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CONTENTS

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INTRODUCTION

The spectral theorem for bounded selfadjoint (and normal) operators in a Hilbert space is a classical result that can be proved in many ways. The most interesting proof is certainly that which makes use of the theory of C^*-algebras. This algebraic approach to spectral theory is very powerfull: many theorems can be proved in a relatively simple way.
The spectral theorem for an unbounded selfadjoint operator is usually derived from the corresponding theorem for bounded operators. An algebraic spectral theory for unbounded operators was not available.
It is our aim to fill up this gap in the theory.

The idea is the following. Instead of considering unbounded operators in a Hilbert space, we consider these operators in a suitable locally convex space in which they act continuously.
This idea is very well-known in the theory of partial differential operators. Instead of considering a partial differential operator in an L^2-space, one considers this operator in a space of test functions or in a space of distributions (or in a chain of Sobolev spaces). In an abstract form this idea was already used by J. SEBASTIÃO E SILVA [31]
So we were led to consider algebras of continuous operators in a locally convex space. In this way locally convex algebras enter into the picture.

Locally convex algebras were examined by many authors. We mention the work of WAELBROECK [33], MARINESCU [18], NEUBAUER [21], ALLAN [1] and [3], DIXON [7] and MOORE [20].
L. WAELBROECK [33] developed a functional calculus for elements of a locally convex algebra. The notion of boundedness plays an important role in Waelbroeck's work.
G. R. ALLAN [1] introduced the notion of a bounded element of a locally convex algebra. He took this notion as a starting point for a spectral theory for locally convex algebras. Furthermore, Allan introduced a new type of locally convex algebras, namely the GB^*-algebra (cf. Allan [3]), which is a

locally convex analogon of a C^*-algebra. Allan developed a Gelfand theory for commutative GB^*-algebras. Further investigations were made by P. G. DIXON [7].

In this thesis we shall show how the theory of GB^*-algebras can be applied to the spectral theory of unbounded selfadjoint operators in a Hilbert space. We were inspired by the work of MOORE [20], who studied spectral theory of operators in a locally convex space and who used formally the same methods as Allan did.

In Chapter 1 we develop the framework which is needed in order to apply the theory of Allan.
Let H_0 be a Hilbert space and let T be an unbounded selfadjoint operator in H_0 with dense domain D(T). Then we consider the space

$$H_\infty := \bigcap_{k=1}^{\infty} D(T^k) .$$

This space can be equipped with a locally convex topology such that $T : H_\infty \longrightarrow H_\infty$ is continuous. The relation between the spaces H_0 and H_∞ is studied. Here triples and chains of Hilbert spaces play a central role.

In Chapter 2 we give a brief survey of Allan's theory on locally convex algebras. In particular attention is paid to the Gelfand theory for commutative GB^*-algebras.

In Chapter 3 the results of Chapter 1 and 2 are applied in order to obtain an algebraic spectral theory for unbounded selfadjoint operators. The idea is as follows.
Let T be an unbounded selfadjoint operator in a Hilbert space H_0. We consider T as an element of $L(H_\infty)$, the algebra of all continuous linear operators in H_∞, and we prove that the bicommutant of T in $L(H_\infty)$ is a commutative GB^*-algebra. Then we study the spectral theory of this algebra. Among other things we obtain a spectral representation theorem for elements of the bicommutant of T. This result is used to derive the spectral theorem for unbounded normal operators in a Hilbert space (Cor. 3.5.6). An application of the theory is given in section 3.6 where we study the spectrum of tensor product operators.
In this context we mention the work of POWERS [24], [25] and POULSEN [26] who studied the representation theory of algebras of unbounded operators. There is some resemblance between their methods and the techniques that we use in Chapter 3.

The second main theme of this thesis is formed by an algebraic theory of generalized eigenvectors (Chapters 4 and 5).

In Chapter 4 we have collected some well-known facts which are need in Chapter 5 (direct integrals of Hilbert spaces and disintegration of measures).

In Chapter 5 we are interested in generalized eigenprojections and generalized eigenvectors associated with a spectral measure. Our starting point is the well-known theorem due to VON NEUMANN concerning the direct integral decomposition of a Hilbert space with respect to a spectral measure. The central result describes the relation between the generalized eigenvectors corresponding to a spectral measure E(.) and the generalized eigenvectors corresponding to a spectral measure F(.) which is the image of E(.) under some continuous mapping. We indicate how this result can be applied to tensor product operators.

Notation

The identity operator in a linear space is denoted by I.
If T is a linear operator, then
D(T) denotes the domain of T and
R(T) denotes the range of T.

CHAPTER I

TRIPLES AND CHAINS OF HILBERT SPACES

In this chapter we consider triples and chains of Hilbert spaces. The first section is introductory; here we introduce the notion of a anti-dual space and the notions of right and left anti-transposed maps.
In section 1.2 the definition of a triple of Hilbert spaces is given. This concept is related to that of a space with "negative norm" which was introduced by LAX [14]. In theorem 1.2.4 we give a characterization of triples.
In section 1.3 we consider selfadjoint operators related to a triple. We make use of a theorem due to BEREZANSKII for which we give a new proof (cf. Theorem 1.3.1). The main result of this section is Theorem 1.3.4.
The definition of a chain of Hilbert spaces is presented in section 1.4. Informally speaking one can say that a chain is built up by triples. Another definition of a chain of Hilbert spaces is given by PALAIS ([22], Ch.VIII, §1). PALAIS' definition is more general, but it is not as rich as our definition. The main result of section 1.4 is Theorem 1.4.3, which is an extension of Theorem 1.3.4.
In section 1.5 we introduce the limit spaces associated with a chain and we study the duality in a chain. In theorem 1.5.2 we establish the anti-duality between the limit spaces.
Finally, in section 1.7 tensor products of triples and chains are studied. The results of this chapter will be used in Chapter 3 where we consider algebras of unbounded operators.

1.1 ANTI-DUAL SPACES

Let E be a Hilbert space over $\mathbb{C}$ with inner product $(.,.)_E$ and norm $\| . \|_E$. The linear space of all continuous anti-linear mappings $f : E \to \mathbb{C}$ is called the anti-dual space of E and is denoted by E'. The canonical mapping $\phi_E : E \to E'$ is defined by

$$(\phi_E(u))(v) := (u,v)_E \qquad (u,v \in E).$$

By the representation theorem of RIESZ, ϕ_E is surjective. Furthermore, E' is a Hilbert space with inner product

$$(\phi_E(u),\phi_E(v))_{E'} := (u,v)_E \qquad (u,v \in E).$$

So ϕ_E is a linear isometry onto.
Let E" be the anti-dual of E'. If we define $\gamma_E : E \to E''$ by

$$(\gamma_E(u))(f) := \overline{f(u)} \qquad (f \in E',\ u \in E),$$

then γ_E is also a linear isometry onto.
Let F be another Hilbert space over $\mathbb{C}$. The space of all continuous linear mappings from E into F is denoted by L(E,F). For $T \in L(E,F)$ the *anti-transposed* T' of T is the map $T' : F' \to E'$ defined by

$$T'(f) := f \circ T \qquad (f \in F').$$

1.1.1 LEMMA. *Let* E *be a Hilbert space. Then*
(i) $\phi_{E'} = (\phi_E^{-1})'$,
(ii) $\gamma_E = \phi_{E'}\phi_E$.

PROOF.
(i) Take $f,g \in E'$. Then $f = \phi_E(u)$ and $g = \phi_E(v)$ for some $u,v \in E$. Then $(\phi_{E'}(f))(g) = (f,g)_{E'} = (u,v)_E$. On the other hand, $(\phi_E^{-1})'(f)(g) = f(\phi_E^{-1}(g)) = (\phi_E(u))(v) = (u,v)_E$. So $\phi_{E'} = (\phi_E^{-1})'$.
(ii) Let $u \in E$ and $f \in E'$. Then $f = \phi_E(v)$ for some $v \in E$. Then $(\gamma_E(u))(f) = \overline{f(u)} = \overline{\phi_E(v)(u)} = (u,v)_E$ and $(\phi_{E'}\phi_E)(u)(f) = (\phi_E(u),f)_{E'} = (u,v)_E$. So $\gamma_E = \phi_{E'}\phi_E$. □

Many theorems for dual spaces and transposed maps also hold for anti-dual spaces and and anti-transposed maps.
The following lemma is easily proved.

1.1.2 LEMMA. *Let* E *and* F *be Hilbert spaces and let* $T \in L(E,F)$.
(i) If T^{-1} *exists and belongs to* L(F,E), *then* $(T')^{-1}$ *exists and* $(T')^{-1} \in L(E',F')$; *moreover* $(T^{-1})' = (T')^{-1}$.

(ii) *Let* $T'' : E'' \to F''$ *be the anti-transposed of* T'*; then* $T'' = \gamma_F T \gamma_E^{-1}$.
(iii) *Let* $T^* : F \to E$ *be the Hilbert space adjoint of* T*; then* $T^* = \phi_E^{-1} T' \phi_F$.

1.1.3 DEFINITION. Let E and F be Hilbert spaces and let $T \in L(E,F)$. The *left anti-transposed* of T is the map $T'_{\ell} : F' \to E$ defined by

$$T'_{\ell} := \phi_E^{-1} T' ;$$

and the *right anti-transposed* of T is the map $T'_{r} : F \to E'$ defined by

$$T'_{r} := T' \phi_F .$$

1.1.4 LEMMA. *Let* E *and* F *be Hilbert spaces and let* $T \in L(E,F)$. *Then*
(i) $(T'_{\ell})'_{r} = \gamma_F T$ *and* $(T'_{r})'_{\ell} = T\gamma_E^{-1}$
(ii) $(T'_{\ell})'_{\ell} = (T'_{r})'_{r}$
(iii) $(T'_{\ell})^* = \phi_F T$ *and* $(T'_{r})^* = T\phi_E^{-1}$.

PROOF. We prove the first statement.
Since $T'_{\ell} = \phi_E^{-1} T' : F' \to E$, we have $(T'_{\ell})'_{r} : E \to F''$ and

$$\begin{aligned}(T'_{\ell})'_{r} = (\phi_E^{-1} T')' \phi_E &= T'' (\phi_E^{-1})' \phi_E = \\ &= T'' \phi_{E'} \phi_E = \\ &= T'' \gamma_E = \\ &= \gamma_F T\end{aligned}$$

(apply 1.1.1 and 1.1.2).
The remaining statements are proved in the same way. □

1.1.5 LEMMA. *Let* E *and* F *be Hilbert spaces and let* $T \in L(E,F)$. *Then*

$$TT^* = (T'_{r})^* T'_{r}$$

and

$$T^*T = (T'_{\ell})(T'_{\ell})^* .$$

PROOF. $(T'_{r})^* T'_{r} = T\phi_E^{-1} T' \phi_F = TT^*$ (apply 1.1.4 and 1.1.2). □

1.2 TRIPLES OF HILBERT SPACES

As an introduction to our definition of triple (Def. 1.2.1) we consider the following situation (cf. BEREZANSKII [4], Chapter 1 and LAX [14]).

Let H_0 be a Hilbert space (with inner product $(.,.)_0$ and norm $\| \, . \, \|_0$) and let H_1 be a dense linear subspace of H_0 which is itself a Hilbert space with inner product $(.,.)_1$ and norm $\| \, . \, \|_1$ such that $\| u \|_0 \leq \| u \|_1$ for all $u \in H_1$. Let

$$i : H_1 \hookrightarrow H_0$$

be the inclusion map. We introduce a third space H_{-1} and an embedding $j : H_0 \hookrightarrow H_{-1}$ as follows.
Let H_{-1} be the anti-dual space of H_1 and let $\phi : H_1 \to H_{-1}$ be the canonical map, i.e. $(\phi(u))(v) = (u,v)_1$ $(u,v \in H_1)$. Then ϕ is unitary, so $\phi^* = \phi^{-1}$.
Define $j : H_0 \to H_{-1}$ by

$$(j(f))(u) := (f,iu)_0 \qquad (f \in H_0,\ u \in H_1).$$

Then $(jf)(u) = (f,iu)_0 = (i^*f,u)_1 = (\phi i^* f)(u)$. So $j = \phi i^*$ and $i = j^*\phi$ (recall that $\phi^* = \phi^{-1}$). Hence $ii^* = j^*j$.
Since i is injective and $R(i)$ is dense, it follows that $R(j)$ is dense and that j is injective. So we can identify H_0 (as a linear space) with the linear subspace $j(H_0)$ of H_{-1}. If $f \in H_0$, then

$$\| jf \|_{-1} = \| i^* f \|_1 = \sup_{0 \neq u \in H_1} \frac{|(f,iu)_0|}{\| u \|_1}.$$

Hence H_{-1} can be considered as the completion of H_0 with respect to the so called "negative norm" $f \mapsto \| jf \|_{-1}$ (cf. LAX [14]).
Now $H_1 \overset{i}{\hookrightarrow} H_0 \overset{j}{\hookrightarrow} H_{-1}$ is called a triple of Hilbert spaces. We shall give another definition of this concept and we prove that our definition is equivalent to the one given above.

1.2.1 <u>DEFINITION</u>. Let E, F and G be Hilbert spaces and let $i : E \hookrightarrow F$ and $j : F \hookrightarrow G$ be dense embeddings of norm ≤ 1 (injective operators with dense range). Then

$$E \overset{i}{\hookrightarrow} F \overset{j}{\hookrightarrow} G$$

is called a *triple of Hilbert spaces* if $ii^* = j^*j$.

1.2.2 PROPOSITION. *If* $E \overset{i}{\hookrightarrow} F \overset{j}{\hookrightarrow} G$ *is a triple of Hilbert spaces then there exists a unique isometry*

$$V : G \longrightarrow E$$

from G *onto* E *such that* $Vj = i^*$ *and* $iV = j^*$.

PROOF. Let $\alpha,\beta \in R(j)$. If $\alpha = jf$ and $\beta = jg$ with $f,g \in F$, then
$(i^*f,i^*g)_E = (f,ii^*g)_F = (f,j^*jg)_F = (jf,jg)_G = (\alpha,\beta)_G$.
Hence the map $i^*j^{-1} : R(j) \to E$ can be extended to an isometry V from G onto E (V is surjective since $R(i^*)$ is dense in E). Clearly $Vj = i^*$. Furthermore, $iV(\alpha) = iVj(f) = ii^*(f) = j^*j(f) = j^*(\alpha)$. So $iV = j^*$. The uniqueness of V is clear. □

COROLLARY. *If* $E \overset{i}{\hookrightarrow} F \overset{j}{\hookrightarrow} G$ *is a triple, then* $R(i) = R(j^*)$.

1.2.3 EXAMPLES. Let E, F and G be Hilbert spaces and let $i : E \hookrightarrow F$ and $j : F \hookrightarrow G$ be dense embeddings (of norm ≤1). Then one can form the following three standard triples.

a) $E \overset{i}{\hookrightarrow} F \overset{i^*}{\hookrightarrow} E$,

b) $E \overset{i}{\hookrightarrow} F \overset{i'_r}{\hookrightarrow} E'$,

c) $G' \overset{j'_\ell}{\hookrightarrow} F \overset{j}{\hookrightarrow} G$.

In example a) we have $V = I$. In b) we have $V = \phi_E^{-1}$ (since $i\phi_E^{-1} = (i'_r)^*$). In c) we have $V = \phi_G$ (since $\phi_G j = (j'_\ell)^*$).

Now we come to the main result which gives a characterization of triples.

1.2.4 THEOREM. *Let* $E \overset{i}{\hookrightarrow} F \overset{j}{\hookrightarrow} G$ *be a triple of Hilbert spaces. This triple can be reduced to any of the three standard triples of* 1.2.3. *This means that there exist unique unitary maps*

$$V : G \longrightarrow E ,$$
$$\Psi : G \longrightarrow E' ,$$
and $$\Phi : E \longrightarrow G'$$

such that (with the notation of 1.2.3*)*

$$Vj = i^* ,$$
$$\Psi j = i'_r ,$$
and $$i = j'_\ell \Phi .$$

Moreover, $\Psi = \phi_E V$ *and* $\Phi = \phi_G V^{-1}$.

PROOF. The existence of V follows from 1.2.2. The mappings Ψ and Φ are defined by

$$\Psi := \phi_E V$$

and

$$\Phi := \phi_G V^{-1} .$$

It is easily verified that $\Psi j = i'_r$ and that $j'_\ell \Phi = i$. □

1.2.5 DEFINITION. Let $E \overset{i}{\hookrightarrow} F \overset{j}{\hookrightarrow} G$ be a triple of Hilbert spaces. Let V, Ψ and Φ be as in 1.2.4. We define a sesquilinear form (.,.) on $E \times G$ as follows.
For $u \in E$ and $\alpha \in G$ (so $\Psi\alpha \in E'$) we define

$$(u,\alpha) := \overline{(\Psi\alpha)(u)} .$$

Then (.,.) is a continuous sesquilinear form on $E \times G$ and

$$|(u,\alpha)| \leq \| u \|_E \, \| \alpha \|_G \qquad (u \in E,\ \alpha \in G).$$

REMARK. Note that

$$(u,\alpha) = \overline{(\Psi\alpha)(u)} = \overline{(\phi_E V\alpha)(u)} =$$

$$= (u,V\alpha)_E = (V^{-1}u,\alpha)_G = (\phi_G V^{-1}u)(\alpha) =$$

$$= (\Phi u)(\alpha) .$$

If $\alpha = jf$ ($\alpha \in G$, $f \in F$), then (u,α) reduces to the inner product $(iu,f)_F$ of iu and f in F. Indeed,

$$(u,\alpha) = (u,V\alpha)_E = (u,i^*f)_E = (iu,f)_F.$$

So if i and j are considered as identifications, one can say that (u,α) reduces to the inner product in F in the case that $\alpha \in F$.

1.3 SELFADJOINT OPERATORS ASSOCIATED WITH A TRIPLE

We start with a simple lemma.

1.3.1 LEMMA. *Let* H *be a Hilbert space and let* $T : H \to H$ *be a bounded hermitian operator with* $0 \le T \le I$ *which means* $0 \le (Tx,x) \le \| x \|^2$ *for all* $x \in H$. *Suppose* T *is injective and has dense range. Then* T^{-1} *is a densely defined operator with* $D(T^{-1}) = R(T)$ *and* T^{-1} *is a selfadjoint operator* $\ge I$, *i.e.* $(T^{-1}x,x) \ge \| x \|^2$ *for all* $x \in D(T^{-1})$.

PROOF. If $x,y \in R(T)$ and $x = Tz$, $y = Tw$, then
$(T^{-1}x,y) = (z,Tw) = (Tz,w) = (x,T^{-1}y)$.
So T^{-1} is symmetric ($T^{-1} \subset (T^{-1})^*$).
Suppose that for some y and w we have $(T^{-1}x,y) = (x,w)$ for all $x \in D(T^{-1})$. Since T^{-1} is surjective, $w = T^{-1}z$ for some $z \in R(T)$. So $(T^{-1}x,y) = (x,T^{-1}z) = (T^{-1}x,z)$, since T^{-1} is symmetric. Hence $y = z \in D(T^{-1})$. This means that T^{-1} is selfadjoint. Since $T \ge 0$, it follows that $(Tx,Tx) \le \| T \| \, (Tx,x)$. Since $0 \le T \le I$ we have $\| T \| \le 1$. Thus $(Tx,Tx) \le (Tx,x)$ for all $x \in H$. Hence $(T^{-1}x,x) \ge \| x \|^2$ for all $x \in D(T^{-1})$. □

Now let E and F be Hilbert spaces and let $i : E \hookrightarrow F$ be a dense embedding of norm ≤ 1. Then $ii^* : F \to F$ is an hermitian operator with $0 \le ii^* \le I$. From the spectral theory for bounded hermitians operators it follows that ii^* has a unique positive square root $(ii^*)^{\frac{1}{2}}$.
Then $(ii^*)^{\frac{1}{2}}$ satisfies the conditions of lemma 1.3.1. Hence

$$A = ((ii^*)^{\frac{1}{2}})^{-1}$$

is a selfadjoint operator $\ge I$. And $A^2 = (ii^*)^{-1}$. Indeed, if $x \in D(A^2)$ and $y = Ax$ and $z = Ay = A^2x$, then $x = (ii^*)^{\frac{1}{2}}y$ and $y = (ii^*)^{\frac{1}{2}}z$, so $x = (ii^*)z$ and $A^2x = (ii^*)^{-1}x$. Since A, A^2 and $(ii^*)^{-1}$ are injective operators with range F, if follows that $A^2 = (ii^*)^{-1}$.

The linear space $D(A)$ equipped with the norm

$$f \longmapsto \| Af \|_F$$

is denoted by D. Then $D(A^2)$ is dense in D. This can be seen as follows. Suppose $f \in D$ is orthogonal (in D) to $D(A^2)$; then $(Af,Ag)_F = 0$ for all $g \in D(A^2)$. Since $R(A^2) = D(ii^*) = F$, it follows that $f = 0$.

The definition of A also appears in BEREZANSKII ([4], Chapter 1). The next theorem can also be found in BEREZANSKII. The proof in [4] is complicated and makes use of the spectral theorem for unbounded selfadjoint operators; our proof is more direct. The uniqueness (cf. 1.3.2) is not proved in [4].

1.3.2 THEOREM. $R(i) = D(A)$ *and* $Ai : E \to F$ *is an isometry from* E *onto* F. *The selfadjoint positive operator* A *is uniquely determined by these properties.*

PROOF. Let $f,g \in R(ii^*) = D(A^2)$. Then

$$(Af,Ag)_F = (A^2f,g)_F = ((ii^*)^{-1}f,g)_F = (i^{-1}f,i^{-1}g)_E.$$

So the mapping

$$i : E \supset R(i^*) \longrightarrow D(A^2) \subset D$$
$$u \longmapsto iu$$

is an isometry. Since $R(i^*)$ is dense in E and $D(A^2)$ is dense in D, it follows that $i : R(i^*) \to D(A^2)$ can be extended continuously to an isometry i_1 from E onto D. We show that this isometrical extension i_1 coincides with the map i when D is considered as a subset of F. For the moment let $i_0 : D \hookrightarrow F$ be the inclusion map. Then

$$i_0 i_1(u) = i(u)$$

for all $u \in R(i^*) \subset E$. Since $R(i^*)$ is dense in E, it follows that $i_0 i_1 = i$. This means that $R(i) = D(A)$ and that $Ai : E \to F$ is an isometry.
The uniqueness is proved as follows. Let B be another selfadjoint positive operator in F such that $R(i) = D(B)$ and $Bi : E \to F$ is an isometry from E onto F.
If $iu \in D(A^2)$, then

$$(Biu,Biv)_F = (Aiu,Aiv)_F = (A^2iu,iv)_F$$

for all $v \in E$. So

$$(Biu,Bf)_F = (A^2iu,f)_F$$

for all $f \in R(i) = D(B)$. This means that $Biu \in D(B^*) = D(B)$ and that $B^2iu = A^2iu$. So $A^2 \subset B^2$. Since A^2 and B^2 are both selfadjoint, we conclude that $A^2 = B^2$. Hence $(B^{-1})^2 = (B^2)^{-1} = ii^*$. Since B^{-1} is the positive square root of B^{-2}, it follows that $A = B$. □

<u>REMARK.</u> Note that we did not use the spectral theorem for unbounded self-adjoint operators.

Now we consider the triple

$$E \overset{i}{\hookrightarrow} F \overset{i^*}{\hookrightarrow} E \ .$$

Let B_0 be the operator

$$B_0 := ((i^*i)^{\frac{1}{2}})^{-1} \ .$$

Then $D(B_0) = R(i^*)$ and

$$B_0 i^* : F \longrightarrow E$$

is an isometry from F onto E.

1.3.3 <u>LEMMA</u>. *The operator* $B_0 i^* Ai$ *is the identity operator on* E *and* $i^*Ai = B_0 i^* i$.

<u>PROOF.</u> The operator i^*Ai is a bounded hermitian operator in E and it is positive. And $(i^*Ai)^2 = i^*Aii^*Ai = i^*i = B_0^{-2}$. Since B_0^{-1} is the positive square root of B_0^{-2}, it follows that $i^*Ai = B_0^{-1}$. Hence $B_0 i^*Ai$ is the identity operator on E. So $(Ai)^{-1} = B_0 i^*$. Since $Aii^*Ax = x$ for all $x \in D(A)$, it follows that $i^*A \subset B_0 i^*$. Hence $i^*Ai = B_0 i^* i$. □

We take now the triple

$$E \overset{i}{\hookrightarrow} F \overset{j}{\hookrightarrow} G \ .$$

Let $V : G \to E$ be the isometry such that $Vj = i^*$. We define

$$B_0 \ := ((i^* i)^{\frac{1}{2}})^{-1}$$

and

$$B \ := ((jj^*)^{\frac{1}{2}})^{-1} .$$

Then $B^{-1} = (V^{-1} i^* iV)^{\frac{1}{2}} = V^{-1}(i^* i)^{\frac{1}{2}} V = V^{-1} B_0^{-1} V$. So

$$B = V^{-1} B_0 V .$$

1.3.4 THEOREM. Ai *is an isometry from* E *onto* F *and* Bj *is an isometry from* F *onto* G. *Moreover,*

$$BjAi = V^{-1}$$

and $jAi = Bji$.

PROOF. The theorem follows from 1.3.3. □

Let us consider two Hilbert spaces E_1 and E_2 and let

$$i_1 : E_1 \hookrightarrow F$$

and

$$i_2 : E_2 \hookrightarrow F$$

be dense embeddings of norm ≤ 1. We shall identify E_1 with its image under i_1 and E_2 with its image under i_2; so E_1 and E_2 will be considered as subspaces of F. Then the linear space $E_1 + E_2$ equipped with the norm

$$\| x \|_{E_1+E_2} \ := \inf\left\{\left(\| x_1 \|_{E_1}^2 + \| x_2 \|_{E_2}^2\right)^{\frac{1}{2}} \middle| x_1 \in E_1,\ x_2 \in E_2,\ x = x_1 + x_2\right\}$$

is a Hilbert space and the identity map $i : E_1 + E_2 \to F$ is a dense embedding of norm ≤ 2. Let A_1 be the positive selfadjoint operator in F such that $D(A_1) = E_1$ and $A_1 : E_1 \to F$ is an isometry from E_1 onto F. Similarly, let A_2 and B be the selfadjoint operators in F corresponding to E_2 and $E_1 + E_2$ respectively. The relation between A_1, A_2 and B is given in the next proposition.

1.3.5 PROPOSITION.

$$B^{-2} = A_1^{-2} + A_2^{-2} .$$

<u>PROOF.</u> Since $B^{-2} = i^*$, $A_1^{-2} = i_1^*$ and $A_2^{-2} = i_2^*$, we have to show that $i^* = i_1^* + i_2^*$.
By $E_1 \oplus E_2$ we denote the direct sum of E_1 and E_2 equipped with the norm

$$\|(x_1,x_2)\| := (\|x_1\|^2 + \|x_2\|^2)^{\frac{1}{2}} \qquad (x_1 \in E_1,\ x_2 \in E_2).$$

Let $A : E_1 \oplus E_2 \to E_1 + E_2$ be the addition map

$$A(x_1,x_2) = x_1 + x_2 \ .$$

It is easy to see that A restricted to the orthogonal complement of the kernel N(A) of A is an isometrical isomorphism.
Let $x \in E_1 + E_2$ and $f \in F$. Take $x_1 \in E_1$ and $x_2 \in E_2$ such that $x = x_1 + x_2$. Then

$$\begin{aligned}(ix,f)_F &= (i_1x_1+i_2x_2,f)_F = \\ &= (x_1,i_1^*f)_{E_1} + (x_2,i_2^*f)_{E_2} = \\ &= ((x_1,x_2),(i_1^*f,i_2^*f))_{E_1\oplus E_2} \ .\end{aligned}$$

It is easily verified that (i_1^*f,i_2^*f) is orthogonal to the kernel N(A) of A. So

$$(ix,f)_F = (x,(i_1^*+i_2^*)f)_{E_1+E_2} \ .$$

We conclude that $i^* = i_1^* + i_2^*$. □

1.4 CHAINS OF HILBERT SPACES

Now we come to our definition of a chain of Hilbert spaces.

1.4.1 <u>DEFINITION.</u> A sequence $\{H_p \mid p \in \mathbb{Z}\}$ of Hilbert spaces together with a sequence of maps $\{i_p \mid p \in \mathbb{Z}\}$ such that

$$i_p : H_{p+1} \hookrightarrow H_p \qquad (p \in \mathbb{Z})$$

is a dense embedding of norm ≤ 1, is called a *chain of Hilbert spaces* if for all $p \in \mathbb{Z}$

$$H_{p+1} \overset{i_p}{\hookrightarrow} H_p \overset{i_{p-1}}{\hookrightarrow} H_{p-1}$$

is a triple of Hilbert spaces.

A chain of Hilbert spaces may be constructed as follows. Let H_1 and H_0 be Hilbert spaces and let $i_0 : H_1 \hookrightarrow H_0$ be a dense embedding of norm ≤ 1. Define by induction

$$H_{p+1} := H'_{p-1} \qquad (p=1,2,3,\ldots)$$

and

$$H_{p-1} := H'_{p+1} \qquad (p=0,-1,-2,\ldots).$$

Furthermore, define

$$H_{p+1} \overset{i_p}{\hookrightarrow} H_p \quad \text{by} \quad i_p := (i_{p-1})'_{\ell} \qquad (p=1,2,3,\ldots)$$

and

$$H_p \overset{i_{p-1}}{\hookrightarrow} H_{p-1} \quad \text{by} \quad i_{p-1} := (i_p)'_{r} \qquad (p=0,-1,-2,\ldots).$$

Note that we use 1.2.3 c) for the definition of H_p in the case $p \geq 1$ and that we use 1.2.3 b) for the definition of H_p in the case $p \leq 0$.

Let us consider a chain of Hilbert spaces $\{H_p \mid p \in \mathbb{Z}\}$. The inner product and the norm in H_p are denoted by $(\cdot,\cdot)_p$ and $\|\cdot\|_p$ respectively ($p \in \mathbb{Z}$). The map

$$i_p : H_{p+1} \hookrightarrow H_p$$

is a dense embedding of norm ≤ 1 ($p \in \mathbb{Z}$). For $p < q$ the map $i_p \circ i_{p+1} \circ \ldots \circ i_{q-1} : H_q \hookrightarrow H_p$ is denoted by $i_{p,q}$. Note that $i_{p,q}$ is also a dense embedding of norm ≤ 1 ($p,q \in \mathbb{Z}$, $p < q$). For $p \in \mathbb{Z}$ the map A_p is defined by

$$A_p := ((i_p i_p^*)^{\frac{1}{2}})^{-1} ,$$

and for $p,q \in \mathbb{Z}$, $p < q$ the map $A_{p,q}$ is defined by

$$A_{p,q} := ((i_{p,q} i^*_{p,q})^{\frac{1}{2}})^{-1} .$$

We want to examine the relation between the maps $A_{p,q}$ and A_p. We need the following lemma.

1.4.2 LEMMA. *Let* $p, q \in \mathbb{Z}$ *and* $p < q$. *Then*

(i) $i_{p,q} i^*_{p,q} = (i_p i^*_p)^{q-p}$ *and*

(ii) $i^*_{p,q} i_{p,q} = (i^*_{q-1} i_{q-1})^{q-p}$.

PROOF. We only prove the first relation. We have to show that for all $p \in \mathbb{Z}$ and $k \in \mathbb{N}$

$$i_{p,p+k} i^*_{p,p+k} = (i_p i^*_p)^k .$$

The proof proceeds by induction with respect to k.
For $k = 1$ the relation holds for all $p \in \mathbb{Z}$.
Assume that the relation holds for some $k \in \mathbb{N}$ and for all $p \in \mathbb{Z}$. For all $p \in \mathbb{Z}$ we have

$$\begin{aligned} i_{p,p+k+1} i^*_{p,p+k+1} &= i_p i_{p+1,p+k+1} i^*_{p+1,p+k+1} i^*_p = \\ &= i_p (i_{p+1} i^*_{p+1})^k i^*_p = \\ &= (i_p i^*_p)^{k+1} . \ \square \end{aligned}$$

1.4.3 THEOREM. *Let* $p \in \mathbb{Z}$ *and* $k \in \mathbb{N}$. *Then*

(i) $A_{p,p+k}$ *is a selfadjoint positive operator in* H_p *with domain* $D(A_{p,p+k}) = i_{p,p+k}(H_{p+k}) \subset H_p$,

(ii) $H_{p+k} \xhookrightarrow{i_{p,p+k}} H_p \xhookrightarrow{i_{p-k,p}} H_{p-k}$ *is a triple of Hilbert spaces,*

(iii) $A_{p,p+k} = A^k_p$,

(iv) $A^k_p \, i_{p,p+k} : H_{p+k} \longrightarrow H_p$ *is an isometry onto.*

(v) *If* $q \in \mathbb{Z}$ *and* $p < q$, *then* $i_{p,q} \, A^k_q \, i_{q,q+k} = A^k_p \, i_{p,q+k}$.

PROOF.

(i) Follows from 1.3.2.

(ii) Follows from 1.4.2.

(iii) From the relation (i) in 1.4.2, it follows that $A^{-1}_{p,p+k} = (A^{-1}_p)^k$.
Hence $A_{p,p+k} = A^k_p$.

(iv) Follows from (iii) and 1.3.2.

(v) This is an easy consequence of 1.3.4. □

Sequences of Hilbert spaces which are in fact chains in our sense appear at several places in the literature. PIETSCH [23] considers the spaces $D(A^k)$ $(k \in \mathbb{N})$ where A is a selfadjoint positive operator in a Hilbert space. And STUMMEL [32] considers an abstract version of Sobolev spaces. PALAIS ([22], Ch. VIII) gives also a definition of a chain; his definition is more general and is not as rich as ours.

1.5 DUALITY IN A CHAIN

We consider a chain of Hilbert spaces $\{H_p \mid p \in \mathbb{Z}\}$. The inner product and norm in H_p are denoted by $(.,.)_p$ and $\|\,.\,\|_p$ respectively. We shall identify H_{p+1} with its image $i_p(H_{p+1})$ in H_p $(p \in \mathbb{Z})$. So $\{H_p \mid p \in \mathbb{Z}\}$ will be considered as a decreasing sequence of Hilbert spaces (decreasing as $p \to \infty$).

Let $\{A_p \mid p \in \mathbb{Z}\}$ be the sequence of operators as defined in 1.4. We define linear spaces H_∞ and $H_{-\infty}$ by

$$H_\infty := \bigcap_{p\in\mathbb{Z}} H_p$$

and

$$H_{-\infty} := \bigcup_{p\in\mathbb{Z}} H_p .$$

Then $H_\infty \subset H_p \subset H_{-\infty}$ $(p \in \mathbb{Z})$.

Furthermore, we define linear operators $A^k : H_{-\infty} \to H_{-\infty}$ $(k \in \mathbb{N})$ by

$$A^k x := A^k_p x$$

for $x \in H_{p+k}$. These operators are well defined by 1.4.3 (v).

And $A^k \mid H_{p+k}$ is an isometry from H_{p+k} onto H_p; so

$$(1) \qquad (u,v)_p = (A^k u, A^k v)_{p-k} \qquad (u,v \in H_p),$$

and A^k leaves invariant H_∞. Moreover, A^k considered as an unbounded operator in H_p with domain $D(A^k) = H_{p+k} \subset H_p$ is selfadjoint and positive.

For the rest of this section we fix some integer $p \in \mathbb{Z}$. For $u \in H_\infty$ and $\alpha \in H_{-\infty}$ we want to define $(u,\alpha)_p$.
For any $k \in \mathbb{N}$

$$H_{p+k} \underset{i_{p,p+k}}{\hookrightarrow} H_p \underset{i_{p-k,p}}{\hookrightarrow} H_{p-k}$$

is a triple of Hilbert spaces. By 1.2.2 there exists an isometry

$$V_{p,k} : H_{p-k} \longrightarrow H_{p+k}$$

such that

$$V_{p,k} i_{p-k,p} = i^*_{p,p+k} .$$

It follows from 1.3.4 that

$$(2) \qquad V^{-1}_{p,k} = A^{2k} .$$

On $H_{p+k} \times H_{p-k}$ a sesquilinear form $(.,.)_{p,k}$ may be defined by (cf. 1.2.5)

$$(u,\alpha)_{p,k} := (u,V_{p,k}\alpha)_{p+k} \qquad (u \in H_{p+k},\ \alpha \in H_{p-k}).$$

From (2) it follows that

$$(u,\alpha)_{p,k} = (V^{-1}_{p,k}u,\alpha)_{p-k} = $$
$$= (A^{2k}u,\alpha)_{p-k} .$$

So

$$(u,\alpha)_{p,k} = (A^k u, A^{-k}\alpha)_p \qquad (u \in H_{p+k},\ \alpha \in H_{p-k}).$$

Now we take $u \in H_\infty$ and $\alpha \in H_{-\infty}$. Suppose that $\alpha \in H_{p-k}$ for some $k \in \mathbb{N}$. We show that $(u,\alpha)_{p,k}$ is independent of the choice of k.
Indeed, if $s \in \mathbb{N}$, then

$$(u,\alpha)_{p,k+s} = (A^{k+s}u, A^{-k-s}\alpha)_p .$$

Since $\alpha \in H_{p-k}$, $A^{-k-s}\alpha \in H_{p+s}$. Now, if we consider A^s as an unbounded self-adjoint operator in H_p with domain H_{p+s}, it follows that

$$(u,\alpha)_{p,k+s} = (A^k u, A^{-k}\alpha)_p =$$

$$= (u,\alpha)_{p,k} .$$

This proves that $(u,\alpha)_{p,k}$ is independent of k.
If $u,\alpha \in H_p$, then $(u,\alpha)_{p,k}$ equals the inner product $(u,\alpha)_p$ of u and α in H_p (cf. 1.2.5). Therefore $(.,.)_{p,k}$ is also denoted by $(.,.)_p$. So we have the relation

(3) $$(u,\alpha)_p = (A^k u, A^{-k}\alpha)_p \qquad (u \in H_\infty,\ \alpha \in H_{-\infty}).$$

We note that for any $k \in \mathbb{N}$ the form $(.,.)_p$ is a continuous sesquilinear form on $H_{p+k} \times H_{p-k}$; in fact

(4) $$|(u,\alpha)_p| \leq \| u \|_{p+k} \ \| \alpha \|_{p-k} \qquad (u \in H_{p+k},\ \alpha \in H_{p-k}).$$

Furthermore, we note that for any $f \in (H_{p-k})'$ there exists a unique $u \in H_{p+k}$ such that

$$f(\alpha) = (u,\alpha)_p \qquad (\alpha \in H_{p-k})$$

(cf. 1.2.5). Similarly, for any $f \in (H_{p+k})'$ there exists a unique $\alpha \in H_{p-k}$ such that

$$f(u) = \overline{(u,\alpha)_p} \qquad (u \in H_{p+k}).$$

1.5.1 <u>LEMMA.</u> *If* $u \in H_\infty$ *and* $\alpha \in H_{-\infty}$ *then*

(5) $$(Au,\alpha)_p = (u,A\alpha)_p .$$

<u>PROOF.</u> Suppose $\alpha \in H_{p-k+1}$ for some $k \in \mathbb{N}$. Then $A\alpha \in H_{p-k}$ and $A^{-k}\alpha \in H_{p+1}$. Since A is a selfadjoint operator in H_p with domain H_{p+1}, it follows that

$$(A^{k+1}u, A^{-k}\alpha)_p = (A^k u, A^{-k+1}\alpha)_p .$$

The left hand member equals $(Au,\alpha)_p$ and the right hand member equals $(u,A\alpha)_p$. This proves the lemma. □

Now the space $H_{-\infty}$ will be equipped with the locally convex inductive limit topology, i.e. the finest locally convex topology on $H_{-\infty}$ such that all inclusion maps

$$H_p \hookrightarrow H_{-\infty} \qquad (p \in \mathbb{Z})$$

are continuous. The anti-dual of $H_{-\infty}$, i.e. the linear space of all continuous anti-linear functionals on $H_{-\infty}$, is denoted by $(H_{-\infty})'$. The space H_∞ will be equipped with the projective limit topology, i.e. the coarsest topology such that all inclusion maps

$$H_\infty \hookrightarrow H_p \qquad (p \in \mathbb{Z})$$

are continuous.
The anti-dual of H_∞ is denoted by $(H_\infty)'$.
The space H_∞ is semi-reflexive (cf. SCHAEFER [29], Ch. IV, 5.8). It is also barrelled, hence it is reflexive (cf. SCHAEFER [29], Ch. IV, 5.5 and 5.6).

1.5.2 THEOREM. *Let* $p \in \mathbb{Z}$.
(i) *The map* $\Phi_p : H_\infty \to (H_{-\infty})'$ *defined by*

$$(\Phi_p u)(\alpha) := (u,\alpha)_p$$

is an algebraic isomorphism from H_∞ *onto* $(H_{-\infty})'$.
(ii) H_∞ *dense in* H_p *and* H_p *is dense in* $H_{-\infty}$ *(so* H_∞ *is dense in* $H_{-\infty}$*).*
(iii) *The map* $\Psi_p : H_{-\infty} \to (H_\infty)'$ *defined by*

$$(\Psi_p \alpha)(u) := \overline{(u,\alpha)_p}$$

is an algebraic isomorphism from $H_{-\infty}$ *onto* $(H_\infty)'$.

PROOF.
(i) First note that $\Phi_p u \in (H_{-\infty})'$. Indeed, if $\alpha \in H_{p-k}$, then

$$|(\Phi_p u)(\alpha)| = |(u,\alpha)_p| \le \| u \|_{p+k} \, \| \alpha \|_{p-k} \qquad (u \in H_\infty).$$

So $\Phi_p u \mid H_{p-k}$ is continuous on H_{p-k} for all $k \in \mathbb{N}$. By the definition of the locally convex inductive limit topology it follows that $\Phi_p u \in (H_{-\infty})'$. So Φ_p is well defined. Moreover, Φ_p is injective (if

$(u,\alpha)_p = 0$ for all $\alpha \in H_{p-k}$, then $u = 0$).

We show that Φ_p is surjective. Suppose $f \in (H_{-\infty})'$. Then again by the definition of the locally convex inductive limit topology, $f \mid H_{p-k}$ is continuous on H_{p-k} for all $k \in \mathbb{N} \cup \{0\}$. So for all $k \in \mathbb{N} \cup \{0\}$ there exists an element $u_k \in H_{p+k}$ such that

$$f(\alpha) = (u_k,\alpha)_p$$

for all $\alpha \in H_{p-k}$. In particular

$$f(\alpha) = (u_o,\alpha)_p$$

for all $\alpha \in H_p$. Hence $u_k = u_0$ is independent of p, and $u_0 \in H_\infty$. And $f = \Phi_p u_0$.

(ii) Let $\phi_p : H_p \to H_p'$ be the canonical map defined by

$$(\phi_p u)(v) := (u,v)_p \qquad (u,v \in H_p).$$

Consider the injections

$$i : H_p \hookrightarrow H_{-\infty}$$

and $\quad j : H_\infty \hookrightarrow H_p$.

Let $\quad i' : (H_{-\infty})' \longrightarrow H_p'$

be the anti-transposed of i. Then it is easily verified that the following diagram is commutive:

$$\begin{array}{ccc} (H_{-\infty})' & \xrightarrow{\ i'\ } & H_p' \\ \Phi_p \uparrow & & \uparrow \phi_p \\ H_\infty & \underset{j}{\hookrightarrow} & H_p \end{array}$$

Since j and hence i' are injective, $R(i) = H_p$ is dense in $H_{-\infty}$. Since i is injective, $R(j) = H_\infty$ is weakly dense in H_p and hence dense in H_p.

(iii) If $\alpha \in H_{p-k}$, then

$$|(\Psi_p\alpha)(u)| = |(u,\alpha)_p| \leq \| u \|_{p+k} \; \| \alpha \|_{p-k} \qquad (u \in H_\infty).$$

From the definition of projective limit topology it follows that $\Psi_p\alpha \in H'_\infty$.
If $\Psi_p\alpha = 0$ $(\alpha \in H_{p-k})$, then $(u,\alpha)_p = 0$ for all $u \in H_\infty$. Since H_∞ is dense in H_p by (ii), it follows that $\alpha = 0$. So Ψ_p is injective. We show that Ψ_p is surjective. Suppose $f \in (H_\infty)'$. Then for some $c \geq 0$ and some $k \in \mathbb{N}$

$$|f(u)| \leq c \, \| u \|_{p+k} \qquad (u \in H_\infty).$$

Since H_∞ is dense in H_{p+k}, there is a unique $\alpha \in H_{p-k}$ such that

$$f(u) = \overline{(u,\alpha)_p} \; .$$

So $f = \Psi_p\alpha$. □

Results similar to 1.5.2 (i) and (iii) also appear in the work of STUMMEL ([32], Kap. III, §1).
The first statement of 1.5.2 (ii) (H_∞ dense in H_p) is well-known but is usually proved with the spectral theorem for unbounded selfadjoint operators.

Let $p \in \mathbb{Z}$. In the next theorem we consider the anti-dual pair $(H_\infty, H_{-\infty})$ with the anti-duality $(\cdot,\cdot)_p$.

1.5.3 THEOREM.

(i) *The locally convex inductive limit topology on* $H_{-\infty}$ *coincides with the strong topology* $\beta(H_{-\infty}, H_\infty)$.

(ii) H_∞ *is reflexive and its strong anti-dual is* $H_{-\infty}$*; also,* $H_{-\infty}$ *is reflexive and its strong anti-dual is* H_∞.

PROOF.

(i) For $k \in \mathbb{N}$ we consider the injection $i : H_\infty \hookrightarrow H_{p+k}$. And let $i' : H'_{p+k} \to H'_\infty$ be the anti-transposed of i.
Let

$$\psi_{p,k} : H_{p-k} \to H'_{p+k}$$

be the map defined by

$$(\psi_{p,k}\alpha)(u) := \overline{(u,\alpha)_p} \qquad (u \in H_{p+k},\ \alpha \in H_{p-k}).$$

Then $\psi_{p,k}$ is an isometry onto (cf. 1.2.4 and 1.2.5).
Furthermore, let $\Psi_p : H_{-\infty} \to H'_\infty$ be as in 1.5.2 (iii). Then it is easily verified that the following diagram is commutative:

$$\begin{array}{ccc} H'_{p+k} & \xrightarrow{\ i'\ } & H'_\infty \\ \psi_{p,k}\Big\uparrow & & \Big\uparrow\Psi_p \\ H_{p-k} & \hookrightarrow & H_{-\infty} \end{array}$$

The map i' is continuous with respect to the strong topologies on H'_{p+k} and H'_∞ (cf. HORVATH [12], p. 256). So

$$H_{p-k} \hookrightarrow (H_{-\infty}, \beta(H_{-\infty}, H_\infty))$$

is continuous. This holds for all $k \in \mathbb{N}$. Hence by the definition of the locally convex inductive limit topology $\mathcal{T}$ on $H_{-\infty}$, it follows that the topology $\mathcal{T}$ is finer than $\beta(H_{-\infty}, H_\infty)$. On the other hand, the topology $\mathcal{T}$ is admissible with respect to the pair $(H_\infty, H_{-\infty})$ by 1.5.2 (i). Hence $\mathcal{T} = \beta(H_{-\infty}, H_\infty)$.

(ii) We have seen already that H_∞ is reflexive.
Since by (i) the space $H_{-\infty}$ is the strong anti-dual of H_∞, it follows that $H_{-\infty}$ is also reflexive and that its strong anti-dual is H_∞ (cf. HORVATH [12], p. 229). □

1.6 THE CHAIN ASSOCIATED WITH A SELFADJOINT OPERATOR

Let H_0 be a Hilbert space (with inner product $(\cdot,\cdot)_0$ and norm $\|\cdot\|_0$) and let T be an unbounded selfadjoint operator in H_0 with domain $D(T)$. Set

$$A := I + T^2 ,$$

where I denotes the identity operator in H_0. It is well-known that A is a selfadjoint positive operator.

With the operator A we can associate a chain as follows.
For H_1 we take the linear space D(A) equipped with the inner product

$$(u,v)_1 := (Au,Av)_0 \qquad (u,v \in H_1).$$

The inclusion map $i_0 : H_1 \hookrightarrow H_0$ is a dense embedding of norm ≤ 1.
A chain of Hilbert spaces $\{H_p^{(A)} \mid p \in \mathbb{Z}\}$ may be constructed in the way indicated at the beginning of section 1.4. This chain is called the chain generated by H_0 and A. The inner product and the norm in $H_p^{(A)}$ are denoted by $(.,.)_p$ and $\| \,.\, \|_p$ respectively.
Furthermore, we define:

$$H_\infty^{(A)} := \bigcap_{p\in\mathbb{Z}} H_p^{(A)}$$

equipped with the projective limit topology, and

$$H_{-\infty}^{(A)} := \bigcup_{p\in\mathbb{Z}} H_p^{(A)}$$

equipped with the locally convex inductive limit topology.
The selfadjoint operator corresponding to $i_0 : H_1 \hookrightarrow H_0$ (which is the operator $((i_0 i_0^*)^{\frac{1}{2}})^{-1}$, see section 1.3) is equal to A; this follows from the uniqueness property in Theorem 1.3.2.
By using 1.4.3 we conclude
(i) $D(A^k) = H_k^{(A)}$ and $\| u \|_k = \| A^k u \|_0 \quad (k \in \mathbb{N})$,
(ii) A can be extended to $H_p^{(A)}$ and this extension which is again denoted by A, maps $H_p^{(A)}$ isometrically onto $H_{p-1}^{(A)}$ $(p \in \mathbb{Z})$.
From (ii) it follows that A leaves invariant $H_\infty^{(A)}$ and that $A : H_\infty^{(A)} \to H_\infty^{(A)}$ is continuous. And furthermore, it follows from (ii) that $A : H_{-\infty}^{(A)} \to H_{-\infty}^{(A)}$ is continuous.
By 1.5.2 (ii) the space

$$H_\infty^{(A)} = \bigcap_{k\in\mathbb{N}} D(A^k) = \bigcap_{k\in\mathbb{N}} D(T^k)$$

is dense in H_0.

Let us consider the space D(T) equipped with the inner product

$$(x,y)_{D(T)} := (x,y)_0 + (Tx,Ty)_0 .$$

Let $i : H_1 \hookrightarrow D(T)$ and $j : D(T) \hookrightarrow H_0$ be the inclusion maps. Then we have the following proposition.

1.6.1 PROPOSITION.

(i) H_1 *is dense in* $D(T)$ *and* $i : H_1 \hookrightarrow D(T)$ *is a continuous map of norm* ≤ 1,

(ii) $H_1 \overset{i}{\hookrightarrow} D(T) \overset{j}{\hookrightarrow} H_0$ *is a triple of Hilbert spaces.*

PROOF.

(i) If $y \in D(T)$ is orthogonal to H_1 in $D(T)$, then $(x,y)_0 + (Tx,Ty)_0 = 0$ for all $x \in H_1 = D(T^2)$. So $((I+T^2)x,y)_0 = 0$ for all $x \in H_1$. Since $A = I + T^2$ maps H_1 onto H_0, it follows that $y = 0$.

For $x \in H_1$ we have

$$\| (I+T^2)x \|_0^2 - \| x \|_0^2 - \| Tx \|_0^2 =$$

$$= ((I+T^2)x,(I+T^2)x)_0 - ((I+T^2)x,x)_0 =$$

$$= ((I+T^2)x,T^2x)_0 =$$

$$= \| Tx \|_0 + \| T^2x \|_0 \geq 0 .$$

So i is continuous and its norm is ≤ 1.

(ii) For $u \in H_1$ and $f \in D(T)$ we have

$$(iu,f)_{D(T)} = (jiu,jf)_0 + (Tjiu,Tjf)_0 =$$

$$= (Ajiu,jf)_0 =$$

$$= (Ai_0u,jf)_0 .$$

On the other hand,

$$(iu,f)_{D(T)} = (u,i^*f)_1 = (Ai_0u,Ai_0i^*f)_0 .$$

Hence $j = Ai_0i^*$.

Since Ai_0 is an isometry onto, it follows that $j^*j = ii^*$. □

1.6.2 LEMMA. T *leaves invariant* $H_\infty^{(A)}$ *and* T *maps* $H_{k+1}^{(A)}$ *continuously into* $H_k^{(A)}$. *So*

$$T : H_\infty^{(A)} \longrightarrow H_\infty^{(A)}$$

is continuous.

PROOF. By 1.6.1 (i)

$$\| x \|_{D(T)} \leq \| x \|_1 \qquad (x \in H_1).$$

Hence

$$\| Tx \|_0 \leq \| Ax \|_0 \qquad (x \in H_1).$$

Hence for $x \in H_{k+1}$ $(k \in \mathbb{N})$

$$\| Tx \|_k = \| A^k Tx \|_0 = \| TA^k x \|_0 \leq \| A^{k+1} x \|_0 = \| x \|_{k+1} .$$

This proves the lemma. □

1.7 TENSOR PRODUCTS OF CHAINS

We recall briefly some facts about tensor products of Hilbert spaces (cf. PALAIS [22], Ch. XIV).

If E_1 and E_2 are Hilbert spaces, then there is a natural inner product on their algebraic tensor product $E_1 \otimes E_2$ characterized by

$$(x_1 \otimes y_1, x_2 \otimes y_2) := (x_1, x_2)_{E_1} (y_1 y_2)_{E_2} \qquad (x_1, x_2 \in E_1 \text{ and } y_1, y_2 \in E_2).$$

The completion of this pre-Hilbert space is denoted by $E_1 \tilde{\otimes} E_2$. It is called the *Hilbert space tensor product* of E_1 and E_2.

If F_1 and F_2 are also Hilbert spaces and

$$T_k : E_k \longrightarrow F_k \qquad (k=1,2)$$

is a bounded linear operator, then it is easily seen that

$$T_1 \otimes T_2 : E_1 \otimes E_2 \longrightarrow F_1 \otimes F_2$$

is bounded and in fact $\| T_1 \otimes T_2 \| = \| T_1 \| \| T_2 \|$. So $T_1 \otimes T_2$ extends to a bounded linear transformation from $E_1 \tilde{\otimes} E_2$ into $F_1 \tilde{\otimes} F_2$; this extension is denoted by $T_1 \tilde{\otimes} T_2$. If T_1 and T_2 are injective, then $T_1 \tilde{\otimes} T_2$ is also injective (its adjoint has dense range).

Now consider a Hilbert space $H_0^{(1)}$ and an (unbounded) selfadjoint operator T with dense domain in $H_0^{(1)}$. Put $A := I + T^2$. Let $\{H_p^{(A)} \mid p \in \mathbb{Z}\}$ be the chain of Hilbert spaces generated by $H_0^{(1)}$ and A. Then A can be extended to a continuous operator on $H_{-\infty}^{(A)} := \bigcup_{p\in\mathbb{Z}} H_p^{(A)}$ (equipped with the locally convex inductive limit topology). And the extension of A maps $H_{p+1}^{(A)}$ isometrically onto $H_p^{(A)}$ $(p \in \mathbb{Z})$
Furthermore, we consider a Hilbert space $H_0^{(2)}$ and a selfadjoint operator S with dense domain in $H_0^{(2)}$. Let $\{H_p^{(B)} \mid p \in \mathbb{Z}\}$ be the chain of Hilbert spaces generated by $B := I + S^2$ (I = identity operator on $H_0^{(2)}$).
If

$$i : H_q^{(A)} \hookrightarrow H_p^{(A)}$$

and

$$j : H_q^{(B)} \hookrightarrow H_p^{(B)}$$

$(p,q \in \mathbb{Z},\ p > q)$ are the inclusion maps, then

$$i \tilde{\otimes} j : H_q^{(A)} \tilde{\otimes} H_q^{(B)} \longrightarrow H_p^{(A)} \tilde{\otimes} H_p^{(B)}$$

is a continuous injection with dense range.
Now it is easily verified that

$$\left\{H_p^{(A)} \tilde{\otimes} H_p^{(B)} \mid p \in \mathbb{Z}\right\}$$

(together with the natural inclusion maps) is again a chain of Hilbert spaces. We want to examine the relation between this chain and the chains $\{H_p^{(A)}\}$ and $\{H_p^{(B)}\}$.

1.7.1 <u>THEOREM</u>. *Let $k,\ell \in \mathbb{N}$. The operator $A^k \otimes B^\ell$ considered as an unbounded operator in $H_0^{(A)} \tilde{\otimes} H_0^{(B)}$ with domain $H_k^{(A)} \otimes H_\ell^{(B)}$ is closable; its closure which is denoted by $A^k \tilde{\otimes} B^\ell$ is selfadjoint and positive and its domain is $H_k^{(A)} \tilde{\otimes} H_\ell^{(B)} \subset H_0^{(A)} \tilde{\otimes} H_0^{(B)}$. Moreover, $A^k \tilde{\otimes} B^\ell$ is an isometry from $H_k^{(A)} \tilde{\otimes} H_\ell^{(B)}$ onto $H_0^{(A)} \tilde{\otimes} H_0^{(B)}$.*

<u>PROOF</u>. Since $A^k : H_k^{(A)} \to H_0^{(A)}$ and $B^\ell : H_\ell^{(B)} \to H_0^{(B)}$ are isometric surjections it follows that

$$C := A^k \tilde{\otimes} B^\ell : H_k^{(A)} \tilde{\otimes} H_\ell^{(B)} \to H_0^{(A)} \tilde{\otimes} H_0^{(B)}$$

is also isometric and onto. If C is considered as an unbounded operator in $H_0^{(A)} \tilde{\otimes} H_0^{(B)}$ with domain $H_k^{(A)} \tilde{\otimes} H_\ell^{(B)}$, it is easily verified that C is symmetric. Since C is also surjective, it follows that C is selfadjoint.
It remains to prove that C is positive. The following argument is taken from SAKAI ([28], p. 60). If

$$\xi = \sum_{i=1}^{n} x_i \otimes y_i \in H_k^{(A)} \otimes H_\ell^{(B)} ,$$

then

$$(C\xi,\xi) = \sum_{i,j=1}^{n} (A^k x_i, x_j)(B^\ell y_i, y_j) .$$

For any family of complex numbers $(\lambda_1,\dots,\lambda_n)$,

$$\sum_{i,j=1}^{n} (B^\ell y_i, y_j)\lambda_i\overline{\lambda_j} = \left(B^\ell\left(\sum_{i=1}^{n} \lambda_i y_i\right), \sum_{i=1}^{n} \lambda_i y_i\right) \geq 0 .$$

Hence the matrix $\left((B^\ell y_i, y_j)\right)_{i,j=1}^n$ is positive and so it is a positive linear combination of one-dimensional projections. Since any one-dimensional projection is of the form $(\lambda_i\overline{\lambda_j})_{i,j=1}^n$, we have

$$\sum_{i,j=1}^{n} (A^k x_i, x_j)(B^\ell y_i, y_j) \geq 0 .$$

This implies that C is positive. This proves the theorem. □

Later on (in Chapter 3) the space $H_p^{(A)} \tilde{\otimes} H_p^{(B)}$ will also be denoted by $H_p^{(A\otimes B)}$ $(p \in \mathbb{Z})$.
Furthermore,

$$H_\infty^{(A\otimes B)} := \bigcap_{p\in\mathbb{Z}} H_p^{(A\otimes B)}$$

and

$$H_{-\infty}^{(A\otimes B)} := \bigcup_{p\in\mathbb{Z}} H_p^{(A\otimes B)} .$$

1.7.2 <u>LEMMA.</u> $H_\infty^{(A)} \otimes H_\infty^{(B)}$ *is dense in* $H_\infty^{(A\otimes B)}$ *where the latter space is equipped with the projective limit topology.*

<u>PROOF.</u> By 1.5.2 (ii) the space $H_\infty^{(A)} \otimes H_\infty^{(B)}$ is dense in $H_p^{(A)} \otimes H_p^{(B)}$. Hence $H_\infty^{(A)} \otimes H_\infty^{(B)}$ is also dense in

$$H_p^{(A)} \tilde{\otimes} H_p^{(B)} = H_p^{(A\otimes B)} \qquad (p \in \mathbb{Z}).$$

So $H_\infty^{(A)} \otimes H_\infty^{(B)}$ is contained in $H_\infty^{(A\otimes B)}$ and is dense in $H_\infty^{(A\otimes B)}$. □

CHAPTER II

LOCALLY CONVEX ALGEBRAS

In this chapter we present some facts of the theory of locally convex algebras as developed by G.R. ALLAN ([1] and [3]). The results will be used in the next chapter.

2.1 LOCALLY CONVEX ALGEBRAS AND GB*-ALGEBRAS

2.1.1 DEFINITION. Let A be an algebra over $\mathbb{C}$ which is a locally convex Hausdorff space. A is called a *locally convex algebra* if the multiplication is separately continuous; this means that the mappings

$$x \longmapsto ax \quad \text{and} \quad x \longmapsto xa \qquad (x \in A)$$

are continuous for all $a \in A$.
An element $x \in A$ is called *bounded* if for some $0 \neq \lambda \in \mathbb{C}$ the set $\{(\lambda x)^n \mid n \in \mathbb{N}\}$ is a bounded subset of A. The set of bounded elements is denoted by $b(A)$.

Let A be a locally convex algebra. By $\mathcal{B}$ we denote the collection of all subsets B of A such that B is absolutely convex, bounded, closed and $B^2 \subset B$ (here $B^2 = \{b_1 b_2 \mid b_1, b_2 \in B\}$).
For each $B \in \mathcal{B}$ let $A(B)$ denote the subalgebra of A generated by B. Then

(1) $$A(B) = \{\lambda x \mid \lambda \in \mathbb{C}, x \in B\} .$$

The relation

(2) $$\| x \|_B = \inf\{\lambda > 0 \mid x \in \lambda B\}$$

defines a norm on $A(B)$ which turns $A(B)$ into a normed algebra.

2.1.2 DEFINITION. The locally convex algebra A is called *pseudocomplete* if each of the normed algebras $A(B)$ $(B \in \mathcal{B})$ is a Banach algebra.

2.1.3 REMARK. If $x \in B \in \mathcal{B}$, then $\{x^n \mid n \in \mathbb{N}\} \subset B$. Hence $x \in b(A)$. So $A(B) \subset b(A)$. So pseudocompleteness is in fact a property of $b(A)$.

2.1.4 PROPOSITION. *Let A be a locally convex algebra.*

(i) *If A is pseudocomplete, then any closed subalgebra of A is pseudocomplete.*

(ii) *If A is sequentially complete, then A is pseudocomplete.*

(iii) *If A is commutative and pseudocomplete, then $\mathcal{B}$ is outer-directed by inclusion, i.e. if $B_1, B_2 \in \mathcal{B}$, then there is some $C \in \mathcal{B}$ such that $B_1 \cup B_2 \subset C$.*

PROOF. See ALLAN ([1], (2.8) and (2.6)). □

2.1.5 THEOREM. *Let A be a pseudocomplete locally convex algebra. If $x, y \in b(A)$ and $xy = yx$, then xy and $x+y \in b(A)$. So, if A is commutative and pseudocomplete, then $b(A)$ is a subalgebra of A.*

PROOF. See ALLAN ([1], (2.10)). □

2.1.6 DEFINITION. Let A be a locally convex algebra with unit e. The *spectrum* of $x \in A$, denoted by $\sigma_A(x)$ (or just $\sigma(x)$) is that subset of $\mathbb{C}^*$ $(: = \mathbb{C} \cup \{\infty\})$ defined as follows:

(a) for $\lambda \neq \infty$, $\lambda \in \sigma(x)$ iff $\lambda e - x$ has no bounded inverse,

(b) $\infty \in \sigma(x)$ iff x is not bounded.

The *resolvent* set $\rho(x) : = \mathbb{C}^* \backslash \sigma(x)$.

2.1.7 THEOREM. *Let A be a locally convex algebra with unit e which is pseudocomplete. If $x \in A$, then $\sigma(x)$ is a non-empty closed subset of $\mathbb{C}^*$ and $\rho(x)$ is precisely the set on which the A-valued function*

$$\lambda \longmapsto (\lambda e - x)^{-1}$$

is locally holomorphic.

PROOF. See ALLAN ([1], (3.10)). □

2.1.8 DEFINITION. Let A be a locally convex algebra. If a continuous involution

$$x \longmapsto x^*$$

is defined in A, then A is called a *locally convex *-algebra*. An element x of a locally convex *-algebra is called
hermitian if $x = x^*$,
and *normal* if $xx^* = x^*x$.
The set $H \subset A$ of all hermitian elements is a closed real-linear subspace of A.
A locally convex *-algebra A with unit e is called *symmetric* if $(e + x^*x)$ has a bounded inverse for all $x \in A$.

2.1.9 PROPOSITION. *Let A be a pseudocomplete symmetric algebra. If $h \in A$ is an hermitian element, then $\sigma(h) \subset \mathbb{R}^*(:= \mathbb{R} \cup \{\infty\})$.*

PROOF. See ALLAN ([3], (2.2)). □

2.1.10 DEFINITION. Let A be a locally convex *-algebra with unit e. By $\mathcal{B}^*$ we denote the collection of all subsets B of A such that
(i) B is absolutely convex, bounded and closed,
(ii) $B^2 \subset B$, $e \in B$,
(iii) $B = B^*(:= \{b^* \mid b \in B\})$.

A locally convex *-algebra A with unit e is called a *GB^*-algebra* (generalized B^*-algebra or generalized C^*-algebra) if
(i) A is pseudocomplete,
(ii) A is symmetric,
(iii) $\mathcal{B}^*$ has a greatest member which is denoted by B_0.

The algebra $A(B_0)$ is a *-subalgebra of A which contains all normal elements of $b(A)$. (If x is a normal element of $b(A)$, put $x = h + ik$ where $h,k \in H$. Then $x^* \in b(A)$, since $x \in b(A)$ and the involution is continuous. Since x is normal, it follows by 2.1.5 that $h,k \in b(A)$. But then, for some $B \in \mathcal{B}$ and some $\lambda > 0$, we have $\lambda h \in B$; hence $\lambda h \in B \cap B^* \in \mathcal{B}^*$ and so $h \in A(B_0)$. Similarly $k \in A(B_0)$ and so $x \in A(B_0)$).
In particular, if A is commutative, then

(3) $$b(A) = A(B_0).$$

2.1.11 THEOREM. *If* A *is a* GB^**-algebra, then* $A(B_0)$ *is a* C^**-algebra with unit (the norm in* $A(B_0)$ *is defined by the Minkowski functional of* B_0*).*

PROOF. See ALLAN ([3], (2.6)). □

The proof makes use of the following theorem which explains the name GB^*-algebra.

2.1.12 THEOREM. *Let* A *be a Banach algebra with unit and with a continuous involution. Then* A *is a* C^**-algebra (=* B^**-algebra) iff*

(i) A *is symmetric,*

(ii) $\mathcal{B}^*$ *has a greatest member.*

PROOF. See ALLAN ([2]). □

2.1.13 THEOREM. *A closed* **-subalgebra of a* GB^**-algebra* A *that contains the unit of* A*, is also a* GB^**-algebra.*

PROOF. See ALLAN ([3], (2.9)). □

2.2 FUNCTIONAL REPRESENTATION THEORY FOR COMMUTATIVE GB^*-ALGEBRAS

In this section A will be a commutative GB^*-algebra with unit e. Then $b(A) = A(B_0)$ where B_0 is the greatest member of the collection $\mathcal{B}^*$ and $b(A) = A(B_0)$ is a commutative C^*-algebra with unit (cf. 2.1.10 and 2.1.11).

We denote by M the spectrum of the commutative C^*-algebra $A(B_0)$; so M is the set of all non zero multiplicative linear functionals on $b(A) = A(B_0)$ in the topology $\sigma(M,b(A))$.
Then M is a compact Hausdorff space.
Let C(M) be the algebra of all continuous complex-valued functions on M topologized by the uniform norm and with complex conjugation as an involution.
The Gelfand map

$$b(A) \longrightarrow C(M)$$
$$x \longmapsto \hat{x}$$

is defined by

$$\hat{x}(\phi) := \phi(x) \qquad (\phi \in M).$$

By the theorem of GELFAND-NAIMARK, the Gelfand map is an isometric *-isomorphism of $b(A)$ onto $C(M)$. It is our aim to extend the Gelfand map to the whole algebra A.

2.2.1 THEOREM. *Let A be a commutative GB^*-algebra with unit. Then corresponding to any $\phi \in M$ there is a unique function*

$$\tilde{\phi} : A \to \mathbb{C}^*$$

such that

(i) *$\tilde{\phi}$ is an extension of ϕ,*

(ii) *$\tilde{\phi}$ is a "partial homomorphism" in the following sense*

- (a) $\tilde{\phi}(\lambda x) = \lambda\tilde{\phi}(x)$ $(\lambda \in \mathbb{C},\ x \in A)$ *with the convention* $0.\infty = 0$,
- (b) $\tilde{\phi}(x_1+x_2) = \tilde{\phi}(x_1) + \tilde{\phi}(x_2)$ $(x_1,x_2 \in A)$ *provided that* $\tilde{\phi}(x_1)$ *and* $\tilde{\phi}(x_2)$ *are not both* ∞,
- (c) $\tilde{\phi}(x_1x_2) = \tilde{\phi}(x_1)\tilde{\phi}(x_2)$ $(x_1,x_2 \in A)$ *provided that the pair* $(\tilde{\phi}(x_1),\tilde{\phi}(x_2))$ *is* $\neq (0,\infty)$ *and* $\neq (\infty,0)$,
- (d) $\tilde{\phi}(x^*) = \overline{\tilde{\phi}(x)}$ $(x \in A)$ *with the convention that* $\overline{\infty} = \infty$.

PROOF. For $h \in H$, choose $\mu \in \rho(h)$, $\mu \neq \infty$. Then $y := (\mu e - h)^{-1} \in b(A)$. Define

$$\phi'(h) := \begin{cases} \mu - \phi(y)^{-1} & \text{if } \phi(y) \neq 0, \\ \infty & \text{if } \phi(y) = 0. \end{cases}$$

If $x \in A$, then $x = h + ik$ with $h,k \in H$. Define

$$\tilde{\phi}(x) := \phi'(h) + i\phi'(k)$$

where the right hand member is interpreted as ∞ if either or both of $\phi'(h)$, $\phi'(k)$ is ∞.

One has to show that these definitions make sense and that $\tilde{\phi}$ satisfies (i) and (ii). For details we refer to ALLAN ([3], (3.1)). □

2.2.2 LEMMA. *Let $x \in A$. Then the function*

$$\hat{x} : M \longrightarrow \mathbb{C}^*$$

defined by

$$\hat{x}(\phi) := \tilde{\phi}(x) \qquad (\phi \in M)$$

is continuous. The set

$$N_x := \{\phi \in M \mid \tilde{\phi}(x) = \infty\}$$

is a nowhere dense closed subset of M.

PROOF. See ALLAN ([3], (3.6) and (3.9)). □

2.2.3 DEFINITION. Let X be a topological space and let F be a collection of continuous $\mathbb{C}^*$-valued functions on X. Then F is said to be a **-algebra of functions* on X provided that

(i) each $f \in F$ takes the value ∞ on at most a nowhere dense subset of X,

(ii) for any $f,g \in F$ and $\lambda \in \mathbb{C}$ the functions

$$\lambda f,\ f+g,\ fg,\ f^*(=\overline{f})$$

defined pointwise on the dense subset of X where f and g are both finite are extendible (necessarily uniquely) to continuous $\mathbb{C}^*$-valued functions on X which also belong to F (these extensions will also be denoted by λf, $f+g$, fg and f^* respectively).

The next theorem is a generalization of the theorem of GELFAND-NAIMARK.

2.2.4 THEOREM. *Let* A *be a commutative* GB**-algebra with unit. Then the Gelfand map*

$$x \longmapsto \hat{x}$$

of A *into the set of* $\mathbb{C}^*$*-valued functions on* M*, given by*

$$\hat{x}(\phi) := \tilde{\phi}(x) \qquad (x \in A,\ \phi \in M),$$

is a **-isomorphism of* A *onto a* **-algebra* $\hat{A}$ *of continuous* $\mathbb{C}^*$*-valued functions on* M.

PROOF. See ALLAN ([3], (3.9)). □

COROLLARY. *Let* $x \in A$. *Then* $\sigma(x) = \{\hat{x}(\phi) \mid \phi \in M\}$.

PROOF. See ALLAN ([3], (3.10)). □

COROLLARY. *An element* $x \in A$ *is hermitian iff* $\sigma(x) \subset \mathbb{R}^*$.

2.2.5 DEFINITION. Let X be a compact subset of $\mathbb{C}^*$. Then $C_0(X)$ denotes the algebra of all continuous complex-valued functions vanishing at ∞ on X. And $C_1(X)$ denotes the algebra of all continuous functions

$$f : X \cap \mathbb{C} \longrightarrow \mathbb{C}$$

such that

$$|f(\lambda)| \leq c(1 + |\lambda|^2)^n \qquad (\lambda \in X \cap \mathbb{C})$$

for some constant c and some $n \in \mathbb{N}$ (which may be different for different f).

2.2.6 THEOREM. *There exists a unique homomorphism of* $C_0(\sigma(x))$ $(x \in A)$ *into* $b(A)$

$$C_0(\sigma(x)) \longrightarrow b(A)$$
$$f \longmapsto f(x)$$

such that

(i) *if* $f(\lambda) = (1 + |\lambda|^2)^{-1}$ *then* $f(x) = (e + x^*x)^{-1}$ *and*
if $g(\lambda) = \lambda(1 + |\lambda|^2)^{-1}$ *then* $g(x) = x(e + x^*x)^{-1}$,

(ii) *for any* $f \in C_0(\sigma(x))$ *and any* $\phi \in M$

$$f(x)^\wedge(\phi) = f(\hat{x}(\phi)) .$$

PROOF. See ALLAN ([3], (3.11)). □

2.2.7 THEOREM. *Let* $x \in A$. *The map* $f \mapsto f(x)$ *of* 2.2.6 *may be extended uniquely to a* **-isomorphism of* $C_1(\sigma(x))$ *into* A

$$C_1(\sigma(x)) \longrightarrow A$$
$$f \longmapsto f(x)$$

such that

(i) *if* $u_0(\lambda) \equiv 1$ *then* $u_0(x) = e$

(ii) *if* $u_1(\lambda) = \lambda$ *then* $u_1(x) = x$

(iii) *for any* $f \in C_1(\sigma(x))$ *and* $\phi \in M$

$$f(x)^\wedge(\phi) = f(\hat{x}(\phi)) .$$

<u>PROOF.</u> If $f \in C_0(\sigma(x))$ then $f \circ \hat{x} \in C(M)$ and $f \circ \hat{x}$ is thus the Gelfand image of a unique element of $b(A)$; this element will be denoted by $f(x)$. For a given $f \in C_1(\sigma(x))$, choose n so that if $g_n(\lambda) := f(\lambda)/(1 + |\lambda|^2)^n$ then $g_n \in C_0(\sigma(x))$. Then $g_n(x)$ has been defined already.
Define $f(x) := g_n(x)(e + x^*x)^n$.
For details, see ALLAN ([3], (3.11) and (3.12)). □

CHAPTER III

SPECTRAL THEORY AND GB-ALGEBRAS*

In this chapter we show how the theory of GB^*-algebras of ALLAN can be applied to spectral theory of unbounded selfadjoint and normal operators.

If T is an unbounded selfadjoint operator in a Hilbert space H_0, then T can be considered as a continuous operator on the locally convex space H_∞ (cf. section 1.6). In section 2 we investigate the structure of the bicommutant $\Gamma''_\infty(T)$ of T in the algebra $L(H_\infty)$ of all continuous linear operators in H_∞. We define an involution and we show that the algebra $\Gamma''_\infty(T)$ is a GB^*-algebra in the sense of Allan. The main tool for characterizing the bounded elements of this algebra will be a lemma due to LAX [15].
In section 3 it is proved that any element C of $\Gamma''_\infty(T)$ is closable as an operator in H_0 and that its closure $\overline{C}$ is a normal operator in H_0. The relation between the spectrum $\sigma(C)$ of C in $\Gamma''_\infty(T)$ and the spectrum $\sigma(\overline{C})$ of $\overline{C}$ (as an operator in H_0) is also given.
In section 4 and 5 we develop the spectral theory of the algebra $\Gamma''_\infty(T)$. In particular a spectral representation theorem is derived (cf. 3.5.2 and 3.5.4).
Finally, in section 6 tensor products of GB^*-algebras are considered. We generalize a result of L. & K. MAURIN [19] concerning the spectrum of tensor products of selfadjoint operators.

3.1 GENERALITIES ON SPACES L(E)

Consider the algebra L(E) of all continuous linear operators on a locally convex Hausdorff space E.
The space L(E) with the topology of uniform convergence on bounded sets of E (the uniform topology) is denoted by $L_\beta(E)$; and L(E) with the topology of pointwise convergence (the strong topology) is denoted by $L_\sigma(E)$.
The following lemma is well-known.

3.1.1 LEMMA. *Let* E *be a barrelled locally convex Hausdorff space and let* $B \subset L(E)$. *Then the following statements are equivalent:*

(i) B *is bounded in* $L_\beta(E)$,

(ii) B *is bounded in* $L_\sigma(E)$,

(iii) B *is equicontinuous in* L(E).

PROOF. See SCHAEFER ([29], Ch.III, 4.1). □

3.1.2 DEFINITION. Let

$$\phi : E \times F \longrightarrow G$$

be a bilinear function where E, F and G are locally convex spaces. For each $u \in E$, write $\phi_u : F \to G$ for the function $\phi_u(v) := \phi(u,v)$ $(v \in F)$. Similarly $\phi_v : E \to G$ $(v \in F)$ is defined by $\phi_v(u) := \phi(u,v)$ $(u \in E)$.
Then ϕ is called *separately continuous* if ϕ_u and ϕ_v are continuous for $u \in E$ and $v \in F$.
And ϕ is called *left* (respectively, *right*) *hypocontinuous* if for every set $B_1 \subset E$ the set $\{\phi_u \mid u \in B_1\}$ is equicontinuous in L(F,G) (respectively, for every bounded set $B_2 \subset F$ the set $\{\phi_v \mid v \in B_2\}$ is equicontinuous in L(E,G)).

3.1.3 LEMMA. *Let* E *be a locally convex Hausdorff space.*

(i) *The multiplication in* $L_\beta(E)$ *and* $L_\sigma(E)$ *is separately continuous. So* $L_\beta(E)$ *and* $L_\sigma(E)$ *are locally convex algebras.*

(ii) *If* E *is barrelled, then the multiplication in* $L_\sigma(E)$ *is left hypocontinuous.*

(iii) *If* E *is barrelled, then the multiplication in* $L_\beta(E)$ *is left and right hypocontinuous.*

PROOF. Let us prove the second part of (iii).
Let N be a bounded set in E and let V be an absolutely convex closed neighbourhood of 0 in E. Then

$$U_{N,V} := \{S \in L(E) \mid S(N) \subset V\}$$

is a neighbourhood of 0 in $L_\beta(E)$. For $T \in L(E)$, define $R_T : L(E) \to L(E)$ by $R_T(S) := ST(S \in L(E))$.
Let B be bounded in $L_\beta(H_\infty)$. Then

$$\bigcap_{T\in B} R_T^{-1}(U_{N,V}) = \{S \mid ST(N) \subset V \text{ for all } T \in B\} =$$

$$= \{S \mid \bigcup_{T\in B} T(N) \subset S^{-1}(V)\} .$$

Since B is equicontinuous, the set $M := \bigcup_{T\in B} T(N)$ is bounded in E. Hence

$$\bigcap_{T\in B} R_T^{-1}(U_{N,V}) = U_{M,V}$$

is a neighbourhood of 0 in $L_\beta(E)$. This proves that the multiplication in $L_\beta(E)$ is right hypocontinuous. □

3.2 THE BICOMMUTANT OF T IN $L(H_\infty)$ AS A GB^*-ALGEBRA

We consider the situation described in section 1.6. Let H_0 be a separable Hilbert space and let T be an unbounded selfadjoint operator in H_0. Put $A := I + T^2$ where I denotes the identity operator on H_0. Let $\{H_p \mid p \in \mathbb{Z}\}$ be the chain of Hilbert spaces generated by H_0 and A (cf. 1.6); the inner product and the norm in H_p are denoted by $(.,.)_p$ and $\|\,.\,\|_p$ respectively. Then for $k \in \mathbb{N}$

$$H_k = D(A^k)$$

and the inner product in H_k is given by

$$(u,v)_k = (A^k u, A^k v)_0 .$$

Furthermore, we introduce

$$H_\infty := \bigcap_{p\in\mathbb{Z}} H_p$$

with the projective limit topology, and

$$H_{-\infty} := \bigcup_{p\in\mathbb{Z}} H_p$$

with the locally convex inductive limit topology.

Let $L(H_p,H_q)$ $(-\infty \le p,q \le \infty)$ be the space of continuous linear transformations of H_p into H_q; we abbreviate $L(H_p,H_p)$ by $L(H_p)$ $(-\infty \le p \le \infty)$.

And let $L_{\pi}(H_p,H_q)$ $(p,q \in \mathbb{Z})$ be the set of linear operators with domain H_∞ (which is dense in each H_p by 1.5.2 (ii)) which extend by continuity to be in $L(H_p,H_q)$; again $L_{\pi}(H_p) := L_{\pi}(H_p,H_p)$.
The norm of an operator S in $L(H_p,H_q)$ or in $L_{\pi}(H_p,H_q)$ $(p,q \in \mathbb{Z})$ is denoted by $\| S \|_{p,q}$; we put $\| S \|_p := \| S \|_{p,p}$ if $S \in L(H_p)$ $(p \in \mathbb{Z})$.

The operator A can be extended to an operator in $L(H_{-\infty})$ (cf. section 1.6); this extension is again denoted by A. Moreover, A maps H_{p+1} isometrically onto H_p $(p \in \mathbb{Z})$. So the restriction of A to H_∞ (also denoted by A) is in $L(H_\infty)$.
Furthermore, T leaves invariant H_∞ and the restriction of T to H_∞, which is also denoted by T, is in $L(H_\infty)$ (see 1.6.2).

3.2.1 <u>DEFINITION.</u> Let $\mathcal{A}$ be a subset of $L(H_p)$ $(-\infty \le p \le \infty)$. Then the *commutant* of $\mathcal{A}$ is the set

$$\Gamma_p'(\mathcal{A}) := \{S \in L(H_p) \mid SC = CS \text{ for all } C \in \mathcal{A}\} .$$

The commutant of $\Gamma_p'(\mathcal{A}) \subset L(H_p)$ is called the *bicommutant* of $\mathcal{A}$ and is denoted by $\Gamma_p''(\mathcal{A})$.

Since $T,A \in L(H_\infty)$, $\Gamma_\infty'(T)$ and $\Gamma_\infty'(A)$ are defined. Since $A = I + T^2$, we have

$$\Gamma_\infty'(T) \subset \Gamma_\infty'(A) .$$

Hence

$$(1) \qquad \Gamma_\infty''(A) \subset \Gamma_\infty''(T) \subset \Gamma_\infty'(T) \subset \Gamma_\infty'(A) .$$

The algebras mentioned in (1) are clearly closed in $L_\sigma(H_\infty)$ and $L_\beta(H_\infty)$.
Note that $\Gamma_\infty''(A)$ and $\Gamma_\infty''(T)$ are commutative subalgebras of $L(H_\infty)$.

It is our aim to define an involution in $\Gamma_\infty'(A)$. Then we shall prove that $\Gamma_\infty''(T)$ is a GB^*-algebra.

Before we prove the next lemma we note the following.
If $S \in L(H_\infty)$ then for all $q \in \mathbb{Z}$ there is some $p \in \mathbb{Z}$ such that

$$S \in L_{\pi}(H_p,H_q) .$$

3.2.2 <u>LEMMA.</u> *If* $S \subset \Gamma'_\infty(A)$ *is an equicontinuous set, then there is some* $k \in \mathbb{N}$ *and some constant* $c \geq 0$ *such that for all* $p \in \mathbb{Z}$

$$S \in L_{h}(H_{p+k}, H_p)$$

and

$$\| S \|_{p+k,p} \leq c \quad \textit{for all } S \in S.$$

<u>PROOF.</u> Since S is equicontinuous, there is some $k \in \mathbb{N}$ and some $c \geq 0$ such that for all $S \in S$

$$\| Sx \|_0 \leq c \| x \|_k \qquad (x \in H_\infty).$$

Hence, for all $S \in S$

$$\| Sx \|_p = \| A^p Sx \|_0 = \| SA^p x \|_0 \leq c \| A^p x \|_k = c \| x \|_{p+k}$$

for all $p \in \mathbb{Z}$ and all $x \in H_\infty$. This proves the lemma. □

<u>COROLLARY.</u> $\Gamma'_\infty(A) \cap L_{h}(H_p) = \Gamma'_\infty(A) \cap L_{h}(H_0) \qquad (p \in \mathbb{Z}).$

3.2.3 <u>DEFINITION.</u> Now we shall define an involution in $\Gamma'_\infty(A)$.
Let $S \in \Gamma'_\infty(A)$ and let $S' : (H_\infty)' \to (H_\infty)'$ be the anti-transposed of S. Let $\Psi_p : H_{-\infty} \to (H_\infty)'$ be as in 1.5.2 (iii).
Then we define

$$S^{*(p)} := \Psi_p^{-1} S' \Psi_p .$$

So the following diagram is commutative:

$$\begin{array}{ccc} (H_\infty)' & \xrightarrow{S'} & (H_\infty)' \\ \Psi_p \uparrow & & \uparrow \Psi_p \\ H_{-\infty} & \xrightarrow[S^{*(p)}]{} & H_{-\infty} \end{array}$$

It is easy to verify that

(2) $$(Sx,y)_p = (x, S^{*(p)} y)_p \qquad (x \in H_\infty,\ y \in H_{-\infty}).$$

This relation can also be taken as a definition of $S^{*(p)}$.

We show that

$$S^{*(p)} = S^{*(q)}$$

for all $p,q \in \mathbb{Z}$. We may assume that $q = p+k$, $k \in \mathbb{N}$. First we note that by 1.5.1 we have $A^{*(p)} = \bar{A}$ where $\bar{A}$ denotes for the moment the extension of A to $H_{-\infty}$. Since $S \in \Gamma'_\infty(A)$, it follows that $S^{*(p)}\bar{A} = \bar{A}\, S^{*(p)}$. So

$$\left(x, S^{*(p+k)}y\right)_{p+k} = \left(Sx,y\right)_{p+k} = \left(A^k Sx, \bar{A}^k y\right)_p =$$
$$= \left(A^k x, S^{*(p)}\bar{A}^k y\right)_p = \left(x, S^{*(p)}y\right)_{p+k} .$$

Hence $S^{*(p)} = S^{*(p+k)}$.

This result means that $S^{*(p)}$ is independent of p, therefore, we denote it by S^*.

The next step is to show that S^* leaves invariant H_∞. First we prove the following lemma.

3.2.4 <u>LEMMA.</u> *If* $S \in L_\hbar(H_{p+k}, H_p)$ *for some* $p \in \mathbb{Z}$ *and some* $k \in \mathbb{N}$, *then* S^* *maps* H_p *continuously into* H_{p-k}.

<u>PROOF.</u> For $x \in H_\infty$ and $y \in H_p \subset H_{-\infty}$ we have

$$|(x,S^*y)_p| = |(Sx,y)_p| \le \|Sx\|_p \|y\|_p \le$$
$$\le \|S\|_{p+k,p} \|x\|_{p+k} \|y\|_p .$$

Hence

$$S^*y \in H_{p-k} \quad \text{and} \quad \|S^*y\|_{p-k} \le \|S\|_{p+k,p}\|y\|_p .$$

This proves the lemma. □

Since $S \in \Gamma'_\infty(A)$ there is some $k \in \mathbb{N}$ such that $S \in L_\hbar(H_{p+k,p})$ for all $p \in \mathbb{Z}$. So S^* maps H_p continuously into H_{p-k} ($p \in \mathbb{Z}$). Hence S^* leaves invariant H_∞. And the restriction of S^* to H_∞, which is also denoted by S^*, is in $L(H_\infty)$.

<u>REMARK.</u> Note that $A = A^*$ (cf. 1.5.1). Since $(Tx,y)_0 = (x,Ty)_0$ for all $x,y \in H_\infty$ (recall that T is a selfadjoint operator in H_0), it follows that $T = T^*$.

3.2.5 <u>PROPOSITION.</u>

(i) *In* $\Gamma'_\infty(A)$ *an involution* $S \longmapsto S^*$ *is defined such that for all* $p \in \mathbb{Z}$

$$(Sx,y)_p = (x,S^*y)_p \qquad (x,y \in H_\infty).$$

(ii) *If* $\mathcal{A}$ *is a commutative* **-subalgebra of* $\Gamma'_\infty(A)$, *then* $\mathcal{A}$ *is a locally convex* **-algebra in the topologies induced by* $L_\beta(H_\infty)$ *and* $L_\sigma(H_\infty)$.

PROOF.

(i) For an $S \in \Gamma'_\infty(A)$, S^* has been defined in 3.2.4. It is clear that S^* is again in $\Gamma'_\infty(A)$. And it is easily verified that $S \longmapsto S^*$ has all the properties of an involution.

(ii) If $S \in \mathcal{A}$, then $(Sx,Sx)_p = (S^*Sx,x)_p = (SS^*x,x)_p = (S^*x,S^*x)_p$. Hence

$$\| Sx \|_p = \| S^*x \|_p \qquad (p \in \mathbb{Z},\ x \in H_\infty).$$

So $S \mapsto S^*$ is continuous with respect to the topology induced by $L_\sigma(H_\infty)$. If $p \in \mathbb{Z}$ and M is a bounded subset of H_∞, then

$$\sup_{x \in M} \| Sx \|_p = \sup_{x \in M} \| S^*x \|_p .$$

Hence the involution is also continuous with respect to the topology induced by $L_\beta(H_\infty)$. □

COROLLARY. $\Gamma''_\infty(A)$ *and* $\Gamma''_\infty(T)$ *are locally convex* **-algebras in the topologies induced by* $L_\beta(H_\infty)$ *and* $L_\sigma(H_\infty)$.

PROOF. $\Gamma''_\infty(A)$ and $\Gamma''_\infty(T)$ are *-subalgebras of $\Gamma'_\infty(A)$ (recall that $A = A^*$ and $T = T^*$) and they are also commutative. □

3.2.6 PROPOSITION. $L_\beta(H_\infty)$ *and* $L_\sigma(H_\infty)$ *are pseudocomplete locally convex algebras.*

PROOF. The space $L_\sigma(H_\infty)$ is sequentially complete since H_∞ is sequentially complete and barrelled (cf. HORVATH [12], p.216). And $L_\beta(H_\infty)$ is sequentially complete since H_∞ is bornological (cf. HORVATH [12], p.223). The proposition now follows by applying 2.1.4 (ii). □

COROLLARY. *If* $\mathcal{A}$ *is a subalgebra of* $L(H_\infty)$ *which is closed in* $L_\beta(H_\infty)$ *or* $L_\sigma(H_\infty)$, *then* $\mathcal{A}$ *is pseudocomplete.*

COROLLARY. $\Gamma''_\infty(A)$ *and* $\Gamma''_\infty(T)$ *are pseudocomplete locally convex* **-algebras in the topologies induced by* $L_\beta(H_\infty)$ *and* $L_\sigma(H_\infty)$.

We proved that $\Gamma''_\infty(T)$ is a pseudocomplete locally convex *-algebra. In order to prove that $\Gamma''_\infty(T)$ is a GB^*-algebra, we have to show that it is symmetric and that the collection $\mathcal{B}^*$ (of $\Gamma''_\infty(T)$) has a greatest element. First we investigate the bounded elements of $\Gamma''_\infty(T)$.

Since H_∞ is a barrelled locally convex space, the bounded elements are the same in the topology induced by $L_\beta(H_\infty)$ and in the topology induced by $L_\sigma(H_\infty)$ (cf. 3.1.1).

The main tool for characterizing the bounded elements is the next lemma, which is a generalization of a lemma due to LAX [15].

3.2.7 LEMMA. *Let* H *be a Hilbert space with inner product* $(.,.)$ *and with norm* $\|\,.\,\|$. *And let* G *be a dense linear subspace of* H *with norm* $\|\,.\,\|^\sim$ *so that*

$$\|u\| \le \|u\|^\sim$$

for all $u \in G$. *Let* S *be a continuous operator on* $(G, \|\,.\,\|^\sim)$ *with norm* ≤ 1. *Suppose there is a continuous operator* T *on* $(G, \|\,.\,\|^\sim)$ *of norm* ≤ 1 *such that*

$$(Su,v) = (u,Tv)$$

for all $u,v \in G$. *Then* S *can be extended to a continuous operator on* H *with norm* ≤ 1.

PROOF. First we assume that S is hermitian, i.e. $(Su,v) = (u,Sv)$ for all $u,v \in G$. This case is in fact the lemma of LAX. Our proof of this case follows the proof given in BEREZANSKII ([4], p.38).
Let $u \in G$, $\|u\| = 1$. Let $s_n := \|S^n u\|^2$ $(n=0,1,2,\ldots)$. Then

$$0 \le \|S^{n-1}u + \lambda S^{n+1}u\|^2 = s_{n-1} + 2\lambda s_n + \lambda^2 s_{n+1}$$

for all $\lambda \in \mathbb{R}$. Hence

$$s_n^2 \le s_{n-1}s_{n+1}$$

and

$$\frac{s_1}{s_0} \le \frac{s_2}{s_1} \le \ldots \le \frac{s_n}{s_{n-1}} .$$

So

$$s_n \ge s_1 s_{n-1} \ge s_1^2 s_{n-2} \ge \ldots \ge s_1^n .$$

Hence

$$s_1 \leq \sqrt[n]{s_n} .$$

Now

$$s_n = \| S^n u \|^2 \leq (\| S^n u \|^\sim)^2 \leq (\| u \|^\sim)^2 .$$

So

$$s_1 \leq \inf_{n \in \mathbb{N}} (\| u \|^\sim)^{2/n} .$$

Since $\| u \| = 1 \leq \| u \|^\sim$, it follows that $s_1 = \| Su \| \leq 1$. This implies the statement for the case that S is hermitian.

For the general case consider the operator TS; this operator is hermitian. Hence by the first part of the proof, TS can be extended to a continuous operator on H with norm ≤ 1. Let $u \in G$, $\| u \| = 1$. Then

$$\| Su \|^2 = (Su, Su) = (u, TSu) \leq \| TSu \| \leq 1 .$$

Hence S can be extended to a continuous operator on H of norm ≤ 1. □

3.2.8 THEOREM.

(i) *If* $S \in \Gamma'_\infty(A) \cap L_\hbar(H_p)$ *for some* $p \in \mathbb{Z}$ *then* S *is a bounded element of* $L_\beta(H_\infty)$ *and* $L_\sigma(H_\infty)$.

(ii) *Let* S *be a normal element of* $\Gamma'_\infty(A)$. *If* $\{S^k \mid k \in \mathbb{N}\}$ *is bounded (in* $L_\beta(H_\infty)$ *or* $L_\sigma(H_\infty)$*), then* $S \in L_\hbar(H_p)$ *and* $\| S \|_p \leq 1$ *for all* $p \in \mathbb{Z}$.

PROOF.

(i) Suppose $S \in \Gamma'_\infty(A) \cap L_\hbar(H_p)$ and $\| S \|_p \leq 1$ for some $p \in \mathbb{Z}$. Then $S \in L_\hbar(H_q)$ and $\| S \|_q \leq 1$ for all $q \in \mathbb{Z}$ (cf. 3.2.3). Hence for all $p \in \mathbb{Z}$ and all $k \in \mathbb{N}$

$$\| S^k x \|_p \leq \| x \|_p$$

for all $x \in H_\infty$. This means that $\{S^k \mid k \in \mathbb{N}\}$ is an equicontinuous set in $L(H_\infty)$; hence it is bounded in $L_\beta(H_\infty)$ and $L_\sigma(H_\infty)$.

(ii) We assume first that S is an hermitian element of $\Gamma'_\infty(A)$. Since $\{S^k \mid k \in \mathbb{N}\}$ is an equicontinuous semigroup of operators on H_∞, there exists for all $p \in \mathbb{Z}$ a $q \in \mathbb{Z}$ and a constant $c \geq 0$ such that for all $k \in \mathbb{N} \cup \{0\}$

$$\| S^k x \|_p \leq c \| x \|_q \qquad (x \in H_\infty).$$

For a fixed $p \in \mathbb{Z}$ we define

$$\| x \|^{\sim} := \sup_{k=0,1,2\ldots} \| S^k x \|_p .$$

Then $\| x \|^{\sim} < \infty$ for all $x \in H_\infty$ and $\| \cdot \|^{\sim}$ is a norm on H_∞.
Moreover,

$$\| x \|_p \leq \| x \|^{\sim} \quad \text{and} \quad \| Sx \|^{\sim} \leq \| x \|^{\sim} \qquad (x \in H_\infty).$$

Applying 3.2.7 we obtain that $S \in L_h(H_p)$ and that $\| S \|_p \leq 1$. This holds for all $p \in \mathbb{Z}$.
Now let S be a normal element of $\Gamma'_\infty(A)$. Then

$$\| S^k x \|_p = \| S^{*k} x \|_p$$

(see the proof of 3.2.5 (ii). So $\{S^{*k} \mid k \in \mathbb{N}\}$ is also an equicontinuous subset of $L(H_\infty)$, hence bounded in $L_\beta(H_\infty)$ and in $L_\sigma(H_\infty)$.
Since $\Gamma'_\infty(A)$ is a closed subalgebra of $L_\sigma(H_\infty)$, it is pseudocomplete (cf. 3.2.6). Then from 2.1.5 it follows that $\{(S^*S)^k \mid k \in \mathbb{N}\}$ is also bounded in $L_\beta(H_\infty)$ and $L_\sigma(H_\infty)$.
Since S^*S is hermitian, it follows from the first part of the proof that $\| S^*S \|_p \leq 1$ for all $p \in \mathbb{Z}$. Hence

$$\| Sx \|_p^2 = (S^*Sx, x)_p \leq \| x \|_p^2 \qquad (x \in H_\infty).$$

So $\| S \|_p \leq 1$ for all $p \in \mathbb{Z}$. □

3.2.9 <u>THEOREM.</u> *Let $\mathcal{A}$ be a commutative *-subalgebra of $\Gamma'_\infty(A)$ which contains the identity operator, then*

$$b(\mathcal{A}) = \mathcal{A} \cap L_h(H_p) \qquad (p \in \mathbb{Z}).$$

Moreover, the collection $\mathcal{B}^*$ of the algebra $\mathcal{A}$ has a greatest member B_0; and an element $S \in \mathcal{A}$ belongs to B_0 iff $\| S \|_p \leq 1$ for some $p \in \mathbb{Z}$ (or equivalently for all $p \in \mathbb{Z}$).

<u>PROOF.</u> From 3.2.8 it follows that an element $S \in \mathcal{A}$ is a bounded element of $\mathcal{A}$ iff $S \in L_h(H_p)$ for some $p \in \mathbb{Z}$ (or equivalently for all $p \in \mathbb{Z}$).
Let us define for some $p \in \mathbb{Z}$

$$B_0 := \{S \in A \mid S \in L_h(H_p),\ \| S \|_p \le 1\}\ .$$

Then B_0 is independent of p. Clearly $B_0 \in \mathcal{B}^*$. It follows from 3.2.8 (ii) that B_0 is the greatest member of $\mathcal{B}^*$. □

3.2.10 PROPOSITION. *Let* $S \in \Gamma'_\infty(A)$ *and let* $C := I + S^*S$ *where* I *denotes the identity operator on* H_∞, *then* C *is invertible and its inverse* C^{-1} *is a bounded element of* $\Gamma'_\infty(A)$.

PROOF. For all $p \in \mathbb{Z}$ and $x \in H_\infty$ we have $(Cx,x)_p = \| x \|_p^2 + \| Sx \|_p^2$; hence $\| x \|_p^2 \le (Cx,x)_p \le \| Cx \|_p \| x \|_p$. Thus

$$\| Cx \|_p \ge \| x \|_p \qquad (p \in \mathbb{Z},\ x \in H_\infty).$$

This proves that $C : H_\infty \to H_\infty$ is injective and that the range R(C) of C is closed in H_∞.
We show that C is surjective.

Consider the operator $C^* = \Psi_p^{-1}C'\Psi_p$ (cf. 3.2.3). Since C is an hermitian element of $\Gamma'_\infty(A)$ it follows that $C^* \mid H_\infty = C$. It is sufficient to show that $C^* : H_{-\infty} \to H_{-\infty}$ is injective.
Since $C \in \Gamma'_\infty(A)$ there is a $k \in \mathbb{N}$ such that for all $p \in \mathbb{Z}$ we have $C \in L_h(H_{p+k},H_p)$. By lemma 3.2.4 it follows that C^* maps H_p continuously into H_{p-k} $(p \in \mathbb{Z})$.
Suppose that $C^*x = 0$ for some $x \in H_{-\infty}$, say $x \in H_p$ for some $p \in \mathbb{Z}$. Let $(x_n)_{n=1}^\infty$ be a sequence in H_∞ such that $x_n \to x$ in H_p (hence in H_{p-k}) as $n \to \infty$. Then $Cx_n = C^*x_n \to C^*x = 0$ in H_{p-k} as $n \to \infty$. Since $\| Cx_n \|_{p-k} \ge \| x_n \|_{p-k}$, it follows that $x_n \to 0$ in H_{p-k} as $n \to \infty$. Hence $x = 0$. This proves that C^* is injective.

So $C : H_\infty \to H_\infty$ is bijective. If C^{-1} is the inverse of C, then

$$\| C^{-1}x \|_p \le \| x \|_p \qquad (p \in \mathbb{Z},\ x \in H_\infty).$$

Also it is clear that $C^{-1} \in \Gamma'_\infty(A)$. Therefore, by 3.2.8 (ii), C^{-1} is a bounded element of $\Gamma'_\infty(A)$. □

3.2.11 DEFINITION. A subalgebra A of $L(H_\infty)$ is called *full* if the following condition is satisfied:
if $C \in A$ and C is invertible in $L(H_\infty)$, then $C^{-1} \in A$.

REMARK. If A is a subset of $L(H_\infty)$ then $\Gamma'_\infty(A)$ is a full subalgebra.

3.2.12 THEOREM. *Let* A *be a full commutative* **-subalgebra of* $\Gamma'_\infty(A)$ *which contains the identity operator. Then* A *is a symmetric locally convex* **-algebra.*

PROOF. Let $S \in A$ and put $C := I + S^*S$, then C is invertible and its inverse C^{-1} is a bounded element of $\Gamma'_\infty(A)$. Since A is a full subalgebra, $C^{-1} \in A$. □

Now we come to the main result of this section.

3.2.13 THEOREM. *Let* A *be a full commutative* **-subalgebra of* $\Gamma'_\infty(A)$ *which contains the identity operator and which is closed in* $L_\beta(H_\infty)$ *or* $L_\sigma(H_\infty)$. *Then* A *is a* GB^**-algebra in the topologies induced by* $L_\beta(H_\infty)$ *and* $L_\sigma(H_\infty)$.

PROOF. The proof follows from 3.2.8 (ii), 3.2.6, 3.2.12 and 3.2.9. □

COROLLARY. $\Gamma''_\infty(A)$ *and* $\Gamma''_\infty(T)$ *are commutative* GB^**-algebras with respect to the topologies induced by* $L_\beta(H_\infty)$ *and* $L_\sigma(H_\infty)$.

3.3 THE ELEMENTS OF THE BICOMMUTANT OF T AS OPERATORS ON H_0.

From 3.2.9 if follows that

$$b(\Gamma''_\infty(T)) = \Gamma''_\infty(T) \cap L_h(H_0) \ .$$

It is possible to give another description of the algebra $b(\Gamma''_\infty(T))$ of the bounded elements of $\Gamma''_\infty(T)$. From this description it will follow that $b(\Gamma''_\infty(T))$ is a W^*-algebra. We need the following definition.

3.3.1 DEFINITION. Let B be a selfadjoint operator in H_0. The *commutant* of B in $L(H_0)$ will be the set

$$\Gamma'_0(B) := \{S \in L(H_0) \mid SB \subset BS\} \ .$$

Here $SB \subset BS$ means: $Sx \in D(B)$ for all $x \in D(B)$ and $BSx = SBx$ for all $x \in D(B)$.

REMARK. If $S \in \Gamma'_0(B)$, then S leaves invariant $D(B^k)$ $(k \in \mathbb{N})$.

3.3.2 PROPOSITION. *Any* $S \in \Gamma_0'(T)$ *leaves invariant* H_∞ *and its restriction to* H_∞ *is in* $\Gamma_\infty'(T) \cap L_h(H_0)$. *And if* $S \in \Gamma_\infty'(T) \cap L_h(H_0)$ *then its continuous extension* $\overline{S}^{(0)}$ *to* H_0 *is in* $\Gamma_0'(T)$. *So (identifying* S *and* $\overline{S}^{(0)}$*) we may write*

$$\Gamma_0'(T) = \Gamma_\infty'(T) \cap L_h(H_0) \ .$$

PROOF. If $S \in \Gamma_0'(T)$, then S leaves invariant $D(T^k)$ $(k \in \mathbb{N})$. So S leaves invariant H_∞. Moreover, $STx = TSx$ for all $x \in H_\infty$. And $S \in L(H_\infty)$ since for all $p \in \mathbb{Z}$

$$\| Sx \|_p = \| A^p Sx \|_0 = \| SA^p x \|_0 \le \| S \|_0 \| x \|_p \qquad (x \in H_\infty).$$

Conversely, let $S \in \Gamma_\infty'(T) \cap L_h(H_0)$. Then by 3.2.2 also $S \in L_h(H_1)$. If $\overline{S}^{(1)}$ and $\overline{S}^{(0)}$ denote the continuous extensions of S to H_1 and H_0 respectively, then it is easily verified that $\overline{S}^{(0)} \mid H_1 = \overline{S}^{(1)}$. So $\overline{S}^{(0)}$ leaves invariant H_1. It is also easy to see that $A\overline{S}^{(0)} = \overline{S}^{(0)}Ax$ for all $x \in H_1$.

Indeed, if $(x_n)_{n=1}^\infty$ is a sequence in H_∞ such that $x_n \to x$ in H_1 as $n \to \infty$, then $Sx_n \to \overline{S}^{(0)}x$ in H_1 and hence $ASx_n \to A\overline{S}^{(0)}x$ in H_0. Since $Ax_n \to Ax$ in H_0, it follows that $ASx_n = SAx_n \to \overline{S}^{(0)}Ax$ in H_0 as $n \to \infty$.

Now suppose that S is an hermitian element of $\Gamma_\infty'(T)$. Consider the space $D(T)$ equipped with the inner product

$$(x,y)_{D(T)} := (x,y)_0 + (Tx,Ty)_0 \ .$$

Then H_1 is dense in $D(T)$ and the inclusion map $H_1 \hookrightarrow D(T)$ has norm ≤ 1 (cf. 1.6.1).

For $x,y \in H_1$ we have

$$(\overline{S}^{(1)}x,y)_{D(T)} = (A\overline{S}^{(1)}x,y)_0 = (x,A\overline{S}^{(1)}y)_0 =$$
$$= (x,\overline{S}^{(1)}y)_{D(T)} \ .$$

By lemma 3.2.7 (with H, G and S replaced by $D(T)$, H_1 and $\overline{S}^{(1)}$ resp.) $\overline{S}^{(1)}$ has a continuous extension to $D(T)$. This extension has to coincide with $\overline{S}^{(0)}$. So $\overline{S}^{(0)}$ leaves invariant $D(T)$. Since $STx = TSx$ for all $x \in H_\infty$, it follows easily that $T\overline{S}^{(0)}x = \overline{S}^{(0)}Tx$ for all $x \in D(T)$. So $\overline{S}^{(0)} \in \Gamma_0'(T)$.

If S is not hermitian, then we can write $S = S_1 + iS_2$ where S_1 and S_2 are hermitian elements of $\Gamma_\infty'(T) \cap L_h(H_0)$. Then the first part of this proof can be applied. □

3.3.3 LEMMA. *Let* $V = (T - iI)(T + iI)^{-1} \in L(H_0)$ *be the Cayley transform of* T. *Then* $\Gamma_0'(V) = \Gamma_0'(T)$.

PROOF. Recall that $I - V$ is injective, $D(T) = R(I - V)$ and that $T = i(I + V)(I - V)^{-1}$.
Let $B \in \Gamma_0'(V)$ and take $x \in D(T)$. Then $x = (I - V)y$ for some $y \in H_0$. So $Bx = B(I - V)y = (I - V)By \in R(I - V) = D(T)$ and $TBx = T(I - V)By =$ $= i(I + V)By = Bi(I + V)y = BTx$. Hence $B \in \Gamma_0'(T)$.
Conversely, if $B \in \Gamma_0'(T)$ then B leaves invariant $D(T)$ so $(T + iI)B(T + iI)^{-1}x$ is well defined $(x \in H_0)$ and $(T + iI)B(T + iI)^{-1}x = Bx$.
Hence $BVx = (T - iI)B(T + iI)^{-1}x = VBx$ $(x \in H_0)$. So $B \in \Gamma_0'(V)$. □

3.3.4 THEOREM. $b(\Gamma_\infty''(T)) = \Gamma_0''(T)$.
Precisely: if $S \in b(\Gamma_\infty''(T))$, *then its continuous extension* $\overline{S}$ *to* H_0 *is in* $\Gamma_0''(T)$ *(= the commutant of* $\Gamma_0'(T) \subset L(H_0)$*) and conversely, if* $\overline{S} \in \Gamma_0''(T)$ *then* $\overline{S}$ *leaves invariant* H_∞ *and its restriction to* H_∞ *is an element of* $b(\Gamma_\infty''(T))$.

PROOF. If $S \in b(\Gamma_\infty''(T)) = \Gamma_\infty''(T) \cap L_h(H_0)$ (cf. 3.2.9), then S commutes with every $R \in \Gamma_\infty'(T) \cap L_h(H_0) = \Gamma_0'(T)$ (3.3.2). Hence the continuous extension of S to H_0 is an element of $\Gamma_0''(T)$.

Conversely, let $S \in \Gamma_0''(T)$. If $V = (T - iI)(T + iI)^{-1} \in L(H_0)$ is the Cayley transform of T, then by 3.3.3

$$\Gamma_0''(T) = \Gamma_0''(V) \subset \Gamma_0'(V) = \Gamma_0'(T) .$$

Using 3.3.2, we obtain

$$S \in \Gamma_0'(T) = \Gamma_\infty'(T) \cap L_h(H_0) \subset \Gamma_\infty'(A) .$$

So S is a bounded element of $L(H_\infty)$ (cf. 3.2.8).
It remains to prove that $S \in \Gamma_\infty''(T)$. It is sufficient to show that S commutes with all hermitian elements $R \in \Gamma_\infty'(T)$. Let R be an hermitian element of $\Gamma_\infty'(T)$ and let A be a maximal commutative *-subalgebra containing R and contained in $\Gamma_\infty'(T)$. Then A is a full subalgebra of $\Gamma_\infty'(T)$ (which is itself a full subalgebra of $L(H_\infty)$ and A is closed in $L_\sigma(H_\infty)$. So, by 3.2.13, A is a GB^*-algebra. Hence, by 2.1.9, $\sigma_A(R) \subset \mathbb{R}^*$.
So $(R - iI)^{-1} \in b(A) = A \cap L_h(H_0)$ (cf. 3.2.9).
Hence $(R - iI)^{-1} \in A \cap L_h(H_0) \subset \Gamma_\infty'(T) \cap L_h(H_0) = \Gamma_0'(T)$ (by 3.3.2). So S

commutes with $(R - iI)^{-1}$ and therefore also with R. This means that $S \in \Gamma''_{\infty}(T)$. □

3.3.5 COROLLARY. *$b(\Gamma''_{\infty}(T))$ is a W^*-algebra (the norm in $b(\Gamma''_{\infty}(T))$ is defined by the Minkowski functional of B_0 where B_0 is the greatest member of the collection B^* of the algebra $\Gamma''_{\infty}(T)$).*

PROOF. Let V be the Cayley transform of T. Then $\Gamma''_0(T) = \Gamma''_0(V)$ is a W^*-algebra. □

REMARK. $\Gamma''_{\infty}(T)$ may be called a GW^*-algebra (generalized W^*-algebra).

If S is a normal element of $\Gamma'_{\infty}(A)$ which is bounded, then S has a continuous extension to H_0. In the next theorem we will see that any normal element S of $\Gamma'_{\infty}(A)$ is closable as an operator on H_0 and that its closure $\overline{S}^{(0)}$ is a normal operator in H_0. Recall that an unbounded operator B in a Hilbert space is called normal if $B^*B = BB^*$. It is easy to show that B is normal iff $D(B) = D(B^*)$ and $\| Bx \| = \| B^*x \|$ (cf. DUNFORD-SCHWARTZ [9], p.1258).

3.3.6 THEOREM. *Any normal element of $\Gamma'_{\infty}(A)$ is closable as an operator in H_p ($p \in \mathbb{Z}$) and its closure is a normal operator in H_p.*

PROOF. We prove the theorem for $p = 0$.

Let $S \in \Gamma'_{\infty}(A)$ be hermitian. Then

$$(Sx,y)_0 = (x,Sy)_0 \qquad (x,y \in H_{\infty}).$$

Since H_{∞} is dense in H_0, this implies that S, considered as an operator in H_0 with domain H_{∞}, is closable. Let $\overline{S}$ be its closure. Then it is easy to prove that $\overline{S}$ is symmetric, i.e. $\overline{S} \subset \overline{S}^{\times}$ where $\overline{S}^{\times}$ denotes the Hilbert space adjoint of $\overline{S}$ in H_0. Therefore, in order to prove that $\overline{S}$ is selfadjoint, it suffices to show that $R(\overline{S} + iI) = H_0$.
Let A be a maximal commutative *-subalgebra containing S and contained in $\Gamma'_{\infty}(A)$. Then by 3.2.13, A is a GB^*-algebra; so, by 2.1.9, $\sigma_A(S) \subset \mathbb{R}^*$. Hence $R(S + iI) = H_{\infty}$ is dense in H_0. Since

$$\| (\overline{S} + iI)x \|_0 \geq \| x \|_0 \qquad (x \in D(\overline{S})),$$

it follows that $R(\overline{S} + iI)$ is closed in H_0.

Hence $R(\overline{S} + iI) = H_0$. This proves that $\overline{S}$ is selfadjoint. In particular, any two selfadjoint extensions of S are equal.

Now let C be a normal element of $\Gamma'_\infty(A)$. Then

$$(Cx,y)_0 = (x,C^*y)_0 \qquad (x,y \in H_\infty).$$

This again implies that C is closable as an operator in H_0. Let $\overline{C}$ be the closure of C in H_0 and let $C^\times = \overline{C}^\times$ be the Hilbert space adjoint of C in H_0. Then $C^\times\overline{C}$ and $\overline{C}C^\times$ are both selfadjoint extensions of the hermitian element CC^*. These extensions are equal, so $C^\times\overline{C} = \overline{C}C^\times$. Thus $\overline{C}$ is normal. □

<u>REMARK.</u> Let S be a normal element of $\Gamma'_\infty(A)$. And let S and $S^\times$ be as in the proof of the theorem. We show that $\overline{S} = S^{*\times}$.
It is trivial that $\overline{S} \subset S^{*\times}$ (since $S^{*\times}$ is closed). Moreover, $D(\overline{S}) = D(\overline{S^*})$ since $\| Sx \|_0 = \| S^*x \|_0$ $(x \in H_\infty)$. And $D(\overline{S^*}) = D(S^{*\times})$ since $\overline{S^*}$ is a normal operator in H_0.
Hence $\overline{S} = S^{*\times}$.

Using this remark, we can give a description of the closure $\overline{S}$ of a normal element $S \in \Gamma'_\infty(A)$ (as an operator in H_0).
Let

$$S^{**} : H_{-\infty} \to H_{-\infty}$$

be defined by (cf. 3.2.3)

$$S^{**} := \Psi_p^{-1} S^{*\prime} \Psi_p .$$

Then S^{**} is continuous and S^{**} leaves invariant H_∞. Moreover, $S^{**} \mid H_\infty = S$.
For $x \in H_0$ such that $S^{**}x \in H_0$, we define

$$\tilde{S}x := S^{**}x .$$

Thus $\tilde{S}$ is an operator in H_0 with dense domain

$$D(\tilde{S}) = \{x \in H_0 \mid S^{**}x \in H_0\} .$$

The graph

$$G(S^{**}) = \{(x,S^{**}x) \mid x \in H_{-\infty}\}$$

is closed in $H_{-\infty} \times H_{-\infty}$. The inverse image of $G(S^{**})$ under the inclusion map $H_0 \times H_0 \hookrightarrow H_{-\infty} \times H_{-\infty}$ is precisely the graph $G(\tilde{S})$ of $\tilde{S}$. Hence $G(\tilde{S})$ is closed in $H_0 \times H_0$. So $\tilde{S}$ is a closed operator in H_0 and thus $\overline{S} \subset \tilde{S}$.
If $x \in D(\tilde{S})$ then

$$(S^*y,x)_0 = (y,S^{**}x)_0 = (y,\tilde{S}x)_0 \qquad (y \in H_\infty).$$

So $x \in D(S^{*\times})$ and $S^{*\times}x = \tilde{S}x$. This means $\tilde{S} \subset S^{*\times}$. By the remark made above we have $\overline{S} = S^{*\times}$. So we obtain $\overline{S} = \tilde{S}$.

Let S be a normal element of $\Gamma'_\infty(A)$. The spectrum of S in $\Gamma'_\infty(A)$ (or, which is the same, in $L(H_\infty)$) is denoted by $\sigma(S)$. The spectrum of the closure of $\overline{S}$ of S as an operator in H_0 is denoted by $\sigma(\overline{S})$. Then we have the following theorem.

3.3.7 <u>THEOREM.</u> *Let* S *be a normal element of* $\Gamma'_\infty(A)$. *Then*

$$\sigma(\overline{S}) = \sigma(S) \cap \mathbb{C}\ .$$

<u>PROOF.</u>

a) If $\lambda \in \rho(\overline{S})$, then $(\lambda I - \overline{S})^{-1} \in L(H_0)$. We have to show that $(\lambda I - \overline{S})^{-1}$ leaves invariant H_∞.
Note that $\lambda I - S : H_\infty \to H_\infty$ is injective and that

$$A(\lambda I - S)^{-1}x = A(\lambda I - S)^{-1}x$$

for all $x \in R(\lambda I - S) \subset H_\infty$.
Since $\lambda I - \overline{S} : D(\overline{S}) \to H_0$ is bijective, $R(\lambda I - S)$ is dense in H_0. We show that $R(\lambda I - S)$ is also dense in H_1.
Since $\lambda I - S$ commutes with A and hence with A^{-1} (on H_∞), it follows that $R(\lambda I - S) = A^{-1}R(\lambda I - S)$. Since $A : H_1 \to H_0$ is an isometry onto, it follows that $R(\lambda I - S)$ is dense in H_1.
Now we prove that $(\lambda I - \overline{S})^{-1} \in \Gamma'_0(A)$. Let $x \in H_1$ and let $(x_n)_{n=1}^\infty$ be a sequence in $R(\lambda I - S)$ such that $x_n \to x$ in H_1 (and hence in H_0) as $n \to \infty$
Then $(\lambda I - \overline{S})^{-1}x_n \to (\lambda I - \overline{S})^{-1}x$ in H_0 as $n \to \infty$ and $A(\lambda I - \overline{S})^{-1}x_n \to A(\lambda I - \overline{S})^{-1}x$ in H_{-1}. On the other hand, since $Ax_n \to Ax$ in H_0, we have

$$A(\lambda I - \overline{S})^{-1} = (\lambda I - \overline{S})^{-1}Ax_n \longrightarrow (\lambda I - \overline{S})^{-1}Ax$$

in H_0 as $n \to \infty$. Hence

$$A(\lambda I - \overline{S})^{-1}x = (\lambda I - \overline{S})^{-1}Ax \in H_0 \qquad (x \in H_1).$$

This relation implies that $(\lambda I - \overline{S})^{-1}$ leaves invariant H_∞. Hence $\lambda I - S : H_\infty \to H_\infty$ is bijective and its inverse $(\lambda I - S)^{-1} \in \Gamma'_\infty(A) \cap L_{\hbar}(H_0)$; so by 3.2.8 (i), $(\lambda I - S)^{-1}$ is a bounded element. Hence $\lambda \in \rho(S)$. So we proved $\rho(\overline{S}) \subset \rho(S)$.

b) Conversely, if $\lambda \in \rho(S) \cap \mathbb{C}$, then $\lambda I - S$ has a bounded inverse U. By 3.2.8 (ii), $U \in L_{\hbar}(H_0)$. Let $\overline{U}$ denote the continuous extension of U to H_0. We show that $\lambda \in \rho(\overline{S})$.

If $x \in D(\overline{S})$ and $(x_n)_{n=1}^{\infty}$ is a sequence in H_∞ such that $x_n \to x$ and $Sx_n \to \overline{S}x$ in H_0 as $n \to \infty$, then $U(\lambda I - S)x_n = x_n \to \overline{U}(\lambda I - \overline{S})x$ in H_0. So $x = \overline{U}(\lambda I - \overline{S})x$ and

$$\| x \|_0 = \| \overline{U}(\lambda I - \overline{S})x \|_0 \le \| \overline{U} \|_0 \, \| (\lambda I - \overline{S})x \|_0 \qquad (x \in D(\overline{S})).$$

So $\lambda I - \overline{S}$ is injective and $R(\lambda I - \overline{S})$ is closed in H_0. Since $R(\lambda I - \overline{S})$ is also dense in H_0, it follows that $\lambda \in \rho(\overline{S})$. □

3.4 SPECTRAL THEORY OF THE BICOMMUTANT OF T

We proved that $b(\Gamma''_\infty(T))$ is a W^*-algebra. If $S \in b(\Gamma''_\infty(T))$ then S has a unique continuous extension $\overline{S}$ to H_0 and $\overline{S} \in \Gamma''_0(T)$. The mapping $S \mapsto S^*$ is an isometric *-isomorphism of $b(\Gamma''_\infty(T))$ onto $\Gamma''_0(T)$. Since $\Gamma''_0(T) = \Gamma''_0(V)$ where V is the Cayley transform of T, $b(\Gamma''_\infty(T))$ is isomorphic to the W^*-algebra $\Gamma''_0(V)$. The spectrum of the W^*-algebra $b(\Gamma''_\infty(T))$ is a hyper-Stonean space (cf. SAKAI [28], 1.18); recall that a compact Hausdorff space is called *Stonean* if the closure of every open set in it is open.
The spectrum of $b(\Gamma''_\infty(T))$ will be denoted by X.

3.4.1 DEFINITION. Let $C^\infty(X)$ be the set of all continuous functions

$$f : X \longrightarrow \mathbb{C}^* = \mathbb{C} \cup \{\infty\}$$

such that f takes the value ∞ only on a nowhere dense set. And $C_r^\infty(X)$ will be the subset of $C^\infty(X)$ consisting of all $\mathbb{R}^*$-valued functions, where $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$.

By 2.2.4 the Gelfand map $S \mapsto \hat{S}$ of $b(\Gamma''_\infty(T))$ onto $C(X)$ extends to a *-isomorphism of $\Gamma''_\infty(T)$ onto a *-subalgebra F of $C^\infty(X)$. Moreover, $\sigma(S) = \hat{S}(X) \subset \mathbb{C}^*$.

We do not introduce a topology in the algebra F, but we consider its order structure.
Recall that a Riesz space is called *Dedekind complete* if every non-empty subset which is bounded from above has a supremum.
Since X is Stonean, we have the following theorem.

3.4.2 THEOREM. *$C_r^\infty(X)$ is a Dedekind complete Riesz space. More precisely: if $f,g \in C_r^\infty(X)$ and $\lambda \in \mathbb{R}$, then the functions*

$$\lambda f, \quad f + g, \quad f \vee g, \quad f \wedge g$$

defined pointwise on the open dense subset of X where f and g are both finite, are extendible to continuous $\mathbb{R}^*$-valued functions on X (these extensions are also denoted by λf, $f + g$, $f \vee g$ and $f \wedge g$ respectively).

PROOF. See LUXEMBURG-ZAANEN [16], §47.

REMARK. Usually, $C_r^\infty(X)$ is defined as the set of all continuous functions $f : X \to \mathbb{R}^\infty := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ which take the values $\pm\infty$ only on a nowhere dense set.

3.4.3 DEFINITION. An element $S \in \Gamma''_\infty(T)$ is called *positive* if $\hat{S} \geq 0$ (or equivalently if $\sigma(S) \subset [0,\infty]$).
Note that any positive element is hermitian (cf. 2.2.4). Let H be the real linear space of all hermitian elements of $\Gamma''_\infty(T)$ (then $\sigma(S) \subset \mathbb{R}^*$ for all $S \in H$).
An order relation on H is defined by

$$S_1 \geq S_2 \quad \text{iff} \quad \hat{S}_1 \geq \hat{S}_2 \qquad (S_1,S_2 \in H).$$

3.4.4 LEMMA. *If A is a commutative C^*-algebra with unit of bounded operators on a Hilbert space H (with inner product $(.,.)$) and if*

$$\begin{aligned} A &\longrightarrow C(Y) \\ S &\longmapsto \hat{S} \end{aligned}$$

denotes the Gelfand map of A onto C(Y) where Y denotes the spectrum of A, *then*

$$\hat{S} \geq 0 \quad iff \quad (Sx,x) \geq 0 \quad for\ all\ x \in H.$$

PROOF. See SAKAI ([28], 1.1.4, p.8).

3.4.5 THEOREM. *Let* $S \in \Gamma''_\infty(T)$. *Then the following statements are equivalent:*
(i) S *is positive*,
(ii) $(Sx,x)_0 \geq 0$ *for all* $x \in H_\infty$,
(iii) $(\overline{S}x,x)_0 \geq 0$ *for all* $x \in D(\overline{S})$, *where* $\overline{S}$ *denotes the closure of* S *in* H_0.

PROOF. The proof of (ii) $\iff$ (iii) is easy.
(i) $\Rightarrow$ (ii). Let $\varepsilon > 0$. Since $-\varepsilon \notin \sigma(S)$, $S + \varepsilon I$ has a bounded inverse $C := (S + \varepsilon I)^{-1} \in L_h(H_0)$. Since $\hat{S} \geq 0$ and $\hat{C} = (\hat{S} + \varepsilon I)^{-1}$, it follows that $\hat{C} \geq 0$. Hence by 3.4.4,

$$0 \leq (Cx,x)_0 = ((S + \varepsilon I)^{-1}x,x)_0 \qquad (x \in H_\infty).$$

Since $S + \varepsilon I$ is a bijection of H_∞ onto itself, if follows that

$$((S + \varepsilon I)x,x)_0 \geq 0 \qquad (x \in H_\infty).$$

Since $\varepsilon > 0$ is arbitrary,

$$(Sx,x)_0 \geq 0 \qquad (x \in H_\infty).$$

(ii) $\Rightarrow$ (i). It follows that for all $p \in \mathbb{Z}$

$$(Sx,x)_p \geq 0 \qquad (x \in H_\infty).$$

So S is hermitian.
Let $\varepsilon > 0$. We have to show that $S + \varepsilon I$ has a bounded inverse. For all $p \in \mathbb{Z}$

$$\varepsilon \| x \|_p^2 \leq ((S + \varepsilon I)x,x)_p \leq \| (S + \varepsilon I)x \|_p \| x \|_p \qquad (x \in H_\infty).$$

So

$$(*) \qquad \| (S + \varepsilon I)x \|_p \geq \varepsilon \| x \|_p \qquad (x \in H_\infty).$$

This implies that $R(S + \varepsilon I)$ is closed in H_∞.
By 3.2.2 there is some $k \in \mathbb{N}$ such that $S + \varepsilon I \in L_\pi(H_{p+k}, H_p)$ for all $p \in \mathbb{Z}$. Then, by 3.2.4, $S^* + \varepsilon I$ maps H_p continuously into H_{p-k} $(p \in \mathbb{Z})$. Since S is hermitian, $S^* + \varepsilon I$ is an extension of $S + \varepsilon I$. It follows from (*) that $S^* + \varepsilon I$ is injective. Hence $R(S + \varepsilon I)$ is dense in H_∞.
Thus $S + \varepsilon I : H_\infty \to H_\infty$ is bijective. From (*) it follows that $(S + \varepsilon I)^{-1} \in L_\pi(H_p)$, so $(S + \varepsilon I)^{-1}$ is bounded. This means that $-\varepsilon \notin \sigma(S)$. So S is positive. □

In the next theorem we give a characterization of the algebra $F \subset C^\infty(X)$,

$$F = \{f \in C^\infty(X) \mid f = \hat{S} \text{ for some } S \in \Gamma''_\infty(T)\} .$$

3.4.6 <u>THEOREM</u>. *Let* $f \in C^\infty(X)$. *Then* $f = \hat{S}$ *for some* $S \in \Gamma''_\infty(T)$ *iff for some* $n \in \mathbb{N}$ *and some constant* $c \geq 0$

$$|f(\gamma)| \leq c|\hat{A}(\gamma)|^n \qquad (\gamma \in X).$$

<u>PROOF.</u> Suppose $S \in \Gamma''_\infty(T)$. Then there is a $k \in \mathbb{N}$ and a constant $c \geq 0$ such that

$$\| Sx \|_0 \leq c\| x \|_k = c\| A^k x \|_0 \qquad (x \in H_\infty).$$

So $(S^*Sx,x)_0 \leq c^2(A^{2k}x,x)_0$ and thus $(c^2A^{2k} - S^*S)^\wedge \geq 0$ by 3.4.5. This means

$$|\hat{S}(\gamma)| \leq c|\hat{A}(\gamma)|^k \qquad (\gamma \in X).$$

Conversely, if $f \in C^\infty(X)$ and $|f(\gamma)| \leq c|\hat{A}(\gamma)|^n$ $(\gamma \in X)$, then $\hat{A}^{-n}f \in C(X)$. Hence $\hat{A}^{-n}f = \hat{S}$ for some bounded element $S \in \Gamma''_\infty(T)$. Thus $f = (SA^n)^\wedge$. □

3.4.7 <u>THEOREM.</u> *H is a Dedekind complete Riesz space.*

<u>PROOF.</u> It follows from 3.4.6 that H is a Riesz subspace (even an ideal) of $C_r^\infty(X)$. We prove it is Dedekind complete.
Let $\{S_\alpha\}$ be a non empty subset which is bounded from above by S. Since $C_r^\infty(X)$ is a Dedekind complete Riesz space, $\sup \hat{S}_\alpha =: g$ exists in $C_r^\infty(X)$. Since $\hat{S}_\alpha \leq g \leq \hat{S}$, it follows by 3.4.6 that $g = \hat{B}$ for some $B \in H$. Then $B = \sup S_\alpha$. □

The relation between the topology and the order of H is given in the next theorem.

3.4.8 THEOREM. *Let* $(S_n)_{n=1}^{\infty}$ *be an increasing sequence of positive elements of* H *with* $\sup S_n = S \in H$. *Then for all* $x \in H_\infty$

$$S_n x \longrightarrow Sx \text{ in } H_\infty \text{ as } n \to \infty \,.$$

PROOF. Since $S_n \geq 0$, we have $S_n S_m \geq S_n^2$ for $m \geq n$. If $p \in \mathbb{Z}$ and $x \in H_\infty$ then for $m \geq n$

$$0 \leq \| S_m x - S_n x \|_p^2 = (S_m^2 x,x)_p - 2(S_n S_m x,x)_p + (S_n^2 x,x)_p \leq$$
$$\leq (S_m^2 x,x)_p - (S_n^2 x,x)_p \,.$$

Since the sequence $\{(S_n^2 x,x)_p \mid n \in \mathbb{N}\}$ converges (in $\mathbb{R}$), it follows that $(S_n x)_{n=1}^{\infty}$ converges in H_∞.
From $S_n^2 \leq S^2$ it follows that $\| S_n x \|_p^2 \leq \| Sx \|_p^2$ $(x \in H_\infty,\ p \in \mathbb{Z})$. So $(S_n)_{n=1}^{\infty}$ is equicontinuous in $L(H_\infty)$.
Hence, by the Banach-Steinhaus theorem, the pointwise limit C of the sequence sequence $(S_n)_{n=1}^{\infty}$ is in $L(H_\infty)$. Also it is clear that $C \in H$. Since $(Cx,x)_p = \lim_{n\to\infty} (S_n x,x)_p = \sup_n (S_n x,x)_p$ $(x \in H_\infty)$, we conclude that $C = \sup_n S_n$. So $C = S$. □

Let us denote by $\sigma_\infty(T)$ the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ has no inverse in $L(H_\infty)$. And let $\sigma(\overline{T})$ be the spectrum of T as an operator on H_0. Then we have the following theorem (cf. 3.3.7).

3.4.9 THEOREM. $\sigma_\infty(T) = \sigma(T) \cap \mathbb{C} = \sigma(\overline{T})$.

PROOF. We have to prove: if $\lambda I - T$ is invertible in $L(H_\infty)$, then $\lambda I - T$ has a bounded inverse in $L(H_\infty)$ $(\lambda \in \mathbb{C})$.
Suppose, $\lambda I - T$ has an inverse $C \in L(H_\infty)$. Then $C \in \Gamma_\infty''(T)$ and $\hat{C} = (\lambda - \hat{T})^{-1}$. Suppose $(\lambda - \hat{T})(\gamma) = 0$ for some $\gamma \in X$. Then $\hat{C}(\gamma) = \infty$. Hence by 3.4.6, $\hat{A}(\gamma) = \infty$. So $\hat{T}(\gamma) = \infty$. Contradiction. This means that $\lambda - \hat{T}$ is nowhere 0 on X. Thus $\lambda I - T$ has a bounded inverse. □

3.5 THE SPECTRAL MEASURE OF THE BICOMMUTANT OF T

Let X again be the spectrum of the commutative W^*-algebra $b(\Gamma''_\infty(T))$. Then X is a Stonean space. Let us recall the construction of the spectral measure E(.) on X (see DOUGLAS-PEARCY [8]).

Let M be a Borel subset of X. Since X is a Stonean space there is a unique compact open set $\tilde{M}$ such that $M \Delta \tilde{M}$ is a set of the first category (i.e., a countable union of nowhere dense sets). Then E(M) is defined to be the unique element of H such that

$$E(M)^{\wedge} = \chi_{\tilde{M}} ,$$

where $\chi_{\tilde{M}}$ denotes the characteristic function of $\tilde{M}$.
In this way a function E(.) is defined whose domain is the σ-algebra of all Borel subsets of X and whose values are idempotents in H. The function E(.) has the following properties:

(i) $E(M) \geq 0$,

(ii) $E(X) = I$ (= identity element of $\Gamma''_\infty(T)$),

(iii) $E\left(\bigcup_{n=1}^{\infty} M_n\right) = \sup_n \sum_{k=1}^{n} E(M_k)$ where $(M_n)_{n=1}^{\infty}$ is a sequence of pairwise disjoint Borel subsets of X.

Note that a Borel subset of the first category has E-measure 0.

In this section we shall make use of the theory of integration with respect to a Riesz space valued measure (see HACKENBROCH [11]).
Let us recall the definition of the integral.

If f is a step function, i.e. $f = \sum_{i=1}^{n} \alpha_i \chi_{M_i} : X \to \mathbb{R}$ where $(M_i)_{i=1}^{n}$ is a pairwise disjoint family of Borel subsets of X, then $\int f dE$ is defined by

$$\int f dE := \sum_{i=1}^{n} \alpha_i E(M_i) .$$

A Borel function $f : X \to [0,\infty]$ is called E-*integrable* if there exists a sequence $(f_n)_{n=1}^{\infty}$ of step functions such that

(i) $0 \leq f_n \uparrow f$ pointwise,

(ii) $\sup_n (\int f_n dE)$ exists in H.

For such a function f we define

$$\int f dE := \sup_n \left(\int f_n dE\right) .$$

It can be shown that $\int f dE$ is independent of the choice of the sequence $(f_n)_{n=1}^{\infty}$ satisfying (i).
A Borel function $f : X \to \mathbb{R}^*$ which is E-almost everywhere finite, is called E-*integrable* when f^+ and f^- (which are defined E-almost everywhere) are E-integrable. The integral is defined by

$$\int f dE := \int f^+ dE - \int f^- dE .$$

A Borel function $f : X \to \mathbb{C}^*$ which is E-almost everywhere finite is called E-*integrable* when Re(f) and Im(f) are E-integrable and the integral is defined by

$$\int f dE := \int \mathrm{Re}(f) dE + i \int \mathrm{Im}(f) dE .$$

Let $L^1(X; E, H)$ denote the set of all functions $f : X \to \mathbb{R}^*$ which are E-integrable; two functions that are E-almost everywhere equal are identified.

3.5.1 THEOREM.
(i) $L^1(X; E, H)$ *is a linear space and the map*

$$f \longmapsto \int f dE$$

of $L^1(X; E, H)$ *into* H *is positive.*
(ii) (BEPPO LEVI) *Let* (f_n) *be a monotone increasing sequence of non negative* E-*integrable functions with pointwise* sup $f_n = f$. *If* $\int f_n dE \leq B$ *for some* $B \in H$ $(n \in \mathbb{N})$ *then* f *is also* E-*integrable and*

$$\int f_n dE \uparrow \int f dE \text{ in } H.$$

PROOF. See HACKENBROCH [11]. □

Now we are able to derive a spectral representation theorem for the hermitian elements of $\Gamma''_{\infty}(T)$.

3.5.2 THEOREM.
(i) *If* $B \in H$ *then* $\hat{B} \in L^1(X; E, H)$ *and* $B = \int \hat{B} dE$.
(ii) *If* $f \in L^1(X; E, H)$, *then* $(\int f dE)^{\wedge} = f$ *except perhaps on a set of the first category.*

PROOF.

(i) First we assume that B is positive and bounded. Then $\hat{B}$ is a positive (bounded) function in C(X). Since X is a Stonean space, there exists an increasing sequence of continuous step functions (g_n) such that $g_n \uparrow \hat{B}$ pointwise. Put $C_n := \int g_n dE$. Then it is a consequence of the definition of the spectral measure E(.) that $\hat{C}_n = g_n$. So $\hat{C}_n \uparrow \hat{B}$. Hence $C_n \uparrow B$ (in H). Since also $C_n = \int g_n dE \uparrow \int \hat{B} dE$ (cf. 3.5.1 (ii)), it follows that $B = \int \hat{B} dE$.

Now let B be an arbitrary positive element. Then $(B \wedge nI)^\wedge = \hat{B} \wedge n1$ (I is the identity operator on H_∞ and 1 is the function on X that is constantly 1). Since clearly $\hat{B} \wedge n1 \uparrow B$ $(n \in \mathbb{N})$, it follows that

$$\int (B \wedge nI)^\wedge dE \uparrow \int \hat{B} dE \quad \text{in } H \qquad (n \in \mathbb{N}).$$

By the first part of the proof

$$B \wedge nI = \int (B \wedge nI)^\wedge dE .$$

Since $B \wedge nI \uparrow B$ (becuase $\hat{B} \wedge n1 \uparrow \hat{B}$), it follows that $B = \int \hat{B} dE$. For an arbitrary $B \in H$ the theorem follows by writing $B = B^+ - B^-$ where B^+ and B^- are positive elements.

(ii) If f is a real-valued function in C(X), then $f = \hat{B}$ for some bounded element $B \in H$. Since $\int \hat{B} dE = B$, it follows that $(\int f dE)^\wedge = f$.

If f is a real-valued bounded Borel function, then there is a unique continuous function g (not taking the value ∞), such that f = g except perhaps on a set of the first category (cf. DOUGLAS-PEARCY [8]). So f = g E-almost everywhere. Hence $\int f dE = \int g dE$. If $g = \hat{B}$ then $\int f dE = \int \hat{B} dE = B$. Thus $(\int f dE)^\wedge = g$, and $(\int f dE)^\wedge = f$ except perhaps on a set of the first category.

If $f : X \to [0,\infty]$ is a Borel function, the theorem follows by considering the sequence (f_n) where

$$f_n := f \wedge n1 \qquad (n \in \mathbb{N}). \; \square$$

3.5.3 <u>COROLLARY.</u> *A Borel subset* $M \subset X$ *has* E*-measure* 0 *iff it is of the first category.*

PROOF. If $E(M) = 0$, then $0 = E(M)^{\wedge} = (\int \chi_M dE)^{\wedge} = \chi_M$ except perhaps on a set of the first category (χ_M denotes the characteristic function of M). □

REMARK. In a hyper-Stonean space every first category set is nowhere dense (cf. BADE [34], p.102). So from 3.5.3 it follows that $E(M) = 0$ iff M is nowhere dense.

3.5.4 THEOREM. (Spectral theorem) *Let* $B \in \Gamma''_\infty(T)$ *and let* $\overline{B}$ *denote the closure of* B *as an operator in* H_0. *Let*

$$B_n = \int_{V_n} \hat{B} dE,$$

where $V_n = \{\gamma \in X \mid |\hat{B}(\gamma)| \le n\}$ *and* E *is considered as a measure with values in* $L(H_0)$. *Let* $\overline{B}$ *be the closure of* B *as an operator in* H_0.
Then $B_n x \to \overline{B}x$ *in* H_0 *as* $n \to \infty$ *for* $x \in D(\overline{B})$.

PROOF. It suffices to prove the theorem for B positive. Since $B_n \uparrow B$ by 3.5.1 (ii), it follows by 3.4.8 that $B_n x \to Bx$ in H_0 for $x \in H_\infty$. We equip $D(\overline{B})$ with the graph norm. Then B and B_n considered as operators from $D(\overline{B})$ into H_0 have norm ≤ 1. Since H_∞ is dense in $D(\overline{B})$ we may conclude that $B_n x \to \overline{B}x$ in H_0 for $x \in D(\overline{B})$. □

3.5.5 DEFINITION. Let $B \in \Gamma''_\infty(T)$. The *spectral measure* E_B of B (defined on $\sigma(B)$) is defined by

$$E_B(M) := E(\hat{B}^{-1}(M))$$

where M is a Borel subset of $\sigma(B)$.

3.5.6 COROLLARY. *Let* $B \in \Gamma''_\infty(T)$. *Then*

$$\overline{B}x = \lim_{n\to\infty} \left(\int_{D_n} \lambda dE_B \right) x \qquad (x \in D(\overline{B})$$

where $D_n \subset \mathbb{C}$ *is the set* $\{z \mid |z| \le n\}$.

If V is a bounded normal operator on a Hilbert space and B is in the bicommutant of V then there exists a bounded Borel function g on $\sigma(V)$ such that $B = g(V)$. This theorem is well-known (see, for example, SCHWARTZ [30], p.14). A similar result can be derived for the bicommutant $\Gamma''_\infty(T)$ of T.

3.5.7 DEFINITION. Let $F_1(\sigma(T))$ be the algebra of all Borel functions $f : \sigma(T) \cap \mathbb{C} \longrightarrow \mathbb{C}$ such that $|f(\lambda)| \leq c(1 + |\lambda|^2)^n$ $(\lambda \in \sigma(T) \cap \mathbb{C})$ for some constant c and some $n \in \mathbb{N}$.
If $f \in F_1(\sigma(T))$ then $g := f \circ \hat{T}$ is a Borel function on X and

$$|g(\gamma)| \leq c|\hat{A}(\gamma)|^n \qquad (\gamma \in X)$$

for some constant c and some $n \in \mathbb{N}$.
From this it follows that g is E-integrable. We define

$$f(T) := \int g dE .$$

REMARK. It follows (cf. 3.5.2 (ii)) that $f(T)^\wedge = f \circ \hat{T}$ except perhaps on a set of the first category. So $f(T)^\wedge = f \circ \hat{T}$ E-almost everywhere.

3.5.8 THEOREM. *The map*

$$\begin{aligned} F_1(\sigma(T)) &\longrightarrow \Gamma''_\infty(T) \\ f &\longmapsto f(T) \end{aligned}$$

is a homomorphism onto. And $f(T) = 0$ *iff* f *is* E_T*-almost everywhere equal to zero.*

PROOF. We prove that the map is surjective.
Let $B \in H \subset \Gamma''_\infty(T)$. Then $(B - iI)^{-1} \in b(\Gamma''_\infty(T)) = \Gamma''_0(V)$ where V is the Cayley transform of T. Then $(\hat{B} - iI)^{-1} = g \circ \hat{V}$ where g is some bounded Borel function on $\sigma(V)$. Hence $(\hat{B} - iI)^{-1} = h \circ \hat{T}$, where h is a bounded Borel function on $\sigma(T)$. So $\hat{B} = f \circ \hat{T}$ for some Borel function f on $\sigma(T)$.
Since $B \in \Gamma''_\infty(T)$, it follows from 3.4.6 that $f \in F_1(\sigma(T))$. □

3.6 TENSOR PRODUCTS

We are interested in spectral properties of operators of the form $P(T \otimes I,\ I \otimes S)$; here T and S are selfadjoint operators and P is a polynomial (in two variables). The case $T \otimes I + I \otimes S$ has been treated by L. & K. MAURIN [19]. In this section we want to show how these results of L. & K. MAURIN can be derived in a systematic way.
The notations are as in section 1.7.

If $P \in L(H_\infty^{(A)})$ and $Q \in L(H_\infty^{(B)})$, then it is easily seen that $P \otimes Q$ extends to a continuous linear operator in $H_\infty^{(A\otimes B)}$. This defines a mapping (the canonical mapping)

$$L\left(H_\infty^{(A)}\right) \otimes L\left(H_\infty^{(B)}\right) \longrightarrow L\left(H_\infty^{(A\otimes B)}\right).$$

3.6.1 LEMMA. *The canonical mapping*

$$L\left(H_\infty^{(A)}\right) \otimes L\left(H_\infty^{(B)}\right) \longrightarrow L\left(H_\infty^{(A\otimes B)}\right)$$

is injective.

PROOF. Suppose $\sum_{i=1}^n P_i \otimes Q_i$ defines the zero operator in $L(H_\infty^{(A\otimes B)})$. If $\alpha \in H_{-\infty}^{(A)}$ and $\beta \in H_{-\infty}^{(B)}$, then $\alpha \otimes \beta$ defines a continuous linear functional on $H_\infty^{(A)} \otimes H_\infty^{(B)}$ by

$$(x \otimes y,\ \alpha \otimes \beta) := (x,\alpha)_0 (y,\beta)_0 \qquad (x \in H_\infty^{(A)},\ y \in H_\infty^{(B)}).$$

Then our assumption implies

$$\begin{aligned} 0 &= \left(\Big(\sum_{i=1}^n P_i \otimes Q_i\Big)(x \otimes y),\ \alpha \otimes \beta\right) = \\ &= \sum_{i=1}^n (P_i x,\alpha)_0 (Q_i y,\beta)_0 = \\ &= \sum_{i=1}^n (P_i,\ x \otimes \alpha)(Q_i,\ y \otimes \beta)\ , \end{aligned}$$

where $x \otimes \alpha \in H_\infty^{(A)} \otimes H_{-\infty}^{(A)}$ is the linear functional on $L(H_\infty^{(A)})$ defined by

$$(P,\ x \otimes \alpha) := (Px,\alpha)_0 \qquad (P \in L(H_\infty^{(A)})),$$

and similarly for $y \otimes \beta$.
Since $H_\infty^{(A)} \otimes H_{-\infty}^{(A)}$ is total over $L(H_\infty^{(A)})$ (and $H_\infty^{(B)} \otimes H_{-\infty}^{(B)}$ is total over $L(H_\infty^{(B)})$), it follows that

$$\left(\sum_{i=1}^{n} P_i \otimes Q_i, \mu\right) = 0$$

for all $\mu \in L(H_\infty^{(A)})^* \otimes L(H_\infty^{(B)})^*$, where $L(H_\infty^{(A)})^*$ and $L(H_\infty^{(B)})^*$ denote the algebraic duals of $L(H_\infty^{(A)})$ and $L(H_\infty^{(B)})$ respectively. Since $\left(L(H_\infty^{(A)}) \otimes L(H_\infty^{(B)}), L(H_\infty^{(A)})^* \otimes L(H_\infty^{(B)})^*\right)$ form a dual pair, it follows that $\sum_{i=1}^n P_i \otimes Q_i = 0$. □

We put $\mathcal{A} := \Gamma''_\infty(T) \subset L(H_\infty^{(A)})$ and $\mathcal{B} := \Gamma''_\infty(S) \subset L(H_\infty^{(B)})$. And we consider $\mathcal{A} \otimes \mathcal{B}$ as a subset of $L(H_\infty^{(A\otimes B)})$. Then we have the following inclusions:

$$\Gamma''_\infty(A \otimes B) \subset \Gamma''_\infty(\mathcal{A} \otimes \mathcal{B}) \subset \Gamma'_\infty(\mathcal{A} \otimes \mathcal{B}) \subset \Gamma'_\infty(A \otimes B) .$$

It follows from 3.2.13 that $\Gamma''_\infty(\mathcal{A} \otimes \mathcal{B})$ is a commutative GB^*-algebra. We shall prove that it is a GW^*-algebra, i.e. the bounded elements of $\Gamma''_\infty(\mathcal{A} \otimes \mathcal{B})$ form a W^*-algebra (cf. 3.3.5).

3.6.2 LEMMA.
(i) $\Gamma'_\infty(\mathcal{A} \otimes \mathcal{B}) = \Gamma'_\infty(b(\mathcal{A}) \otimes b(\mathcal{B}))$
(ii) $\Gamma'_\infty(b(\mathcal{A}) \otimes b(\mathcal{B})) \cap L_h(H_0^{(A\otimes B)}) = \Gamma'_0(b(\mathcal{A}) \otimes b(\mathcal{B}))$;
$\Gamma'_0(b(\mathcal{A}) \otimes b(\mathcal{B}))$ *is the commutant of* $b(\mathcal{A}) \otimes b(\mathcal{B})$ *in* $L(H_0^{(A\otimes B)})$; *so* $b(\mathcal{A}) \otimes b(\mathcal{B})$ *has to be considered here as a subset of* $L(H_0^{(A\otimes B)})$.

PROOF.
(i) Clearly, $\Gamma'_\infty(\mathcal{A} \otimes \mathcal{B}) \subset \Gamma'_\infty(b(\mathcal{A}) \otimes b(\mathcal{B}))$.
Conversely, let $S \in \Gamma'_\infty(b(\mathcal{A}) \otimes b(\mathcal{B}))$. It is sufficient to show that S commutes with all elements of the form $P \otimes Q$ where $P \in \mathcal{A}$ and $Q \in \mathcal{B}$ are hermitian elements.
Since $(P - iI)^{-1} \in b(\mathcal{A})$, it follows that S commutes with $(P - iI)^{-1} \otimes I$, hence with $(P - iI) \otimes I$ and with $P \otimes I$. Similarly S commutes with $I \otimes Q$. So S commutes with $P \otimes Q$.
(ii) If $C \in \Gamma'_\infty(b(\mathcal{A}) \otimes b(\mathcal{B})) \cap L_h(H_0^{(A\otimes B)})$ and $S \in b(\mathcal{A}) \otimes b(\mathcal{B})$, then $CS = SC$ on $H_\infty^{(A\otimes B)}$. Hence the continuous extensions of C and S to $H_0^{(A\otimes B)}$ commute with each other.
Conversely, let $C \in \Gamma'_0(b(\mathcal{A}) \otimes b(\mathcal{B}))$. Since $A^{-1} \otimes B^{-1} \in b(\mathcal{A}) \otimes b(\mathcal{B})$,

C commutes with $A^{-1} \otimes B^{-1}$. Hence C leaves invariant $H_\infty^{(A\otimes B)}$ and $C \in \Gamma'_\infty(b(A) \otimes b(B))$. □

3.6.3 THEOREM. $b(\Gamma''_\infty(A \otimes B)) = \Gamma''_0(b(A) \otimes b(B))$.

PROOF. If $C \in \Gamma''_0(b(A) \otimes b(B))$, then C leaves $H_0^{(A\otimes B)}$ invariant. In order to show that $C \in \Gamma''_\infty(A \otimes B)$, it suffices to show that C commutes with every hermitian element $S \in \Gamma'_\infty(A \otimes B)$.
The same reasoning as in the proof of 3.3.4 gives that $(S - i)^{-1}$ is a normal element of $\Gamma'_\infty(A \otimes B) \cap L_h(H_0^{(A\otimes B)}) = \Gamma'_0(b(A) \otimes b(B))$ (cf. 3.6.2 (ii)). So C commutes with $(S - i)^{-1}$ and hence with S.
Conversely, if $C \in b(\Gamma''_\infty(A \otimes B) = \Gamma''_\infty(A \otimes B) \cap L_h(H_0^{(A\otimes B)})$ (cf. 3.2.9), then C commutes with $\Gamma'_\infty(A \otimes B) = \Gamma'_\infty(b(A) \otimes b(B))$ (cf. 3.6.2 (i)).
So it follows by 3.6.2 (ii) that $C \in \Gamma''_0(b(A) \otimes b(B))$. □

3.6.4 THEOREM.

(i) *Let* $C \in \Gamma''_\infty(A \otimes B)$, *then* C *is closable as an operator in* $H_0^{(A\otimes B)}$ *and its closure* $\overline{C}$ *is a normal operator in* $H_0^{(A\otimes B)}$. *Moreover,* $\sigma(\overline{C}) = \sigma(C) \cap \mathbb{C}$ (cf. 3.3.7).

(ii) *Let* P *be a polynomial in two variables. Then the spectrum of* $P(T \otimes I, I \otimes S)$ *is the closure (in* $\mathbb{C}$*) of the range of* P *on the product of the spectra* $\sigma(\overline{T}) \times \sigma(\overline{S})$, *where* $\sigma(\overline{T})$ *and* $\sigma(\overline{S})$ *are the spectra of* T *and* S *considered as selfadjoint operators on* $H_0^{(A)}$ *and* $H_0^{(B)}$ *resp.; that is*

$$\sigma(\overline{P(T \otimes I, I \otimes S)}) = \overline{P(\sigma(\overline{T}), \sigma(\overline{S}))} .$$

PROOF.

(i) This follows from 3.3.7.

(ii) Let X, Y and Z be the spectra of $b(A)$, $b(B)$ and $b(\Gamma''_\infty(A \otimes B))$ respectively. Let

$$\hat{T} : X \longrightarrow \mathbb{C}^* \text{ and } \hat{S} : Y \longrightarrow \mathbb{C}^*$$

denote the Gelfand images of T and S respectively. And let

$$(T \otimes I)^\wedge : Z \longrightarrow \mathbb{C}^* \text{ and } (I \otimes S)^\wedge : Z \longrightarrow \mathbb{C}^*$$

denote the Gelfand images of $T \otimes I$ and $I \otimes S$ respectively.

Let

$$p : Z \longrightarrow X \times Y$$

be the canonical surjection (recall that $b(A)$, $b(B)$ and $b(\Gamma''_\infty(A \otimes B))$ are isomorphic to $C(X)$, $C(Y)$ and $C(Z)$ resp.).

Then, for $\gamma \in Z$

$$(T \otimes I)^\wedge(\gamma) = \hat{T}(\alpha) \quad \text{and} \quad (I \otimes S)^\wedge(\gamma) = \hat{S}(\beta) \ ,$$

where $(\alpha,\beta) = p(\gamma)$. Now let

$$M_1 := \{\alpha \in X \mid \hat{T}(\alpha) \neq \infty\} \quad \text{and} \quad M_2 := \{\beta \in Y \mid \hat{S}(\beta) \neq \infty\} \ .$$

Then (by 3.3.7)

$$\hat{T}(M_1) = \sigma(\overline{T}) \quad \text{and} \quad \hat{S}(M_2) = \sigma(\overline{S}) \ .$$

Let

$$M := \{\gamma \in Z \mid (T \otimes I)^\wedge(\gamma) \neq \infty \text{ and } (I \otimes S)^\wedge(\gamma) \neq \infty\} \ .$$

Then M is an open dense subset of Z and

$$p(M) = M_1 \times M_2 \ .$$

So $\sigma(P(T \otimes I, I \otimes S))$ is equal to the closure in $\mathbb{C}^*$ of the set

$$\begin{aligned}&\{P((T \otimes I)^\wedge(\gamma), (I \otimes S)^\wedge(\gamma) \mid \gamma \in M\} = \\ &= \{P(\hat{T}(\alpha),\hat{S}(\beta)) \mid \alpha \in M_1, \beta \in M_2\} = \\ &= P(\sigma(\overline{T}),\sigma(\overline{S})) \ .\end{aligned}$$

This implies the statement. □

<u>REMARK.</u> A spectral mapping theorem for tensor products of closed operators on Banach spaces was proved by REED & SIMON [27]. Their method of proof is also an algebraic one. However, they consider only algebras of bounded operators.

CHAPTER IV

DIRECT INTEGRALS OF HILBERT SPACES AND DISINTEGRATION OF MEASURES

In this chapter we present some facts which will be needed in the next chapter. The first section contains some basic facts about direct integrals of Hilbert spaces. In the second section we prove the main theorem concerning disintegration of measures.

4.1 DIRECT INTEGRALS OF HILBERT SPACES

As for the theory of direct integrals we follow DIXMIER [6] and SAKAI [28].

Let Z be a compact Hausdorff space and let μ be a positive Radon measure on Z. A collection $\{H(z) \mid z \in Z\}$ of separable Hilbert spaces is called a *field of Hilbert spaces*. The norm and the inner product in H(z) are denoted by $\|\,.\,\|_z$ and $(.,.)_z$ respectively. Elements of $\prod_{z\in Z} H(z)$ are called *vector fields*.

4.1.1 <u>DEFINITION.</u> A sequence $(x_n)_{n=1}^{\infty}$ in $\Pi H(z)$ is called a *measurable fundamental sequence* if

(i) the functions $z \mapsto (x_n(z),x_m(z))_z$ are μ-measurable for all n and m,

(ii) the sequence $(x_n(z))_{n=1}^{\infty}$ is total in H(z) for every $z \in Z$.

The field $\{H(z)\}$ is called a *measurable field* if there exists a measurable fundamental sequence (x_n) in $\Pi H(z)$.

A linear subspace $V \subset \Pi H(z)$ is called *measurable* if

(i) the functions $z \mapsto (x(z),y(z))_z$ are μ-measurable for all $x,y \in V$,

(ii) V contains a measurable fundamental sequence (x_n).

Let $\{H(z)\}$ be a measurable field. Let $\{V_\alpha\}$ be the set of all measurable linear subspaces. We define an order in $\{V_\alpha\}$ by set inclusion. Then, by Zorn's lemma there exists a maximal measurable subspace V_0.

Let $M(Z,\mu)$ be the linear space of all measurable complex-valued functions on Z. If $f \in M(Z,\mu)$ and x is a vector field, then fx will be the vector-field

$$fx : z \longmapsto f(z)x(z) \qquad (z \in Z).$$

If V is a measurable subspace then the linear space generated by $\{fx \mid f \in M(z,\mu),\ x \in V\}$ is again measurable. Hence V_0 is invariant under $M(Z,\mu)$.

4.1.2 <u>THEOREM</u>. *Let* V_0 *be a maximal measurable subspace. Then there exists a countable family* $(e_n)_{n=1}^{\infty}$ *in* V_0 *such that* $(e_n(z))_{n=1}^{\infty}$ *is a complete orthonormal system in* $H(z)$ *for every* $z \in Z$.

<u>PROOF.</u> See SAKAI ([28], p.138). □

Such a sequence (e_n) is called a *measurable field of orthonormal bases.*

4.1.3 <u>REMARK.</u> The sequence (e_n) is constructed from a measurable fundamental sequence (x_n). Let ν be another measure on Z such that (x_n) is also a ν-measurable fundamental sequence for the field $\{H(z)\}$.
An analysis of the proof of 4.1.2 shows that the sequence (e_n) is also a ν-measurable field of orthonormal bases.

4.1.4 <u>DEFINITION.</u> Let K_0 be the set of all elements x in V_0 with

$$\int \| x(z) \|_z^2 \, d\mu(z) < \infty .$$

For $x,y \in K_0$ we define

$$(x,y) := \int (x(z),y(z))_z \, d\mu(z) .$$

Two elements of K_0 are identified when they are almost everywhere equal. Then we obtain an inner product space which is denoted by H_0.

4.1.5 <u>PROPOSITION.</u>

(i) H_0 *is a Hilbert space.*

(ii) *If* (x_n) *is a measurable fundamental sequence in* K_0*, then the linear space generated by* $\{fx_n \mid f \in C(Z),\ n \in \mathbb{N}\}$ *is dense in* H_0.

<u>PROOF.</u> See SAKAI ([28], p.139). □

4.1.6 THEOREM. *Let* $\{H(z)\}$ *be a measurable field of Hilbert spaces and let* V_0 *and* V_1 *be two maximal measurable subspaces of* $\prod H(z)$. *Let* H_0 *and* H_1 *be the Hilbert spaces associated with* V_0 *and* V_1 *respectively. Then for each* $z \in Z$ *there exists a unitary map* $V(z)$ *of* $H(z)$ *onto itself such that the map which sends* $x \in H_0$ *into the field* $z \mapsto V(z)x(z)$ *is a unitary map of* H_0 *onto* H_1.

PROOF. See SAKAI ([28], p.140). □

4.1.7 DEFINITION. Theorem 4.1.6 shows that we obtain an essentially unique Hilbert space H_0 from a given measurable field $\{H(z)\}$. The Hilbert space H_0 is called *the direct integral of the field* $\{H(z)\}$.
It is denoted by

$$\int H(z)\, d\mu(z) \ .$$

The following definition is given for later use.

4.1.8 DEFINITION. For $f \in L^\infty(Z,\mu)$ the multiplication operator M_f on $\int H(z)d\mu(z)$ is defined by

$$M_f : x \longmapsto fx \qquad (x \in \int H(z)d\mu(z)).$$

An operator of the form M_f is called a *diagonalizable operator*. If $f \in C(Z)$, then M_f is called *continuously diagonalizable*.

4.2 DISINTEGRATION OF MEASURES

Let (Z,E,μ) be a complete measure space. And denote by $M^\infty(Z,\mu)$ the algebra of all complex-valued bounded measurable functions on Z, by $N^\infty(Z,\mu)$ the ideal of all $f \in M^\infty(Z,\mu)$ which are locally μ-almost everywhere negligible and by $L^\infty(Z,\mu)$ the quotient algebra $M^\infty(Z,\mu)/N^\infty(Z,\mu)$. The canonical image of $f \in M^\infty(Z,\mu)$ in $L^\infty(Z,\mu)$ is denoted by $\tilde{f}$. By abuse of notation we shall sometimes not distinguish between f and $\tilde{f}$.
A *linear lifting* of $L^\infty(Z,\mu)$ is a linear positive map

$$\rho : L^\infty(Z,\mu) \longrightarrow M^\infty(Z,\mu)$$

satisfying $\rho(\tilde{1}) = 1$ and $\rho(\tilde{f})^\sim = \tilde{f}$ for all $f \in L^\infty(Z,\mu)$.
A *lifting* of $L^\infty(Z,\mu)$ is a linear lifting of $L^\infty(Z,\mu)$ which is also multi-

plicative.
A theorem of MAHARAM [17] asserts that every finite measure space admits a lifting. Using this result one can prove that a measure space $(Z,\mathcal{E},\mu)$ admits a lifting if it is a disjoint union of μ-summable sets. In particular, if Z is locally compact and μ is a positive Radon measure on Z, then a lifting exists. For the proofs we refer to A. & C. IONESCU TULCEA [13].
Assume that Z is locally compact and μ is a positive Radon measure on Z with $\mathrm{supp}(\mu) = Z$. A linear lifting of $L^\infty(Z,\mu)$ is called a *strong linear lifting* if $\rho(f) = f$ for all real-valued continuous and bounded functions f on Z.
If X is a locally compact metrizable space and μ is a positive Radon measure on X with $\mathrm{supp}(\mu) = X$, then (X,μ) has the strong lifting property (see A. & C. IONESCU TULCEA [13], Ch.VIII, §4).

To avoid complications of technical nature we restrict ourselves to compact spaces with positive Radon measures.
The proof of the theorem about disintegration of measures is based on the Dunford-Pettis theorem. The existence of (strong) liftings permits to remove the separability assumptions which are usually made.

Let T be a compact Hausdorff space and let ν be a positive Radon measure on T. Let E be a Banach space and let E' be its dual. We denote by $M^\infty(T,\nu;\, E')$ the space of all mappings $h : T \to E'$ with the following properties

(i) $h(T) \subset E'$ is bounded,

(ii) $\langle h,x\rangle \in M^\infty(T,\nu)$ for every $x \in E$, where $\langle h,x\rangle$ denotes the function $t \mapsto \langle h(t),x\rangle$.

Note that $\|h\|_\infty := \sup_{t\in T} \|h(t)\| < \infty$.

The next theorem is a representation theorem for continuous linear maps U of $L^1(T,\nu)$ into E'. In the case that $E = \mathbb{C}$, this representation is well-known. In fact, if U is a continuous linear functional on $L^1(T,\nu)$ then there exists an element $h \in L^\infty(T,\nu)$ such that

$$U(g) = \int_T g(t)\, h(t)\, d\nu(t) \qquad (g \in L^1(T,\nu)).$$

4.2.1 THEOREM (DUNFORD-PETTIS). *Let* ρ *be a linear lifting of* $L^\infty(T,\nu)$. *Let*

$$U : L^1(T,\nu) \longrightarrow E'$$

be a continuous linear map. Then there exists a unique mapping $h : T \to E'$, $h \in M^{\infty}(T,\nu; E')$ *such that*

(i) $\rho\langle h,x\rangle = \langle h,x\rangle$ *for all* $x \in E$,

(ii) $\langle Ug,x\rangle = \int_T g(t)\langle h(t),x\rangle \, d\nu(t)$ *for all* $g \in L^1(T, \ )$ *and all* $x \in E$,

(iii) $\| h \|_{\infty} = \| U \|$.

PROOF. Let

$$U' : E \longrightarrow L^{\infty}(T,\nu)$$

be the adjoint of U (restricted to E). For a $t \in T$ we consider the linear functional

$$x \longmapsto (\rho U'(x))(t) \qquad (x \in E).$$

This is a continuous linear functional on E which we call h(t). Thus $h(t) \in E'$ and

$$\langle h(t),x\rangle = (\rho U'(x))(t) \ .$$

Now one has to verify that h satisfies the required properties. We refer to A. & C. IONESCU TULCEA ([13], p.87). □

REMARK. If E is a Banach lattice and U is a positive linear map, then U' is also positive. So h(t) is a positive linear functional on E for all $t \in T$.

We apply the Dunford-Pettis theorem in the case that $E = C(S)$, the space of all complex-valued continuous functions on the compact Hausdorff space S. The dual of C(S) is M(S), the space of Radon measures on S. The set of all positive Radon measures is denoted by $M^+(S)$.
The next theorem about disintegration of measures is an adapted version of two theorems in the book of A. & C. IONESCU TULCEA [13] (namely Th.2 and Th.5 of Ch.IX).

4.2.2 THEOREM (Disintegration of measures).
Let S *and* T *be two compact Hausdorff spaces and let* $\pi : S \to T$ *be a continuous surjection. Let* $\mu \neq 0$ *be a positive Radon measure on* S *and let* $\nu = \pi(\mu)$ *be the image of* μ *under the map* π.

Suppose that r *is a strong lifting of* $L^\infty(T,\nu)$. *Then there is a mapping* $t \mapsto \lambda_t$ *of* T *into* $M^+(S)$ *with the following properties*

(i) $\langle f,\lambda\rangle \in M^\infty(T,\nu)$ *for all* $f \in C(S)$ *and* $r\langle f,\lambda\rangle = \langle f,\lambda\rangle$ *for all* $f \in C(S)$,

(ii) $\int_S (g\circ\pi)(s)\ f(s)\ d\mu(s) = \int_T g(t)\ \langle f,\lambda_t\rangle\ d\nu(t)$ *for all* $f \in C(S)$ *and all* $g \in L^1(T,\nu)$,

(iii) $\langle g\circ\pi,\lambda\rangle = g$ *for all* $g \in C(T)$; *hence* $\mathrm{supp}(\lambda_t) \subset \pi^{-1}(\{t\})$ *and* $\|\lambda_t\| = 1$ *for all* $t \in T$.

PROOF. Consider the mapping

$$U : L^1(T,\nu) \longrightarrow M(S)$$

defined by

$$U(g) := (g \circ \pi)\mu \qquad (g \in L^1(T,\nu)).$$

This mapping is well defined (see BOURBAKI [5], Ch.V, §6, no 2, Th.1). Moreover, U is continuous and positive. So by the Dunford-Pettis theorem, there exists a mapping $t \mapsto \lambda_t$ of T into $M^+(S)$ satisfying (i) and (ii). Let $g \in C(T)$. Then for all $h \in C(T)$ we have

$$\int h(t)\ \langle g\circ\pi,\lambda_t\rangle\ d\nu(t) = \int h\circ\pi(s)\ g\circ\pi(s)\ d\mu(s) = \\ = \int h(t)\ g(t)\ d\nu(t)\ .$$

So $\langle g\circ\pi,\lambda_t\rangle = g(t)$ for μ-almost all t. Thus $r(\langle g\circ\pi,\lambda_t\rangle) = (rg)(t) = g(t)$ for all $t \in T$. Since also $r(\langle g\circ\pi,\lambda_t\rangle) = \langle g\circ\pi,\lambda_t\rangle$ for all $t \in T$, it follows that $\langle g\circ\pi,\lambda_t\rangle = g(t)$ for all $t \in T$. □

REMARK 1. It follows that

$$\int_S f(s)\ d\mu(s) = \int_T \langle f,\lambda_t\rangle\ d\nu(t)$$

for all $f \in C(S)$. This relation can also be written as

$$\mu = \int \lambda_t\ d\nu$$

where the integral has to be considered as a weak-star integral. Thus we have obtained a disintegration of μ with respect to ν.

REMARK 2. The mapping $t \mapsto \lambda_t$ is uniquely determined by (i), (ii) and (iii). For, let $t \mapsto \lambda'_t$ be another mapping satisfying (i), (ii) and (iii), then

$$\int g(t)\ \langle f,\lambda_t\rangle\ d\mu(t) = \int g(t)\ \langle f,\lambda'_t\rangle\ d\mu(t)$$

for all $f \in C(S)$ and $g \in C(T)$. Hence $\langle f,\lambda_t\rangle = \langle f,\lambda'_t\rangle$ for μ-almost all t. So, for all t we have

$$\langle f,\lambda_t\rangle = r(\langle f,\lambda_t\rangle) = r(\langle f,\lambda'_t\rangle) = \langle f,\lambda'_t\rangle \qquad (f \in C(S)).$$

Hence $\lambda_t = \lambda'_t$ for all $t \in T$.

We consider the situation of 4.2.2. Relation (ii) can be generalized; this relation also holds for bounded measurable functions.

4.2.3 THEOREM. *If* $f \in L^\infty(S,\mu)$, *then*

(i) f *is* λ_t*-measurable for* ν*-almost all* t,

(ii) *the map* $t \mapsto \langle f,\lambda_t\rangle$ *is bounded and* ν*-integrable; and*

$$\int_S g\circ\pi(s)\ f(s)\ d\mu(s) = \int_T g(t)\ \langle f,\lambda_t\rangle\ d\nu(t)$$

for all $g \in L^\infty(T,\nu)$.

PROOF. See A. & C. IONESCU TULCEA ([13], Ch.IX, Prop.6, p.147). □

CHAPTER V

EIGENFUNCTION EXPANSIONS

The idea of using triples of spaces $\Phi \subset H_0 \subset \Phi'$ to construct generalized eigenvectors for selfadjoint (and normal) operators in a Hilbert space H_0 is to be found at several places. We mention BEREZANSKII [4] and GELFAND & WILENKIN [10]. In this chapter we shall develop this theory from an algebraic point of view.

In the first section we consider the question whether the spaces H_∞ and $H_{-\infty}$ (associated with some selfadjoint operator T) are suitable for eigenfunction expansions. It turns out that they are not; generalized eigenvectors of T which are to be found in $H_{-\infty}$, are already in the space H_0 and are thus eigenvectors of T.

The starting point for the algebraic theory is an adapted version of the well-known theorem due to VON NEUMANN concerning the direct integral decomposition of a Hilbert space H_0 with respect to some spectral measure E(.). The usual proof of this theorem (cf. DIXMIER [6], Ch.II, §6) has been adapted to our purpose, namely eigenfunction expansions. A discussion of our proof is given at the end of section 2.

In section 3 we consider a second spectral measure F(.) which is the image of E(.) under some continuous map. We examine the relation between the direct integral decomposition of H_0 induced by E(.) and F(.). Here we make use of the theory of disintegration of measures.

In section 4 we come to the main subject of this chapter: generalized eigenvectors. With any spectral measure E(.) we associate generalized eigenvectors. Again, we consider a second spectral measure F(.) which is the image of E(.) under some continuous map. Then the results of section 3 are used to express the generalized eigenvectors associated with F(.) in terms of the generalized eigenvectors associated with E(.).

The results of section 4 can be applied in order to express the generalized eigenvectors of an operator in a GB^*-algebra A in terms of the generalized eigenvectors corresponding to the spectral measure of the C^*-algebra $b(A)$. This is indicated in section 5, where VON NEUMANN's theorem is generalized to GB^*-algebras.

In section 6 we indicate how the generalized eigenvectors of an operator of the form $P(T_1 \otimes I, I \otimes T_2)$ (cf. section 3.6) can be expressed in terms of the generalized eigenvectors of T_1 and T_2. This generalizes a result of L. & K. MAURIN [19] who considered the case $T_1 \otimes I + I \otimes T_2$.

5.1 GENERALIZED EIGENVECTORS

Let T be a selfadjoint operator in a separable Hilbert space H_0 and let $A := I + T^2$. Let $\{H_p \mid p \in \mathbb{Z}\}$ be the chain of Hilbert spaces associated with H_0 and A. Let $H_\infty := \bigcap_{p\in\mathbb{Z}} H_p$ and $H_{-\infty} := \bigcup_{p\in\mathbb{Z}} H_p$.

The operator T can be described in terms of its generalized eigenvectors. Generalized eigenvectors are not in the space H_0, but they are contained in a bigger space. We show that the space $H_{-\infty}$ is not big enough; in other words, a generalized eigenvector which is not in H_0, is also not in $H_{-\infty}$.

Let Φ be a locally convex space and let $\Phi \hookrightarrow H_0$ be a continuous injection such that Φ is dense in H_0. Suppose $\Phi \subset D(T)$ and $T\Phi \subset \Phi$. And assume that T maps Φ continuously into itself.

5.1.1 <u>LEMMA.</u> $\Phi \subset H_\infty$ *and the injection* $\Phi \hookrightarrow H_\infty$ *is continuous.*

<u>PROOF.</u> It is clear that $\Phi \subset D(T^k)$ for all $k \in \mathbb{N}$. So $\Phi \subset H_\infty$. Let $k \in \mathbb{N}$. Since A^k maps Φ continuously into itself, there is a continuous semi-norm q on Φ such that

$$\| A^k u \|_0 \le q(u) \qquad (u \in \Phi).$$

Thus $\| u \|_k = \| A^k u \|_0 \le q(u)$ $(u \in \Phi)$. This means that the injection $\Phi \hookrightarrow H_\infty$ is continuous. □

Now we want to analyse the condition that Φ is dense in H_∞ and we derive some consequences of this condition. We need the following lemma.

5.1.2 LEMMA. *Let* n *be a fixed natural number. For a linear subspace* $\Psi \subset D(T^n)$ *the following assertions are equivalent:*

(i) *there is a* $\lambda \in \mathbb{C}$ *with* $\mathrm{Im}(\lambda) \neq 0$ *such that* $(T - \lambda I)^n \Psi$ *is dense in* H_0,

(ii) *for all* $\lambda \in \mathbb{C}$ *with* $\mathrm{Im}(\lambda) \neq 0$, $(T - \lambda I)^n \Psi$ *is dense in* H_0,

(iii) *for every* n*-tuple* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *with* $\lambda_i \notin \sigma_p(T)$ *(= the point spectrum of* T*) the space* $(T - \lambda_1 I)(T - \lambda_2 I)\ldots(T - \lambda_n I)\Psi$ *is dense in* H_0.

PROOF. The implications (iii) ⟹ (ii) ⟹ (i) are trivial.
The implication (i) ⟹ (iii) is proved by induction.

First we consider the case $n = 1$. Suppose $\Psi \subset D(T)$ and $(T - \lambda I)\Psi$ is dense in H_0 for some $\lambda \in \mathbb{C}$ with $\mathrm{Im}(\lambda) \neq 0$. Let $\lambda_1 \notin \sigma_p(T)$ $(\lambda_1 \neq \lambda)$. Let

$$V := (T - \overline{\lambda} I)(T - \lambda I)^{-1}$$

be the Cayley transform of T (with respect to λ). Then V is a unitary operator, $D(T) = R(V - I)$ and $T = (\lambda V - \overline{\lambda} I)(V - I)^{-1}$.
Suppose x is orthogonal in H_0 to $(T - \lambda_1 I)\Psi$. Let $u \in (T - \lambda I)\Psi$. Then $u = (T - \lambda I)v$ for some $v \in \Psi$ and

$$
\begin{aligned}
(u, V^*x)_0 &= (Vu, x)_0 = ((T - \overline{\lambda} I)v, x)_0 = \\
&= (\lambda_1 - \overline{\lambda})(v, x)_0 , \qquad \text{since } ((T - \lambda_1 I)v, x)_0 = 0, \\
&= \frac{\lambda_1 - \overline{\lambda}}{\lambda_1 - \lambda}((T - \lambda I)v, x)_0 = \frac{\lambda_1 - \overline{\lambda}}{\lambda_1 - \lambda}(u, x)_0 .
\end{aligned}
$$

This holds for all $u \in (T - \lambda I)\Psi$. Since $(T - \lambda I)\Psi$ is dense, this implies that

$$V^*x = \frac{\overline{\lambda}_1 - \lambda}{\overline{\lambda}_1 - \overline{\lambda}} x .$$

Suppose $x \neq 0$. Then $\lambda_1 \in \mathbb{R}$,

$$Vx = \frac{\lambda_1 - \overline{\lambda}}{\lambda_1 - \lambda} x \quad \text{and} \quad x = \frac{\lambda_1 - \lambda}{\lambda - \overline{\lambda}}(V - I)x .$$

So $x \in D(T)$ and $Tx = \lambda_1 x$. This is a contradiction since $\lambda_1 \notin \sigma_p(T)$.

Now suppose that the implication (i) ⟹ (iii) holds for some $n \geq 1$. Let $\Psi \subset D(T^{n+1})$ and let $(T - \lambda I)^{n+1}\Psi$ be dense in H_0 for some $\lambda \in \mathbb{C}$ with $\mathrm{Im}(\lambda) \neq 0$. Let $\lambda_1, \ldots, \lambda_n, \lambda_{n+1} \in \mathbb{C} \setminus \sigma_p(T)$. Let $\Psi_n := (T - \lambda_2 I)\ldots(T - \lambda_n I)(T - \lambda_{n+1} I)\Psi$. By applying the induction hypothesis to $(T - \lambda I)\Psi$, we conclude that

$(T - \lambda I)\Psi_n = (T - \lambda_2 I)\ldots(T - \lambda_{n+1}I)(T - \lambda I)\Psi$ is dense in H_0. Since the assertion holds for $n = 1$, it follows that

$(T - \lambda_1 I)\Psi_n = (T - \lambda_1 I)(T - \lambda_2 I)\ldots(T - \lambda_{n+1}I)\Psi$ is also dense in H_0. □

5.1.3 COROLLARY. *$(T - iI)^n\Phi$ is dense in H_0 for all $n \in \mathbb{N}$ iff $A^n\Phi$ is dense in H_0 for all $n \in \mathbb{N}$.*

PROOF. Use $A = (T + iI)(T - iI)$. □

5.1.4 LEMMA. *Φ is dense in H_∞ iff $(T - iI)^n\Phi$ is dense in H_0 for all $n \in \mathbb{N}$.*

PROOF. Suppose Φ is dense in H_∞. Since A^n is a topological isomorphism of H_∞ onto itself, $A^n\Phi$ is dense in H_∞ and hence in H_0 $(n \in \mathbb{N})$. Then from 5.1.3 it follows that $(T - iI)^n\Phi$ is dense in H_0 for all $n \in \mathbb{N}$.
Conversely, let $(T - iI)^n\Phi$ be dense in H_0 for all $n \in \mathbb{N}$. Suppose that Φ is not dense in H_∞. Then there is an element $0 \neq f \in H_{-\infty}$ such that $(u,f)_0 = 0$ for all $u \in \Phi$. Assume $f \in H_{-n}$ $(n \in \mathbb{N})$; then $f = A^n x$ for some $x \in H_0$. Then it follows that

$$(u,f)_0 = (u,A^n x)_0 = (A^n u,x)_0 = 0 \qquad (u \in \Phi).$$

Since $A^n\Phi$ is dense in H_0 (cf. 5.1.3), it follows that $x = 0$. Contradiction. □

5.1.5 COROLLARY. *If Φ is dense in H_∞, then $A^p(T - \lambda_1 I)\ldots(T - \lambda_n I)\Phi$ is dense in H_0 ($p \in \mathbb{N}$ and $\lambda_1,\ldots,\lambda_n \notin \sigma_p(T)$ (= the point spectrum of T)).*

PROOF. Apply 5.1.2 and 5.1.4. □

Now assume that Φ is dense in H_∞. Let $i : \Phi \hookrightarrow H_\infty$ be the injection. Let Φ' be the anti-dual of Φ.
Let $\Psi_0 : H_{-\infty} \to (H_\infty)'$ be as in 1.5.2 (ii) (take $p = 0$ in 1.5.2). Then

$$j := i'\Psi_0 : H_{-\infty} \longrightarrow \Phi'$$

is injective. We shall identify $H_{-\infty}$ with its image under j. Then we obtain the following chain:

$$\Phi \hookrightarrow H_\infty \hookrightarrow H_0 \hookrightarrow H_{-\infty} \hookrightarrow \Phi' .$$

The continuous extension of T to $H_{-\infty}$ is also denoted by T. Let $T' : \Phi' \longrightarrow \Phi'$ be the anti-transposed of $T : \Phi \longrightarrow \Phi$. Then it is easily verified that T' is an extension of T.

We are interested in eigenvalues and eigenvectors of T'. If $T'\phi = \lambda\phi$ $(0 \neq \phi \in \Phi',\ \lambda \in \mathbb{C})$, then ϕ and λ are called a *generalized eigenvector* and a *generalized eigenvalue* of T respectively.

The kernel of $T' - \lambda I$ is denoted by $N(T' - \lambda I)$. And the kernel of $T - \lambda I$ (considered as an unbounded operator in H_0) is denoted by $N_0(T - \lambda I)$.

5.1.6 LEMMA. *If* Φ *is dense in* H_∞, *then* $N(T' - \lambda I) \cap H_{-\infty} = N_0(T - \lambda I)$.

PROOF. It is sufficient to prove that $N(T' - \lambda I) \cap H_{-\infty} \subset N_0(T - \lambda I)$. Take $f \in N(T' - \lambda I) \cap H_{-\infty}$ and suppose $f \in H_{-n}$ for some $n \in \mathbb{N}$. Then $f = A^n x$ with $x \in H_0$. And $T'A^n x = \lambda A^n x$. Hence $Tx = \lambda x$. So $x \in D(T)$ and $x \in N_0(T - \lambda I)$. Since

$$(u,f)_0 = (u,A^n x)_0 = (\lambda^2 + 1)^n (u,x)_0$$

for all $u \in \Phi$, it follows that

$$f = (\lambda^2 + 1)^n x\ .$$

So $f \in N_0(T - \lambda I)$. □

REMARK. This lemma says the following. If $f \in H_{-\infty}$ is a generalized eigenvector of T, then $f \in H_0$ (hence $f \in H_\infty$).

We can prove a little more.

5.1.7 PROPOSITION. *Let* Φ *be dense in* H_∞. *Let* E *denote the linear span of all eigenvectors of* T' *belonging to generalized eigenvalues* $\lambda \notin \sigma_p(T)$. *Then* $E \cap H_{-\infty} = (0)$.

PROOF. Let $\phi = \sum_{k=1}^n \phi_k$ where $\phi_k \in N(T' - \lambda_k I)$ and $\lambda_k \notin \sigma_p(T)$ $(k=1,\dots,n)$. Suppose $\phi \in H_{-\infty}$. Then $\phi = A^p x$ for some $x \in H_0$ and some $p \in \mathbb{N}$. Furthermore,

$$((T - \lambda_1 I)\dots(T - \lambda_n I)u,\phi)_0 = 0$$

for all $u \in \Phi$. Thus

$$(v,A^p x)_0 = (A^p v,x)_0 = 0$$

for all $v \in (T - \lambda_1 I)\dots(T - \lambda_n I)\Phi$. Since $A^p(T - \lambda_1))\dots(T - \lambda_n I)\Phi$ is dense in H_0 (by 5.1.5), it follows that $x = 0$. So $\phi = 0$. □

5.2 DECOMPOSITION OF A HILBERT SPACE WITH RESPECT TO A SPECTRAL MEASURE

The theorem which is presented in this section is a very well-known theorem due to VON NEUMANN; a proof is to be found in DIXMIER ([6], Ch.II, §6). The proof we shall give differs from the proof in DIXMIER at several points. The reason is that we adapted the proof to our needs in the next sections. At the end of this section we shall discuss the main points of difference.

Let H_0 be a separable Hilbert space with inner product $(.,.)_0$ and norm $\| . \|_0$.
Let A be a commutative C^*-algebra of bounded operators in H_0 containing the identity operator. For A one can take for example the algebra of bounded elements of GB^*- algebra.
The spectrum of A is denoted by S and $E(.)$ will be the spectral measure on S of the algebra A.

5.2.1 <u>THEOREM</u> (VON NEUMANN). *There exists a finite positive Borel measure* μ *on* S *and a* μ*-measurable field of Hilbert spaces* $\{H(s) \mid s \in S\}$ *and an isometric isomorphism*

$$H_0 \longrightarrow \int H(s)\, d\mu(s)$$

of H_0 *onto* $\int H(s) d\mu(s)$ *which transforms* A *onto the set of all continuously diagonalizable operators on* $\int H(s) d\mu(s)$.

<u>PROOF.</u> The proof is divided into several parts.

a) For the measure μ one has to take a basic measure, i.e. a measure which is equivalent to the spectral measure $E(.)$ (a measure μ is said to be equivalent to $E(.)$ if μ and $E(.)$ have the same zero-sets). We give an explicit expression for a basic measure. In order to do so we consider a second Hilbert space H_1 which is densely embedded in H_0 such that the inclusion map

$$i : H_1 \hookrightarrow H_0$$

is of Hilbert-Schmidt type. The inner product and the norm in H_1 are denoted by $(.,.)_1$ and $\| . \|_1$ respectively.

Let Δ be a Borel subset of S. Then $i^*E(\Delta)i$ is a positive nuclear operator; so its trace $tr(i^*E(\Delta)i)$ is finite. We define μ by

$$\mu(\Delta) := tr(i^*E(\Delta)i) \ .$$

Then μ is a finite positive Borel measure on S. And it is clear that μ is a basic measure.

b) Now we introduce the spaces $H(s)$ $(s \in S)$.

For $x,y \in H_0$, let $\mu_{x,y}$ be the measure defined by

$$\mu_{x,y}(\Delta) := (E(\Delta)x,y)_0$$

(Δ is a Borel subset of S).

Then $\mu_{x,y}$ is absolutely continuous with respect to μ. Hence by the Radon-Nikodym theorem, there is a function $h_{x,y} \in L^1(S,\mu)$ such that

$$\mu_{x,y}(\Delta) = \int_\Delta h_{x,y}(s)\, d\mu(s) \qquad (\Delta \text{ a Borel set in S}).$$

If $x,y \in H_1$, then $\mu_{x,y}(\Delta) = (E(\Delta)ix,iy)_0 = (i^*E(\Delta)ix,y)_1$. Hence

$$|\mu_{x,y}(\Delta)| \leq \mu(\Delta)\ \| x \|_1\ \| y \|_1 \ .$$

This implies that $h_{x,y} \in L^\infty(S,\mu)$ and that

$$(1) \qquad \| h_{x,y} \|_\infty \leq \| x \|_1\ \| y \|_1 \qquad (x,y \in H_1).$$

Now let ρ be a linear lifting of $L^\infty(S,\mu)$. Then we may assume that for $x,y \in H_1$

$$h_{x,y}(s) = (\rho h_{x,y})(s) \qquad (s \in S).$$

Since ρ is a positive linear map it follows that

$$(2) \qquad |h_{x,y}(s)| \leq \| h_{x,y} \|_\infty \qquad (s \in S).$$

From (1) and (2) it follows that for $x,y \in H_1$ and $s \in S$

(3) $|h_{x,y}(s)| \le \| x \|_1 \ \| y \|_1$.

The map

(4) $$\begin{aligned} H_1 \times H_1 &\longrightarrow \mathbb{C} \\ (x,y) &\longmapsto h_{x,y}(s) \end{aligned}$$

is for fixed $s \in S$ a sesquilinear form. We show that it is linear in the first variable.

Since $\mu_{x',y} + \mu_{x'',y} = \mu_{x'+x'',y}$, it follows that $h_{x',y} + h_{x'',y} = h_{x'+x'',y}$ μ-almost everywhere $(x',x'',y \in H_1)$. Since $h_{x,y}(s) = (\rho h_{x,y})(s)$ for all $x,y \in H_1$, we obtain

$$h_{x',y}(s) + h_{x'',y}(s) = h_{x'+x'',y}(s) \qquad (s \in S).$$

So, for a fixed $s \in S$, the map (4) is a continuous sesquilinear form on H_1.

Let $N(s) := \{x \in H_1 \mid h_{x,x}(s) = 0\}$ and let

$$F(s) : H_1 \longrightarrow H_1/N(s)$$

be the quotient map. The space $H_1/N(s)$ is an inner product space in a natural way; the inner product $(.,.)_s$ is defined by

$$(F(s)x, F(s)y)_s := h_{x,y}(s) \qquad (x,y \in H_1).$$

Let $H(s)$ be the completion of $H_1/N(s)$. So $H(s)$ is a Hilbert space with inner product $(.,.)_s$ and norm $\| \cdot \|_s$.

In this way we have obtained a field $\{H(s)\}$ of Hilbert spaces. If $x \in H_1$ then $F(s)x \in H(s)$ is also denoted by $\hat{x}(s)$ and the vector field $s \longmapsto \hat{x}(s)$ is denoted by $\hat{x}$.

c) The field $\{H(s)\}$ is a measurable field. Indeed, if (x_n) is a sequence which is dense in H_1, then the sequence $(\hat{x}_n)$ is a measurable fundamental sequence. So the direct integral $\int H(s)d\mu(s)$ is defined.
Note that $x \in \int H(s)d\mu(s)$ for $x \in H_1$ since $s \longmapsto \| \hat{x}(s) \|_s^2 = h_{x,x}(s)$ is bounded.

d) Now we define the isomorphism of H_0 onto $\int H(s)d\mu(s)$.

Let $y = \sum_{i=1}^{n} E(\Delta_i)z_i$ where $z_i \in H_1$ and Δ_i is a Borel subset of S (i=1,..,n). Then

$$d\mu_{y,y} = \sum_{i,j} \chi_{\Delta_i} \chi_{\Delta_j} d\mu_{z_i,z_j} =$$

$$= \sum_{i,j} \chi_{\Delta_i}(s) \chi_{\Delta_j}(s) h_{z_i,z_j} d\mu(s) .$$

Hence

$$\| y \|_0^2 = \int d\mu_{y,y} = \sum_{i,j} \int \chi_{\Delta_i}(s) \chi_{\Delta_j}(s) (\hat{z}_i(s),\hat{z}_j(s))_s d\mu(s) =$$

$$= \| \sum_{i=1}^{n} \chi_{\Delta_i} \hat{z}_i \|^2 .$$

Hence $\sum_{i=1}^{n} \chi_{\Delta_i}\hat{z}_i$ depends only on y and not on the respresentation of y in de form $y = \sum_{i=1}^{n} E(\Delta_i)z_i$. So the map $U_0 : \sum E(\Delta_i)z_i \mapsto \sum \chi_{\Delta_i}\hat{z}_i$ is well defined. And U_0 maps a dense subspace of H_0 isometrically onto a dense subspace of $\int H(s)d\mu(s)$.

Thus U_0 can be extended in a unique way to an isometry U of H_0 onto $\int H(s)d\mu(s)$.

e) Finally it is proved that U transforms A onto the set of all continuously diagonalizable operators.

The Gelfand map $A \to C(S)$ is an isometrical *-isomorphism. Its inverse

$$C(S) \longrightarrow A$$
$$f \longmapsto T_f$$

is given by

$$(T_f x,y)_0 = \int f(s)\, d\mu_{x,y}(s) \qquad (x,y \in H_0).$$

For $f \in C(S)$ and $x,y \in H_1$ we have

$$(U_0T_f x,\hat{y}) = (T_f x,y)_0 = \int f(s)\, d\mu_{x,y}(s) =$$

$$= \int f(s)(\hat{x}(s),\hat{y}(s))_s\, d\mu(s) =$$

$$= (M_f U_0 x,\hat{y}) ,$$

where M_f denotes the multiplication by f in $\int H(s)d\mu(s)$.
So $U_0T_f x = M_f U_0 x$ $(x \in H_1)$. Hence $M_f = UT_fU^{-1}$.
This completes the proof of the theorem. □

This decomposition of H_0 as a direct integral is called *the direct integral decomposition* of H_0 with respect to the C^*-algebra A or the decomposition with respect to the spectral measure $E(.)$.

5.2.2 REMARK. We show that

$$F(s) : H_1 \longrightarrow H(s)$$

is a Hilbert-Schmidt map for μ-almost all $s \in S$.

Let (e_n) be an orthonormal basis in H_1. Then

$$\sum_{n=1}^{\infty} \int_S \| F(s)e_n \|_s^2 \, d\mu(s) = \sum_{n=1}^{\infty} \int h_{e_n,e_n}(s) \, d\mu(s) = $$
$$= \sum_{n=1}^{\infty} \| ie_n \|_0^2 \, .$$

Since $i : H_1 \hookrightarrow H_0$ is a Hilbert-Schmidt map, it follows that $\sum_{n=1}^{\infty} \| ie_n \|_0^2 < \infty$. So

$$\sum_{n=1}^{\infty} \| F(s)e_n \|_s^2 < \infty$$

for μ-almost all $s \in S$.

This means that $F(s)$ is a Hilbert-Schmidt map for μ-almost all $s \in S$.

5.2.3 REMARK. The new point in the proof of VON NEUMANN's theorem is the fact that we considered a second Hilbert space H_1. This Hilbert space is used to define the basic measure μ. This definition also appears in BEREZANSKII ([4], Ch.V, §1) who needs it for other purposes. A consequence of this definition is that the functions $h_{x,y}$ $(x,y \in H_1)$ are in $L^{\infty}(S,\mu)$. This means that the theory of liftings can be applied.

5.3 DECOMPOSITION WITH RESPECT TO THE IMAGE OF A SPECTRAL MEASURE

Let H_0 be a separable Hilbert space and let $A \subset L(H_0)$ be a commutative C^*-algebra containing the identity operator. The spectrum of A is denoted by S and the spectral measure of A is denoted by $E(.)$. Then the inverse of the Gelfand map $A \mapsto \hat{A}$ of A onto $C(S)$ is the map

$$(1) \qquad \begin{aligned} C(S) &\longrightarrow A \\ f &\longmapsto T_f := \int f dE \, . \end{aligned}$$

Let T be another compact Hausdorff space and let $p : S \to T$ be a continuous surjection. Then the set

$$\left\{ T_{(g\circ p)} \mid g \in C(T) \right\}$$

is a commutative C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$. The inclusion map is denoted by $j : \mathcal{B} \hookrightarrow \mathcal{A}$.

The Gelfand map $B \mapsto \hat{B}$ of $\mathcal{B}$ onto $C(T)$ is then given by

$$\hat{B}(p(\phi)) = (jB)^{\wedge}(\phi) \qquad (B \in \mathcal{B},\ \phi \in S). \tag{2}$$

The spectral measure of $\mathcal{B}$ is denoted by $F(.)$.

We show that $F(.)$ is the image under the map p of the spectral $E(.)$. For $x,y \in H_0$ the measures $\mu_{x,y}$ and $\nu_{x,y}$ are defined by

$$\mu_{x,y}(.) := (E(.)x,y)_0 \quad \text{and} \quad \nu_{x,y}(.) := (F(.)x,y)_0 .$$

Let $x,y \in H_0$ and $B \in \mathcal{B}$. Then

$$(Bx,y)_0 = \int_T \hat{B}\, d\nu_{x,y} .$$

Since also

$$(Bx,y)_0 = \int_S (jB)^{\wedge} d\mu_{x,y} = \int_T \hat{B}\circ p\, d\mu_{x,y} ,$$

it follows that $\nu_{x,y}$ is the image under p of $\mu_{x,y}$. So

$$\mu_{x,y}(p^{-1}(\Delta)) = \nu_{x,y}(\Delta) \qquad (\Delta \text{ a Borel subset of } T)$$

Hence

$$E(p^{-1}(\Delta)) = F(\Delta) \qquad (\Delta \text{ a Borel subset of } T), \tag{3}$$

which means that $F(.)$ is the image under p of $E(.)$.

Again we consider a second Hilbert space H_1 which is densely embedded in H_0 such that the inclusion map $i : H_1 \hookrightarrow H_0$ is of Hilbert-Schmidt type. The measures μ and ν are defined by

$$\mu(.) := \operatorname{tr}(i^*E(.)i) \quad \text{and} \quad \nu(.) := \operatorname{tr}(i^*F(.)i) .$$

From (3) it follows that ν is the image of μ under the map p.

For $x,y \in H_1$ there exist functions $h_{x,y} \in L^\infty(S,\mu)$ and $g_{x,y} \in L^\infty(T,\nu)$ such that

$$d\mu_{x,y} = h_{x,y}\, d\mu \quad \text{and} \quad d\nu_{x,y} = g_{x,y}\, d\nu \ .$$

We have established the relation between $\mu_{x,y}$ and $\nu_{x,y}$ and between μ and ν. Now, using disintegration of measures, we derive a relation between $h_{x,y}$ and $g_{x,y}$.

Let ρ be a linear lifting of $L^\infty(S,\mu)$. Now we assume that there exists a *strong lifting* r of $L^\infty(T,\nu)$; if T is metrizable, then a strong lifting r certainly exists (cf. A. & C. IONESCU TULCEA [13], Ch.VIII, §4). We may suppose that for $x,y \in H_1$

$$(4) \qquad h_{x,y}(s) = \rho\, h_{x,y}(s) \ , \quad g_{x,y}(t) = r\, g_{x,y}(t) \qquad (s \in S,\ t \in T).$$

5.3.1 <u>THEOREM</u>. *Suppose that there exists a strong lifting* r *of* $L^\infty(T,\nu)$ *and assume that* (4) *holds. Then there exists a mapping* $t \mapsto \lambda_t$ *of* T *into* $M^+(S)$ *and a* ν-*zero-set* $N \subset T$ *such that for all* $t \notin N$ *and for all* $x,y \in H_1$

$$\langle h_{x,y}, \lambda_t \rangle = g_{x,y}(t) \ .$$

<u>PROOF</u>. By 4.2.2 there exists a mapping $t \mapsto \lambda_t$ of T into $M^+(S)$ such that (i), (ii), (iii) of 4.2.2 and 4.2.3 hold.
For $x,y \in H_1$, let $N'_{x,y}$ be the set of all $t \in T$ for which $h_{x,y}$ is not λ_t-measurable. Then by 4.2.3 it follows that $\nu(N'_{x,y}) = 0$ and that $t \mapsto \langle h_{x,y}, \lambda_t \rangle$ (which is defined for $t \notin N'_{x,y}$) is bounded and ν-measurable. Also

$$\int_T f(t)\langle h_{x,y}, \lambda_t \rangle d\nu(t) = \int_S (f\circ p)(s) h_{x,y}(s) d\mu(s) \qquad (f \in C(T)).$$

Since

$$\begin{aligned} \int_S (f\circ p)(s) h_{x,y}(s) d\mu(s) &= \int_S (f\circ p)(s) d\mu_{x,y}(s) = \\ &= \int_T f(t) d\nu_{x,y}(t) = \\ &= \int_T f(t) g_{x,y}(t) d\nu(t) \ , \end{aligned}$$

it follows that

$$\langle h_{x,y}, \lambda_t \rangle = g_{x,y}(t)$$

for all $t \notin N_{x,y}$, where $N_{x,y} \subset T$ is a subset of ν-measure 0.
Now, let D be a countable dense set in H_1. Then there exists a set $N \subset T$ with $\nu(N) = 0$ such that for all $t \notin N$ and for all $x,y \in D$

$$(5) \qquad \langle h_{x,y}, \lambda_t \rangle = g_{x,y}(t) \qquad (t \notin N).$$

We show that (5) holds for all $x,y \in H_1$.

Let $x,y \in H_1$ and let (x_n) and (y_n) be sequences in D such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$ in H_1 as $n \to \infty$. Now recall that $(x,y) \longmapsto h_{x,y}(s)$ is a continuous sesquilinear form on H_1 and that

$$|h_{x,y}(s)| \le \| x \|_1 \; \| y \|_1 \qquad (s \in S).$$

So

$$|h_{x,y}(s) - h_{x_n,y_n}(s)| \le |h_{x-x_n,y}(s)| + |h_{x_n,y-y_n}(s)| \le$$
$$\le \| x - x_n \| \; \| y \|_1 + \| x_n \|_1 \; \| y - y_n \|_1 \qquad (s \in S,\ n \in \mathbb{N}).$$

Hence h_{x_n,y_n} converges to $h_{x,y}$ uniformly on S. In the same way it follows that g_{x_n,y_n} converges to $g_{x,y}$ uniformly on T. This implies that (5) holds for all $x,y \in H_1$. □

Let $\int H(s)d\mu(s)$ be the direct integral decomposition of H_0 with respect to E(.) (as constructed in the proof of 5.2.1). Similarly, let $\int G(t)d\nu(t)$ be the direct integral decomposition of H_0 with respect to F(.).
Let $U : H_0 \longrightarrow \int H(s)d\mu(s)$ and $V : H_0 \longrightarrow \int G(t)d\nu(t)$ be the corresponding isomorphisms. The image of an element $x \in H_0$ under U and V are denoted by

$$(s \longmapsto \hat{x}(s)) \quad \text{and} \quad (t \longmapsto \hat{x}(t))$$

respectively.

5.3.2 <u>THEOREM</u> (continuation of 5.3.1). *For* $t \notin N$*, the field* $\{H(s) \mid s \in S\}$ *is a* λ_t*-measurable field of Hilbert spaces. The map*

$$U(t) : G(t) \longrightarrow \int H(s)\, d\lambda_t \qquad (t \notin N)$$

which takes $\hat{x}(t) \in G(t)$ $(x \in H_1)$ *into the vector field* $s \longmapsto \hat{x}(s)$ *is well defined and is an isometry of* $G(t)$ *into* $\int H(s)d\lambda_t$.

<u>PROOF.</u> Let D be the set introduced in the proof of 5.3.1; so $D = (x_n)_{n=1}^{\infty}$ is a countable dense set in H_1. Then $(\hat{x}_n(s))_{n=1}^{\infty}$ is dense in H(s) $(s \in S)$. And the functions $s \mapsto (\hat{x}_n(s), \hat{x}_m(s))_s = h_{x_n,x_m}(s)$ $(n,m \in \mathbb{N})$ are λ_t-measurable for $t \notin N$. Thus for $t \notin N$, the field $\{H(s)\}$ is λ_t-measurable. So the direct integral $\int H(s)d\lambda_t$ is defined for all $t \notin N$.

If $x \in H_1$, then $s \mapsto \hat{x}(s)$ is an element of $\int H(s)d\lambda_t$ $(t \notin N)$. Indeed,

$$(6) \qquad \int \| x(s) \|_s^2 \, d\lambda_t = \int h_{x,x}(s)\, d\lambda_t = g_{x,x}(t) = \| \hat{x}(t) \|_t^2 \quad (t \notin N).$$

So the vector field $s \mapsto \hat{x}(s)$ (as an element of $\int H(s)d\lambda_t$) depends only on $\hat{x}(t)$. This means that the map which (for fixed $t \notin N$) takes $\hat{x}(t) \in G(t)$ into the vector field $s \mapsto \hat{x}(s)$ is well defined. It follows from (6) that this map is an isometry of a dense subspace of G(t) into $\int H(s)d\lambda_t$ $(t \notin N)$. So it can be extended to an isometry U(t) of G(t) into $\int H(s)d\lambda_t$ $(t \notin N)$. □

5.3.3 <u>REMARK.</u> The question arises whether U(t) is surjective $(t \notin N)$. In the following special case the answer is affirmative.

Suppose that $\mathcal{A}$ is the C^*-algebra generated by an hermitian operator $A \in L(H_0)$ which leaves invariant H_1. Then there exists a sequence (f_k) which is dense in C(S) such that T_{f_k} leaves invariant H_1 $(k \in \mathbb{N})$. Since S is metrizable we may suppose that ρ is a strong lifting. A consequence is that for all $x \in H_1$

$$(T_{f_k}x)^{\wedge}(s) = f_k(s)\hat{x}(s) \qquad (s \in S,\ k \in \mathbb{N}).$$

This means that U(t) maps $(T_{f_k}x_n)^{\wedge}(t) \in G(t)$ into the vector field $f_k\hat{x}_n \in \int H(s)d\lambda_t$ $(t \notin N)$. Since the set $\{f_k\hat{x}_n \mid k,n \in \mathbb{N}\}$ is dense in $\int H(s)d\lambda_t$ (cf. 4.1.5 (ii)) it follows that U(t) is surjective $(t \notin N)$.

5.4 APPLICATION TO EIGENFUNCTION EXPANSIONS

The notations in this section will be the same as in the previous section.

Let H_{-1} be the anti-dual of H_1; we obtain the triple $H_1 \hookrightarrow H_0 \hookrightarrow H_{-1}$. Then $(\alpha,u)_0$ is well defined for $\alpha \in H_{-1}$ and $u \in H_1$ (cf. 1.2.5).

We proved that $(x,y) \mapsto h_{x,y}(s)$ $(x,y \in H_1)$ is a continuous sesquilinear form on H_1 for all $s \in S$. Hence there exist continuous linear operators

$$P(s) : H_1 \longrightarrow H_{-1} \qquad (s \in S)$$

such that

(1) $$h_{x,y}(s) = (P(s)x,y)_0 \qquad (x,y \in H_1).$$

The meaning of these operators $P(s)$ is the following.
Suppose $A \in \mathcal{A}$ leaves invariant H_1, then for $x,y \in H_1$

$$\mu_{Ax,y}(\Delta) = (AE(\Delta)x,y)_0 = \int_\Delta \hat{A}(s)d\mu_{x,y} = \int_\Delta \hat{A}(s)(P(s)x,y)_0 d\mu$$

is equal to

$$\mu_{Ax,y}(\Delta) = \int_\Delta h_{Ax,y}(s)d\mu = \int_\Delta (P(s)Ax,y)_0 d\mu$$

(Δ is a Borel subset of S). Hence

(2) $$(P(s)Ax,y)_0 = \hat{A}(s)(P(s)x,y)_0$$

for μ-almost all $s \in S$. Now recall that for all $x,y \in H_1$

$$(P(s)x,y)_0 = h_{x,y}(s) = (\rho h_{x,y})(s) \qquad (s \in S)$$

where ρ is a linear lifting of $L^\infty(S,\mu)$. If ρ is supposed to be a *strong lifting*, then it follows from (2) that for all $x,y \in H_1$

$$(P(s)Ax,y)_0 = \hat{A}(s)(P(s)x,y)_0$$

for all $s \in S$. Thus, for all $A \in \mathcal{A}$ leaving invariant H_1,

(3) $$P(s)A = \hat{A}(s)\ P(s) \qquad (s \in S).$$

For this reason the operators $\{P(s) \mid s \in S\}$ are called *generalized eigenprojections* corresponding to the spectral measure $E(.)$.

<u>REMARK.</u> The operators $\{P(s)\}$ were introduced in another way by BEREZANSKII [4], Ch.V, §2.

Let $\{Q(t) \mid t \in T\}$ be the set of generalized eigenprojections corresponding to the spectral measure $F(.)$. Then the next proposition is an immediate consequence of 5.3.1.

5.4.1 PROPOSITION. *Under the same hypotheses as in* 5.3.1 *the following holds:*

$$Q(t) = \int P(s)\, d\lambda_t \qquad (t \notin N),$$

where the integral has to be interpreted as a weak integral, i.e. $(Q(t)x,y)_0 = \int (P(s)x,y)_0 d\lambda_t$ *for* $x,y \in H_1$ $(t \notin N)$.

PROOF. By 5.3.1 we have

$$g_{x,y}(t) = \int h_{x,y}(s)\, d\lambda_t \qquad (t \notin N).$$

The result follows since

$$g_{x,y}(t) = (Q(t)x,y)_0 \quad \text{and} \quad h_{x,y}(s) = (P(s)x,y)_0. \qquad \square$$

Let $D = (x_n)_{n=1}^{\infty}$ be the set (dense in H_1) which was introduced in the proof of 5.3.1. Then the sequence $s \mapsto \hat{x}_n(s)$ $(n \in \mathbb{N})$ is a μ-measurable fundamental sequence for the field $\{H(s)\}$ on the measure space (S,μ). Let $e_n : s \mapsto e_n(s)$ $(n \in \mathbb{N})$ be the μ-measurable field of orthonormal bases that can be constructed from the sequence $(s \mapsto \hat{x}_n(s))$ (cf. 4.1.3). The sequence $(s \mapsto \hat{x}_n(s))$ is also a λ_t-measurable fundamental sequence $(t \notin N)$ (cf. the proof of 5.3.2). Hence (e_n) is also a λ_t-measurable field of orthonormal bases.
The sequence of fields (e_n) is used now to obtain generalized eigenvectors.

Let $F(s) : H_1 \to H(s)$ be the canonical map and let $F(s)'_h : H(s) \to H_{-1}$ be the right anti-transposed of $F(s)$. Since $F(s)$ has dense range, $F(s)'_h$ is injective. Since $(P(s)x,y)_0 = (F(s)x,F(s)y)_s = (F(s)'_h F(s)x,y)_0$ $(x,y \in H_1)$ it follows that

$$(4) \qquad P(s) = F(s)'_h F(s) \qquad (s \in S).$$

The ranges of $P(s)$ and $F(s)'_h$ are denoted by $E(s)$ and $\tilde{H}(s)$ respectively. Then $E(s)$ is dense in $\tilde{H}(s)$. So

$$(5) \qquad E(s) \subset \tilde{H}(s) \subset \overline{E(s)} \qquad (s \in S).$$

Now $\phi_n(s) \in H_{-1}$ is defined by

(6) $$\phi_n(s)(x) := (e_n(s),\hat{x}(s))_s \qquad (s \in S,\ x \in H_1,\ n \in \mathbb{N}).$$

Thus $\phi_n(s) = F(s)'_{\hbar}\, e_n(s)$. Since $F(s)'_{\hbar}$ is injective the sequence $(\phi_n(s))_{n=1}^{\infty}$ is linearly independent. Furthermore, the functions

$$S \ni s \longmapsto \phi_n(s) \in H_{-1} \qquad (n \in \mathbb{N})$$

are weakly μ-measurable, because $\phi_n(s)(x) = (e_n(s),\hat{x}(s))_s$ is μ-measurable for all $x \in H_1$ $(n \in \mathbb{N})$.

Assume that $A \in \mathcal{A}$ leaves invariant H_1. Then $A : H_1 \to H_1$ is continuous. This is a consequence of the closed graph theorem (the graph of $A : H_1 \to H_1$ is the inverse image of the graph of $A : H_0 \to H_0$ under the inclusion map $H_1 \times H_1 \hookrightarrow H_0 \times H_0$; so the graph of $A : H_1 \to H_1$ is closed). The anti-transposed of $A : H_1 \to H_1$ is denoted by $A' : H_{-1} \to H_{-1}$.
Let us assume that (3) holds. Since $F(s)'_{\hbar}$ is injective, it follows from (3) and (4) that

$$F(s)Ax = \hat{A}(s)F(s)x \qquad (x \in H_1,\ s \in S).$$

Thus

(7) $$(Ax)^{\wedge}(s) = \hat{A}(s)\hat{x}(s) \qquad (x \in H_1,\ s \in S).$$

Hence

$$(A'\phi_n(s))(x) = \phi_n(s)(Ax) = (e_n(s),\hat{A}(s)\hat{x}(s))_s =$$
$$= \overline{\hat{A}}(s)\phi_n(s)(x) \ .$$

So

(8) $$A'\phi_n(s) = \overline{\hat{A}}(s)\phi_n(s) \qquad (s \in S,\ n \in \mathbb{N}).$$

For this reason the elements $\{\phi_n(s) \mid n \in \mathbb{N},\ s \in S\}$ are called *generalized eigenvectors* corresponding to the spectral measure $E(.)$.

Now let (f_n) be the ν-measurable field of orthonormal bases constructed from the sequence $(t \mapsto x_n(t))$. And let $\{\psi_m(t) \mid m \in \mathbb{N},\ t \in T\}$ be the corresponding set of generalized eigenvectors.
The relation between the sets $\{\phi_n(s)\}$ and $\{\psi_m(t)\}$ is given in the next theorem.

5.4.2 THEOREM. *Suppose that the conditions of* 5.3.1 *are fulfilled. Let* $t \mapsto \lambda_t$ *and* N *be as in* 5.3.1. *Then for* $t \notin N$ *and* $m \in \mathbb{N}$ *there exists a sequence* $(\gamma_n(s))$ *of* λ_t*-measurable functions (depending on* t *and* m*) such that*

$$\psi_m(t) = \sum_n \int_S \gamma_n(s)\ \phi_n(s)\ d\lambda_t(s) \qquad (t \notin N)$$

where the summation and integration are to be taken in the weak sense.

PROOF. Let $U(t) : G(t) \to \int H(s) d\lambda_t$ be as in 5.3.2. The vector field $U(t) f_m(t) \in \int H(s) d\lambda_t$ is denoted by $z : s \mapsto z(s)$. Then

$$z = U(t) f_m(t) = \lim_{k\to\infty} \sum_{n=1}^{k} (z(s), e_n(s))_s\ e_n(s)$$

where the limit is taken in the space $\int H(s) d\lambda_t$ (cf. DIXMIER [6], Ch.II, §1, Prop.6).

Let $(z, \hat{x})$ denote the inner product in $\int H(s) d\lambda_t$ of z and $s \mapsto \hat{x}(s)$ $(x \in H_1)$. Since U(t) is an isometry it follows that

$$\begin{aligned}\psi_m(t)(x) &= (f_m(t), x)_t = (z, \hat{x}) = \\ &= \sum_n \int_S (z(s), e_n(s))_s (e_n(s), \hat{x}(s))_s\ d\lambda_t \\ &= \sum_n \int_S (z(s), e_n(s))_s \phi_n(s)(x) d\lambda_t\ .\end{aligned}$$

□

5.5 EIGENFUNCTION EXPANSIONS FOR UNBOUNDED OPERATORS

Consider an unbounded selfadjoint operator B in the separable Hilbert space H_0. Let $A := I + B^2$. We form the chain of Hilbert spaces generated by H_0 and A and we consider the corresponding space H_∞ (cf. 1.6).
Then by 3.2.13 the algebra $A := \Gamma''_\infty(B)$ is a GB^*-algebra. The algebra $b(A)$ of bounded elements of A is a W^*-algebra (see 3.3.5). Its spectrum is denoted by S.
Let $\int H(s) d\mu(s)$ be the decomposition of H_0 with respect to $b(A)$ and let $U : H_0 \to \int H(s) d\mu(s)$ be the corresponding isomorphism (see 5.2.1).
Then VON NEUMANN's theorem can be extended to the GB^*-algebra $\Gamma''_\infty(B)$ as follows.

5.5.1 THEOREM. *Let* $C \in \Gamma''_\infty(B)$ *and let* $\overline{C}$ *denote the closure of* C *as an operator in* H_0. *Then* $U\,\overline{C}\,U^{-1}$ *is the multiplication by* $\hat{C}$ *in the space* $\int H(s) d\mu(s)$.

PROOF. If C is an hermitian element of $\Gamma''_\infty(B)$ and $\lambda \notin \sigma(C)$ $(\lambda \in \mathbb{C})$, then $C_\lambda := (C - \lambda I)^{-1} \in b(\Gamma''_\infty(B))$. If $\overline{C}_\lambda$ denotes the continuous extension of C_λ to H_0, then $U\,\overline{C}_\lambda\,U^{-1}$ is the multiplication operator

$$z \longmapsto \hat{C}_\lambda z \qquad (z \in \int H(s)d\mu(s)).$$

Since $\hat{C} = \lambda + 1/\hat{C}_\lambda$ and $\overline{C}_\lambda = (\overline{C} - \lambda I)^{-1}$ (cf. 3.3.7), it follows that $U\,\overline{C}\,U^{-1}$ is the multiplication operator

$$z \longmapsto \hat{C}z \qquad (z \in UD(\overline{C})).$$

If C is an arbitrary element of $\Gamma''_\infty(B)$, we write $C = C_1 + iC_2$ where C_1 and C_2 are hermitian; then we apply the preceding result. □

Let T be the compact space $\sigma(B)$ $(\subset \mathbb{R}^*)$ where $\sigma(B)$ denotes the spectrum of B in $\Gamma''_\infty(B)$. Then

$$\hat{B} : S \longrightarrow T$$

is continuous. Let F(.) be the image under $\hat{B}$ of the spectral measure E(.) of $b(\Gamma''_\infty(B))$.
The spectral measure F(.) is called the *spectral measure of* B.
Let $\int G(t)d\nu(t)$ be the decomposition of H_0 with respect to F(.) and let $V : H_0 \longrightarrow \int G(t)d\nu(t)$ be the corresponding isomorphism (cf. 5.2.1).

5.5.2 LEMMA. *The operator* VBV^{-1} *is the multiplication operator* M *in* $\int G(t)d\nu(t)$ *given by*

$$M : z \longmapsto tz$$

(z is a vector field in $\int G(t)d\nu(t)$ *for which the vector field* $tz : t \longmapsto tz(t)$ *belongs to* $\int G(t)d\nu(t)$*).*

PROOF. The space T and the spectral measure F(.) can be viewed as the spectrum and the spectral measure of some C^*-subalgebra of $b(\Gamma''_\infty(B))$. So, if $f \in C(T)$, then the multiplication operator $M_f : z \longrightarrow fz$ on $\int G(t)d\nu(t)$ corresponds to the multiplication by $g = f \circ \hat{B}$ on $\int H(s)d\mu(s)$.
Taking $g := 1/(\lambda - \hat{B})$ (with $\mathrm{Im}(\lambda) \neq 0$) then it follows that $(\lambda I - B)^{-1}$ corresponds the multiplication by $f(t) = 1/(\lambda - t)$ on $\int G(t)d\nu(t)$. Now take inverses and the lemma follows. □

Since T is metrizable, the space $L^{\infty}(T,\nu)$ possesses the strong lifting property. So we may assume that the conditions of 5.4.2 (or 5.3.1) are fullfilled. This means that the generalized eigenvectors corresponding to F(.) can be expressed in terms of the generalized eigenvectors corresponding to E(.).

5.6 EIGENFUNCTION EXPANSIONS AND TENSOR PRODUCTS

The results of the preceding sections and of section 3.6 can be used to obtain information about generalized eigenvectors of a tensor product operator.

Let $H_0^{(1)}$ and $H_0^{(2)}$ be separable Hilbert spaces and let $A_1 \subset L(H_0^{(1)})$ and $A_2 \subset L(H_0^{(2)})$ be commutative C^*-algebras containing the identity operator. The spectrum and the spectral measure of A_i are denoted by S_i and $E_i(.)$ respectively (i=1,2).
The spectrum of the C^*-tensor product $A_1 \tilde{\otimes} A_2$ (which is the closure of $A_1 \otimes A_2$ in $L(H_0^{(1)} \tilde{\otimes} H_0^{(2)})$) is $S_1 \times S_2$ (cf. SAKAI [28], p.62). The spectral measure of $A_1 \tilde{\otimes} A_2$ is denoted by E(.). If $\Delta_i \subset S_i$ is a Borel subset of S_i (i=1,2) then $E(\Delta_1 \times \Delta_2) = E_1(\Delta_1) \tilde{\otimes} E_2(\Delta_2)$.
The spectral measure E is called the *tensor product of* E_1 *and* E_2.
Let $i_1 : H_1^{(1)} \hookrightarrow H_0^{(1)}$ and $i_2 : H_1^{(2)} \hookrightarrow H_0^{(2)}$ be Hilbert-Schmidt embeddings with dense range. Then

$$i_1 \tilde{\otimes} i_2 : H_1^{(1)} \tilde{\otimes} H_1^{(2)} \longrightarrow H_0^{(1)} \tilde{\otimes} H_0^{(2)}$$

is injective, has dense range and is of Hilbert-Schmidt type.
Let $\int H(s_1,s_2)d\mu(s_1,s_2)$ be the direct integral decomposition of $H_0^{(1)} \tilde{\otimes} H_0^{(2)}$ with respect to the spectral measure E(.) as constructed in 5.2.1 (replace H_0 and H_1 by $H_0^{(1)} \tilde{\otimes} H_0^{(2)}$ and $H_1^{(1)} \tilde{\otimes} H_1^{(2)}$ respectively).
The direct integral decompositions of $H_0^{(1)}$ (with respect to $E_1(.)$) and $H_0^{(2)}$ (with respect to $E_2(.)$) are denoted by $\int H^{(1)}(s_1)d\mu^{(1)}(s_1)$ and $\int H^{(2)}(s_2)d\mu^{(2)}(s_2)$ respectively.

The notations in the next proposition are as in 5.2.1.

5.6.1 PROPOSITION. *For μ-almost all $(s_1,s_2) \in S_1 \times S_2$ there exists a surjective isometry*

$$V(s_1,s_2) : H^{(1)}(s_1) \tilde{\otimes} H^{(2)}(s_2) \longrightarrow H(s_1,s_2)$$

such that

$$V(s_1,s_2)(\hat{x}(s_1) \otimes \hat{y}(s_2)) = (x \otimes y)^{\wedge}(s_1,s_2)$$

($x \in H_1^{(1)}$ *and* $y \in H_1^{(2)}$).

PROOF. Since for nuclear operators C_1 and C_2, $tr(C_1 \tilde{\otimes} C_2) = tr(C_1)tr(C_2)$, it follows that $\mu(\Delta_1 \times \Delta_2) = \mu^{(1)}(\Delta_1)\mu^{(2)}(\Delta_2)$ (Δ_1 and Δ_2 are Borel subsets of S_1 and S_2 resp.). So

$$(1) \qquad \mu = \mu^{(1)} \otimes \mu^{(2)} .$$

It is also easily verfied that for $x_i \in H_0^{(1)}$ and $y_i \in H_0^{(2)}$ (i=1,2)

$$(2) \qquad \mu_{x_1\otimes y_1,x_2\otimes y_2} = \mu^{(1)}_{x_1,x_2} \otimes \mu^{(2)}_{y_1,y_2} .$$

For $x_i \in H_1^{(1)}$ and $y_i \in H_1^{(2)}$ (i=1,2), let

$$(3) \qquad \begin{aligned} &d\mu^{(1)}_{x_1,x_2} = g^{(1)}_{x_1,x_2} d\mu^{(1)} \quad , \quad d\mu^{(2)}_{y_1,y_2} = g^{(2)}_{y_1,y_2} d\mu^{(2)} \quad \text{and} \\ &d\mu_{x_1\otimes y_1,x_2\otimes y_2} = g_{x_1\otimes y_1,x_2\otimes y_2} d\mu . \end{aligned}$$

It follows from (2) and (3) that

$$(4) \qquad g^{(1)}_{x_1,x_2}(s_1)\, g^{(2)}_{y_1,y_2}(s_2) = g_{x_1\otimes y_1,x_2\otimes y_2}(s_1,s_2)$$

for all (s_1,s_2) outside some set of μ-measure 0, depending on x_1,x_2,y_1,y_2. Using the separability of $H_1^{(1)}$ and $H_1^{(2)}$, one can show that there exists a subset K of $S_1 \times S_2$ of μ-measure 0 such that (4) holds for all $x_1,x_2 \in H_1^{(1)}$ and $y_1,y_2 \in H_1^{(2)}$ and for all $(s_1,s_2) \notin K$ (cf. the proof of 5.3.1).
Let $(s_1,s_2) \notin K$. Consider the elements

$$\phi := \sum_{i=1}^{n} x_i \otimes y_i$$

and

$$\hat{\psi} := \sum_{i=1}^{n} \hat{x}_i(s_1) \otimes \hat{y}_i(s_2)$$

where $x_i \in H_1^{(1)}$ and $y_i \in H_1^{(2)}$ (i=1,...,n). We show that $\| \hat{\psi} \| = \| \hat{\phi}(s_1,s_2) \|$; the norm of $\hat{\psi}$ is taken in $H^{(1)}(s_1) \tilde{\otimes} H^{(2)}(s_2)$ and the norm of $\hat{\phi}(s_1,s_2)$ is taken in $H(s_1,s_2)$.

Indeed,

$$\|\hat{\psi}\|^2 = \sum_{i,j} (\hat{x}_i(s_1),\hat{x}_j(s_1))_{s_1} \ (y_i(s_2),y_j(s_2))_{s_2} =$$

$$= \sum_{i,j} g^{(1)}_{x_i,x_j}(s_1)\ g^{(2)}_{y_i,y_j}(s_2) = \sum_{i,j} g_{x_i\otimes y_i,x_j\otimes y_j}(s_1,s_2) =$$

$$= g_{\phi,\phi}(s_1,s_2) = \|\hat{\phi}(s_1,s_2)\| \ .$$

So it follows that the map

$$\hat{x}(s_1) \otimes \hat{y}(s_2) \longmapsto (x \otimes y)\hat{\ }(s_1,s_2) \qquad (x \in H_1^{(1)},\ y \in H_1^{(2)})$$

defines a linear isometry of a dense set of $H^{(1)}(s_1) \tilde{\otimes} H^{(2)}(s_2)$ onto a dense set of $H(s_1,s_2)$. So this map has a unique continuous extension to an isometry of $H^{(1)}(s_1) \tilde{\otimes} H^{(2)}(s_2)$ onto $H(s_1,s_2)$. □

Let $\{\phi_n^{(1)}(s_1) \mid s_1 \in S_1,\ n \in \mathbb{N}\}$ and $\{\phi_m^{(2)}(s_2) \mid s_2 \in S_2,\ m \in \mathbb{N}\}$ be the generalized eigenvectors corresponding to E_1 and E_2 respectively.
Then it follows from 5.6.1 that

$$\left\{\phi_n^{(1)}(s_1) \otimes \phi_m^{(2)}(s_2) \mid (s_1,s_2) \in S_1 \times S_2 \setminus K,\ n,m \in \mathbb{N}\right\}$$

is a set of generalized eigenvectors corresponding to E. Note that

$$\phi_n^{(1)}(s_1) \otimes \phi_m^{(2)}(s_2) \in H_{-1}^{(1)} \tilde{\otimes} H_{-1}^{(2)} \ .$$

These results can be used to obtain information about the generalized eigenvectors for a tensor product operator.

Let B_1 and B_2 be selfadjoint operators in $H_0^{(1)}$ and $H_0^{(2)}$ respectively. Let $S_1 := \sigma(B_1)$ and $S_2 := \sigma(B_2)$ denote the spectra of B_1 and B_2 in the GB^*-algebras $\Gamma''_\infty(B_1)$ and $\Gamma''_\infty(B_2)$ respectively (cf. section 5.5).
The spectral measures of B_1 and B_2 are denoted by $E_1(.)$ and $E_2(.)$ resp.
Let $E(.)$ be the tensor product of $E_1(.)$ and $E_2(.)$.

Now, if P is a polynomial in two variables, then

$$C := \overline{P(B_1 \otimes I,\ I \otimes B_2)}$$

is defined as in 3.6.4. The spectral measure F(.) of C is the image of E(.) under the map

$$P : S_1 \times S_2 \longrightarrow \mathbb{C}^*$$

(in general this map is not continuous).

Combining 5.4.2 with the results of this section one proves that the generalized eigenvectors corresponding to F(.) can be expressed in terms of the generalized eigenvectors corresponding to $E_1(.)$ and $E_2(.)$. This generalizes a result of L. & K. MAURIN [19].

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