in Sections 8.2 and 8.4. The weights a_{ij} must be determined in an "optimal" way; we use the optimum property "everywhere asymptotically most stringent".

Unfortunately, the computations involved in the construction of the MS-linear test for the limiting problem become very complex as m increases; the solution is known only for $m = 3$. In that case, the alternative hypothesis can be formulated as

$$A : p_{31} \geq p_{13},\ p_{21} + p_{31} \geq p_{12} + p_{13},\ p_{31} + p_{32} \geq p_{13} + p_{23},$$

$$p_{21} + p_{31} + p_{32} \geq p_{12} + p_{13} + p_{23}\ ,$$

with at least one inequality strict.

The EAMS - asymptotically - linear level α test for testing H against A in the case $m = 3$ rejects the null hypothesis if

$$(9.5.1) \qquad (S^2)^{-\frac{1}{2}} \sum_{1 \leq j < i \leq 3} a_{ij} (N_{ij} - N_{ji}) > u_\alpha$$

where

$$a_{21} = \hat{p}_{12}^{-\frac{1}{2}}\ , \qquad a_{32} = \hat{p}_{23}^{-\frac{1}{2}}\ ,$$

$$a_{31} = \hat{p}_{13}^{-\frac{1}{2}} \left(\{\min(\hat{p}_{12}, \hat{p}_{23})\}^{-\frac{1}{2}} + [\hat{p}_{13}^{-1} + \{\min(\hat{p}_{12}, \hat{p}_{23})\}^{-1}]^{\frac{1}{2}} \right)$$

$$\hat{p}_{ij} = \frac{N_{ij} + N_{ji}}{2n}$$

$$S^2 = \sum_{1 \leq j < i \leq 3} a_{ij}^2 (N_{ij} + N_{ji}) \quad .$$

The test statistic has under the null-hypothesis, conditionally given $(n_{ij} + n_{ji})_{i \leq j}$, mean 0 and variance 1.

In practice, the outcomes (i,j) are often reduced to the differences $i - j$. This is relevant especially if the arithmetical difference $i - j$ is regarded as a good measure for the "size of the conceptual difference between outcome i and outcome j". However, the reduction to the arithmetical differences is often applied not on conceptual grounds, but in order to facilitate the statistical treatment. As this reduction may obscure conclusions which could be drawn from the experimental data, one should be careful with it and not apply it too soon.

If the reduction to the arithmetical differences is applied, then the resulting data can be condensed into a one-dimensional table

outcome							total
$-m$	$-m+1$	.	.	.	$m-1$	m	
n_{-m}	n_{-m+1}	.	.	.	n_{m-1}	n_m	n

where n_j is the frequency count for outcome j and n is the sample size. (The upper bound m corresponds to $m-1$ in the bivariate problem discussed above.) The probability model states that $(n_{-m}, n_{-m+1}, \ldots, n_m)$ is the outcome of a random variable $(N_{-m}, N_{-m+1}, \ldots, N_m)$ which has the multinomial distribution with parameters n and $p = (p_{-m}, p_{-m+1}, \ldots, p_m)$. The parameter p satisfies

$$p_j > 0, \qquad \sum_{j=-m}^{m} p_j = 1.$$

The null hypothesis of symmetry

$$H_u : p_j = p_{-j} \qquad \text{for all } j$$

is to be tested against the alternative hypothesis of "asymmetry towards high values"

$$A_u : \sum_{j=h}^{m} p_j \geq \sum_{j=h}^{m} p_{-j} \qquad \text{for } 1 \leq h \leq m,$$

with at least one inequality strict.

This testing problem sometimes occurs in its own right, not after a reduction from bivariate data to arithmetical differences.

We restrict attention to one-sided tests, based on test statistics of the form

$$\Sigma_j\, a_j\, N_j$$

where the weights a_j are allowed to depend on the "marginal" outcomes $n_{-1} + n_1, n_{-2} + n_2, \ldots, n_{-m} + n_m$ in such a way that the resulting tests are asymptotically - linear and conditionally - linear, as defined in Sections 8.2 and 8.4. The weights a_j must be determined in an "optimal" way; we use the optimum property "everywhere asymptotically most stringent".

The EAMS - asymptotically - linear level α test for testing H_u against A_u rejects the null hypothesis if

$$(9.5.2) \qquad (S^2)^{-\frac{1}{2}} \sum_{j=1}^{m} a_j (N_j - N_{-j}) > u_\alpha$$

where

$$a_j = \hat{p}_1^{-\frac{1}{2}} + \sum_{h=1}^{j-1} (\hat{p}_h^{-1} + \hat{p}_{h+1}^{-1})^{\frac{1}{2}} \ , \quad a_1 = \hat{p}_1^{-\frac{1}{2}}$$

$$\hat{p}_j = \frac{N_{-j} + N_j}{2n}$$

$$S^2 = \sum_{j=1}^{m} a_j^2 (N_{-j} + N_j) \ .$$

The test (9.5.2) can be regarded as a version of the Wilcoxon signed ranks test, with an EAMS treatment of ties. The test statistic has under the null hypothesis, conditionally given $(n_{-1} + n_1, \ldots, n_{-m} + n_m)$ mean 0 and variance 1.

For the testing problems which are the two-sided analogues of the testing problems of this section, the two-sided versions of the tests (9.5.1) and (9.5.2) are EAMS - asymptotically -linear level α tests.

APPENDIX 1

A.1. THE WEAK TOPOLOGY ON THE CLASS OF PROBABILITY MEASURES ON $\mathbb{R}^m$

The weak topology on the class of probability measures in $\mathbb{R}^m$ has been studied extensively, e.g., in PARTHASARATHY (1967). In this appendix, some well-known definitions and results which are basic to this study are presented.

The definition below will be used only for $X = \mathbb{R}^m$ and for $X = [-\infty,+\infty]^m$.

DEFINITION A.1.1. (i) Let X be a metric space. The class of all probability measures on X is denoted by $M_1(X)$. The weak topology on $M_1(X)$ is the weakest topology with respect to which the functions

$$P \mapsto \int f dP$$

are continuous, for all bounded continuous functions $f: X \to \mathbb{R}$.
(ii) Let $\{P_\nu\}$ be a sequence of probability measures on $[-\infty,+\infty]^m$. $\{P_\nu\}$ is tight if for every $\varepsilon > 0$ a number k exists such that

$$\liminf_\nu P_\nu\{x \mid \|x\| \le k\} \ge 1 - \varepsilon.$$

It can be proved that the weak topology on $M_1(\mathbb{R}^m)$ is identical to the relative topology which $M_1(\mathbb{R}^m)$ has as a subset of $M_1([-\infty,+\infty]^m)$, the latter space being considered with its weak topology. This is pleasant for the avoidance of confusion. It may be noted that in most texts, a family A of finite measures on the metric space X is defined to be tight, or uniformly tight, if for every $\varepsilon > 0$ there exists a compact $K \subset X$ with $P(X\backslash K) < \varepsilon$ for all $P \in A$. Definition A.1.1(ii) is more convenient for the purposes of Appendix A.2; for sequences of probability measures on $\mathbb{R}^m$, Definition A.1.1(ii) is equivalent to the usual one.

THEOREM A.1.1.

(i) *If X is a separable metric space, then the weak topology on $M_1(X)$ is separable and metrizable.*

(ii) *A sequence $\{P_\nu\}$ of probability measures on $[-\infty,+\infty]^m$ is tight iff every subsequence of $\{P_\nu\}$ has a further subsequence which converges weakly in $M_1([-\infty,+\infty]^m)$ to a probability measure concentrated on*

$(-\infty,+\infty)^m$.

(iii) *A sequence* $\{P_\nu\}$ *of probability measures on* $\mathbb{R}^m$ *is tight iff it is weakly relatively compact in* $M_1(\mathbb{R}^m)$.

PROOF. See PARTHASARATHY (1967). Note that (iii) is an immediate corollary of (i) and (ii). □

LEMMA A.1.1.

(i) *A sequence* $\{P_\nu\}$ *of probability measures on* $[-\infty,+\infty]$ *is tight iff for every sequence* $\{z_\nu\} \subset \mathbb{R}$ *with* $z_\nu \to \infty$*, one has*

$$P_\nu[-z_\nu,z_\nu] \to 1.$$

(ii) *Let* $\{P_\nu\}$ *be a sequence of probability distributions on* $\mathbb{R}$ *with moments*

$$\mu_\nu = \int x dP_\nu(x), \qquad \sigma_\nu^2 = \int (x-\mu_\nu)^2 dP_\nu(x).$$

Suppose that $\{\sigma_\nu^2\}$ *is a bounded sequence. Then* $\{P_\nu\}$ *is tight iff* $\{\mu_\nu\}$ *is a bounded sequence.*

PROOF. (i) Immediately from the definition.
(ii) Follows from (i) with Chebychev's inequality. □

PROPOSITION A.1.1. *If* X *is a metric space, then* $P_\nu \to P$ *weakly in* $M_1(X)$ *if and only if*

$$P_\nu(A) \to P(A)$$

for all Borel-measurable $A \subset X$ *with* $P(\partial A) = 0$.

PROOF. See PARTHASARATHY (1967). □

APPENDIX 2

A.2. CONTIGUITY

The concept of contiguity has been introduced in Section 4.1. In this appendix some well known results are derived, which are essentially present already in LE CAM (1960). The other references mentioned in Section 4.1 also contain proofs of most of these results, or of slightly different versions of them.

In the following theorem, for every $\nu \in \mathbb{N}$, P_ν and Q_ν will be probability measures on $(\mathcal{X}_\nu, \mathcal{F}_\nu)$ and L_ν is a version of the likelihood ratio statistic dQ_ν/dP_ν defined in the following way. For every ν, a set $C_\nu \in \mathcal{F}_\nu$ exists such that $P_\nu(C_\nu) = 1$ and

$$Q_\nu|_{C_\nu} << P_\nu|_{C_\nu},$$

where $|_{C_\nu}$ denotes restriction to C_ν. Define the function $\ell_\nu: \mathcal{X}_\nu \to [0,\infty]$ by

$$\ell_\nu(x) = \begin{cases} dQ_\nu/dP_\nu(x) & x \in C_\nu \\ \infty & x \in \mathcal{X}_\nu \backslash C_\nu. \end{cases}$$

Then $\ell_\nu(x)$ is defined up to equivalence a.e. $[P_\nu+Q_\nu]$. The distribution of the random variable $L_\nu = \ell_\nu(X_\nu)$, when X_ν is a random variable with values in $\mathcal{X}_\nu$, is completely determined when X_ν has probability distribution P_ν and also when X_ν has probability distribution Q_ν.

THEOREM A.2.1. *The following four statements are equivalent.*

(i) $\{Q_\nu\} \lhd \{P_\nu\}$

(ii) *for every sequence of random variables* $T_\nu = t_\nu(X_\nu)$ *with values in* $[-\infty,+\infty]$, $\{\mathcal{L}_{P_\nu}(T_\nu)\}$ *is tight implies that* $\{\mathcal{L}_{Q_\nu}(T_\nu)\}$ *is tight*

(iii) $\{\mathcal{L}_{Q_\nu}(L_\nu)\}$ *is tight*

(iv) $\{\mathcal{L}_{Q_\nu}(\log L_\nu)\}$ *is tight.*

PROOF. The characterization of tightness given in Lemma A.1.1(i) will be used.

(i) ⇒ (ii) If $\{\mathcal{L}_{P_\nu}(T_\nu)\}$ is tight, then

$$P_\nu\{|T_\nu| > z_\nu\} \to 0 \qquad \text{for all } \{z_\nu\} \subset \mathbb{R} \text{ with } z_\nu \to \infty.$$

Together with (i), this implies

$$Q_\nu\{|T_\nu| > z_\nu\} \to 0 \quad \text{for all } \{z_\nu\} \subset \mathbb{R} \text{ with } z_\nu \to \infty.$$

Hence $\{\mathcal{L}_{Q_\nu}(T_\nu)\}$ is tight.

(ii) ⇒ (iii) Note that for all ν and all $z > 0$,

$$(1) \qquad P_\nu\{L_\nu > z\} = \int_{\{L_\nu > z\}} L_\nu^{-1}\, dQ_\nu \le z^{-1}.$$

Hence $\{\mathcal{L}_{P_\nu}(L_\nu)\}$ is tight, and the implication is trivial.

(iii) ⇒ (iv) An application of (1) with interchanged roles of P_ν and Q_ν yields

$$Q_\nu\{|\log L_\nu| > z_\nu\} = Q_\nu\{L_\nu^{-1} > \exp z_\nu\} + Q_\nu\{L_\nu > \exp z_\nu\}$$

$$\le \exp(-z_\nu) + Q_\nu\{L_\nu > \exp z_\nu\}.$$

Condition (iii) and $z_\nu \to \infty$ imply that the right hand side tends to 0.

(iv) ⇒ (i) If $B_\nu \in \mathcal{F}_\nu$ and $0 < z_\nu < \infty$, then

$$Q_\nu(B_\nu) = \int_{B_\nu \cap \{L_\nu \le z_\nu\}} L_\nu dP_\nu + Q_\nu(B_\nu \cap \{L_\nu > z_\nu\})$$

$$\le z_\nu P_\nu(B_\nu) + Q_\nu\{\log L_\nu > \log z_\nu\}.$$

Suppose $P_\nu(B_\nu) \to 0$. Let $\{z_\nu\}$ be a sequence with $z_\nu P_\nu(B_\nu) \to 0$ and $z_\nu \to \infty$. Condition (iv) implies that

$$Q_\nu\{\log L_\nu > \log z_\nu\} \to 0,$$

so that also $Q_\nu(B_\nu) \to 0$. □

PROPOSITION A.2.1. *Let* $(\mathbb{R}^m, \{P_\theta | \theta \in \Theta\})$ *be a canonical* m*-dimensional exponential family, and denote the* ν*-fold product measure of* P_θ *by* P_θ^ν. *Let* $\{\eta_\nu\}$ *be a sequence in* Θ *which is relatively compact in* Θ. *Then for* $\{\theta_\nu\} \subset \Theta$,

$$\{P^\nu_{\theta_\nu}\} \lhd \{P^\nu_{\eta_\nu}\} \quad \textit{iff} \quad \{\nu^{\frac{1}{2}}\|\theta_\nu - \eta_\nu\|\} \textit{ is bounded.}$$

Denote the sample mean for P_θ^ν *(regarded as a sample of size* ν *from* P_θ*) by* $X_\cdot^{(\nu)}$. *If* $\nu^{\frac{1}{2}}\|\theta_\nu-\eta_\nu\| \to \infty$, *then there is a sequence* $\{y_\nu\} \subset \mathbb{R}^m$ *with* $\|y_\nu\| = 1$ *such that* $\nu^{\frac{1}{2}} y_\nu'(X_\cdot^{(\nu)}-E_{\eta_\nu}X_\cdot^{(\nu)}) \to \infty$ *in* $\{\theta_\nu\}$*-prob..*

PROOF. A σ-finite measure λ on $\mathbb{R}^m$ exists which is not concentrated on a hyperplane, such that

$$dP_\theta/d\lambda(x) = \exp\{\theta'x-\psi(\theta)\}$$

for a norming function $\psi: \Theta \to \mathbb{R}$. Let $X_1,\ldots,X_\nu$ be a random sample from P_θ, i.e.

$$L_\theta(X_1,\ldots,X_\nu) = P_\theta^\nu \quad ,$$

and let $X_{\nu+} = \sum_{j=1}^{\nu} X_j$, $\mu(\theta) = E_\theta X_j$, $\Sigma_\theta = \mathrm{cov}_\theta X_j$.

Corollary 2.3.1 shows that ψ is infinitely often differentiable and

$$\mu_i(\theta) = \partial\psi(\theta)/\partial\theta_i$$

$$(\Sigma_\theta)_{ij} = \partial^2\psi(\theta)/\partial\theta_i\partial\theta_j.$$

Let $\{\eta_\nu\}$ and $\{\theta_\nu\}$ be sequences in Θ, and let $\{\eta_\nu\}$ be relatively compact in Θ.

"If". Suppose that $\{\nu^{\frac{1}{2}}\|\theta_\nu-\eta_\nu\|\}$ is bounded. One has

$$\log L_\nu = \log(dP_{\theta_\nu}^\nu/dP_{\eta_\nu}^\nu) = (\theta_\nu-\eta_\nu)X_{\nu+}-\nu(\psi(\theta_\nu)-\psi(\eta_\nu))$$

$$E_{\theta_\nu}\log L_\nu = \nu[(\theta_\nu-\eta_\nu)'\mu(\theta_\nu)-\psi(\theta_\nu)+\psi(\eta_\nu)]$$

$$\mathrm{var}_{\theta_\nu}\log L_\nu = \nu(\theta_\nu-\eta_\nu)'\ \Sigma_{\theta_\nu}(\theta_\nu-\eta_\nu).$$

Taylor's Theorem yields the existence, for every ν, of a $t_\nu \in (0,1)$ such that for $\zeta_\nu = t_\nu\theta_\nu+(1-t_\nu)\eta_\nu$,

$$E_{\theta_\nu}\log L_\nu = -\tfrac{1}{2}\nu(\theta_\nu-\eta_\nu)'\ \Sigma_{\zeta_\nu}(\theta_\nu-\eta_\nu).$$

As $\{\eta_\nu\}$ is relatively compact and $\{\nu^{\frac{1}{2}}\|\theta_\nu-\eta_\nu\|\}$ is bounded, $\{\theta_\nu\}$ and $\{\zeta_\nu\}$ are also relatively compact. With the continuity of Σ_θ, this shows that $\{\Sigma_{\theta_\nu}\}$

and $\{\Sigma_{\zeta_\nu}\}$ are bounded. Hence $\{E_{\theta_\nu}\log L_\nu\}$ and $\{\mathrm{var}_{\theta_\nu}\log L_\nu\}$ are bounded sequences. Lemma A.1.1(ii) shows that $\{L_{\theta_\nu}(\log L_\nu)\}$ is tight, which according to Theorem A.2.1, implies that $\{P^\nu_{\theta_\nu}\} \lhd \{P^\nu_{\eta_\nu}\}$.

"Only if". Let $s_\nu = \|\theta_\nu - \eta_\nu\|$, $y_\nu = s_\nu^{-1}(\theta_\nu - \eta_\nu)$, and suppose that $\nu^{\frac{1}{2}} s_\nu \to \infty$. It must be shown that $\{P^\nu_{\theta_\nu}\}$ is not contiguous to $\{P^\nu_{\eta_\nu}\}$. Define

$$Z_\nu = \nu^{-\frac{1}{2}}\, y_\nu'(X_{\nu+} - \nu\mu(\eta_\nu)).$$

Then

$$E_\theta Z_\nu = \nu^{\frac{1}{2}}\, y_\nu'(\mu(\theta) - \mu(\eta_\nu))$$

$$\mathrm{var}_\theta Z_\nu = y_\nu' \Sigma_\theta y_\nu.$$

As Σ_θ is a continuous function and $\|y_\nu\| = 1$, Lemma A.1.1(ii) yields that $\{L_{\eta_\nu}(Z_\nu)\}$ is tight. It will be proved that $Z_\nu \to \infty$ in $\{\theta_\nu\}$-prob., which by Theorem A.2.1 contradicts $\{P^\nu_{\theta_\nu}\} \lhd \{P^\nu_{\eta_\nu}\}$.

First consider the case where $s_\nu \to 0$. The mean value theorem yields the existence, for every ν, of a $t_\nu \in (0, s_\nu)$ such that for $\zeta_\nu = \eta_\nu + t_\nu y_\nu$,

$$E_{\theta_\nu} Z_\nu = \nu^{\frac{1}{2}} s_\nu y_\nu' \Sigma_{\zeta_\nu} y_\nu.$$

It follows from $\theta_\nu - \eta_\nu \to 0$ that $\{\zeta_\nu\}$ is relatively compact in Θ; hence $\liminf_\nu y_\nu' \Sigma_{\zeta_\nu} y_\nu > 0$ and therefore

$$E_{\theta_\nu} Z_\nu \to \infty.$$

With the boundedness of $\{\mathrm{var}_{\theta_\nu} Z_\nu\}$ and Chebychev's inequality, this shows that $Z_\nu \to \infty$ in $\{\theta_\nu\}$-prob.

Now consider the general case. Note that the family of probability distributions $\{L_{\eta_\nu + t y_\nu}(Z_\nu) \mid 0 \le t \le s_\nu\}$ has monotone likelihood ratio. Therefore, $0 \le t_\nu \le s_\nu$ implies that $L_{\eta_\nu + t_\nu y_\nu}(Z_\nu)$ is stochastically smaller than $L_{\eta_\nu + s_\nu y_\nu}(Z_\nu) = L_{\theta_\nu}(Z_\nu)$. If $\nu^{\frac{1}{2}} s_\nu \to \infty$, there exists a sequence $\{t_\nu\}$ with $0 \le t_\nu \le s_\nu$, $\nu^{\frac{1}{2}} t_\nu \to \infty$ and $t_\nu \to 0$. The result proved above shows that $Z_\nu \to \infty$ in $\{\eta_\nu + t_\nu y_\nu\}$-prob. Therefore also $Z_\nu \to \infty$ in $\{\theta_\nu\}$-prob.. □

Although Proposition A.2.1 belongs to "common statistical knowledge", I have not found explicit proofs of the "only if" part in the literature. This may be related to the fact that contiguity for product measures is a "local" property, whereas the "only if" statement expresses a "global" property of the family $\{P_\theta | \theta \in \Theta\}$.

COROLLARY A.2.1. *Let* $\{P_\theta | \theta \in \Theta\}$ *and* P_θ^ν *be as in the proposition above, and let* $\{\eta_\nu\}$ *be a sequence in* Θ *which is relatively compact in* Θ. *Then for every* $\{\theta_\nu\} \subset \Theta$,

$$\{P^\nu_{\theta_\nu}\} \lhd \{P^\nu_{\eta_\nu}\} \quad \textit{iff} \quad \{P^\nu_{\eta_\nu}\} \lhd \{P^\nu_{\theta_\nu}\} \quad \textit{iff} \quad \{\nu^{\frac{1}{2}}\|\theta_\nu-\eta_\nu\|\} \textit{ is bounded.}$$

If $\nu^{\frac{1}{2}}\|\theta_\nu-\eta_\nu\| \to \infty$, *then there is a sequence* $\{B_\nu\}$ *of measurable sets with*

$$P^\nu_{\theta_\nu}(B_\nu) \to 0, \qquad P^\nu_{\eta_\nu}(B_\nu) \to 1.$$

PROOF. Proposition A.2.1 implies that if $\{\nu^{\frac{1}{2}}\|\theta_\nu-\eta_\nu\|\}$ is bounded, then $\{P^\nu_{\theta_\nu}\} \lhd\rhd \{P^\nu_{\eta_\nu}\}$. The proof of the "only if" part of the proposition shows that if $\nu^{\frac{1}{2}}\|\theta_\nu-\eta_\nu\| \to \infty$, then a sequence $\{z_\nu\} \subset \mathbb{R}$ exists such that for $B_\nu = \{Z_\nu \le z_\nu\}$ one has

$$P^\nu_{\theta_\nu}(B_\nu) \to 0, \qquad P^\nu_{\eta_\nu}(B_\nu) \to 1.$$

But this implies that neither of the one-sided contiguities can hold. □

The regularity property for sequences of product measures $\{P^\nu_\nu\}$ and $\{Q^\nu_\nu\}$ that either $\{P^\nu_\nu\} \lhd\rhd \{Q^\nu_\nu\}$, or a sequence $\{B_\nu\}$ exists with $P^\nu_\nu(B_\nu) \to 1$, $Q^\nu_\nu(B_\nu) \to 0$, is by no means restricted to exponential families; see, e.g., ROUSSAS (1972). But there are examples which do not exhibit this regularity property: see Example 4.1.1. Some further results about contiguity for sequences of product measures can be found in OOSTERHOFF and VAN ZWET (1975).

LEMMA A.2.1. *For* $1 \le i \le k$, *let* $\{P_{i\nu}\}$ *and* $\{Q_{i\nu}\}$ *be sequences of probability measures. Define* $P_\nu = \otimes_{i=1}^k P_{i\nu}$ *and* $Q_\nu = \otimes_{i=1}^k Q_{i\nu}$. *Then* $\{Q_\nu\} \lhd \{P_\nu\}$ *iff*

$$\{Q_{i\nu}\} \lhd \{P_{i\nu}\} \quad \textit{for } i = 1,\dots,k.$$

PROOF. "Only if" follows immediately from the definition of contiguity. "If". Define $L_{i\nu} = dQ_{i\nu}/dP_{i\nu}$, $L_\nu = dQ_\nu/dP_\nu$ as in the beginning of this appendix. Then $\log L_\nu = \sum_{i=1}^k \log L_{i\nu}$ a.e. $[P_\nu+Q_\nu]$. For every t one has

$$\{|\log L_\nu| \le kt\} \supset \bigcap_{i=1}^{k} \{|\log L_{i\nu}| \le t\}.$$

The desired implication follows from Theorem A.2.1. □

APPENDIX 3

A.3. TOPOLOGIES ON THE SPACE 2^X

In accordance with KURATOWSKI (1966), the space of all closed subsets of the topological space X will be denoted by 2^X.

In Section 4.2, the concept of topological convergence of a sequence of subsets of a pseudo-metrizable topological space plays an important role. In Appendix A.4, the equivalence of the weak* topology on the space $\Phi_{\mathcal{C}}$ of tests with convex acceptance region with the H-topology (to be defined below) on the space $\mathcal{C}$ of convex acceptance regions is used. This appendix is devoted to a study of some properties of topological convergence and the H-topology, and of their relation with the exponential topology and the Hausdorff metric.

In this appendix, X will be a pseudo-metrizable topological space, and d will be a bounded pseudo-metric on X generating its topology. The ε-neighbourhood of $x \in X$ will be denoted by

$$S(x;\varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\}.$$

<u>DEFINITION A.3.1</u>. The exponential topology on 2^X is the topology with the base

$$\{B(H;G_1,\dots,G_n) \mid n \in \mathbb{N}\,;\ H \in 2^X,\ G_h \text{ open subsets of } X \text{ for } h = 1,\dots,n\}$$

where

$$B(H;G_1,\dots,G_n) = \{F \in 2^X \mid F \cap H = \emptyset,\ \ F \cap G_h \neq \emptyset \text{ for } h = 1,\dots,n\}.$$

The H-topology on 2^X is the topology with the base

$$\{B(H;G_1,\dots,G_n) \mid n \in \mathbb{N}\,;\ H \in 2^X,\ H \text{ compact, } G_h \text{ open subsets of } X \text{ for } h = 1,\dots,n\}.$$

Convergence of $\{F_\nu\}$ to F in the exponential and the H-topology will be denoted by $F = \text{e-lim}_\nu F_\nu$ and $F = \text{H-lim}_\nu F_\nu$, respectively.
The Hausdorff metric on 2^X is defined by

$$\rho(F,H) = \begin{cases} \max\{\sup_{x\in F}\inf_{y\in H} d(x,y),\ \sup_{y\in H}\inf_{x\in F} d(x,y)\} & F,H \neq \emptyset \\ 1 & F = \emptyset \neq H \text{ or } F \neq \emptyset = H \\ 0 & F = H = \emptyset. \end{cases}$$

Topological convergence of a sequence $\{F_\nu\}$ in 2^X has been defined in Definition 4.2.1.

It follows from Definition A.3.1 that if $\{F_\nu\}$ is a sequence in 2^X, then $F = \text{e-lim}_\nu F_\nu$ iff

(i) for every closed $H \subset X$ with $F \cap H = \emptyset$ it holds that $F_\nu \cap H = \emptyset$ for ν sufficiently large, and

(ii) for every open $G \subset X$ with $F \cap G \neq \emptyset$ it holds that $F_\nu \cap G \neq \emptyset$ for ν sufficiently large;

and that $F = \text{H-lim}_\nu F_\nu$ iff (i) and (ii) hold with "closed" in (i) replaced by "compact".

The exponential topology is sometimes called the finite topology, the Vietoris topology (after its originator) or the closed-open topology. Some books in which results about topological convergence, the exponential topology and the Hausdorff metric can be found are HAUSDORFF (1927) (does not mention the exponential topology) and KURATOWSKI (1966). The H-topology was introduced by FELL (1962); it is sometimes called the compact-open topology. Topologies on 2^X which are very closely related to the H-topology were introduced by WATSON (1953) and MRÓWKA (1957). A historical exposition about these and related concepts, including some further references, is given by McALLISTER (1978). The theorems A.3.1,2 and 4 are known, and can be found in the references mentioned. First, the implications between the four convergence concepts will be treated.

<u>THEOREM A.3.1.</u>

(i) $\rho(F_\nu,F) \to 0$ *implies that* $F = \mathrm{Lt}_\nu F_\nu$.

(ii) $F = \text{e-lim}_\nu F_\nu$ *implies that* $F = \mathrm{Lt}_\nu F_\nu$.

(iii) $F = \mathrm{Lt}_\nu F_\nu$ *implies that* $F = \text{H-lim}_\nu F_\nu$.

PROOF.

(i) Suppose that $\rho(F_\nu,F) \to 0$. It must be proved that $\mathrm{Ls}_\nu F_\nu \subset F \subset \mathrm{Li}_\nu F_\nu$. First let $x \in \mathrm{Ls}_\nu F_\nu$; then a subsequence $\{\xi\}$ of $\{\nu\}$ exists, and a sequence $\{x_\xi\}$ with $x_\xi \in F_\xi$ and $d(x_\xi,x) \to 0$. The definition of ρ implies that for every ξ, a $y_\xi \in F$ exists with $d(x_\xi,y_\xi) \le \rho(F_\xi,F) + \xi^{-1}$. Since F is closed and $d(x,y_\xi) \le d(x_\xi,x) + \rho(F_\xi,F) + \xi^{-1} \to 0$, $x \in F$. Secondly let $x \in F$; for every ν, a $x_\nu \in F_\nu$ exists with $d(x_\nu,x) \le \rho(F_\nu,F) + \nu^{-1}$. Therefore $d(x_\nu,x) \to 0$, showing that $x \in \mathrm{Li}_\nu F_\nu$.

(ii) Suppose that $F = \text{e-lim}_\nu F_\nu$. It must be proved that $\mathrm{Ls}_\nu F_\nu \subset F \subset \mathrm{Li}_\nu F_\nu$. First suppose that $x \notin F$; since F is closed, $S(x;\varepsilon) \cap F = \emptyset$ for some $\varepsilon > 0$. Define

$$H = \{y \in X \mid d(x,y) \le \tfrac{1}{2}\varepsilon\}.$$

Then $H \in 2^X$ and $H \cap F = \emptyset$. Since $B(H;X)$ is a H-neighbourhood of F, one has that $F_\nu \in B(H;X)$ for ν sufficiently large, which is equivalent to $F_\nu \cap H = \emptyset$. This shows that $x \notin \mathrm{Ls}_\nu F_\nu$. Secondly suppose that $x \in F$. For every $\varepsilon > 0$, $B(\emptyset;S(x;\varepsilon))$ is a H-neighbourhood of F, so that $F_\nu \cap S(x;\varepsilon) \ne \emptyset$ for ν sufficiently large. Therefore, $\inf_{y\in F_\nu} d(x,y) \to 0$. For every ν, let $x_\nu \in F_\nu$ be such that $d(x,x_\nu) \le \inf_{y\in F_\nu} d(x,y) + \nu^{-1}$. Then $x_\nu \to x$, which shows that $x \in \mathrm{Li}_\nu F_\nu$.

(iii) Suppose that $F = \mathrm{Lt}_\nu F_\nu$. First let G be an open subset of X with $F \cap G \ne \emptyset$. Let $x \in F \cap G$; since $F = \mathrm{Li}_\nu F_\nu$, a sequence $\{x_\nu\}$ exists with $x_\nu \in F_\nu$, $x_\nu \to x$. Since G is open, $x_\nu \in F_\nu \cap G$ for ν sufficiently large, showing that $F_\nu \cap G \ne \emptyset$. Secondly let $H \in 2^X$ be compact with $F \cap H = \emptyset$. Let $x \in H$; then $x \notin F$ and there exist $\varepsilon(x) > 0$ and $\nu(x)$ with $S(x;\varepsilon(x)) \cap F_\nu = \emptyset$ for all $\nu \ge \nu(x)$. Since H is compact, there exist $x_1,\dots,x_n \in H$, for some n, with

$$H \supset \bigcup_{h=1}^{n} S(x_h;\varepsilon(x_h)).$$

This shows that $H \cap F_\nu = \emptyset$ for all $\nu \ge \max_{h=1}^{n} \nu(x_h)$. □

Other implications than those mentioned in Theorem A.3.1 do not exist for all spaces X. For example, let $X = \mathbb{R}$ with $d(x,y) = \min\{|x-y|,1\}$.

(a) Let $F_\nu = [-\nu,\nu]$; then $\rho(F_\nu,\mathbb{R}) = 1$ for all ν, but $\mathbb{R} = \text{e-lim}_\nu F_\nu$.

(b) Let $F_\nu = \bigcup_{h\in\mathbb{N}} [h-\nu^{-1},h+\nu^{-1}]$; then $\rho(F_\nu,\mathbb{N}) = \nu^{-1} \to 0$, but $G = \bigcup_{h\in\mathbb{N}} (h-h^{-1},h+h^{-1})$ is an open set with $\mathbb{N} \subset G$ and $F_\nu \not\subset G$ for all ν.

Hence $\mathbb{N}$ is not the exponential limit of F_ν.

These examples show that the topology generated by the Hausdorff metric and the exponential topology are not comparable, and also that the implications of Theorem A.3.1(i,ii) cannot be reversed. An example showing that the implication of Theorem A.3.1(iii) cannot be reversed is given by KURATOWSKI (1966), §29 IX, Remark 1.

THEOREM A.3.2. *If X is compact, then the exponential topology and the H-topology coincide, and can be metrized by the Hausdorff metric, while topological convergence is the associated convergence concept.*

PROOF. Every closed subset of a compact space is compact. Hence the equivalence of the exponential topology and the H-topology follows immediately from the definition. Theorem A.3.1(ii,iii) shows that topological convergence is the associated convergence for sequences.

In order to show that the exponential topology can be metrized by the Hausdorff metric, first let $\varepsilon > 0$ and $F \in 2^X$. Then $x_1,\ldots,x_n \in F$ exist for some n, such that

$$F \subset \bigcup_{h=1}^{n} S(x_h;\tfrac{1}{2}\varepsilon).$$

Define

$$H = X\setminus \bigcup_{x\in F} S(x;\varepsilon).$$

Then $B(H;S(x_1;\frac{1}{2}\varepsilon),\ldots,S(x_n;\frac{1}{2}\varepsilon))$ is an exponential neighbourhood of F contained in $\{F' \in 2^X \mid \rho(F,F') < \varepsilon\}$. Secondly let $F,H \in 2^X$, let $n \in \mathbb{N}$ and let $G_1,\ldots,G_n$ be open subsets of X, with $H \cap F = \emptyset$ and $F \cap G_h \neq \emptyset$ for $h = 1,\ldots,n$. Let $x_h \in F \cap G_h$ for $1 \leq h \leq n$; then there exists an $\varepsilon_1 > 0$ such that $S(x_h,\varepsilon_1) \subset G_h$ for all h. The compactness of H implies the existence of an $\varepsilon_2 > 0$ such that $(\bigcup_{x\in H} S(x;\varepsilon_2)) \cap F = \emptyset$. Let $\varepsilon = \min\{\varepsilon_1,\varepsilon_2\}$; then $\{F' \in 2^X \mid \rho(F',F) < \varepsilon\} \subset B(H;G_1,\ldots,G_n)$. □

For spaces X with "enough" compact subsets, H-convergence is in some way equivalent to "exponential convergence of the intersections with compact subsets". A first conjecture could be that $F = \text{H-lim}_\nu F_\nu$ iff $F \cap K = \text{e-lim}_\nu F_\nu \cap K$ for every compact $K \subset X$. This conjecture is refuted by the counterexample where X is a metric space, K a compact subset of X and $x \in \partial K$, where $\{x_\nu\}$ is a sequence in X with $x_\nu \notin K$ and $x_\nu \to x$, and where $F = \{x\}$,

F_ν is the set consisting of only the element x_ν. Then $F = \text{H-lim}_\nu F_\nu$ but $F \cap K = \{x\}$ and $F_\nu \cap K = \emptyset$ for all ν. The boundaries of the compact sets K produce complications, which are dealt with by the formulations in the following theorem.

<u>THEOREM A.3.3</u>. *Suppose that* X *is locally compact, and that* $F \in 2^X$ *and* $\{F_\nu\}$ *is a sequence in* 2^X. *The following statements are equivalent.*

(i) $F = \text{Lt}_\nu F_\nu$

(ii) $F = \text{H-lim}_\nu F_\nu$

(iii) $F \cap K = \text{e-lim}_\nu((F_\nu \cap K) \cup (F \cap \partial K))$ *for every compact* $K \in 2^X$

(iv) $(F \cap K) \cup \partial K = \text{e-lim}_\nu((F_\nu \cap K) \cup \partial K)$ *for every compact* $K \in 2^X$

(v) *for every compact* $K \in 2^X$, *an* $L \in 2^X$ *exists with* $K \subset \text{int } L$ *and* $(F \cap L) \cup \partial L = \text{e-lim}_\nu((F_\nu \cap L) \cup \partial L)$.

<u>PROOF</u>. (i) ⇒ (ii) This is Theorem A.3.1(iii).
(ii) ⇒ (iii) Suppose that $F = \text{H-lim}_\nu F_\nu$ and that $K \in 2^X$ is compact. First let $H \in 2^X$ be such that $H \cap (F \cap K) = \emptyset$. Then $H \cap K$ is compact, and $F_\nu \cap (H \cap K) = \emptyset$ for ν sufficiently large. As $H \cap F \cap K = \emptyset$ and $\partial K \subset K$ one has that $H \cap ((F_\nu \cap K) \cup (F \cap \partial K)) = \emptyset$ for ν sufficiently large. Secondly let G be an open subset of X with $G \cap (F \cap K) \neq \emptyset$. Then $F \cap (G \cap \partial K) \neq \emptyset$ or $F \cap (G \cap \text{int} K) \neq \emptyset$. In the first case, $G \cap ((F_\nu \cap K) \cup (F \cap \partial K)) \neq \emptyset$ for all ν. In the second case, $G \cap \text{int } K$ is an open set with $F \cap (G \cap \text{int } K) \neq \emptyset$, implying that $G \cap ((F_\nu \cap K) \cup (F \cap \partial K)) \neq \emptyset$ for ν sufficiently large.
(iii) ⇒ (iv) It is easy to see that the function $F \mapsto F \cup L$, for $L \in 2^X$, is continuous in the exponential topology. Apply this with $L = \partial K$.
(iv) ⇒ (v) As X is locally compact, for every compact $K \in 2^X$ there is a compact $L \in 2^X$ with $K \subset \text{int } L$.
(v) ⇒ (i) Suppose that (v) holds; it must be demonstrated that $\text{Ls}_\nu F_\nu \subset F \subset \text{Li}_\nu F_\nu$. First let $x \notin F$. As X is locally compact and F is closed, a compact K exists with $x \in \text{int } K$, $K \cap F = \emptyset$. Let $L \in 2^X$ satisfy $K \subset \text{int } L$ and $((F \cap L) \cup \partial L) = \text{e-lim}_\nu ((F_\nu \cap L) \cup \partial L)$. As K is closed and $K \cap ((F \cap L) \cup \partial L) = \emptyset$, $K \cap ((F_\nu \cap L) \cup \partial L) = \emptyset$ for ν sufficiently large. This implies that $K \cap F_\nu = \emptyset$ for ν sufficiently large, so that $x \notin \text{Ls}_\nu F_\nu$. Secondly let $x \in F$, and let $K = \text{cl}\{x\}$; then K is compact. Let $L \in 2^X$ satisfy $K \subset \text{int } L$ (that is, $x \in \text{int } L$) and $(F \cap L) \cup \partial L = \text{e-lim}_\nu((F_\nu \cap L) \cup \partial L)$. For every $\varepsilon > 0$ one has $(S(x;\varepsilon)) \cap ((F \cap L) \cup \partial L) \neq \emptyset$, implying that $(S(x;\varepsilon)) \cap ((F_\nu \cap L) \cup \partial L) \neq \emptyset$ for ν sufficiently large. There is an $\varepsilon_0 > 0$ so that $S(x;\varepsilon_0) \subset \text{int } L$, implying that $S(x;\varepsilon) \cap \partial L = \emptyset$ for all $\varepsilon < \varepsilon_0$. Hence for every $\varepsilon \in (0,\varepsilon_0)$, it holds that

$F_\nu \cap S(x;\varepsilon) \neq \emptyset$ for ν sufficiently large. This shows that $\inf_{y\in F_\nu} d(x,y) \to 0$, so that $x \in \mathrm{Li}_\nu F_\nu$. □

THEOREM A.3.4. *Suppose that* X *is separable. Then every sequence in* 2^X *has a topologically convergent subsequence.*

PROOF. See HAUSDORFF (1927) §28 or KURATOWSKI (1966) §29. □

A corollary of Theorems A.3.1(iii) and A.3.4 is that for separable X, the H-topology is sequentially compact. The following corollary will be used in Appendix A.4.

COROLLARY A.3.1. *If* E *is an open subset of* $\mathbb{R}^m$*, then the* H*-topology on* 2^E *is metrizable and compact.*

Let d *be the Euclidean metric on* $\mathbb{R}^m$*, and* ρ *the associated Hausdorff metric (which can assume the value* $+\infty$*).*

Let F *be a non-empty closed convex set in* $\mathbb{R}^m$ *and* $\{F_\nu\} \subset 2^{\mathbb{R}^m}$*, and let* $S_r = \{x \in \mathbb{R}^m \mid \|x\| \le r\}$*. Then* $F = \text{H-lim}_\nu F_\nu$ *iff*

$$\rho(F\cap S_r, F_\nu \cap S_r) \to 0$$

for all r *with* $F \cap \operatorname{int} S_r \neq \emptyset$.

PROOF. If $E \subset \mathbb{R}^m$ is open, then there is an increasing sequence $\{K_r\}$ of compact subsets of E such that for every compact $K \subset E$, there is an r with $K \subset \operatorname{int} K_r$. Theorems A.3.2,3 show that the H-topology can be metrized by the metric

$$\tilde{\rho}(F,F') = \sum_r 2^{-r} \rho((F\cap K_r)\cup\partial K_r, (F'\cap K_r)\cup\partial K_r).$$

The compactness follows from Theorems A.3.1(iii) and A.3.4.

Of the last assertion, "if" follows from Theorems A.3.2 and A.3.3 (v) ⇒ (ii), together with the continuity in the exponential topology of the function $F' \mapsto F' \cup \partial S_r$. To prove "only if", suppose that $F = \text{H-lim}_\nu F_\nu$ and that $x \in F \cap \operatorname{int} S_r$. Because of Theorem A.3.2 it is sufficient to prove that $F \cap S_r = \text{e-lim}_\nu (F_\nu \cap S_r)$. First let H be a closed subset of $\mathbb{R}^m$ and $H \cap F \cap S_r = \emptyset$. Then $H \cap S_r$ is compact, so that $H \cap F_\nu \cap S_r = \emptyset$ for ν sufficiently large.

Secondly let G be an open subset of $\mathbb{R}^m$ with $G \cap F \cap S_r \neq \emptyset$; say, $y \in G \cap F \cap S_r$. For $\varepsilon > 0$ sufficiently small, the convexity of F, and G being open, imply that $\varepsilon x + (1-\varepsilon)y \in G \cap F \cap (\text{int } S_r)$. Hence $F_\nu \cap (G \cap \text{int } S_r) \neq \emptyset$ for ν sufficiently large, so that certainly $G \cap (F_\nu \cap S_r) \neq \emptyset$ for ν sufficiently large. □

The exponential topology and the H-topology have been studied also for topological spaces X which are not metrizable: e.g., by MRÓWKA (1957,1970) and by FELL (1962).

Theorem A.3.3 (i)⟺ (ii) shows that for metric, locally compact X the concept of topological convergence corresponds to a topology on 2^X. The reader may be interested to know that WATSON (1953) and MRÓWKA (1957,1970) proved converse results: if topological convergence for sequences of subsets of X corresponds to a topology on 2^X, then X is locally compact.

APPENDIX 4

A.4. TESTS WITH CONVEX ACCEPTANCE REGION

The class of all test functions $\phi\colon \mathbb{R}^m \to [0,1]$ will be denoted by Φ_1. According to Definition 2.7.2, the test function $\phi \in \Phi_1$ has acceptance region C if C is a closed subset of $\mathbb{R}^m$ with

$$\phi(x) = \begin{cases} 1 & x \notin C \\ 0 & x \in \text{int } C. \end{cases}$$

Note that not all tests have an acceptance region. If the test ϕ has acceptance region C, this is denoted by $\text{acc}\phi = C$. For a class $\mathcal{F}$ of closed subsets of $\mathbb{R}^m$, the class of all tests ϕ with $\text{acc}\phi \in \mathcal{F}$ is denoted by $\Phi_{\mathcal{F}}$. The class of all closed convex subsets of $\mathbb{R}^m$, including $\emptyset$, is denoted by $\mathcal{C}$. Thus, $\Phi_{\mathcal{C}}$ is the class of all tests with convex acceptance region. The class $\Phi_{\mathcal{C}}$ plays an important role, because it is an essentially complete class for testing problems for exponential families with a simple null hypothesis (Theorem 2.7.2). In this appendix some properties of $\Phi_{\mathcal{C}}$ are studied.

<u>LEMMA A.4.1</u>. *Let* $\mathcal{F} \subset 2^{\mathbb{R}^m}$ *be closed in the* H-*topology of* $2^{\mathbb{R}^m}$, *and let* λ *be a* σ-*finite measure on* $\mathbb{R}^m$ *with* $\lambda(\partial F) = 0$ *for all* $F \in \mathcal{F}$. *Suppose that if* $\{F_\nu\} \subset \mathcal{F}$, $F = \text{H-lim}_\nu F_\nu$ *then* $I_{F_\nu}(x) \to I_F(x)$ a.e. $[\lambda]$.

Consider the weak* *topology in* $\Phi_{\mathcal{F}}$ *with respect to* λ, *and define the* H-*topology in* $\Phi_{\mathcal{F}}$ *as the topology induced by the* H-*topology in* $\mathcal{F}$ *through the map* $\phi \mapsto \text{acc}\phi$. *Then the following conclusions hold.*

(i) *Let* $\{\phi_\nu\} \subset \Phi_{\mathcal{F}}$. *If* $\phi_\nu \overset{H}{\to} \phi$, *then* $\phi_\nu(x) \to \phi(x)$ a.e. $[\lambda]$. *If* $\phi_\nu(x) \to \phi(x)$ a.e. $[\lambda]$, *then* $\phi_\nu \overset{*}{\to} \phi$.

(ii) *Let* $f\colon \mathbb{R}^m \to (0,\infty)$ *be* λ-*integrable. A pseudo-metric for the weak** *topology on* $\Phi_{\mathcal{F}}$ *is given by*

$$\rho_f(\phi,\psi) = \int f|\phi-\psi|d\lambda.$$

(iii) $\Phi_{\mathcal{F}}$ *is* H-*compact and weakly** *compact.*

PROOF.

(i) First suppose that $\mathrm{acc}\phi = \text{H-lim}_\nu\ \mathrm{acc}\phi_\nu$. It follows immediately from the assumptions that $\phi_\nu(x) \to \phi(x)$ a.e. $[\lambda]$. Secondly suppose that $\phi_\nu(x) \to \phi(x)$ a.e. $[\lambda]$. Then Lebesgue's dominated convergence theorem shows that $\phi_\nu \overset{*}{\to} \phi$.

(iii) Corollary A.3.1 shows that $2^{\mathbb{R}^m}$ is H-metrizable and H-compact. Since $\mathcal{F}$ is a H-closed subset of $2^{\mathbb{R}^m}$, $\Phi_{\mathcal{F}}$ is H-compact. (i) shows that the weak* topology on $\Phi_{\mathcal{F}}$ is weaker than the H-topology, so that $\Phi_{\mathcal{F}}$ is also weakly* compact.

(ii) Suppose that $\phi_\nu \overset{*}{\to} \phi$ *or* $\rho_{\mathcal{F}}(\phi_\nu,\phi) \to 0$, for $\{\phi_\nu\} \subset \Phi_{\mathcal{F}}$, $\phi \in \Phi_{\mathcal{F}}$. According to (i) and (iii), every subsequence of $\{\phi_\nu\}$ has a further subsequence $\{\phi_\xi\}$ such that $\phi_\xi(x) \to \psi(x)$ a.e. $[\lambda]$, for some $\psi \in \Phi_{\mathcal{F}}$. Lebesgue's dominated convergence theorem shows that $\phi_\xi \overset{*}{\to} \psi$ and $\rho_{\mathcal{F}}(\phi_\xi,\psi) \to 0$. This implies that $\phi(x) = \psi(x)$ a.e. $[\lambda]$. Hence $\phi_\nu \overset{*}{\to} \phi$ *and* $\rho_{\mathcal{F}}(\phi_\nu,\phi) \to 0$. □

Note that if λ is Lebesgue measure, then $\lambda(\partial C) = 0$ for every **convex** set C.

LEMMA A.4.2. *Let* $\mathcal{F} \subset 2^{\mathbb{R}^m}$ *be closed in the* H*-topology of* $2^{\mathbb{R}^m}$. *Suppose that if* $\{F_\nu\} \subset \mathcal{F}$ *and* $F = \text{H-lim}_\nu F_\nu$, *then* $I_{F_\nu}(x) \to I_F(x)$ *for all* $x \in \mathbb{R}^m\backslash\partial F$. *Then* $\Phi_{\mathcal{F}}$ *is weakly** *compact with respect to every* σ*-finite measure* λ *on* $\mathbb{R}^m$.

PROOF. Since the weak* topology on Φ_1 is pseudo-metrizable (Theorem 2.4.2), it is sufficient to demonstrate that $\Phi_{\mathcal{F}}$ is weakly* sequentially compact. Let $\{\phi_\nu\}$ be a sequence in $\Phi_{\mathcal{F}}$. According to Corollary A.3.1 there is a subsequence $\{\phi_\xi\}$ of $\{\phi_\nu\}$ and a $F \in \mathcal{F}$ such that $F = \text{H-lim}_\xi\ \mathrm{acc}\phi_\xi$. By assumption, $\phi_\xi(x) \to 1 - I_F(x)$ for all $x \in \mathbb{R}^m\backslash\partial F$. Since the weak* topology on Φ_1 is compact (Theorem 2.4.2), $\{\phi_\xi\}$ has a further subsequence $\{\phi_\zeta\}$ such that $\phi_\zeta \overset{*}{\to} \phi$, for some $\phi \in \Phi_1$. Then $\phi_\zeta(x) \to 1 - I_F(x)$ for all $x \in \mathbb{R}^m\backslash\partial F$. Define

$$\psi(x) = \begin{cases} \phi(x) & x \in \partial F \\ 0 & x \in \operatorname{int} F \\ 1 & x \in \mathbb{R}^m\backslash F. \end{cases}$$

Then $\psi \in \Phi_{\mathcal{F}}$. For every λ-integrable function f,

$$\int_{\partial F} f\phi_\zeta\, d\lambda \to \int_{\partial F} f\phi d\lambda = \int_{\partial F} f\psi d\lambda,$$

$$\int_{\mathbb{R}^m\backslash\partial F} f\phi_\zeta d\lambda \to \int_{\mathbb{R}^m\backslash\partial F} f(1-I_F)d\lambda = \int_{\mathbb{R}^m\backslash\partial F} f\psi d\lambda.$$

This implies that $\int f\phi_\zeta d\lambda \to \int f\psi d\lambda$, so that $\phi_\zeta \overset{*}{\to} \psi$. This demonstrates the weak* sequential compactness of Φ_F. □

<u>LEMMA A.4.3</u>. *C is a* H-*closed subset of* $2^{\mathbb{R}^m}$. *If* $\{C_\nu\} \subset C$ *and* $C = \text{H-lim}_\nu\, C_\nu$, *then every* $x \in \text{int}\, C$ *has a neighbourhood* $U \subset C$ *with* $U \subset C_\nu$ *for* ν *sufficiently large, while every* $x \notin C$ *has a neighbourhood* U *with* $U \cap C = \emptyset$ *and* $U \cap C_\nu = \emptyset$ *for* ν *sufficiently large; in particular,* $I_{C_\nu}(x) \to I_C(x)$ *for all* $x \in \mathbb{R}^m \setminus \partial C$.

<u>PROOF</u>.

(i) Suppose that $B \subset \mathbb{R}^m$ is closed but not convex. Then there exist $x_1, x_2 \in B$ and $t \in (0,1)$ with $y = tx_1 + (1-t)x_2 \notin B$. There is a compact neighbourhood K of y with $K \cap B = \emptyset$; and open neighbourhoods G_1 of x_1 and G_2 of x_2 such that for every $z_1 \in G_1$, $z_2 \in G_2$, the line segment joining z_1 and z_2 intersects K. Hence

$$\{F \in 2^{\mathbb{R}^m} \mid F \cap G_1 \neq \emptyset,\ F \cap G_2 \neq \emptyset,\ F \cap K = \emptyset\}$$

is a H-neighbourhood of B which does not intersect C.

(ii) Suppose that $\{C_\nu\} \subset C$ and $C = \text{H-lim}_\nu\, C_\nu$. First let $x \notin C$ and let K be a compact neighbourhood of x with $K \cap C = \emptyset$; take $U = \text{int}\, K$. Then $K \cap C_\nu = \emptyset$, and hence $U \cap C_\nu = \emptyset$, for ν sufficiently large. Secondly let $x \in \text{int}\, C$. Then for $1 \le i \le m+1$ there exist $x_i \in C$ and $t_i > 0$ with $x = \Sigma_i t_i x_i$ and $\Sigma_i t_i = 1$. There exists an $\varepsilon > 0$ so that if $\|y-x\| < \varepsilon$ and $\|y_i - x_i\| < \varepsilon$ $(1 \le i \le m+1)$, then $y \in \text{conv}\{y_1, \ldots, y_{m+1}\}$. Let $U = \{y \mid \|y-x\| < \varepsilon\}$ and $G_i = \{y \mid \|y-x_i\| < \varepsilon\}$. Then $C \cap G_i \neq \emptyset$; hence a ν_0 exists with $C_\nu \cap G_i \neq \emptyset$ for all i and all $\nu \ge \nu_0$. The convexity of C_ν and the choice of ε imply that $U \subset C_\nu$ for all $\nu \ge \nu_0$; similarly one has that $U \subset C$. □

<u>THEOREM A.4.1</u>.

(i) Φ_C *is weakly** *compact with respect to every* σ*-finite measure* λ *on* $\mathbb{R}^m$.

(ii) *Let* λ *be Lebesgue measure on* $\mathbb{R}^m$, *and consider the weak** *topology with respect to* λ. *Let* $f\colon \mathbb{R}^m \to (0,\infty)$ *be* λ*-integrable. For* $\{\phi_\nu\} \subset \Phi_C$, $\phi \in \Phi_C$, *one has that* $\phi_\nu \overset{*}{\to} \phi$ *iff* $\int f|\phi_\nu - \phi| d\lambda \to 0$. *If* ϕ *is not a.e.* $[\lambda]$ *equal to* 1, *then* $\phi_\nu \overset{*}{\to} \phi$ *is equivalent to* $\phi_\nu(x) \to \phi(x)$ *a.e.* $[\lambda]$, *and also to* $\text{acc}\phi = \text{H-lim}_\nu\, \text{acc}\phi_\nu$.

PROOF. (i) This follows immediately from Lemmas A.4.2,3.
(ii) Lemmas A.4.1(ii) and A.4.3 imply that $\phi_\nu \overset{*}{\to} \phi$ is equivalent to $\int f|\phi_\nu - \phi| d\lambda \to 0$. Lemmas A.4.1(i) and A.4.3 show that if $\phi_\nu(x) \to \phi(x)$ a.e. $[\lambda]$ or $\mathrm{acc}\phi = \text{H-lim}_\nu\, \mathrm{acc}\phi_\nu$, then $\phi_\nu \overset{*}{\to} \phi$. Now suppose that $\phi_\nu \overset{*}{\to} \phi$ and that ϕ is not a.e. $[\lambda]$ equal to 1. Lemmas A.4.1,3 show that every subsequence of $\{\phi_\nu\}$ has a further subsequence $\{\phi_\xi\}$ with $C = \text{H-lim}_\xi\, \mathrm{acc}\phi_\xi$, for some $C \in \mathcal{C}$; and with $\phi_\xi(x) \to 1 - I_C(x)$ for all $x \in \mathbb{R}^m \backslash \partial C$, so that $\phi_\xi \overset{*}{\to} 1 - I_C$. This implies that $\phi(x) = 1 - I_C(x)$ a.e. $[\lambda]$, so that int C = int $\mathrm{acc}\phi$. EGGLESTON (1958) Corollary 1.3.3 states that for every closed convex $C \subset \mathbb{R}^m$ with non-empty interior, one has that C = cl int C. Since ϕ is not equal to 1 a.e. $[\lambda]$, one has that int $\mathrm{acc}\phi \neq \emptyset$; therefore $C = \mathrm{acc}\phi$.

Thus it has been shown that every subsequence of $\{\phi_\nu\}$ has a further subsequence $\{\phi_\xi\}$ with $\phi_\xi(x) \to \phi(x)$ for all $x \in \mathbb{R}^m \backslash \partial\mathrm{acc}\phi$ and with $\mathrm{acc}\phi =$ $= \text{H-lim}_\xi\, \mathrm{acc}\phi_\xi$. This implies that $\phi_\nu(x) \to \phi(x)$ for all $x \in \mathbb{R}^m \backslash \partial\mathrm{acc}\phi$ and $\mathrm{acc}\phi = \text{H-lim}_\nu\, \mathrm{acc}\phi_\nu$. □

The weak* compactness of $\Phi_{\mathcal{C}}$ was proved by MATTHES and TRUAX (1967), who used a method similar to the approach followed here. They employed the convergence concept in $\mathcal{C}$ which is mentioned in Corollary A.3.1 together with the Blaschke Selection Theorem. The latter theorem is a relative of Theorem A.3.4. See also EATON (1970).

It follows from Theorem A.4.1(ii) that if λ' is a finite measure on $\mathbb{R}^m$ which is equivalent to Lebesgue measure, then the weak* topology on $\Phi_{\mathcal{C}}$ coincides with the L_1 topology with respect to λ'.

Another example of a class $\mathcal{F}$ which satisfies the conditions of Lemmas A.4.1,2 is the class of all closed increasing subsets of $\mathbb{R}^m$. A subset B of $\mathbb{R}^m$ is called increasing if $(x_1,\ldots,x_m) \in B$, $x_i \le y_i$ for all i, implies that $(y_1,\ldots,y_m) \in B$.

The following results are used in Section 4.4. It will be assumed that F is an open subset of $\mathbb{R}^m$, and that $f\colon F \to f(F) \subset \mathbb{R}^m$ is one-to-one and twice continuously differentiable, with non-vanishing Jacobian $|(\partial f/\partial x)|$, and with a continuous inverse. λ will denote Lebesgue measure on $\mathbb{R}^m$ in the remainder of this appendix.

LEMMA A.4.4. *Let $\{t_\nu\} \subset (0,\infty)$ be a sequence with $t_\nu \to \infty$, let $\{\mu_\nu\} \subset F$ be a sequence with $\mu_\nu \to \mu_0 \in F$ and let $\{C_\nu\} \subset \mathcal{C}$. Let ϕ_ν be a test with*

$$\mathrm{acc}\phi_\nu = t_\nu(\mathrm{cl}\ f(C_\nu \cap F) - f(\mu_\nu)).$$

(i) *If* $\phi_\nu \overset{*}{\to} \phi$, *then* ϕ *is a.e.* $[\lambda]$ *equal to a test* ψ *with convex acceptance region.*

(ii) *Suppose that* $\phi_\nu \overset{*}{\to} \phi$, $\phi \in \Phi_C$ *and* ϕ *is not a.e.* $[\lambda]$ *equal to* 1. *Then for every compact* $K \subset \text{int acc}\phi$ *one has that* $K \subset \text{int acc}\phi_\nu$ *for* ν *sufficiently large; and for every compact* K *with* $K \cap \text{acc}\phi = \emptyset$ *one has that* $K \cap \text{acc}\phi_\nu = \emptyset$ *for* ν *sufficiently large. In particular,* $\phi_\nu(x) \to \phi(x)$ *a.e.* $[\lambda]$.

(iii) *Suppose that* $\phi_\nu \overset{*}{\to} 1$. *Then for every compact* K *one has that* $\text{diam}(K \cap \text{acc}\phi_\nu) \to 0$.

(iv) *For every* $C \in \mathcal{C}$, $\{C_\nu\}$ *can be chosen so that* $\phi_\nu \overset{*}{\to} 1 - I_C$.

<u>PROOF</u>. Denote by D_ν $[D]$ the matrix of partial derivatives $(\partial f/\partial x)$ in $x = \mu_\nu$ $[x=\mu_0]$. For $x \in F$, define

$$g_{1\nu}(x) = t_\nu(f(x)-f(\mu_\nu))$$

$$g_{2\nu}(x) = t_\nu D_\nu(x-\mu_\nu).$$

It will be shown first, that $g_{1\nu}(x_\nu) \to y$ implies that $g_{2\nu}(x_\nu) \to y$. Since f is twice continuously differentiable and $\mu_\nu \to \mu_0$, one has that $D_\nu \to D$ and that for $x \to \mu_0$

$$f(x)-f(\mu_\nu) = D_\nu(x-\mu_\nu)+\mathcal{O}(\|x-\mu_\nu\|^2),$$

uniformly in ν. If $g_{1\nu}(x_\nu) \to y$, then the continuity of the inverse of f implies that $x_\nu \to \mu_0$. Further,

$$g_{1\nu}(x_\nu) = t_\nu D_\nu(x_\nu-\mu_\nu)+\mathcal{O}(t_\nu\|x_\nu-\mu_\nu\|^2).$$

Since $D_\nu \to D$ and $|D| \neq 0$, this shows that $t_\nu\|x_\nu-\mu_\nu\| = \mathcal{O}(1)$ and that $t_\nu\|x_\nu-\mu_\nu\|^2 \to 0$. Hence

$$g_{1\nu}(x_\nu)-g_{2\nu}(x_\nu) = \mathcal{O}(t_\nu\|x_\nu-\mu_\nu\|^2) \to 0,$$

implying that $g_{2\nu}(x_\nu) \to y$.

Define

$$F_{1\nu} = \mathrm{acc}\phi_\nu = t_\nu(\mathrm{cl}\ f(C_\nu \cap F) - f(\mu_\nu)) = g_{1\nu}(C_\nu \cap F)$$

$$F_{2\nu} = t_\nu D_\nu(C_\nu - \mu_\nu) = g_{2\nu}(C_\nu).$$

(i) Suppose that $\phi_\nu \overset{*}{\to} \phi$. Corollary A.3.1 implies that $\{F_{2\nu}\}$ has a subsequence $\{F_{2\xi}\}$ such that $C = \text{H-lim}_\xi F_{2\xi}$ for some closed $C \subset \mathbb{R}^m$. As every $F_{2\nu}$ is convex, Lemma A.4.3 yields that C is convex. Define $\psi = 1 - I_C$. Let K be compact and $K \cap C = \emptyset$. It will be proved that $K \cap \mathrm{acc}\phi_\xi = \emptyset$ for ξ sufficiently large. There exists a compact K_1 with $K \subset \mathrm{int}\ K_1$, $K_1 \cap C = \emptyset$. As $C = \text{H-lim}_\xi F_{2\xi}$, one has that $K_1 \cap F_{2\xi} = \emptyset$ for ξ sufficiently large. Argue by contradiction, and suppose that a subsequence $\{\zeta\}$ of $\{\xi\}$ exists with $K \cap F_{1\zeta} \neq \emptyset$ for all ζ. It may as well be supposed that $y_\zeta = g_{1\zeta}(x_\zeta)$ with $x_\zeta \in C_\zeta$ and $y_\zeta \to y \in K$. But then $g_{2\zeta}(x_\zeta) \in F_{2\zeta}$ and $g_{2\zeta}(x_\zeta) \to y \in \mathrm{int}\ K_1$. This is a contradiction with $K_1 \cap F_{2\xi} = \emptyset$ for ξ sufficiently large.

Now let K be compact and $K \subset \mathrm{int}\ C$. It will be proved that $K \subset \mathrm{int}\ \mathrm{acc}\phi_\xi$ for ξ sufficiently large. Argue by contradiction, and suppose that a subsequence $\{\zeta\}$ of $\{\xi\}$ exists with $K \not\subset \mathrm{int}\ F_{1\zeta}$. It may as well be supposed that $y_\zeta \to y$, $y \in K$ and $y_\zeta \notin F_{1\zeta}$. As $y_\zeta \in g_{1\zeta}(F)$ for ζ sufficiently large, we have that $y_\zeta = g_{1\zeta}(x_\zeta)$ with $x_\zeta \notin C_\zeta$ for ζ sufficiently large. As $y \in \mathrm{int}\ C$ and $C = \text{H-lim}_\xi F_{2\xi}$, Lemma A.4.3 implies that y has a neighbourhood $U \subset \mathrm{int}\ C$ with $U \subset F_{2\xi}$ for ξ sufficiently large. But $g_{2\zeta}(x_\zeta) \to y$ and hence $g_{2\zeta}(x_\zeta) \in U \backslash F_{2\zeta}$ for ζ sufficiently large. This is the desired contradiction.

It has been demonstrated in particular, that $\phi_\xi(x) \to \psi(x)$ for all $x \in \mathbb{R}^m \backslash \partial C$. Since also $\phi_\xi \overset{*}{\to} \phi$, it follows that $\phi(x) = \psi(x)$ a.e. $[\lambda]$.

(ii) Suppose that $\phi_\nu \overset{*}{\to} \phi$, and that ϕ is not a.e. $[\lambda]$ equal to 1. Then the argument in (i) shows that for every subsequence of $\{\phi_\nu\}$ a further subsequence $\{\phi_\xi\}$ exists and a set $C \in \mathcal{C}$ such that for every compact $K \subset \mathrm{int}\ C$ one has $K \subset \mathrm{int}\ \mathrm{acc}\phi_\xi$ for ξ sufficiently large, and for every compact K with $K \cap C = \emptyset$ one has $K \cap \mathrm{acc}\phi_\xi = \emptyset$ for ξ sufficiently large; while $\phi(x) = 1 - I_C(x)$ a.e. $[\lambda]$. As ϕ is not a.e. $[\lambda]$ equal to 1, $C = \mathrm{acc}\phi$ independently of the subsequence (see the proof of Lemma A.4.3). This establishes (ii).

(iii) Let $\phi_\nu \overset{*}{\to} 1$, let K be compact and let $\{\xi\}$ be a sequence of $\{\nu\}$ with $\mathrm{diam}(K \cap \mathrm{acc}\phi_\xi) \to \limsup_\nu \mathrm{diam}(K \cap \mathrm{acc}\phi_\nu)$. According to the proof of (i), it may be assumed that $\text{H-lim}_\xi\ \mathrm{acc}\phi_\xi = C$, for some convex set C with $1 = 1 - I_C$ a.e. $[\lambda]$. This implies that $C = \emptyset$ or $C = \{x\}$, for some $x \in \mathbb{R}^m$. In both cases, for every $\varepsilon > 0$ a compact $K' \subset K$ exists with $\mathrm{diam}(K \backslash K') < \varepsilon$ and

$K' \cap \mathrm{acc}\phi_\xi = \emptyset$ for ξ sufficiently large.
(iv) As $|D_\nu| \neq 0$, there exist convex sets C_ν with $F_{2\nu} = t_\nu D_\nu(C_\nu - \mu_\nu) = C$ for every ν. Let $\phi_\nu = 1 - I_{F_{1\nu}}$, with $F_{1\nu} = t_\nu(\mathrm{cl}\ f(C_\nu \cap F) - f(\mu_\nu))$. The argument in the proof of (i) shows that $\phi_\nu(x) \to 1 - I_C(x)$ for all $x \in \mathbb{R}^m \setminus \partial C$. This implies that $\phi_\nu \overset{*}{\to} 1 - I_C$. □

THEOREM A.4.2. *Suppose that ϕ_ν is as in Lemma A.4.4, and that $\phi_\nu \overset{*}{\to} \phi$. Let $\{P_\nu\} \subset M_1(\mathbb{R}^m)$ be a sequence with $P_\nu \to P$ weakly, where $P \ll \lambda$. Then $E_{P_\nu}\phi_\nu \to E_P\phi$.*

PROOF. First suppose that $\phi(x) = 1$ a.e. $[\lambda]$, and let $\varepsilon > 0$. Theorem A.1.1 (iii) shows that a compact K exists with $P_\nu(K) > 1-\frac{1}{2}\varepsilon$ for all ν. Lemma A.4.4(iii) shows that sequences $\{x_\nu\} \subset K$ and δ_ν with $\delta_\nu > 0$, $\delta_\nu \to 0$ exist such that $\phi_\nu(x) = 1$ for all $x \in K$ with $\|x-x_\nu\| \geq \delta_\nu$. Hence $E_{P_\nu}\phi_\nu > 1-\frac{1}{2}\varepsilon - P_\nu\{x \mid \|x-x_\nu\| < \delta_\nu\}$. It follows from $P_\nu \to P_\nu$ and $P \ll \lambda$ that $P_\nu\{x \mid \|x-x_\nu\| < \delta_\nu\} < \frac{1}{2}\varepsilon$ for ν sufficiently large. Hence $E_{P_\nu}\phi_\nu \to 1 = E_P\phi$.

Secondly suppose that ϕ is not a.e. $[\lambda]$ equal to 1. According to Lemma A.4.4(i), a closed convex set C exists with $\phi(x) = 1 - I_C(x)$ a.e. $[\lambda]$. Let $\varepsilon > 0$. There exist compact sets K_1 and K_2 with $K_1 \subset \mathrm{int}\ C$, $K_2 \cap C = \emptyset$, and $P(K_1 \cup K_2) > 1-\varepsilon$. It follows from Lemma A.4.4(ii) that for ν sufficiently large, one has that $I_{K_2}(x) \leq \phi_\nu(x) \leq 1 - I_{K_1}(x)$ for all $x \in \mathbb{R}^m$. Hence for ν sufficiently large,

$$P_\nu(K_2) \leq E_{P_\nu}\phi_\nu \leq 1 - P_\nu(K_1).$$

As $P(\partial K_1) = P(\partial K_2) = 0$ and $P_\nu \to P$ weakly, one has that $P_\nu(K_1) \to P(K_1)$, $P_\nu(K_2) \to P(K_2)$. Furthermore, $P(K_2) \leq E_P\phi \leq 1-P(K_1)$ and $(1-P(K_1))-P(K_2) < \varepsilon$. This shows that $|E_{P_\nu}\phi_\nu - E_P\phi| < \varepsilon$ for ν sufficiently large. □

When Theorem A.4.2 is applied with f being the identity function, it yields the result that if $\{\phi_\nu\} \subset \Phi_C$ and $\phi_\nu \overset{*}{\to} \phi$ while $P_\nu \to P \ll \lambda$, then $E_{P_\nu}\phi_\nu \to E_P\phi$. Combined with the weak* compactness of Φ_C this shows that if $P_\nu \to P \ll \lambda$ then $P_\nu(C) \to P(C)$ uniformly in $C \in \mathcal{C}$. The latter result was proved first by RANGA RAO (1962); more general results were given by BILLINGSLEY and TOPSØE (1967) and by TOPSØE (1977), while FABIAN (1970) provided a simpler proof. The proof given here is related to Fabian's proof.

The following three lemmas are used in several chapters. Lemma A.4.7 can be regarded as a stochastic version of Lemma A.4.6.

LEMMA A.4.5. *Let* (X_ν, Y_ν) *be pairs of random variables on* $\mathbb{R}^m$ *with* $X_\nu - Y_\nu \to 0$ *in prob. and with* $L(X_\nu) \to P_0$ *weakly for a* P_0 *with* $P_0 \ll \lambda$. *Then* $E|\phi(X_\nu)-\phi(Y_\nu)| \to 0$ *for every* $\phi \in \Phi_C$.

PROOF. Let $\mathrm{acc}\phi = C$. Define $\partial^\varepsilon C = \{y \in \mathbb{R}^m \mid \|x-y\| \le \varepsilon$ for some $x \in \partial C\}$, for $\varepsilon > 0$. Then $\partial^\varepsilon C$ is closed, and $\bigcap_{\varepsilon>0} \partial^\varepsilon C = \partial C$. For every $\varepsilon > 0$,

$$\{|\phi(X_\nu)-\phi(Y_\nu)| > 0\} \subset \{|X_\nu - Y_\nu| \ge \varepsilon\} \cup \{X_\nu \in \partial^\varepsilon C\}.$$

One has that $P\{|X_\nu-Y_\nu| \ge \varepsilon\} \to 0$ and $P\{X_\nu \in \partial^\varepsilon C\} \to P_0(\partial^\varepsilon C)$ for every $\varepsilon > 0$. Let $\delta > 0$. There is an $\varepsilon > 0$ with $P_0(\partial^\varepsilon C) < \delta/3$. If ν is so large that $P\{|X_\nu-Y_\nu| \ge \varepsilon\} < \delta/3$ and that $|P\{X_\nu \in \partial^\varepsilon C\} - P_0(\partial^\varepsilon C)| < \delta/3$, then $E|\phi(X_\nu)-\phi(Y_\nu)| < \delta$. □

LEMMA A.4.6. *Let* $\{P_\nu\} \subset M_1(\mathbb{R}^m)$ *with* $P_\nu \to P$ *weakly and* $P \ll \lambda \ll P$. *Let* $\{\phi_\nu\} \subset \Phi_C$, $\phi \in \Phi_C$. *Then* $E_{P_\nu}|\phi_\nu - \phi| \to 0$ *iff* $\phi_\nu \overset{*}{\to} \phi$.

PROOF. "if". First suppose that $\phi(x) = 1$ a.e. $[\lambda]$; then $\phi(x) < 1$ in at most one point x_0. Therefore

$$E_{P_\nu}|\phi_\nu - \phi| \le E_{P_\nu}(1-\phi_\nu) + P_\nu\{x_0\} = 1 - E_{P_\nu}\phi_\nu + P_\nu\{x_0\} \to 0,$$

according to Theorem A.4.2. Secondly suppose that ϕ is not a.e. $[\lambda]$ equal to 1, and let $\varepsilon > 0$. The proof of Theorem A.4.2 shows that a compact K exists such that $P(K) > 1-\varepsilon$ and $\phi_\nu(x) = \phi(x)$ for all $x \in K$, for ν sufficiently large. Then

$$\limsup_\nu E_{P_\nu}|\phi_\nu - \phi| \le \limsup_\nu (1-P_\nu(K)) < \varepsilon.$$

"only if". Because of the compactness of Φ_C, it is sufficient to prove that if $\{\xi\}$ is a subsequence of $\{\nu\}$ and $\phi_\xi \overset{*}{\to} \psi$ for some $\psi \in \Phi_C$, then $\phi(x) = \psi(x)$ a.e. $[\lambda]$. If $\phi_\xi \overset{*}{\to} \psi$, then the "if" part shows that

$$E_{P_\xi}|\phi - \psi| \le E_{P_\xi}|\phi_\xi - \phi| + E_{P_\xi}|\phi_\xi - \psi| \to 0.$$

As $|\phi-\psi|$ is a.e. $[\lambda]$ continuous, this implies that $E_P|\phi-\psi| = \lim_\xi E_{P_\xi}|\phi-\psi| = 0$. As $\lambda \ll P$, $\phi(x) = \psi(x)$ a.e. $[\lambda]$. □

COROLLARY A.4.1. *Let* $\{P_\nu\}$, $\{Q_\nu\} \subset M_1(\mathbb{R}^m)$ *with* $P_\nu \to P$ *and* $Q_\nu \to P$ *weakly and* $P \ll \lambda$. *Let* $\{\phi_\nu\} \subset \Phi_C$. *Then* $E_{P_\nu}\phi_\nu - E_{Q_\nu}\phi_\nu \to 0$.

PROOF. This follows from Theorem A.4.1(i) and Lemma A.4.6. □

LEMMA A.4.7. *Let* (X_ν, T_ν) *be pairs of random variables with values in* $\mathbb{R}^m \times T$ *for some measurable space* T. *Let* $\phi_\nu : \mathbb{R}^m \times T \to [0,1]$ *be measurable functions so that* $\phi_\nu(\cdot,t) \in \Phi_C$ *for every* ν *and every* $t \in T$. *Let* $\phi \in \Phi_C$, ϕ *not a.e.* $[\lambda]$ *equal to* 1, *and suppose that for every weak** *neighbourhood* W *of* ϕ *in* Φ_C *one has that* $P\{\phi_\nu(\cdot,T_\nu) \in W\} \to 1$. *(In other words,* $\phi_\nu(\cdot,T_\nu) \overset{*}{\to} \phi$ *in probability.) Let* $L(X_\nu) \to P_0$ *weakly for a* P_0 *with* $P_0 \ll \lambda$. *Then* $E|\phi_\nu(X_\nu,T_\nu)-\phi(X_\nu)| \to 0$.

PROOF. Lemma A.4.4(ii) shows that for all compact sets $K_1 \subset \text{int acc}\phi$ and K_2 with $K_2 \cap \text{acc}\phi = \emptyset$,

$$W(K) = \{\psi \in \Phi_C \mid \phi(x) = \psi(x) \quad \text{for } x \in K\}$$

is for $K = K_1 \cup K_2$ a weak* neighbourhood of ϕ. Let $\varepsilon > 0$. Then compact sets $K_1 \subset \text{int acc}\phi$ and K_2 with $K_2 \cap \text{acc}\phi = \emptyset$ exist such that $P_0(K_1 \cup K_2) > 1-\varepsilon$. Let $K = K_1 \cup K_2$. One has that

$$E|\phi_\nu(X_\nu,T_\nu)-\phi(X_\nu)| \le P\{\phi_\nu(\cdot,T_\nu) \notin W(K)\} +$$

$$+ E\{|\phi_\nu(X_\nu,T_\nu)-\phi(X_\nu)| \mid \phi_\nu(\cdot,T_\nu) \in W(K)\}.$$

The first term goes to 0. The second term is majorized by

$$P\{X_\nu \notin K \mid \phi_\nu(\cdot,T_\nu) \in W(K)\}.$$

It follows from $P\{\phi_\nu(\cdot,T_\nu) \in W(K)\} \to 1$ and $L(X_\nu) \to P_0$ that

$$L(X_\nu | \phi_\nu(\cdot,T_\nu) \in W(K)) \to P_0.$$

Since $P_0(\partial K) = 0$, this implies that

$$P\{X_\nu \notin K \mid \phi_\nu(\cdot,T_\nu) \in W(K)\} \to 1 - P_0(K) < \varepsilon. \quad \square$$

The result of Lemma A.4.7 is not valid for $\phi(x) = 1$ a.e. $[\lambda]$. To see this, let $T = \mathbb{R}^m$,

$$\phi_\nu(x,t) = \begin{cases} 0 & \|x-t\| \le \nu^{-1} \\ 1 & \|x-t\| > \nu^{-1} \end{cases}$$

and $T_\nu = X_\nu$. Then $\phi_\nu(\cdot, X_\nu) \to \phi$ in probability in Φ_C for the function ϕ with $\phi(x) \equiv 1$. But $\phi_\nu(X_\nu, X_\nu) \equiv 0$.

The last results of this appendix are used in consistency proofs. They use the concept of the recession cone 0^+C of a convex set C, defined in Definition 2.8.1. Lemma A.4.8(i) is a generalization of Proposition 2.8.1(iii) and its proof is hardly different from the proof of that result.

LEMMA A.4.8.

(i) *Let* $\{C_\nu\} \subset \mathcal{C}$, $\text{H-lim}_\nu\, C_\nu = C \neq \emptyset$. *Let* $\{x_\nu\} \subset \mathbb{R}^m$, $\|x_\nu\| \to \infty$, $x_\nu/\|x_\nu\| \to x \notin 0^+C$. *Then* $\text{H-lim}_\nu (C_\nu - x_\nu) = \emptyset$.

(ii) *Let* M *be a closed cone in* $\mathbb{R}^m$. *Then*

$$\mathcal{C}_M = \{C \in \mathcal{C} \mid C \neq \emptyset,\ M \cap 0^+C = \{0\}\}$$

is a H-*open subset of* $\mathcal{C}$.

PROOF. (i) By Theorem A.3.1(iii), it is sufficient to prove that $\text{Ls}_\nu(C_\nu - x_\nu) = \emptyset$. Argue by contradiction, and suppose that $y_\xi \in C_\xi$, $y_\xi - x_\xi \to z$ for some subsequence $\{\xi\}$ of $\{\nu\}$. As $\text{H-lim}_\nu\, C_\nu = C \neq \emptyset$, there exist $p_\xi \in C_\xi$, $p \in C$ with $p_\xi \to p$. As $x \notin 0^+C$, there is a $t > 0$ with $p + tx \notin C$. Let $t_\xi = t/\|x_\xi\|$; then

$$(1-t_\xi)p_\xi + t_\xi y_\xi \to p + tx.$$

The convexity of C_ξ implies that $(1-t_\xi)p_\xi + t_\xi y_\xi \in C_\xi$ for $\|x_\xi\| \ge t$. With $C = \text{H-lim}_\nu\, C_\nu$ this implies that $p + tx \in C$, a contradiction.

(ii) If $M = \{0\}$, the assertion is trivial. Now suppose that $M \neq \{0\}$ and let $C \in \mathcal{C}_M$. Argue by contradiction and suppose that $C_\nu \in \mathcal{C}\backslash\mathcal{C}_M$, $C = \text{H-lim}_\nu\, C_\nu$. Let $y \in C$; there exist $y_\nu \in C_\nu$ with $y_\nu \to y$. As $M \cap 0^+C_\nu \neq \{0\}$ for every ν, there exist $x_\nu \in M \cap 0^+C_\nu$ with $\|x_\nu\| \to \infty$. Let $\{\xi\}$ be a subsequence of $\{\nu\}$ with $x_\xi/\|x_\xi\| \to x$ for some x. Then $x \in M$, hence $x \notin 0^+C$. With (i), this implies that $H\text{-}\lim_\xi(C_\xi - x_\xi) = \emptyset$. But $y_\xi + x_\xi \in C_\xi$ and hence $y \in Li_\xi(C_\xi - x_\xi)$. This is contradictory (use Theorem A.3.3(ii) $\Rightarrow$ (i)). □

<u>COROLLARY A.4.2</u>. *Let* $L_\mu(X) = N_m(\mu,\Sigma)$ *with* $|\Sigma| \neq 0$. *Let* $\|\mu_\nu\| \to \infty$, $\mu_\nu/\|\mu_\nu\| \to \mu$ *and let* $\phi \in \Phi_C$ *be a test with* $\mu \notin 0^+(\mathrm{acc}\phi)$, *and which is not a.e.* $[\lambda]$ *equal to* 1. *If* $\{\phi_\nu\} \subset \Phi_C$, $\phi_\nu \overset{*}{\to} \phi$ *then* $E_{\mu_\nu}\phi_\nu(X) \to 1$.

<u>PROOF</u>. Theorem A.4.1(ii) implies that $H\text{-}\lim_\nu \mathrm{acc}\phi_\nu = \mathrm{acc}\phi$. With Lemma A.4.8(i) this shows that $H\text{-}\lim_\nu(\mathrm{acc}\phi_\nu - \mu_\nu) = \emptyset$. Hence

$$E_{\mu_\nu}\phi_\nu(X) = 1 - P_0\{X \in \mathrm{acc}\phi_\nu - \mu_\nu\} \to 1. \qquad \square$$

<u>COROLLARY A.4.3</u>. *Make the assumptions of Lemma* A.4.7. *Let* $\{x_\nu\}$ *be a sequence with* $\|x_\nu\| \to \infty$, $x_\nu/\|x_\nu\| \to x \notin 0^+(\mathrm{acc}\phi)$. *Then* $E\phi_\nu(X_\nu + x_\nu, T_\nu) \to 1$.

<u>PROOF</u>. For compact $K \subset \mathbb{R}^m$ and $\nu_0 \in \mathbb{N}$ define

$$W(K,\nu_0) = \{\psi \in \Phi_C \mid K \cap (\mathrm{acc}\psi - x_\nu) = \emptyset \quad \text{for all } \nu \geq \nu_0\}.$$

Then $W(K,\nu) \subset W(K,\nu+1)$ for all K and ν. It will be proved first that for every K, there exists a ν such that $W(K,\nu)$ is a weak* neighbourhood of ϕ. It follows from Theorem A.4.1(ii) that the weak* topology on Φ_C is metrizable; let ρ be a metric for the weak* topology. Argue by contradiction, and suppose that there exists a compact K such that for every ν there is a $\phi_\nu \in \Phi_C \backslash W(K,\nu)$ and $\rho(\phi,\phi_\nu) \leq \nu^{-1}$. Theorem A.4.1(ii) implies that $\mathrm{acc}\phi = H\text{-}\lim_\nu \mathrm{acc}\phi_\nu$; Lemma A.4.8(i) implies that $H\text{-}\lim_\nu (\mathrm{acc}\phi_\nu - x_\nu) = \emptyset$, which is in contradiction with $\phi_\nu \notin W(K,\nu)$ for every ν.

Hence for every compact set $K \subset \mathbb{R}^m$, there exists a ν_K such that $W(K) = W(K,\nu_K)$ is a weak* neighbourhood of ϕ. For every K and $\nu \geq \nu_K$ one has

$$E\phi_\nu(X_\nu + x_\nu, T_\nu) \geq E\{\phi_\nu(X_\nu + x_\nu, T_\nu) \mid \phi_\nu(\cdot,T_\nu) \in W(K)\}P\{\phi_\nu(\cdot,T_\nu) \in W(K)\}$$

$$\geq P\{X_\nu \in K \mid \phi_\nu(\cdot,T_\nu) \in W(K)\}P\{\phi_\nu(\cdot,T_\nu) \in W(K)\}.$$

For K arbitrarily large, the right hand side can be made arbitrarily close to 1 (see the proof of Lemma A.4.7). □

In this study the (relative) weak* topology has been used as the basic topology on Φ_C. From this appendix it can be concluded that it would also be possible to use as the basic topology on Φ_C the topology generated by the base

$$\{W(K_1,K_2) \mid K_1 \text{ and } K_2 \text{ are disjoint compact subsets of } \mathbb{R}^m\}$$

of open sets $W(K_1,K_2)$ defined by

$$W(K_1,K_2) = \{\phi \in \Phi_C \mid K_1 \subset \text{int acc}\phi,\ K_2 \cap \text{acc}\phi = \emptyset\}.$$

This topology could be called the K-topology. It is clear that the K-topology can also be defined on the class of all tests with an acceptance region.

Lemmas A.4.3 and A.4.4(ii) demonstrate the relation between the K-topology and the weak* topology. The K-topology is related to the H-topology in the following way: $\phi_\nu \overset{K}{\to} \phi$ iff $\text{acc}\phi = \text{H-lim}_\nu\ \text{acc}\phi_\nu$ and $(\mathbb{R}^m \backslash \text{int acc}\phi) = \text{H-lim}_\nu (\mathbb{R}^m \backslash \text{int acc}\phi_\nu)$.

APPENDIX 5

A.5. THREE LEMMAS ON DOUBLE SEQUENCES

<u>LEMMA A.5.1</u>. *Let* $\{a(n,\nu)\}$ *be a double sequence with* $a(n,\nu) \le a(n+1,\nu)$ *for every* (n,ν) *and with* $\sup_n \limsup_\nu a(n,\nu) = a$. *Then a non-decreasing sequence* $\{\bar{n}(\nu)\}$ *exists with* $\bar{n}(\nu) \to \infty$ *and* $\limsup_\nu a(\bar{n}(\nu),\nu) = a$.

<u>PROOF</u>. Define

$$\bar{\nu}(n) = \min\{\nu_0 \mid a(n,\nu) \le a+n^{-1} \quad \text{for all } \nu \ge \nu_0\}.$$

Then for all $\nu \ge \bar{\nu}(n+1)$ one has that

$$a(n,\nu) \le a(n+1,\nu) \le a + (n+1)^{-1} \le a + n^{-1},$$

so that $\bar{\nu}(n) \le \bar{\nu}(n+1)$. Define $\bar{n}(\nu)$ by

$$\bar{n}(\nu) = \nu \quad \text{if } \lim_n \bar{\nu}(n) < \infty$$

$$\bar{n}(\nu) = \max\{n \mid \bar{\nu}(n) \le \nu\} \quad \text{if } \lim_n \bar{\nu}(n) = \infty.$$

The sequence $\{\bar{n}(\nu)\}$ satisfies the requirements. □

<u>LEMMA A.5.2</u>. *Let* $\{a(n,\nu)\}$ *be a double sequence with* $a(n,\nu) \le a(n+1,\nu)$ *for every* (n,ν) *and with* $\sup_n \liminf_\nu a(n,\nu) = a$. *Then a subsequence* $\{\xi\}$ *of* $\{\nu\}$ *exists with* $\sup_n \limsup_\xi a(n,\xi) = a$.

<u>PROOF</u>. For every h define

$$\bar{\nu}(h) = \min\{\nu \ge h \mid a(h,\nu) \le a + h^{-1}\}.$$

Then $h \le \bar{\nu}(h) \le \bar{\nu}(h+1) < \infty$, and $a(h,\bar{\nu}(h)) \le a + h^{-1}$. Hence for every n,

$$\limsup_h a(n,\bar{\nu}(h)) \le \limsup_h a(h,\bar{\nu}(h)) \le a.$$

Take $\{\xi\} = \{\bar{\nu}(h)\}$. □

<u>LEMMA A.5.3</u>. *Let A be a collection of sequences* $\{a_\nu\} \subset [0,1]$. *Then a subsequence* $\{\xi\}$ *of* $\{\nu\}$ *exists with*

$$\sup_{\{a_\nu\}\in A} \liminf_{\xi} a_\xi = \sup_{\{a_\nu\}\in A} \limsup_{\xi} a_\xi .$$

<u>PROOF</u>. For every subsequence $\{\zeta\}$ of $\{\nu\}$, define

$$a_-\{\zeta\} = \sup_{\{a_\nu\}\in A} \liminf_{\zeta} a_\zeta$$

$$a_+\{\zeta\} = \sup_{\{a_\nu\}\in A} \limsup_{\zeta} a_\zeta .$$

For every subsequence $\{\zeta\}$ and every $\varepsilon > 0$, there exists an $\{a_\nu\} \in A$ with

$$\limsup_{\zeta} a_\zeta > a_+\{\zeta\} - \varepsilon$$

and therefore also a further subsequence $\{\zeta'\}$ with

$$\liminf_{\zeta'} a_{\zeta'} > a_+\{\zeta\} - \varepsilon .$$

For this $\{\zeta'\}$ one has that

$$a_+\{\zeta\} - \varepsilon < a_-\{\zeta'\} \le a_+\{\zeta'\} \le a_+\{\zeta\},$$

so that

$$0 \le a_+\{\zeta'\} - a_-\{\zeta'\} < \varepsilon .$$

Now the subsequence $\{\xi\}$ will be constructed. Define $\{\zeta^{(1)}\} = \{\nu\}$ and, inductively, let $\{\zeta^{(h+1)}\}$ be a subsequence of $\{\zeta^{(h)}\}$ with

$$0 \le a_+\{\zeta^{(h+1)}\} - a_-\{\zeta^{(h+1)}\} < (h+1)^{-1} .$$

Let $\{\xi\}$ be a diagonal sequence of the $\{\zeta^{(h)}\}$; then for every h, $\{\xi\}$ is a subsequence of $\{\zeta^{(h)}\}$ apart from finitely elements of $\{\xi\}$. Hence for every h

$$a_+\{\xi\} \le a_+\{\zeta^{(h)}\}, \qquad a_-\{\xi\} \ge a_-\{\zeta^{(h)}\},$$

so that $\{\xi\}$ does the job. □

APPENDIX 6

A.6. THE EAMS - ASYMPTOTICALLY - SQUARED MEANS TEST FOR SECTION 9.3

Consider the limiting problem for the testing problem of Section 9.3 in the form (8.1.1). This limiting problem is denoted by $\mathcal{T}(p_0)$, where $p_0 = (p_{01}, \ldots, p_{0m})$ is the outcome of $n^{-1}(N_{+1}, \ldots, N_{+m})$. It is the testing problem where $Y_1, \ldots, Y_k$ are independent random variables,

$$\mathcal{L}_\eta(Y_i) = \mathcal{N}_{m-1}(\eta_i, \rho_i^{-1}\,\Sigma(p_0))$$

$$H : \eta_1 = \eta_2 = \ldots = \eta_k$$

$$H \vee A : \text{a permutation } (i_1, \ldots, i_k) \text{ of } (1,2, \ldots, k) \text{ exists such that } \Sigma_{j=1}^h \eta_{i_{r+1}j} \leq \Sigma_{j=1}^h \eta_{i_r j}\,, \text{ for } 1 \leq h \leq m-1, \quad 1 \leq r \leq k-1\,.$$

The covariance matrix $\Sigma(p_0)$ has diagonal elements $p_{0j}(1-p_{0j})$ and off-diagonal elements $-p_{0j}p_{0h}$. Recall that the set of all $\eta = (\eta_1', \eta_2', \ldots, \eta_k')'$ satisfying $H \vee A$ is denoted by $V + K$. Define $Y_{im} = n^{\frac{1}{2}} - \Sigma_{j=1}^{m-1} Y_{ij}$, $\eta_{im} = n^{\frac{1}{2}} - \Sigma_{j=1}^{m-1} \eta_{ij}$. Attention is restricted to tests based on test statistics

$$T_a = \sum_{i=1}^{k} \rho_i \{\sum_{j=1}^{m} a_j (Y_{ij} - Y_{.j})\}^2$$

where

$$Y_{.j} = \sum_{i=1}^{k} \rho_i Y_{ij}\,.$$

A family of transformations B_Λ exists, satisfying (8.2.1). The class of level α tests for $\mathcal{T}(p_o)$ rejecting for large values of T_a, where $a \in \mathbb{R}^m$, is transformed by B_Λ (Λ depending on p_o and ρ) to a class Ψ satisfying the assumptions made in Section 8.2. Ψ is called the class of squared means tests. The transformations B_Λ will not be specified explicitly, as it is more convenient to study the limiting problem in the form $\mathcal{T}(p_o)$.

As $\Sigma_{j=1}^m (Y_{ij} - Y_{.j}) \equiv 0$, attention may be restricted to all a with

$$\sum_{j=1}^{m} a_j p_{0j} = 0, \qquad \sum_{j=1}^{m} a_j p_{0j}^2 = 1.$$

Denote the class of all $a \in \mathbb{R}^m$ satisfying these restrictions by $A(p_0)$. Let

$$Z_i(a) = \rho_i^{\frac{1}{2}} \sum_{j=1}^{m} a_j Y_{ij} \ , \ Z_.(a) = \sum_{i=1}^{k} \rho_i^{\frac{1}{2}} Z_i(a) \ .$$

Then

$$T_a = \sum_{i=1}^{k} \{Z_i(a) - \rho_i^{\frac{1}{2}} Z_.(a)\}^2 \ ,$$

and standard methods for normal variables yield that, if $a \in A(p_0)$,

$$L_\eta(T_a) = \chi^2_{k-1,\delta^2(a,\eta)}$$

where

$$\delta^2(a,\eta) = \sum_{i=1}^{k} \rho_i \{\sum_{j=1}^{m} a_j(\eta_{ij} - \eta_{.j})\}^2$$

$$\eta_{.j} = \sum_{i=1}^{k} \rho_i \eta_{ij} \ .$$

Let ϕ_a be the test which rejects for $T_a > \chi^2_{k-1;\alpha}$. It follows from Corollary 8.3.1 that if $\phi_{a^*(p_0)}$ is the MS - $\{\phi_a \mid a \in A(p_0)\}$ test for $T(p_0)$, then the test which rejects for

$$T_{a^*(p_0)} > \chi^2_{k-1;\alpha} \ ,$$

with $p_0 = n^{-1}(N_{+1}, \ldots, N_{+m})$ and

$$Y_{ij} = \rho_i^{-\frac{1}{2}} n_i^{-\frac{1}{2}} N_{ij} \ ,$$

is an EAMS - asymptotically - squared means test for the testing problem of Section 9.3. The vector $a^*(p_0)$ will be determined; consider a fixed p_0 and let $a^* = a^*(p_0)$, $A = A(p_0)$.

Let $G(\delta^2)$ be the probability that a random variable with the χ^2_{k-1,δ^2} distribution exceeds the value $\chi^2_{k-1;\alpha}$. Then the power of ϕ_a in η is $G(\delta^2(a,\eta))$. The vector a^* satisfies

$$\sup_{\eta \in V+K} \gamma(a^*,\eta) = \inf_{a \in A} \sup_{\eta \in V+K} \gamma(a,\eta) ,$$

where $\gamma(a,.)$ is the shortcoming function of ϕ_a with respect to $\{\phi_c \mid c \in A\}$:

$$\gamma(a,\eta) = \sup_{c \in A} G(\delta^2(c,\eta)) - G(\delta^2(a,\eta)) .$$

Note that $V + K$ is a cone and $\delta^2(a,\eta)$ a homogeneous function in η. The following lemma shows that the maximum shortcoming of ϕ_a on the half-line $\{t\eta \mid t > 0\}$ depends on (a,η) as a decreasing function of

$$\tilde{\delta}^2(a,\eta) = \delta^2(a,\eta) / \{\sup_{c \in A} \delta^2(c,\eta)\} .$$

<u>LEMMA A.6.1</u>. *The function*

$$\sup_{t>0} \gamma(a,t\eta)$$

is a decreasing function of $\tilde{\delta}^2(a,\eta)$.

<u>PROOF</u>. This result follows from

$$\sup_{t>0} \gamma(a,t\eta) =$$

$$= \sup_{t>0} \{ \sup_{c \in A} G(\delta^2(c,t\eta)) - G(\delta^2(a,t\eta))\}$$

$$= \sup_{t>0} \{ \sup_{c \in A} G(t^2\delta^2(c,\eta)) - G(t^2\delta^2(a,\eta))\}$$

$$= \sup_{t>0} \{G(t^2 \sup_{c \in A} \delta^2(c,\eta)) - G(t^2\delta^2(a,\eta))\}$$

$$= \sup_{s>0} \{G(s) - G(s\tilde{\delta}^2(a,\eta)) \quad ;$$

note that G is an increasing function. □

Hence the desired vector a^* is the solution of

$$\inf_{\eta \in V+K} \tilde{\delta}^2(a^*,\eta) = \sup_{a \in A} \inf_{\eta \in V+K} \tilde{\delta}^2(a,\eta) .$$

The following lemma demonstrates that a^* can be obtained as the solution of the simpler maximin problem

$$\text{(A.6.1)} \qquad \inf_{\theta \in K_0} d^2(a^*,\theta) = \sup_{a \in A} \inf_{\theta \in K_0} d^2(a,\theta),$$

where

$$K_0 = \{\theta \in \mathbb{R}^m \mid \sum_{j=1}^{h} \theta_j \leq 0 \quad \text{for } 1 \leq h \leq m-1, \ \sum_{j=1}^{m} \theta_j = 0\}$$

$$d^2(a,\theta) = (a'\theta)^2 / \{\sup_{c \in A}(c'\theta)^2\} \qquad .$$

LEMMA A.6.2. *For all* $a \in A$ *one has*

$$\text{(A.6.2)} \qquad \inf_{\eta \in V+K} \tilde{\delta}^2(a,\eta) = \inf_{\theta \in K_0} d^2(a,\theta) \ .$$

PROOF. (i) Let $t_1, \ldots, t_k$ be numbers with $\Sigma_i \rho_i t_i = 0$. For every $\theta \in K_0$, there exists an $\eta \in V+K$ with $\eta_{ij} - \eta_{.j} = t_i \theta_j$ for all (i,j). For this η,

$$\delta^2(a,\eta) = \sum_{i=1}^{k} \rho_i \{\sum_{j=1}^{m} a_j(\eta_{ij} - \eta_{.j})\}^2 = \sum_{i=1}^{k} \rho_i t_i^2 \ (a'\theta)^2 \ ,$$

implying that $\tilde{\delta}^2(a,\eta) = d^2(a,\theta)$. Hence the $\leq$ - sign in (A.6.2) is trivial; it remains to prove the $\geq$ - sign.

(ii) Note that if a is not monotone (i.e., if there exist indices j_1, j_2, j_3 with $j_1 < j_2 < j_3$ and $(a_{j_3} - a_{j_2})(a_{j_2} - a_{j_1}) < 0$), then both sides are equal to 0. Note also that both sides are invariant under the transformation $a \mapsto -a$. Hence it is sufficient to consider an arbitrary $a_0 \in A$ with $a_{01} \leq a_{02} \leq \cdots \leq a_{0m}$, and prove that the $\geq$ - sign in (A.6.2) holds for this a_0.

(iii) It may be assumed that (after a permutation of the indices i) $\tilde{\delta}^2(a_0,.)$ assumes its infimum in some $\eta \in (V+K)_+$ where

$$(V+K)_+ = \{\eta \in V+K \mid \sum_{j=1}^{h} (\eta_{i+1,j} - \eta_{ij}) \leq 0 \quad \text{for all h and i}\} \ .$$

It will be convenient to transform η to a variable for which the inequalities defining $(V+K)_+$ are more simply related to the expression for the function δ^2. Define

$$b_j(a) = a_{j+1} - a_j \qquad (1 \le j \le m-1)$$

$$u_{ih}(\eta) = \sum_{j=h+1}^{m} (\eta_{ij} - \eta_{.j}), \qquad u_i(\eta) = (u_{i1}, \ldots, u_{i,m-1})'.$$

Then

$$\sum_{j=1}^{m} a_j(\eta_{ij} - \eta_{.j}) = b'(a)u_i(\eta) \;, \quad \sum_{i=1}^{k} \rho_i u_i(\eta) = 0 \;.$$

Now let

$$v_i(\eta) = u_{i+1}(\eta) - u_i(\eta) \qquad (1 \le i \le k-1) \;.$$

Then $v_{ij}(\eta) \ge 0$ for all $\eta \in (V+K)_+$. Some computations show that

$$\delta^2(a,\eta) = \sum_{i,h=1}^{k-1} r_{ih}(b'(a)v_i(\eta))(b'(a)v_h(\eta)) \tag{1}$$

where $r_{ih} = r_{hi}$ and

$$r_{ih} = (\sum_{g=1}^{h} \rho_g)(\sum_{g=i+1}^{k} \rho_g)(1 + 2\sum_{g=h+1}^{i} \rho_g) > 0 \qquad 1 \le h \le i \le k.$$

(iv) For $v \in \mathbb{R}^{m-1}$ define

$$||v||^2 = \sup_{a \in A} (b'(a)v)^2 \;.$$

Denote $b(a_0)$ by b_0. Let $v_i = v_i(\eta)$ for some $\eta \in (V+K)_+$, and let i_0 be the index with

$$(b_0' v_{i_0})^2 / ||v_{i_0}||^2 = \min_{i=1}^{k-1} (b_0' v_i)^2 / ||v_i||^2 \;. \tag{2}$$

Denote v_{i_0} by v. Note that $b_0'v_i \ge 0$ for all i. We have

$$(b_0'v)^2 \sup_{a \in A} \delta^2(a,\eta) \overset{(1)}{\leq}$$

$$\leq (b_0'v)^2 \sum_{i,h} r_{ih} \, ||v_i|| \, ||v_h|| \overset{(2)}{\leq}$$

$$\leq ||v||^2 \sum_{i,h} r_{ih} (b_0'v_i)(b_0'v_h) \overset{(1)}{=}$$

$$= ||v||^2 \, \delta^2(a_0,\eta),$$

which is equivalent to

$$(3) \qquad (b_0'v)^2 / ||v||^2 \leq \tilde{\delta}^2(a_0,\eta) .$$

Let $v = (v_{01}, \ldots, v_{0,m-1})$ and $v_{00} = v_{0m} = 0$; define

$$\theta_j = v_{0,j-1} - v_{0j} \qquad 1 \leq j \leq m .$$

Then $\Sigma_{j=1}^{h} \theta_j = -v_{0h}$, implying that $\theta \in K_0$; furthermore we have

$$a'\theta = b'(a)v \qquad \text{for all } a \in A .$$

Hence (3) is equivalent to $d^2(a_0,\theta) \leq \tilde{\delta}^2(a_0,\eta)$. This implies that the $\geq$ - sign holds in (A.6.2) for $a = a_0$. □

The maximin problem (A.6.1) will be solved by means of the method of ABELSON and TUKEY (1963) (see part 4 of Section 3.3). Note that we consider a fixed p_0. Define the inner product in $\mathbb{R}^m$

$$[x,y] = \sum_{j=1}^{m} x_j y_j / p_{oj} , \quad ||x|| = \{[x,x]\}^{\frac{1}{2}} ,$$

and let

$$\tilde{a}_j = a_j p_{oj}, \quad B = \{\tilde{a} \mid a \in A\}.$$

Then

$$a'\theta = [\tilde{a},\theta]$$

$$B = \{x \in \mathbb{R}^m \mid \sum_{j=1}^{m} x_j = 0, \; ||x|| = 1\} .$$

The maximin problem (A.6.1) amounts to finding $x^* \in B$ with

(A.6.3) $$\inf_{\theta \in K_0} [x^*,\theta] \,/\, ||\theta|| = \sup_{x \in B} \inf_{\theta \in K_0} [x,\theta] \,/\, ||\theta|| .$$

The edges of the cone K_0 are $e_1, \ldots, e_{m-1}$ where

$$e_j = \{x \in \mathbb{R}^m \mid x_j = -t,\ x_{j+1} = t \text{ for some } t > 0;\ x_h = 0 \text{ for } h \neq j, j+1\} .$$

The method of ABELSON and TUKEY (1963) implies that if $x^* \in K_0$ is a vector which makes equal angles with the edges $e_1, \ldots, e_{m-1}$, then x^* is the solution of (A.6.3). So we try to solve the system of equations

$$(p_{0j}^{-1} + p_{0,j+1}^{-1})^{-\frac{1}{2}} (p_{0,j+1}^{-1} x_{j+1} - p_{0j}^{-1} x_j) = c \qquad 1 \leq j \leq m-1$$

or

$$x_{j+1} = p_{0,j+1} \{p_{0j}^{-1} x_j + c(p_{0j}^{-1} + p_{0,j+1}^{-1})^{\frac{1}{2}}\} \qquad 1 \leq j \leq m-1 .$$

With the restriction $\Sigma_j x_j = 0$, we obtain the solution

$$x_j^* = c\, p_{0j}(\pi_j - \pi.)$$

where

$$\pi_j = \sum_{h=1}^{j-1} (p_{0h}^{-1} + p_{0,h+1}^{-1})^{\frac{1}{2}} , \quad \pi. = \sum_{j=1}^{m} p_{0j}\, \pi_j ,$$

while c is determined by the requirement $\Sigma_j (x_j^*)^2\, p_{0j} = 1$. Hence the maximin vector a^*, with $a_j^* = p_{0j}^{-1} x_j^*$, is given by

$$a_j^* = c(\pi_j - \pi.)$$

$$c = \{\sum_{j=1}^{m} p_{0j}(\pi_j - \pi.)^2\}^{-\frac{1}{2}} .$$

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