MEASURES AND INDICES OF REFLECTION SYMMETRY FOR CONVEX POLYHEDRA

ALEXANDER V. TUZIKOV *
Institute of Engineering Cybernetics
Surganova 6, 220012 Minsk, Belarus

JOS B.T.M. ROERDINK
University of Groningen
P.O. Box 800, 9700 AV Groningen, The Netherlands

HENK J.A.M. HEIJMANS
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

and

STANISLAV A. SHEYNIN
Institute of Engineering Cybernetics
Surganova 6, 220012 Minsk, Belarus

Abstract. This paper discusses measures of reflection symmetry for 3D convex sets which are based on Minkowski addition and Brunn-Minkowski inequalities for volume and mixed volume. These measures are invariant to similitude transformations. It is also shown how these measures can be computed efficiently for convex polyhedra.

Key words: Reflection symmetry measure, Minkowski addition, Brunn-Minkowski inequality, Index of reflection symmetry, Convex polyhedra

1. Introduction

For many objects, presence or absence of symmetry is a major feature, and therefore the problem of object symmetry identification is of great interest in image analysis and recognition, computer vision and computational geometry. There exists a vast literature dealing with all kinds of symmetry of shapes and grey-scale images. There exist algorithms for the identification of exact symmetries as well as techniques devoted to the computation of quantitative information about the amount of symmetry in shapes and images. Efficient algorithms for detection of exact symmetries of point sets, polygons and polyhedra can be found e.g. in [9, 19].

Since real images are always disturbed by noise it is useful to have a tool to measure the amount of symmetry in them. Towards this goal Grünbaum [6] introduced the notion of *symmetry measure*. Most of the theoretical results concerning symmetry measures are obtained for convex sets. Some interesting results concerning measures of central symmetry (point reflections) can be found in [6]. Related results for reflection and rotation symmetry of convex sets can be found in [1, 2]. Studies of central symmetry measures for convex sets using morphological transformations

* A. Tuzikov and S. Sheynin were supported by the INTAS grant N 96-785

are reported in [10, 11, 13]. Symmetrization transformations based on Minkowski addition were used in [17] to study rotation and reflection symmetry measures of convex sets.

Most practical algorithms developed for symmetry measurement are applied to the 2D case and only some of them can be extended to 3D (see, for example, the symmetry distance introduced in [20]). Since this paper deals with the 3D case we discuss only some literature concerning the 3D case. The extended Gaussian image representation was used in [15] for finding axes of reflection and rotation symmetries of 3D objects. Octree representations were used in [12] to measure the symmetry degree of 3D objects.

In this paper we extend the approach described in [8] for finding symmetry measures and indices of 2D convex sets to the 3D case. We restrict ourselves to reflection symmetry measures. The measures discussed here are based on Minkowski addition and properties of volume and mixed volume functionals.

As was shown in [18] for the case of convex polygons, the reflection symmetry measure can be computed in polynomial time. Here it is conjectured that finding reflection symmetry indices for convex polyhedra is reduced to the computation of the symmetry measure for a finite number of reflection planes only. This conjecture is based on a more general result obtained in [16] for comparing convex polyhedra.

2. Symmetry measures

Denote by $\mathcal{K}(\mathbb{R}^3)$, or briefly $\mathcal{K}$, the family of all nonempty compact subsets of $\mathbb{R}^3$. The compact convex subsets of $\mathbb{R}^3$ are denoted by $\mathcal{C} = \mathcal{C}(\mathbb{R}^3)$. For two subsets A and B we write $A \equiv B$ if these sets differ only by translation. The group of all linear transformations on $\mathbb{R}^3$ is denoted by G. If $g \in G$ and $A \subseteq \mathbb{R}^3$, then $g(A) = \{g(a) \mid a \in A\}$. Furthermore, we denote by $E \subseteq G$ the reflections in the planes through the origin, and by R the rotations around axes through the origin.

Let u be a vector on the unit sphere S^2 in $\mathbb{R}^3$ and let Π_u be the plane in $\mathbb{R}^3$ orthogonal to the vector u passing through the origin. The reflection w.r.t. the plane Π_u is denoted by e_u.

Definition 1 *A set $A \subset \mathbb{R}^3$ is called* reflection symmetric *if there exists a reflection $e_u \in E$ such that $e_u(A) \equiv A$. We say that e_u is a symmetry of A, and we call Π_u the* plane of reflection symmetry.

To access the symmetry of sets we need a tool to measure the amount of symmetry. Several years ago, Grünbaum [6] introduced the notion of *symmetry measure.* We adapt his definition of central symmetry measure for the case of reflection symmetry in the following way.

Definition 2 *Let $\mathcal{J} \subset \mathcal{K}$. A function $\mu : \mathcal{J} \times E \to [0,1]$ is called a* reflection symmetry measure *if for every $e \in E$ the function $\mu(\cdot, e)$ is continuous on $\mathcal{J}$ with respect to the Hausdorff topology, and if the following conditions hold:*

1. $\mu(A, e) = \mu(A', e)$ *if* $A \equiv A'$;
2. $\mu(A, e) = \mu(e(A), e)$;
3. $\mu(A, e) = 1$ *iff* $e(A) \equiv A$.

Let $H \subseteq G$ be a set such that heh^{-1} is a reflection if e is a reflection and $h \in H$. We say that a reflection symmetry measure μ is H-invariant *if*

$$\mu(A, e) = \mu(h(A), heh^{-1}) \quad \textit{for all } h \in H.$$

Below we introduce two reflection symmetry measures based on Minkowski addition and properties of volume and mixed volume functionals. The *Minkowski addition* of two sets $A, B \subseteq \mathbb{R}^3$ is defined by

$$A \oplus B = \{a + b \mid a \in A,\ b \in B\}.$$

It is obvious that for two sets $A, B \subseteq \mathbb{R}^3$ and $g \in G$, we have

$$g(A \oplus B) = g(A) \oplus g(B).$$

Denote by $V(A)$ the volume (Lebesgue measure) of the set $A \subset \mathbb{R}^3$. Given convex sets $A, B \subset \mathbb{R}^3$ and $\alpha, \beta \geq 0$ one gets from the Minkowski theorem on mixed volumes [3, p.353] the following relation:

$$V(\alpha A \oplus \beta B) = \alpha^3 V(A) + 3\alpha^2 \beta V(A, A, B) + 3\alpha\beta^2 V(A, B, B) + \beta^3 V(B). \quad (1)$$

Here $V(A, A, B)$ and $V(A, B, B)$ are called *mixed volumes*. The following inequalities are used below; see Hadwiger [7] or Schneider [14] for a comprehensive discussion.

Brunn-Minkowski inequality. For two arbitrary compact sets $A, B \subset \mathbb{R}^3$ the following inequality holds:

$$V(A \oplus B)^{\frac{1}{3}} \geq V(A)^{\frac{1}{3}} + V(B)^{\frac{1}{3}}, \quad (2)$$

with equality if and only if A and B are convex and homothetic modulo translation, i.e., $B \equiv \alpha A$ for some $\alpha > 0$.

Minkowski inequality. For convex sets $A, B \in \mathcal{C}(\mathbb{R}^3)$

$$V(A, A, B)^3 \geq V(A)^2 V(B), \quad (3)$$

and as before equality holds if and only if $B \equiv \alpha A$ for some $\alpha > 0$.

Given a plane reflection e_u define the transformation $b_u : \mathcal{K} \longrightarrow \mathcal{K}$ by

$$b_u(A) = \frac{1}{2}(A \oplus e_u(A)).$$

It is easy to see that the set $b_u(A)$ is reflection symmetric with respect to the plane Π_u. In the literature the transformation b_u is called *Blaschke symmetrization* [14].

Proposition 1 *Given a set $A \in \mathcal{C}$ and a plane reflection e_u, the inequality $V(b_u(A)) \geq V(A)$ always holds. Furthermore, the following statements are equivalent:*
(i) $e_u(A) \equiv A$, *i.e.,* e_u *is a symmetry of* A*;*
(ii) $b_u(A) \equiv A$*;*
(iii) $V(b_u(A)) = V(A)$.

For a proof we refer to [8, Prop.7.2].

Let us introduce the functionals $\mu_1, \mu_2 : \mathcal{C} \times E \to \mathbb{R}_+$ defined for compact convex sets as follows

$$\mu_1(A, e) = \frac{8V(A)}{V(A \oplus e(A))}, \tag{4}$$

$$\mu_2(A, e) = \frac{V(A)}{V(A, A, e(A))}. \tag{5}$$

It is known from properties of mixed volumes that for every linear transformation g, the following identity holds: $V(g(A), g(B), g(C)) = |\det g| V(A, B, C)$. Here $\det g$ denotes the determinant of g. Now, using relation (1) it follows that

$$\mu_2 = \frac{3\mu_1}{4 - \mu_1}. \tag{6}$$

Moreover the following proposition is true.

Proposition 2 *The functionals μ_1 and μ_2 are reflection symmetry measures which are invariant under rotations and scalings.*

Sometimes, one is not so much interested in the symmetry measure for a specific reflection plane Π_u, but rather in the maximum of these values over all planes. We call this maximum the *index of reflection symmetry*. Thus we define:

$$\iota_1(A) = \sup_{e \in E} \frac{8V(A)}{V(A \oplus e(A))}, \tag{7}$$

$$\iota_2(A) = \sup_{e \in E} \frac{V(A)}{V(A, A, e(A))}. \tag{8}$$

Evidently, both indices are related to each other by a formula analogous to (6). It is obvious that both indices ι_1 and ι_2 are invariant under similitude transformations, i.e., $\iota_1(g(A)) = \iota_1(A)$ and $\iota_2(g(A)) = \iota_2(A)$ for every similitude transformation g.

If the supremum in (7) is achieved for $e = e_u$, then we call Π_u the *ι_1-optimal plane of reflection symmetry* (note that because of relation (6), the ι_1-optimal and ι_2-optimal planes coincide).

To find the ι_1-optimal plane of reflection symmetry it is necessary to maximize $\mu_1(A, e_u)$ over all possible positions of reflection planes passing through the coordinate origin. In general this is a time consuming problem. Therefore, one often restricts oneselves to a finite number of reflection planes. Following ideas from classical mechanics [5] one can associate with every body its ellipsoid of inertia. It is known that symmetry planes of reflection symmetrical bodies are orthogonal to the principal axes of this ellipsoid. Therefore, to reduce the computation complexity of the optimization, one can define approximate measures by considering only planes orthogonal to the principal axes.

If one restricts attention to the class of convex polyhedra then it is possible to get additional results. In Section 3 we show that it is possible to reduce the time complexity of the given optimization problem.

3. Convex polyhedra

To compute the indices of reflection symmetry ι_1, ι_2 for a convex polyhedron P it is necessary to find minima of the following functionals

$$V(P \oplus e_u(P)) \quad \text{and} \quad V(P, P, e_u(P)) \tag{9}$$

for $u \in S^2_+$, the hemisphere containing unit vectors with non-negative z-coordinate.

Denote by $r_{u,\alpha}$ the rotation in $\mathbb{R}^3$ about the oriented axis directed along vector u over an angle α (counter-clockwise direction). Given two unit vectors $u_1, u_2 \in S^2_+$, denote by $\alpha(u_1, u_2)$ the angle smaller than 180° between them. The composition of plane reflections can be expressed as a rotation in the following way

$$e_{u_2} e_{u_1} = r_{u_1 \times u_2,\ 2\alpha(u_1,u_2)}, \tag{10}$$

where $u_1 \times u_2$ denotes the outer product of vectors u_1 and u_2.

Given any vector $v \in S^2$ denote by $S_+(v)$ the upper part of the great circle in S^2 which is orthogonal to v. Let us investigate the functional $V(P \oplus e_u(P))$ for $u \in S_+(v)$ (see Fig. 1).

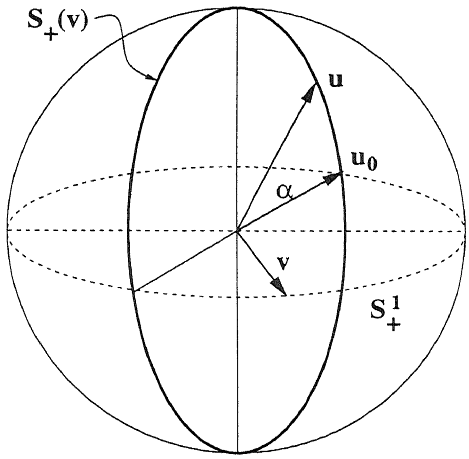

Fig. 1. Vector v is orthogonal to (upper part of) the great circle $S_+(v)$ and $u_0, u \in S_+(v)$.

Proposition 3 *For a vector $v \in S^2$ the functionals $V(P \oplus e_u(P))$ and $V(P, P, e_u(P))$ are piecewise concave in u, when u runs over $S_+(v)$.*

Proof. To prove this result, fix $u_0 \in S_+(v)$ as in Fig. 1. For $u \in S_+(v)$ we have $u \times u_0 = v$, hence

$$e_u e_{u_0} = r_{v,2\alpha},$$

where $\alpha = \alpha(u_0, u)$. Applying e_{u_0} at the right of both expressions and using that $e^2_{u_0} = \mathsf{id}$, we get

$$e_u = r_{v,2\alpha} e_{u_0}.$$

Therefore, putting $Q = e_{u_0}(P)$, we find

$$V(P \oplus e_u(P)) = V(P \oplus r_{v,2\alpha}(Q)).$$

When u runs over $S_+(v)$, the angle α runs over $(0, \pi]$. The concavity of the functional at the right hand-side has been established in [16], and the proof follows. Q.E.D.

Observe that u_0 is the $90°$-rotation of v around the z-axis. Thus $Q = e_{u_0}(P)$ depends on v: $Q = Q(v)$. We have shown the following relation:

$$\min_{u \in S^2_+} V(P \oplus e_u(P)) = \min_{v \in S^1_+} \min_{0 < \alpha \le \pi} V(P \oplus r_{v,2\alpha}(Q(v))).$$

A similar relation holds for the functional $V(P, P, e_u(P))$.

Therefore the optimization problem on the hemisphere S^2_+ is reduced to the optimization problem on the semi-equator S^1_+.

In our report [16] we have dealt with the computation of

$$\min_{u \in S^2,\ 0 \le \alpha < 2\pi} V(P \oplus r_{u,\alpha}(Q)),$$

$$\min_{u \in S^2,\ 0 \le \alpha < 2\pi} V(P, P, r_{u,\alpha}(Q)),$$

for arbitrary convex polyhedra P and Q. It was proven that only finitely many vectors $u \in S^2$ have to be checked to compute the minimum. Based on these results we formulate the following conjectures:

Conjecture 1 *There exist a finite number of vectors $v_1, \ldots, v_k \in S^1_+$ such that*

$$\min_{u \in S^2_+} V(P \oplus e_u(P)) = \min_{i=1,\ldots,k} \min_{0 \le \alpha < \pi} V(P \oplus r_{v_i,2\alpha}(Q(v_i))),$$

$$\min_{u \in S^2_+} V(P, P, e_u(P)) = \min_{i=1,\ldots,k} \min_{0 \le \alpha < \pi} V(P, P, r_{v_i,2\alpha}(Q(v_i))).$$

4. Conclusion

The 3D case is essentially more difficult than the 2D case: for the latter it has been shown [8] that only finitely many cases have to be checked. Basically, the reason for this difference is that in the 2D case, the composition of a reflection and a rotation is a reflection; this no longer true in the 3D case. The conjecture formulated above is still to be proven. Our present work concerns the implementation of algorithms for computing indices of reflection symmetry for convex polyhedra. Such algorithms are based on the slope diagram representation of convex polyhedra [4].

The problem of computation of indices of reflection symmetry is more difficult than computation of central symmetry measures which indicate the amount of central symmetry. Write $\check{A} = \{-x,\ x \in A\}$ and introduce the functionals $\mu_3, \mu_4 : \mathcal{C} \longrightarrow \mathbb{R}_+$ defined for compact convex sets as follows

$$\mu_3(A) = \frac{8V(A)}{V(A \oplus \check{A})},$$

$$\mu_4(A) = \frac{V(A)}{V(A, A, \breve{A})}.$$

One can show that the functionals μ_3 and μ_4 define affine invariant central symmetry measures for convex sets in $\mathbb{R}^3$.

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