Saffman-Taylor streamers: mutual finger interaction in an electric breakdown

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Bunches of streamers form the early stages of sparks and lightning but theory presently concentrates on single streamers or on coarse approximations of whole breakdown trees. Here a periodic array of interacting streamer discharges in a strong homogeneous electric field is studied in PDE approximation in two dimensions. If the period of the streamer array is small enough, the streamers do not branch, but approach uniform translation. When the streamers are close to the branching regime, the enhanced field at the tip of the streamer is close to $2E_\infty$, where E_∞ is the homogeneous field applied between the electrodes. We discuss a moving boundary approximation to the set of PDEs. This moving boundary model turns out to be essentially the same as the one for two-fluid Hele-Shaw flows. In two dimensions, this model possesses a known analytical solution. The shape of the 2D interacting streamers in uniform motion obtained from the PDE simulations is actually well fitted by the analytically known "selected Saffman-Taylor finger". This finding helps to understand streamer interactions and raises new questions on the general theory of finger selection in moving boundary problems.

I. INTRODUCTION

Streamers are growing ionized fingers that appear in electric breakdown whenever non-ionized matter is suddenly exposed to strong electric fields, therefore they are very common in nature and technology in gases, liquids and solids [1, 2, 3]. They occur for instance in early stages of atmospheric discharges such as sparks and lightning or in sprite discharges high above thunderclouds [4, 5, 6, 7, 8]. Streamers are characterized by a thin space charge layer around their tip that enhances the local electric field; this enhanced field in turn creates a very active impact ionization region.

Most experiments produce many streamers, certainly when the emitting electrode is a long wire [9], and frequently also when it is the point of a needle [10]. Simulations, on the other hand, concentrate almost exclusively either on single streamers within a microscopic discharge model, or on the complete streamer branching tree in quite coarse phenomenological models. Only in [11], the electrostatic interaction of narrow streamers within a widely spaced streamer array is studied in relatively low electric fields within a microscopic model; as the streamer radius is fixed, the numerical implementation is essentially one-dimensional. In the present paper, we mimic a similar periodic array of identical parallel streamers, but in a higher field, see Fig. 1. Furthermore, rather than fixing radius and shape of the streamers a priori, we let it emerge dynamically within the simulation. Such arrays of streamers can be created experimentally by an array of needles inserted into a plate electrode [12, 13]. Bunches of parallel streamers have also been observed in sprite discharges above thunderclouds [6, 8].

The problems addressed in this paper are multiple: What is the charge distribution and velocity of an array of streamers, depending on their distance and on the applied electric field? Do they approach a state of uniform translation, in contrast to single streamers? And

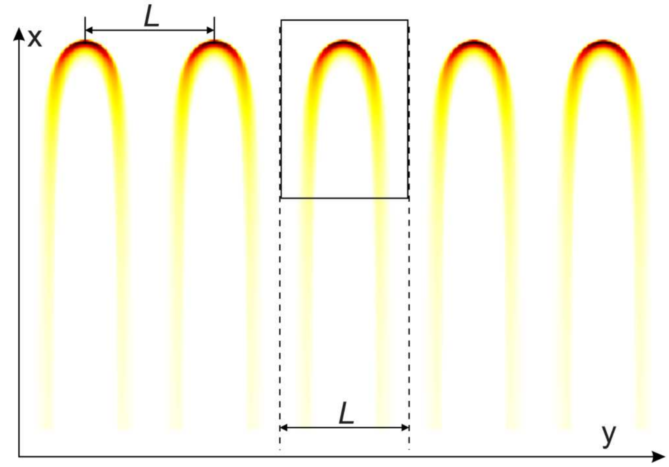

FIG. 1: Periodic array of negative streamers (net charge density) in a strong homogeneous background electric field E_∞ pointing downwards. L is the period of the array. The dash lines represent two symmetry lines. The box around a part of the central streamer indicates the part presented in Fig. 3.

how can the dynamical evolution of their shape be placed in the context of other moving boundary problems in nature? Giving already a major conclusion of the paper, we find that uniformly translating streamer arrays in the microscopic discharge model in two spatial dimensions are very well fitted by a classical solution of two-fluid-flow [14], namely by the so-called selected Saffman-Taylor finger [15], cf. Fig. 3. Therefore the velocity of the streamer array is about twice the electron drift motion in the background field, and their diameter approaches half the period of the array. The observation also raises a theoretical question on pattern selection, namely why the same finger shape is selected in the hydrodynamic and in the discharge problem, given the fact that the problems are similar but not identical.

The paper is organized as follows: Sections II A and II B introduce the minimal PDE model for streamers adapted to describe the evolution of an array of streamers. The general behaviour and properties of these interacting streamers are discussed in Sec. II C. Section III A presents a moving boundary approximation of the minimal model used in the simulations. In Sec. III B we present an analytical solution of this approximation for the shape of the streamer. This solution is known as the selected Saffman-Taylor finger and fits well the charge distribution of the streamer, cf. Fig. 3. In Sec. III C we briefly discuss some open issues related to this boundary observation. We conclude by shortly summarizing and discussing our study in Sec. IV.

II. DENSITY APPROXIMATION AND SIMULATION RESULTS

A. Minimal streamer model

We analyze negative streamers in simple media like pure nitrogen within the minimal streamer model [1, 16, 17, 18, 19, 20] that includes electron diffusion and drift in a self-consistent electric field, while ions are taken as immobile due to their much larger mass. New charge carriers are generated by an impact ionization term in Townsend approximation that depends nonlinearly on the local electric field. In dimensionless units, the model is

$$\partial_t \sigma \;=\; D\nabla^2\sigma + \nabla\cdot(\sigma\mathbf{E}) + \sigma\,|\mathbf{E}|\,e^{-1/|\mathbf{E}|}, \qquad (1)$$

$$\partial_t \rho \;=\; \sigma\,|\mathbf{E}|\,e^{-1/|\mathbf{E}|}, \qquad (2)$$

$$\nabla^2\phi \;=\; \sigma - \rho, \quad \mathbf{E} = -\nabla\phi, \qquad (3)$$

where σ and ρ are the electron and ion densities, ϕ is the electrostatic potential, $\mathbf{E}$ the electric field, and D is a diffusion coefficient, taken as $D = 0.1$ [3, 19]. The intrinsic length scale of the model is the mean free path of an electron between two ionizing collisions in fields $|\mathbf{E}| \gg 1$, for nitrogen at standard temperature and pressure, it is $2.3\,\mu$m; the scale of time is 3 ps and the scale of the electric field is ≈ 200 kV/cm in this case. A general discussion of dimensions can be found, e.g., in [3, 19, 21]. There it is argued that the main advantage of working with dimensionless quantities is that all basic results are immediately generalized to any gas pressure, temperature and composition.

The model is solved numerically on adaptively refined comoving grids as described in detail in [22]; the finest grid in our simulations was $1/4$.

Notice that the only ionization source in our model is impact ionization. We assumed this for the sake of simplicity and in order to emphasize the elementary processes that are common between streamers and two-phase hydrodynamic systems, as will be discussed below. In exchange for this simplicity we restricted ourselves to negative streamers in media where photo-ionization is absent or negligible, such as pure nitrogen, argon or GaAs. Moreover, in [21] it was shown that under certain conditions the effects of photo-ionization on the propagation of negative streamers are negligible even in ambient air.

B. Implementing an array of streamers

Another simplifying assumption is the restriction of the problem to two-dimensions. Indeed, we believe that the main characteristics of the dynamics of the interacting streamers in two-dimensions, as described below, will be qualitatively the same as those in three-dimensions. This is supported by past simulations of single streamers in two-dimensions [23] which are qualitatively very similar to three-dimensional simulations [21]. Moreover the generalization to a three-dimensional geometry is not straightforward since the numerical implementation of streamers in a 2D periodic lattice is non-trivial due mainly to the implementation of the boundary conditions. Another reason to consider only a two-dimensional geometry is that it allows the construction of moving boundary approximations with a known analytical solution. The agreement between that explicit solution and the actual shape of the front is remarkable and detailed below. However, two-dimensional streamers are not just interesting from an academic point of view, they also occur in experiments in thin semiconductor wafers [24].

An infinite, periodic array of streamers can be reduced to the simulation of a single streamer in a channel with Neumann conditions on the lateral boundaries. This is done as follows. If the streamer is centered at $y = 0$ and propagates along the x direction, and if the period of the streamer array is L, there are two symmetry lines at $y = \pm L/2$ where all normal derivatives vanish; therefore Neumann conditions for potential and electron density $\partial_y\phi = 0$, $\partial_y\sigma = 0$, at $y = \pm L/2$ can substitute the other streamers in the array.

This array of streamers is now studied in a constant electric field $\mathbf{E} = -E_\infty\hat{\mathbf{x}}$ far ahead of the streamers; this field is imposed as an inhomogeneous Neumann boundary condition on ϕ at the boundary at $x \gg 1$ while at $x = 0$ the electrostatic potential is fixed. These conditions can be used when planar electrodes are first charged and then insulated; for more general electric circuits, they also approximate streamers that are much shorter than the inter-electrode distance. For the particle densities, we used homogeneous Neumann boundary conditions on all boundaries. As initial conditions we used in this paper an electrically neutral Gaussian seed centered at $(x, y) = (0, 0)$ of width 16 and height $1/4.7$, except in Fig. 6b, where the width is larger but the total number of particles is not changed. We verified that the same attractor of the dynamics is approached after sufficiently long time when lateral position, width and height of the initial seed were varied.

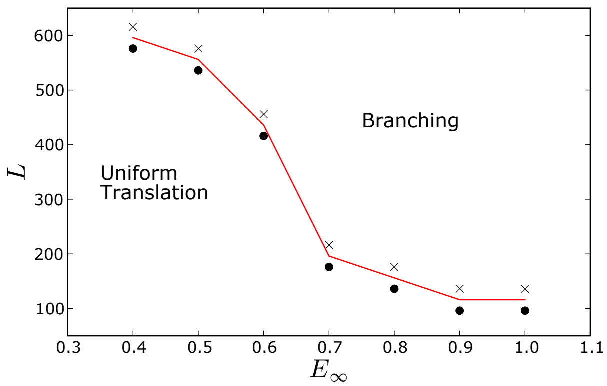

FIG. 2: Simulated streamers that branch (×) or translate uniformly (•) as a function of the period L of the array and of the uniform field E_∞ ahead of it. The line interpolates the phase transition.

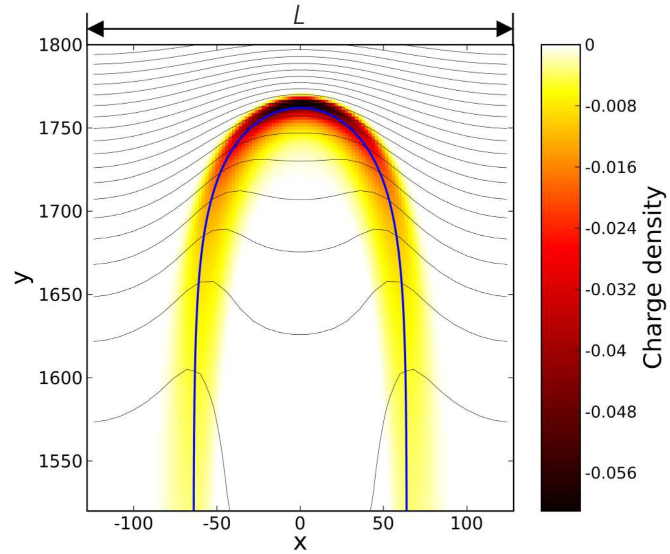

FIG. 3: (Color online) The thin space charge layer $\rho - \sigma$ around the uniformly translating streamer discharge (density color coded) with the Saffman-Taylor finger of width $L/2$ superimposed (thick solid line). L is the width of the Hele-Shaw cell for the Saffman-Taylor finger or the period of the array for streamers, the lateral boundaries then being lines of mirror symmetry between the streamers. Here $L = 256$, and the electric field far ahead is $E_\infty = 0.5$; this corresponds to $\simeq 0.059$ cm and $\simeq 100$ kV/cm for nitrogen under normal conditions. Equipotential lines are also plotted (thin solid lines).

C. Branching versus uniform translation

After a transient evolution, the simulated streamers either reach a state of uniform translation, i.e. they propagate with constant velocity and unchanged shape, or they branch like single streamers [25, 26]. Two parameters control the two regimes for the evolution of the streamers: the period of the array, L, and the background electric field applied between the electrodes, E_∞. Fig. 2 shows a phase diagram spanned by the electric field E_∞ and the spatial period L; here $L = 96$ to 616 was explored in steps of $\Delta L = 40$ and $E_\infty = 0.4$ to 1.0 in steps of $\Delta E_\infty = 0.1$. Below the transition line, i.e., for small period L, the proximity of the other streamers suppresses branching and the whole streamer array propagates uniformly after some transient stage, while above the line the streamers branch eventually. We remark that in general, there can be uniformly translating solutions in the part of the phase diagram marked as “branching”; however, the set of initial conditions for which those solutions emerge (their basin of attraction) is so small that they are not reached from our initial conditions.

We now analyze in detail the uniformly translating streamer array that emerges for sufficiently small E_∞ and/or L (the lower part of the phase diagram, see Fig. 2). After initial transients of duration $t \cong 100$ or less, these streamer heads reach a constant velocity and a constant shape for the rest of the evolution: this is the attractor of the dynamics, namely, the solution reached after a sufficiently long evolution from a large set of initial conditions. Therefore this attractor does not depend on the particular choice of the initial seed used. Only the transient evolution and its duration can depend on this choice. But all these various transient regimes lead to the same final uniformly translating state, with the same shape and velocity. Fig. 3 shows the space charge distribution $\rho - \sigma$ of the attractor for $E_\infty = 0.5$ and $L = 256$ at the time $t = 1800$ (long after the transient evolution ended). Figure 4 shows the electric field and the net charge profiles from the same simulation. The electric field along the streamer axis ($y = 0$) is presented at times 1400, 1600 and 1800 together with the net charge density at time 1800. The three profiles of the electric field show that the propagation indeed is uniform. The thin space charge layer creates a strong field enhancement immediately ahead of the ionization fronts like in a freely propagating streamer. However, behind the space charge layer, the electric field profile inside the streamer array shows characteristic differences to the field profile within a single streamer [21, 22, 23, 25, 26]. Immediately behind the space charge layer, the electric field decays very rapidly like in a single streamer. Then a transition to a slower field decay sets in. Finally, far behind the streamer head, the electric field vanishes completely, in contrast to the nonvanishing residual field inside a single streamer. These observations require further studies. However, one conclusion can already be drawn by applying the Poisson equation $\nabla \cdot \mathbf{E} = \rho - \sigma$ to the streamer head front as a whole. As the field has a constant value $-E_\infty$ far ahead of the streamer array and vanishes far behind the streamer heads, the streamer heads must carry an average charge $-E_\infty$ per unit area, i.e., each streamer head must carry a total charge overshoot of $-E_\infty \cdot L$ to collectively screen the electric field completely behind the array of heads.

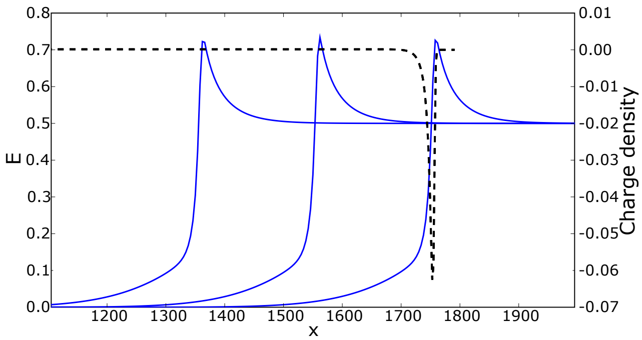

FIG. 4: Absolute value of the electric field (solid lines) for times $t = 1400$, 1600, and 1800, and space charge density (dotted line) at time $t = 1800$ on the streamer axis for $L = 256$ and $E_\infty = 0.5$. Field and density at time $t = 1800$ correspond to the uniformly translating finger in Fig. 3.

These properties of an array of streamers contrast strongly with those of a single streamer, discussed extensively in [23]. For example, a single streamer in a strong homogeneous background electric field never reaches a state of uniform translation. The radius of curvature of the head of the streamer expands during its motion up to the time where instabilities grow and branching occurs. Furthermore, the electric field inside of a single streamer is not as perfectly screened. There are, therefore, remarkable qualitative changes in the propagation of a streamer when the interaction of neighbouring streamers is significant. These effects are expected to persist also in the three-dimensional case.

However, global considerations on the charge content of the streamer head do not fix the shape of the finger and the spatial charge distribution within each uniformly translating streamer head. These density distributions and the consecutive field enhancement and velocity are problems of dynamical selection that will be addressed in the remainder of the paper.

III. MOVING BOUNDARY APPROXIMATION AND SAFFMAN-TAYLOR SOLUTION

A. Moving boundary approximation

As shown in Figs. 1 and 3, after a sufficiently long evolution, during the steady evolution of the streamer, the width of the ionization front can be much smaller than its radius of curvature. Similarly to other pattern forming systems, such as solidification fronts, this separation of scales enables one to consider the front as an infinitesimally thin, sharp moving interface. The original nonlinear dynamics is then replaced by a set of linear field equations (typically Laplace) on both sides of the interface, with appropriate boundary conditions at the interface and further away from it. The interface dynamics is then typically related to gradients of the Laplacian fields at its vicinity.

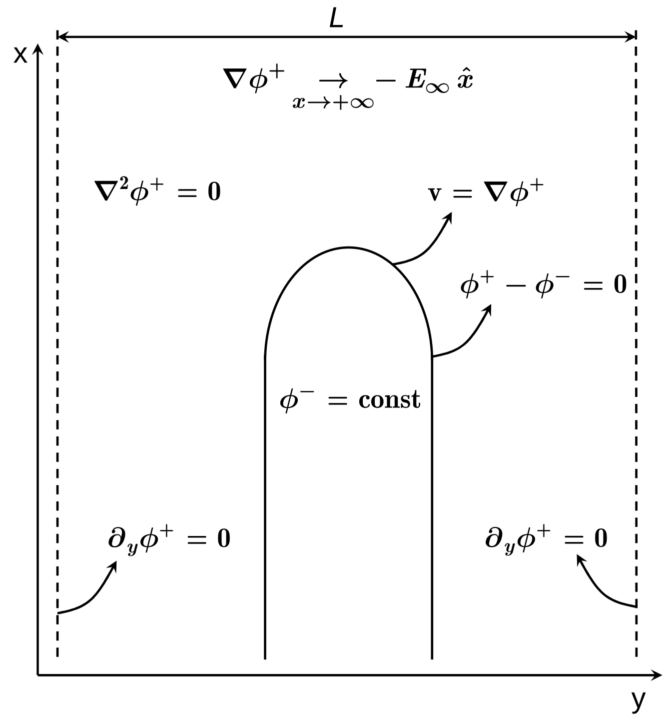

FIG. 5: Schematic view of the mathematical setup for the moving boundary approximation. ϕ^+ and ϕ^- stand for the electric potential outside and inside the streamer respectively.

In [23, 25, 27, 28, 29], a moving boundary approximation was proposed and elaborated for the thin space charge layer and associated sharp ionization front that separates the ionized from the non-ionized region. This model was proposed to describe the evolution of a single streamer but it is straightforward to adapt it to the evolution of an array of streamers since we just need to add homogeneous Neumann boundary conditions on the symmetry lines. See Fig. 5 for a schematic view of the mathematical setup of the moving boundary approximation.

The non-ionized, electrically neutral region outside the streamer is fully described by $\nabla^2\phi = 0$, and $\phi \to \phi_0 + E_\infty x$ fixes the homogeneous field E_∞ far ahead of the streamers at $x \gg 1$. The symmetry line between two streamers is represented by a Neumann boundary condition for the electric potential, $\partial_y\phi = 0$, at $y = \pm L/2$. If the boundary motion is approximated by the local electron drift velocity $\mathbf{v} = \nabla\phi$, the interior of the streamer as ideally conducting $\phi = \text{const.}$ (where the constant can be set to 0 due to electrostatic gauge invariance), and the electric potential across the boundary as continuous, we arrive precisely at the unregularized moving boundary problem for a Saffman-Taylor finger after simply substituting the electric potential ϕ by the pressure field p. This is a classical problem where a very viscous fluid is penetrated by a much less viscous one within the narrow spacing of a Hele-Shaw cell.

B. Comparison with the Saffman-Taylor solution

An explicit uniformly translating solution for this moving boundary problem was found long ago by Saffman and Taylor [15]. The solution for the interface $x = x(y,t)$ in a channel of width L is given by

$$x = \frac{L(1-\lambda)}{2\pi} \ln\left[\frac{1}{2}\left(1+\cos\left(\frac{2\pi y}{\lambda L}\right)\right)\right] + vt, \qquad (4)$$

where the velocity is $v = E_\infty/\lambda$ in our notation and the field at the tip is enhanced by a factor $1/\lambda$. The parameter λ is the ratio between the width of the finger and the width of the channel; λ can take any value between 0 and 1, parametrizing a continuous family of finger solutions. However, experiments only showed fingers with $\lambda = 1/2$. This selection problem was understood only three decades later by different groups [30, 31, 32, 33, 34]. They included surface tension into the boundary condition for the pressure p on the interface. This boundary condition also prevents cusp formation within a finite time [35, 36]; this leads to a regularized moving boundary problem. It was shown, using expansion beyond all orders and reduction to a nonlinear eigenvalue problem, that in the limit of small surface tension only the finger with $\lambda = 1/2$ is stable. Recently it was found that the so-called "kinetic undercooling" boundary condition also leads to regularization and dynamical selection of the Saffman-Taylor finger with width $\lambda = 1/2$ for infinitesimally weak regularization [37]. We recently have proposed a similar regularization mechanism for streamers [23, 28].

We therefore have superimposed the Saffman-Taylor finger with width $L/2$ as a solid line on the streamer in Fig. 3. The agreement is convincing. In Fig. 6, the comparison is further elaborated for three different cases, where (b) differs from (a) by the initial density distribution, and (c) by E_∞ and L. Here solid lines represent stages of evolution of the density model from initial transients to uniform translation; they indicate the position $y(x)$ of the maximal charge density for every x. The dashed lines are the Saffman-Taylor solution (4) with the selected width $\lambda = 1/2$. No adjustments are possible, except for an arbitrary translation of the Saffman-Taylor finger along the x-axis. This is chosen to overlap with the latest stage of the density evolution which is the attractor of the dynamics (at a later stage the shape of the front stays identical and it moves at constant speed). Again the agreement is very convincing. A direct consequence of this agreement is that we expect the field to be enhanced by a factor of 2 immediately ahead of the front, and the finger velocity to be $2E_\infty$, independently of the values of L and E_∞. Indeed we observe that this value of the enhanced field is reached when the moving boundary approximation is most accurate, i.e. when the width of

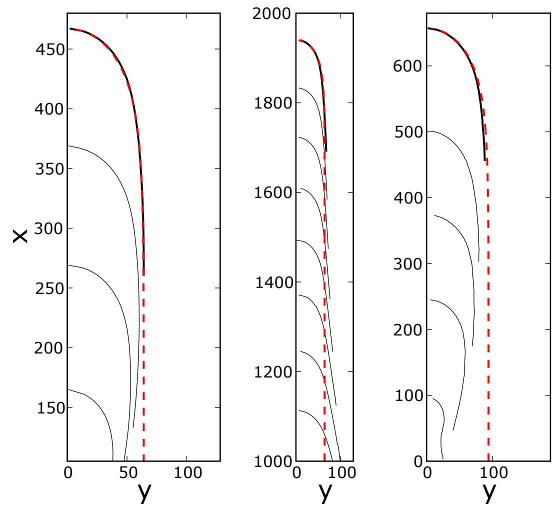

FIG. 6: Solid lines: contour lines characterizing the simulations at time steps of $\Delta t = 100$; dashed curves: the uniformly translating Saffman-Taylor finger solution (4) with $\lambda = 1/2$. (a) background electric field $E_\infty = 0.5$ and width $L = 256$ with an initial seed smaller than the steady state solution (same as in Fig. 3), (b) same as in (a) but with an initial seed wider than the asymptotic solution (here it takes longer to reach uniform translation) and (c) $E_\infty = 0.6$, $L = 376$.

the space-charge layer is much smaller than the radius of curvature of the front. The former is rather independent on L and E_∞, while the latter is of order L, since the width of streamer approaches $L/2$. However, since large L also leads to branching, this behavior is observed only for parameters slightly to the left of the phase-separation curve of Fig. 2.

C. Open problems for boundary analysis

However, this apparently very successful interfacial model relies on four approximations.

1. For the front motion, the electron drift velocity v in the local electric field E is increased by a diffusion-reaction correction [19]. The present simulations show that the streamer velocity in the maximal electric field E^+ can be linearly interpolated by $v = 1.312\, E^+ + 6 \cdot 10^{-4}$ within the explored field range, giving values closely below the analytical result $v = |E| + 2\sqrt{D|E|e^{-1/|E|}}$ for planar fully relaxed fronts [19]. Such a velocity correction $v = c\, E^+$ can be absorbed completely into rescaling time with c.

2. The streamer interior is not field free immediately behind the ionization front as Fig. 4 shows. Consequently, in contrast to the prediction of the moving boundary approximation the front obtained from

the minimal streamer model (1)-(3) is not completely equipotential, as Fig. 3 shows.

3. The space charge layer has a finite width. As a consequence, it also can be seen in this figure that the electric field is not enhanced by a factor of 2 but somewhat less, while the interface position agrees very well.

4. The interfacial approximation breaks down at the sides of the streamer finger where the local electric field $\mathbf{E}$ is too low to sustain substantial ionization, $e^{-1/|\mathbf{E}|} \ll 1$, while the interface between two fluids in the Saffman-Taylor finger, of course, continues along the whole channel length. Therefore the mathematical similarity between Saffman-Taylor fingers and streamer fingers holds only close to their tips, while the analytical construction of fingers requires their whole length.

These observations pose new challenges to the theoretical understanding of finger selection in moving boundary problems.

IV. SUMMARY AND CONCLUSIONS

This paper presents, up to our knowledge, the first studies on the full dynamics of multiple interacting streamers. By using a simplified but physically relevant model, we were able to focus on the main effects of the interaction and stress the most general electro-dynamic properties of a bunch of streamers. We obtained a phase diagram spanned by the electric field E_∞ and the spatial period L, see Fig. 2. For L and/or E_∞ large enough, the streamers branch similarly to single streamers. For L and/or E_∞ small enough, the streamers do not branch and approach the width $L/2$. Furthermore, we used a moving boundary approximation to derive surprisingly accurate predictions. We showed that close to the braching line of the phase diagram, the enhanced field at the tip of the streamer is close to $2E_\infty$, where E_∞ is the background electric field applied between the electrode. Moreover we showed that the shape of the front is well fitted by the selected Saffman-Taylor finger derived analytically from the moving boundary approximation.

Certainly there are still many open questions about this topic. Further investigations should extend our model to three spatial dimensions and to a wider variety of media, including nonlocal ionization mechanisms [21]. A rigorous analysis of the problem of finger selection in this context of interacting streamers would also prove valuable both for the pattern formation community and for an improved understanding of streamers.

Acknowledgments

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