Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.

The Mathematical Centre, founded 11 February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

MATHEMATICAL CENTRE TRACTS 105

EDGEWORTH EXPANSIONS FOR LINEAR COMBINATIONS OF ORDER STATISTICS

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Second edition

MATHEMATISCH CENTRUM AMSTERDAM 1984

1980 Mathematics subject classification: 62E20, 62G20, 62G30

ISBN 90 6196 174 2

Firts printing 1982
Second edition 1984

ACKNOWLEDGEMENTS

This monograph is a slightly revised version of a Ph.D. thesis written at the Department of Mathematical Statistics of the Mathematical Centre in Amsterdam under the supervision of professor W.R. van Zwet. I am very grateful to him for suggesting the problem and for his excellent guidance throughout the preparation of this work. Also I have benefited from a number of comments made by dr. W. Albers.

I thank the Mathematical Centre for the opportunity to publish this monograph in the series Mathematical Centre Tracts and all those at the Mathematical Centre who have contributed to its technical realization.

CONTENTS

GENERAL INTRODUCTION

Let $X_1, X_2, \ldots$ denote a sequence of independent and identically distributed random variables with common distribution function F. Statistics of the form

$$(0.1) \qquad T_n = n^{-1} \sum_{i=1}^{n} c_{in} X_{i:n} \qquad n = 1,2,\ldots,$$

where $X_{i:n}$ $(1 \le i \le n)$ denotes the i^{th} order statistic of $X_1,\ldots,X_n$ and the c_{in}, $i = 1,2,\ldots,n$ are real numbers (weights) are said to be *linear combinations (functions) of order statistics*, or *L-estimators*. Many authors have established the asymptotic normality of T_n under different sets of conditions (see section 1.2); e.g. in STIGLER (1974) it is assumed that the weights are given by

$$(0.2) \qquad c_{in} = J\left(\frac{i}{n+1}\right), \qquad i = 1,2,\ldots,n, \quad n = 1,2,\ldots,$$

where J is a smooth bounded function on (0,1), the second moment of F is finite and $\sigma^2(J,F) > 0$ where

$$(0.3) \qquad \sigma^2(J,F) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(F(x))J(F(y))(\min(F(x),F(y)) - F(x)F(y))dxdy.$$

Under these assumptions Stigler shows that

$$(0.4) \qquad \sup_x |F_n^*(x) - \Phi(x)| = o(1), \qquad \text{as } n \to \infty,$$

where

$$(0.5) \qquad F_n^*(x) = P(\{(T_n - E(T_n))/\sigma(T_n) \le x\})$$

and Φ denotes the standard normal distribution function. In addition these assumptions imply that

$$\lim_{n\to\infty} n\sigma^2(T_n) = \sigma^2(J,F). \tag{0.6}$$

The question which first aroused the author's interest was to obtain precise information about the rate of convergence in (0.4). Assuming now that the third absolute moment of F is finite and imposing a stronger smoothness condition on J we prove in chapter 3 that $\sigma^2(J,F) > 0$ implies in this case that

$$\sup_x \left| F_n^*(x) - \Phi(x) \right| = \mathcal{O}(n^{-\frac{1}{2}}), \qquad \text{as } n \to \infty, \tag{0.7}$$

i.e. we establish *Berry-Esseen bounds* of order $n^{-\frac{1}{2}}$ for linear combinations of order statistics with smooth weights. Similar results employing a different and more practical standardization and for a studentized version of these statistics are also proved.

For several reasons, to be explained in the sequel, it is of interest to go a step further and to derive *Edgeworth expansions* for linear combinations of order statistics. General theorems according to which statistics of the form (0.1) possess valid Edgeworth expansions will require, of course, stronger conditions than before. We now assume that the fourth moment of F is finite, we impose an even stronger smoothness condition on J, and, in addition, we impose a local smoothness condition on F. The latter condition, which is due to VAN ZWET (1977) (see lemma 2.1.2), will do what Cramér's condition (C) does in the classical case of sums of independent random variables: it guarantees that F_n^* is sufficiently smooth. In chapter 4 we prove that $\sigma^2(J,F) > 0$ implies in this case that

$$\begin{aligned}\sup_x \Big| F_n^*(x) - \Phi(x) + \phi(x)\Big\{ &\frac{\kappa_3}{6n^{\frac{1}{2}}}(x^2-1) + \frac{\kappa_4}{24n}(x^3-3x) + \\ &+ \frac{\kappa_3^2}{72n}(x^5-10x^3+15x)\Big\}\Big| = o(n^{-1}), \qquad \text{as } n \to \infty,\end{aligned} \tag{0.8}$$

i.e. we establish an uniformly valid Edgeworth expansion for linear combinations of order statistics with a remainder $o(n^{-1})$. The function ϕ denotes the standard normal density; the quantities $\kappa_3 n^{-\frac{1}{2}}$ and $\kappa_4 n^{-1}$ are the leading terms in asymptotic expansions for the third and fourth cumulant of

$T_n^* = (T_n - ET_n)/\sigma(T_n)$. Similar results generalizing the type of weights and employing a different and more practical standardization of T_n are also proved.

It is a well-known phenomenon that to every asymptotic result for linear combinations of order statistics with smooth weights, like (0.7) and (0.8), there corresponds a similar result for these statistics with smooth F. The Berry-Esseen bound (0.7) for smooth F was derived by BJERVE (1977). Edgeworth expansions for the case of smooth F are established in chapter 5. However, to obtain such results, one is forced to restrict attention to trimmed linear combinations of order statistics; i.e. instead of (0.2) one has to assume that

$$c_{in} = 0 \quad \text{for} \quad i < n\alpha \quad \text{or} \quad i > n\beta, \tag{0.9}$$

for all $n \geq 1$ and some $0 < \alpha < \beta < 1$. These results include trimmed and Winsorized means (see the examples (1.2.2) and (1.2.5)) as important special cases. An Edgeworth expansion for α-trimmed means (i.e. for the special case that $c_{in} = (n-2[n\alpha])^{-1}n$ for $[n\alpha]+1 \leq i \leq n-[n\alpha]$) was derived by BJERVE (1974). He exploits a special property of trimmed means which does not carry over to the more general statistics we consider.

There are several reasons to establish Berry-Esseen bounds and Edgeworth expansions for linear combinations of order statistics. In the first place we note that from the standpoint of probability theory the type of results discussed so far can be viewed as a contribution to the problem of extending the classical theory of Edgeworth expansions for sums of independent random variables to certain sums of dependent random variables. However, also from a statistical point of view, there are several reasons to be interested in such results. First there is the possibility to use these expansions to obtain better numerical approximations to the distribution functions of linear combinations of order statistics than can be provided by the usual normal approximation. A second and perhaps more compelling reason is the fact that Edgeworth expansions can be used to compute higher order efficiencies of L-estimators. The introduction of the concept of *deficiency* by HODGES & LEHMANN in 1970 has been the starting point of much work in this direction. Let us briefly introduce the concept of deficiency and indicate the kind of deficiency computations we shall perform. Let T_1 and T_2 be two point estimators. If T_1 has a better performance than T_2 and T_1 is based on n observations we need $k_n = n+d_n$ observations for T_2 to perform equally well. We may think of the expected mean square error or some other reasonable measure of dispersion as a criterion of performance. Here k_n and d_n have to be treated as continuous variables the performance of T_2 being defined for real n by linear interpolation between consecutive integers. The quantity d_n - the number of additional observations needed by T_2 to perform equally well as T_1 - is called the deficiency of T_2 with respect to T_1. In general, however, d_n cannot be determined exactly for fixed n and we have to rely on its asymptotic behaviour for $n \to \infty$. Such an investigation is useful in particular when for $n \to \infty$ the ratio n/k_n tends to 1; i.e. when the asymptotic relative efficiency of T_2 with respect to T_1 is equal to 1. In this case T_1 and T_2 are, at least to first order, equally efficient, and the asymptotic behaviour of d_n - which may now be anything from $O(1)$ to $O(n)$ - does provide important additional information about the relative performances of the estimators involved. Of special interest is the case where d_n tends to a finite limit, the asymptotic deficiency of T_2 with respect to T_1. Of course an asymptotic evaluation of d_n is a more delicate matter than showing that the asymptotic relative efficiency of T_2 with respect to T_1 is equal to 1. What is needed is an expansion of the type we discussed above. With the aid of such expansions we obtain expressions for d_n with remainder $O(1)$. In chapter 6 we compute a number of asymptotic deficiencies of L-estimators with respect to two other types of estimators: M-estimators which are of maximum likelihood type and R-estimators derived from rank tests.

The organization of this study is as follows. In chapter 1 we review the literature on Edgeworth expansions and on linear combinations of order statistics. A number of preliminary results are collected in chapter 2. Chapter 3 is devoted to the problem of establishing Berry-Esseen type bounds for linear combinations of order statistics. In chapter 4 we establish Edgeworth expansions for these statistics for the case of smooth weights, whereas in chapter 5 we do the same for the case of a smooth distribution function. Chapter 6 contains deficiency computations for L-estimators with respect to M- and R-estimators. The numerical aspects of the expansions are briefly discussed in chapter 7.

CHAPTER I

INTRODUCTION

1.1. EDGEWORTH EXPANSIONS

The purpose of this section is twofold. In the first place we present a brief survey of some of the main results of the classical theory of Edgeworth expansions for sums of independent random variables. Secondly the problem of extending the theory of Edgeworth expansions for sums of independent random variables to more general statistics is briefly considered and a review of a number of the more recent results in this area is given.

We begin by introducing some notations that will be used throughout this study. Let $(\Omega,\mathcal{A},P)$ be a probability space on which a random variable (rv) X is defined, having distribution function (df)

$$(1.1.1) \qquad F(x) = P(\{X \le x\})$$

for all $-\infty < x < \infty$. The inverse F^{-1} of a df F will always be defined as

$$(1.1.2) \qquad F^{-1}(t) = \inf\{x: F(x) \ge t\}$$

for all $0 < t < 1$. We shall assume that all rv's will be defined on the above mentioned probability space. For any positive integer k the k^{th} moment and the k^{th} central moment of X are EX^k and $E(X-EX)^k$ respectively, whenever well-defined; for any positive number k the k^{th} absolute moment of X is $E|X|^k$. The variance $E(X-EX)^2$ will also be written as $\sigma^2(X)$. For any rv X with $0 < \sigma(X) < \infty$ we introduce

$$(1.1.3) \qquad \hat{X} = X - E(X)$$

and

(1.1.4) $$X^* = \hat{X}/\sigma(X) = (X-E(X))/\sigma(X).$$

The characteristic function (ch.f.) of a rv X is defined as

(1.1.5) $$Ee^{itX} = \int_{-\infty}^{\infty} e^{itx} dF(x)$$

for all $-\infty < t < \infty$. All integrals will be understood to be Lebesgue-Stieltjes integrals. In the notation of these integrals we always write dF for integration with respect to the measure corresponding to F. Finally let Φ and ϕ denote the standard normal df and its density.

The classical theory of Edgeworth expansions is concerned with sums of independent rv's. This theory is a well-established part of probability theory and there exist a number of excellent accounts of the theory of Edgeworth expansions for such sums; e.g. CRAMÉR (1962), GNEDENKO & KOLMOGOROV (1954), PETROV (1972) and BHATTACHARYA & RAO (1976). The latter reference contains the extensions of the classical theory to the multi-dimensional case: i.e. to sums of independent random vectors. A nice introduction can be found in FELLER (1966).

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) rv's with common df F. Let us indicate the expectation and variance of X_1 by μ and σ^2 respectively. We assume that $\sigma^2 > 0$. Consider, for each $n \geq 1$, the normalized sum

(1.1.6) $$T_n^* = n^{-\frac{1}{2}}\sigma^{-1} \sum_{i=1}^{n} (X_i-\mu)$$

and let us denote the df of T_n^* by

(1.1.7) $$F_n^*(x) = P(\{T_n^* \leq x\})$$

for all $-\infty < x < \infty$. The Lindeberg-Lévy central limit theorem asserts that

(1.1.8) $$\sup_x |F_n^*(x) - \Phi(x)| = o(1), \qquad \text{as } n \to \infty,$$

provided $0 < \sigma^2 < \infty$. When higher moments of X_1 exist precise information concerning the rate of convergence of F_n^* to Φ can be obtained. More specifically if we assume that $E|X_1|^3 < \infty$, the Berry-Esseen theorem states that

(1.1.9) $$\sup_x |F_n^*(x) - \Phi(x)| = O(n^{-\frac{1}{2}}), \qquad \text{as } n \to \infty,$$

i.e. the order of the normal approximation to the exact df of a normalized sum of i.i.d. rv's is $n^{-\frac{1}{2}}$. One way to improve upon the normal approximation is to establish Edgeworth expansions. The main result in this direction is due to Cramér. Suppose that

(1.1.10) $$EX_1^4 < \infty$$

and let $\kappa_3 = E(X_1-\mu)^3/\sigma^3$ and $\kappa_4 = E(X_1-\mu)^4/\sigma^4 - 3$ denote the third and fourth cumulant of $(X_1-\mu)/\sigma$. Moreover we assume that *Cramér's condition* (C) (CRAMÉR (1962)) is satisfied; i.e.

(1.1.11) $$\limsup_{|t| \to \infty} |\rho(t)| < 1$$

where ρ denotes the ch.f. of X_1. We remark that (1.1.11) implies that for every $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\sup_{|t| \geq \delta} |\rho(t)| \leq 1 - \varepsilon.$$

<u>THEOREM 1.1</u>. (Cramér). *Suppose that the assumptions* (1.1.10) *and* (1.1.11) *are satisfied. Then* $\sigma^2 > 0$ *implies that*

(1.1.12) $$\sup_x |F_n^*(x) - \tilde{F}_n(x)| = o(n^{-1}), \qquad \text{as } n \to \infty$$

with

(1.1.13) $$\tilde{F}_n(x) = \Phi(x) - \phi(x)\left\{\frac{\kappa_3}{6n^{\frac{1}{2}}}(x^2-1) + \frac{\kappa_4}{24n}(x^3-3x) + \frac{\kappa_3^2}{72n}(x^5-10x^3+15x)\right\}$$

for all $-\infty < x < \infty$.

It may be useful to comment briefly on Cramér's result. In the first place we remark that the quantities $\kappa_3 n^{-\frac{1}{2}}$ and $\kappa_4 n^{-1}$ are the third and fourth cumulant of the normalized sum (1.1.6) and that the polynomials appearing in (1.1.13) are the Hermite polynomials of order 2, 3 and 5. Secondly we note that Cramér's condition (C) (cf. (1.1.11)) is satisfied if F possesses an absolutely continuous component. Finally we remark that, although we have restricted attention to the case of an Edgeworth expansion with remainder

$O(n^{-1})$ (cf. (1.1.12)) Edgeworth expansions for sums of i.i.d. rv's to any order can be obtained at cost of a stronger moment condition in essentially the same way. Edgeworth expansions with remainder $O(n^{-1})$ will be sufficient for our purposes. The proof of Cramér's result is well-known (see, e.g., FELLER (1966)). Because it contains in essence already a few crucial ideas, which will be of great importance in the more general problem we consider, we shall briefly sketch the proof. We follow mainly the one-page version of Cramér's proof as given in VAN ZWET (1977). The starting point of the proof is a famous result proved by ESSEEN (1945).

<u>LEMMA 1.2.</u> (Esseen smoothing lemma). *Let* m *be a positive number,* F *a df on* $\mathbb{R}$ *and* $\tilde{F}$ *a differentiable function of bounded variation on* $\mathbb{R}$ *with* $\tilde{F}(-\infty) = 0$, $\tilde{F}(\infty) = 1$ *and* $|\tilde{F}'| \leq m$ *(the prime denoting differentiation). Define the Fourier-Stieltjes transforms* $\psi(t) = \int_{-\infty}^{\infty} e^{itx} dF(x)$ *and* $\tilde{\psi}(t) = \int_{-\infty}^{\infty} e^{itx} d\tilde{F}(x)$. *Then there exists a constant* C *such that for every* $T > 0$

$$(1.1.14) \qquad \sup_x |F(x) - \tilde{F}(x)| \leq \frac{1}{\pi} \int_{-T}^{T} \left|\frac{\psi(t) - \tilde{\psi}(t)}{t}\right| dt + \frac{Cm}{T}.$$

<u>SKETCH OF THE PROOF OF THEOREM 1.1.</u> Let ρ_n^* denote the ch.f. of $n^{-\frac{1}{2}}\sigma^{-1}\Sigma_{i=1}^n (X_i - \mu)$, i.e.

$$(1.1.15) \qquad \rho_n^*(t) = \rho^n(tn^{-\frac{1}{2}}\sigma^{-1})e^{-itn^{\frac{1}{2}}\mu\sigma^{-1}} \qquad \text{for } -\infty < t < \infty.$$

It follows from assumption (1.1.10) that for $|t| = O(n^{\frac{1}{2}})$

$$(1.1.16) \qquad \log \rho_n^*(t) = -\frac{t^2}{2} - \frac{i}{6}\kappa_3 n^{-\frac{1}{2}}t^3 + \frac{1}{24}\kappa_4 n^{-1}t^4 + O(n^{-1}t^4) \qquad \text{as } n \to \infty.$$

This expansion of $\log \rho_n^*(t)$ can be converted into an expansion for $\rho_n^*(t)$:

$$(1.1.17) \qquad \rho_n^*(t) = \tilde{\rho}_n(t) + O(n^{-1}|t|e^{-\frac{t^2}{4}}),$$

where

$$(1.1.18) \qquad \tilde{\rho}_n(t) = e^{-\frac{t^2}{2}}\{1 - \frac{i}{6}\kappa_3 n^{-\frac{1}{2}}t^3 + \frac{1}{24}\kappa_4 n^{-1}t^4 - \frac{1}{72}\kappa_3^2 n^{-1}t^6\}.$$

For any sufficiently small $\delta > 0$ this expansion remains valid for all $|t| \leq \delta n^{\frac{1}{2}}$ because

$$(1.1.19)\qquad |\rho_n^*(t)| \le \left(1-\frac{t^2}{3n}\right)^n \le e^{-\frac{1}{3}t^2} \qquad \text{for } |t| \le \delta n^{\frac{1}{2}}.$$

Hence it follows that

$$(1.1.20)\qquad \int_{-\delta n^{\frac{1}{2}}}^{\delta n^{\frac{1}{2}}} \left|\frac{\rho_n^*(t)-\tilde{\rho}_n(t)}{t}\right| dt = o(n^{-1}), \qquad \text{as } n \to \infty$$

and also that

$$(1.1.21)\qquad \int_{|t|\ge\delta n^{\frac{1}{2}}} \left|\frac{\tilde{\rho}_n(t)}{t}\right| dt = o(n^{-1}), \qquad \text{as } n \to \infty.$$

It remains to show that also

$$(1.1.22)\qquad \int_{\delta n^{\frac{1}{2}}\le|t|\le n^{\frac{3}{2}}} \left|\frac{\rho_n^*(t)}{t}\right| dt = o(n^{-1}) \qquad \text{as } n \to \infty.$$

This, however, is a direct consequence of the product-structure (cf. (1.1.15)) present in $\rho_n^*(t)$ and the fact that Cramér's condition (C) (cf. (1.1.11) and the remark following it) can be applied. Since $\tilde{F}_n$ (cf. (1.1.13)) is the Fourier-Stieltjes transform of $\tilde{\rho}_n$ (cf. (1.1.18)) it follows now from (1.1.20), (1.1.21), (1.1.22) in combination with an application of Esseen smoothing lemma, taking $T = n^{3/2}$, that the theorem is proved. □

The problem to extend the classical theory of Edgeworth expansions for sums of independent rv's to more general statistics has been the subject of much work in recent years. Let us first briefly indicate that such an extension is plausible and then survey some of the more recent results obtained in this area.

Suppose that a sequence of statistics T_n^* with df F_n^*, $n = 1,2,\ldots$ converges in distribution to the standard normal distribution. If we write

$$(1.1.23)\qquad \rho_n^*(t) = Ee^{itT_n^*}$$

we are simply saying that

$$(1.1.24)\qquad \rho_n^*(t) \to e^{-\frac{t^2}{2}} \qquad \text{as } n \to \infty$$

for all $-\infty < t < \infty$. Suppose now that T_n^* has cumulants κ_{jn} $(1 \le j \le 4)$. Typically we will have

$$(1.1.25) \qquad \kappa_{1n} = 0, \quad \kappa_{2n} = 1, \quad \kappa_{3n} = \mathcal{O}(n^{-\frac{1}{2}}) \quad \text{and} \quad \kappa_{4n} = \mathcal{O}(n^{-1}).$$

We can now formally expand $\log \rho_n^*$ in a Taylor series of which the first terms are given by

$$(1.1.26) \qquad -\frac{t^2}{2} - \frac{i}{6} t^3 \kappa_{3n} + \frac{1}{24} t^4 \kappa_{4n}.$$

Again expanding formally, we approximate ρ_n^* itself by

$$(1.1.27) \qquad e^{-\frac{t^2}{2}} \left(1 - \frac{it^3}{6} \kappa_{3n} + \frac{3\kappa_{4n} t^4 - \kappa_{3n}^2 t^6}{72}\right)$$

which is the Fourier-Stieltjes transform of

$$(1.1.28) \qquad \tilde{F}_n(x) = \Phi(x) - \phi(x)\left\{\frac{\kappa_{3n}}{6}(x^2-1) + \frac{\kappa_{4n}}{24}(x^3-3x) + \frac{\kappa_{3n}^2}{72}(x^5-10x^3+15x)\right\}.$$

In view of this formal argument it seems reasonable to hope that $\tilde{F}_n$ will indeed provide an approximation to F_n^*. Note that in the case of theorem 1.1. $\kappa_{3n} = \kappa_3 n^{-\frac{1}{2}}$ and $\kappa_{4n} = \kappa_4 n^{-1}$. Of course this heuristic argument will have to be verified in each particular case; more precisely one has to show that

$$(1.1.29) \qquad \sup_x |F_n^*(x) - \tilde{F}_n(x)| = o(n^{-1}), \qquad \text{as } n \to \infty,$$

with the aid of lemma 1.2.

The validity of (1.1.29) has been established for quite a number of estimators and test statistics arising in statistical models. Concerning statistics arising in parametric models we mention the work of CHIBISOV (1972), (1973a), (1973b), (1973c), (1974) and PFANZAGL (1972), (1973), (1974a), (1974b). These authors established Edgeworth expansions for maximum likelihood estimators and also for the more general class of minimum contrast estimators. We also refer to a recent paper of BHATTACHARYA & GHOSH (1978) who obtained some related results. In non-parametric statistics ALBERS, BICKEL & VAN ZWET (1976) have established asymptotic expansions for the power of linear rank tests for the one-sample symmetry problem.

In a parallel paper BICKEL & VAN ZWET (1978) established similar results for two-sample rank statistics. Extension of these results to the case of general linear rank statistics is an interesting unsolved problem. A review of these developments was given by BICKEL (1974). The problem to establish Berry-Esseen type bounds and Edgeworth expansions for linear combinations of order statistics was an open problem at the time of Bickel's 1974 review paper, although a number of partial results were known. ROSENKRANTZ & O'REILLY (1972) found a rate of convergence not better than $n^{-\frac{1}{4}}$ for the normal approximation to the df of linear combinations of order statistics, using the Skorohod embedding method. They also showed that nothing more can be obtained by this approach. A nearly optimal error bound of order $n^{-\frac{1}{2}} \ell n\, n$ for the same problem was derived by EGOROV & NEVZOROV (1976) using an exponential bound due to PETROV (1972) as an important tool. A related result was obtained by DE WET (1976). An important stimulus to obtain the optimal rate of convergence $n^{-\frac{1}{2}}$ for the normal approximation to the df's of linear combinations of order statistics was given by BICKEL (1974). By an ingenious method based on the martingale structure of U-statistics BICKEL (1974) was able to use Esseen's smoothing lemma to establish a Berry-Esseen bound of order $n^{-\frac{1}{2}}$ for U-statistics of order 2 with a non-degenerate bounded kernel. The method of proof of BICKEL (1974) was then used by BJERVE (1977) and HELMERS (1977) to obtain Berry-Esseen type bounds of order $n^{-\frac{1}{2}}$ for linear combinations of order statistics. We may also mention in this connection two papers of HÚSKOVA (1977), (1979) who obtained, also applying Bickel's method, a Berry-Esseen bound of order $n^{-\frac{1}{2}}$ for general linear rank statistics, both under the hypothesis, contiguous and fixed alternatives. Bickel's result concerning U-statistics was further improved by CHAN & WIERMAN (1977) and CALLAERT & JANSSEN (1978), using the martingale structure inherent in U-statistics in a different way. Using the Callaert & Janssen result the author (HELMERS (1981)) was able to weaken the conditions in HELMERS (1977). These results on Berry-Esseen bounds for linear combinations of order statistics are contained in chapter 3.

The problem to go from these Berry-Esseen bounds to Edgeworth expansions for linear combinations of order statistics was considered by VAN ZWET (1977). He was able to derive a bound on the characteristic function of a linear combination of order statistics which solves a crucial part of the problem to establish Edgeworth expansions for these statistics. Using this result of VAN ZWET (1977) (reproduced here as lemma 2.1.2) the author obtained Edgeworth expansions for linear combinations of order statistics with a remainder term of order $O(n^{-1})$ for $n \to \infty$. Based on totally different representations of a linear combination of order statistics these expansions were derived for the case of smooth weights (HELMERS (1980)) and for the case of a smooth distribution function (HELMERS (1979)). Edgeworth expansions for the special case of trimmed means were obtained by BJERVE (1974). These results concerning Edgeworth expansions for linear combinations of order statistics are contained in the chapters 4 and 5.

1.2. LINEAR COMBINATIONS OF ORDER STATISTICS

In this section we review the extensive literature on linear combinations of order statistics. We begin by introducing some more notation that will be used throughout this study.

Let $X_1, X_2, \ldots$ denote a sequence of i.i.d. rv's with common df F and let for each $n \geq 1$

(1.2.1) $$X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$$

denote $X_1, \ldots, X_n$ ordered in ascending order of magnitude. $X_{i:n}$ $(1 \leq i \leq n)$ is called the i^{th} order statistic of a sample of size n.

Furthermore let for each $n \geq 1$

(1.2.2) $$c_{1n}, c_{2n}, \ldots, c_{nn}$$

be a sequence of real numbers called *weights*. Frequently but not always, it will be assumed that these real numbers are generated in one way or another by a fixed real-valued measurable function J - called the *weight function* - defined in (0.1). One such way of generating weights is the following: Suppose that for each $n \geq 1$

(1.2.3) $$c_{in} = J\left(\frac{i}{n+1}\right) \qquad i = 1,2,\ldots,n.$$

Weights of the form (1.2.3) are the ones which are most frequently studied in the literature. In chapter 4 a quite general way of generating weights by means of weight functions is introduced and studied. We also refer to that chapter for a discussion of the various ways of generating weights found in the literature. *Linear combinations (functions) of order statistics*, or *L-estimators*, are statistics of the form

$$(1.2.4) \qquad T_n = n^{-1} \sum_{i=1}^{n} c_{in} X_{i:n}$$

Several authors (e.g. SHORACK (1972)) consider the somewhat larger class of statistics of the form

$$(1.2.5) \qquad T_n = n^{-1} \sum_{i=1}^{n} c_{in} h(X_{i:n}) + \sum_{k=1}^{K} d_{kn} X_{i_k:n}$$

where h is some function on the support of F, the d_{kn} form a double sequence of real numbers and the indices $i_1,\ldots,i_K$ satisfy $1 \le i_1 \le i_2 \le \ldots \le i_K \le n$. Though not indicated in the notation the function h and the indices i_k $(1 \le k \le K)$ may depend on n. K is fixed.

We present a few examples. For any real number x the largest integer smaller or equal than x will be denoted by [x].

<u>EXAMPLE 1.2.1</u>. *The sample mean*. If we take $c_{in} = 1$ for $i = 1,2,\ldots,n$ and $n \ge 1$, we see that $T_n = n^{-1} \Sigma_{i=1}^{n} X_i$, the sample mean.

<u>EXAMPLE 1.2.2</u>. *The α-trimmed mean*. Let $T_{n\alpha}$ denote the α-trimmed mean,

$$(1.2.6) \qquad T_{n\alpha} = (n - 2[n\alpha])^{-1} \sum_{i=[n\alpha]+1}^{n-[n\alpha]} X_{i:n}, \qquad 0 < \alpha < \frac{1}{2},$$

i.e. we take $c_{in} = (n-2[n\alpha])^{-1} n$ for $i = [n\alpha]+1,\ldots,n-[n\alpha]$, $n = 1,2,\ldots,$ and $c_{in} = 0$ otherwise.

<u>EXAMPLE 1.2.3</u>. *L-estimator for logistic location* (see, e.g., DAVID (1970), page 224). Let

$$(1.2.7) \qquad c_{in} = 6 \frac{i}{n+1} \left(1 - \frac{i}{n+1}\right)$$

for $i = 1,2,\ldots,n$ and $n \ge 1$. Then $T_n = n^{-1} \Sigma_{i=1}^{n} c_{in} X_{i:n}$ is the L-estimator for logistic location.

<u>EXAMPLE 1.2.4</u>. *Gini's mean difference* (see, e.g., STIGLER (1974)). Gini's mean difference is defined by

$$(1.2.8) \qquad G_n = (n(n-1))^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} |X_i - X_j|$$

but it can also be written as

$$(1.2.9) \qquad G_n = \frac{4(n+1)}{n(n-1)} \sum_{i=1}^{n} \left(\frac{i}{n+1} - \frac{1}{2}\right) X_{i:n}.$$

EXAMPLE 1.2.5. *The* α*-Winsorized mean*. Let $W_{n\alpha}$ denote the α-Winsorized mean,

$$(1.2.10) \qquad W_{n\alpha} = n^{-1}\Big([n\alpha]X_{[n\alpha]+1:n} + \sum_{i=[n\alpha]+1}^{n-[n\alpha]} X_{i:n} + [n\alpha]X_{n-[n\alpha]:n}\Big), \qquad 0 < \alpha < \frac{1}{2}.$$

This example falls into the wider class (1.2.5). We take $K = 2$, $c_{in} = 1$ for $i = [n\alpha]+1,\dots,n-[n\alpha]$, $n = 1,2,\dots$, $c_{in} = 0$ otherwise and $d_{1n} = d_{2n} = [n\alpha]n^{-1}$ for all $n \geq 1$.

The above examples illustrate a number of weights that may occur. More examples will be given in the subsequent chapters.

Statistics of the form (1.2.4) were already studied by P. Daniell in 1920 in an interesting paper "Observations Weighted According to Order" published in the American Journal of Mathematics. Daniell was the first to give a mathematical treatment of the class of statistics which are linear combinations of order statistics. His results include a derivation of the optimal weights in the linear combination for estimating location and scale parameters and an expression for the asymptotic variance of trimmed means. We refer to a paper of STIGLER (1972) for a nice account of these historical developments.

The work of Daniell was not noticed by the mathematicians of his time and it was in the early fifties that several people became interested again in the problem. BENNETT (1952) was concerned with least squares estimation of location and scale parameters by means of order statistics. Using the Gauss-Markov theorem Bennett was able to derive, for fixed sample size n and a fixed family of distributions depending only on location and scale, unbiased estimators for location and scale which have minimum variance in the class of all unbiased estimators which are of the form (1.2.4). We also refer to the work of LLOYD (1952), who obtained these results independently of Bennett. The computation of Bennett's estimators, however, is very difficult because it requires knowledge of the expectation of any single order statistic (up to a location-scale transformation) and the covariance of any two of them. For this reason BLOM (1958) and JUNG (1955) have attempted to derive large sample approximations to the best unbiased estimators of Bennett and Lloyd. They obtained estimators which are "nearly unbiased, nearly best" by using asymptotic approximations to the expectations of the order statistics and to their covariances. We refer to DAVID (1970) for a recent discussion of these results.

It seems useful to say a bit more about the work of JUNG (1955). He considers weights of the form (1.2.3). Assuming that J is four times differentiable with bounded derivatives on (0,1) he first derives asymptotic integral approximations for the expectation and variance of $n^{-1} \sum_{i=1}^{n} J(\frac{i}{n+1}) X_{i:n}$. He then proceeds, by using a calculus of variation argument, to find the linear combination of order statistics which is asymptotically optimal in the sense that the estimator is asymptotically normally distributed, with asymptotic mean equal to the location or scale parameter to be estimated and asymptotic variance attaining the Cramer-Rao bound. In fact he does not prove the asymptotic normality of his estimator but he only shows that these estimators are asymptotically unbiased and have minimum asymptotic variance.

However, the comparison of the performance of two estimators (or rather two sequences of estimators), with the asymptotic variances as the criterion of performance, seems only to be justified when these asymptotic variances can be considered as reasonable measures of dispersion of the two estimators considered. The classical situation in which this is the case arises, of course, when both estimators are asymptotically normally distributed. Thus motivated by the work of JUNG (1955) several authors became interested in the problem to find sufficient conditions for the asymptotic normality of linear combinations of order statistics.

BICKEL (1967) and CHERNOFF, GASTWIRTH & JOHNS (1967) seem to be first to consider this important problem. We shall review very briefly their approaches to the problem as well as that of the other contributors to this problem who came after them, notably MOORE (1968), STIGLER (1969), (1973), (1974) and SHORACK (1969), (1972), (1974).

Let us start by remarking that the problem of proving asymptotic normality for statistics of the form (1.2.4) (or (1.2.5)) has no easy answer. Several sets of sufficient conditions which guarantee that statistics of the form (1.2.4) - when appropriately normalized - are asymptotically normally distributed are possible: there exists a kind of balance between the restrictions put on the weights and the conditions imposed upon the df F. Either heavy restrictions are required for the c_{in} and rather mild conditions for F or the other way around. There is also another dichotomy present in the problem: although a number of different approaches to the problem of providing sufficient conditions for the asymptotic normality of statistics of the form (1.2.4) (or (1.2.5)) can be found in the literature, essentially two methods of proof appear to exist.

The first method is to decompose $n^{-1} \Sigma_{i=1}^{n} c_{in} X_{i:n}$ as follows

$$(1.2.11) \qquad n^{-1} \sum_{i=1}^{n} c_{in} X_{i:n} = S_n + R_n$$

such that nS_n is a sum of independent rv's to which - when appropriately normalized - a form of the central limit theorem can be applied and R_n is a remainder term which turns out to be of negligible order of magnitude; i.e. $n^{\frac{1}{2}} R_n$ converges in probability to zero, as $n \to \infty$. Slutsky's theorem can then be applied to conclude the proof. Though this idea is attractive because it is simple, the technical problems in carrying out this idea are not easy at all. First a decomposition of the form (1.2.11) has to be found. Then the program indicated above has to be carried out. There are several ways available in the literature to do this. CHERNOFF, GASTWIRTH & JOHNS (1967) exploit special properties of exponential order statistics and use a Taylor type argument (assuming a smooth df F) to find a decomposition of the form (1.2.11). Applying the Lindeberg - Feller central limit theorem to their S_n and making an intricate analysis of $E|R_n|$, the first absolute moment of their remainder term, they succeed in proving asymptotic normality for statistics of the form (1.2.4). Their conditions require a smooth F, but rather arbitrary weights are allowed.

A perhaps more elegant idea was used by STIGLER (1969), (1974). His approach is to apply Hájek's projection lemma (HÁJEK (1968)) to find a sum of independent rv's - the projection - which approximates a linear combination of order statistics T_n in mean square and show that this sum, when appropriately normalized, and T_n^* are mean square equivalent. As a consequence of using two different techniques of treating the remainder term STIGLER (1969) results require smooth df's, whereas STIGLER's (1974) results require a smooth weight function. To conclude our discussion of the various approaches based on a decomposition of the form (1.2.11) let us mention that an elegant short proof of the asymptotic normality of statistics of the form (1.2.4) was given by MOORE (1968). Moore took advantage of the possibility to represent T_n in terms of the empirical df. Assuming rather restrictive smoothness conditions for his weight function (the weights are of the form (1.2.3)) he can apply a Taylor type argument to complete his proof. Note, however, that the theorem of MOORE (1968) is false as stated (see STIGLER (1974)).

The second method of proving asymptotic normality for linear combinations of order statistics is to relate the problem to the weak convergence of certain processes on [0,1] with values in certain functions spaces. BICKEL (1967) was the first to follow this line of attack and his proof was based on the weak convergence of suitably defined "quantile" or "inverse empirical" processes. He then writes T_n (cf. (1.2.4)) in terms of these processes, notes the weak convergence of these processes to a Brownian bridge process, and then verifies that the convergence in distribution of T_n follows from the weak convergence of the processes on which T_n is a functional. BICKEL's (1967) results are somewhat restricted because he does not allow the more extreme observations to be weighted more than in the case of the sample mean. SHORACK (1969), (1972) has overcome this drawback by using the weak convergence of suitable quantile processes in stronger metrics than the usual uniform metric. His results allow the weight functions to be unbounded and are of the approximately equal strength as the various results obtained by Chernoff, Gastwirth & Johns and Stigler. An important disadvantage of the approach of proving asymptotic normality via the weak convergence of associated processes is that it does not seem suitable to derive optimal rate of convergence results from it.

We conclude this review of the problem of the asymptotic normality of linear combinations of order statistics by discussing very briefly a few special cases and some extensions. First of all we have, of course, the traditional sample mean (see example 1.2.1). It is well-known that the sample mean is, for any fixed sample size n, the best estimator for the expectation of a normal distribution in almost every conceivable sense. When F is not normal, but its variance is finite it is also best (in the sense of minimum variance) in the class of all unbiased estimators which are linear functions of the observations. The special case of trimmed means was considered in detail by STIGLER (1973). He shows that suitably normalized trimmed means are asymptotically normally distributed if and only if the population quantiles corresponding to the trimming percentages are uniquely determined. Another well-known special case is that of a single order statistic. It is well-known that "central" order statistics are asymptotically normally distributed under certain conditions. SMIRNOV (1944) gives necessary and sufficient conditions for this being the case. BALKEMA & DE HAAN (1978) have given a detailed description of all possible limitlaws which may arise. REISS (1974) (see also VAN ZWET (1964)) has proved that the error of the normal approximation for central order statistics is of order $n^{-\frac{1}{2}}$ if the underlying df F possesses a bounded non-zero second derivative. Edgeworth expansions for sample quantiles and also for the joint distribution of a finite or slowly increasing number of sample quantiles were recently obtained by REISS (1976), (1977). We shall not go into this any further because in this study we shall restrict attention to the case when essentially all the observations, or at least a positive fraction of them, will contribute to the linear combination of order statistics we consider. This, of course, includes the sample mean as a special case, but rules out sample quantiles and statistics based on a finite or slowly increasing number of order statistics. Finally we remark that for the special case that F is the uniform df HECKER (1976) has given necessary and sufficient conditions for the asymptotic normality of linear combinations of uniform order statistics. The same problem for the case of an exponential df is trivial, because then any linear combination of order statistics reduces to a sum of independent rv's.

The case of non-i.i.d but independent rv's was considered by SHORACK (1973), STIGLER (1974) and more recently by RUYMGAART & VAN ZUYLEN (1977). Known theorems on the asymptotic normality of linear combinations of order statistics are extended to the non-i.i.d. case by each of these authors. MEHRA & RAO (1975) proved asymptotic normality for linear combinations of order statistics when the observations possess a certain dependence structure.

Although linear combinations of order statistics of a simple type like e.g., trimmed means were already in use in the 19^{th} century (see, e.g. HUBER (1972)) it was mainly through the work of TUKEY (1960), (1962) that it became clear that the main reason to study and to apply linear combinations of order statistics is the usefulness of these statistics in robust estimation problems. Whereas the sample mean may behave very badly when estimating location with observations which are not normally distributed, L-estimators as well as estimators of different type were constructed which are robust under departure of normality and have high efficiency to the sample mean under normality. A sophisticated theory of robust estimation was developed during the past 15 years by P.J. Huber, F. Hampel and several others. We refer to HUBER (1977) for an account of this theory and a number of references. In particular in the case of estimating the centre of a symmetric distribution it was shown that there are several methods of estimation leading to estimators which are both robust and efficient. Besides estimation by means of linear combinations of order statistics (L-estimators), estimators can be constructed by the method of maximum likelihood (M-estimators) and by the method of deriving estimators from rank tests (R-estimators) which are "first order efficient" in the sense that these estimators are asymptotically normally distributed, with asymptotic mean equal to the parameter to be estimated and with asymptotic variance equal to the Cramér-Rao bound. JAECKEL (1971) has proved a related, somewhat more general, result. He shows that for fixed F there corresponds to each L-estimator (efficient or not) an M-estimator and an R-estimator having, under appropriate conditions, the same asymptotic variance. We also refer a paper of SCHOLZ (1974) who has shown that, when one compares the asymptotic variances of first order efficient L- and R-estimators (when estimating location) the R-estimator has a better performance when the supposed underlying df is not the true one. In a recent paper BICKEL & LEHMANN (1975) considered what happens when the distribution is no longer assumed to be symmetric. They defined measures of location, without assuming symmetry, as functionals satisfying certain equivariance and order conditions. They discuss classes of such measures which can be estimated by L-, R- or M-estimators. Of these three methods of estimation it is found that trimmed L-estimators are the only ones which are both robust and have guaranteed high efficiency with respect to the sample mean for all underlying distributions.

CHAPTER 2

PRELIMINARIES

In this chapter we shall present a number of results which we shall need in the subsequent chapters. We also introduce some more notation which will be used throughout this study. Section 2.1 contains two lemma's which will be basic tools in our proofs. In the sections 2.2 and 2.3 we present a number of rather technical results which we shall frequently use in the chapters 3, 4 and 5.

2.1. TWO BASIC TOOLS

Let $X_1, X_2, \ldots$ denote a sequence of i.i.d. rv's with common df F and let $X_{i:n}$ $(1 \le i \le n)$ denote the i^{th} order statistic of $X_1, \ldots, X_n$. Furthermore let $U_1, U_2 \ldots$ denote a sequence of independent uniform (0,1) rv's and let $U_{i:n}$ $(1 \le i \le n)$ be the i^{th} order statistic of $U_1, \ldots, U_n$. It is well-known that the joint distribution of $X_1, X_2, \ldots$ is the same as that of $F^{-1}(U_1)$, $F^{-1}(U_2), \ldots$ for any df F. Since F is monotone this implies that the joint df of $X_{i:n}$, $i = 1,2,\ldots,n$, $n = 1,2,\ldots$ is the same as that of $F^{-1}(U_{i:n})$, $i = 1,2,\ldots,n$, $n = 1,2,\ldots$. The empirical df based on $U_1, \ldots, U_n$ will be denoted by Γ_n; i.e.

$$(2.1.1) \qquad \Gamma_n(s) = n^{-1} \sum_{i=1}^{n} \chi_{(0,s]}(U_i) \qquad \text{for } 0 < s < 1$$

Here and elsewhere χ_E denotes the indicator of a set E.

The first lemma of this section will be used in the estimation of certain (small) remainder terms.

<u>LEMMA 2.1.1</u>. *Let* $\{X_n,\ n = 1,2,\ldots\}$ *and* $\{Y_n,\ n = 1,2,\ldots\}$ *be two sequences of rv's and let there exist positive numbers* A *and* b *and a number* $\eta > 1$ *such that for all* $n \ge 1$

(i) $\sigma^2(X_n - Y_n) \le An^{-\eta}$ *and*

(ii) $\sigma^2(X_n) \ge bn^{-1}$ *holds.*

Then there exists a positive number C *depending only on* A,b *and* η *but not on* n *such that for all* n

(2.1.2) $$\sigma^2(X_n^* - Y_n^*) \le Cn^{-\eta+1}$$

PROOF. Note first that

(2.1.3) $$\sigma^2(X_n - Y_n) = (\sigma(X_n) - \sigma(Y_n))^2 + 2(1-\rho_n)\sigma(X_n)\sigma(Y_n)$$

where ρ_n denotes the correlation coefficient of X_n and Y_n. Because of assumption (i) and the fact that each of the terms on the right of (2.1.3) is non-negative we find that

(2.1.4) $$\sigma(X_n) - \sigma(Y_n) \le A^{\frac{1}{2}}n^{-\frac{\eta}{2}}$$

and

(2.1.5) $$2(1-\rho_n)\sigma(X_n)\ \sigma(Y_n) \le An^{-\eta}.$$

Using now assumption (ii) and (2.1.4) and noting that $\eta > 1$ we see that $\sigma^2(Y_n) \ge \frac{1}{2}bn^{-1}$ for $n \ge n_0$, n_0 depending only on A,b and η. Combining this and assumption (ii) with (2.1.5) we find that

(2.1.6) $$2(1-\rho_n) \le \frac{A}{b}\sqrt{2}\ n^{-\eta+1}$$

for all $n \ge n_0$. Because $\sigma^2(X_n^* - Y_n^*) = 2(1-\rho_n)$ we have proved the lemma. □

The second lemma of this section is due to W.R. Van Zwet. In VAN ZWET (1977) he obtains a bound on the characteristic function of a linear combination of order statistics, which solves a crucial part of the problem of establishing Edgeworth expansions for these statistics.

Let h be a real-valued measurable function on (0,1) and let $U_{1:n} \le U_{2:n} \le \ldots \le U_{n:n}$ denote the order statistics of a sample of size n from the uniform (0,1) distribution. Let c_{in}, $i = 1,2,\ldots,n$, $n = 1,2,\ldots$ be real numbers and let T_n be a linear combination of functions of order statistics of the form

$$(2.1.7) \qquad T_n = n^{-1} \sum_{i=1}^{n} c_{in} h(U_{i:n}).$$

Note that in the important case $h = F^{-1}$, (2.1.7) reduces to a statistic of the form (1.2.4).

<u>LEMMA 2.1.2</u>. (VAN ZWET). *Suppose that there exist numbers* $0 \le t_1 < t_2 \le 1$ *and positive numbers* m, M, c *and* C *such that*

(i) h *is twice differentiable on* (t_1, t_2) *with first and second derivative* h' *and* h" *such that*

$$h' \ge m \quad \textit{and} \quad |h"| \le M \quad \textit{on} \quad (t_1, t_2)$$

(ii) $$c \le c_{in} \le C \quad \textit{for all } i \textit{ with} \quad t_1 < \frac{i}{n} < t_2$$

Then for every positive integer r *there exist a positive number* A_1 *depending only on* t_1, t_2, m, M, c, C *and* r *and positive numbers* A_2 *and* γ *depending only on* t_1, t_2 *and* r *such that*

$$(2.1.8) \qquad |\mathcal{E} e^{itn^{\frac{1}{2}} T_n}| \le A_1 |t|^{-r} + A_2 e^{-\gamma n} \qquad \textit{for all } t \ne 0.$$

<u>PROOF</u>. See VAN ZWET (1977). □

2.2. SOME LEMMAS

The first lemma of this section is an obvious result concerning the finiteness of certain integrals. For any positive number ℓ the ℓ^{th} absolute moment of a distribution F will sometimes be denoted by β_ℓ

<u>LEMMA 2.2.1</u>.

(a) *Let* ℓ *be a number* >1 *and let, for some* $\delta > 0$, $\beta_{\ell+\delta} < \infty$. *Then there exists* $A > 0$ *depending only on* ℓ *and* δ *such that*

$$(2.2.1) \qquad \int_0^1 (s(1-s))^{\frac{1}{\ell}} \, dF^{-1}(s) \le A \, \beta_{\ell+\delta}^{\frac{1}{\ell+\delta}} < \infty$$

(b) *If* $\ell = 1$ *and* $\delta = 0$ *then* (2.2.1) *holds with* $A = 1$.

PROOF. Applying integration by parts we obtain

$$(2.2.2)\qquad \int_0^1 (s(1-s))^{\frac{1}{\ell}}\, dF^{-1}(s) = (s(1-s))^{\frac{1}{\ell}} F^{-1}(s) \Big|_0^1 - $$

$$- \ell^{-1} \int_0^1 F^{-1}(s)(s(1-s))^{\frac{1}{\ell}-1}(1-2s)\,ds.$$

Both under the assumptions a and b the first term on the right of (2.2.2) is easily seen to be zero. To conclude the proof of part a we apply Hölder's inequality to the second term on the right of (2.2.2):

$$|\ell^{-1} \int_0^1 F^{-1}(s)(s(1-s))^{\frac{1}{\ell}-1}(1-2s)\,ds| \le \int_0^1 |F^{-1}(s)|\,(s(1-s))^{\frac{1}{\ell}-1}\,ds \le$$

$$\le \beta_{\ell+\delta}^{\frac{1}{\ell+\delta}} \left(\int_0^1 (s(1-s))^{-1+\frac{\delta}{\ell(\ell+\delta-1)}}\,ds\right)^{\frac{\ell+\delta-1}{\ell+\delta}} < \infty.$$

The proof of part b is immediate from (2.2.2) and the remark made after it. This completes the proof of the lemma. □

The second lemma of this section will enable us to estimate certain moments.

LEMMA 2.2.2. *Let* ℓ *be a positive integer and let, for some* $\delta > 0$, $\beta_{\ell+\delta} < \infty$. *Then for any number* p *for which* $p\ell \ge 2$, *there exists* $A > 0$ *depending only on* p, ℓ *and* δ, *such that*

$$(2.2.3)\qquad E\left(\int_0^1 |\Gamma_n(s)-s|^p dF^{-1}(s)\right)^{\ell} \le A\, \beta_{\ell+\delta}^{\frac{\ell}{\ell+\delta}}\, n^{-\frac{p\ell}{2}}$$

PROOF. By Fubini's theorem we have

$$E\left(\int_0^1 |\Gamma_n(s)-s|^p dF^{-1}(s)\right)^{\ell} =$$

$$= \int_0^1 \cdots \int_0^1 E \prod_{i=1}^{\ell} |\Gamma_n(s_i)-s_i|^p dF^{-1}(s_1),\ldots,dF^{-1}(s_\ell).$$

Application of Hölder's inequality shows that

$$E \prod_{i=1}^{\ell} |\Gamma_n(s_i)-s_i|^p \le \prod_{i=1}^{\ell} (E\,|\,\Gamma_n(s_i)-s_i|^{p\ell})^{\frac{1}{\ell}}$$

for all $0 < s_1,\dots,s_p < 1$. Hence we know that

$$E(\int_0^1 |\Gamma_n(s)-s|^p dF^{-1}(s))^{\ell} \le (\int_0^1 (E\,|\,\Gamma_n(s)-s|^{p\ell})^{\frac{1}{\ell}} dF^{-1}(s))^{\ell}.$$

At this point we use an inequality due to MARCINKIEVITZ, ZYGMUND & CHUNG (see CHUNG (1951)): If $Y_1,\dots,Y_n$ are independent rv's with expectation zero, we have for all $k \ge 1$

$$(2.2.4)\qquad E\,|\sum_{i=1}^{n} Y_i|^{2k} \le Cn^{k-1} \sum_{i=1}^{n} E|Y_i|^{2k},$$

where the constant C only depends on k. By taking

$$Y_i = \chi_{(0,s]}(U_i) - s, \qquad i = 1,2,\dots,n$$

with $0 < s < 1$ we find, taking $k = p\ell/2$, that

$$(2.2.5)\qquad E|\Gamma_n(s)-s|^{p\ell} \le Bn^{-\frac{p\ell}{2}} s(1-s)$$

for all $0 < s < 1$ and $n \ge 1$. The constant B depends only on p and ℓ. It follows that

$$E(\int_0^1 |\Gamma_n(s)-s|^p dF^{-1}(s))^{\ell} \le Bn^{-\frac{p\ell}{2}} (\int_0^1 (s(1-s))^{\frac{1}{\ell}} dF^{-1}(s))^{\ell}$$

An application of lemma 2.2.1 completes the proof. □

To formulate the next lemma we need some more notation. Let m be a function on (0,1). In certain cases the function m is defined on (0,1) outside a set of F^{-1}-measure zero in (0,1). Define $\|m\|_\infty = \text{ess sup}|m|$ where the ess sup is taken with respect to the measure induced by F^{-1}. Consider for a positive integer k, the function

$$(2.2.6)\qquad m_k(u_1,\dots,u_k) = \int_0^1 m(s) \prod_{i=1}^{k} (\chi_{(0,s]}(u_i) - s)\, d\,F^{-1}(s)$$

which is properly defined for $0 < u_1,\dots,u_k < 1$ whenever $\beta_1 < \infty$ and $\|m\|_\infty < \infty$. Define a function H by

(2.2.7) $$H(u) = \int_0^1 |\chi_{(0,s]}(u) - s| d\, F^{-1}(s)$$

for $0 < u < 1$. Note that m_k is symmetric in its k arguments and that

(2.2.8) $$|m_k(u_1,\ldots,u_k)| \le \|m\|_\infty \cdot H(u_i)$$

for $i = 1,2,\ldots,k$.

LEMMA 2.2.3.
(a) *Let* ℓ *be a positive integer and suppose that* $\beta_\ell < \infty$. *Then*

(2.2.9) $$EH^\ell(U_1) \le 4^\ell \beta_\ell < \infty$$

(b) *Suppose that* $\|m\|_\infty < \infty$ *and* $\beta_1 < \infty$. *Then*

(2.2.10) $$Em_1(U_i) = 0$$

for any i *and with probability one*

(2.2.11) $$E(m_k(U_{i_1},\ldots,U_{i_k})|U_{i_1},\ldots,U_{i_{k-1}}) = 0$$

for any positive integers $i_1,\ldots,i_k$ *provided* $i_k \notin \{i_1,\ldots,i_{k-1}\}$.

PROOF. (a) We prove (2.2.9). It is immediate from (2.2.7) that

$$H(U_1) \le \int_{(0,U_1)} s d\, F^{-1}(s) + \int_{[U_1,1)} (1-s) d\, F^{-1}(s)$$

Applying the c_r-inequality (see, e.g., LOÉVE (1955), page 155) we find

$$EH^\ell(U_1) \le 2^{\ell-1}[E(\int_{(0,U_1)} s d\, F^{-1}(s))^\ell + E(\int_{[U_1,1)} (1-s) d\, F^{-1}(s))^\ell]$$

Using integration by parts and the finiteness of β_ℓ and applying the c_r-inequality once more we see that

$$E(\int_{(0,U_1)} s d\, F^{-1}(s))^\ell = E|U_1 F^{-1}(U_1) - \int_0^{U_1} F^{-1}(s) ds|^\ell \le$$

$$\leq 2^{\ell-1}(E|F^{-1}(U_1)|^{\ell} + (\int_0^1 |F^{-1}(s)|ds)^{\ell})$$

$$\leq 2^{\ell-1}(E|X_1|^{\ell} + (E|X_1|)^{\ell}) \leq 2^{\ell}E|X_1|^{\ell}.$$

Similarly we can show that

$$E(\int_{[U_1,1)} (1-s)d\, F^{-1}(s))^{\ell} \leq 2^{\ell}E|X_1|^{\ell}$$

so that

$$EH^{\ell}(U_1) \leq 4^{\ell}E|X_1|^{\ell} = 4^{\ell}\beta_{\ell} < \infty$$

which proves (2.2.9).

(b) By Fubini's theorem we see that with probability one

$$E(\int_0^1 |m(s)|\prod_{j=1}^{k}|\chi_{(0,s]}(U_{i_j}) - s|d\, F^{-1}(s)|U_{i_1},\ldots,U_{i_{k-1}}) \leq$$

$$\leq \|m\|_{\infty}\cdot EH(U_1) < \infty.$$

Therefore the conditional expectation in (2.2.11) is well-defined and Fubini's theorem can be applied once more to find that

$$E(m_k(U_{i_1},\ldots,U_{i_k})|U_{i_1},\ldots,U_{i_{k-1}}) = 0$$

with probability one. Of course (2.2.10) follows similarly. $\square$

The next lemma gives conditions which guarantee that the quantity $\sigma^2(J,F)$ (cf. (0.3)) given by

$$\sigma^2(J,F) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(F(x))J(F(y))(\min(F(x),F(y)) - F(x)F(y))dxdy \tag{2.2.12}$$

is bounded away from zero. We remark that a different expression for $\sigma^2(J,F)$ is given by

$$\sigma^2(J,F) = \int_0^1 h_1^2(u)du \tag{2.2.13}$$

where the function h_1 is given by

$$h_1(u) = -\int_0^1 J(s)(\chi_{(0,s]}(u) - s)\,d\,F^{-1}(s) \tag{2.2.14}$$

for $0 < u < 1$.

LEMMA 2.2.4. *Let J be bounded on (0,1) and let $\beta_1 < \infty$. Suppose that positive numbers M_1 and c and numbers $0 \le t_1 < t_2 \le 1$ exist such that on $(F^{-1}(t_1), F^{-1}(t_2))$ F possesses a density f, such that on $(F^{-1}(t_1), F^{-1}(t_2))$, $f \le M_1$ and on (t_1,t_2), $J \ge c$. Then there exists $\sigma_0^2 > 0$ depending only on M_1, c, t_1 and t_2 such that*

$$\sigma^2(J,F) \ge \sigma_0^2. \tag{2.2.15}$$

PROOF. Note first that h_1 is well-defined and finite for every $0 < u < 1$. Secondly we remark that

$$\sigma^2(J,F) = \int_0^1 h_1^2(u)du \ge \int_{t_1}^{t_2} h_1^2(u)du.$$

It follows directly from (2.2.14) and the assumptions of the lemma that

$$h_1(u_2) - h_1(u_1) \ge c\, M_1^{-1}(u_2-u_1)$$

for $t_1 < u_1 < u_2 < t_2$. The geometry of the situation ensure now that $\int_{t_1}^{t_2} h_1^2(u)du$ is minimized for

$$h_1(u_1) = (u_1 - \frac{t_1}{2} - \frac{t_2}{2})\frac{c}{M_1}$$

for $t_1 < u_1 < t_2$. Hence

$$\sigma^2(J,F) \ge \frac{c^2(t_2-t_1)^3}{12\, M_1^2}$$

This completes the proof of the lemma. □

2.3. BOUNDS FOR MOMENTS OF CENTRAL ORDER STATISTICS

The first lemma of this section gives conditions which guarantee that the k^{th} absolute moment of a trimmed linear combination of order statistics is finite.

LEMMA 2.3.1. *Let, for some* $\delta > 0$, $\beta_\delta < \infty$. *Suppose that numbers* $0 < \alpha < \beta < 1$ *and real numbers* c_{in}, $i = 1,2,\ldots,n$, $n = 1,2,\ldots$ *exist such that*

$$c_{in} = 0 \quad \text{for} \quad i < [n\alpha] \quad \textit{and} \quad i > [n\beta]. \tag{2.3.1}$$

Then, for any number $k > 0$, *there exists a positive integer* n_1, *depending only on* k, α, β *and* δ, *such that*

$$E\left|\sum_{i=1}^{n} c_{in}X_{i:n}\right|^k < \infty \qquad \text{for } n \geq n_1. \tag{2.3.2}$$

PROOF. The proof is essentially contained in BICKEL (1967). Note that assumption (2.3.1) implies that

$$\left|\sum_{i=1}^{n} c_{in}X_{i:n}\right|^k \leq (|X_{[n\alpha]:n}|^k + |X_{[n\beta]:n}|^k)\left(\sum_{i=[n\alpha]}^{[n\beta]} |c_{in}|\right)^k.$$

Application of theorem 2.2a of BICKEL (1967) implies that there exists a natural number n_1, depending only on k, α, β and δ, such that for $n \geq n_1$ both $E|X_{[n\alpha]:n}|^k$ and $E|X_{[n\alpha]:n}|^k$ are finite. Hence we have proved the lemma. □

Next we collect some well-known useful facts about order statistics from an exponential df. Let $Z_1, Z_2, \ldots$ denote a sequence of independent rv's with common exponential df E given by

$$E(z) = 1 - e^{-z} \qquad \text{for } 0 \leq z < \infty. \tag{2.3.3}$$

Let, for each $n \geq 1$, $Z_{i:n}$ denote the i^{th} order statistic of $Z_1,\ldots,Z_n$. It is well-known (see, e.g., DAVID (1970)) that $Z_{i:n}$ $(1 \leq i \leq n)$ has the same distribution as the rv

$$\sum_{j=1}^{i} \frac{Z_j}{(n-j+1)} \tag{2.3.4}$$

$(1 \leq i \leq n)$; i.e. $Z_{i:n}$ is distributed as a sum of independent rv's.

In the second lemma of this section we obtain estimates for the absolute central moments of exponential order statistics. Note that $EZ_{i:n} = \nu_{in}$ $(1 \leq i \leq n)$ where

$$\nu_{in} = \sum_{j=1}^{i} \frac{1}{(n-j+1)} \qquad i = 1,2,\ldots,n. \tag{2.3.5}$$

LEMMA 2.3.2. *Let* $0 < \alpha < \beta < 1$ *and let* $p > 0$. *Then there exists a positive constant* A, *depending only on* α, β *and* p, *but not on* n, *such that for all* $n \geq 1$

$$\max_{[n\alpha]\leq i\leq[n\beta]} E|Z_{i:n} - \nu_{in}|^p \leq An^{-\frac{p}{2}}. \qquad (2.3.6)$$

PROOF. The proof is an immediate consequence of lemma A.2.4 of ALBERS, BICKEL & VAN ZWET (1976). □

REMARK. The order bound (2.3.6) holds only true for "central" exponential order statistics. The "upper" exponential order statistics are of a larger order of magnitude. It is exactly for this reason that we have to restrict attention to trimmed linear combinations of order statistics in chapter 5.

CHAPTER 3

BERRY - ESSEEN THEOREMS

3.1. INTRODUCTION AND MAIN RESULTS

The purpose of this chapter is to obtain precise information about the rate of convergence to the normal limit distribution of the df's of linear combinations of order statistics. In our main results - stated in the form of three theorems - we establish Berry-Esseen bounds of order $n^{-\frac{1}{2}}$ for these statistics. Before listing the assumptions needed for the theorems let us introduce some notation. Let $X_1, X_2, \ldots$ denote a sequence of i.i.d. rv's with common df F. Consider, for each $n \geq 1$, statistics of the form

(3.1.1) $$T_n = n^{-1} \sum_{i=1}^{n} c_{in} X_{i:n}$$

(cf. (1.2.4)) . Furthermore define, for each $n \geq 1$ and real x,

(3.1.2) $$F_n^*(x) = P(\{T_n^* \leq x\})$$

where (cf. (1.1.4))

(3.1.3) $$T_n^* = (T_n - E(T_n))/\sigma(T_n).$$

Let J denote a real-valued bounded measurable function on (0,1). The first two assumptions will be needed to prove the first and second main result of this chapter.

<u>ASSUMPTION 3.1.1</u>. As $n \to \infty$

$$\max_{\substack{1 \leq i \leq n \\ i \neq j_1, \ldots, j_k}} \left| c_{in} - n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s)ds \right| = O(n^{-1})$$

In addition the weights $c_{j_\ell n}$ $(1 \le \ell \le k)$ are uniformly bounded in n, $j_\ell = [ns_\ell] + 1$, $\ell = 1,\ldots,k$, $n \ge 1$, $0 < s_1,\ldots,s_k < 1$ and the inverse F^{-1} satisfies a Lipschitz condition of order $\alpha_1 \ge \frac{1}{2}$ on neighbourhoods of $s_1,\ldots,s_k$. k is fixed.

ASSUMPTION 3.1.2. The function J satisfies a Lipschitz condition of order 1 on (0,1).

The third assumption is a strengthened version of assumption 3.1.2 which we shall need to prove the third main result of this chapter.

ASSUMPTION 3.1.3. The function J is bounded and continuous on (0,1). The derivative $J^{(1)}$ exists except possibly at a finite number of points; $J^{(1)}$ satisfies a Lipschitz condition of order $\alpha_2 > \frac{1}{2}$ on the open intervals where it exists. The inverse F^{-1} satisfies a Lipschitz condition of order $\alpha_3 > \frac{1}{2}$ on neighbourhoods of the points where $J^{(1)}$ does not exist.

THEOREM 3.1.1. *Let* $E|X_1|^3 < \infty$ *and suppose that the assumptions* 3.1.1 *and* 3.1.2 *are satisfied. Then* $\sigma^2(J,F) > 0$ *(cf.* (2.2.12)*) implies that*

$$\sup_x |F_n^*(x) - \Phi(x)| = O(n^{-\frac{1}{2}}), \qquad \text{as } n \to \infty \tag{3.1.4}$$

Our second theorem is a modification of theorem 3.1.1 in which we shall employ a different and more practical standardization. Let us introduce the quantity $\mu = \mu(J,F)$ by

$$\mu = \mu(J,F) = \int_0^1 J(s)F^{-1}(s)ds \tag{3.1.5}$$

and define, for each $n \ge 1$ and real x, the df G_n by

$$G_n(x) = P(\{n^{\frac{1}{2}}(T_n - \mu)/\sigma \le x\}) \tag{3.1.6}$$

with $\sigma^2 = \sigma^2(J,F)$ as in (2.2.12).

THEOREM 3.1.2. *Suppose that the assumptions of theorem* 3.1.1 *are satisfied. Then* $\sigma^2(J,F) > 0$ *implies that*

$$\sup_x |G_n(x) - \Phi(x)| = O(n^{-\frac{1}{2}}), \qquad \text{as } n \to \infty \tag{3.1.7}$$

In the third and final main result of this chapter we establish a Berry-Esseen bound for a studentized version of $n^{\frac{1}{2}}(T_n - \mu)/\sigma$; i.e. $\sigma = \sigma(J,F)$ is estimated by its natural estimator which is given by

$$(3.1.8) \qquad s_n = \sigma(J,F_n)$$

where F_n denotes the empirical df based on $X_1,\ldots,X_n$:

$$(3.1.9) \qquad F_n(x) = n^{-1} \sum_{i=1}^{n} \chi_{(-\infty,x]}(X_i)$$

for $-\infty < x < \infty$. Introduce, for each $n \geq 1$ and real x, the df H_n by

$$(3.1.10) \qquad H_n(x) = P(\{n^{\frac{1}{2}}(T_n - \mu)/s_n \leq x\})$$

THEOREM 3.1.3. *Let* $E|X_1|^6 < \infty$ *and suppose that the assumptions* 3.1.1 *and* 3.1.3 *are satisfied. Then* $\sigma^2(J,F) > 0$ *implies that*

$$(3.1.11) \qquad \sup_x |H_n(x) - \Phi(x)| = \mathcal{O}(n^{-\frac{1}{2}}) \qquad \textit{as } n \to \infty$$

Weights of the form (cf. (1.2.3))

$$(3.1.12) \qquad c_{in} = J(\tfrac{i}{n+1})$$

$i = 1,2,\ldots,n$, $n \geq 1$ are frequently studied in the literature, (see, e.g., STIGLER (1974)). The following proposition ensures that we may replace assumption 3.1.1 by (3.1.12) in each of the theorems 3.1.1 - 3.1.3.

PROPOSITION 3.1.4. *Let either assumption* 3.1.2 *or assumption* 3.1.3 *be satisfied. Then assumption* 3.1.12 *implies assumption* 3.1.1.

PROOF. As in either case J is Lipschitz of order 1 on (0,1) we immediately find that

$$\max_{1\leq i\leq n} |J(\tfrac{i}{n+1}) - n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s)ds| = \mathcal{O}(n^{-1}) \qquad \text{as } n \to \infty$$

which completes the proof. □

It is useful to comment on these results. In the first place we remark that, except for possibly finitely many weights, the weights are approximated, up to an error of order $\mathcal{O}(n^{-1})$, by a smooth weight function. An important example in which this is the case is provided by proposition 3.1.4. In the theorems 3.1.1 and 3.1.2 the function J must be Lipschitz of order 1. In theorem 3.1.3 we need a stronger smoothness condition, but still we allow points of non-differentiability. The price for this is a local smoothness condition on the inverse F^{-1}. In the second place we require the finiteness of the absolute third moment of the underlying df F in the theorems 3.1.1 and 3.1.2. In view of the classical Berry-Esseen theorem this seems a natural condition. In theorem 3.1.3, on the other hand, we assume the finiteness of the sixth moment of the df F. Note that, if we take $J \equiv 1$ and multiply the statistic in (3.1.10) by the harmless factor $(\frac{n-1}{n})^{\frac{1}{2}}$, a Berry-Esseen bound of order $n^{-\frac{1}{2}}$ for the Student t-statistic follows as an important special case. In CHUNG (1946) the same doubling of the order of the required moment is needed to obtain an Edgeworth expansion for the t-statistic. In section 3.5 we indicate that theorem 3.1.3 remains valid when the sixth moment assumption is replaced by a 4.5^{th} absolute moment for the underlying df F.

It may be remarked that trimmed and Winsorized means are not included as special cases in the theorems 3.1.1 and 3.1.2. However, BJERVE (1977) has obtained a Berry-Esseen bound of order $n^{-\frac{1}{2}}$ for trimmed linear combinations of order statistics. His result admits quite general weights on the observations between the α^{th} and β^{th} sample percentiles $(0 < \alpha < \beta < 1)$ but he does not allow weights to be put on the remaining observations. In addition the underlying df F must satisfy a rather restrictive smoothness condition. It is worth noting that in contrast with Bjerve's result we allow weights to be put on all observations and the underlying df need not even be continuous. Theorem 3.1.1 was proved for weights of the form (3.1.12) assuming a finite third absolute moment, assumption 3.1.3 and the rather restrictive requirement $\int_0^1 |J^{(1)}(s)|d\, F^{-1}(s) < \infty$ in HELMERS (1977). This latter requirement was removed in HELMERS (1981). The present chapter extends the latter paper.

To conclude this section let us give an example which illustrates the importance of allowing points of non-differentiability in the condition for the weight function. Although our results cannot be applied to trimmed means they apply to the linearized smooth trimmed means which were advocated by STIGLER (1974) for use in estimation problems when, e.g., the observations are drawn from discrete populations. These smooth trimmed means are generated by the function J, according to (3.1.12), where

$$
\begin{aligned}
J(s) &= (s - \tfrac{\alpha}{2})\, 2\, \frac{h}{\alpha} && \tfrac{\alpha}{2} \le s \le \alpha \\
&= h && \alpha < s < 1-\alpha \\
&= (1 - \tfrac{\alpha}{2} - s)\, 2\, \frac{h}{\alpha} && 1-\alpha \le s \le 1-\tfrac{\alpha}{2} \\
&= 0 && \text{otherwise}
\end{aligned}
$$

with $h = 2(2 - 3\alpha)^{-1}$.

In section 3.2 we prove theorem 3.1.1. Theorem 3.1.2 is proved in section 3.3 and theorem 3.1.3 in section 3.4. A refinement of theorem 3.1.3 is indicated in section 3.5.

3.2. PROOF OF THEOREM 3.1.1.

The purpose of this section is to provide a proof for theorem 3.1.1. We shall need four lemma's. In the first lemma we shall approximate T_n by a rv V_n given by

$$(3.2.1) \qquad V_n = \int_0^1 J(s) F_n^{-1}(s)\,ds = \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s)\,ds\; X_{i:n}$$

where F_n is as in (3.1.9). Let $\|h\| = \sup_{0<s<1} |h(s)|$ for any function h on (0,1). In certain cases the function h is defined on (0,1) except at a finite number of points. Then $\|h\|$ will denote the supremum of $|h|$ on the domain of h. For notation see also section 3.1.

<u>LEMMA 3.2.1</u>. *Let* $EX_1^2 < \infty$. *Suppose that assumption* 3.1.1 *is satisfied and that* J *is bounded and continuous on* (0,1). *Then* $\sigma^2(J,F) > 0$ *implies that as* $n \to \infty$.

$$(3.2.2) \qquad \sigma^2(T_n^* - V_n^*) = O(n^{-\frac{3}{2}})$$

<u>PROOF</u>. It follows from $EX_1^2 < \infty$ that $EX_{i:n}^2 < \infty$ for any $1 \le i \le n$. Furthermore it is well-known (see, e.g., BICKEL (1967)) that the conditional expectation of $X_{j:n}$ is non-decreasing in $X_{i:n}$ $(1 \le i < j \le n)$ with probability

one. This result directly implies that the covariance between $X_{i:n}$ and $X_{j:n}$ is non-negative for all $1 \le i \ne j \le n$. Obviously this implies that

$$(3.2.3) \qquad \sigma^2\Big(\sum_{i=1}^{n} a_i X_{i:n}\Big) \le \sigma^2\Big(\sum_{i=1}^{n} b_i X_{i:n}\Big)$$

holds, provided $a_i a_j \le b_i b_j$ for all $1 \le i,j \le n$. To prove (3.2.2) we first note that without loss of generality we assume that $k = 1$ in assumption 3.1.1. Using inequality (3.2.3) twice we see that

$$(3.2.4) \qquad \sigma^2(T_n - V_n) \le 2\sigma^2\Big(\sum_{\substack{i=1\\ i\ne j_1}}^{n} X_{i:n}\Big|\frac{c_{in}}{n} - \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s)ds\Big|\Big)$$

$$+ 2\sigma^2\Big(X_{j_1:n}\Big|\frac{c_{j_1 n}}{n} - \int_{\frac{j_1-1}{n}}^{\frac{j_1}{n}} J(s)ds\Big|\Big)$$

Using assumption 3.1.1 and applying (3.2.3) once more we obtain

$$(3.2.5) \qquad \sigma^2(T_n - V_n) \le 2n^{-3}\,\sigma^2(X_1) +$$

$$+ 2n^{-2}\big[\max_{n\ge 1} |\,c_{j_1 n}| + \|J\|\big]^2 \sigma^2(X_{j_1:n})$$

To proceed we prove that $\sigma^2(X_{j_1:n}) = O(n^{-\alpha_1})$ as $n \to \infty$. Let γ_n denote the beta-density of the uniform order statistic $U_{j_1:n}$ ($j_1 = [ns_1]+1$) and let E_n be the set

$$(3.2.6) \qquad E_n = \{u: \Big|u - \frac{[ns_1]+1}{n+1}\Big| \le (mn^{-1}\ell n\ n)^{\frac{1}{2}},\ 0 < u < 1\}$$

for some fixed $m > 0$. The complement of E_n in (0,1) will be denoted by E_n^c. Then we have that

$$(3.2.7) \qquad \sigma^2(X_{j_1:n}) \le E\Big(X_{j_1:n} - F^{-1}\Big(\frac{j_1}{n+1}\Big)\Big)^2 =$$

$$= \int_{E_n} \Big(F^{-1}(u) - F^{-1}\Big(\frac{j_1}{n+1}\Big)\Big)^2 \gamma_n(u)du +$$

$$+ \int_{E_n^c} (F^{-1}(u) - F^{-1}(\tfrac{j_1}{n+1}))^2 \gamma_n(u)du$$

Because $EX_1^2 < \infty$ we can use lemma 4 of STIGLER (1969) to see that the second integral on the right hand side of (3.2.7) is $O(n^{-r})$ for any $r > 0$, as $n \to \infty$, provided we choose m sufficiently large (depending on r). The Lipschitz condition of F^{-1} on a neighbourhood of s_1 can be used to treat the first integral on the righthand side of (3.2.7). Since $\frac{j_1-1}{n} \le s_1 < \frac{j_1}{n}$ we have for sufficiently large n and some constant $C > 0$ that

$$(3.2.8) \qquad \int_{E_n} (F^{-1}(u) - F^{-1}(\tfrac{j_1}{n+1}))^2 \gamma_n(u)du$$

$$\le C \cdot E|U_{j_1:n} - \tfrac{j_1}{n+1}|^{2\alpha_1}$$

It follows from this and the well-known fact that, as $\lim_{n\to\infty} \frac{j_1}{n} = s_1$ for $0 < s_1 < 1$, $E|U_{j_1:n} - \frac{j_1}{n+1}|^{2\alpha_1} = O(n^{-\alpha_1})$ as $n \to \infty$, that the first integral on the righthand side of (3.2.7) is $O(n^{-\alpha_1})$ as $n \to \infty$. This and (3.2.5) together imply that

$$(3.2.9) \qquad \sigma^2(T_n - V_n) = O(n^{-\frac{5}{2}}) \qquad \text{as } n \to \infty$$

To complete the proof of the lemma we remark that it is not difficult to check from theorem 1 and remark 2 of STIGLER (1974) that $\lim_{n\to\infty} n\sigma^2(V_n) = \sigma^2(J,F) > 0$ holds under the assumptions of the lemma. Combining this and (3.2.9) with lemma 2.1.1 we see that (3.2.2) holds. □

Define for $0 \le u \le 1$ the function

$$(3.2.10) \qquad \psi(u) = \int_u^1 J(s)ds - (1-u)\int_0^1 J(s)ds$$

and let $c = \int_0^1 J(s)ds$. Note that $\psi(0) = \psi(1) = 0$. Now we can write

$$(3.2.11) \qquad V_n = \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s)ds\, F^{-1}(U_{i:n}) =$$

$$= \sum_{i=1}^{n} (\psi(\tfrac{i-1}{n}) - \psi(\tfrac{i}{n}))F^{-1}(U_{i:n}) + cn^{-1} \sum_{i=1}^{n} F^{-1}(U_{i:n})$$

$$= \sum_{i=1}^{n} \psi(\tfrac{i}{n})(F^{-1}(U_{i+1:n}) - F^{-1}(U_{i:n})) + cn^{-1} \sum_{i=1}^{n} F^{-1}(U_i)$$

$$= \int_0^1 \psi(\Gamma_n(s))d\,F^{-1}(s) + cn^{-1} \sum_{i=1}^{n} F^{-1}(U_i)$$

where the last inequality holds with probability 1. We use the fact that, almost surely, none of the rv's $U_1, U_2, \ldots$ take values in the discontinuity set of F^{-1}.

To proceed we note that, as J is Lipschitz of order 1 on (0,1) (cf. assumption 3.1.2), we can approximate V_n from above and below by

$$(3.2.12) \qquad W_{n+} = \int_0^1 \{\psi(s) + (\Gamma_n(s) - s)\psi'(s)\}d\,F^{-1}(s) + cn^{-1} \sum_{i=1}^{n} F^{-1}(U_i)$$
$$+ K \int_0^1 (\Gamma_n(s) - s)^2 d\,F^{-1}(s)$$

and

$$(3.2.13) \qquad W_{n-} = \int_0^1 \{\psi(s) + (\Gamma_n(s) - s)\psi'(s)\}d\,F^{-1}(s) + cn^{-1} \sum_{i=1}^{n} F^{-1}(U_i)$$
$$- K \int_0^1 (\Gamma_n(s) - s)^2 d\,F^{-1}(s)$$

for some fixed $K > 0$ and all $n \geq 1$; i.e. for all $n \geq 1$

$$(3.2.14) \qquad W_{n-} \leq V_n \leq W_{n+}$$

It will be convenient to have

LEMMA 3.2.2. *Let* $E|X_1|^{2+\varepsilon} < \infty$ *for some* $\varepsilon > 0$ *and suppose that assumption 3.1.2 is satisfied. Then* $\sigma^2(J,F) > 0$ *implies that as* $n \to \infty$

$$(3.2.15) \qquad \frac{\sigma(W_{n+})}{\sigma(V_n)} = 1 + O(n^{-\frac{1}{2}}), \qquad \frac{E(V_n - W_{n+})}{\sigma(V_n)} = O(n^{-\frac{1}{2}})$$

and

(3.2.16) $$\frac{\sigma(W_{n-})}{\sigma(V_n)} = 1 + \mathcal{O}(n^{-\frac{1}{2}}), \qquad \frac{E(V_n - W_{n-})}{\sigma(V_n)} = \mathcal{O}(n^{-\frac{1}{2}})$$

PROOF. It is immediate from (3.2.11), (3.2.12) and assumption 3.1.2 that

(3.2.17) $$|V_n - W_{n+}| = \mathcal{O}(\int_0^1 (\Gamma_n(s) - s)^2 d\, F^{-1}(s))$$

as $n \to \infty$. Application of lemma 2.2.2 (with $p = 2$ and $\ell = 1$ and 2 respectively) implies that

(3.2.18) $$E|V_n - W_{n+}| = \mathcal{O}(n^{-1})$$

and

(3.2.19) $$\sigma^2(V_n - W_{n+}) \le E(V_n - W_{n+})^2 = \mathcal{O}(n^{-2})$$

as $n \to \infty$. As in the proof of lemma 3.2.1 we also have that $\lim_{n\to\infty} n\sigma^2(V_n) = \sigma^2(J,F) > 0$ under the present assumptions (cf. STIGLER (1974)). The Cauchy-Schwarz inequality implies that $|\sigma(W_{n+}) - \sigma(V_n)| \le \sigma(W_{n+} - V_n)$ and (3.2.15) follows. The proof of (3.2.16) is similar. □

In the following lemma we relate W_{n+} and W_{n-} to appropriate U-statistics $\mathcal{U}_{n+}$ and $\mathcal{U}_{n-}$. Define, for each $n \ge 1$

(3.2.20) $$\mathcal{U}_{n+} = \binom{n}{2}^{-1} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \tilde{h}_+(U_i, U_j)$$

and

(3.2.21) $$\mathcal{U}_{n-} = \binom{n}{2}^{-1} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \tilde{h}_-(U_i, U_j)$$

where the functions $\tilde{h}_+$ and $\tilde{h}_-$ are given by

(3.2.22) $$\tilde{h}_+(u,v) = h_1(u) + h_1(v) + h_{2,K}(u,v)$$

and

$$(3.2.23)\qquad \tilde{h}_{-}(u,v) = h_1(u) + h_1(v) - h_{2,K}(u,v)$$

for $0 < u,v < 1$, with (cf. (2.2.14))

$$(3.2.24)\qquad h_1(u) = -\int_0^1 J(s)(\chi_{(0,s]}(u) - s)d\,F^{-1}(s)$$

and

$$(3.2.25)\qquad h_{2,K}(u,v) = +2K\int_0^1 (\chi_{(0,s]}(u) - s)(\chi_{(0,s]}(v) - s)d\,F^{-1}(s)$$

for $0 < u,v < 1$ and K as in (3.2.12) and (3.2.13).

LEMMA 3.2.3. *Let* $EX_1^2 < \infty$ *and suppose that assumption* 3.1.2 *is satisfied. Then* $\sigma^2(J,F) > 0$ *implies that as* $n \to \infty$

$$(3.2.26)\qquad \sigma^2(W^*_{n+} - U^*_{n+}) = O(n^{-2})$$

and

$$(3.2.27)\qquad \sigma^2(W^*_{n-} - U^*_{n-}) = O(n^{-2})$$

PROOF. We first prove (3.2.26). In view of (3.2.10) and (3.2.12) we can rewrite W_{n+} as

$$(3.2.28)\qquad W_{n+} = \int_0^1 \psi(s)d\,F^{-1}(s) - \int_0^1 J(s)(\Gamma_n(s) - s)d\,F^{-1}(s)$$
$$+ c\int_0^1 (\Gamma_n(s) - s)d\,F^{-1}(s) + cn^{-1}\sum_{i=1}^n F^{-1}(U_i)$$
$$+ K\int_0^1 (\Gamma_n(s) - s)^2 d\,F^{-1}(s)$$

Because of the definition of Γ_n (cf. (2.1.1)) we have

$$(3.2.29)\qquad \int_0^1 (\Gamma_n(s) - s)d\,F^{-1}(s) = n^{-1}\sum_{i=1}^n \Big(\int_{(0,U_i)} (-s)d\,F^{-1}(s) +$$
$$\int_{[U_i,1)} (1-s)d\,F^{-1}(s)\Big)$$

Now integration by parts, the finiteness of $E|X_1|$ and the fact that, almost surely, none of the rv's $U_1, U_2, \ldots$ take values corresponding to the discontinuities of F^{-1}, shows that

$$(3.2.30)\qquad \int_0^1 (\Gamma_n(s) - s)\,d\,F^{-1}(s) = -n^{-1}\sum_{i=1}^{n} F^{-1}(U_i) + \int_0^1 F^{-1}(s)ds$$

holds with probability 1. Thus (cf. (3.2.24) and (3.2.25))

$$(3.2.31)\qquad W_{n+} - EW_{n+} = n^{-1}\sum_{i=1}^{n} h_1(U_i) + $$

$$+2^{-1}n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} h_{2,K}(U_i,U_j) - Kn^{-1}\int_0^1 s(1-s)d\,F^{-1}(s)$$

with probability 1. In view of this, (3.2.20) - (3.2.25), we easily check that

$$(3.2.32)\qquad \frac{1}{2}\left(1-\frac{1}{n}\right)U_{n+} = W_{n+} - E(W_{n+}) - n^{-2}\sum_{i=1}^{n} h_1(U_i)$$

$$- Kn^{-2}\sum_{i=1}^{n}\int_0^1 (\chi_{(0,s]}(U_i) - s)^2 d\,F^{-1}(s)$$

$$+ Kn^{-1}\int_0^1 s(1-s)d\,F^{-1}(s)$$

Thus

$$(3.2.33)\qquad \sigma^2\left(\frac{1}{2}\left(1-\frac{1}{n}\right)U_{n+} - W_{n+}\right) \le 2\sigma^2\left(n^{-2}\sum_{i=1}^{n} h_1(U_i)\right)$$

$$+ 2K^2\sigma^2\left(n^{-2}\sum_{i=1}^{n}\int_0^1 (\chi_{(0,s]}(U_i) - s)^2 d\,F^{-1}(s)\right)$$

$$= 2n^{-3}\sigma^2(J,F) + 2n^{-3}K^2\sigma^2\left(\int_0^1 (\chi_{(0,s]}(U_1) - s)^2 d\,F^{-1}(s)\right).$$

Define H as in (2.2.7). Then

$$(3.3.34)\qquad \sigma^2\left(\int_0^1 (\chi_{(0,s]}(U_1) - s)^2 d\,F^{-1}(s)\right) \le EH^2(U_1) < \infty$$

because of lemma 2.2.3.a. This proves that

$$\sigma^2(\tfrac{1}{2}(1-\tfrac{1}{n})\mathcal{U}_{n+} - W_{n+}) = \mathcal{O}(n^{-3}) \qquad \text{as } n \to \infty$$

As it is easily verified that $\lim_{n\to\infty} n\sigma^2(W_{n+}) = \sigma^2(J,F) > 0$ we have, in view of lemma 2.1.1, proved (3.2.26). The proof of (3.2.27) is similar. □

In the fourth and final lemma of this section we establish Berry-Esseen bounds for $\mathcal{U}^*_{n+}$ and U^*_{n-}.

<u>LEMMA 3.2.4</u>. *Let* $E|X_1|^3 < \infty$ *and suppose that* J *is bounded on* (0,1). *Then* $\sigma^2(J,F) > 0$ *implies that as* $n \to \infty$

$$\sup_x |P(\{U^*_{n+} \le x\}) - \Phi(x)| = \mathcal{O}(n^{-\frac{1}{2}}) \tag{3.2.35}$$

and

$$\sup_x |P(\{U^*_{n-} \le x\}) - \Phi(x)| = \mathcal{O}(n^{-\frac{1}{2}}) \tag{3.2.36}$$

<u>PROOF</u>. It follows from lemma 2.2.3.b that (cf. (3.2.22))

$$E(\tilde{h}_+(U_1,U_2) \mid U_1) = h_1(U_1) \tag{3.2.37}$$

with probability 1. Also note that $Eh_1^2(U_1) = \sigma^2(J,F) > 0$ (cf. (2.2.13)) so that we find that the conditional expectation (3.2.37) has a positive variance. Moreover lemma 2.2.3(a) yields

$$E|h_1(U_1)|^3 \le \|J\|^3\, EH^3(U_1) < \infty$$

$$E|h_{2,K}(U_1,U_2)|^3 \le 8K^3 EH^3(U_1) < \infty$$

and therefore

$$E|\tilde{h}_+(U_1,U_2)|^3 < \infty$$

The conditions of the Berry-Esseen theorem of U-statistics (CALLAERT & JANSSEN (1978)) are therefore satisfied and (3.2.35) follows. The proof of (3.2.36) is similar. □

We are now in a position to prove theorem 3.1.1. In the first place we use lemma 3.2.1 and Chebychev's inequality to find that

$$(3.2.38) \qquad P(\{|T_n^* - V_n^*| \geq n^{-\frac{1}{2}}\}) \leq n\sigma^2(T_n^* - V_n^*) = \mathcal{O}(n^{-\frac{1}{2}}) \qquad \text{as } n \to \infty$$

Using this we see that

$$(3.2.39) \qquad F_n^*(x) = P(\{T_n^* \leq x\}) =$$

$$= P(\{T_n^* \leq x \wedge |T_n^* - V_n^*| < n^{-\frac{1}{2}}\})$$

$$+ P(\{T_n^* \leq x \wedge |T_n^* - V_n^*| \geq n^{-\frac{1}{2}}\})$$

$$\leq P(\{V_n^* \leq x + n^{-\frac{1}{2}}\}) + P(\{|T_n^* - V_n^*| \geq n^{-\frac{1}{2}}\})$$

$$= P(\{V_n^* \leq x + n^{-\frac{1}{2}}\}) + \mathcal{O}(n^{-\frac{1}{2}}) \qquad \text{as } n \to \infty$$

uniformly in x. A similar argument yields the opposite inequality

$$(3.2.40) \qquad F_n^*(x) \geq P(\{V_n^* \leq x - n^{-\frac{1}{2}}\}) + \mathcal{O}(n^{-\frac{1}{2}}) \qquad \text{as } n \to \infty$$

uniformly in x. Secondly we remark that, because of (3.2.14),

$$(3.2.41) \qquad P(\{V_n^* \leq x + n^{-\frac{1}{2}}\}) \leq P(\{W_{n-}^* \frac{\sigma(W_{n-})}{\sigma(V_n)} + \frac{E(W_{n-}-V_n)}{\sigma(V_n)} \leq x + n^{-\frac{1}{2}}\})$$

and similarly

$$(3.2.42) \qquad P(\{V_n^* \leq x - n^{-\frac{1}{2}}\}) \geq P(\{W_{n+}^* \frac{\sigma(W_{n+})}{\sigma(V_n)} + \frac{E(W_{n+}-V_n)}{\sigma(V_n)} \leq x - n^{-\frac{1}{2}}\})$$

for $-\infty < x < \infty$ and $n \geq 1$. This, together with lemma 3.2.2 yields that

$$(3.2.43) \qquad P(\{V_n^* \leq x + n^{-\frac{1}{2}}\}) \leq P(\{W_{n-}^* \leq x_{n+}\})$$

and

$$(3.2.44) \qquad P(\{V_n^* \leq x - n^{-\frac{1}{2}}\}) \geq P(\{W_{n+}^* \leq x_{n-}\})$$

for appropriate sequences x_{n+}, n = 1,2,... and x_{n-}, n = 1,2,... satisfying

$$(3.2.45) \qquad x_{n\pm} = x(1 + \mathcal{O}(n^{-\frac{1}{2}})) + \mathcal{O}(n^{-\frac{1}{2}})$$

as $n \to \infty$. We can now simply repeat the argument leading to (3.2.39) and (3.2.40), using lemma 3.2.3 and Chebychev's inequality, to find that

$$(3.2.46) \qquad P(\{W^*_{n-} \le x_{n+}\}) \le P(\{\mathcal{U}^*_{n-} \le x_{n+} + n^{-\frac{2}{3}}\}) + \mathcal{O}(n^{-\frac{2}{3}})$$

and

$$(3.2.47) \qquad P(\{W^*_{n+} \le x_{n-}\}) \ge P(\{\mathcal{U}^*_{n+} \le x_{n-} - n^{-\frac{2}{3}}\}) + \mathcal{O}(n^{-\frac{2}{3}})$$

as $n \to \infty$, uniformly in x. Combining all these inequalities we obtain that

$$(3.2.48) \qquad P(\{T^*_n \le x\}) \le P(\{\mathcal{U}^*_{n-} \le x_{n+} + n^{-\frac{2}{3}}\}) + \mathcal{O}(n^{-\frac{1}{2}})$$

and

$$(3.2.49) \qquad P(\{T^*_n \le x\}) \ge P(\{\mathcal{U}^*_{n+} \le x_{n-} - n^{-\frac{2}{3}}\}) + \mathcal{O}(n^{-\frac{1}{2}})$$

as $n \to \infty$, uniformly in x. Applying now lemma 3.2.4 we see that the first terms on the right of (3.2.48) and (3.2.49) are equal to $\Phi(x_{n+} + n^{-2/3}) + \mathcal{O}(n^{-\frac{1}{2}})$ and $\Phi(x_{n-} + n^{-2/3}) + \mathcal{O}(n^{-\frac{1}{2}})$ respectively for $n \to \infty$, uniformly in x. As these two terms are easily seen to be equal to $\Phi(x) + \mathcal{O}(n^{-\frac{1}{2}})$, as $n \to \infty$, uniformly in x, the proof of the theorem is complete.

3.3. PROOF OF THEOREM 3.1.2.

To start with we remark that for each $n \ge 1$ and real x

$$(3.3.1) \qquad G_n(x) = F^*_n(x\sigma n^{-\frac{1}{2}}\sigma^{-1}(T_n) + (\mu - E(T_n))\sigma^{-1}(T_n))$$

with $\mu = \mu(J,F)$ and $\sigma^2 = \sigma^2(J,F)$ as in (3.1.5) and (2.2.12). Using this identity and applying theorem 3.1.1 we find

$$(3.3.2) \qquad \sup_x |\, G_n(x) - \Phi(x\sigma n^{-\frac{1}{2}}\sigma^{-1}(T_n) + (\mu - E(T_n))\sigma^{-1}(T_n))| = \mathcal{O}(n^{-\frac{1}{2}})$$

as $n \to \infty$

To proceed we shall need asymptotic approximations for $\sigma n^{-\frac{1}{2}}\sigma^{-1}(T_n)$ and $(\mu - E(T_n))\sigma^{-1}(T_n)$.

LEMMA 3.3.1. *Let* $E|X_1|^{2+\varepsilon} < \infty$ *for some* $\varepsilon > 0$ *and suppose that the assumptions* 3.1.1 *and* 3.1.2 *are satisfied. Then* $\sigma^2(J,F) > 0$ *implies that as* $n \to \infty$

$$(3.3.3) \qquad |\sigma n^{-\frac{1}{2}}\sigma^{-1}(T_n) - 1| = O(n^{-\frac{1}{2}})$$

and

$$(3.3.4) \qquad |(\mu - E(T_n))\sigma^{-1}(T_n)| = O(n^{-\frac{1}{2}}).$$

PROOF. We first prove (3.3.3). It was already shown in the proof of lemma 3.2.1 (cf. (3.2.9)) that $\sigma^2(T_n - V_n) = O(n^{-5/2})$ and $\lim_{n\to\infty} n\sigma^2(V_n) = \sigma^2(J,F) > 0$ holds for $n \to \infty$. Also note that, in view of (3.2.19), $\sigma^2(V_n - W_{n+}) = O(n^{-2})$ as $n \to \infty$. Hence

$$(3.3.5) \qquad \sigma^2(T_n) = \sigma^2(W_{n+}) + O(\sigma(T_n)\sigma(T_n - W_{n+})) = \sigma^2(W_{n+}) + O(n^{-\frac{3}{2}}),$$

as $n \to \infty$

This and a simple computation using (3.2.31) and lemma 2.2.3 yields

$$(3.3.6) \qquad \sigma^2(T_n) = n^{-1}\sigma^2(J,F) + O(n^{-\frac{3}{2}}) \qquad \text{as } n \to \infty$$

and a simple Taylor expansion argument completes the proof of (3.3.3). To prove (3.3.4) we first use assumption 3.1.1 and (3.2.18) to see that

$$(3.3.7) \qquad ET_n = EW_{n+} + O(n^{-1}), \qquad \text{as } n \to \infty$$

This and relation (3.2.28) gives

$$(3.3.8) \qquad ET_n = \int_0^1 \psi(s)\,d\,F^{-1}(s) + cEX_1 + O(E(\int_0^1 (\Gamma_n(s) - s)^2 d\,F^{-1}(s))) + O(n^{-1}) \qquad \text{as } n \to \infty$$

Applying lemma 2.2.2 (with $p = 2$ and $\ell = 1$) to the third term on the right and integration by parts (cf. (3.2.10)) to the first term on the right of

(3.3.8) yields

$$ET_n = \mu(J,F) + \mathcal{O}(n^{-1}), \qquad \text{as } n \to \infty,$$

with $\mu = \mu(J,F)$ as in (3.1.5). This combined with (3.3.3) proves (3.3.4). □

To complete the proof of theorem 3.1.2 we use (3.3.3), and (3.3.4) and apply a simple Taylor argument to find that

$$(3.3.9) \qquad \Phi(x\sigma n^{-\frac{1}{2}}\sigma^{-1}(T_n) + (\mu - E(T_n))\sigma^{-1}(T_n)) = \Phi(x) + \mathcal{O}(n^{-\frac{1}{2}})$$

as $n \to \infty$, uniformly in x. This combined with (3.3.2) completes the proof of theorem 3.1.2.

3.4. PROOF OF THEOREM 3.1.3.

To prove theorem 3.1.3 we first need two lemma's. To start with we remark that s_n^2 (cf. (3.1.8)) can also be written as

$$(3.4.1) \qquad s_n^2 = \int_0^1\int_0^1 J(\Gamma_n(s))J(\Gamma_n(t))(\Gamma_n(s) \wedge \Gamma_n(t) - \Gamma_n(s)\Gamma_n(t))dF^{-1}(s)dF^{-1}(t)$$

Using this and (2.2.12) we arrive at the following decomposition of s_n^2:

$$(3.4.2) \qquad s_n^2 = \sigma^2 + \int_0^1\int_0^1 (J(\Gamma_n(s))J(\Gamma_n(t)) - J(s)J(t)) \cdot$$

$$\cdot (\Gamma_n(s) \wedge \Gamma_n(t) - \Gamma_n(s)\Gamma_n(t))d\,F^{-1}(s)d\,F^{-1}(t) +$$

$$+ \int_0^1\int_0^1 J(s)J(t)(\Gamma_n(s) \wedge \Gamma_n(t) - \Gamma_n(s)\Gamma_n(t) - s \wedge t + st)\cdot$$

$$\cdot d\,F^{-1}(s)d\,F^{-1}(t) = \sigma^2 + Y_n + R_n$$

where

$$(3.4.3) \qquad Y_n = Y_{n1} + Y_{n2}$$

with

$$(3.4.4)\qquad Y_{n1} = n^{-1} \sum_{i=1}^{n} g_1(U_i)$$

and

$$(3.4.5)\qquad Y_{n2} = n^{-1} \sum_{i=1}^{n} g_2(U_i)$$

The functions g_1 and g_2 are given by

$$(3.4.6)\qquad g_1(u) = 2 \int_0^1 \int_0^1 J^{(1)}(s)J(t)(\chi_{(0,s]}(u) - s)(s \wedge t - st)\,d\,F^{-1}(s)\,d\,F^{-1}(t)$$

and

$$(3.4.7)\qquad g_2(u) = 2 \int_0^1 \int_0^t J(s)J(t)\{(\chi_{(0,s]}(u) - s)(1-t) - (\chi_{(0,t]}(u) - t)s\}\cdot$$

$$\cdot d\,F^{-1}(s)\,d\,F^{-1}(t)$$

for $0 < u < 1$. Finally

$$(3.4.8)\qquad R_n = R_{n1} + R_{n2} + R_{n3}$$

where

$$(3.4.9)\qquad R_{n1} = \int_0^1 \int_0^1 \{J(\Gamma_n(s))J(\Gamma_n(t)) - J(s)J(t) -$$

$$- J^{(1)}(s)J(t)(\Gamma_n(s) - s) - J(s)J^{(1)}(t)(\Gamma_n(t) - t)\}\cdot$$

$$\cdot\{s \wedge t - st\}d\,F^{-1}(s)\,d\,F^{-1}(t)$$

$$(3.4.10)\qquad R_{n2} = \int_0^1 \int_0^1 (J(\Gamma_n(s))J(\Gamma_n(t)) - J(s)J(t))\cdot$$

$$\cdot(\Gamma_n(s) \wedge \Gamma_n(t) - \Gamma_n(s)\Gamma_n(t) - s \wedge t + st)\,d\,F^{-1}(s)\,d\,F^{-1}(t)$$

and

(3.4.11) $$R_{n3} = -\left(\int_0^1 J(s)(\Gamma_n(s) - s)\,d\,F^{-1}(s)\right)^2$$

Note that the first double integral on the right of (3.4.2) is equal to $Y_{n1} + R_{n1} + R_{n2}$ and that the second double integral on the right of (3.4.2) is precisely $Y_{n2} + R_{n3}$.

<u>LEMMA 3.4.1</u>. *Let* $E|X_1|^{2+\varepsilon} < \infty$ *for some* $\varepsilon > 0$ *and suppose that assumption 3.1.3 is satisfied. Then* $\sigma^2(J,F) > 0$ *implies that*

(3.4.12) $$|R_n| = O(n^{-\frac{1}{2}}(\ell n n)^{-1})$$

except on a set with probability $O(n^{-\frac{1}{2}})$ *as* $n \to \infty$.

<u>PROOF</u>. The proof will consist of two parts. In the first place we shall prove that

(3.4.13) $$|R_{n1}| = O(n^{-\frac{1}{2}}(\ell n n)^{-1})$$

(cf. (3.4.9)) except on a set with probability $O(n^{-\frac{1}{2}})$ as $n \to \infty$. To prove this it will be no loss of generality to assume that $J^{(1)}$ does not exist at only one point, say $s = u_1$. By the Markov inequality it clearly suffices to show that

(3.4.14) $$ER_{n1}^2 = O(n^{-\frac{3}{2}}(\ell n n)^{-2}), \qquad \text{as } n \to \infty$$

Let, for each $n \geq 1$, A_n be the random set

(3.4.15) $$A_n = \{s\colon \Gamma_n(s) \leq u_1 \leq s\} \cup \{s\colon s \leq u_1 \leq \Gamma_n(s)\}$$

with Γ_n as in (2.1.1). The complement of A_n in (0,1) will be denoted by A_n^c. We begin by remarking that the first factor (within curly brackets) in the integrand of (3.4.9) is in absolute value

(i) $\quad O(|\Gamma_n(s) - s| + |\Gamma_n(t) - t|^{1+\alpha_2}) \qquad$ when $s \in A_n$, $t \in A_n^c$

(ii) $\quad O(|\Gamma_n(s) - s|^{1+\alpha_2} + |\Gamma_n(t) - t|) \qquad$ when $s \in A_n^c$, $t \in A_n$

(iii) $\quad O(|\Gamma_n(s) - s| + |\Gamma_n(t) - t|) \qquad$ when $s,t \in A_n$

(iv) $\qquad O(|\Gamma_n(s)-s|^{\frac{1+\alpha}{2}} + |\Gamma_n(t)-t|^{\frac{1+\alpha}{2}}) \qquad$ when $s,t \in A_n^c$

where the order symbol is uniform with respect to the values of s and t considered in each case. A further simplifying remark is that the second factor (within curly brackets) is the integrand of (3.4.9) can be bounded above by $(s(1-s))^{\frac{1}{2}}(t(1-t))^{\frac{1}{2}}$ for all $0 < s,t < 1$. Also note that $\int_0^1 (s(1-s))^{\frac{1}{2}} d\, F^{-1}(s) < \infty$ by lemma 2.2.1 and the moment condition of the lemma. Combining the above considerations we can easily verify that to prove (3.4.14) it suffices to show that

$$(3.4.16) \qquad E\Big(\int_{A_n} |\Gamma_n(s)-s|\, d\, F^{-1}(s)\Big)^2 = O(n^{-\frac{3}{2}}(\ell nn)^{-2})$$

and

$$(3.4.17) \qquad E\Big(\int_{A_n^c} |\Gamma_n(s)-s|^{\frac{1+\alpha}{2}}\, d\, F^{-1}(s)\Big)^2 = O(n^{-\frac{3}{2}}(\ell nn)^{-2})$$

holds as $n \to \infty$.

It is convenient to introduce at this point the well-known Kolmogorov-Smirnov statistic

$$(3.4.18) \qquad D_n = n^{\frac{1}{2}} \sup_{0<s<1} |\Gamma_n(s)-s|$$

It was shown by DVORETSKY, KIEFER & WOLFOWITZ (1956) that

$$(3.4.19) \qquad P(\{D_n \geq \lambda\}) \leq c\, \exp(-2\lambda^2)$$

for all $n \geq 1$, $\lambda \geq 0$ and a positive constant c independent of n and λ. This obviously implies that

$$(3.4.20) \qquad ED_n^m = \int_0^\infty P(\{D_n \geq x^{\frac{1}{m}}\})dx$$

$$\leq c \int_0^\infty \exp(-2x^{\frac{2}{m}})dx = O(1), \qquad \text{as } n \to \infty$$

for any fixed $m > 0$. Hence we obtain that

$$(3.4.21) \qquad E(\sup_{0<s<1} |\Gamma_n(s)-s|)^m = O(n^{-\frac{m}{2}}), \qquad \text{as } n \to \infty.$$

Let U_δ be the neighbourhood of the point u_1 on which F^{-1} satisfies a Lipschitz condition:

$$(3.4.22) \qquad U_\delta = \{s: |s-u_1| < \delta,\ 0 < s < 1\}$$

Let U_δ^c denote the complement of U_δ in (0,1). To treat the expectation in (3.4.16) we remark that

$$(3.4.23) \qquad E(\int_{A_n} |\Gamma_n(s) - s| d\, F^{-1}(s))^2 \le 2E(\int_{A_n \cap U_\delta} |\Gamma_n(s) - s| d\, F^{-1}(s))^2 + 2E(\int_{A_n \cap U_\delta^c} |\Gamma_n(s) - s| d\, F^{-1}(s))^2$$

The first expectation on the right of (3.4.23) is bounded above by $E(\sup_{0<s<1}|\Gamma_n(s) - s|)^{2+2\alpha_3}$, which is $O(n^{-1-\alpha_3})$ for $n \to \infty$, in view of (3.4.21). If χ_n denotes the indicator of the set $\{\sup_{0<s<1}|\Gamma_n(s) - s| > \delta\}$ we see that the second integral on the right of (3.4.23) is bounded above by $\chi_n \cdot \int_0^1 |\Gamma_n(s) - s| d\, F^{-1}(s)$. Using this and the Cauchy-Schwarz inequality we find that

$$(3.4.24) \qquad E(\int_{A_n \cap U_\delta^c} |\Gamma_n(s) - s| d\, F^{-1}(s))^2 \le (P(\{\chi_n = 1\}))^{\frac{1}{2}} \cdot (E(\int_0^1 |\Gamma_n(s) - s| d\, F^{-1}(s))^4)^{\frac{1}{2}}.$$

Application of (3.4.19) with $\lambda = \delta n^{\frac{1}{2}}$ and lemma 2.2.2 yields that the second expectation on the right of (3.4.23) is $O(n^{-1} \exp(-2\delta^2 n))$ for $n \to \infty$. This completes the proof of (3.4.16). To establish (3.4.17) we replace the set A_n^c by (0,1) and we apply lemma 2.2.2 once more to find that this expectation is $O(n^{-1-\alpha_2})$ as $n \to \infty$. This proves (3.4.17) and the first part of the proof is complete.

Next we shall prove that

$$(3.4.25) \qquad |R_{ni}| = O(n^{-\frac{1}{2}}(\ell n n)^{-1}) \qquad \text{for } i = 2,3,$$

(cf. (3.4.10), (3.4.11)) except on set with probability $O(n^{-\frac{1}{2}})$ as $n \to \infty$. We first prove (3.4.25) for $i = 2$. As J is Lipschitz of order 1 on (0,1) we clearly have

$$(3.4.26)\qquad |J(\Gamma_n(s))J(\Gamma_n(t)) - J(s)J(t)| = \mathcal{O}(|\Gamma_n(s) - s| + |\Gamma_n(t) - t|)$$

as $n \to \infty$, uniformly for all $0 < s,t < 1$. Also note that we may restrict, for reasons of symmetry, integration in (3.4.10) to $0 < s \le t \le 1$ and then

$$(3.4.27)\qquad \Gamma_n(s) \wedge \Gamma_n(t) - \Gamma_n(s)\Gamma_n(t) - s \wedge t + st =$$

$$= (\Gamma_n(s) - s)(1-t) - (\Gamma_n(s) - s)(\Gamma_n(t) - t) - (\Gamma_n(t) - t)s$$

Now (3.4.26) and (3.4.27) together ensures that it suffices to prove (instead of (3.4.25) for $i = 2$)

$$(3.4.28)\qquad \int_0^1\int_0^t \{|\Gamma_n(s) - s| + |\Gamma_n(t) - t|\}.$$

$$\cdot\{|(\Gamma_n(s) - s)(1-t) - (\Gamma_n(s) - s)(\Gamma_n(t) - t) -$$

$$- (\Gamma_n(t) - t)s|\}d\,F^{-1}(s)d\,F^{-1}(t) = \mathcal{O}(n^{-\frac{1}{2}}(\ell nn)^{-1})$$

except on a set with probability $\mathcal{O}(n^{-\frac{1}{2}})$ as $n \to \infty$. Because the integrand in (3.4.28) can be bounded by $4|\Gamma_n(s) - s||\Gamma_n(t) - t| + (\Gamma_n(s) - s)^2(1-t) + (\Gamma_n(t) - t)^2 s$, it is easily inferred from the moment condition of the lemma and two applications of integration by parts that the left-hand side of (3.4.28) is of order

$$(3.4.29)\qquad \mathcal{O}\Big(\Big(\int_0^1 |\Gamma_n(s) - s|d\,F^{-1}(s)\Big)^2$$

$$+ \int_0^{\frac{1}{2}} s^{-\frac{1}{(2+\varepsilon)}}(\Gamma_n(s) - s)^2 d\,F^{-1}(s)$$

$$+ \int_{\frac{1}{2}}^1 (1-s)^{-\frac{1}{(2+\varepsilon)}}(\Gamma_n(s) - s)^2 d\,F^{-1}(s)\Big),\qquad \text{as } n \to \infty$$

Application of lemma 2.2.2 (with $\ell = 1$, $p = 2+2\varepsilon$) yields that the $(1+\varepsilon)$th absolute moment of the first term in (3.4.29) is $\mathcal{O}(n^{-1-\varepsilon})$, so that, using

Markov's inequality for $(1 + \varepsilon)$th absolute moments, this term is of order $\mathcal{O}(n^{-\frac{1}{2}}(\ell nn)^{-1})$, except on a set with probability $\mathcal{O}(n^{-\frac{1}{2}-\varepsilon/2}(\ell nn)^{1+\varepsilon})$ as $n \to \infty$. To treat the second term in (3.4.29) we first note that, because of the moment assumption of the lemma, this term can also be written as $c \int_0^{\frac{1}{2}} s^{-1+\varepsilon/(4+2\varepsilon)}(\Gamma_n(s) - s)^2 d\, K(s)$ where K is the df on $(0,\frac{1}{2})$ determined by the equation $dK(s) = c^{-1}s^{\frac{1}{2}}d\, F^{-1}(s)$ for $0 < s < \frac{1}{2}$ with $c = \int_0^{\frac{1}{2}} s^{\frac{1}{2}}d\, F^{-1}(s)$. Using this, Jensen's inequality, Fubini's theorem and the fact that we know from (2.2.5) that

$$(3.4.30) \qquad E|\Gamma_n(s) - s|^{2+\frac{\varepsilon}{4}} = \mathcal{O}(n^{-1-\frac{\varepsilon}{8}}s(1-s))$$

as $n \to \infty$, uniformly in $0 < s < 1$, we obtain

$$(3.4.31) \qquad E\left(\int_0^{\frac{1}{2}} s^{-1+\frac{\varepsilon}{(4+2\varepsilon)}}(\Gamma_n(s) - s)^2 dK(s)\right)^{1+\frac{\varepsilon}{8}}$$

$$\le E\int_0^{\frac{1}{2}} \{s^{-1+\frac{\varepsilon}{(4+2\varepsilon)}}(\Gamma_n(s) - s)^2\}^{1+\frac{\varepsilon}{8}} dK(s)$$

$$\le \int_0^{\frac{1}{2}} s^{-1-\frac{\varepsilon}{8}} E|\Gamma_n(s) - s|^{2+\frac{\varepsilon}{4}} dK(s)$$

$$= \mathcal{O}(n^{-1-\frac{\varepsilon}{8}}\int_0^{\frac{1}{2}} s^{\frac{1}{2}-\frac{\varepsilon}{8}} d\, F^{-1}(s)), \qquad \text{as } n \to \infty.$$

Using now the moment condition of the lemma (taking $\varepsilon < 1$) we can apply lemma 2.2.1 (with $\ell = (\frac{1}{2} - \frac{\varepsilon}{8})^{-1}$) to find that the $(1 + \frac{\varepsilon}{8})$th absolute moment of the second term in (3.4.29) is $\mathcal{O}(n^{-1-\varepsilon/8})$, so that, applying Markov's inequality for $(1 + \frac{\varepsilon}{8})$th absolute moments, this term is $\mathcal{O}(n^{-\frac{1}{2}}(\ell nn)^{-1})$, except on a set with probability $\mathcal{O}(n^{-\frac{1}{2}-\varepsilon/16}(\ell nn)^{1+\varepsilon/8})$ as $n \to \infty$. The third term in (3.4.29) can be treated likewise, and the proof of (3.4.28) and hence of (3.4.25) for the case $i = 2$ is now complete. Because $|R_{n3}|$ (cf. (3.4.11)) is almost identical with the first term in (3.4.29) we have also proved (3.4.25) for the case $i = 3$. In view of (3.4.8) the proof of the lemma is now complete. □

In the second lemma of this section we convert (3.4.2) into a stochastic expansion for σs_n^{-1}.

LEMMA 3.4.2. *Let* $\mathcal{E}|X_1|^{4+\varepsilon} < \infty$ *for some* $\varepsilon > 0$. *Suppose that* J *is continuous on* (0,1), *differentiable except possibly at a finite number of points, and that* $J^{(1)}$ *is bounded on the open intervals where it exists. The inverse* F^{-1} *puts mass zero at the points where* $J^{(1)}$ *remains undefined. Then* $\sigma^2(J,F) > 0$ *implies that*

$$(3.4.32) \qquad |\sigma s_n^{-1} - 1 + 2^{-1}\sigma^{-2}Y_n| = \mathcal{O}(n^{-\frac{1}{2}}(\ell n n)^{-1})$$

except on a set with probability $\mathcal{O}(n^{-\frac{1}{2}})$ *as* $n \to \infty$. *In addition we have that*

$$(3.4.33) \qquad \sigma s_n^{-1} \le 2$$

also except on a set with probability $\mathcal{O}(n^{-\frac{1}{2}})$ *as* $n \to \infty$.

PROOF. In view of (3.4.12) we may rewrite (3.4.2) as

$$(3.4.34) \qquad s_n^2\sigma^{-2} = 1 + \sigma^{-2}Y_n + \mathcal{O}(n^{-\frac{1}{2}}(\ell n n)^{-1})$$

except on a set with probability $\mathcal{O}(n^{-\frac{1}{2}})$ as $n \to \infty$. Since $(1+x)^{-\frac{1}{2}} = 1 - 2^{-1}x + \mathcal{O}(x^2)$ for $x \to 0$ this implies (3.4.32) provided we can show that

$$(3.4.35) \qquad Y_n^2 = \mathcal{O}(n^{-\frac{1}{2}}(\ell n n)^{-1})$$

except on a set with probability $\mathcal{O}(n^{-\frac{1}{2}})$ as $n \to \infty$. To see this we first note that the function g_1 (cf. (3.4.6)) is bounded on (0,1). In second place we remark that a simple computation using the conditions of the lemma and applying integration by parts yields that

$$(3.4.36) \qquad |g_2(u)| \le A_1(1 + (F^{-1}(u))^2)$$

for $0 < u < 1$ and some constant $A_1 > 0$. Using this and the Marcinkievitz Zygmund, Chung inequality (cf. (2.2.4)) we obtain

$$(3.4.37) \qquad \mathcal{E}|Y_n|^{2+\frac{\varepsilon}{2}} \le A_2 n^{-1-\frac{\varepsilon}{4}}(1 + \mathcal{E}|X_1|^{4+\varepsilon})$$

where the constant $A_2 > 0$ depends only on A_1 and ε. Together with the moment assumption of the lemma this ensures that $\mathcal{E}|Y_n|^{2+\varepsilon/2} = \mathcal{O}(n^{-1-\varepsilon/4})$ as $n \to \infty$, so that by Markov's inequality $Y_n^2 = \mathcal{O}(n^{-\frac{1}{2}}(\ell n n)^{-1})$, except on a set with probability $\mathcal{O}(n^{-\frac{1}{2}-\varepsilon/8}(\ell n n)^{1+\varepsilon/4})$ as $n \to \infty$. This proves (3.4.35) and hence (3.4.32). Obviously (3.4.33) is a consequence of (3.4.32) and the fact that $P(\{|Y_n| > d\}) = \mathcal{O}(n^{-1-\varepsilon/4})$ for any fixed $d > 0$ and $n \to \infty$. This completes the proof of the lemma. □

We are now in a position to prove theorem 3.1.3. To begin with we note that in the proof of theorem 3.1.1 two types of arguments occur. The df of T_n^* is approximated by that of V_n^* by showing that $P(\{|T_n^* - V_n^*| \geq n^{-\frac{1}{2}}\}) = \mathcal{O}(n^{-\frac{1}{2}})$ and the same reasoning is involved later in the transition from W_{n+} (or W_{n-}) to $\mathcal{U}_{n+}$ (or $\mathcal{U}_{n-}$). In view of (3.4.33) this type of argument remains valid if we multiply T_n^*, V_n^*, $W_{n\pm}^*$, and $\mathcal{U}_{n\pm}^*$ by $\frac{\sigma}{s_n}$. The second type of argument is based on the inequality $W_{n-} \leq V_n \leq W_{n+}$ which leads to (3.2.43) and (3.2.44). We can duplicate this part of the proof also to show that

$$(3.4.38) \qquad P(\{V_n^*\sigma s_n^{-1} \leq x + n^{-\frac{1}{2}}\}) \leq P(\{W_{n-}^*\sigma s_n^{-1} \leq x_{n+}\}) + \mathcal{O}(n^{-\frac{1}{2}})$$

and

$$(3.4.39) \qquad P(\{V_n^*\sigma s_n^{-1} \leq x - n^{-\frac{1}{2}}\}) \geq P(\{W_{n+}^*\sigma s_n^{-1} \leq x_{n-}\}) + \mathcal{O}(n^{-\frac{1}{2}})$$

as $n \to \infty$, with $x_{n\pm}$, $n = 1,2,\ldots$ as in (3.2.45). Together all this leads to

$$(3.4.40) \qquad P(\{T_n^*\sigma s_n^{-1} \leq x\}) \geq P(\{U_{n-}^*\sigma s_n^{-1} \leq x_{n+} + n^{-\frac{2}{3}}\}) + \mathcal{O}(n^{-\frac{1}{2}})$$

and

$$(3.4.41) \qquad P(\{T_n^*\sigma s_n^{-1} \leq x\}) \leq P(\{U_{n+}^*\sigma s_n^{-1} \leq x_{n-} + n^{-\frac{2}{3}}\}) + \mathcal{O}(n^{-\frac{1}{2}})$$

as $n \to \infty$, uniformly in x. As an example of the computations involved we prove

$$(3.4.42) \qquad P(\{V_n^*\sigma s_n^{-1} \leq x + n^{-\frac{1}{2}}\}) \leq P(\{W_{n-}^*\sigma s_n^{-1} \leq x_{n+}\}) + \mathcal{O}(n^{-\frac{1}{2}})$$

as $n \to \infty$, for sequences x_{n+}, $n = 1,2,\ldots$ satisfying (3.2.45). Using (3.2.41) and (3.4.33) and lemma 3.2.2 we see that

$$(3.4.43)\qquad P(\{V^*_n\sigma s_n^{-1} \le x + n^{-\frac{1}{2}}\}) \le$$

$$\le P(\{W^*_{n-}\sigma s_n^{-1} \frac{\sigma(W_{n-})}{\sigma(V_n)} + \sigma s_n^{-1} \frac{E(W_{n-}-V_n)}{\sigma(V_n)} \le x + n^{-\frac{1}{2}}\}) =$$

$$= P(\{W^*_{n-}\sigma s_n^{-1} \frac{\sigma(W_{n-})}{\sigma(V_n)} + \sigma s_n^{-1} \frac{E(W_{n-}-V_n)}{\sigma(V_n)} \le x + n^{-\frac{1}{2}} \cap \sigma s_n^{-1} \le 2\}) +$$

$$+ P(\{W^*_{n-}\sigma s_n^{-1} \frac{\sigma(W_{n-})}{\sigma(V_n)} + \sigma s_n^{-1} \frac{E(W_{n-}-V_n)}{\sigma(V_n)} \le x + n^{-\frac{1}{2}} \cap \sigma s_n^{-1} > 2\}) \le$$

$$\le P(\{W^*_{n-}\sigma s_n^{-1} \le \{(x + n^{-\frac{1}{2}}) + 2 \frac{|E(W_{n-}-V_n)|}{\sigma(V_n)}\} \frac{\sigma(V_n)}{\sigma(W_{n-})}\}) +$$

$$+ P(\{\sigma s_n^{-1} > 2\})$$

$$= P(\{W^*_{n-}\sigma s_n^{-1} \le x_{n+}\}) + O(n^{-\frac{1}{2}}), \qquad \text{as } n \to \infty$$

uniformly in x. This proves (3.4.42).

Starting with (3.4.40), (3.4.41) we begin by proving a Berry-Esseen bound for $T^*_n\sigma s_n^{-1}$ by establishing one for $U^*_{n+}\sigma s_n^{-1}$ and $U^*_{n-}\sigma s_n^{-1}$. In view of (3.4.32), lemma 3.2.4 and Mill's ratio we find that

$$(3.4.44)\qquad P(\{|U^*_{n\pm}(\sigma s_n^{-1} - 1 + 2^{-1}\sigma^{-2}Y_n)| \ge n^{-\frac{1}{2}}\})$$

$$\le P(\{|U^*_{n\pm}| \ge \ell n n\}) + O(n^{-\frac{1}{2}}) = O(n^{-\frac{1}{2}}), \qquad \text{as } n \to \infty$$

Thus instead of $U^*_{n\pm}\sigma s_n^{-1}$ we may consider $U^*_{n\pm}\ (1 - 2^{-1}\sigma^{-2}Y_n)$, which can be written as

$$(3.4.45)\qquad \sigma^{-1}(U_{n\pm})\cdot\{2n^{-1} \sum_{i=1}^{n} h_1(U_i) \pm \binom{n}{2}^{-1} \sum_{i=1}^{n}\sum_{j=1}^{i-1} h_{2,K}(U_i,U_j)\}\cdot$$
$$\cdot\{1 - 2^{-1}\sigma^{-2}n^{-1} \sum_{i=1}^{n} g(U_i)\},$$

where g is the sum of g_1 and g_2 (see (3.4.6) and (3.4.7)). It is clear from the proofs of the lemma's 2.2.3 and 3.4.2 that $h_1(u) = O(|F^{-1}(u)|)$ and $g(u) = O(|F^{-1}(u)|^2)$ for $u \to 0$ and 1. Also note that (cf. the remark preceding lemma 3.2.3) that $2K\cdot H(u)$ majorizes $h_{2,K}(u,v)$ and that $H(u) = O(|F^{-1}(u)|)$ for $u \to 0$ and 1. Using all this together with $Eh_1(U_1) = Eg(U_1) = 0$ and $Eh_{2,K}(U_1,U_2) = 0$ and exploiting the independence present, we arrive at

$$(3.4.46)\qquad \sigma^2(n^{-3} \sum_{i=1}^{n}\sum_{j=1}^{i-1}\sum_{k=1}^{n} h_{2,K}(U_i,U_j)g(U_k)) = O(n^{-3})$$

assuming a finite fourth moment of F, and

$$(3.4.47) \qquad \mathcal{E}(n^{-2} \sum_{i=1}^{n} h_1(U_i)g(U_i)) = \mathcal{O}(n^{-1})$$

and

$$(3.4.48) \qquad \sigma^2(n^{-2} \sum_{i=1}^{n} h_1(U_i)g(U_i)) = \mathcal{O}(n^{-3}), \qquad \text{as } n \to \infty$$

where we have to assume the sixth moment assumption of the theorem for (3.4.48) to hold. Combining these results with an application of Chebychev's inequality we find that the terms in (3.4.45) corresponding to the sums considered in (3.4.46), (3.4.47) and (3.4.48) are $\mathcal{O}(n^{-\frac{1}{2}})$ except on a set with probability $\mathcal{O}(n^{-\frac{1}{2}})$ as $n \to \infty$.

To conclude our proof of a Berry-Esseen bound for $\mathcal{U}^*_{n\pm}\sigma s_n^{-1}$ we have to consider the rv's

$$(3.4.49) \qquad \sigma^{-1}(U_{n\pm})\cdot\{2n^{-1} \sum_{i=1}^{n} h_1(U_i) \pm \binom{n}{2}^{-1} \sum_{i=1}^{n} \sum_{j=1}^{i-1} h_{2,K}(U_i,U_j) - \sigma^{-2}n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{i-1} (h_1(U_i)g(U_j) + h_1(U_j)g(U_i))\}.$$

Upon multiplication with a harmless factor $1 + \mathcal{O}(n^{-1})$, because of the non-exact standardization in (3.4.49), these rv's are U-statistics with kernels

$$(3.4.50) \qquad h_1(u) + h_1(v) \pm h_{2,K}(u,v) - 2\sigma^{-2}n(n-1)^{-1}(h_1(u)g(v) + h_1(v)g(u))$$

for $0 < u,v < 1$, to which the Berry-Esseen theorem for U-statistics (CALLAERT & JANSSEN (1978)) can be applied. We argue as in the proof of lemma 3.2.4 to validate this application of the Callaert-Janssen result. Note again that the sixth moment assumption of the theorem is needed to ensure a finite third absolute moment of $h_1(U_1)g(U_2)$. Hence a Berry-Esseen bound for $\mathcal{U}^*_{n\pm}\sigma s_n^{-1}$ follows, and this obviously implies a Berry Esseen bound for $T^*_n\sigma s_n^{-1}$.

To conclude our proof of theorem 3.1.3 let us note that

$$(3.4.51) \qquad n^{\frac{1}{2}}(T_n - \mu)/s_n = \{T^*_n\sigma^{-1}n^{\frac{1}{2}}\sigma(T_n) + (\mathcal{E}T_n - \mu)\sigma^{-1}n^{\frac{1}{2}}\}\sigma s_n^{-1}.$$

Combining now the argument leading to the proof of theorem 3.1.2 (cf. lemma 3.3.1 and the remark made after it) with the bound for σs_n^{-1} given in

(3.4.33) we can complete our proof of theorem 3.1.3.

3.5. A REFINEMENT

Going through the proof of theorem 3.1.3 we see that the sixth moment condition is really needed at only two points in the proof. First we need the sixth moment condition in (3.4.48). However, application of an inequality of VON BAHR & ESSEEN (1965) for the p^{th} absolute moments of sums of i.i.d. rv's ($1 \le p \le 2$) (see also PETROV (1975), page 60) shows (we take $p = \frac{3}{2}$) that the term considered in (3.4.48) is of sufficiently small order of magnitude, whenever the finiteness of a 4.5^{th} absolute moment is assumed.

The second place in the proof we need to reconsider is the application of the Berry-Esseen theorem of CALLAERT & JANSSEN (1978) to the U-statistic with kernel (3.4.50). In HELMERS & VAN ZWET (1982) the conditions needed in the Callaert-Janssen result are relaxed. Application of this stronger result shows that only a fourth moment of F is needed to establish a Berry-Esseen bound for the U-statistic with kernel (3.4.50). Hence theorem 3.1.3 remains valid when the sixth moment assumption is replaced by a 4.5^{th} absolute moment for the underlying df F.

CHAPTER 4

EDGEWORTH EXPANSIONS FOR LINEAR COMBINATIONS OF ORDER STATISTICS WITH SMOOTH WEIGHT FUNCTIONS

4.1. INTRODUCTION AND MAIN RESULTS

In the previous chapter we have obtained Berry-Esseen bounds of order $n^{-\frac{1}{2}}$ for the accuracy of the normal approximation for linear combinations of order statistics. In this chapter we investigate higher order approximations to the df's of these statistics. We shall establish Edgeworth expansions for linear combinations of order statistics with remainder $O(n^{-1})$ in the case of smooth weights. These have been derived in HELMERS (1980); the present chapter contains the results of this paper.

Let $X_1, X_2, \ldots$ be a sequence of i.i.d rv's with common df F and let us consider statistics of the form

$$(4.1.1) \qquad T_n = n^{-1} \sum_{i=1}^{n} c_{in} X_{i:n}$$

(cf. (1.2.4), (3.1.1)), where $X_{i:n}$ $(1 \le i \le n)$ denotes the i^{th} order statistic of $X_1, \ldots, X_n$ and the c_{in}, $i = 1,2,\ldots,n$, $n = 1,2,\ldots$ are real numbers. Let, furthermore, J_1 and J_2 be real-valued bounded measurable functions on $(0,1)$. We begin by listing the assumptions needed to prove the main results of this chapter. We recall that $\|h\| = \sup_{0<s<1} |h(s)|$ for any function h defined on $(0,1)$.

<u>ASSUMPTION 4.1.1</u>. There exists a number $\gamma > \frac{3}{2}$ such that as $n \to \infty$

$$\max_{1\le i\le n} \left| c_{in} - n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_1(s)\,ds - \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_2(s)\,ds \right| = O(n^{-\gamma}).$$

<u>ASSUMPTION 4.1.2</u>.

(i) The function J_1 is twice differentiable on $(0,1)$ with first and second bounded derivative $J_1^{(1)}$ and $J_1^{(2)}$ on $(0,1)$. The function J_2 is bounded on (0,1).

(ii) The functions $J_1^{(2)}$ and J_2 satisfy Lipschitz conditions of order $\alpha_1 > 0$ and $\alpha_2 > 0$ respectively on (0,1).

ASSUMPTION 4.1.3. There exists numbers $0 \le t_1 < t_2 \le 1$ such that

$$J_1(s) > 0 \qquad \text{for } t_1 < s < t_2$$

and such that on $(F^{-1}(t_1), F^{-1}(t_2))$ F is twice differentiable with positive density f and bounded second derivative f'.

Before we formulate the first main result of this chapter let us introduce some more notation. Introduce functions h_1, h_2 and h_3 (cf. (2.2.6), (2.2.14)) by

$$h_1(u) = -\int_0^1 J_1(s)(\chi_{(0,s]}(u) - s)\,d\,F^{-1}(s) \tag{4.1.2}$$

$$h_2(u,v) = -\int_0^1 J_1^{(1)}(s)(\chi_{(0,s]}(u) - s)(\chi_{(0,s]}(v) - s)\,d\,F^{-1}(s) \tag{4.1.3}$$

$$h_3(u,v,w) = -\int_0^1 J_1^{(2)}(s)(\chi_{(0,s]}(u) - s)(\chi_{(0,s]}(v) - s)\cdot (\chi_{(0,s]}(w) - s)\,d\,F^{-1}(s) \tag{4.1.4}$$

for $0 < u,v,w < 1$. Furthermore define, for each $n \ge 1$ and real x, the function $\tilde{F}_n$ by

$$\tilde{F}_n(x) = \Phi(x) - \phi(x)\left\{\frac{\kappa_3}{6n^{\frac{1}{2}}}(x^2 - 1) + \frac{\kappa_4}{24n}(x^3 - 3x) + \frac{\kappa_3^2}{72n}(x^5 - 10x^3 + 15x)\right\} \tag{4.1.5}$$

where Φ and ϕ are the df and density of the standard normal distribution. The quantities $\kappa_3 = \kappa_3(J_1,F)$ and $\kappa_4 = \kappa_4(J_1,F)$ are given by

$$(4.1.6) \qquad \kappa_3 = \kappa_3(J_1,F) = \frac{1}{\sigma^3(J_1,F)} \left[\int_0^1 h_1^3(u)\,du + \right.$$

$$\left. + 3\int_0^1\int_0^1 h_1(u)h_1(v)h_2(u,v)\,dudv \right]$$

and

$$(4.1.7) \qquad \kappa_4 = \kappa_4(J_1,F) = \frac{1}{\sigma^4(J_1,F)} \left[\int_0^1 h_1^4(u)\,du - \right.$$

$$- 3\sigma^4(J_1,F) + 12 \int_0^1\int_0^1 h_1^2(u)h_1(v)h_2(u,v)\,dudv +$$

$$+ \int_0^1\int_0^1\int_0^1 (4h_1(u)h_1(v)h_1(w)h_3(u,v,w) +$$

$$\left. + 12h_1(u)h_1(v)h_2(u,w)h_2(v,w))\,dudvdw \right]$$

where (cf. (2.2.13))

$$(4.1.8) \qquad \sigma^2 = \sigma^2(J_1,F) = \int_0^1 h_1^2(u)\,du$$

In the first theorem of this chapter we establish an asymptotic expansion with remainder $O(n^{-1})$ for (cf. (3.1.2))

$$(4.1.9) \qquad F_n^*(x) = P(\{T_n^* \le x\}), \qquad -\infty < x < \infty$$

where

$$(4.1.10) \qquad T_n^* = (T_n - E(T_n))/\sigma(T_n)$$

for the case of smooth weights.

<u>THEOREM 4.1.1</u>. *Let* $EX_1^4 < \infty$ *and suppose that the assumptions* 4.1.1, 4.1.2 *and* 4.1.3 *are satisfied. Then,*

$$\text{(4.1.11)} \qquad \sup_x |F_n^*(x) - \tilde{F}_n(x)| = o(n^{-1}), \qquad \textit{as } n \to \infty.$$

Note that the expansion $\tilde{F}_n$ does not depend on the function J_2. This is due to the exact standardization we have employed in theorem 4.1.1.

The second theorem in this chapter is a modification of theorem 4.1.1 which lends itself better to applications. Since a different standardization is used in this case, our expansion will not only depend on J_1 and F but also on J_2. We shall establish an asymptotic expansion with remainder $o(n^{-1})$ for the df (cf. (3.1.6))

$$\text{(4.1.12)} \qquad G_n(x) = P(\{n^{\frac{1}{2}}(T_n - \mu)/\sigma \le x\})$$

for $-\infty < x < \infty$ where (cf. (3.1.5))

$$\text{(4.1.13)} \qquad \mu = \mu(J_1,F) = \int_0^1 J_1(s)F^{-1}(s)ds$$

and $\sigma^2 = \sigma^2(J_1,F)$ as in (4.1.8). Introduce a function h_4 by

$$\text{(4.1.14)} \qquad h_4(u) = -\int_0^1 J_2(s)(\chi_{(0,s]}(u) - s)d\, F^{-1}(s)$$

for $0 < u < 1$. Furthermore quantities $a = a(J_1,J_2,F)$ and $b = b(J_1,J_2,F)$ are given by

$$\text{(4.1.15)} \qquad a = a(J_1,J_2,F) = \frac{1}{\sigma(J_1,F)}\left[2^{-1}\int_0^1 s(1-s)J_1^{(1)}(s)d\, F^{-1}(s) - \int_0^1 J_2(s)F^{-1}(s)ds\right]$$

and

$$\text{(4.1.16)} \qquad b = b(J_1,J_2,F) = \frac{1}{2\sigma^2(J_1,F)}\left[\int_0^1 (h_1(u)h_2(u,u) + 2h_1(u)h_4(u))du + \int_0^1\int_0^1 (2^{-1}h_2^2(u,v) + h_1(u)h_3(u,v,v))dudv\right]$$

Finally define, for each $n \ge 1$ and real x, the function $\tilde{G}_n$ by

$$\tilde{G}_n(x) = \tilde{F}_n(x) - \phi(x)\left\{ \frac{-a}{n^{\frac{1}{2}}} + \frac{(a\kappa_3+a^2+2b)}{2n} x - \frac{a\kappa_3}{6n} x^3 \right\} \tag{4.1.17}$$

THEOREM 4.1.2. *Suppose that the assumptions of theorem* 4.1.1 *are satisfied. Then,*

$$\sup_x |G_n(x) - \tilde{G}_n(x)| = O(n^{-1}), \qquad \text{as } n \to \infty. \tag{4.1.18}$$

It is useful to comment on these results. In the first place we remark that in spite of its unusual appearance assumption 4.1.1 covers a number of interesting situations, whenever assumption 4.1.2(i) is also satisfied. Four examples of the validity of these assumptions are provided by

$$c_{in} = J_1\left(\frac{i}{n+1}\right) \tag{4.1.19}$$

$$c_{in} = J_1\left(\frac{i}{n}\right) \tag{4.1.20}$$

$$c_{in} = n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_1(s)\,ds \tag{4.1.21}$$

and

$$c_{in} = EJ_1(U_{i:n}) \tag{4.1.22}$$

where J_1 is a function on (0,1) satisfying assumption 4.1.2(i). In each of these four cases it is easy to verify that assumption 4.1.1 holds with $\gamma = 2$ and

$$J_2(s) = (\tfrac{1}{2} - s)J_1^{(1)}(s) \tag{4.1.23}$$

$$J_2(s) = \tfrac{1}{2} J_1^{(1)}(s) \tag{4.1.24}$$

$$J_2(s) = 0 \tag{4.1.25}$$

$$J_2(s) = (\tfrac{1}{2} - s)J_1^{(1)}(s) + \tfrac{1}{2}s(1-s)J_1^{(2)}(s) \tag{4.1.26}$$

respectively. The weights (4.1.19) were considered by CHERNOFF et al. (1967)

and STIGLER (1974). MOORE (1968) studied weights of the type (4.1.20) and BICKEL (1967) investigated weights of the form (4.1.21). The weights given in (4.1.22) do not seem to appear in the literature, but weights of this form are of course well-known in the theory of rank tests.

We note that it is clear from the proof of theorem 4.1.1 (see (4.2.16)) that theorem 4.1.1 remains valid if we weaken assumption 4.1.1 slightly by requiring $\gamma > 1$. On the other hand, to prove theorem 4.1.2 we need assumption 4.1.1 as stated. Since assumption 4.1.1 is satisfied in all of the above cases, we have preferred to formulate theorem 4.1.1 in its present form.

In the second place we may remark that the assumptions 4.1.1 and 4.1.2 together put a rather restrictive smoothness requirement upon the weights. In particular the results of this chapter do not include trimmed means and the more general class of trimmed linear combinations of order statistics. For complementary results for these statistics the reader is referred to chapter 5.

In the third place we note that assumption 4.1.3 is needed to ensure sufficient smoothness of F_n^* and G_n, which is what Crámer's condition (C) (cf. (1.1.11)) does in the classical case of sums of independent rv's (cf. lemma 2.1.2; see also theorem 4.1 of VAN ZWET (1977)). Finally we require the finiteness of the fourth moment of the underlying df F. In view of Cramér's result for sums of i.i.d rv's (cf. theorem 1.1) this seems a natural condition.

Next we give a few applications of theorem 4.1.2. First of all we have, of course, the sample mean (cf. example 1.2.1). As in this case $J_1(s) \equiv 1$, $J_2(s) \equiv 0$ the assumptions of theorem 4.1.2 concerning the weights are trivially satisfied, we obtain Cramér's result (cf. theorem 1.1) as a very special case under a slightly stronger smoothness condition for the df F.

As a second application of theorem 4.1.2 we consider the L-estimator (cf. example 1.2.3)

$$T_n = 6n^{-1} \sum_{i=1}^{n} \frac{i}{n+1} \left(1 - \frac{i}{n+1}\right) X_{i:n} \tag{4.1.27}$$

in the case of the logistic distribution $F(x) = (1 + e^{-x})^{-1}$ for $-\infty < x < \infty$. In this case $J_1(s) = 6s(1-s)$, $J_2(s) = 3(1-2s)^2$, $F^{-1}(s) = \ell n(s(1-s)^{-1})$ and the conditions of theorem 4.1.2 are easily verified; we find $\mu = \mu(J_1,F) = 0$, $\sigma^2 = \sigma^2(J_1,F) = 3$ and after a number of computations

$$(4.1.28)\qquad P(\{2.3^{\frac{1}{2}}n^{-\frac{1}{2}} \sum_{i=1}^{n} \frac{i}{n+1}(1 - \frac{i}{n+1})X_{i:n} \le x\}) =$$

$$= \Phi(x) - \phi(x)\left[\frac{1}{20n}(x^3 - 3x) + \frac{(11-\pi^2)}{n}x\right] + o(n^{-1})$$

as $n \to \infty$. As a third application we consider Gini's mean difference (example 1.2.4) in the case of the uniform distribution $F(x) = x$ for $0 \le x \le 1$. We now have $J_1(s) = J_2(s) = 4(s - \frac{1}{2})$, $F^{-1}(s) = s$ and the conditions of theorem 4.1.2 are again satisfied. We find $\mu = \mu(J_1,F) = \frac{1}{3}$, $\sigma^2 = \sigma^2(J_1,F) = \frac{1}{45}$ and after a number of computations

$$(4.1.29)\qquad P(\{3.5^{\frac{1}{2}}n^{\frac{1}{2}}(\frac{4(n+1)}{n(n-1)} \sum_{i=1}^{n} (\frac{i}{n+1} - \tfrac{1}{2})X_{i:n} - \frac{1}{3}) \le x\}) =$$

$$= \Phi(x) - \phi(x)\left[\frac{-2\cdot 5^{\frac{1}{2}}}{21n^{\frac{1}{2}}}(x^2-1) + \frac{1}{28n}(x^3-3x) + \frac{10}{441n}(x^5-10x^3+15x) + \frac{2}{n}x\right] +$$
$$+ o(n^{-1}), \qquad \text{as } n \to \infty.$$

We note that there is no term of order $n^{-\frac{1}{2}}$ in the expansion (4.1.28). This is due to the fact that in this case F is symmetric about its expectation and the weight functions are both symmetric about $\frac{1}{2}$. In the expansion (4.1.29), on the other hand, there is a term of order $n^{-\frac{1}{2}}$ present because the weight functions are no longer symmetric. Recently CALLAERT, JANSSEN & VERAVERBEKE (1980) (see also JANSSEN (1978)) derived Edgeworth expansions for U-statistics. As Gini's mean difference in the case of an uniform distribution is a U-statistic satisfying the conditions of their theorem the expansion (4.1.29) can also be obtained from their results.

In section 4.2 we prove theorem 4.1.1. Theorem 4.1.2 is proved in section 4.3. Extensions are given in section 4.4,

4.2. PROOF OF THEOREM 4.1.1.

The purpose of this section is to provide a proof of theorem 4.1.1. Since our proofs will depend on characteristic function arguments let us denote by $\rho_n^*(t)$ the ch.f. of T_n^* and by $\tilde{\rho}_n(t)$ the Fourier-Stieltjes transform $\tilde{\rho}_n(t) = \int_{-\infty}^{\infty} e^{itx} d\,\tilde{F}_n(x)$ of $\tilde{F}_n$ (see (4.1.5)).

We shall show that for some sufficiently small $\varepsilon > 0$

$$(4.2.1)\qquad \int_{|t|\le n^{\varepsilon}} |\rho_n^*(t) - \tilde{\rho}_n(t)|\,|t|^{-1}dt = o(n^{-1})$$

$$(4.2.2)\qquad \int_{n^{\varepsilon}<|t|<n^{\frac{3}{2}}} |\rho_n^*(t)|\,|t|^{-1}dt = o(n^{-1})$$

and

$$(4.2.3)\qquad \int_{|t|>\log(n+1)} |\tilde{\rho}_n(t)|\,|t|^{-1}dt = o(n^{-1})$$

holds as $n \to \infty$. An application of Esseen's smoothing lemma (lemma 1.2) will then complete our proof.

We first prove (4.2.1). We shall essentially have to expand $\rho_n^*(t)$ for these "small" values of $|t|$. To start with we define for $0 \le u \le 1$ (cf. (3.2.10)) the functions

$$(4.2.4)\qquad \psi_i(u) = \int_u^1 J_i(s)ds - (1-u)\bar{J}_i$$

where $\bar{J}_i = \int_0^1 J_i(s)ds$ for $i = 1,2$. Then, by following the argument given in (3.2.11), we find that with probability one

$$(4.2.5)\qquad T_n = \int_0^1 (\psi_1(\Gamma_n(s)) + n^{-1}\psi_2(\Gamma_n(s)))d\,F^{-1}(s) +$$

$$+ (\bar{J}_1 + n^{-1}\bar{J}_2)n^{-1} \sum_{i=1}^{n} F^{-1}(U_i) +$$

$$+ n^{-1} \sum_{i=1}^{n} (c_{in} - n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_1(s)ds - \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_2(s)ds)F^{-1}(U_{i:n}),$$

Let J_1 be twice differentiable with first and second derivative $J_1^{(1)}$ and $J_1^{(2)}$ on (0,1). Let $J_1^{(2)}$ and J_2 be bounded on (0,1) and let $\beta_1 = E|X_1| < \infty$. Introduce for each $n \ge 1$ the rv S_n by (the superscript denoting differentiation)

(4.2.6) $$S_n = \int_0^1 \Big\{\psi_1(s) + n^{-1}\psi_2(s) + (\Gamma_n(s)-s)(\psi_1^{(1)}(s) + n^{-1}\psi_2^{(1)}(s)) + \frac{(\Gamma_n(s)-s)^2}{2}\psi_1^{(2)}(s) + \frac{(\Gamma_n(s)-s)^3}{6}\psi_1^{(3)}(s)\Big\} d\,F^{-1}(s) + (\bar{J}_1 + n^{-1}\bar{J}_2)n^{-1}\sum_{i=1}^{n} F^{-1}(U_i)$$

Note that $|\psi_i(u)| \le 4\|J_i\| u(1-u)$ for $0 < u < 1$, $i = 1,2$, and that $\psi_1^{(1)} = -J_1 + \bar{J}_1$, $\psi_2^{(1)} = -J_2 + \bar{J}_2$, $\psi_1^{(2)} = -J_1^{(1)}$ and $\psi_1^{(3)} = -J_1^{(2)}$ on $(0,1)$, so that it is easily verified that S_n is a well-defined rv. Later on in this section it will become clear that $T_n^* - S_n^*$ is, under appropriate conditions, of negligible order for our purposes.

It is convenient to introduce some more notation. Define rv's I_{mn} for $m = 1,2,3,4$ and $n \ge 1$ by

(4.2.7) $$I_{1n} = -\int_0^1 J_1(s)(\Gamma_n(s)-s)d\,F^{-1}(s) = n^{-1}\sum_{i=1}^{n} h_1(U_i)$$

(4.2.8) $$I_{2n} = -\int_0^1 J_1^{(1)}(s)\frac{(\Gamma_n(s)-s)^2}{2} d\,F^{-1}(s) = 2^{-1}n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} h_2(U_i,U_j)$$

(4.2.9) $$I_{3n} = -\int_0^1 J_1^{(2)}(s)\frac{(\Gamma_n(s)-s)^3}{6} d\,F^{-1}(s) = 6^{-1}n^{-3}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} h_3(U_i,U_j,U_k)$$

and

(4.2.10) $$I_{4n} = -n^{-1}\int_0^1 J_2(s)(\Gamma_n(s)-s)d\,F^{-1}(s) = n^{-2}\sum_{i=1}^{n} h_4(U_i)$$

where the functions h_1, h_2, h_3 and h_4 are given by (4.1.2) - (4.1.4) and (4.1.14). It is easily checked that

(4.2.11) $$\hat{S}_n = S_n - ES_n = \sum_{m=1}^{4}\hat{I}_{mn} = \sum_{m=1}^{4}(I_{mn} - EI_{mn})$$

Furthermore define rv's J_{mn} for $m = 1,2,3,4$ and $n \ge 1$ by

(4.2.12) $$J_{mn} = \hat{I}_{mn}/\sigma(S_n) = (I_{mn} - E(I_{mn}))/\sigma(S_n)$$

so that

$$(4.2.13) \qquad S_n^* = \sum_{m=1}^{4} J_{mn}$$

The proof of (4.2.1) will be split up in a number of lemma's. In the first lemma in this section we derive an asymptotic expansion for the variance of S_n.

<u>LEMMA 4.2.1</u>. *Let* $EX_1^2 < \infty$ *and suppose that assumption* 4.1.2(i) *is satisfied. Then,*

$$(4.2.14) \qquad |\sigma^2(S_n) - n^{-1}\sigma^2 - 2n^{-2}\sigma^2 b| = O(n^{-\frac{5}{2}}), \qquad \text{as } n \to \infty$$

where $\sigma^2 = \sigma^2(J_1,F)$ *is as in* (4.1.8) *and* $b = b(J_1,J_2,F)$ *as in* (4.1.16). *In addition* σ^2 *and* $\sigma^2 b$ *are finite.*

<u>PROOF</u>. In view of (4.2.11) $\sigma^2(S_n) = \sigma^2(\Sigma_{m=1}^4 I_{mn})$. It follows directly from (4.1.8) and (4.2.7) that $\sigma^2(I_{1n}) = n^{-1}\sigma^2$. Also note that it is immediate from (4.2.7), (4.2.8) and an application of lemma 2.2.3.b that

$$2\,\mathrm{cov}(I_{1n},I_{2n}) = 2EI_{1n}I_{2n} = n^{-3}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} Eh_1(U_i)h_2(U_j,U_k) =$$

$$= n^{-2}\int_0^1 h_1(u)h_2(u,u)\,du.$$

Next we consider $\sigma^2(I_{2n})$. Using lemma 2.2.3.b once more we directly find that

$$EI_{2n}^2 = 4^{-1}n^{-2}\,(Eh_2(U_1,U_1))^2 + 2^{-1}n^{-2}Eh_2^2(U_1,U_2) + O(n^{-3}), \qquad \text{as } n \to \infty.$$

Because we also know that $(EI_{2n})^2 = 4^{-1}n^{-2}(Eh_2(U_1,U_1))^2$ we have shown that

$$\sigma^2(I_{2n}) = 2^{-1}n^{-2}\int_0^1\int_0^1 h_2^2(u,v)\,du\,dv + O(n^{-3}), \qquad \text{as } n \to \infty.$$

Similarly we can prove that

$$2\,\mathrm{cov}(I_{1n},I_{3n}) = n^{-2}\int_0^1\int_0^1 h_1(u)h_3(u,v,v)\,du\,dv + O(n^{-3})$$

as $n \to \infty$, and also that

$$2\,\mathrm{cov}(I_{1n}, I_{4n}) = 2n^{-2} \int_0^1 h_1(u) h_4(u)\,du.$$

Finally we remark that it is easy to prove using similar arguments as above that

$$\sigma^2(I_{3n}) + \sigma^2(I_{4n}) = \mathcal{O}(n^{-3}), \qquad \text{as } n \to \infty$$

and also that in view of the Cauchy-Schwarz inequality

$$|\mathrm{cov}(I_{2n}, I_{3n}) + \mathrm{cov}(I_{2n}, I_{4n}) + \mathrm{cov}(I_{3n}, I_{4n})| = \mathcal{O}(n^{-\frac{5}{2}}), \quad \text{as } n \to \infty.$$

Combining all these results we here proved (4.2.14). The assertion that σ^2 and $\sigma^2 b$ are finite is a simple consequence of lemma 2.2.3(a) and the formulas for σ^2 and b given in (4.1.8) and (4.1.16). □

LEMMA 4.2.2. *Let* $EX_1^2 < \infty$ *and suppose that assumption* 4.1.2(i) *is satisfied. Then* $\sigma^2(J_1,F) > 0$ *implies that for any fixed real number* m

$$(4.2.15) \qquad |\sigma^{-m}(S_n) - n^{\frac{m}{2}}\sigma^{-m}| = \mathcal{O}(n^{\frac{m}{2}-1}), \qquad \text{as } n \to \infty$$

where $\sigma^2 = \sigma^2(J_1,F)$ *is as in* (4.1.8).

PROOF. The statement is immediate from lemma 4.1.1. □

The next lemma will enable us to show that $T_n^* - S_n^*$ is of negligible order for our purposes. Let τ_n^* denote the ch.f. of S_n^*.

LEMMA 4.2.3. *Let, for some* $\delta > 0$, $E|X_1|^{2+\delta} < \infty$ *and suppose that the assumptions* 4.1.1 *and* 4.1.2 *are satisfied. Then* $\sigma^2(J_1,F) > 0$ *implies that for every* $\varepsilon > 0$

$$(4.2.16) \qquad \int_{|t| \le n^{\varepsilon}} |\rho_n^*(t) - \tau_n^*(t)|\,|t|^{-1}\,dt = \mathcal{O}(n^{-1-\min(\frac{\alpha_1}{2}, \frac{\alpha_2}{2}, \gamma - 1)+\varepsilon})$$

as $n \to \infty$.

PROOF. It follows from lemma X.V.4.1 of FELLER (1966) that

(4.2.17) $$|\rho_n^*(t) - \tau_n^*(t)| \le |t| E |T_n^* - S_n^*|$$

for all t and $n \ge 1$. Using (4.2.5), (4.2.6), assumption 4.1.2(ii) and applying Taylor's theorem we see directly that

(4.2.18) $$\sigma^2(T_n - S_n) = \mathcal{O}(E(\int_0^1 |\Gamma_n(s) - s|^{1+\alpha_1} d\, F^{-1}(s))^2 + n^{-2} E(\int_0^1 |\Gamma_n(s) - s|^{1+\alpha_2} d\, F^{-1}(s))^2 + \sigma^2(n^{-1} \sum_{i=1}^{n} (c_{in} - n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_1(s)ds - \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_2(s)ds) \cdot F^{-1}(U_{i:n}))).$$

Application of lemma 2.2.2 with $\ell = 2$ and $p = 3 + \alpha_1$ and $p = 1 + \alpha_2$ respectively implies that the sum of the first two terms on the right of (4.2.18) is

(4.2.19) $$\mathcal{O}(n^{-3-\min(\alpha_1,\alpha_2)}), \qquad \text{as } n \to \infty.$$

To treat the third term on the right of (4.2.18) we need inequality (3.2.3). Using this inequality and assumption 4.1.1 we see directly that

$$\sigma^2(n^{-1} \sum_{i=1}^{n} (c_{in} - n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_1(s)ds - \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_2(s)ds) F^{-1}(U_{i:n}))$$

$$= \mathcal{O}(n^{-1-2\gamma}), \qquad \text{as } n \to \infty.$$

Combining this result with (4.2.19) it is easy to conclude that

(4.2.20) $$\sigma^2(T_n - S_n) = \mathcal{O}(n^{-3-\min(\alpha_1,\alpha_2)}) + \mathcal{O}(n^{-1-2\gamma})$$

as $n \to \infty$. To complete our proof we remark that it follows from an application of the lemma's 2.1.1 and 4.2.2 (with m = -2) that (4.2.20) implies that

$$(4.2.21) \qquad \sigma^2(T_n^* - S_n^*) = \mathcal{O}(n^{-2-\min(\alpha_1,\alpha_2)}) + \mathcal{O}(n^{-2\gamma})$$

as $n \to \infty$. This combined with (4.2.17) proves (4.2.16). □

Next define for real t and $n \geq 1$

$$(4.2.22) \qquad \tau_{1n}(t) = Ee^{itJ_{1n}}(1 + it(J_{2n} + J_{3n} + J_{4n}) + \frac{(it)^2}{2} J_{2n}^2).$$

In the following lemma we shall approximate τ_n^* by τ_{1n} for all $|t| \leq n^{\varepsilon}$.

<u>LEMMA 4.2.4</u>. *Let, for some* $\delta > 0$, $E|X_1|^{3+\delta} < \infty$ *and suppose that assumption* 4.1.2(i) *is satisfied. Then* $\sigma^2(J_1,F) > 0$ *implies that*

$$(4.2.23) \qquad \int_{|t|\leq n^{\varepsilon}} |\tau_n^*(t) - \tau_{1n}(t)||t|^{-1}dt = \mathcal{O}(n^{-\frac{3}{2}+3\varepsilon})$$

as $n \to \infty$.

<u>PROOF</u>. Application of lemma X.V.4.1 of FELLER (1966) yields that

$$|\tau_n^*(t) - \tau_{1n}(t)| = |Ee^{it J_{1n}}(e^{it(J_{2n}+J_{3n}+J_{4n})} - 1 -$$
$$- it(J_{2n} + J_{3n} + J_{4n}) - \frac{(it)^2}{2} J_{2n}^2)| \leq$$
$$\leq t^2(E|J_{2n}J_{3n}| + E|J_{2n}J_{4n}| + E|J_{3n}J_{4n}| +$$
$$+ EJ_{3n}^2 + EJ_{4n}^2) + |t|^3E|J_{2n} + J_{3n} + J_{4n}|^3,$$

for all t and $n \geq 1$. It is not difficult to verify from the proof of lemma 4.2.1 and from lemma 4.2.2 that the coefficient of t^2 on the right in the above inequality is $\mathcal{O}(n^{-3/2})$, as $n \to \infty$. An application of the c_r-inequality, lemma 2.2.2 with $\ell = 3$ and p = 2,3 and 4 respectively and of lemma 4.2.2 shows that also $E|J_{2n} + J_{3n} + J_{4n}|^3 = \mathcal{O}(n^{-3/2})$, as $n \to \infty$. Combining these results we easily check that (4.2.23) is proved. □

We continue with the analysis of $\tau_{1n}(t)$. For convenience we write σ_n^2 to indicate $n\sigma^2(S_n)$ and we denote the ch.f. of $h_1(U_1)$ by ρ. To start with we remark that it follows from (4.2.22) that (cf. (1.1.3))

(4.2.24)
$$\tau_{1n}(t) = \rho^n\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) +$$

$$+ \frac{it}{2n^{\frac{3}{2}}\sigma_n}\,\rho^{n-2}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n(n-1)\, E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))}\cdot h_2(U_1,U_2) +$$

$$+ \frac{it}{2n^{\frac{3}{2}}\sigma_n}\,\rho^{n-1}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n} h_1(U_1)}\,\hat{h}_2(U_1,U_1)$$

$$+ \frac{it}{6n^{\frac{5}{2}}\sigma_n}\,\rho^{n-3}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n(n-1)(n-2)\, E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2)+h_1(U_3))}\cdot$$

$$\cdot\, h_3(U_1,U_2,U_3) +$$

$$+ \frac{it}{6n^{\frac{5}{2}}\sigma_n}\,\rho^{n-2}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) 3n(n-1)\, E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))}\, h_3(U_1,U_1,U_2)$$

$$+ \frac{it}{6n^{\frac{5}{2}}\sigma_n}\,\rho^{n-1}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n} h_1(U_1)}\,\hat{h}_3(U_1,U_1,U_1) +$$

$$+ \frac{it}{n^{\frac{3}{2}}\sigma_n}\,\rho^{n-1}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n} h_1(U_1)}\, h_4(U_1) +$$

$$+ \frac{(it)^2}{8n^3\sigma_n^2}\,\rho^{n-4}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n(n-1)(n-2)(n-3)\cdot$$

$$\cdot\left(E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))}\, h_2(U_1,U_2)\right)^2 +$$

$$+ \frac{(it)^2}{8n^3\sigma_n^2}\,\rho^{n-3}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) 4n(n-1)(n-2)\cdot$$

$$\cdot E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2)+h_1(U_3))} h_2(U_1,U_2)h_2(U_1,U_3) \; +$$

$$+ \frac{(it)^2}{8n^3\sigma_n^2} \rho^{n-3}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) 2n(n-1)(n-2) \cdot$$

$$\cdot E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2)+h_1(U_3))} \hat{h}_2(U_1,U_1)H_2(U_2,U_3) +$$

$$+ \frac{(it)^2}{8n^3\sigma_n^2} \rho^{n-2}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) 4n(n-1) \cdot$$

$$\cdot E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))} \hat{h}_2(U_1,U_1)h_2(U_1,U_2) \; +$$

$$+ \frac{(it)^2}{8n^3\sigma_n^2} \rho^{n-2}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) 2n(n-1) E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))} (h_2(U_1,U_2))^2 \; +$$

$$+ \frac{(it)^2}{8n^3\sigma_n^2} \rho^{n-2}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n(n-1) \left(E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n} h_1(U_1)} \hat{h}_2(U_1,U_1)\right)^2 +$$

$$+ \frac{(it)^2}{8n^3\sigma_n^2} \rho^{n-1}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n} h_1(U_1)} (\hat{h}_2(U_1,U_1))^2.$$

In the next lemma we derive an asymptotic expansion for the factors $\rho^{n-m}(t/(n^{\frac{1}{2}}\sigma_n))$ appearing in the terms on the right of (4.2.24).

<u>LEMMA 4.2.5</u>. *Let* $EX_1^4 < \infty$ *and suppose that assumption* 4.1.2(i) *is satisfied. Then* $\sigma^2(J_1,F) > 0$ *implies that there exists* $a > 0$ *and a fixed polynomial* P *in* t, *such that for any fixed integer* $m \geq 0$ *and uniformly for* $|t| \leq an^{\frac{1}{2}}$.

$$(4.2.25) \qquad |\rho^{n-m}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) - e^{-\frac{t^2}{2}}\left(1 - \frac{(it)^2}{n}\left(\frac{m}{2}+b\right) + \frac{(it)^3 \int_0^1 h_1^3(u)du}{6n^{\frac{1}{2}}\sigma^3} + \right.$$

$$+ \frac{(it)^4(\int_0^1 h_1^4(u)du-3\sigma^4)}{24n\sigma^4} + \frac{(it)^6(\int_0^1 h_1^3(u)du)^2}{72n\sigma^6})|$$

$$= o(n^{-1}|t|P(t)e^{-\frac{t^2}{4}}), \qquad \text{as } n \to \infty$$

where $\sigma^2 = \sigma^2(J_1,F)$ *is as in* (4.1.8) *and* $b = b(J_1,J_2,F)$ *as in* (4.1.16).

PROOF. Since $\sigma^{-1}(n-m)^{-\frac{1}{2}} \Sigma_{i=1}^{n-m} h_1(U_i)$ is a properly standardized sum of independently and identically distributed rv's with expectation zero, variance one, and finite fourth moment, it follows directly from the classical theory of Edgeworth expansions for such sums (see, e.g., GNEDENKO - KOLMOGOROV (1954), §41, theorem 2.1, inequality (b)) that there exist a' > 0 such that uniformly for $|t| \le a'n^{\frac{1}{2}}$

$$(4.2.26) \qquad |\rho^{n-m}(\frac{t}{(n-m)^{\frac{1}{2}}\sigma}) - e^{-\frac{t^2}{2}}(1 + \frac{(it)^3 \int_0^1 h_1^3(u)du}{6n^{\frac{1}{2}}\sigma^3} +$$

$$+ \frac{(it)^4(\int_0^1 h_1^4(u)du-3\sigma^4)}{24n\sigma^4} + \frac{(it)^6(\int_0^1 h_1^3(u)du)^2}{72n\sigma^6})| =$$

$$= o(n^{-1}|t|P(t)e^{-\frac{t^2}{4}}), \qquad \text{as } n \to \infty,$$

where P is a fixed polynomial in t. We now replace t by $t_n = t(n-m)^{\frac{1}{2}}\sigma/(n^{\frac{1}{2}}\sigma_n)$. It follows after expanding $e^{-t_n^2/2}$ around t and using the result of lemma 4.2.1 that we obtain (4.2.25). □

The expectations appearing on the right of (4.2.24) are expanded in the following lemma.

LEMMA 4.2.6. *Let* $EX_1^4 < \infty$ *and suppose that assumption* 4.1.2(i) *is satisfied. Then* $\sigma^2(J_1,F) > 0$ *implies that uniformly for all* t

$$(4.2.27) \qquad |Ee^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))} h_2(U_1,U_2) -$$

$$- \frac{(it)^2}{n\sigma^2}\int_0^1\int_0^1 h_1(u)h_1(v)h_2(u,v)dudv -$$

$$- \frac{(it)^3}{n^{\frac{3}{2}}\sigma^3} \int_0^1 \int_0^1 h_1^2(u) h_1(v) h_2(u,v)\,dudv| = O(n^{-2}(t^2+t^4) + n^{-\frac{5}{2}} |t|^3)$$

(4.2.28) $$|Ee^{\frac{it}{n^{\frac{1}{2}}\sigma_n} h_1(U_1)} \hat{h}_2(U_1,U_1) - \frac{it}{n^{\frac{1}{2}}\sigma} \int_0^1 h_1(u) h_2(u,u)\,du| =$$

$$= O(n^{-1}t^2 + n^{-\frac{3}{2}} |t|)$$

(4.2.29) $$|Ee^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2)+h_1(U_3))} h_3(U_1,U_2,U_3) -$$

$$- \frac{(it)^3}{n^{\frac{3}{2}}\sigma^3} \int_0^1 \int_0^1 \int_0^1 h_1(u) h_1(v) h_1(w) h_3(u,v,w)\,dudvdw| \leq$$

$$= O(n^{-2}t^4 + n^{-\frac{5}{2}} |t|^3)$$

(4.2.30) $$|Ee^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))} h_3(U_1,U_1,U_2) -$$

$$- \frac{it}{n^{\frac{1}{2}}\sigma} \int_0^1 \int_0^1 h_1(u) h_3(u,v,v)\,dudv| = O(n^{-1}t^2 + n^{-\frac{3}{2}} |t|)$$

(4.2.31) $$|Ee^{\frac{it}{n^{\frac{1}{2}}\sigma_n} h_1(U_1)} \hat{h}_3(U_1,U_1,U_1)| = O(n^{-\frac{1}{2}}|t|)$$

(4.2.32) $$|Ee^{\frac{it}{n^{\frac{1}{2}}\sigma_n} h_1(U_1)} h_4(U_1) - \frac{it}{n^{\frac{1}{2}}\sigma} \int_0^1 h_1(u) h_4(u)\,du| =$$

$$= O(n^{-1}t^2 + n^{-\frac{3}{2}} |t|)$$

(4.2.33) $$\left|\left(E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))} h_2(U_1,U_2)\right)^2 - \right.$$

$$\left. - \frac{(it)^4}{n^2\sigma^4}\left(\int_0^1\int_0^1 h_1(u)h_1(v)h_2(u,v)\,du\,dv\right)^2\right| = \mathcal{O}(n^{-\frac{5}{2}}|t|^5 + n^{-3}t^4)$$

(4.2.34) $$\left|E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2)+h_1(U_3))} h_2(U_1,U_2)h_2(U_1,U_3) - \right.$$

$$\left. - \frac{(it)^2}{n\sigma^2}\int_0^1\int_0^1\int_0^1 h_1(u)h_1(v)h_2(u,w)h_2(v,w)\,du\,dv\,dw\right| = \mathcal{O}(n^{-\frac{3}{2}}|t|^3 + n^{-2}t^2)$$

(4.2.35) $$\left|E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2)+h_1(U_3))} \hat{h}_2(U_1,U_1)h_2(U_2,U_3)\right| = \mathcal{O}(n^{-\frac{3}{2}}|t|^3)$$

(4.2.36) $$\left|E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))} \hat{h}_2(U_1,U_1)h_2(U_1,U_2)\right| = \mathcal{O}(n^{-\frac{1}{2}}|t|)$$

(4.2.37) $$\left|E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))} (h_2(U_1,U_2))^2 - \int_0^1\int_0^1 h_2^2(u,v)\,du\,dv\right| =$$

$$= \mathcal{O}(n^{-\frac{1}{2}}|t|)$$

(4.2.38) $$\left|\left(E e^{\frac{it}{n\,\sigma_n}h_1(U_1)} \hat{h}_2(U_1,U_1)\right)^2\right| = \mathcal{O}(n^{-1}t^2)$$

(4.2.39) $$\left|E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}h_1(U_1)} (\hat{h}_2(U_1,U_1))^2\right| = \mathcal{O}(1), \qquad \text{as } n \to \infty.$$

PROOF. Because the statements (4.2.27) - (4.2.39) are all proved in essentially the same manner we shall only prove (4.2.27), by way of example. Expanding $\exp(it/(n^{\frac{1}{2}}\sigma_n)(h_1(U_1)+h_1(U_2))$ around $t = 0$ we find that for all t and $n \geq 1$

$$(4.2.40)\qquad |Ee^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))} h_2(U_1,U_2) -$$

$$- \frac{it}{n^{\frac{1}{2}}\sigma_n}(Eh_1(U_1)h_2(U_1,U_2) + Eh_1(U_2)h_2(U_1,U_2)) -$$

$$- \frac{(it)^2}{2n\sigma_n^2}(Eh_1^2(U_1)h_2(U_1,U_2) + 2Eh_1(U_1)h_1(U_2)h_2(U_1,U_2) + $$

$$+ Eh_1^2(U_2)h_2(U_1,U_2))$$

$$- \frac{(it)^3}{6n^{\frac{3}{2}}\sigma_n^3}(Eh_1^3(U_1)h_2(U_1,U_2) + 3Eh_1^2(U_1)h_1(U_2)h_2(U_1,U_2) +$$

$$+ 3Eh_1^2(U_2)h_1(U_1)h_2(U_1,U_2) + Eh_1^3(U_2)h_2(U_1,U_2))| \le$$

$$\le \frac{t^4}{n^2\sigma_n^4} E|h_1(U_1) + h_1(U_2)|^4|h_2(U_1,U_2)|.$$

We next show that $Eh_1(U_i)h_2(U_1,U_2) = Eh_1^2(U_i)h_2(U_1,U_2) = Eh_1^3(U_i)h_2(U_1,U_2) = 0$ for $i = 1,2$. We first prove that $Eh_1^3(U_1)h_2(U_1,U_2) = 0$. It follows directly from (4.1.2), (4.1.3), (2.2.8), the independence of U_1 and U_2, and lemma 2.2.3.a that

$$|Eh_1^3(U_1)h_2(U_1,U_2)| \le \|J_1\|^3\|J_1^{(1)}\| EH^3(U_1)EH(U_2) < \infty.$$

Hence we can write

$$Eh_1^3(U_1)h_2(U_1,U_2) = E[E(h_1^3(U_1)h_2(U_1,U_2)|U_1)] =$$

$$= E[h_1^3(U_1)E\{h_2(U_1,U_2)|U_1\}] = 0$$

because of lemma 2.2.3.b. This proves the assertion. The other statements can be proved in the same way. It follows that

$$(4.2.41) \qquad |Ee^{\frac{it}{n^{\frac12}\sigma_n}(h_1(U_1)+h_1(U_2))} h_2(U_1,U_2) -$$

$$- \frac{(it)^2}{n\sigma_n^2}\int_0^1\int_0^1 h_1(u)h_1(v)h_2(u,v)\,dudv -$$

$$- \frac{(it)^3}{n^{\frac32}\sigma_n^3}\int_0^1\int_0^1 h_1^2(u)h_1(v)h_2(u,v)\,dudv| \le$$

$$\le \frac{t^4}{n\sigma_n^4} E|h_1(U_1) + h_1(U_2)|^4|h_2(U_1,U_2)|.$$

Using now (2.2.8) (with $m_1 = h_1$ and $m_2 = h_2$) and lemma 2.2.3.a once more, we see that

$$E|h_1(U_1) + h_1(U_2)|^4|h_2(U_1,U_2)| \le$$

$$\le 8Eh_1^4(U_1)|h_2(U_1,U_2)| + 8Eh_1^4(U_2)|h_2(U_1,U_2)| =$$

$$= 16Eh_1^4(U_1)|h_2(U_1,U_2)| \le 16\|J^{(1)}\|Eh_1^4(U_1)H(U_2) =$$

$$= 16\|J^{(1)}\|Eh_1^4(U_1)EH(U_2) \le 4^7\|J\|^4\|J^{(1)}\|_{\beta_1\beta_4} < \infty$$

so that the term on the right of (4.2.41) is $O(n^{-2}\sigma^{-4}t^4)$ as $n \to \infty$. Next we remark that lemma 4.2.2 implies that $\sigma_n^{-1} = \sigma^{-1} + O(n^{-1})$, as $n \to \infty$. Inserting this result in (4.2.41) we have proved (4.2.27). □

We are now in a position to prove (4.2.1). We first apply lemma 4.2.3 with $0 < \varepsilon < \min(\frac{\alpha_1}{2}, \frac{\alpha_2}{2}, \gamma - 1)$ to see that the integral on the left of (4.2.16) is $o(n^{-1})$, as $n \to \infty$. Secondly we use lemma 4.2.4 with $0 < \varepsilon < \frac{1}{6}$ to find that the integral on the left of (4.2.23) is also $o(n^{-1})$ as $n \to \infty$. To proceed let us note that we can write down $\tilde{\rho}_n(t)$ explicitly as

$$(4.2.42) \qquad \tilde{\rho}_n(t) = e^{-\frac{t^2}{2}}\left(1 - \frac{it^3\kappa_3}{6n^{\frac12}} + \frac{3\kappa_4 t^4 - \kappa_3^2 t^6}{72n}\right)$$

Next we apply (4.2.42) and the results of the lemma's 4.2.5 and 4.2.6 to check that for $n \to \infty$

$$(4.2.43) \qquad \int_{|t| \le an^{\frac{1}{2}}} |\tau_{1n}(t) - \tilde{\rho}_n(t)|\,|t|^{-1} dt = O(n^{-1})$$

with a as in lemma 4.2.5. Hence we can conclude that (4.2.1) holds for $0 < \varepsilon < \min(\frac{\alpha_1}{2}, \frac{\alpha_2}{2}, \gamma - 1, \frac{1}{6})$ under the assumptions 4.1.1, 4.1.2, the finiteness of $\beta_4 = EX_1^4$, and the positivity of $\sigma^2(J_1, F)$. According to lemma 2.2.4, $\sigma^2(J_1, F) > 0$ follows from the assumptions 4.1.2 and 4.1.3, so that (4.2.1) holds under the conditions of theorem 4.1.1.

To prove (4.2.2) we remark first that application of lemma 2.1.2 with $h = F^{-1}$ and $r > -1 + \frac{5}{2\varepsilon}$ implies that

$$(4.2.44) \qquad \int_{n^{\varepsilon} < |t| < n^{\frac{3}{2}}} |\rho_n^*(t)|\,|t|^{-1} dt = O(n^{-1})$$

as $n \to \infty$, provided positive numbers e and E exist such that $e \le n^{\frac{1}{2}}\sigma(T_n) \le E$. To see that this is true we first apply the lemma's 2.2.4 and 4.2.1 to find that $n^{\frac{1}{2}}\sigma(S_n)$ is bounded away from zero and infinity and then apply (4.2.20). Hence (4.2.2) is shown to hold if we assume that, for some $\delta > 0$, $\beta_{2+\delta} < \infty$ and that the assumptions 4.1.1, 4.1.2 and 4.1.3 are all satisfied.

To prove (4.2.3) we simply use (4.2.42) and lemma 2.2.4 to find that, under the assumptions of theorem 4.1.1, κ_3 and κ_4 are finite. This completes the proof. □

4.3. PROOF OF THEOREM 4.1.2

In this section we prove theorem 4.1.2. The idea of the proof is the same as that of theorem 3.1.2, but in this case a more precise evaluation of the effect of changing the standardization is needed. To start with we remark that for each $n \ge 1$ and real x

$$(4.3.1) \qquad G_n(x) = F_n^*(x\sigma n^{-\frac{1}{2}}\sigma^{-1}(T_n) + (\mu - E(T_n))\sigma^{-1}(T_n)).$$

Using this identity and applying theorem 4.1.1 we find that

$$(4.3.2)\qquad \sup_x |G_n(x) - \tilde{F}_n(x\sigma n^{-\frac12}\sigma^{-1}(T_n) + (\mu - ET_n)\sigma^{-1}(T_n))| = o(n^{-1})$$

as $n \to \infty$.

To proceed we shall need expansions for $\sigma n^{-\frac12}\sigma^{-1}(T_n)$ and $(\mu - ET_n)\sigma^{-1}(T_n)$.

<u>LEMMA 4.3.1</u>. *Let, for some* $\delta > 0$, $E|X_1|^{2+\delta} < \infty$ *and suppose that the assumptions* 4.1.1 *and* 4.1.2 *are satisfied. Then* $\sigma^2(J_1,F) > 0$ *implies that*

$$(4.3.3)\qquad |(\mu - ET_n)\sigma^{-1}(T_n) - an^{-\frac12}| = o(n^{-1})$$

and

$$(4.3.4)\qquad |\sigma n^{-\frac12}\sigma^{-1}(T_n) - 1 + bn^{-1}| = o(n^{-1}), \quad \text{as } n \to \infty$$

with $a = a(J_1,J_2,F)$ *and* $b = b(J_1,J_2,F)$ *as in* (4.1.15) *and* (4.1.16).

<u>PROOF</u>. We first prove (4.3.4). Application of lemma 4.2.1, (4.2.20), and the Cauchy-Schwarz inequality yields

$$(4.3.5)\qquad \frac{\sigma^2}{n\sigma^2(T_n)} = \frac{\sigma^2}{n\sigma^2(S_n)}\left(1 + O\left(n^{-1-\min(\frac{\alpha_1}{2}, \frac{\alpha_2}{2}, \gamma - 1)}\right)\right), \qquad \text{as } n \to \infty.$$

Lemma 4.2.1 implies that

$$(4.3.6)\qquad \frac{\sigma^2}{n\sigma^2(S_n)} = 1 - 2\frac{b}{n} + O(n^{-\frac32}), \qquad \text{as } n \to \infty$$

Combining (4.3.5) and (4.3.6) we find

$$(4.3.7)\qquad \frac{\sigma^2}{n\sigma^2(T_n)} = 1 - 2\frac{b}{n} + O\left(n^{-1-\min(\frac12, \frac{\alpha_1}{2}, \frac{\alpha_2}{2}, \gamma - 1)}\right)$$

as $n \to \infty$. Inequality (4.3.4) is an immediate consequence of (4.3.7). To prove (4.3.3) we first use (4.2.5), (4.2.6), the assumptions 4.1.1 and 4.1.2 and Taylor's theorem to find that

$$(4.3.8)\qquad E|T_n - S_n| = O(E\int_0^1 |\Gamma_n(s) - s|^{3+\alpha_1} d\, F^{-1}(s) +$$

$$+ n^{-1}E\int_0^1 |\Gamma_n(s) - s|^{1+\alpha_2} d\,F^{-1}(s) + n^{-\gamma}E|X_1|), \qquad \text{as } n \to \infty.$$

Application of lemma 2.2.2 with $\ell = 1$ and $p = 3 + \alpha_1$ implies that the first term on the right of (4.3.8) is $O(n^{-3/2-\alpha_1/2})$ as $n \to \infty$. To treat the second term on the right of (4.3.8) we first note that this term is at most $n^{-1}(E(\int_0^1 |\Gamma_n(s) - s|^{1+\alpha_2} d\,F^{-1}(s))^2)^{\frac{1}{2}}$ and then we apply lemma 2.2.2 once more (with $\ell = 2$ and $p = 1 + \alpha_2$) to find that this term is $O(n^{-3/2-\alpha_2/2})$ as $n \to \infty$. Combining these results we obtain

$$(4.3.9) \qquad ET_n = ES_n + O(E|T_n - S_n|) = ES_n + O(n^{-\frac{3}{2}-\min(\frac{\alpha_1}{2}, \frac{\alpha_2}{2})}) + O(n^{-\gamma})$$

Using the definition of S_n (see (4.2.6)) and noting that

$$E(\Gamma_n(s) - s)^3 = n^{-2}s(1-s)(1-2s), \qquad 0 < s < 1$$

we can easily check that

$$ES_n = \mu - a\sigma n^{-1} + O(n^{-2}), \qquad \text{as } n \to \infty$$

so that (4.3.9) implies that

$$(4.3.10) \qquad \mu - ET_n = a\sigma n^{-1} + O(n^{-\frac{3}{2}-\min(\frac{1}{2}, \frac{\alpha_1}{2}, \frac{\alpha_2}{2})} + n^{-\gamma})$$

as $n \to \infty$. Because (4.3.7) directly implies that

$$\sigma^{-1}(T_n) = n^{\frac{1}{2}}\sigma^{-1} + O(n^{-\frac{1}{2}}), \qquad \text{as } n \to \infty$$

we have proved (4.3.4). □

To complete the proof of theorem 4.1.2 we use (4.1.5), (4.1.17), (4.3.3), (4.3.4) and apply a Taylor expansion argument to find that

$$\tilde{F}_n(xn^{-\frac{1}{2}}\sigma^{-1}(T_n)\sigma + (\mu - ET_n)\sigma^{-1}(T_n)) = \tilde{G}_n(x) + o(n^{-1}), \quad \text{as } n \to \infty$$

uniformly in x. Combining this with (4.3.2) completes the proof. □

4.4. EXTENSIONS

In the theorems 4.1.1 and 4.1.2 we have established asymptotic expansions for the df's of linear combinations of order statistics with remainder $o(n^{-1})$. However, no new difficulties will be encountered when showing that under somewhat stronger conditions the remainder is $\mathcal{O}(n^{-3/2})$, which is of course the natural order of the remainder term. To do this for theorem 4.1.1 we need a strengthened version of assumption 4.1.2.

ASSUMPTION 4.1.2.*

(i) The function J_1 is three-times differentiable on $(0,1)$ with bounded first, second and third derivative $J_1^{(1)}$, $J_1^{(2)}$ and $J_1^{(3)}$ on $(0,1)$. The function J_2 is differentiable on $(0,1)$ with bounded derivative $J_2^{(1)}$ on $(0,1)$.

(ii) The functions $J_1^{(3)}$ and $J_2^{(1)}$ satisfy Lipschitz conditions of order $\alpha_1 > 0$ and $\alpha_2 > 0$ respectively on $(0,1)$.

We shall state the results without further proof.

THEOREM 4.4.1. *Let* $\mathcal{E}|X_1|^5 < \infty$ *and suppose that the assumptions* 4.1.1, 4.1.2* *and* 4.1.3 *are satisfied. Then,*

$$\sup_x |F_n^*(x) - \tilde{F}_n(x)| = \mathcal{O}(n^{-\frac{3}{2}}), \qquad \text{as } n \to \infty. \tag{4.4.1}$$

with F_n^* *and* $\tilde{F}_n$ *as in* (4.1.9) *and* (4.1.5).

To obtain the corresponding result for theorem 4.1.2 we need also a strengthened version of assumption 4.1.1. Let J_3 be a bounded real-valued measurable function on $(0,1)$.

ASSUMPTION 4.1.1.* There exist a number $\gamma > 2$ such that

$$\max_{1\le i\le n} |c_{in} - n\int_{\frac{i-1}{n}}^{\frac{i}{n}} J_1(s)ds - \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_2(s)ds - n^{-1}\int_{\frac{i-1}{n}}^{\frac{i}{n}} J_3(s)ds| =$$

$$= \mathcal{O}(n^{-\gamma}), \qquad \text{as } n \to \infty.$$

<u>THEOREM 4.4.2</u>. *Let* $E|X_1|^5 < \infty$ *and suppose that the assumptions* 4.1.1*, 4.1.2* *and* 4.1.3 *are satisfied. Then*

(4.4.2) $$\sup_x |G_n(x) - \tilde{G}_n(x)| = \mathcal{O}(n^{-\frac{3}{2}}), \qquad \text{as } n \to \infty$$

with G_n *and* $\tilde{G}_n$ *as* (4.1.12) *and* (4.1.17).

CHAPTER 5

EDGEWORTH EXPANSIONS FOR TRIMMED LINEAR COMBINATIONS OF ORDER STATISTICS

5.1. INTRODUCTION AND MAIN RESULTS

In this chapter the results of the preceding chapter will be supplemented by considering the case of trimmed linear combinations of order statistics. We establish Edgeworth expansions with remainder $o(n^{-1})$ for these statistics in the case of a smooth underlying distribution. Again we consider suitably standardized statistics of the form (cf. (4.1.1))

$$T_n = n^{-1} \sum_{i=1}^{n} c_{in} X_{i:n} . \tag{5.1.1}$$

To prove the first main result of this chapter we shall suppose that numbers $0 < \alpha < \beta < 1$ exist for which the following assumptions are satisfied.

ASSUMPTION 5.1.1. There exist positive numbers c and C and numbers t_1 and t_2 satisfying $0 < \alpha \le t_1 < t_2 \le \beta < 1$ such that

(i) $c_{in} = 0$ for all i with $\frac{i}{n} < \alpha$ or $\frac{i}{n} > \beta$

(ii) $\sum_{i=1}^{n} |c_{in}| = O(n)$ as $n \to \infty$

(iii) $c \le c_{in} \le C$ for all i with $t_1 < \frac{i}{n} < t_2$.

ASSUMPTION 5.1.2. There exist numbers a and b satisfying $0 \le F(a) < \alpha < \beta < F(b) \le 1$ such that

(i) F is three times differentiable on $[a,b]$ with positive density f and bounded second and third derivative f' and f'' on $[a,b]$.

(ii) the function f'' satisfies a Lipschitz condition of order $\alpha_1 > 0$ on $[a,b]$.

Before we state the first main result of this chapter we need some more notation. Introduce a function H by

$$H(x) = F^{-1}(1 - e^{-x}), \qquad 0 \le x < \infty. \tag{5.1.2}$$

Furthermore define, for $j = 1,2,\dots,n$, $n = 1,2,\dots$ quantities $\alpha_{j,n}$, $\beta_{j,n}$, $\gamma_{j,n}$ by

$$\alpha_{j,n} = (n - j + 1)^{-1} \sum_{i=j}^{n} c_{in} H'(\nu_{in}) \tag{5.1.3}$$

$$\beta_{j,n} = (n - j + 1)^{-1} \sum_{i=j}^{n} c_{in} H''(\nu_{in}) \tag{5.1.4}$$

$$\gamma_{j,n} = (n - j + 1)^{-1} \sum_{i=j}^{n} c_{in} H'''(\nu_{in}) \tag{5.1.5}$$

where (see (2.3.5))

$$\nu_{in} = \sum_{j=1}^{i} (n - j + 1)^{-1}, \qquad i = 1,2,\dots,n, \quad n \ge 1, \tag{5.1.6}$$

and H', H" and H"' are the first, second and third derivative of H on the interval where these derivatives exist. Note that, under the assumptions 5.1.1 (i) and 5.1.2 (i), the quantities $\alpha_{j,n}$, $\beta_{j,n}$, $\gamma_{j,n}$ are properly defined for all $n \ge n_0$ (n_0 being a sufficiently large positive integer).

Finally define, for each $n \ge n_0$ and real x, the function

$$\bar{F}_n(x) = \Phi(x) - \phi(x)\{\frac{\bar{\kappa}_{3n}}{6}(x^2-1) + \frac{\bar{\kappa}_{4n}}{24}(x^3 - 3x) + \frac{\bar{\kappa}_{3n}^2}{72}(x^5 - 10x^3 + 15x)\} \tag{5.1.7}$$

The quantities $\bar{\kappa}_{3n}$ and $\bar{\kappa}_{4n}$ are given by

$$\bar{\kappa}_{3n} = (\sum_{j=1}^{n} \alpha_{j,n}^2)^{-\frac{3}{2}}[2 \sum_{j=1}^{n} \alpha_{j,n}^3 + 3 \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\alpha_{i,n}\alpha_{j,n}\beta_{i\vee j,n}}{(n-(i\wedge j)+1)}] \tag{5.1.8}$$

and

$$\bar{\kappa}_{4n} = (\sum_{j=1}^{n} \alpha_{j,n}^2)^{-2}[6 \sum_{j=1}^{n} \alpha_{j,n}^4 + 24 \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\alpha_{i,n}^2\alpha_{j,n}\beta_{i\vee j,n}}{(n-(i\wedge j)+1)} + \tag{5.1.9}$$

$$+ 4 \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\alpha_{i,n}\alpha_{j,n}\alpha_{k,n}\beta_{i\vee j\vee k,n}}{(n-((i\vee j)\wedge(i\vee k)\wedge(j\vee k))+1)(n-(i\wedge j\wedge k)+1)} +$$

$$+ 12 \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\alpha_{i,n}\alpha_{j,n}\beta_{i\vee k,n}\beta_{j\vee k,n}}{(n-(i\wedge k)+1)(n-(j\wedge k)+1)}]$$

Here and elsewhere $p \vee q$ $(p \wedge q)$ denotes the maximum (minimum) of two integers p and q; note that $(i \vee j) \wedge (i \vee k) \wedge (j \vee k)$ is the middle one of i, j and k.

In the first theorem of this chapter we establish an asymptotic expansion with remainder $O(n^{-1})$ for (cf. (4.1.9))

$$(5.1.10) \qquad F_n^*(x) = P(\{T_n^* \leq x\}), \qquad -\infty < x < \infty$$

where (cf. (4.1.10))

$$(5.1.11) \qquad T_n^* = (T_n - E(T_n))/\sigma(T_n)$$

for the case of a smooth underlying df F.

<u>THEOREM 5.1.1</u>. *Let, for some* $\delta > 0$, $E|X_1|^{\delta} < \infty$ *and suppose that there exist numbers* $0 < \alpha < \beta < 1$ *for which both assumption* 5.1.1 *and assumption* 5.1.2 *are satisfied. Then,*

$$(5.1.12) \qquad \sup_x |F_n^*(x) - \bar{F}_n(x)| = O(n^{-1}) \qquad \text{as } n \to \infty.$$

It is useful to comment briefly on this result. In the first place we note that assumption 5.1.1(i) requires that there are no weights in the tails. The basic function of this requirement is to control the order of the remainder terms in our expansions. Technically speaking this is reflected in the proof at those points where lemma 2.3.2 (cf. also the remark following this lemma) is used to show that certain moments are of a required order. The parts (ii) and (iii) of assumption 5.1.1 are rather harmless, because they are satisfied for almost every conceivable linear combination of order statistics which may arise in practice.

In the second place we may mention that assumption 5.1.2 puts a rather severe smoothness condition upon the underlying df F. This, in contrast with the results of chapter 4 where a rather stringent smoothness condition is required for the weights. Finally, we assume the finiteness of a δ-th absolute moment of the underlying df F to ensure that the expectation and variance of a trimmed linear combination of order statistics is finite for all sufficiently large n (cf. lemma 2.3.1). We need this because of the exact standardization we have employed in theorem 5.1.1.

In the third place we remark that trimmed and Winsorized means (see the examples 1.2.2 and 1.2.5) are included as important special cases in theorem 5.1.1. BJERVE (1974) has derived an Edgeworth expansion for trimmed means for the case of a symmetric underlying df F. Because he exploits the very special structure of trimmed means his proof needs weaker smoothness conditions for the underlying df F than ours. Theorem 5.1.1 was proved in HELMERS (1979). The present chapter extends the latter paper.

As the second main result of this chapter we shall give a modification of theorem 5.1.1 which lends itself better to applications. To obtain such a result we replace assumption 5.1.1 by one which requires rather regular weights. Let J be a bounded real-valued measurable function on (0,1). We shall restrict attention to weights of the form $c_{in} = J[i/(n+1)]$, so that

$$(5.1.13) \qquad T_n = n^{-1} \sum_{i=1}^{n} J\left(\frac{i}{n+1}\right) X_{i:n}.$$

We shall suppose that numbers $0 < \alpha < \beta < 1$ exist for which both the assumptions 5.1.2 and 5.1.3 are satisfied.

<u>ASSUMPTION 5.1.3</u>. There exist numbers t_1 and t_2 satisfying $0 < \alpha \le t_1 < t_2 \le \beta < 1$ such that

(i) $\qquad J(s) = 0 \qquad$ for $0 < s < \alpha$ and $\beta < s < 1$

(ii) the function J is differentiable on (α,β) with bounded derivative $J^{(1)}$ on (α,β); the function $J^{(1)}$ satisfies a Lipschitz condition of order $\alpha_2 > \frac{1}{2}$ on (α,β).

(iii) $\qquad J(s) > 0 \qquad$ for $t_1 < s < t_2$.

Introduce the quantity $\mu = \mu(J,F)$ (cf. (4.1.13))

$$(5.1.14) \qquad \mu(J,F) = \int_0^1 J(s)F^{-1}(s)\,ds$$

and define, for each $n \ge 1$ and real x, the df G_n (cf. (4.1.12))

$$(5.1.15) \qquad G_n(x) = P(\{n^{\frac{1}{2}}(T_n - \mu)/\sigma \le x\})$$

with $\sigma^2 = \sigma^2(J,F)$ as in (2.1.12) (cf. (4.1.8)). Introduce functions $\bar{h}_1$, $\bar{h}_2$, $\bar{h}_3$ and $\bar{h}_4$ by

$$(5.1.16) \qquad \bar{h}_1(u) = \int_0^1 J(s)(F^{-1}(s))^{(1)}(\chi_{(0,s]}(u) - s)ds$$

$$(5.1.17) \qquad \bar{h}_2(u,v) = \int_0^1 J(s)(F^{-1}(s))^{(2)}(\chi_{(0,s]}(u) - s)(\chi_{(0,s]}(v) - s)ds$$

$$(5.1.18) \qquad \bar{h}_3(u,v,w) = \int_0^1 J(s)(F^{-1}(s))^{(3)}(\chi_{(0,s]}(u) - s)(\chi_{(0,s]}(v) - s)\cdot(\chi_{(0,s]}(w) - s)ds$$

$$(5.1.19) \qquad \bar{h}_4(u) = \int_0^1 (\tfrac{1}{2} - s)(J(s)(F^{-1}(s))^{(1)})^{(1)}(\chi_{(0,s]}(u) - s)ds$$

for $0 < u,v,w < 1$, where $(F^{-1})^{(k)}$ denotes the k-th derivative of F^{-1}. Furthermore quantities $\bar{\kappa}_3 = \bar{\kappa}_3(J,F)$, $\bar{\kappa}_4 = \bar{\kappa}_4(J,F)$, $\bar{a} = \bar{a}(J,F)$ and $\bar{b} = \bar{b}(J,F)$ are given by

$$(5.1.20) \qquad \bar{\kappa}_3 = \bar{\kappa}_3(J,F) = \frac{1}{\sigma^3(J,F)}[2\int_0^1 \bar{h}_1^3(u)du + 3\int_0^1\int_0^1 \bar{h}_1(u)\bar{h}_1(v)\bar{h}_2(u,v)dudv]$$

$$(5.1.21) \qquad \bar{\kappa}_4 = \bar{\kappa}_4(J,F) = \frac{1}{\sigma^4(J,F)}[6\int_0^1 \bar{h}_1^4(u)du - 12\sigma^4(J,F) + 24\int_0^1\int_0^1 \bar{h}_1^2(u)\bar{h}_1(v)\bar{h}_2(u,v)dudv + \int_0^1\int_0^1\int_0^1 (4\bar{h}_1(u)\bar{h}_1(v)\bar{h}_1(w)\bar{h}_3(u,v,w) + 12\bar{h}_1(u)\bar{h}_1(v)\bar{h}_2(u,w)\bar{h}_2(v,w))dudvdw]$$

$$(5.1.22) \qquad \bar{a} = \bar{a}(J,F) = \frac{1}{\sigma(J,F)}[2^{-1}\int_0^1 s(1-s)J^{(1)}(s)(F^{-1}(s))^{(1)}ds -$$

$$- \int_0^1 (\tfrac{1}{2} - s) J^{(1)}(s) F^{-1}(s) ds]$$

and

$$(5.1.23) \qquad \bar{b} = \bar{b}(J,F) = \frac{1}{2\sigma^2(J,F)} [-3\sigma^2 + 2^{-1}h_1^{-2}(1) + 2^{-1}\bar{h}_1^{2}(0) + \int_0^1 (2\bar{h}_1(u)\bar{h}_2(u,u) + 2\bar{h}_1(u)\bar{h}_4(u))du + \int_0^1\int_0^1 (2^{-1}\bar{h}_2^2(u,v) + \bar{h}_1(u)\bar{h}_3(u,v,v))dudv]$$

where

$$(5.1.24) \qquad \sigma^2(J,F) = \int_0^1 \bar{h}_1^2(u)du.$$

Finally define, for each $n \geq 1$ and real x, the function $\bar{G}_n$ by

$$(5.1.25) \qquad \bar{G}_n(x) = \Phi(x) - \phi(x)\{\frac{\bar{\kappa}_3}{6n^{\frac{1}{2}}}(x^2-1) + \frac{\bar{\kappa}_4}{24n}(x^3-3x) + \frac{\bar{\kappa}_3^2}{72n}(x^5 - 10x^3 + 15x) + - \frac{\bar{a}}{n^{\frac{1}{2}}} + (\frac{\bar{a}\bar{\kappa}_3 + \bar{a}^2 + 2\bar{b}}{2n})x - \frac{\bar{a}\bar{\kappa}_3}{6n}x^3\}$$

<u>THEOREM 5.1.2</u>. *Suppose that there exist numbers* $0 < \alpha < \beta < 1$ *for which both assumption* 5.1.2 *and assumption* 5.1.3 *are satisfied. Then,*

$$(5.1.26) \qquad \sup_x |G_n(x) - \bar{G}_n(x)| = o(n^{-1}), \qquad \textit{as } n \to \infty.$$

Note that theorem 5.1.2 supplements theorem 4.1.2. The present theorem covers a class of trimmed linear combinations of order statistics with smooth weights, whereas theorem 4.1.2 does not include these statistics. To conclude this section we remark that, in case both the assumptions of theorems 4.1.2 and the assumptions of theorem 5.1.2 are satisfied, the expansions G_n and $\bar{G}_n$ given in these theorems are identical. This affords a welcome check on the laborious calculations leading to $\bar{\kappa}_3$ and $\bar{\kappa}_4$. Straightforward but lengthy computations show that indeed $\bar{\kappa}_3 = \kappa_3$ and $\bar{\kappa}_4 = \kappa_4$ in this case.

Theorem 5.1.1 is proved in section 5.2, theorem 5.1.2 in section 5.3. Some extensions are indicated in section 5.4.

5.2. PROOF OF THEOREM 5.1.1.

The proof of theorem 5.1.1 will parallel that of theorem 4.1.1. Again our proof will depend on ch.f. arguments. Denote by $\rho_n^*(t)$ the ch.f. of T_n^* and by $\bar{\rho}_n(t)$ the Fourier-Stieltjes transform

$$(5.2.1) \qquad \bar{\rho}_n(t) = \int_{-\infty}^{\infty} e^{itx} d\, \bar{F}_n(x)$$

of $\bar{F}_n$ (cf. (5.1.7)). As in section 4.2 we shall show that for some sufficiently small $\varepsilon > 0$

$$(5.2.2) \qquad \int_{|t|\le n^{\varepsilon}} |\rho_n^*(t) - \bar{\rho}_n(t)|\,|t|^{-1} dt = o(n^{-1})$$

$$(5.2.3) \qquad \int_{n^{\varepsilon}<|t|<n^{\frac{3}{2}}} |\rho_n^*(t)|\,|t|^{-1} dt = o(n^{-1})$$

$$(5.2.4) \qquad \int_{|t|>\log(n+1)} |\bar{\rho}_n(t)|\,|t|^{-1} dt = o(n^{-1})$$

as $n \to \infty$. An application of Esseen's smoothness lemma (lemma 1.2) will then complete our proof. We first prove (5.2.2). In section 4.2 the proof of the corresponding relation (4.2.1) depends very much on the fact that T_n can be written in terms of the empirical df in such a way that a stochastic expansion of the rv T_n itself can be obtained. This expansion of the rv T_n is used to establish an expansion for $\rho_n^*(t)$ for $|t| \le n^{\varepsilon}$ for sufficiently small $\varepsilon > 0$ from which (4.2.1) then follows. To establish (5.2.2) we follow another line of attack, though the structure of the proof remains essentially the same. Rather than representing a linear combination of order statistics in terms of the empirical df we shall exploit a different technique based on representing the order statistics in terms of independent exponentially distributed rv's. The same idea was used by CHERNOFF et.al. (1967) and BJERVE (1977) in proving asymptotic normality and Berry-Esseen bounds for linear combinations of order statistics.

To start with the proof of (5.2.2) we note that the joint distribution of $X_{i:n}$, $i = 1,2,\dots,n$, $n = 1,2,\dots$ is the same as that of $H(Z_{i:n})$, $i = 1,2,\dots,n$, $n = 1,2,\dots$ with H as in (5.1.2). Recall (cf. (2.3.3) and the remark following it) that the $Z_{i:n}$'s are the order statistics of a sample of size n from the exponential df $E(z) = 1 - e^{-z}$ for $0 \le z < \infty$. Hence we may identify T_n with $n^{-1}\Sigma_{i=1}^{n} c_{in}H(Z_{i:n})$.

Introduce, for each $n \ge 1$, the rv $\bar{S}_n$ by

(5.2.5) $$\bar{S}_n = n^{-1} \sum_{i=1}^{n} c_{in}\{H(\nu_{in}) + (Z_{i:n} - \nu_{in})H'(\nu_{in}) + \frac{(Z_{i:n}-\nu_{in})^2}{2} H''(\nu_{in}) + \frac{(Z_{i:n}-\nu_{in})^3}{6} H'''(\nu_{in})\}$$

with ν_{in} $(1 \le i \le n)$ as in (5.1.6). Here H', H" and H"' denote the first, second and third derivative of H on the interval where these derivatives are defined. Note that the assumptions of theorem 5.1.1 guarantee that $\bar{S}_n$ is well-defined for all sufficiently large n. Now the $Z_{i:n}$'s are replaced by $\Sigma_{j=1}^{i} Z_j/(n-j+1)$ (cf. (2.3.4)). It follows that $\bar{S}_n$ can be written as

(5.2.6) $$\bar{S}_n = n^{-1} \sum_{i=1}^{n} c_{in}H(\nu_{in}) + \bar{I}_{1n} + \bar{I}_{2n} + \bar{I}_{3n}$$

where

(5.2.7) $$\bar{I}_{1n} = n^{-1} \sum_{j=1}^{n} \alpha_{j,n}(Z_j - 1)$$

(5.2.8) $$\bar{I}_{2n} = 2^{-1}n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\beta_{i\vee j,n}}{(n-(i\wedge j)+1)} (Z_i - 1)(Z_j - 1)$$

(5.2.9) $$\bar{I}_{3n} = 6^{-1}n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\gamma_{i\vee j\vee k,n}}{(n-((i\vee j)\wedge(i\vee k)\wedge(j\vee k))+1)(n-(i\wedge j\wedge k)+1)} \cdot (Z_i - 1)(Z_j - 1)(Z_k - 1)$$

The quantities $\alpha_{j,n}$, $\beta_{j,n}$ and $\gamma_{j,n}$ are given in (5.1.3) - (5.1.5). Finally introduce rv's J_{mn}, for $m = 1,2,3$ and $n \ge n_0$ by

(5.2.10) $$J_{mn} = (\bar{I}_{mn} - E(\bar{I}_{mn}))/\sigma(\bar{S}_n)$$

and the rv $\bar{S}_n^*$ by

$$(5.2.11) \qquad \bar{S}_n^* = (\bar{S}_n - E(\bar{S}_n))/\sigma(\bar{S}_n) = \bar{J}_{1n} + \bar{J}_{2n} + \bar{J}_{3n}.$$

The proof of (5.2.2) will be split up in a number of lemma's. In the first lemma we obtain an asymptotic expansion for the variance of $\bar{S}_n$.

LEMMA 5.2.1. *Suppose there exist numbers* $0 < \alpha < \beta < 1$ *for which both the assumptions* 5.1.1(i) *and* (ii) *and* 5.1.2(i) *are satisfied. Then,*

$$(5.2.12) \qquad |\sigma^2(\bar{S}_n) - n^{-2} \sum_{j=1}^{n} \alpha_{j,n}^2 - n^{-2}\{2 \sum_{i=1}^{n} \frac{\alpha_{i,n}\beta_{i,n}}{(n-i+1)} +$$

$$+ 2^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\beta_{i\vee j,n}^2}{(n-(i\wedge j)+1)^2} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\alpha_{i\vee j,n}\gamma_{i\vee j,n}}{(n-(i\wedge j)+1)^2}$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\alpha_{i\wedge j,n}\gamma_{i\vee j,n}}{(n-(i\vee j)+1)(n-(i\wedge j)+1)}\}| = O(n^{-\frac{5}{2}}), \qquad \text{as } n \to \infty.$$

PROOF. In view of (5.2.6) we have that

$$\sigma^2(\bar{S}_n) = \sigma^2(\sum_{m=1}^{3} \bar{I}_{mn}).$$

It follows from (5.2.7) that

$$\sigma^2(\bar{I}_{1n}) = n^{-2} \sum_{j=1}^{n} \alpha_{j,n}^2.$$

Also note that it is immediate from (5.2.7) and (5.2.8) that

$$2 \operatorname{cov}(\bar{I}_{1n},\bar{I}_{2n}) = 2E\bar{I}_{1n}\bar{I}_{2n} =$$

$$= n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\alpha_{i,n}\, \beta_{j\vee k,n}}{(n-(j\wedge k)+1)} E(Z_i - 1)(Z_j - 1)(Z_k - 1) =$$

$$= 2n^{-2} \sum_{i=1}^{n} \frac{\alpha_{i,n}\beta_{i,n}}{(n-i+1)} .$$

Next we consider $\sigma^2(\bar{I}_{2n})$. Note first that

$$E\bar{I}_{2n}^2 = 4^{-1}n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{m=1}^{n} \frac{\beta_{i\vee j,n}\beta_{k\vee m,n}}{(n-(i\wedge j)+1)(n-(k\wedge m)+1)} \cdot$$

$$\cdot\ E(Z_i-1)(Z_j-1)(Z_k-1)(Z_m-1) =$$

$$= 4^{-1}n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\beta_{i,n}\beta_{j,n}}{(n-i+1)(n-j+1)} + 2^{-1}n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\beta_{i\vee j,n}^2}{(n-i\wedge j+1)^2}$$

$$+ O(n^{-3}), \qquad \text{as } n \to \infty.$$

Because we also know that

$$(E\bar{I}_{2n})^2 = 4^{-1}n^{-2}\Big(\sum_{i=1}^{n} \frac{\beta_{i,n}}{(n-i+1)}\Big)^2$$

we have proved that

$$\sigma^2(\bar{I}_{2n}) = 2^{-1}n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\beta_{i\vee j,n}^2}{(n-(i\wedge j)+1)^2} + O(n^{-3}), \qquad \text{as } n \to \infty.$$

Similarly we can show that

$$2\,\mathrm{cov}(\bar{I}_{1n},\bar{I}_{3n}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\alpha_{i\vee j,n}\gamma_{i\vee j,n}}{(n-(i\wedge j)+1)^2} +$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\alpha_{i\wedge j,n}\gamma_{i\vee j,n}}{(n-(i\vee j)+1)(n-(i\wedge j)+1)} + O(n^{-3}), \qquad \text{as } n \to \infty.$$

Finally we remark that it is easily inferred from lemma 2.3.2 and the Cauchy-Schwarz inequality that

$$|\sigma^2(\bar{I}_{3n}) + \mathrm{cov}(\bar{I}_{2n},\bar{I}_{3n})| = O(n^{-\frac{5}{2}}), \qquad \text{as } n \to \infty,$$

under the assumptions of the lemma. Combining all these results we see that (5.2.12) holds. □

LEMMA 5.2.2. *Suppose there exist numbers* $0 < \alpha < \beta < 1$ *for which both the assumptions* 5.1.1 *and* 5.1.2(i) *are satisfied.*

(i) *There exist a number* $\theta > 0$ *such that*

$$(5.2.13) \qquad n^{-1} \sum_{j=1}^{n} \alpha_{j,n}^2 > \theta$$

for all sufficiently large n.

(ii) *For any fixed real number* m

$$(5.2.14) \qquad |\sigma^{-m}(\bar{S}_n) - n^m(\sum_{j=1}^{n} \alpha_{j,n}^2)^{-\frac{m}{2}}| = O(n^{\frac{m}{2}-1}), \qquad \textit{as } n \to \infty.$$

PROOF. We first prove (5.2.13). The idea of the proof is the same as that of lemma 2.2.4. It was already noted in section 5.1 that the quantities $\alpha_{j,n}$, $j = 1,2,\ldots,n$, $n \geq 1$ are properly defined for all sufficiently large n. To proceed we remark first that

$$n^{-1} \sum_{j=1}^{n} \alpha_{j,n}^2 \geq n^{-1} \sum_{j=[nt_1]+1}^{[nt_2]} \alpha_{j,n}^2 \geq$$

$$\geq n^{-1}(n-[nt_1])^{-2} \sum_{j=[nt_1]+1}^{[nt_2]} (\sum_{i=j}^{n} c_{in} H'(\nu_{in}))^2 .$$

Using the assumptions of the lemma we see that for $[nt_1]+1 \leq j < k \leq [nt_2]$ and sufficiently large n,

$$\sum_{i=j}^{n} c_{in}H'(\nu_{in}) - \sum_{i=k}^{n} c_{in}H'(\nu_{in}) = \sum_{i=j}^{k-1} c_{in}H'(\nu_{in})$$

$$\geq (k-j)c\, M^{-1}(1-t_2)$$

where $M = \max_{a \leq x \leq b} f(x)$. Hence

$$\sum_{j=[nt_1]+1}^{[nt_2]} (\sum_{i=j}^{n} c_{in}H'(\nu_{in}))^2$$

is minimized for

$$\sum_{i=j}^{n} c_{in}H'(\nu_{in}) = (j - \frac{([nt_1]+[nt_2]+1)}{2})c\, M^{-1}(1-t_2).$$

A simple summation completes the proof of (5.2.13). Part (ii) of the lemma is immediate from lemma 5.2.1 and (5.2.13). □

The next lemma will enable us to show that $T_n^* - \bar{S}_n^*$ is of negligible order for our purposes. Let $\bar{\tau}_n^*$ denote the ch.f. of $\bar{S}_n^*$.

LEMMA 5.2.3. *Let, for some* $\delta > 0$, $E|X_1|^\delta < \infty$ *and suppose that there exist numbers* $0 < \alpha < \beta < 1$ *for which both the assumptions* 5.1.1 *and* 5.1.2 *are satisfied. Then we have for every* $\varepsilon > 0$

$$(5.2.15) \qquad \int_{|t|\le n^\varepsilon} |\rho_n^*(t) - \bar{\tau}_n^*(t)||t|^{-1}dt = O(n^{-1-\frac{\alpha_1}{2}+\varepsilon}), \qquad \text{as } n \to \infty.$$

PROOF. We start by noting that, in view of lemma 2.3.1, the moment assumption ensures that every moment of T_n is finite for sufficiently large values of n. An application of lemma X.V.4.1 of FELLER (1966) implies that

$$(5.2.16) \qquad |\rho_n^*(t) - \bar{\tau}_n^*(t)| \le |t|E|T_n^* - \bar{S}_n^*|$$

for all t and sufficiently large n. Replacing T_n by $n^{-1}\Sigma_{i=1}^n c_{in}H(Z_{i:n})$, using the formula for $\bar{S}_n$ (cf. (5.2.5)), Taylor's theorem and an exponential bound for exponential central order statistics we see directly that

$$(5.2.17) \qquad \sigma^2(T_n - \bar{S}_n) \le E(T_n - \bar{S}_n)^2 = O(n^{-2}E(\sum_{i=1}^n |c_{in}||Z_{i:n} - \nu_{in}|^{3+\alpha_1})^2 + O(e^{-\eta_1 n})$$

for some constant $\eta_1 > 0$. Application of lemma 2.3.2 yields now that

$$(5.2.18) \qquad \sigma^2(T_n - \bar{S}_n) = O(n^{-3-\alpha_1}), \qquad \text{as } n \to \infty$$

Combining (5.2.18) with the lemma's 2.1.1 and 5.2.1 we see that

$$(5.2.19) \qquad \sigma^2(T_n^* - \bar{S}_n^*) = O(n^{-2-\alpha_1}), \qquad \text{as } n \to \infty.$$

This together with (5.2.16) proves (5.2.15). □

Next define for real t and all sufficiently large n

$$(5.2.20) \qquad \bar{\tau}_{1n}(t) = Ee^{itJ_{1n}}(1 + it(\bar{J}_{2n} + \bar{J}_{3n}) + \frac{(it)^2}{2}\bar{J}_{2n}^2).$$

In the following lemma we shall approximate $\bar{\tau}_n^*$ by $\bar{\tau}_{1n}$ for all $|t| \le n^\varepsilon$.

LEMMA 5.2.4. *Suppose that there exist numbers* $0 < \alpha < \beta < 1$ *for which both the assumptions* 5.1.1 *and* 5.1.2(i) *are satisfied. Then we have for every* $\varepsilon > 0$.

$$(5.2.21) \qquad \int_{|t|\le n^{\varepsilon}} |\bar{\tau}_n^*(t) - \bar{\tau}_{1n}(t)|\,|t|^{-1}dt = O(n^{-\frac{3}{2}+3\varepsilon}), \qquad \text{as } n \to \infty.$$

PROOF. Application of lemma X.V.4.1 of FELLER (1966) yields that

$$|\bar{\tau}_n^*(t) - \bar{\tau}_{1n}(t)| = |E e^{it\bar{J}_{1n}}\{e^{it(\bar{J}_{2n}+\bar{J}_{3n})} - 1 -$$

$$- it(\bar{J}_{2n}+\bar{J}_{3n}) - \frac{(it)^2}{2}\bar{J}_{2n}^2\}| \le$$

$$\le t^2(E|\bar{J}_{2n}\bar{J}_{3n}| + E\bar{J}_{3n}^2) + |t|^3 E|\bar{J}_{2n}+\bar{J}_{3n}|^3,$$

for all t and sufficiently large n. It follows easily from the proof of lemma 5.2.1 and from lemma 5.2.2(ii) that the coefficient t^2 on the right in the above inequality is $O(n^{-3/2})$, as $n \to \infty$. An application of the c_r-inequality and of lemma 2.3.2 shows that also $E|\bar{J}_{2n}+\bar{J}_{3n}|^3 = O(n^{-3/2})$ as $n \to \infty$. Combining these results we easily check that (5.2.21) is proved. □

We continue with the analysis of $\bar{\tau}_{1n}(t)$. For convenience we write $\bar{\sigma}_n^2$ to indicate $n\sigma^2(\bar{S}_n)$ and we denote the ch.f. of $Z_1 - 1$ by η; i.e.

$$(5.2.22) \qquad \eta(t) = (e^{it}(1-it))^{-1}$$

To start with we remark that it follows from (5.2.20) that

$$(5.2.23) \qquad \bar{\tau}_{1n}(t) = \prod_{j=1}^{n} \eta\left(\frac{\alpha_{j,n}t}{n^{\frac{1}{2}}\bar{\sigma}_n}\right) +$$

$$+ \frac{it}{2n^{\frac{1}{2}}\bar{\sigma}_n}\sum_{k=1}^{n}\frac{\beta_{k,n}}{(n-k+1)}\prod_{\substack{j=1\\ j\ne k}}^{n}\eta\left(\frac{\alpha_{j,n}t}{n^{\frac{1}{2}}\bar{\sigma}_n}\right)E e^{\frac{it}{n^{\frac{1}{2}}\bar{\sigma}_n}\alpha_{k,n}(Z_k-1)}((Z_k-1)^2-1)$$

$$+ \frac{it}{2n^{\frac{1}{2}}\bar{\sigma}_n}\sum_{k=1}^{n}\sum_{\substack{\ell=1\\ \ell\ne k}}^{n}\frac{\beta_{k\vee\ell,n}}{(n-(k\wedge\ell)+1)}\prod_{\substack{j=1\\ j\ne k,\ell}}^{n}\eta\left(\frac{\alpha_{j,n}t}{n^{\frac{1}{2}}\bar{\sigma}_n}\right).$$

$$\cdot E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(\alpha_{k,n}(Z_k-1)+\alpha_{\ell,n}(Z_\ell-1))} (Z_k-1)(Z_\ell-1) \; +$$

$$+ \frac{it}{6n^{\frac{1}{2}}\sigma_n} \sum_{k=1}^{n} \frac{\gamma_{k,n}}{(n-k+1)^2} \prod_{\substack{j=1 \\ j\neq k}}^{n} \eta\left(\frac{\alpha_{j,n}t}{n^{\frac{1}{2}}\sigma_n}\right) E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}\alpha_{k,n}(Z_k-1)} ((Z_k-1)^3-2) \; +$$

$$+ \frac{it}{2n^{\frac{1}{2}}\sigma_n} \sum_{k=1}^{n} \sum_{\substack{\ell=1 \\ \ell\neq k}}^{n} \frac{\gamma_{k\vee\ell,n}}{(n-(k\wedge\ell)+1)(n-(k\vee\ell)+1)} \prod_{\substack{j=1 \\ j\neq k,\ell}}^{n} \eta\left(\frac{\alpha_{j,n}t}{n^{\frac{1}{2}}\sigma_n}\right) \cdot$$

$$\cdot E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(\alpha_{k,n}(Z_k-1)+\alpha_{\ell,n}(Z_\ell-1))} (Z_{k\wedge\ell}-1)(Z_{k\vee\ell}-1)^2 \; +$$

$$+ \frac{it}{2n^{\frac{1}{2}}\sigma_n} \sum_{k=1}^{n} \sum_{\substack{\ell=1 \\ \ell\neq k}}^{n} \frac{\gamma_{k\vee\ell,n}}{(n-(k\wedge\ell)+1)^2} \prod_{\substack{j=1 \\ j\neq k,\ell}}^{n} \left(\frac{\alpha_{j,n}t}{n^{\frac{1}{2}}\sigma_n}\right)$$

$$\cdot E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(\alpha_{k,n}(Z_k+1)+\alpha_{\ell,n}(Z_\ell-1))} (Z_{k\wedge\ell}-1)^2(Z_{k\vee\ell}-1) \; +$$

$$+ \frac{it}{6n^{\frac{1}{2}}\sigma_n} \sum_{k=1}^{n} \sum_{\substack{\ell=1 \\ \ell\neq k}}^{n} \sum_{\substack{m=1 \\ m\neq\ell,k}}^{n} \frac{\gamma_{k\vee\ell\vee m,n}}{(n-((k\vee\ell)\wedge(\ell\vee m)\wedge(k\vee m))+1)(n-(k\wedge\ell\wedge m)+1)}$$

$$\cdot \prod_{\substack{j=1 \\ j\neq k,\ell,m}}^{n} \eta\left(\frac{\alpha_{j,n}}{n^{\frac{1}{2}}\sigma_n}\right) \cdot E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(\alpha_{k,n}(Z_k-1)+\alpha_{\ell,n}(Z_\ell-1)+\alpha_{m,n}(Z_m-1))} \cdot$$

$$\cdot (Z_k-1)(Z_\ell-1)(Z_m-1) \; +$$

$$+ \frac{(it)^2}{8n\sigma_n^2} \sum_{k=1}^{n} \frac{\beta_{k,n}^2}{(n-k+1)^2} \prod_{\substack{j=1 \\ j\neq k}}^{n} \eta\left(\frac{\alpha_{j,n}t}{n^{\frac{1}{2}}\sigma_n}\right) \cdot E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}\alpha_{k,n}(Z_k-1)} ((Z_k-1)^2-1)^2$$

$$+\frac{(it)^2}{4n\bar{\sigma}_n^2}\sum_{k=1}^{n}\sum_{\substack{\ell=1\\ \ell\neq k}}^{n}\frac{\beta^2_{k\vee\ell,n}}{(n-(k\wedge\ell)+1)^2}\prod_{\substack{j=1\\ j\neq k,\ell}}^{n}\eta\left(\frac{\alpha_{jn}t}{n^{\frac{1}{2}}\bar{\sigma}_n}\right)Ee^{\frac{it}{n^{\frac{1}{2}}\bar{\sigma}_n}(\alpha_{k,n}(Z_k-1)+\alpha_{\ell,n}(Z_\ell-1))}$$

$$\cdot(Z_k-1)^2(Z_\ell-1)^2+$$

$$+\frac{(it)^2}{8n\bar{\sigma}_n^2}\sum_{k=1}^{n}\sum_{\substack{\ell=1\\ \ell\neq k}}^{n}\frac{\beta_{k,n}\beta_{\ell,n}}{(n-k+1)(n-\ell+1)}\prod_{\substack{j=1\\ j\neq k,\ell}}^{n}\eta\left(\frac{\alpha_{j,n}t}{n^{\frac{1}{2}}\bar{\sigma}_n}\right)\cdot$$

$$\cdot Ee^{\frac{it}{n^{\frac{1}{2}}\bar{\sigma}_n}(\alpha_{k,n}(Z_k-1)+\alpha_{\ell,n}(Z_\ell-1))}\cdot((Z_k-1)^2-1)((Z_\ell-1)^2-1)+$$

$$+\frac{(it)^2}{4n\bar{\sigma}_n^2}\sum_{k=1}^{n}\sum_{\substack{\ell=1\\ \ell\neq k}}^{n}\frac{\beta_{k,n}\beta_{k\vee\ell,n}}{(n-k+1)(n-(k\wedge\ell)+1)}\prod_{\substack{j=1\\ j\neq k,\ell}}^{n}\eta\left(\frac{\alpha_{j,n}t}{n^{\frac{1}{2}}\bar{\sigma}_n}\right)$$

$$\cdot Ee^{\frac{it}{n^{\frac{1}{2}}\bar{\sigma}_n}(\alpha_{k,n}(Z_k-1)+\alpha_{\ell,n}(Z_\ell-1))}((Z_k-1)^2-1)(Z_k-1)(Z_\ell-1)$$

$$+\frac{(it)^2}{4n\bar{\sigma}_n^2}\sum_{k=1}^{n}\sum_{\substack{\ell=1\\ \ell\neq k}}^{n}\sum_{\substack{m=1\\ m\neq k\\ m\neq\ell}}^{n}\frac{\beta_{m,n}\beta_{k\vee\ell,n}}{(n-m+1)(n-(k\wedge\ell)+1)}\prod_{\substack{j=1\\ j\neq k,\ell,m}}^{n}\eta\left(\frac{\alpha_{j,n}t}{n^{\frac{1}{2}}\bar{\sigma}_n}\right)\cdot$$

$$\cdot Ee^{\frac{it}{n^{\frac{1}{2}}\bar{\sigma}_n}(\alpha_{k,m}(Z_k-1)+\alpha_{\ell,n}(Z_\ell-1)+\alpha_{m,n}(Z_m-1))}\cdot((Z_m-1)^2-1)(Z_k-1)(Z_\ell-1)+$$

$$+\frac{(it)^2}{2n\bar{\sigma}_n^2}\sum_{k=1}^{n}\sum_{\substack{\ell=1\\ \ell\neq k}}^{n}\sum_{\substack{m=1\\ m\neq k\\ m\neq\ell}}^{n}\frac{\beta_{k\vee m,n}\beta_{\ell\vee m,n}}{(n-(k\wedge m)+1)(n-(\ell\wedge m)+1)}\prod_{\substack{j=1\\ j\neq k,\ell,m}}^{n}\eta\left(\frac{\alpha_{j,n}t}{n^{\frac{1}{2}}\bar{\sigma}_n}\right)\cdot$$

$$\cdot Ee^{\frac{it}{n^{\frac{1}{2}}\bar{\sigma}_n}(\alpha_{k,n}(Z_k-1)+\alpha_{\ell,n}(Z_\ell-1)+\alpha_{m,n}(Z_m-1))}\cdot(Z_k-1)(Z_\ell-1)(Z_m-1)^2+$$

$$+\frac{(it)^2}{8n\bar{\sigma}_n^2}\sum_{k=1}^{n}\sum_{\substack{\ell=1\\ \ell\neq k}}^{n}\sum_{\substack{m=1\\ m\neq k\\ m\neq \ell}}^{n}\sum_{\substack{p=1\\ p\neq k\\ p\neq \ell\\ p\neq m}}^{n}\frac{\beta_{k\vee\ell,n}\beta_{m\vee p,n}}{(n-(k\wedge\ell)+1)(n-(m\wedge p)+1)}\prod_{\substack{j=1\\ j\neq k,\ell,m,p}}^{n}\eta\left(\frac{\alpha_{j,n}t}{n^{\frac{1}{2}}\bar{\sigma}_n}\right)\cdot$$

$$\cdot Ee^{\frac{it}{n^{\frac{1}{2}}\bar{\sigma}_n}(\alpha_{k,n}(Z_k-1)+\alpha_{\ell,n}(Z_\ell-1)+\alpha_{m,n}(Z_m-1)+\alpha_{p,n}(Z_p-1))}\cdot$$

$$\cdot(Z_k-1)(Z_\ell-1)(Z_m-1)(Z_p-1).$$

To proceed we have to expand each of the fourteen terms on the right hand side of (5.2.23). Note that $\bar{\rho}_n(t)$, the Fourier-Stieltjes transform of $\bar{F}_n$, can be written down explicitly as

$$(5.2.24)\qquad \bar{\rho}_n(t) = e^{-\frac{t^2}{2}}\left(1-\frac{it^3}{6}\bar{\kappa}_{3n}+\frac{3\bar{\kappa}_{4n}t^4-\bar{\kappa}_{3n}^2t^6}{72}\right)$$

with $\bar{\kappa}_{3n}$ and $\bar{\kappa}_{4n}$ as in (5.1.8) and (5.1.9). Now the same kind of argument that was used to prove the lemma's 4.2.5, 4.2.6 and relation (4.2.43) can also be applied to prove the following lemma.

<u>LEMMA 5.2.5</u>. *Suppose there exist numbers* $0 < \alpha < \beta < 1$ *for which both the assumptions* 5.1.1 *and* 5.1.2 *are satisfied. Then there exist a number* $a > 0$ *such that*

$$(5.2.25)\qquad \int_{|t|\leq an^{\frac{1}{2}}} |\bar{\tau}_{1n}(t)-\bar{\rho}_n(t)||t|^{-1}dt = O(n^{-\frac{3}{2}}),\qquad \textit{as } n\to\infty.$$

<u>PROOF</u>. Let us illustrate the type of computation involved by deriving expansions for the first and third term on the right of (5.2.23). To start with we remark that

$$\prod_{j=1}^{n}\left(\frac{\alpha_{j,n}t}{n^{\frac{1}{2}}\bar{\sigma}_n}\right)$$

is the ch.f. of $\bar{J}_{1n} = n^{-\frac{1}{2}}\bar{\sigma}_n^{-1}\Sigma_{j=1}^n\alpha_{j,n}(Z_j-1)$ (cf. (5.2.10)). Note that $(\Sigma_{j=1}^n\alpha_{j,n}^2)^{-\frac{1}{2}}\Sigma_{j=1}^n\alpha_{j,n}(Z_j-1)$ is a properly standardized sum of independent, non-identically, distributed rv's with expectation zero, and finite absolute moment of any order. As the assumptions of the lemma easily imply that

$$\max_{1\le j\le n} |\alpha_{j,n}| \cdot \left(\sum_{j=1}^{n} \alpha_{j,n}^2\right)^{-\frac{1}{2}} = \mathcal{O}(n^{-\frac{1}{2}}), \qquad \text{as } n \to \infty$$

it follows directly from the classical theory of Edgeworth expansions for sums of independent rv's that for some number $a' > 0$ and uniformly in $|t| \le a'n^{\frac{1}{2}}$

$$\left| \prod_{j=1}^{n} \eta\left(\frac{\alpha_{j,n} t}{(\Sigma_{j=1}^{n} \alpha_{j,n}^2)^{\frac{1}{2}}}\right) - e^{-\frac{t^2}{2}} \left(1 + \frac{(it)^3 \Sigma_{j=1}^{n} \alpha_{j,n}^3}{3(\Sigma_{j=1}^{n} \alpha_{j,n}^2)^{\frac{3}{2}}} + \frac{(it)^4 \Sigma_{j=1}^{n} \alpha_{j,n}^4}{4(\Sigma_{j=1}^{n} \alpha_{j,n}^2)^2} + \frac{(it)^6 (\Sigma_{j=1}^{n} \alpha_{j,n}^3)^2}{18(\Sigma_{j=1}^{n} \alpha_{j,n}^2)^3}\right)\right| =$$

$$= \mathcal{O}(n^{-\frac{3}{2}} |t| P(t) e^{-\frac{t^2}{4}}), \qquad \text{as } n \to \infty$$

where P is a fixed polynomial in t. We now replace t by $t_n = t(\Sigma_{j=1}^{n} \alpha_{j,n}^2)^{\frac{1}{2}}/(n^{\frac{1}{2}}\bar{\sigma}_n)$. It follows after expanding $e^{-t_n^2/2}$ around t and using the result of lemma 5.2.1 that for some number $a > 0$ and uniformly in $|t| \le an^{\frac{1}{2}}$

$$\left| \prod_{j=1}^{n} \eta\left(\frac{\alpha_{j,n} t}{n^{\frac{1}{2}}\bar{\sigma}_n}\right) - e^{-\frac{t^2}{2}} \left(1 - \frac{(it)^2 b_n}{\Sigma_{j=1}^{n} \alpha_{j,n}^2} + \frac{(it)^3 \Sigma_{j=1}^{n} \alpha_{j,n}^3}{3(\Sigma_{j=1}^{n} \alpha_{j,n}^2)^{\frac{3}{2}}} + \frac{(it)^n \Sigma_{j=1}^{n} \alpha_{j,n}^4}{4(\Sigma_{j=1}^{n} \alpha_{j,n}^2)^2} + \frac{(it)^6 (\Sigma_{j=1}^{n} \alpha_{j,n}^3)^2}{18(\Sigma_{j=1}^{n} \alpha_{j,n}^2)^3}\right)\right| = \mathcal{O}(n^{-\frac{3}{2}} |t| P(t) e^{-\frac{t^2}{4}}) \tag{5.2.26}$$

as $n \to \infty$, where P is a fixed polynomial in t (different from above) and $2b_n$ denotes the coefficient of n^{-2} in the expansion for $\sigma^2(\bar{S}_n)$ (cf. (5.2.12))

As a second example of the computations involved we expand the third term on the right hand side of (5.2.23). We shall show that uniformly for $|t| \le an^{\frac{1}{2}}$.

(5.2.27)
$$\Big|\frac{it}{2n^{\frac12}\sigma_n}\sum_{k=1}^{n}\sum_{\substack{\ell=1\\ \ell\neq k}}^{n}\frac{\beta_{k\vee\ell,n}}{(n-(k\wedge\ell)+1)}\prod_{\substack{j=1\\ j\neq k,\ell}}^{n}\eta\Big(\frac{\alpha_{j,n}t}{n^{\frac12}\sigma_n}\Big)\cdot$$

$$\cdot E e^{\frac{it}{n^{\frac12}\sigma_n}(\alpha_{k,n}(Z_k-1)+\alpha_{\ell,n}(Z_\ell-1))}(Z_k-1)(Z_\ell-1)\;-$$

$$-\Big(\frac{(it)^3}{2(\Sigma_{j=1}^n\alpha_{j,n}^2)^{\frac32}}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\frac{\alpha_{k,n}\alpha_{\ell,n}\beta_{k\vee\ell,n}}{(n-k\wedge\ell+1)}+$$

$$+\frac{(it)^4}{(\Sigma_{j=1}^n\alpha_{j,n}^2)^2}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\frac{\alpha_{k,n}^2\alpha_{\ell,n}\beta_{k\vee\ell,n}}{(n-k\wedge\ell+1)}+$$

$$+\frac{(it)^6}{6(\Sigma_{j=1}^n\alpha_{jn}^2)^3}\sum_{j=1}^{n}\alpha_{jn}^3\sum_{k=1}^{n}\sum_{\ell=1}^{n}\frac{\alpha_{k,n}\alpha_{\ell,n}\beta_{k\vee\ell,n}}{(n-k\wedge\ell+1)}\Big)e^{-\frac{t^2}{2}}\Big|=$$

$$=O\big(n^{-\frac32}|t|P(t)e^{-\frac{t^2}{4}}\big),\qquad \text{as } n\to\infty.$$

To prove this we first expand $\exp(\frac{it}{n^{\frac12}\sigma_n}(\alpha_{k,n}(Z_k-1)+\alpha_{\ell,n}(Z_\ell-1)))$ around $t=0$ to find uniformly for all t

$$\Big|E e^{\frac{it}{n^{\frac12}\sigma_n}(\alpha_{k,n}(Z_k-1)+\alpha_{\ell,n}(Z_\ell-1))}(Z_k-1)(Z_\ell-1)-\frac{(it)^2}{n\sigma_n^2}\alpha_{k,n}\alpha_{\ell,n}-$$

$$-\frac{(it)^3}{n^{\frac32}\sigma_n^3}(\alpha_{k,n}^2\alpha_{\ell,n}+\alpha_{k,n}\alpha_{\ell,n}^2)\Big|=O(t^4n^{-2})$$

as $n\to\infty$. Next we observe that it is easily inferred from (5.2.26) that for fixed positive integers k and ℓ and uniformly for all $|t|\le an^{\frac12}$

$$\Big|\prod_{\substack{j=1\\ j\neq k,\ell}}^{n}\eta\Big(\frac{\alpha_{j,n}t}{n^{\frac12}\sigma_n}\Big)-e^{-\frac{t^2}{2}}\Big(1+\frac{(it)^3\,\Sigma_{j=1}^n\alpha_{j,n}^3}{3(\Sigma_{j=1}^n\alpha_{j,n}^2)^{\frac32}}\Big)\Big|=O\big(n^{-1}|t|P(t)e^{-\frac{t^2}{4}}\big)$$

as $n \to \infty$. Combining these results with an application of lemma 5.2.2(ii) to check that

$$\sigma_n^{-1} = n^{\frac{1}{2}}\left(\sum_{j=1}^{n} \alpha_{j,n}^2\right)^{-\frac{1}{2}} + O(n^{-\frac{1}{2}}), \qquad \text{as } n \to \infty,$$

we find that (5.2.27) holds. □

We are now in a position to prove (5.2.2). We first apply lemma 5.2.3 with $0 < \varepsilon < \alpha_1/2$ to see that the integral on the left of (5.2.15) is $o(n^{-1})$ as $n \to \infty$. Next we use lemma 5.2.4 with $0 < \varepsilon < \frac{1}{6}$ to find that the integral on the left of (5.2.21) is $o(n^{-1})$ as $n \to \infty$. Combining these results with lemma 5.2.5 we can conclude that (5.2.2) holds for $0 < \varepsilon < \min(\alpha_1/2, \frac{1}{6})$ under the assumptions of theorem 5.1.1. To see that (5.2.3) and (5.2.4) are also true we simply note that the argument leading to (4.2.2) and (4.2.3) also goes through (with obvious minor modifications) under the assumptions of theorem 5.1.1. This completes the proof of theorem 5.1.1 □

5.3. PROOF OF THEOREM 5.1.2.

To prove theorem 5.1.2 we first need three lemma's. In the first lemma we show that $\bar{\kappa}_{3n}$ and $\bar{\kappa}_{4n}$ (cf. (5.1.8) and (5.1.9)) are the leading terms in asymptotic expansions for the third and fourth cumulant κ_{3n}^* and κ_{4n}^* of T_n^* (cf. (5.1.11)).

<u>LEMMA 5.3.1</u>. *Let, for some* $\delta > 0$, $E|X_1|^{\delta} < \infty$ *and suppose that there exist numbers* $0 < \alpha < \beta < 1$ *for which both the assumptions* 5.1.1 *and* 5.1.2 *are satisfied. Then,*

$$(5.3.1) \qquad \kappa_{3n}^* = \bar{\kappa}_{3n} + o(n^{-1})$$

$$(5.3.2) \qquad \kappa_{4n}^* = \bar{\kappa}_{4n} + o(n^{-1}), \qquad \textit{as } n \to \infty,$$

with $\bar{\kappa}_{3n}$ *and* $\bar{\kappa}_{4n}$ *as in* (5.1.8) *and* (5.1.9).

<u>PROOF</u>. We first note that by several applications of Hölder's inequality and an argument as in the proof of (5.2.17), we can show that $T_n^* - \bar{S}_n^*$ (cf. (5.2.11)) is negligible for our purposes. Secondly, we remark that a relatively straightforward computation using (5.2.11) and applying the lemma's 2.3.2 and 5.2.2(ii) shows that

$$(5.3.3)\qquad E\bar{S}_n^{*3} = E\bar{J}_{1n}^3 + 3E\bar{J}_{1n}^2\bar{J}_{2n} + o(n^{-1})$$

$$(5.3.4)\qquad E\bar{S}_n^{*4} = E\bar{J}_{1n}^4 + 4E\bar{J}_{1n}^3\bar{J}_{2n} + 6E\bar{J}_{1n}^2\bar{J}_{2n}^2 + 4E\bar{J}_{1n}^3\bar{J}_{3n} + o(n^{-1}),\ \text{as } n\to\infty.$$

Rewriting the quantities on the right of (5.3.3) and (5.3.4) with the aid of (5.2.5) - (5.2.9) and (5.2.14) gives the desired results after a number of computations. □

In the second lemma of this section we show that $\bar{\kappa}_{3n}$ and $\bar{\kappa}_{4n}$ can be replaced by $\bar{\kappa}_3\, n^{-\frac{1}{2}}$ and $\bar{\kappa}_4\, n^{-1}$ in (5.3.1) and (5.3.2).

<u>LEMMA 5.3.2</u>. *Let, for some* $\delta > 0$, $E|X_1|^{\delta} < \infty$ *and suppose that there exists numbers* $0 < \alpha < \beta < 1$ *for which both the assumptions* 5.1.2 *and* 5.1.3 *are satisfied. Then,*

$$(5.3.5)\qquad \kappa_{3n}^{*} = \bar{\kappa}_3\, n^{-\frac{1}{2}} + o(n^{-1})$$

$$(5.3.6)\qquad \kappa_{4n}^{*} = \bar{\kappa}_4\, n^{-1} + o(n^{-1}),\qquad \text{as } n\to\infty,$$

with $\bar{\kappa}_3$ *and* $\bar{\kappa}_4$ *as in* (5.1.20) *and* (5.1.21).

<u>PROOF</u>. As an example of the computations involved we prove (5.3.5). We begin by remarking that $T_n = n^{-1}\,\Sigma_{i=1}^n\, J(\frac{i}{n+1})F^{-1}(U_{i:n})$ (cf. (5.1.13)) can be written as

$$(5.3.7)\qquad T_n = n^{-1}\sum_{i=1}^{n} J(\tfrac{i}{n+1})\{F^{-1}(\tfrac{i}{n+1}) + (U_{i:n} - \tfrac{i}{n+1})(F^{-1})^{(1)}(\tfrac{i}{n+1}) + \frac{(U_{i:n}-\frac{i}{n+1})^2}{2}(F^{-1})^{(2)}(\tfrac{i}{n+1}) + \frac{(U_{i:n}-\frac{i}{n+1})^3}{6}(F^{-1})^{(3)}(\tfrac{i}{n+1})\} + R_n$$

where R_n is a remainder, which is easily seen (the argument leading to (5.2.17) goes through with obvious modifications) to have moments of sufficiently low order of magnitude, so that this term can be neglected for our purposes. Next we observe that this fact, (5.3.7) and several applications of Hölder's inequality yields

$$(5.3.8)\qquad E(T_n - ET_n)^3 = n^{-3}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} J(\tfrac{i}{n+1})J(\tfrac{j}{n+1})J(\tfrac{k}{n+1}).$$

$$\cdot (F^{-1})^{(1)}(\tfrac{i}{n+1})(F^{-1})^{(1)}(\tfrac{j}{n+1})(F^{-1})^{(1)}(\tfrac{k}{n+1}) \cdot$$

$$\cdot\ E(U_{i:n} - \tfrac{i}{n+1})(U_{j:n} - \tfrac{j}{n+1})(U_{k:n} - \tfrac{k}{n+1})$$

$$+ \frac{3}{2} n^{-3} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} J(\tfrac{i}{n+1}) J(\tfrac{j}{n+1}) J(\tfrac{k}{n+1}) (F^{-1})^{(1)}(\tfrac{i}{n+1}) \cdot$$

$$\cdot\ (F^{-1})^{(1)}(\tfrac{j}{n+1})(F^{-1})^{(2)}(\tfrac{k}{n+1}) \,.$$

$$\cdot \mathrm{cov}[(U_{i:n} - \tfrac{i}{n+1})(U_{j:n} - \tfrac{j}{n+1}), (U_{k:n} - \tfrac{k}{n+1})^2] + O(n^{-\frac{5}{2}}), \qquad \text{as } n \to \infty.$$

Inserting the relations (cf. DAVID & JOHNSON (1954))

$$(5.3.9) \qquad E(U_{i:n} - \tfrac{i}{n+1})(U_{j:n} - \tfrac{j}{n+1})(U_{k:n} - \tfrac{k}{n+1}) =$$

$$= 2\, \frac{(i \wedge j \wedge k)(n+1-2((i \vee j) \wedge (i \vee k) \wedge (j \vee k)))(n+1-i \vee j \vee k)}{(n+1)^3 (n+2)(n+3)}$$

and

$$(5.3.10) \qquad \mathrm{cov}[(U_{i:n} - \tfrac{i}{n+1})(U_{j:n} - \tfrac{j}{n+1}), (U_{k:n} - \tfrac{k}{n+1})^2] =$$

$$= 2\, \frac{(i \wedge k)(n+1-i \vee k)(j \wedge k)(n+1-(j \vee k))}{(n+1)^6} + O(n^{-3})$$

as $n \to \infty$, into (5.3.8) and replacing the resulting Riemann sums by the corresponding Riemann integrals, we arrive at

$$(5.3.11) \qquad E(T_n - ET_n)^3 = n^{-2}[2 \int_0^1 \int_0^1 \int_0^1 J(s)J(t)J(v)(F^{-1}(s))^{(1)} (F^{-1})^{(1)}(t) \cdot$$

$$\cdot (F^{-1})^{(1)}(v)(s \wedge t \wedge v)(1 - 2((s \wedge t) \vee (s \wedge v) \vee (t \vee v)))(1-(s \vee t \vee v))\,ds\,dt\,dv$$

$$+ 3 \int_0^1 \int_0^1 \int_0^1 J(s)J(t)J(v)(F^{-1})^{(1)}(s)(F^{-1})^{(1)}(t)(F^{-1})^{(2)}(v) \cdot$$

$$\cdot (s \wedge v - sv)(t \wedge v - tv)\,ds\,dt\,dv] + O(n^{-\frac{5}{2}}) =$$

$$= n^{-2}\{2\int_0^1 \bar{h}_1^{\,3}(u)du + 3\int_0^1\int_0^1 \bar{h}_1(u)\bar{h}_1(v)\bar{h}_2(u,v)dudv\} + O(n^{-\frac{5}{2}})$$

$$\text{as } n \to \infty,$$

where we have used (5.1.16) and (5.1.17) in the last line. Because it is easily inferred from (5.3.7) and the argument following it that $\sigma^{-1}(T_n) = n^{\frac{1}{2}}\sigma^{-1}(J,F) + O(n^{-\frac{1}{2}})$ as $n \to \infty$ we have proved (5.3.5). The proof of (5.3.6) is similar but more laborious. The formula for the fourth cumulant of $n^{-1}\Sigma_{i=1}^{n} J(\frac{i}{n+1})(F^{-1})^{(1)}(\frac{i}{n+1})(U_{i:n} - \frac{i}{n+1})$ (cf. VAN ZWET (1979), p.100) and relations similar to (5.3.9) & (5.3.10) (cf. DAVID & JOHNSON (1954), p 238) are employed. □

In the third and final lemma of this section we derive expansions for $\sigma n^{-\frac{1}{2}}\sigma^{-1}(T_n)$ and $(\mu - E(T_n))\sigma^{-1}(T_n)$. The lemma and its proof are parallel to that of lemma 4.3.1.

<u>LEMMA 5.3.3</u>. *Let, for some* $\delta > 0$, $E|X_1|^{\delta} < \infty$ *and suppose that there exists numbers* $0 < \alpha < \beta < 1$ *for which the assumptions* 5.1.2 *and* 5.1.3 *are satisfied. Then,*

$$(5.3.12) \qquad |(\mu - ET_n)\sigma^{-1}(T_n) - \bar{a}n^{-\frac{1}{2}}| = O(n^{-1})$$

and

$$(5.3.13) \qquad |\sigma n^{-\frac{1}{2}}\sigma^{-1}(T_n) - 1 + \bar{b}n^{-1}| = O(n^{-1}), \qquad \text{as } n \to \infty,$$

with $\bar{a} = \bar{a}(J,F)$ *and* $\bar{b} = \bar{b}(J,F)$ *as in* (5.1.22) *and* (5.1.23).

<u>PROOF</u>. We first prove (5.3.13). Starting with (5.3.7) we first note that (cf. the argument given after (5.3.7))

$$(5.3.14) \qquad \sigma^2(T_n) = \sigma^2(n^{-1}\sum_{i=1}^{n} J(\tfrac{i}{n+1})(F^{-1})^{(1)}(\tfrac{i}{n+1})(U_{i:n} - \tfrac{i}{n+1})) +$$

$$+ n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} J(\tfrac{i}{n+1})J(\tfrac{j}{n+1})(F^{-1})^{(1)}(\tfrac{i}{n+1})(F^{-1})^{(2)}(\tfrac{j}{n+1})\cdot$$

$$\cdot E(U_{i:n} - \tfrac{i}{n+1})(U_{j:n} - \tfrac{j}{n+1})^2 +$$

$$+ 4^{-1}\sigma^2(n^{-1}\sum_{i=1}^{n} J(\tfrac{i}{n+1})(F^{-1})^{(2)}(\tfrac{i}{n+1})(U_{i:n} - \tfrac{i}{n+1})^2) +$$

$$+ 3^{-1}n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} J(\tfrac{i}{n+1})J(\tfrac{j}{n+1})(F^{-1})^{(1)}(\tfrac{i}{n+1})(F^{-1})^{(3)}(\tfrac{j}{n+1})\cdot$$

$$\cdot E(U_{i:n} - \tfrac{i}{n+1})(U_{j:n} - \tfrac{j}{n+1})^3 + o(n^{-\frac{5}{2}}), \qquad \text{as } n \to \infty.$$

To approximate the first term on the right of (5.3.14), we first note that this term is equal to

$$(5.3.15) \qquad (n+2)^{-1}n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} J(\tfrac{i}{n+1})J(\tfrac{j}{n+1})(F^{-1})^{(1)}(\tfrac{i}{n+1})(F^{-1})^{(1)}(\tfrac{j}{n+1})\cdot$$

$$\cdot((\tfrac{i}{n+1} \wedge \tfrac{j}{n+1}) - \tfrac{i}{n+1}\tfrac{j}{n+1}).$$

A simple analysis shows that this can be written as

$$(5.3.16) \qquad (n+2)^{-1}\int_0^1\int_0^1 \phi(s,t)\,ds\,dt + n^{-2}\int_0^1\int_0^1 \frac{\partial}{\partial s}\phi(s,t)(1-2s)\,ds\,dt$$

$$+ o(n^{-\frac{5}{2}}), \qquad \text{as } n \to \infty$$

where $\phi(s,t) = J(s)(F^{-1})^{(1)}(s)J(t)(F^{-1})^{(1)}(t)(s\wedge t - st)$ on the unit square. Note that the fact that ϕ is not differentiable at points (s,s) causes no problems. After a little calculation it follows from (5.3.16) that

$$(5.3.17) \qquad \sigma^2(n^{-1}\sum_{i=1}^{n} J(\tfrac{i}{n+1})(F^{-1})^{(1)}(\tfrac{i}{n+1})(U_{i:n} - \tfrac{i}{n+1})) =$$

$$= n^{-1}\sigma^2 + n^{-2}[-3\sigma^2 + 2\int_0^1 \bar{h}_1(u)\bar{h}_4(u)\,du + 2^{-1}\bar{h}_1^2(1) + 2^{-1}\bar{h}_1^2(0)] +$$

$$+ o(n^{-\frac{5}{2}}), \qquad \text{as } n \to \infty.$$

Next we obtain approximations for the other three terms on the right of (5.3.14). Now only first order approximations are needed because these terms are of a lower order than the term considered in (5.3.15). Argueing similarly as in the proof of lemma 5.3.2 we find

$$(5.3.16)\qquad n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} J(\tfrac{i}{n+1})\, J(\tfrac{j}{n+1})\, (F^{-1})^{(1)}(\tfrac{i}{n+1})\, (F^{-1})^{(2)}(\tfrac{j}{n+1})\,\cdot$$

$$\cdot E(U_{i:n}-\tfrac{i}{n+1})(U_{j:n}-\tfrac{j}{n+1})^2 = 2n^{-2}\int_0^1 \bar{h}_1(u)\bar{h}_2(u,u)\,du + o(n^{-2})$$

$$(5.3.19)\qquad 4^{-1}\sigma^2(n^{-1}\sum_{i=1}^{n} J(\tfrac{i}{n+1})\,(F^{-1})^{(2)}(\tfrac{i}{n+1})(U_{i:n}-\tfrac{i}{n+1})^2) =$$

$$= \tfrac{1}{2}n^{-2}\int_0^1\int_0^1 \bar{h}_2^2(u,v)\,du\,dv + o(n^{-2})$$

and

$$(5.3.20)\qquad 3^{-1}n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} J(\tfrac{i}{n+1})J(\tfrac{j}{n+1})\,(F^{-1})^{(1)}(\tfrac{i}{n+1})\,(F^{-1})^{(3)}(\tfrac{j}{n+1})\,\cdot$$

$$\cdot E(U_{i:n}-\tfrac{i}{n+1})(U_{j:n}-\tfrac{j}{n+1})^3 = n^{-2}\int_0^1\int_0^1 \bar{h}_1(u)\bar{h}_3(u,v,v)\,du\,dv + o(n^{-2})$$

as $n \to \infty$. Combining all these results we see that

$$(5.3.21)\qquad \sigma^2(T_n) = n^{-1}\sigma^2 + 2n^{-2}\sigma^2\,\bar{b} + o(n^{-2})$$

from which (5.3.13) is immediate.

To prove (5.3.12) we first remark that it is immediate from (5.3.7) and the remark made after it that

$$(5.3.22)\qquad ET_n = n^{-1}\sum_{i=1}^{n} J(\tfrac{i}{n+1})\{F^{-1}(\tfrac{i}{n+1}) + \tfrac{1}{2}n^{-1}\,\tfrac{i}{n+1}(1-\tfrac{i}{n+1})\,(F^{-1})^{(2)}(\tfrac{i}{n+1})\} +$$

$$+ o(n^{-\frac{3}{2}}),\qquad \text{as } n \to \infty.$$

It follows after replacing these Riemann sums by integrals that

$$(5.3.23)\qquad ET_n = \mu + n^{-1}\{\int_0^1 (\tfrac{1}{2}-s)(JF^{-1})^{(1)}(s)ds + $$
$$+ 2^{-1}\int_0^1 J(s)s(1-s)(F^{-1})^{(2)}(s)ds\} + o(n^{-\frac{3}{2}})$$

from which

$$(5.3.24)\qquad ET_n = \mu - \bar{a}\sigma n^{-1} + o(n^{-\frac{3}{2}})$$

follows by integration by parts. Because (5.3.13) directly implies that

$$(5.3.25)\qquad \sigma^{-1}(T_n) = n^{\frac{1}{2}}\sigma^{-1} + O(n^{-\frac{1}{2}}),\qquad \text{as } n \to \infty,$$

we have proved (5.3.12). □

We are now in a position to prove theorem 5.1.2. We first apply theorem 5.1.1 and the lemma's 5.3.1 - 5.3.3 to find, after a simple Taylor argument that $\sup_x|G_n(x) - \bar{G}_n(x)| = o(n^{-1})$ (cf. (5.1.26)) under the assumptions of theorem 5.1.2 and the additional requirement that $\beta_\delta < \infty$ for some $\delta > 0$. Finally we show that this moment assumption is in fact superfluous. To see this we simply note that as both the expansion $\bar{G}_n$ and the standardization we have employed (cf. (5.1.15)) do not depend on F^{-1} outside some closed subinterval of (0,1), we may modify F^{-1} on neighbourhoods of 0 and 1 appropriately so that the moment assumption is satisfied. This completes the proof of theorem 5.1.2.

5.4. EXTENSIONS

In the theorems 5.1.1 and 5.1.2 we have established expansions for the df's of linear combinations of order statistics with remainder $o(n^{-1})$. Again, as in section 4.4, we remark that we shall encounter no new difficulties when showing that under somewhat stronger conditions the remainder is $O(n^{-3/2})$. To do this for theorem 5.1.1 we need a strengthened version of assumption 5.1.2. We suppose that numbers $0 < \alpha < \beta < 1$ exist for which the assumptions 5.1.1 and 5.1.2* are satisfied.

ASSUMPTION 5.1.2.* There exist numbers a and b satisfying $0 \le F(a) < \alpha < \beta < F(b) \le 1$ such that

(i) F is four times differentiable on [a,b] with positive density f and bounded fourth derivative f"' on [a,b].

(ii) The function f"' satisfies a Lipschitz condition of order $\alpha_1 > 0$ on [a,b].

We shall state the results without further proof.

THEOREM 5.4.1. *Let, for some* $\delta > 0$, $E|X_1|^\delta < \infty$ *and suppose that there exist numbers* $0 < \alpha < \beta < 1$ *for which the assumptions* 5.1.1 *and* 5.1.2* *are satisfied. Then,*

$$\sup_x |F_n^*(x) - \bar{F}_n(x)| = O(n^{-\frac{3}{2}}), \qquad \text{as } n \to \infty$$

with F_n^* *and* $\bar{F}_n$ *as in* (5.1.10) *and* (5.1.7).

To obtain the corresponding result for theorem 5.1.2 we need also a strengthened version of assumption 5.1.3. We shall suppose that numbers $0 < \alpha < \beta < 1$ exist for which the assumptions 5.1.2* and 5.1.3* are satisfied.

ASSUMPTION 5.1.3.* There exist numbers t_1 and t_2 satisfying $0 < \alpha \le t_1 < t_2 \le \beta < 1$ such that

(i) $J(s) = 0 \qquad$ for $0 < s < \alpha$ and $\beta < s < 1$

(ii) the function J is differentiable on (α,β) with bounded derivative $J^{(1)}$ on (α,β); the function $J^{(1)}$ satisfies a Lipschitz condition of order 1 on (α,β).

(iii) $J(s) > 0 \qquad$ for $t_1 < s < t_2$.

THEOREM 5.4.2. *Suppose that there exist numbers* $0 < \alpha < \beta < 1$ *for which both assumtion* 5.1.2* *and assumption* 5.1.3* *are satisfied. Then,*

$$\sup_x |G_n(x) - \bar{G}_n(x)| = \mathcal{O}(n^{-\frac{3}{2}}), \qquad \textit{as } n \to \infty$$

with G_n *and* $\bar{G}_n$ *as in* (5.1.15) *and* (5.1.25).

We conclude this section with two remarks concerning the results obtained in this and the preceding chapter. In the first place we remark that, although we have presented our results for a fixed array of weights and a fixed df F, it is easy to construct classes of weights and distributions for which the expansions are valid uniformly. As the remainder terms depend on the weights and F only through certain constants, upperbounds and lower bounds, occurring in our conditions, the order of the remainder - $o(n^{-1})$ or $\mathcal{O}(n^{-3/2})$ - will always be uniform for fixed values of the constants, upperbounds and lower bounds appearing in the conditions of the statement we are proving.

In the second place we conjecture the existence of valid Edgeworth expansions for linear combinations of order statistics in the case where the weight functions may exhibit a finite number of discontinuities. Such a result would contain the theorems 4.1.1, 4.1.2 and 5.1.2 as special cases. The weakening of the smoothness conditions for the weight functions (cf. the assumptions 4.1.2 and 5.1.3) will then naturally entail a local smoothness condition on the underlying df F. There will be no need to trim. Such a more general result would be obtained by establishing an expansion for the conditional characteristic function of a linear combination of order statistics, where conditioning is on order statistics $X_{i-1:n}$ and $X_{i:n}$ when the weight functions possess a discontinuity in the interval $[\frac{i-1}{n}, \frac{i}{n}]$. By exploiting the independence created in this way and by drawing heavily on the techniques developed in chapter 4 we can - in principle - derive an expansion for the conditional ch.f. An expansion for the ch.f of a linear combination of order statistics then follows by taking the expectation. A main source of technical difficulties will be that the conditioning would change the standardization of the statistics considered. Although a proof along these lines appears to be very technical and laborious it would be interesting to obtain the conjectured more general results.

CHAPTER 6

DEFICIENCIES OF L-ESTIMATORS

In the two preceding chapters we derived expansions to $O(n^{-1})$ for the df's of linear combinations of order statistics. In this chapter we compute deficiencies of L-estimators with the aid of these expansions. In section 6.1 we obtain asymptotic deficiencies of first order efficient L-estimators, for estimating the centre of a symmetric distribution, with respect to maximum likelihood estimators and R-estimators derived from rank tests. In section 6.2 the distribution of the observations is no longer assumed to be symmetric. We show that in the asymmetric location case a phenomenon, first noted by PFANZAGL (1979), that "first order efficiency implies second order efficiency" also holds true for L-estimators.

6.1. DEFICIENCIES OF EFFICIENT L-ESTIMATORS FOR THE CENTRE OF SYMMETRY

Let $X_1, X_2, \ldots$ be i.i.d rv's with df $F(x-\theta)$, where F is known and has a density f that is positive on R' and symmetric about zero. Let f be five times differentiable and let us define functions

$$\psi_i(x) = f^{(i)}(x)/f(x), \qquad i = 1,2,\ldots,5 \tag{6.1.1}$$

$$\zeta_i(x) = (\log f(x))^{(i)}, \qquad i = 0,1,\ldots,5 \tag{6.1.2}$$

where $\zeta_0 = \log f$. Let J_1 and J_2 denote real-valued bounded measurable functions on $(0,1)$. L-estimators $\theta_L = \theta_L(X_1,\ldots,X_n)$ for estimating the centre of symmetry θ are given by

$$\theta_L = \theta_L(X_1,\ldots,X_n) = n^{-1} \sum_{i=1}^{n} c_{in} X_{i:n} \; . \tag{6.1.3}$$

As in chapter 4 we shall suppose that

$$\max_{1\le i\le n} |c_{in} - n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_1(s)ds - \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_2(s)ds| = \mathcal{O}(n^{-\gamma}) \tag{6.1.4}$$

as $n \to \infty$, with $\gamma > \frac{3}{2}$ (cf. assumption 4.1.1). We now add the assumption

$$n^{-1} \sum_{i=1}^{n} c_{in} = 1 \tag{6.1.5}$$

for all $n \ge 1$, by which we simply restrict attention to translation invariant L-estimators. Without loss of generality we may therefore assume that $\theta = 0$. Probabilities are then denoted by P_0.

L-estimators for the centre of symmetry θ which are - at least to first order - efficient are obtained if we choose

$$J_1(s) = -(I(f))^{-1}\zeta_2(F^{-1}(s)), \qquad 0 < s < 1 \tag{6.1.6}$$

where

$$0 < I(f) = \int_{-\infty}^{\infty} \psi_1^2(x)dF(x) < \infty \tag{6.1.7}$$

is the Fisher information number. Note that (6.1.6) and (6.1.7) together ensure that $\int_0^1 J_1(s)ds = 1$ whenever

$$\int_{-\infty}^{\infty} |f^{(2)}(x)|dx < \infty. \tag{6.1.8}$$

We also note that J_1 is symmetric around $\frac{1}{2}$. We add the assumption

$$J_2(s) = J_2(1-s), \qquad 0 < s < 1. \tag{6.1.9}$$

Note that (6.1.4) and (6.1.5) together imply that $\int_0^1 J_2(s)ds = 0$.

Define, for each $n \ge 1$ and real x,

$$L_n(x) = P_0(\{n^{\frac{1}{2}}(I(f))^{\frac{1}{2}}\theta_L \le x\}) \tag{6.1.10}$$

$$\tilde{L}_n(x) = \Phi(x) - \phi(x)\{\frac{(-5\eta_1+12\eta_2-9)}{72n}(x^3-3x) + \frac{\eta_3}{n}x\} \tag{6.1.11}$$

where the quantities η_1, η_2 and η_3 are given by

$$(6.1.12)\qquad \eta_1 = (I(f))^{-2}\cdot \int_{-\infty}^{\infty} \psi_1^4(x)\,dF(x)$$

$$(6.1.13)\qquad \eta_2 = (I(f))^{-2}\cdot \int_{-\infty}^{\infty} \psi_2^2(x)\,dF(x)$$

and

$$(6.1.14)\qquad \eta_3 = 4^{-1}(I(f))^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \zeta_3(x)\zeta_3(y)(F(x)\wedge F(y) - F(x)F(y))^2\cdot \cdot (f(x)f(y))^{-1}dxdy.$$

THEOREM 6.1.1. *Let the assumptions* (6.1.5) - (6.1.9) *as well as the assumptions of theorem* 4.1.2 *be satisfied. Then,*

$$(6.1.15)\qquad \sup_x |L_n(x) - \tilde{L}_n(x)| = o(n^{-1}), \qquad \text{as } n \to \infty.$$

PROOF. We begin by noting that the symmetry of F, J_1 and J_2 ensures that the quantities $\mu = \mu(J,F)$ (cf. (4.1.13)), $a = a(J_1,J_2,F)$ (cf. (4.1.15)) and $\kappa_3 = \kappa_3(J_1,F)$ (cf. (4.1.6)) are easily seen to be equal to zero. It follows, in view of theorem 4.1.2, that

$$(6.1.16)\qquad L_n(x) = \Phi(x) - \phi(x)\{\frac{\kappa_4}{24n}(x^3 - 3x) + \frac{b}{n}x\} + o(n^{-1}), \qquad \text{as } n \to \infty$$

with $\kappa_4 = \kappa_4(J,F)$ and $b = b(J_1,J_2,F)$ as in (4.1.7) and (4.1.16). It remains to compute κ_4 and b. We start the computation by remarking that a simple integration by parts yields (cf. (4.1.2))

$$(6.1.17)\qquad h_1(u) = -\int_0^1 J_1(s)(\chi_{(0,s]}(u) - s)\,d\,F^{-1}(s) =$$

$$= \int_0^u J_1(s)\,s\,dF^{-1}(s) - \int_u^1 J_1(s)(1-s)\,d\,F^{-1}(s)$$

$$= -(I(f))^{-1}\int_{-\infty}^{F^{-1}(u)} \psi_1^{(1)}(x)F(x)\,dx +$$

$$+ (I(f))^{-1}\int_{F^{-1}(u)}^{\infty} \psi_1^{(1)}(x)(1 - F(x))\,dx =$$

$$= -(I(f))^{-1}\psi_1(x)F(x)\Big|_{-\infty}^{F^{-1}(u)} + (I(f))^{-1}\int_{-\infty}^{F^{-1}(u)} \psi_1(x)f(x)dx$$

$$+ (I(f))^{-1}\psi_1(x)(1 - F(x))\Big|_{F^{-1}(u)}^{\infty} + (I(f))^{-1}\int_{F^{-1}(u)}^{\infty} \psi_1(x)f(x)dx$$

$$= -(I(f))^{-1}\psi_1(F^{-1}(u))u - (I(f))^{-1}\psi_1(F^{-1}(u))(1-u) +$$

$$+ (I(f))^{-1}\int_{-\infty}^{\infty} \psi_1(x)f(x)dx = -(I(f))^{-1}\psi_1(F^{-1}(u)), \qquad 0 < u < 1,$$

where we have used that $\int_{-\infty}^{\infty} \psi_1(x)f(x)dx = \int_{-\infty}^{\infty} f^{(1)}(x)dx = 0$. It follows directly from (6.1.17), that

$$\sigma^2(J_1,F) = \int_0^1 h_1^2(u)du = (I(f))^{-2}\int_{-\infty}^{\infty} \psi_1^2(x)dF(x) = (I(f))^{-1}. \tag{6.1.18}$$

$$\int_0^1 h_1^4(u)du = (I(f))^{-4}\int_{-\infty}^{\infty} \psi_1^4(x)dF(x) \tag{6.1.19}$$

Similarly, after a number of tedious computations, we obtain (cf. (4.1.2), (4.1.3) and (4.1.4)).

$$\int_0^1\int_0^1 h_1^2(u)h_1(v)h_2(u,v)dudv = \tag{6.1.20}$$

$$= (I(f))^{-4}\cdot\{-\frac{1}{3}\int_{-\infty}^{\infty} \psi_1^4(x)dF(x) + (\int_{-\infty}^{\infty} \psi_1^2(x)dF(x))^2\}$$

$$\int_0^1\int_0^1\int_0^1 h_1(u)h_1(v)h_1(w)h_3(u,v,w)dudvdw = \tag{6.1.21}$$

$$= (I(f))^{-4}\{-2\int_{-\infty}^{\infty} \psi_2^2(x)dF(x) + \frac{4}{3}\int_{-\infty}^{\infty} \psi_1^4(x)dF(x)\}$$

$$\int_0^1\int_0^1\int_0^1 h_1(u)h_1(v)h_2(u,w)h_2(v,w)dudvdw = \tag{6.1.22}$$

$$= (I(f))^{-4}\{\int_{-\infty}^{\infty} \psi_2^2(x)dF(x) - \frac{1}{3}\int_{-\infty}^{\infty} \psi_1^4(x)dF(x) - (\int_{-\infty}^{\infty} \psi_1^2(x)dF(x))^2\}.$$

Combining all these results we have obtained, in view of the definition of κ_4 (cf. (4.1.7)),

$$\kappa_4 = -\frac{5}{3}\eta_1 + 4\eta_2 - 3 \tag{6.1.23}$$

where η_1 and η_2 are given in (6.1.12) and (6.1.13). Next we have to compute b. In the same way as above we can show that

$$\int_0^1 h_1(u)h_2(u,u)du = -\int_0^1\int_0^1 h_1(u)h_3(u,v,v)dudv \tag{6.1.24}$$

$$= (I(f))^{-1}(\zeta(F^{-1}(0)) + \zeta(F^{-1}(1)) + 2$$

$$\int_0^1\int_0^1 h_2^2(u,v)dudv = \tag{6.1.25}$$

$$= (I(f))^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta_3(x)\zeta_3(y)(F(x)\wedge F(y)-F(x)F(y))^2(f(x)f(y))^{-1}dxdy.$$

and

$$\int_0^1 h_1(u)h_4(u)du = (I(f))^{-1}\int_0^1 \psi_1(F^{-1}(u))h_4(u)du = \tag{6.1.26}$$

$$= (I(f))^{-1}\int_0^1 J_2(s)\int_0^1 \psi_1(F^{-1}(u))(\chi_{(0,s]}(u)-s)du\,dF^{-1}(s)$$

$$= (I(f))^{-1}\int_0^1 J_2(s)ds = 0.$$

where (6.1.26) is easily inferred from (6.1.6) and the fact that $\int_0^1 J_2(s)ds = 0$. Combining these results we find that (cf. (4.1.16)).

$$b = \eta_3 \tag{6.1.27}$$

where η_3 is given in (6.1.14). This completes the proof. □

The L-estimators considered in theorem 6.1.1 are efficient and a natural competitor is of course the maximum likelihood estimator (MLE) $\theta_M = \theta_M(X_1,\ldots,X_n)$ which solves the equation

$$\sum_{i=1}^{n} \psi_1(X_i - \theta_M) = 0 \tag{6.1.28}$$

with ψ_1 as in (6.1.1); note that θ_M is uniquely determined whenever the density is strongly unimodal; i.e. log f is concave.

Define, for each $n \geq 1$ and real x

$$M_n(x) = P_0(\{n(I(f))^{\frac{1}{2}}\theta_M \leq x\}) \tag{6.1.29}$$

and

$$\tilde{M}_n(x) = \Phi(x) + \frac{x\phi(x)}{n}\{-\frac{(\eta_1-3)}{24} + \frac{x^2}{72}(5\eta_1 - 12\eta_2 + 9)\}. \tag{6.1.30}$$

THEOREM 6.1.2. (ALBERS, BICKEL & VAN ZWET (1976)). *Suppose that* f *is positive, symmetric about zero and strongly unimodal and*

$$\limsup_{y \to 0} \int_{-\infty}^{\infty} |\psi_j(x+y)|^{\frac{5}{j}} f(x)dx < \infty, \qquad j = 1,\ldots,5. \tag{6.1.31}$$

Then for every $C > 0$

$$\sup_{|x|\leq C} |M_n(x) - \tilde{M}_n(x)| = \mathcal{O}(n^{-\frac{3}{2}}), \qquad n \to \infty. \tag{6.1.32}$$

PROOF. see lemma 7.1 of ALBERS, BICKEL & VAN ZWET. □

HODGES and LEHMANN (1963) have introduced R-estimators $\theta_R = \theta_R(X_1,\ldots,\ldots,X_n)$ derived from rank tests. Let $0 \leq Z_1 \leq Z_2 \leq \ldots \leq Z_n$ be the ordered absolute values of $X_1,\ldots,X_n$ and define $V_j = 1$ if the X_i corresponding to Z_j is positive and $V_j = 0$ otherwise for $j = 1,2,\ldots,n$. Consider a vector of scores $a = (a_1,\ldots,a_n)$ and let $T_R = T_R(X_1,\ldots,X_n)$ be given by $T_R = \Sigma_{j=1}^{n} a_j V_j$. We assume that the scores a_i are non-negative and non-decreasing in $j = 1,2,\ldots,n$. Rank tests for the hypothesis $\theta = 0$ against $\theta > 0$, which are based on T_R with either $a_j = -E\psi_1(F^{-1}(\frac{1}{2}(1+U_{j:n}))$ or $a_j = -\psi_1(F^{-1}(\frac{1}{2}(1+\frac{j}{n+1}))$, where $U_{1:n}\leq\ldots\leq U_{n:n}$ are order statistics from the uniform df on (0,1), are known to be first order efficient against contiguous location alternatives $F(x-\theta)$, $\theta = \mathcal{O}(n^{-\frac{1}{2}})$ (see, e.g., HÁJEK & ŠIDÁK (1967)).

From these results efficient R-estimators can be obtained by defining

$$(6.1.33)\qquad \theta_R = \tfrac{1}{2}\sup\{t: 2T_R(X_1-t,\dots,X_n-t) > \sum_{j=1}^{n} a_j\}$$

$$+ \tfrac{1}{2}\inf\{t: 2T_R(X_1-t,\dots,X_n-t) < \sum_{j=1}^{n} a_j\}$$

i.e. θ_R is the midpoint of the interval between the upper and lower 0.5 confidence bounds for θ induced by the rank tests T_R.

Define, for each $n \geq 1$ and real x,

$$(6.1.34)\qquad R_n(x) = P_0(\{(nI(f))^{\frac{1}{2}}\theta_R \leq x\})$$

$$(6.1.35)\qquad \tilde{R}_n(x) = \Phi(x) + \frac{x\phi(x)}{n}\{\frac{\eta_1}{12} - \frac{\Sigma_{j=1}^{n}\,\sigma^2(\Psi_1(U_{j:n}))}{2I(f)} +$$

$$+ \frac{x^2}{72}(5\eta_1 - 12\eta_2 + 9)\}$$

where $\Psi_1(t) = \psi_1(F^{-1}(\frac{1+t}{2}))$.

THEOREM 6.1.3. (ALBERS (1974)). *Suppose that* f *is positive, symmetric about zero and strongly unimodal and such that*

$$(6.1.36)\qquad \limsup_{y\to 0}\int_{-\infty}^{\infty}|\psi_j(x+y)|^{m_j}f(x)\,dx < \infty, \qquad j = 1,\dots,4$$

with $m_1 = 6$, $m_2 = 3$, $m_3 = \frac{4}{3}$, $m_4 = 1$, *and*

$$(6.1.37)\qquad \limsup_{t\to 0,1} t(1-t)\left|\frac{\Psi_1''(t)}{\Psi_1'(t)}\right| < \frac{3}{2}\,.$$

Then for every $C > 0$

$$(6.1.38)\qquad \sup_{|x|\leq C}|R_n(x) - \tilde{R}_n(x)| = o(n^{-1}), \qquad \text{as } n\to\infty.$$

PROOF. see lemma 5.3.1 of ALBERS (1974). □

We are now in a position to compute deficiencies of L-estimators θ_L with respect to MLE's θ_M and R-estimators θ_R. Since we are only considering estimators $\hat{\theta}$ that are distributed symmetrically about the centre of symmetry we may take (cf. ALBERS, BICKEL & VAN ZWET (1976)) the s-quantile $\xi(\hat{\theta},s)$ of $\hat{\theta}-\theta$, for any fixed $\frac{1}{2} < s < 1$, as a measure of performance of the estimator $\hat{\theta}$. For any fixed value of s, we define the deficiency $d_{n,s}$ of a sequence of estimators $\{\hat{\theta}_{2,n}\}$ with respect to an estimator $\hat{\theta}_{1,n}$ by the equation

$$(6.1.39) \qquad \xi(\hat{\theta}_{2,n+d_{n,s}},s) = \xi(\hat{\theta}_{1,n},s)$$

with the convention that ξ is determined by linear interpolation for non-integral values of $n+d_{n,s}$.

Define

$$(6.1.40) \qquad \bar{d}(L,M) = \frac{1}{3}\eta_1 - \eta_2 + 2\eta_3 + 1$$

and

$$(6.1.41) \qquad \bar{d}_n(L,R) = \frac{7}{12}\eta_1 - \eta_2 + 2\eta_3 + \frac{3}{4} - \frac{\Sigma_{j=1}^n \sigma^2(\Psi_1(U_{j:n}))}{I(f)}$$

<u>THEOREM 6.1.4(i)</u>. *Let* $d_{n,s}(L,M)$ *be the deficiency of any* L*-estimator* (6.1.3) *satisfying* (6.1.4) - (6.1.9) *with respect to the maximum likelihood estimator for estimating* θ *in* $F(x-\theta)$. *Suppose that the assumption of the theorems* 6.1.1 *and* 6.1.2 *are satisfied. Then, for* $\frac{1}{2} < s < 1$,

$$(6.1.42) \qquad |d_{n,s}(L,M) - \bar{d}(L,M)| = o(1), \qquad \text{as } n \to \infty$$

(ii) *Let* $d_{n,s}(L,R)$ *be the deficiency of any L-estimator* (6.1.3) *satisfying* (6.1.4) - (6.1.9) *with respect to an efficient* R*-estimator* θ_R *for estimating* θ *in* $F(x-\theta)$. *Suppose that the assumptions of the theorems* 6.1.1 *and* 6.1.3 *are satisfied. Then, for* $\frac{1}{2} < s < 1$,

$$(6.1.43) \qquad |d_{n,s}(L,R) - \bar{d}_n(L,R)| = o(1), \qquad \text{as } n \to \infty.$$

PROOF. (i) Writing $\theta_{L,n}$ and $\theta_{M,n}$ for θ_L and θ_M we see that for some ζ

$$(6.1.44) \qquad P_0(\{(nI(f))^{\frac{1}{2}}\theta_{L,n+d_{n,s}} \le \xi\}) = s + o(n^{-1})$$

$$(6.1.45) \qquad P_0(\{(nI(f))^{\frac{1}{2}}\theta_{M,n} \le \xi\}) = s + o(n^{-1})$$

as $n \to \infty$. The theorems 6.1.1 and 6.1.2 now provide expansions for the probabilities in (6.1.44) and (6.1.45). To find $d_{n,s}$ we replace n by $n + d_{n,s}$ and x by $\xi(1+d_{n,s}n^{-1})^{\frac{1}{2}}$ in the expansion $\tilde{L}_n$ (cf. (6.1.11)) and equate the result to the expansion $\tilde{M}_n$ (cf. (6.1.30)) in the point $x = \xi$. Taylor expansion with respect to $d_n n^{-1}$ in $\tilde{L}_{n+d_{n,s}}(\xi(1+d_{n,s}n^{-1})^{\frac{1}{2}})$ yields

$$(6.1.46) \qquad L_{n+d_{n,s}}(\xi(1 + d_n n^{-1})^{\frac{1}{2}}) =$$

$$= \Phi(\xi) + \frac{\xi\phi(\xi)}{72n}\left[36d_n + (5\eta_1 - 12\eta_2 + 9)\xi^2 - 15\eta_1 + 36\eta_2 - 27 - 72\eta_3\right] +$$

$$+ o(n^{-1}), \qquad \text{as } n \to \infty.$$

Relation (6.1.42) now follows after some simple algebra.
(ii) Relation (6.1.43) follows similar, now using the theorems 6.1.1 and 6.1.3. □

We remark that the asymptotic expressions $\bar{d}(L,M)$ and $\bar{d}_n(L,R)$ are independent of s. Thus, to the order $o(1)$, the deficiencies $d_{n,s}(L,M)$ and $d_{n,s}(L,R)$ are asymptotically independent of the particular choice of the quantile used to measure the performance of the estimators. Another interesting property of the asymptotic expressions (6.1.40) and (6.1.41) is that they are independent of the weight function J_2. The reason for this phenomenon is of course that the expression $\tilde{L}_n$ does not depend on J_2 (cf. (6.1.26)).

We now briefly reconsider the various types of weights discussed in section 4.1 and show how our results apply. L-estimators θ_L with weights of the form

$$(6.1.47) \qquad c_{in} = n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_1(s)\,ds$$

or

(6.1.48) $$c_{in} = EJ_1(U_{i:n})$$

(cf. (4.1.21) and (4.1.22)) are translation invariant, whenever J_1 is chosen according to (6.1.6). Also note that the function J_2, determined by the relation (6.1.4), is symmetric around $\frac{1}{2}$ in each of these two cases. L-estimators θ_L with weights of the form

(6.1.49) $$c_{in} = J_1(\frac{i}{n+1})$$

or

(6.1.50) $$c_{in} = J_1(\frac{i}{n})$$

(cf. (4.1.19) and (4.1.20)), on the other hand, are not translation invariant whereas in the case (6.1.50) the function J_2 (cf. (4.1.24)) is not symmetric around $\frac{1}{2}$. However these L-estimators are easily modified to satisfy the requirements of translation invariance and symmetry of the weight functions involved.

It follows from theorem 6.1.4 that L-estimators with weights of the form (6.1.47) and (6.1.48) have asymptotic deficiency zero with respect to each other. The same result does not hold for L-estimators with weights of the form (6.1.49) and (6.1.50). We should note however that, after due modification, the asymptotic deficiency will be zero with respect to each other for L-estimators with these type of weights as well.

To conclude this section let us give one example of theorem 6.1.4. We consider the problem of estimating the centre θ of the logistic distribution

(6.1.51) $$F(x-\theta) = [1 + e^{-(x-\theta)}]^{-1}, \qquad -\infty < x < \infty.$$

We compare first-order efficient translation invariant L-estimators $\theta_L = \theta_L(X_1,\ldots,X_n)$ given by the weight function

(6.1.52) $$J_1(s) = 6s(1-s), \qquad 0 < s < 1,$$

with the maximum likelihood estimator $\theta_M = \theta_M(X_1,\ldots,X_n)$, which is the solution of equation (6.1.28), where $\psi_1(x) = \tanh(x/2)$. We also compare θ_L with the first order efficient Hodges-Lehmann R-estimator $\theta_R = \theta_R(X_1,\ldots,X_n)$, which is in this case given by

$$(6.1.53)\qquad \theta_R = \tfrac{1}{2} \underset{1\le i,j\le n}{\text{median}} \{(X_i + X_j)\}$$

As the assumptions of theorem 6.1.4 are satisfied in this case we find after a number of computations

$$(6.1.54)\qquad \bar{d}(L,M) = 2(10-\pi^2) - 0.2 \approx 0{,}06$$

$$(6.1.55)\qquad \bar{d}_n(L,R) = 2(10-\pi^2) - 0.5 \approx -0{,}24\ .$$

6.2. THE ASYMMETRIC LOCATION PROBLEM

Let $X_1, X_2, \ldots$ be i.i.d. rv's with df $F(x-\theta)$, where F is known and has a density f that is positive on R'. In the previous section we investigated the higher order performance of efficient L-estimators of θ in the case of a symmetric distribution. Here we consider briefly what happens if the distribution F is no longer symmetric. In this asymmetric case we shall compare efficient L-estimators of the location parameter θ to the maximum likelihood estimator of θ.

The purpose of this section is to show that the Edgeworth expansions of the df's of efficient L-estimators and of the maximum likelihood estimator agree not only in their leading terms of order 1 but also in their second order terms of order $n^{-\frac{1}{2}}$, provided these estimators are adjusted in such a way that they are median-unbiased to order $o(n^{-\frac{1}{2}})$. It is only in the third order terms of order n^{-1} that differences begin to show up. This phenomenon "first order efficiency implies second order efficiency" was shown to hold for estimators admitting a certain stochastic expansion by PFANZAGL (1973; (1979). (see also CHIBISOV (1972)). We shall prove that the same phenomenon holds true for adjusted L-estimators of the form

$$(6.2.1)\qquad \tilde{\theta}_L = \tilde{\theta}_L(X_1,\ldots,X_n) = \theta_L(X_1,\ldots,X_n) - \mu + n^{-1}\left(a\sigma + \frac{\kappa_3\sigma}{6}\right)$$

where $\theta_L = n^{-1} \Sigma_{i=1}^{n} c_{in} X_{i:n}$ (cf. (6.1.3)) and $\mu = \mu(J_1,F)$, $a = a(J_1,J_2,F)$, $\sigma^2 = \sigma^2(J,F)$ and $\kappa_3 = \kappa_3(J_1,F)$ are defined in (4.1.13), (4.1.15), (4.1.8) and (4.1.6).
As in section 6.1 J_1 and J_2 are bounded real-valued measurable functions and we again suppose that the assumptions (6.1.4) - (6.1.8) are satisfied. Of course J_1 and J_2 are no longer symmetric. Let

$$\eta_4 = (I(f))^{-\frac{3}{2}} \int_{-\infty}^{\infty} \psi_1^3(x)\,dF(x) \tag{6.2.2}$$

where $I(f)$ and ψ_1 are defined in (6.1.7) and (6.1.1).

<u>THEOREM 6.2</u>. *Let the assumptions* (6.1.5) - (6.1.8) *as well as the assumptions of theorem* 4.1.2 *be satisfied. Then,*

$$\sup_x |P_0(\{(n(I(f)))^{\frac{1}{2}}\tilde{\theta}_L \le x\}) - \Phi(x) + \frac{\eta_4}{12n^{\frac{1}{2}}} x^2\phi(x)| = o(n^{-\frac{1}{2}}) \tag{6.2.3}$$

as $n \to \infty$

<u>PROOF</u>. From the construction of $\tilde{\theta}_L$ it follows that

$$\begin{aligned} P_0(\{(n(I(f)))^{\frac{1}{2}}\tilde{\theta}_L \le x\}) &= P_0(\{(n(I(f)))^{\frac{1}{2}}\theta_L \le \\ &\le x + (n(I(f)))^{\frac{1}{2}}(\mu - n^{-1}(a\sigma + \frac{\kappa_3\sigma}{6})\}) = \\ &= P_0(\{(n(I(f)))^{\frac{1}{2}}(\theta_L - \mu) \le x - n^{-\frac{1}{2}}a - n^{-\frac{1}{2}}\frac{\kappa_3}{6}\}) \end{aligned} \tag{6.2.4}$$

where in the last line we have used the fact that θ_L is first order efficient. Theorem 4.1.2 now provides an expansion for the probabilities in (6.2.4).

$$\begin{aligned} P_0(\{(nI(f))^{\frac{1}{2}}(\theta_L - \mu) \le x - n^{-\frac{1}{2}}a - n^{-\frac{1}{2}}\frac{\kappa_3}{6}\}) &= \\ = \Phi(x - n^{-\frac{1}{2}}a - n^{-\frac{1}{2}}\frac{\kappa_3}{6}) - \phi(x)\{\frac{\kappa_3}{6} n^{-\frac{1}{2}}(x^2 - 1) - an^{-\frac{1}{2}}\} + o(n^{-\frac{1}{2}}) &= \\ = \Phi(x) - \frac{\kappa_3}{6} n^{-\frac{1}{2}}x^2\phi(x) + o(n^{-\frac{1}{2}}), \qquad \text{as } n \to \infty. \end{aligned} \tag{6.2.5}$$

It remains to compute κ_3. To begin with we use (6.1.17) to see that

$$(6.2.6) \qquad \int_0^1 h_1^3(u)\,du = -(I(f))^{-3} \int_{-\infty}^{\infty} \psi_1^3(x)\,dF(x)$$

where h_1 and ψ_1 are defined in (4.1.2) and (6.1.1). Secondly, we remark that

$$(6.2.7) \qquad 3\int_0^1\int_0^1 h_1(u)h_1(v)h_2(u,v)\,du\,dv =$$

$$= -3(I(f))^{-2} \int_0^1 J_1^{(1)}(s)\{\int_0^1 \psi_1(F^{-1}(u))(\chi_{(0,s]}(u) - s)\,du\}^2 dF^{-1}(s)$$

$$= -3(I(f))^{-2} \int_0^1 J_1^{(1)}(s)f(F^{-1}(s))\,ds$$

where we have used a simple integration by parts in the third line. Again applying integration by parts we see that (cf. (6.1.6))

$$(6.2.8) \qquad -\int_0^1 J_1^{(1)}(s)f(F^{-1}(s))\,ds = (I(f))^{-1} \int_{-\infty}^{\infty} \psi_1^{(2)}(x)\,dF(x) =$$

$$= (I(f))^{-1}\psi_1^{(1)}(x)f(x)\Big|_{-\infty}^{\infty} - (I(f))^{-1} \int_{-\infty}^{\infty} \psi_1^{(1)}(x)f^{(1)}(x)\,dx =$$

$$= -(I(f))^{-1}\psi_1(x)f^{(1)}(x)\Big|_{-\infty}^{\infty} + (I(f))^{-1} \int_{-\infty}^{\infty} \psi_1(x)f^{(2)}(x)\,dx =$$

$$= (I(f))^{-1} \int_{-\infty}^{\infty} \frac{f^{(1)}(x)f^{(2)}(x)}{f^2(x)} f(x)\,dx = (I(f))^{-1} \int_{-\infty}^{\infty} \psi_1(x)\psi_2(x)\,dF(x)$$

$$= \tfrac{1}{2}(I(f))^{-1} \int_{-\infty}^{\infty} \psi_1^3(x)\,dF(x)$$

where ψ_1 and ψ_2 are defined in (6.1.1). Combining (6.2.6), (6.2.8) with (6.1.18) we find, in view of the formula for κ_3 (cf. (4.1.6))

$$(6.2.9) \qquad \kappa_3 = \frac{\eta_4}{2}\,.$$

This completes the proof of (6.2.3). □

We remark that in theorem 6.2 we have established the second order term of order $n^{-\frac{1}{2}}$ of the Edgeworth expansion for the adjusted L-estimator $\tilde{\theta}_L$. Note that $\tilde{\theta}_L$ is median-unbiased up to an error $o(n^{-\frac{1}{2}})$; i.e.

$$\text{(6.2.10)} \qquad P_0(\{(n(I(f)))^{\frac{1}{2}}\tilde{\theta}_L \leq 0\}) = \tfrac{1}{2} + o(n^{-\frac{1}{2}}), \qquad \text{as } n \to \infty.$$

We also remark that (6.2.3) and (6.2.10) even holds with $o(n^{-\frac{1}{2}})$ replaced by $\mathcal{O}(n^{-1})$. The corresponding relation with $o(n^{-\frac{1}{2}})$ replaced by $o(n^{-1})$ does not hold true anymore in general. Because, to the order considered, the expansion (6.2.3) coincides with the Edgeworth expansion for the "adjusted" maximum likelihood estimator $\tilde{\theta}_M$ for θ (see PFANZAGL (1973) p. 1006-1007), the df's of $(nI(f))^{\frac{1}{2}}(\tilde{\theta}_L-\theta)$ and $(nI(f))^{\frac{1}{2}}(\tilde{\theta}_M-\theta)$ agree not only in their leading terms but also in their second order terms. Using formal expansions only TAKEUCHI and AKAHIRA (1976) arrived at the same result.

CHAPTER 7

FINITE SAMPLE COMPUTATIONS

In the chapters 4 and 5 we derived asymptotic expansions for the df's of linear combinations of order statistics under various sets of conditions. In the sections 7.1 and 7.2 we investigate the performance of these expansions as approximations for the finite sample distributions. In particular we compare these expansions with the usual normal approximation.

7.1. AN L-ESTIMATOR FOR LOGISTIC LOCATION

In this section we consider (cf. example 1.2.3) the L-estimator

$$(7.1.1)\qquad T_n = 6n^{-1} \sum_{i=1}^{n} \frac{i}{n+1} \left(1 - \frac{i}{n+1}\right) X_{i:n}$$

in the case of the logistic distribution $F(x) = (1 + e^{-x})^{-1}$ for $-\infty < x < \infty$. From section 4.1. we know that

$$(7.1.2)\qquad P(\{2.3^{\frac{1}{2}} n^{-\frac{1}{2}} \sum_{i=1}^{n} \frac{i}{n+1}\left(1 - \frac{i}{n+1}\right) X_{i:n} \leq x\}) =$$

$$= \Phi(x) - \phi(x)\left[\frac{1}{20n}(x^3 - 3x) + \frac{(11-\pi^2)}{n} x\right] + o(n^{-1})$$

as $n \to \infty$. We shall investigate how well the exact df is approximated by the expansion in (7.1.2) for small samples. We shall also compare this approximation with the usual normal approximation. For sample sizes $n = 3$ and $n = 4$ we have computed the multiple integrals involved in the computation of the exact df. For larger sample sizes the amount of computation that is necessary for this method becomes prohibitive and we have relied on Monte-Carlo simulation. For sample sizes $n = 3,4,10$ and 25 we have performed a Monte-Carlo estimation based on 25.000 samples. The agreement between the results from the numerical integration and the Monte-Carlo results for sample sizes n = 3 and n = 4 was satisfactory. The results of the simulation are given in the following table. We give the Monte-Carlo estimate $\hat{G}_n$ for the exact df in (7.1.2), the expansion $\tilde{G}_n$ and the normal approximation, for n = 3,4,10,25 and various values of the argument.

TABLE 7.1

x	$\hat{G}_3$	$\tilde{G}_3$	$\hat{G}_4$	$\tilde{G}_4$	$\hat{G}_{10}$	$\tilde{G}_{10}$	$\hat{G}_{25}$	$\tilde{G}_{25}$	Φ
0.0	.5000	.5000	.5000	.5000	.5000	.5000	.4991	.5000	.5000
0.2	.5640	.5536	.5663	.5601	.5734	.5716	.5758	.5762	.5793
0.4	.6262	.6069	.6307	.6190	.6445	.6409	.6492	.6495	.6554
0.6	.6850	.6592	.6919	.6759	.7089	.7058	.7152	.7177	.7257
0.8	.7391	.7100	.7469	.7318	.7680	.7647	.7728	.7788	.7881
1.0	.7875	.7583	.7963	.7790	.8196	.8164	.8295	.8314	.8413
1.2	.8248	.8032	.8391	.8236	.8629	.8604	.8756	.8752	.8849
1.4	.8658	.8439	.8752	.8627	.8985	.8966	.9100	.9102	.9192
1.6	.8958	.8797	.9049	.8960	.9275	.9256	.9376	.9374	.9452
1.8	.9202	.9100	.9287	.9234	.9486	.9478	.9580	.9576	.9641
2.0	.9397	.9347	.9474	.9454	.9646	.9645	.9732	.9711	.9772
2.2	.9550	.9543	.9618	.9622	.9764	.9766	.9830	.9824	.9861
2.4	.9669	.9691	.9726	.9748	.9845	.9850	.9895	.9890	.9918
2.6	.9758	.9798	.9807	.9837	.9905	.9907	.9942	.9934	.9953
2.8	.9825	.9873	.9865	.9899	.9937	.9945	.9963	.9963	.9974
3.0	.9875	.9863	.9907	.9939	.9959	.9968	.9982	.9979	.9987

Inspection of this table shows that the agreement between the estimated exact df $\hat{G}_n$ and the expansion $\tilde{G}_n$ (cf. (7.1.2)) is already quite reasonable for n = 3. It also shows that the expansion performs much better than the normal approximation as approximations of the finite sample exact df's.

7.2. GINI'S MEAN DIFFERENCE FOR THE UNIFORM DISTRIBUTION

In the previous section we have investigated a case in which there is no $n^{-\frac{1}{2}}$ term present in the expansion. It seems of interest to consider also situations where a $n^{-\frac{1}{2}}$-term has to be taken into account. As an example in which this is the case we consider Gini's mean difference (cf. example 1.2.4) which is given by

$$(7.2.1)\qquad T_n = \frac{4(n+1)}{n(n-1)} \sum_{i=1}^{n} \left(\frac{i}{n+1} - \frac{1}{2}\right) X_{i:n}$$

in case of the uniform distribution $F(x) = x$ for $0 \le x \le 1$. From section 4.1 we know that

$$(7.2.2)\qquad P(\{3.5^{\frac{1}{2}} n^{\frac{1}{2}} (\frac{4(n+1)}{n(n-1)} \sum_{i=1}^{n} (\frac{i}{n+1} - \frac{1}{2}) X_{i:n} - \frac{1}{3}) \le x\}) =$$

$$= \Phi(x) - \phi(x)\left[\frac{-2.5^{\frac{1}{2}}}{21n^{\frac{1}{2}}}(x^2-1) + \frac{1}{28n}(x^3-3x) + \frac{10}{441n}(x^5-10x^3+15x) + \frac{2}{n}x\right] +$$

$$+ o(n^{-1})$$

as $n \to \infty$. For sample size $n = 3$ the exact df is easily obtained. For sample sizes $n = 3,4,10$ and 25 we have performed a Monte-Carlo simulation based on 25.000 samples. The agreement between the exact df and the Monte-Carlo result for $n = 3$ was satisfactory. The results of the simulation are given in table 7.2. Again $\hat{G}_n$ denotes the Monte-Carlo estimate of the exact df in (7.2.2); $\tilde{G}_{n,1}$ and $\tilde{G}_{n,2}$ are the expansion with remainder $o(n^{-\frac{1}{2}})$ and $o(n^{-1})$ respectively. Inspection of this table shows that already for sample size $n = 3$ the expansion $\tilde{G}_{n,2}$ performs better than the expansion $\tilde{G}_{n,1}$ and the normal approximation.

TABLE 7.2

x	$\hat{G}_3$	$\tilde{G}_{3,1}$	$\tilde{G}_{3,2}$	$\hat{G}_4$	$\tilde{G}_{4,1}$	$\tilde{G}_{4,2}$	$\hat{G}_{10}$	$\tilde{G}_{10,1}$	$\tilde{G}_{10,2}$	$\hat{G}_{25}$	$\tilde{G}_{25,1}$	$\tilde{G}_{25,2}$	Φ
-3.0	.0332	.0057	.0155	.0277	.0051	.0125	.0093	.0037	.0067	.0046	.0029	.0040	.0013
-2.6	.0715	.0143	.0394	.0548	.0130	.0318	.0212	.0099	.0175	.0116	.0080	.0110	.0047
-2.2	.1182	.0307	.0844	.0884	.0284	.0687	.0417	.0231	.0392	.0251	.0197	.0262	.0139
-1.8	.1744	.0577	.1528	.1339	.0548	.1261	.0752	.0478	.0764	.0525	.0435	.0549	.0359
-1.4	.2358	.0984	.2356	.1926	.0961	.1989	.1281	.0904	.1316	.1006	.0869	.1033	.0808
-1.0	.3035	.1587	.3142	.2639	.1587	.2753	.2029	.1587	.2053	.1755	.1587	.1773	.1587
-0.6	.3760	.2480	.3750	.3451	.2515	.3468	.2983	.2599	.2980	.2810	.2652	.2804	.2743
-0.2	.4522	.3746	.4240	.4360	.3808	.4178	.4104	.3955	.4103	.4151	.4048	.4107	.4207
0	.4922	.4509	.4509	.4818	.4575	.4575	.4730	.4731	.4731	.4858	.4830	.4830	.5000
0.2	.5335	.5331	.4837	.5306	.5393	.5022	.5390	.5540	.5392	.5571	.5633	.5573	.5793
0.6	.6113	.6995	.5725	.6191	.7030	.6078	.6684	.7114	.6733	.7027	.7167	.7014	.7257
1.0	.6869	.8413	.6858	.7095	.8413	.7247	.7868	.8413	.7947	.8211	.8413	.8227	.8413
1.4	.7583	.9369	.7998	.7957	.9345	.8317	.8770	.9289	.8878	.9088	.9254	.9089	.9192
2.8	.8210	.9858	.8907	.8706	.9829	.9115	.9409	.9760	.9474	.9602	.9716	.9602	.9641
2.2	.8774	1.003	.9491	.9310	1.001	.9603	.9781	.9953	.9791	.9862	.9919	.9854	.9861
2.6	.9254	1.005	.9798	.9682	1.004	.9848	.9936	1.001	.9931	.9957	.9987	.9957	.9953
3.0	.9642	1.003	.9932	.9868	1.002	.9951	.9988	1.001	.9981	.9994	1.000	.9990	.9987

REFERENCES

[1] ALBERS, W. (1974), *Asymptotic expansions and the deficiency concept in statistics,* Mathematical Centre Tracts 58, Amsterdam.

[2] ALBERS, W., P.J. BICKEL & W.R. VAN ZWET (1976), Asymptotic expansions for the power of distribution free tests in the one-sample problem, Ann. Statist. 4, 108-156.

[3] BAHR, VON B. & C.G. ESSEEN (1965), Inequalities for the rth absolute moment of a sum of random variables, $1 \leq r \leq 2$, Ann. Math. Statist. 36, no. 1, 299-303.

[4] BALKEMA, A.A. & L. DE HAAN (1978), Limit distributions for order statistics, Teoriya Veroyatnostei iee Primeneniya, Vol. XXIII, no. 1, 80-96 (Part I), no. 2, 358-375 (Part II).

[5] BENNETT, C.A. (1952), Asymptotic properties of ideal linear estimators, unpublished dissertation, University of Michigan, Ann Arbor.

[6] BHATTACHARYA, R.N. & R. RANGA RAO (1976), *Normal approximation and asymptotic expansions,* Wiley & Sons, New-York.

[7] BHATTACHARYA, R.N. & J.K. GHOSH (1978), On the validity of the formal Edgeworth expansion, Ann. Statist. 6, no. 2, 434-451.

[8] BICKEL, P.J. (1967), Some contributions to the theory of order statistics, Proceedings of the fifth Berkeley Symposium on Mathematical Statistics and Probability, Vol. 1, 575-591.

[9] BICKEL, P.J. (1974), Edgeworth expansions in non-parametric statistics, Ann. Statist. 2, 1-20.

[10] BICKEL, P.J. & E.L. LEHMANN (1975), Descriptive statistics for nonparametric models, II, Location, Ann. Stat. 3, 1045-1069.

[11] BICKEL, P.J. & W.R. VAN ZWET (1978), Asymptotic expansions for the power of distributionfree tests in the two-sample problem, Ann. Stat. 6, 937-1004.

[12] BJERVE, S. (1974), Error bounds and Asymptotic Expansions for linear combinations of order statistics, unpublished Ph.D. thesis, Berkeley.

[13] BJERVE, S. (1977), Error bounds for linear combinations of order statistics, Ann. Statist., 5, 237-255.

[14] BLOM, G. (1958), *Statistical estimates and transformed beta-variables*, Wiley & Sons, New York.

[15] CALLAERT, H. & P. JANSSEN (1978), The Berry-Esseen theorem for U-statistics, Ann. Stat. 6, 417-421.

[16] CALLAERT, H., P. JANSSEN & N. VERAVERBEKE (1980), An Edgeworth expansion for U-statistics, Ann. Stat. 8, 299-312.

[17] CHIBISOV, D.M. (1972), An asymptotic expansion for the distribution of a statistic admitting an asymptotic expansion, Theor. Prob. & Appl. 17, 620-630.

[18] CHIBISOV, D.M. (1973a), An asymptotic expansion for a class of estimators containing maximum likelihood estimators, Theor. Prob. & Appl. 18, 295-303.

[19] CHIBISOV, D.M. (1973b), An asymptotic expansion for the distribution of sums of a special form with an application to minimum contrast estimators, Theor. Prob. & Appl. 18, 649-661.

[20] CHIBISOV, D.M. (1973c), Asymptotic expansions for Neyman's C(α) tests, Lecture notes in mathematics 330, Springer, 16-45.

[21] CHIBISOV, D.M. (1974), Asymptotic expansions for some asymptotically optimal tests, Proc. Prague Symp. on Asymptotic Statistics, Vol. II, 37-68.

[22] CRAMÉR, H. (1962), *Random Variables and Probability Distributions*, 2nd. Ed. Cambridge University Press.

[23] CHUNG, K.L. (1946), The Approximate Distribution of Student's Statistic, Ann. Math. Stat. 17, 442-465.

[24] CHUNG, K.L. (1951), The strong law of large numbers, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, vol. 2, 341-352.

[25] CHAN, Y. & J. WIERMAN (1977), On the Berry-Esseen theorem for U-statistics, Ann. Probability 5, 136-139.

[26] CHERNOFF, H., J. GASTWIRTH & M.V. JOHNS (1967), Asymptotic distribution of linear combinations of functions of order statistics with applications to estimation, Ann. Math. Statist. 38, 52-72.

[27] DAVID, F.N. & N.L. JOHNSON (1954), Statistical treatment of censored data, Part I, Fundamental formulae, Biometrika, Vol. 41, 228-240.

[28] DAVID, H.A. (1970), *Order Statistics,* Wiley & Sons, New York.

[29] DVORETZKY, A., J. KIEFER & J. WOLFOWITZ (1956), Asymptotic minimax character of sample distribution function and of the classical multinomial estimator. Ann. Math. Statist. 27, 642-669.

[30] EGOROV, V.A. & V.B. NEVZOROV (1976), Limit theorems for linear combinations of order statistics, Lecture notes in mathematics 550, 63-79.

[31] ESSEEN, C.G. (1945), Fourier analysis of distribution functions, A mathematical study of the Laplace-Gaussian law, Acta. Mathematica 77, 1-125.

[32] FELLER, W. (1966), *An introduction to Probability Theory and its Applications,* Vol. 2, Wiley, New York.

[33] GNEDENKO, B.V. & A.N. KOLMOGOROV (1954), *Limit Distributions for sums of independent random variables,* Addison-Wesley Publishing Company Inc., Cambridge, Mass.

[34] HÁJEK, J. & Z. ŠIDÁK (1967), *Theory of Rank Tests,* Academic Press, New York.

[35] HÁJEK, J. (1968), Asymptotic normality of simple linear rank statistics under alternatives, Ann. Math. Statist. 39, 325-346.

[36] HECKER, H. (1976), A characterization of the asymptotic normality of linear combinations of order statistics from the uniform distribution, Ann. Stat. 4, 1244-1246.

[37] HELMERS, R. (1977), The order of the normal approximation for linear combinations of order statistics with smooth weight functions, Ann. Probability, vol 5, no. 6, 940-953.

[38] HELMERS, R. (1979), Edgeworth expansions for trimmed linear combinations of order statistics. In. *Proc. Second. Prague Symp. Asymptotic. Statist.,* 221-232. (P.Mandl and M. Hušková, eds), North Holland, Amsterdam.

[39] HELMERS, R. (1980), Edgeworth expansions for linear combinations of order statistics with smooth weight functions, Ann. Statist. vol 8, no. 6, 1361-1374.

[40] HELMERS, R. (1981), A Berry-Esseen theorem for linear combinations of order statistics, Ann. Probability, Vol 9, no 2, 342-347.

[41] HELMERS, R. & W.R. VAN ZWET (1982), The Berry-Esseen bound for U-statistics, *Proc. Symp. on Statistical Decision Theory and Related Topics III,* Vol. I., 497-512, S.S. Gupta & J.O. Berger (eds), Academic Press, New York.

[42] HODGES, J.L. & E.L. LEHMANN (1963), Estimates of location based on rank tests, Ann. Math. Statist. 34, 598-611.

[43] HODGES, J.L. & E.L. LEHMANN (1970), Deficiency, Ann. Math. Statist. 41, 783-801.

[44] HUBER, P.J. (1972), Robust statistics: A review, Ann. Math. Statist. 43, 1041-1067.

[45] HUBER, P.J. (1977), *Robust Statistical Procedures*; Regional Conference Series in Applied Mathematics, 27.

[46] HUŠKOVÁ M. (1977), The rate of convergence of simple linear rank statistics under hypothesis and alternatives, Ann. Stat. 5, 658-670.

[47] HUŠKOVA, M. (1979), The rate of convergence of simple rank statistics under alternatives. In *Contributions to Statistics* (Jaroslav Hájek Memorial Volume) 99-108 (ed. J. Jurečkova) Academia, Prague.

[48] JANSSEN, P. (1978), The Berry-Esseen theorem and an asymptotic expansion for U-statistics, Ph.D. Thesis, Leuver. (in Dutch).

[49] JAECKEL, L.A. (1971), Robust estimates of location: symmetry and asymmetric contamination, Ann. Math. Statist. 42, 1020-1034.

[50] JUNG, J. (1955), On linear estimates defined by a continuous weight function, Ark. Mat. 3, 199-209.

[51] LOEVÈ, M. (1955), *Probability Theory*, Von Nostrand, Princeton.

[52] LLOYD, E.H. (1952), Least squares estimation of location and scale parameters using order statistics, Biometrika 39, 88-95

[53] MEHRA, K.L. & M.S. RAO (1975), On functions of order statistics for mixing processes, Ann. Stat. 3, 874-883.

[54] MOORE, D.S. (1968), An elementary proof of asymptotic normality of linear combinations of order statistics, Ann. Math. Statist. 39, 263-265.

[55] PETROV, V. (1972), *Sums of independent Random Variables*, Springer-Verlag, Berlin.

[56] PFANZAGL, J. (1972), The Berry-Esseen bound for minimum contrast estimators, Metrika 17, 82-91.

[57] PFANZAGL, J. (1973), Asymptotic expansions related to minimum contrast estimators, Ann. Stat. 1, 993-1026.

[58] PFANZAGL, J. (1974a), Asymptotically optimum estimation and test procedures, Proc. Prague Symp. on Asymptotic Statistics, Vol. I, 201-272.

[59] PFANZAGL, J. (1974b), Nonexistence of tests with deficiency zero, Preprints in Statistics 8, Univ. of Köln.

[60] PFANZAGL, J. (1979), First order efficiency implies second order efficiency. In *Contributions to Statistics* (Jaroslav Hájek Memorial Volume) 167-196 (J. Jurečková ed), Academia, Prague.

[61] REISS, R.D. (1974), On the accuracy of the normal approximation for quantiles, Ann. Prob. 2, 741-744.

[62] REISS, R.D. (1976), Asymptotic expansions for sample quantiles, Ann. Prob. 4, 249-258.

[63] REISS, R.D. (1977), The asymptotic normality and asymptotic expansions for the joint distribution of several order statistics, Limit Theorems of Probability Theory. Colloquia Mathematica Societatis János Bolyai, 11, 297-340.

[64] ROSENKRANTZ, W. & N. O'REILLY (1972), Application of the Shorohod representation theorem to rates of convergence for linear combinations of order statistics, Ann. Math. Statist. 43, 1204-1212.

[65] RUYMGAART, F.H. & M.C.A. VAN ZUYLEN (1977), Asymptotic normality of linear combinations of order statistics in the non-i.i.d. case, Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen, Series A, Mathematical Sciences Vol. 80 - no. 5, 432-447.

[66] SCHOLZ, F.W. (1974), A comparison of efficient location estimators, Ann. Stat. 2, 1323-1326.

[67] SHORACK, G.R. (1969), Asymptotic normality of linear functions of order statistics, Ann. Math. Statist. 40, 2041-2050.

[68] SHORACK, G.R. (1972), Functions of order statistics, Ann. Math. Statist. 40, 2041-2050.

[69] SHORACK, G.R. (1973), Convergence of reduced empirical and quantile processes with applications to functions of order statistics in the non-i.i.d. case, Ann. Statist. 1, 146-152.

[70] SHORACK, G.R. (1974), Random Means, Ann. Stat. 2, 661-675.

[71] SMIRNOV, N.V. (1944), Limit distributions for the terms of a variational series, (English transl.), Amer. Math. Soc. Transl. Ser. (1), 11, (1962), 82-143.

[72] STIGLER, S.M. (1969), Linear functions of order statistics, Ann. Math. Statist. 40, 770-788.

[73] STIGLER, S.M. (1972), Simon Newcomb, Percy Daniell, and the history of robust estimation 1885-1920, J. Amer. Statist. Assoc. 68, 872-879.

[74] STIGLER, S.M. (1973), The asymptotic distribution of the trimmed mean, Ann. Stat. I, 472-477.

[75] STIGLER, S.M. (1974), Linear functions of order statistics with smooth weight functions, Ann. Stat. 2, 4, 676-693.

[76] TUKEY, J.W. (1960), A survey of sampling from contaminated distributions, Contributions to Probability and Statistics, I. Olkin, ed., Stanford University Press, Stanford CA.

[77] TUKEY, J.W. (1962), The future of data analysis, Ann. Math. Statist. 33, 1-67.

[78] TAKEUCHI, K. & M. AKAHIRA (1976), On the second order asymptotic efficiencies of estimators, Lecture Notes in Mathematics 550, Springer-Verlag, Berlin.

[79] WET, T. DE (1976), Berry-Esseen results for the trimmed mean, South African Journal, Vol. 10, 77-96.

[80] ZWET, W.R. VAN (1964), *Convex transformations of random variables*, Mathematical Centre Tracts 7, Amsterdam.

[81] ZWET, W.R. VAN (1977), Asymptotic expansions for the distribution functions of linear combinations of order statistics, *Proc. Symp on Statistical Decision Theory and Related Topics* II, 421-437, (ed. D.S. Moore & S.S. Gupta).

[82] ZWET, W.R. VAN. (1979), The Edgeworth expansion for linear combinations of uniform order statistics, In. *Proc. Second Prague Symp Asymptotic Statist*. 93-101. (P. Mandl and M. Hušková, eds). North-Holland, Amsterdam.

SUBJECT INDEX

AUTHOR INDEX

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1 T. van der Walt. *Fixed and almost fixed points*. 1963.

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26 European Meeting 1968. *Selected statistical papers, part I*. 1968.

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38 W.A. Verloren van Themaat. *Automatic analysis of Dutch compound words*. 1972.

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46 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 1: the language*. 1973.

47 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 2: the compiler*. 1973.

48 F.E.J. Kruseman Aretz, P.J.W. ten Hagen, H.L. Oudshoorn. *An ALGOL 60 compiler in ALGOL 60, text of the MC-compiler for the EL-X8*. 1973.

49 H. Kok. *Connected orderable spaces*. 1974.

50 A. van Wijngaarden, B.J. Mailloux, J.E.L. Peck, C.H.A. Koster, M. Sintzoff, C.H. Lindsey, L.G.L.T. Meertens, R.G. Fisker (eds.). *Revised report on the algorithmic language ALGOL 68*. 1976.

51 A. Hordijk. *Dynamic programming and Markov potential theory*. 1974.

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81 K.R. Apt, J.W. de Bakker (eds.). *Foundations of computer science II, part 1*. 1976.

82 K.R. Apt, J.W. de Bakker (eds.). *Foundations of computer science II, part 2*. 1976.

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84 H.L.L. Busard. *The translation of the elements of Euclid from the Arabic into Latin by Hermann of Carinthia (?), books vii-xii*. 1977.

85 J. van Mill. *Supercompactness and Wallman spaces*. 1977.

86 S.G. van der Meulen, M. Veldhorst. *Torrix I, a programming system for operations on vectors and matrices over arbitrary fields and of variable size*. 1978.

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100 P.C. Baayen, D. van Dulst, J. Oosterhoff (eds.). *Proceedings bicentennial congress of the Wiskundig Genootschap, part 1.* 1979.

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