Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.
J. DE VRIES

TOPOLOGICAL TRANSFORMATION GROUPS 1

A CATEGORICAL APPROACH

MATHEMATISCH CENTRUM AMSTERDAM 1975
AMS(MOS) subject classification scheme (1970): 54-02, 54H15, 18B99, 18C15, 43A15

ISBN 90 6196 113 0
The theory of topological transformation groups (ttgs) forms a fascinating and comprehensive realm in the world of mathematics, bordering on the domains of topology, abstract harmonic analysis, ergodic theory, geometry and differential equations. We intend to lead the reader over a more or less artificial path between a number of "vantage points" which afford a "scenic view" of this landscape. These "points" are the notes which conclude the subsections. Without pretending completeness we indicate there the relationship of the material in the subsection to other literature. We shall now present a short description of the path mentioned above which the reader must follow. First of all, we have chosen to take a mainly categorical point of view, with the aim of unifying parts of the existing theory of ttgs. However, our aim could not fully be realized since more category theory had to be included than was originally planned. In a subsequent volume we intend to deal with a number of topics which could only be mentioned in the notes of this volume.

In Chapter I we give some basic material on ttgs. We claim no originality for its content, except for theorem 2.3.15: for any locally compact Hausdorff group \( G \) the dimension of \( L^2(G) \) equals the weight of \( G \).

After this prelude, we describe several categories of ttgs in Chapter II. As in a fugue, we always deal systematically with the same theme in each category. This theme can be described as follows: facts about a certain category of ttgs should be expressed in terms of the underlying categories of topological groups and topological spaces. (Although this tactic will probably hurt the feelings of every pure category theorist, it is a consequence of their wish that each "working mathematician" should try to use category theory for the description of the objects he is studying.) Thus, in §3 we describe the category \( \text{TTG} \) of all ttgs. In accordance with the
policy indicated above, we try to do this by proving certain preservation and reflection properties of the obvious forgetful functor $K: \text{TTG} \to \text{TOPGRP} \times \text{TOP}$. For limits and monomorphisms everything works well, because the functor $K$ is monadic, i.e. $\text{TTG}$ can be identified with the category of all algebras over a certain monad in $\text{TOPGRP} \times \text{TOP}$. For colimits, things go wrong; nevertheless, epimorphisms are preserved and reflected by $K$. In addition, $\text{TTG}$ is cocomplete. In order to prove cocompleteness, we had to generalize a known construction which is related to "induced representations".

In §4, we consider some subcategories of $\text{TTG}$, defined by imposing certain restrictions on the phase groups and the phase spaces of ttgs. It is felt to be a serious omission that in this volume sparse attention could be paid to the much more interesting subcategories which arise from restrictions on actions. This subject is only touched upon in connection with certain reflective subcategories of $\text{TTG}$.

In $\text{TTG}$, the "group component" and the "space component" of a morphism have the same direction. In §6, we define a category $\text{TTG}_*$, which has the same object class as $\text{TTG}$, but in which the two components of a morphism have opposite directions. Thus, we obtain a functor $K_*: \text{TTG}_* \to \text{TOPGRP}^{\text{OP}} \times \text{TOP}$. Although $K_*$ has quite nice properties with respect to colimits, the category $\text{TTG}_*$ is not cocomplete; neither is it complete. Therefore, we consider in §6 also the category $k-\text{TTG}_*$, which is defined similarly to $\text{TTG}_*$, except that its objects are $k$-ttgs. (Roughly speaking, all cartesian products have to be replaced by products in the category $\text{KR}$ of all $k$-spaces.) The study of these objects is initiated in §5, where we consider the category $k-\text{TTG}$ (morphisms similar to those in $\text{TTG}$). There we show that $k-\text{TTG}$ is well-behaved with respect to limits; it is a category of algebras over a certain monad in $\text{KR}$. In §6, we show that $k-\text{TTG}_*$ (morphisms similar to those in $\text{TTG}_*$) is well-behaved with respect to colimits; it is a category of coalgebras over a certain comonad in $\text{KR}$. Combining these results, it follows that the category $k-\text{KR}^\text{G}$ (for a fixed $k$-group $G$) is well-behaved in both respects. This result extends to the category $\text{TOP}^\text{G}$ of all (ordinary) ttgs with phase group a fixed locally compact group $G$.

The final remarks in Chapter II concern the existence of cogenerators in $\text{TTG}_*$. This is the starting point for the considerations in Chapter III. However, the categorical point of view in this chapter is hidden under the surface of variations on another theme. This theme is the question of whether
a given class of G-spaces (i.e. ttgs with a common phase group G, mostly locally compact) admits a comprehensive object, i.e. an object (not necessarily in the given class) in which all G-spaces in question can be embedded in one way or another. Here the main difference between our results and earlier ones (among others, by P.C. BAYEN and J. DE GROOT) is that we obtain comprehensive objects which are G-spaces, whereas in the older results only $G_d$-spaces were obtained. To be honest, most of those $G_d$-spaces turned out to be G-spaces, and that is just what we prove in §8. In other words, the methods used in §8 are modifications of older ones. In addition, our methods and our categorical starting point made it possible to give a more unified treatment of the subject.

The internal reference system is self-explaining; a reference like "cf. p.q.r" means that the reader may find some useful information (or sometimes, material for comparison) in the r'th sub-subsection of subsection q in section p. For references to the literature, we used two methods. References to research papers are by the name(s) of the author(s), followed by the year of publication between brackets. On the other hand, references to books and monographs are by a two-letter abbreviation of the author's name between square brackets. This is due to the fact that originally we planned to refer only to a limited number of standard text books, namely the following ones: for topology, the books by Bourbaki [Bo], Dugundji [Du] and Engelking [En], for topological groups the work of Hewitt and Ross [HR], for topological transformation groups the book by Gottschalk and Hedlund [GH] and, finally, for category theory the text book of Mac Lane [ML]. Unfortunately, the list of books expanded, and it took some efforts to keep it at a reasonable length.

This book is a revised version of the author's Ph.D. thesis, written at the Free University at Amsterdam under the supervision of prof.dr. P.C. Baayen. The author wishes to thank dr. A.B. Paalman-De Miranda for reading large parts of the manuscript and for her valuable suggestions. The author is indebted to the "Wiskundig Seminarium" of the Free University and to the Mathematisch Centrum at Amsterdam for giving him the opportunity to do this research. Finally, I would like to thank the reproduction staff of the Mathematisch Centrum for the excellent way in which they realized this book.

In particular, I mention Mrs. C.J. Klein Velderman-Los and Mrs T. Bays-Renforth, who typed the manuscript.
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CHAPTER I
BASICAL CONCEPTS

0 - PREREQUISITES

Although all terminology and notation in this treatise will be essentially standard, we inevitably have to include some notational conventions and state some known facts for easy reference. In subsection 0.1 some general remarks about notation are made. Then, in subsections 0.2, 0.3 and 0.4 we collect some notions from topology, topological groups and category theory, respectively. Here the choice of what has been inserted is mainly governed by the desire to avoid as much as possible repetitions of similar arguments. This principle is responsible for a number of trivial remarks in subsections 0.2, 0.3 and 0.4. Sometimes this tendency to unification will give rise to slightly sophisticated proofs in the main text, or to references which are, strictly speaking, superfluous. In addition, we tried to make this treatise as self-contained as possible by limitation of the number of text-books and research papers which are referred to in the main text. A few specialized topics are dealt with in appendices.

0.1. General remarks and conventions

The following logical symbols will be used: \( \Rightarrow \) (implication), \( \Leftrightarrow \) (equivalence), \& (conjunction), \( \exists \) (existential quantifier), \( \forall \) (universal quantifier). In order to reduce the number of parentheses, we shall often write \( \forall x, y \in X \) instead of \( \forall (x,y) \in X \times X \), and \( \forall x : \phi(x) \) instead of \( \forall x[\phi(x)] \). The sign \( ! \) is read "such that"; if it immediately precedes a quantifier, it will be omitted.

Expressions like \( P := Q \) or \( Q =: P \) are used when the expression \( Q \) defines \( P \).

Next we make some agreements with respect to sets and functions. In the following list, \( A, X \) and \( Y \) denote sets.
A ⊂ X : A is a proper subset of X;
A ≤ X : A ⊂ X or A = X;
X ~ A : = {x ∈ X : x ∉ A};
1_X : identity mapping on X;
|X| : the cardinality of the set X.

If f is a function on X with values in Y, we write f: X → Y or x ↦ f(x): X → Y.

We shall use the following terminology when f: X → Y is a function:

- domain of f : X;
- codomain of f : Y;
- f(x), f(x), fx : value of f at x ∈ X;
- range of f : f(X) := {f(x) : x ∈ X};
- image of A under f : f(A) := {f(x) : x ∈ A};
- inverse image of Z under f : f⁻¹[Z] := {x ∈ X : f(x) ∈ Z};
- restriction of f to A : f|_A : A → Y.

If f: X → Y is a function and A ⊂ X, then for typographical reasons the restriction of f to A will sometimes be denoted f|_A : A → Y. We shall not introduce a special notation for corestrictions. If we wish to consider a corestriction, we shall use phrases like "consider f: X → Z" (corestriction to Z ⊂ Y), or "consider f: A → Z" (restriction to A ⊂ X and corestriction to Z ⊂ Y, where f(A) ⊂ Z).

Compositions of functions are denoted f · g or fg; to be sure, if f: X → Y and g: Z → X, then f · g(z) := f(g(z)) for z ∈ Z.

If π: X × Y → Z is a function then for each (x, y) ∈ X × Y we write

π²(y) := π(x, y) =: π_y(x).

Obviously this convention defines functions π²: Y → Z and π_y: X → Z.

The cartesian product of a family \{X_j : j ∈ J\} of sets will be denoted by \(\prod \{X_j : j ∈ J\}\) or simply \(\prod_j X_j\); its elements are denoted by \((x_j)_J\). Notations like \(\prod_j X_j\) are reserved for products (in the categorical sense) in the category under consideration.

Some fixed symbols denoting sets are:

- \(\mathbb{C}\) : the set of complex numbers;
- \(\mathbb{R}\) : the set of real numbers;
- \(\mathbb{Z}\) : the set of all integers;
- \(\mathbb{N}\) : = \{n : n ∈ \mathbb{Z} & n ≥ 1\};
- \(\mathbb{Q}\) : the set of rational numbers;
\[ T : = \{ z : z \in \mathbb{C} \land |z| = 1 \}; \]
\[ \mathbb{F} : \text{the unspecified scalar field of a vector space} \]
\[ \text{(in this treatise, always } \mathbb{F} = \mathbb{R} \text{ or } \mathbb{F} = \mathbb{C} ). \]

Parts of the text between braces can be skipped without further ado. Braces are also used to indicate alternative reading: if in a certain passage there are several pairs of braces, then all these pairs can be replaced by the word "respectively". Thus the phrase "if \( P(Q) \) then \( R(S) \)" means "if \( P, \) resp. \( Q, \) then \( R, \) resp. \( S. \)"

0.2. Topology

0.2.1. In this section we embody some definitions and notational conventions which are not universally agreed upon. Otherwise, the reader is referred to [Bo], [Du] or [En].

In general a topological space \( (X,T) \), i.e. a set \( X \) endowed with a topology \( T \), will be denoted only by the symbol \( X \). A similar convention holds for uniform spaces.

Some notations (where \( A \subseteq X \) and \( x \in X \)):

\[ T\text{-cl}_X(A), \text{cl}_X(A), \text{cl}(A) : \text{closure of } A \text{ in } (X,T), \]
\[ T\text{-int}_X(A), \text{int}_X(A), \text{int}(A) : \text{interior of } A \text{ in } (X,T), \]
\[ \text{neighbourhood of } A \text{ (of } x) : \text{a set } U \subseteq X \text{ with } A \subseteq \text{int}(U) \{ x \in \text{int}(U) \}, \]
\[ \mathcal{V}_A := \{ U : U \subseteq X \land A \subseteq \text{int}(U) \} , \]
\[ \mathcal{V}_x := \{ U : U \subseteq X \land x \in \text{int}(U) \} . \]

Our use of the concepts regular, completely regular, normal, paracompact, compact and locally compact is, that they do not incorporate the \( T_1 \)-separation axiom. So \( T_3 = \text{regular}\&T_1, T_{3\frac{1}{2}} = \text{completely regular}\&T_1, T_4 = \text{normal}\&T_1 \).
A \( T_{3\frac{1}{2}} \)-space will also be called a Tychonov space. In contradistinction to the above convention, a \( k \)-space will always be assumed to be a Hausdorff space. Thus, a \( k \)-space is just the same as a compactly generated space in the terminology of [ML].

0.2.2. For easy reference we present here some well-known properties of continuous functions with respect to compactness. The proofs are completely standard, hence they are omitted:

\[ \text{---} \]

\[ {\text{---}} \]

\[ {\text{---}} \]
Let $X$, $Y$ and $Z$ be topological spaces, let $f: X \times Y \rightarrow Z$ be a continuous function, and let $A$ and $B$ be compact subsets of $X$ and $Y$, respectively. Then:

(i) \[ \forall W \subseteq V \subseteq f[A \times B] \exists U \subseteq V_A \times V_B : f[U \times V] \subseteq W. \]

(ii) If the topology of $Z$ is generated by a uniformity $U$, then for all $Y \subseteq Y$ and $\gamma \subseteq U$ there exists $V \subseteq V_Y$ such that

\[ (f(x,y), f(x,y)) \in \gamma \text{ for all } x \in X \text{ and } y \in Y. \]

Thus, \[ f^X : xA \] is an equicontinuous set of mappings on $Y$.

0.2.3. If $T_1$ and $T_2$ are topologies on a set $X$ then we say that $T_2$ is finer than $T_1$ (or $T_1$ is weaker than $T_2$) if $T_1 \subseteq T_2$.

If $X$ is a set and if for each $i \in J$ we have a mapping $f_i : X + Y_i$ (a set, each $Y_i$ a topological space), then the weak topology defined by the mappings $f_i$ is, by definition, the weakest topology on $X$ making each $f_i$ continuous.

If $X$ is endowed with this weak topology, then it is well-known that for any topological space $Z$, a mapping $g: Z + X$ is continuous iff $f_i g: Z + Y_i$ is continuous for each $i \in J$.

The "dual" notion of a weak topology is the finest topology on a set $X$ making all members of a set of functions $f_i : Y_i + X$ continuous (each $Y_i$ a topological space). Then a function $g: X + Z$ is continuous iff each $g(f_i): Y_i + Z$ is (any topological space).

In this context it is useful to recall that a continuous surjection $f: X + Z$ is called a quotient mapping if the topology of $Z$ is the finest one making $f$ continuous.

0.2.4. Let $X$ and $Y$ be topological spaces, $R$ and $S$ equivalence relations on $X$ and $Y$, respectively, and let $r: X + X/R$, $s: Y + Y/S$ denote the quotient mappings. Then $R \times S = \{ ((x,y),(x',y')) : (x,x') \in R \text{ and } (y,y') \in S \}$ is an equivalence relation on $X \times Y$, and there is an obvious bijection $f: X \times Y / R \times S \rightarrow (X/R) \times (Y/S)$ such that $f q = r x s$, $q$ denoting the quotient mapping of $X \times Y$ onto $X \times Y / R \times S$. Plainly, $f$ is continuous. Moreover, it is easy to see that $f$ is a homeomorphism iff $r x s: X \times Y \rightarrow (X/R) \times (Y/S)$ is a quotient mapping. Now a repeated application of the lemma below shows the following:

If $r$ and $s$ are either open or perfect (one of them open and the other perfect)\(^1\) is also allowed then $r x s$ is a quotient mapping, and $f$ is a

\(^1\) A perfect mapping is what [Bo] calls a proper mapping: a continuous function $f: X + Y$ is perfect whenever it is closed and, in addition, each fiber $f^{-1}(y)$ is compact ($y \in Y$).
homeomorphism. This holds true also if the codomain of one of the maps \( r \) and \( s \) and the domain of the other one are locally compact Hausdorff spaces.

Incidentally, conditions implying that a quotient space is \( T_2 \) are given in [Du], Chap. VII, 1.6 and 1.7; cf. also [En], Theorem 2.4.5.

**Lemma.** Let \( f : X \to Y \) be a quotient mapping and let \( Z \) be any topological space. Each of the following conditions implies that \( 1_Z \times f : Z \times X \to Z \times Y \) is a quotient mapping:

(i) \( f \) is an open mapping;
(ii) \( f \) is a perfect mapping;
(iii) \( Z \) is a locally compact \( T_2 \)-space;
(iv) \( Z \times Y \) is a k-space.

**Proof.** We give only brief indications or references to proofs.

(i): This is easy (cf. [Bo], Chap. I, §5.3, Cor. to Prop. 8).

(ii): This is a straightforward application of 0.2.2(i). Alternatively, by [Bo], Chap. I, §10.1, Prop. 4, \( 1_Z \times f \) is perfect, hence a quotient mapping.

(iii): Cf. [Du], Chap. XII, 4.1.

(iv): Replace in the proof referred to in (iii) the application of [Du], Chap. XII, Theorem 3.1 by its Corollary 3.2. Alternatively, see the proof of Theorem 4.4 in N.E. Steenrod [1967].

0.2.5. The following example shows that \( 1_Z \times f \) is not necessarily a quotient mapping if \( f \) is. There exist other examples in the literature, but we shall need this particular example, where \( Z \) is a topological group.

Take \( Z := \mathbb{Q} \), with its usual topology. Let \( X := [0,1] \times \mathbb{N} \), and let \( Y := X/R \), where \( R \) is the equivalence relation defined by

\[
((x,n),(x',n')) \in R \iff \begin{cases} 
\text{either } x=x', \ n=n' \\
\text{or } x = x' = 0.
\end{cases}
\]

So \( X \) is a countable disjoint union of unit intervals, and \( Y \) consists of a countable set of unit intervals with a common begin point \( p \). Let \( f : X \to Z \) denote the quotient mapping, and let \( Y \) be given the quotient topology. We claim that on \( \mathbb{Q} \times Y \) the quotient topology induced by \( 1_{\mathbb{Q}} \times f : \mathbb{Q} \times X \to \mathbb{Q} \times Y \) is strictly finer than the product topology on \( \mathbb{Q} \times Y \). Thus, with this product topology on \( \mathbb{Q} \times Y \), \( 1_{\mathbb{Q}} \times f \) is not a quotient mapping.

\[1\] Thus, \( p = f(0,n) \) for all \( n \in \mathbb{N} \).
{Indication of proof: for each }n \in \mathbb{N}, \text{ take as } V_n \subseteq \mathbb{Q} \times [0,1] \text{ the set that can be pictured as follows:}

Identifying } \mathbb{Q} \times X \text{ with } (\mathbb{Q} \times [0,1]) \times \mathbb{N}, \text{ we see that the set } V := \{(t,x,n) \mid n \in \mathbb{N} \text{ and } (t,x) \in V_n\} \text{ is open in } \mathbb{Q} \times X. \text{ Since } (\mathbb{Q} \times X)^{\sim}(\mathbb{Q} \times X)[V] = V, \text{ it follows that } (\mathbb{Q} \times X)(V) \text{ is a neighbourhood of the point } (0,p) \text{ in } \mathbb{Q} \times Y \text{ with respect to the quotient topology induced by } \mathbb{Q} \times f. \text{ However, if it were a neighbourhood of this point in the product topology, then there would be a real number } a > 0 \text{ and an open subset } W \text{ of } Y \text{ such that } \{t \in \mathbb{Q} \mid -a < t < a\} \times f^*\{W\} \subseteq V. \text{ But this is impossible.}

0.2.6. \text{ Let } X \text{ and } Y \text{ be topological spaces. Then } C(X,Y) \text{ shall denote the set of all continuous functions on } X \text{ with values in } Y. \text{ In this set we shall consider the following topologies:}

(i) \text{ The point-open topology: this is the weakest topology in } C(X,Y) \text{ making each evaluation } \delta_x : f \mapsto f(x) : C(X,Y) \to Y \text{ continuous } (xeX), \text{ i.e. it is the relative topology of } C(X,I) \text{ in } Y^X \text{ with its product topology. If } C(X,Y) \text{ is endowed with this topology, we write } C_p(X,Y).

(ii) \text{ The compact-open topology: This is the topology having as a subbase all sets of the form }

\[ N(K,U) := \{f \in C(X,Y) \mid f[K] \subseteq U\}, \]

where } K \subseteq X \text{ is compact and } U \subseteq Y \text{ is open. If } C(X,Y) \text{ is endowed with this topology, we write } C_c(X,Y).

If the topology in } Y \text{ is generated by a uniformity } U, \text{ then the topology of } C_c(X,Y) \text{ is generated by the uniformity in } C(X,Y) \text{ having as a subbase all sets of the form }

\[ M(K,u) := \{(f,g) \in C(X,Y) \times C(X,Y) \mid (f(x),g(x)) \in a \text{ for all } x \in K\}, \]

where } K \subseteq X \text{ is compact and } a \in U. \text{ In this case, a local base at } f \in C_c(X,Y) \text{ is formed by the family of all sets } M(K,u)f \text{ (} K \text{ and } a \text{ as before). For simplicity, we shall use the following notation:}
U_f(x,a) := M(x,a)[f]
= \{g \in C(X,Y) : (f(x),g(x)) \in a \text{ for all } x \in X\}.

(iii) The topology of uniform convergence: Y is supposed to be a uniform space with uniformity \(U\), and the topology under consideration is the one generated by the uniformity in \(C(X,Y)\) having as a subbase the family of all sets \(M(x,a)\) with \(a \in U\) (where \(M(x,a)\) is defined as in (ii) with \(K = X\)). If \(C(X,Y)\) is endowed with this topology, we shall write \(C_u(X,Y)\).

If \(A \subseteq C(X,Y)\), then we shall always write \(A_p\), \(A_c\) or \(A_u\) if we consider the point-open topology, the compact-open topology or the topology of uniform convergence on \(A\), respectively. Sometimes, this notation will slightly be modified. Thus, we write \(C^p(X,Y)\), \(C^c(X,Y)\), etc. instead of \((C^p(X,Y))_p\), \((C^c(X,Y))_c\), etc. Here \(C^c(X,Y)\) is defined as follows:

\[ C^c(X,Y) := \{f \in C(X,Y) : \text{cl}_Y f[X] \text{ is compact}\} \]

0.2.7. If \(X, Y\) and \(Z\) are topological spaces, then we shall need several times the following facts. For proofs, cf. [Du], Chap. XII.

(i) If \(Y\) is locally compact Hausdorff, then the composition-mapping

\[ \omega : (f,g) \mapsto f \circ g : C^c(Y,Z) \times C(X,Y) \to C(X,Z) \]

is continuous. If \(Y\) is not locally compact, then \(\omega\) is separately continuous.

(ii) If \(Y\) is locally compact Hausdorff, then the evaluation mapping

\[ \delta : (f,y) \mapsto f(y) : C^c(Y,Z) \times Y \to Z \]

is continuous.\(^2\)

(iii) If \(f \in C(X\times Y,Z)\), then the mapping

\[ \tilde{f} : X \mapsto f^2 : X \times C^c(Y,Z) \]

is continuous.\(^3\) Conversely, if either \(Y\) is locally compact \(T_2\) or \(X \times Y\) is a \(k\)-space and \(f : X \times Y \to Z\) is any function such that \(f^2 \in C^c(X,Y)\) for all \(x \in X\), then continuity of \(\tilde{f} : X \mapsto f^2 : X \times C^c(Y,Z)\) implies that \(f \in C(X\times Y,Z)\). In addition, then the mapping

\(^1\) Equations like \(A_p = A_c\) express that the corresponding topologies on \(A\) coincide.

\(^2\) This result is an immediate consequence of (iii) below.

\(^3\) This is an easy corollary of 0.2.2(i).
\[ f \mapsto \pi : C_c(X \times Y, Z) \rightarrow C_c(X, C_c(Y, Z)) \]

is a homeomorphism; in particular, it is a bijection.

Concerning (ii) above we have to make the following remark, which is due to R. ARENS [1946b]: if \( Y \) is completely regular and \( \delta : (f, y) \mapsto f(y) : C_c(Y, [0, 1]) \times Y \rightarrow [0, 1] \) is continuous, then \( Y \) is locally compact. A close examination of the proof reveals that \( C(Y, [0, 1]) \) can be replaced by any of its subsets which separates points and closed subsets of \( Y \). Consequently, if \( Y \) is a uniform space with uniformity \( U \), and if \( UC(Y, [0, 1]) \) denotes the set of all uniformly continuous functions from \( Y \) into \( [0, 1] \), then the lemma below can be used in order to prove that continuity of the restricted evaluation \( \delta : UC(Y, [0, 1]) \times Y \rightarrow [0, 1] \) already implies that \( Y \) is locally compact.

**Lemma.** Let \( Y \) be a uniform space. Then for any \( y \in Y \) and any closed set \( S \subseteq Y \) such that \( y \not\in S \) there exists \( f \in UC(Y, [0, 1]) \) such that \( f(y) = 1 \) and \( f(s) = 0 \) for all \( s \in S \).

**Proof.** Cf. [Is], Theorem I.13. One may also have a close examination of the proof of Cor. 3 to [En], Theorem 8.1.4. Cf. also [Ke], Th. 6.15. [\( \square \)]

0.2.8. It will be convenient to have the following well-known statements at hand, leading up to a proof of the ASCOLI theorem. Let \( X \) be any topological space and let \( Y \) be a uniform space. Then:

(i) If \( A \subseteq C(X, Y) \) is equicontinuous at \( x \in X \), then the closure of \( A \) in \( Y^X \) is equicontinuous at \( x \) as well.

(ii) If \( A \subseteq C(X, Y) \) is equicontinuous on \( X \), then \( A_p = A_c \).

(iii) If \( A \subseteq C(X, Y) \) satisfies the condition that

\[ x \in X \Rightarrow A \text{ equicontinuous at } x \Rightarrow \text{cl}_p A(x) \text{ is compact}, \]

then the closure of \( A \) in \( C_c(X, Y) \) is compact. In this situation the closure of \( A \) in \( C_c(X, Y) \) equals the closure of \( A \) in \( Y^X \) and the point-open and compact-open topologies coincide on it.

There is a converse to (iii), namely the following one: if \( X \) has the property that the evaluation mapping \( \delta : C_c(X, Y) \times X \rightarrow Y \) is continuous on all
sets \( B \times X \), where \( B \) is a compact subset of \( C_c(X,Y) \), then every relatively compact subset \( A \) of \( C_c(X,Y) \) satisfies the condition in (iii). \( \{ \text{In fact, only equicontinuity of } A \text{ needs a proof, and this is just 0.2.2(ii).} \}

Notice that the above mentioned condition on \( X \) is fulfilled if \( X \) is a locally compact \( T_2 \)-space, or more generally, if \( X \) is a \( k \)-space; then \( B \times X \) is a \( k \)-space for every compact \( B \subseteq C_c(X,Y) \) (cf. [Du], Chap. XII, Th. 4.4), so that 0.2.7(iii) gives the desired result).  

0.2.9. Let \( \mathbb{F} \) denote either \( \mathbb{R} \) or \( \mathbb{C} \). Then we shall write \( C(X) := C(X,\mathbb{F}) \) and \( C^*(X) := C^*(X,\mathbb{F}) \) for any topological space \( X \). Thus, \( C^*(X) \) is just the set of bounded continuous \( \mathbb{F} \)-valued functions on \( X \). The topology in \( C^*(X) \) is generated by the norm \( \| \cdot \|_X \). Here we define, for each \( A \subseteq X \) and \( f \in C^*(X) \),

\[
\| f \|_A := \sup\{|f(x)| : x \in A\};
\]

instead of \( \| \cdot \|_X \) we write mostly \( \| \cdot \|_X \).

If \( f \in C(X) \), then the support of \( f \) is defined to be \( \text{supp}(f) := \text{cl}_X \{ x : x \in X \text{ and } f(x) \neq 0 \} \). The following subspace of \( C^*(X) \) will be of importance to us: \( C_{00}(X) := \{ f \in C(X) : \text{supp}(f) \text{ is compact} \} \).

0.2.10. A cardinal invariant in topology is an assignment of a cardinal number to each topological space in such a way that equal cardinal numbers are assigned to homeomorphic spaces. We shall use the following cardinal invariants: the weight \( \omega \), the local weight \( \ell \), the density \( d \) and the Lindelöf degree \( l \). They are defined as follows: if \( X \) is a topological space, then

the weight of \( X \) is

\[
\omega(X) := \min\{|B| : B \text{ is an open base for } X\};
\]

the local weight of \( X \) at \( x \) is

\[
\ell_w(X,x) := \min\{|V| : V \text{ is a local base at } x\};
\]

the local weight of \( X \) is

\[
\ell_w(X) := \sup\{\ell_w(X,x) : x \in X\};
\]

the density of \( X \) is

\[
d(X) := \min\{|A| : \text{cl} A = X\};
\]

(1) See also R.W. BAGLEY & J.S. YANG [1966].
the Lindelöf degree of $X$ is

\[ L(X) := \min\{ \kappa : \text{each open covering of } X \text{ has a subcovering of cardinality } \kappa \} \]

For a systematical treatment of these and other cardinal invariants we refer to [Ju]. However, the following relations are well-known and easy to prove (for (2), see [En], Theorem 1.1.6):

1. \[ d(X) \leq w(X) \leq 2|X| \]
2. \[ L(X) \leq w(X) \]
3. \[ d(X) \cdot \omega(X) \leq w(X) \]

It can be shown by examples that the inequality in (3) may be strict. However, in metrisable spaces $X$ one has always equality in (3).

If $A$ is a dense subspace of a $T_3$-space $X$, then it is not difficult to prove that $\omega(A) = \omega(X)$. In general however, $\omega(A) \leq w(X)$, and $d(A) \geq d(X)$. Consequently, $d(X) \cdot \omega(X) \leq d(A) \cdot \omega(A) \leq w(A) \leq w(X)$. If $X$ is metrisable, then $d(X) \cdot \omega(X) = w(X)$, and we obtain the following result:

If $A$ is a dense subspace of a metrisable space $X$, then $w(A) = w(X)$.

0.3. Topological groups

0.3.1. For all basical knowledge about topological groups we refer to [HR], Vol. I, or to [Bo], Chap. III. In particular, results from [HR], §1-§8 will be used mostly without explicit reference. Some notational conventions in a topological group $G$ which we shall use:

- $e_G, e$: the unit of $G$;
- $\lambda(G), \lambda$: the multiplication mapping $(s,t) \mapsto st: G \times G \to G$;
- $G_d$: the group $G$ endowed with its discrete topology;
- $G/H(G\setminus H)$: the space of all right (left) cosets of a subgroup $H$ of $G$.

Note the difference between the expressions $G \setminus H$ and $G \sim H$ (cf. 0.1).

\[1] ^{1}$ In locally compact $T_2$-spaces $X$ one has ever $w(X) \leq |X|$; cf. [En], Theorem 3.6.9. See also [Ju], 2.2.
0.3.2. The left uniformity in a topological group $G$ is the uniformity generated by all sets of the form $\{(s,t) \in G \times G : s^{-1} t \in U\}$ with $U \in \mathcal{V}_e$. The topology of $G$ is generated by its left uniformity. Similarly, the right uniformity is the uniformity generated by all sets $\{(s,t) \in G \times G : s t^{-1} \in U\}$ with $U \in \mathcal{V}_e$. The right uniformity of $G$ is also compatible with the topology of $G$. In general, the left and the right uniformity in a topological group do not coincide.

We shall use these uniformities mainly in order to introduce the following classes of functions. Let $X$ be a uniform space with uniformity $U$. Then a function $f : G \to X$ is said to be left(right) uniformly continuous whenever it is uniformly continuous with respect to the left(right) uniformity on $G$. The set of all left(right) uniformly continuous functions on $G$ to $X$ is denoted $\text{LUC}(G,X)$ ($\text{RUC}(G,X)$). Thus, if $f : G \to X$ is a function, then $f \in \text{LUC}(G,X)$ iff

$$\forall a \in U, \exists u \in \mathcal{V}_e : s^{-1} t \in U \Rightarrow (f(s), f(t)) \in a,$$

and $f \in \text{RUC}(G,X)$ iff

$$\forall a \in U, \exists u \in \mathcal{V}_e : s t^{-1} \in U \Rightarrow (f(s), f(t)) \in a.$$

We write $\text{LUC}(G)$ and $\text{RUC}(G)$ instead of $\text{LUC}(G,F)$ and $\text{RUC}(G,F)$, where $F$ is either $\mathbb{R}$ or $\mathbb{C}$. In addition, $\text{LUC}^*(G,X) := \text{LUC}(G,X) \cap \mathcal{C}^*(G,X)$ and similarly, $\text{RUC}^*(G,X) := \text{RUC}(G,X) \cap \mathcal{C}^*(G,X)$.

0.3.3. We shall use integration theory on locally compact Hausdorff groups with respect to Haar measure. Here we shall follow [HR], Chap. II, III. Some notations and conventions about this topic will be presented in 2.3.1.

Only a modest amount of knowledge is required about this topic: we need existence of Haar measure and some elementary facts from integration theory, up to the FUBINI theorem and the LEIBNIZ theorem on interchangeability of limits and integrals.

In this context, a certain knowledge of functional analysis is needed. For this, we refer to [Sc] or [HR], Appendix B. The following notation will be used: if $X$ is a Banach space, then $L(X)$ ($\mathcal{L}(X)$) denotes the set of all bounded (invertible bounded) linear operators on $X$.​
0.4. Category theory

0.4.1. For all undefined notions from category theory the reader is referred to [ML], the first six chapters. One of the most notable conventions is, that the set of all morphisms with domain $X$ and codomain $Y$ in a category $\text{LTG}$ is denoted $\text{LTG}(X,Y)$. No rule without exceptions, and the most important one here is the category $\text{TOP}$, where morphism sets are denoted $\text{C}(X,Y)$. Other deviations from the notation used in [ML] are the shape of certain arrows (we use only $\rightarrow$, also if [ML] writes $\rightarrow$ or $\twoheadrightarrow$) and brackets (e.g. we denote monads with $(H,\eta,\mu)$ where [ML] writes $<H,\eta,\mu>$, etc.). However, the author believes that this will cause no confusion. Yet another deviation from [ML] is that by a diagram we always understand a small diagram. Consequently, our term {co}complete means small {co}complete.

For the reader's convenience, we present here a listing of the categories which we shall use frequently:

- **SET**: Objects, all (small) sets; morphisms, all functions between them.
- **TOP**: Objects, all topological spaces; morphisms, all continuous functions between them.
- **HAUS**: Full subcategory of TOP with as its objects all Hausdorff spaces.
- **COMP**: Full subcategory of HAUS with as its objects all compact Hausdorff spaces.
- **KR**: Full subcategory of HAUS determined by all k-spaces.
- **GRP**: Objects, all groups; morphisms, all homomorphisms.
- **TOPGRP**: Objects, all topological groups; morphisms, all continuous morphisms of groups.
- **HAUSGRP**: The full subcategory of TOPGRP determined by all Hausdorff groups.
- **COMPGRP**: The full subcategory of HAUSGRP determined by all compact Hausdorff groups.
- **KRGRP**: cf. 5.1.7 for its definition.

0.4.2. For easy reference we present here some of the various equivalent formulations of adjointness (cf. also [ML], pp. 78-81).

Let $A$ and $X$ be categories. An adjunction from $X$ to $A$ is a triple $(F,G,\varphi)$, where $F: X \to A$ and $G: A \to X$ are functors, while $\varphi$ is a function which assigns to each pair of objects $X \in X$, $A \in A$ a bijection

\[ \varphi_{X,A}: A(FX,A) \to X(X,GA) \]
which is natural in $X$ and $A$. Given such an adjunction, the functor $F$ is said to have a right adjoint $G$, and $G$ is said to have a left adjoint $F$. The following characterizations of adjointness will be used:

(i) A functor $G: A \to X$ has a left adjoint $F$ iff for each object $X \in X$ there exist an object $F_0X \in A$ and a universal arrow $\eta_X: X \to GF_0X$ from $X$ to $G$. This means that $\eta_X$ is a morphism in $X$ such that for every object $A \in A$ and every morphism $f: X \to GA$ in $X$ there exists a unique morphism $f'$ in $A$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & GF_0X \\
\downarrow f & & \downarrow Gf' \\
GA & \xrightarrow{f'} & A
\end{array}
$$

(2)

(ii) A functor $F: X \to A$ has a right adjoint $G$ iff for each object $A \in A$ there exist an object $G_0A \in X$ and a universal arrow $\epsilon_A: FG_0A \to A$ from $F$ to $A$. This means that $\epsilon_A$ is a morphism in $A$ such that for every object $X \in X$ and every morphism $g: FX \to A$ in $A$ there exists a unique morphism $g'$ in $X$ making the following diagram commutative:

$$
\begin{array}{ccc}
G_0A & \xrightarrow{g'} & FG_0A \\
\downarrow \epsilon_A & & \downarrow Gg' \\
X & \xrightarrow{g} & FX
\end{array}
$$

(3)

In this way, a natural transformation $\epsilon: FG \Rightarrow IA$ is obtained.

In the above situations, $\phi$ and $\phi'$ can be determined either by

$$
\phi_{X,A}g = Gg \circ \eta_X: X \to GA
$$

if $g: FX \to A$, or by

(4)
(5) \[ \psi_{X,A}^C f = \varepsilon_A \circ Ff : FX \rightarrow A \]

if \( f : X \rightarrow GA \). In fact, in diagram (2) we have \( f' = \psi_{X,A}^C f \), and in diagram (3), \( g' = \psi_{X,A}^C g \).

The natural transformations \( \eta \) and \( \varepsilon \) are called the unit and the counit of the adjunction. The adjunction \((F,G,\Phi)\) will often be denoted \((F,G,\eta,\varepsilon)\).  

0.4.3. If \( A \) is a subcategory of \( X \) and the inclusion functor \( G : A \rightarrow X \) has a left adjoint \( F \), then \( A \) is called a reflective subcategory of \( X \). If \( \eta \) is the unit of adjunction, then for each \( X \in X \), \( \eta_X : X \rightarrow GFX \) is called the reflection of \( X \) in \( A \) (in concrete categories, the inclusion functor \( G \) is usually suppressed here). If \( E \) is a class of epimorphisms in \( X \) and each \( \eta_X \) is in \( E \), then \( A \) is called an \( E \)-reflective subcategory of \( X \).

We wish to formulate a theorem giving sufficient conditions for a subcategory \( A \) of a "nice" category \( X \) to be reflective. Let \( E(M) \) denote a class of epimorphisms (monomorphisms) in \( X \) that is closed under composition with isomorphisms. Then \( X \) is said to have the \( E-M \)-factorization property whenever each morphism \( f \) in \( X \) factorizes as \( f = me \) with \( m \in M \) and \( e \in E \). The category \( X \) is said to be \( co-E \)-small whenever for each object \( X \in X \) there exists a subset \( E_X \) of \( E \) with the property that for each \( e' \in E \) with domain \( X \) there are an element \( e' \in E_X \) and an isomorphism \( f \) in \( X \) such that \( e' = fe \). Finally, a subcategory \( A \) of \( X \) is said to be closed under the formation of \( M \)-subobjects (of products) in \( X \) provided for each \( m : X \rightarrow A \) in \( M \) the condition \( a \in A \) implies \( X \in A \) (each product in \( X \) of \( A \)-objects is again in \( A \)). Now the following theorem can be proved (cf. [He], 10.2.2(c); see also [HS], 37.1):

**THEOREM.** Let \( A \) be a full subcategory of \( X \) closed under isomorphisms in \( X \). Let \( E \) and \( M \) be as above and suppose that \( X \) is \( co-E \)-small and has the \( E-M \)-factorization property. Assume that \( X \) has all products and consider the following conditions:

(i) \( A \) is \( E \)-reflective in \( X \).
(ii) \( A \) is closed under the formation of products and \( M \)-subobjects in \( X \).

Then always (ii) \( \Rightarrow \) (i). If, in addition, the \( E-M \)-factorization is unique (up to isomorphism) and if both \( E \) and \( M \) are closed under compositions, then we have also (i) \( \Rightarrow \) (ii).

**PROOF.** (outline).

(i) \( \Rightarrow \) (ii): That \( A \) is closed under the formation of \( M \)-subobjects in \( X \) follows from [HS], 36.13. For products, cf. 0.4.4 below.
(ii) ⇒ (i): Since $X$ is co-$E$-small, there exists for any $X \in X$ a set $E_X$ with the above mentioned property. For $f \in E_X$, let $A_f$ be the codomain of $f$. Let $A_0 := \Pi(A_f : f \in E_X \& A_f \in A)$ (product in $X$). The induced morphism $g: X \to A_0$ admits an $E-M$-factorization, say $X \overset{e}{\longrightarrow} A_1 \overset{m}{\longrightarrow} A_0$. Then $e: X \to A_1$ is the desired reflection of $X$ in $A$. □

Of course, the above theorem is an extension of the FREYD adjoint functor theorem for the particular case of an inclusion functor. The condition of co-$E$-smallness replaces the "solution set condition" in FREYD's theorem. Cf. [ML], p.117.

0.4.4. For easy reference we collect here some preservation and reflection properties of functors. First, notice that monomorphisms and limits are related in the following way: a morphism $f: X \to Y$ in a category $X$ is monic iff

$$
\begin{array}{ccc}
X & \xrightarrow{1_X} & X \\
\downarrow & & \downarrow & \text{is a limiting core for} & \downarrow \\
X & \xrightarrow{1_X} & f \\
\end{array}
$$

Thus, if a functor preserves all limits, it preserves all monomorphisms. Similar for colimits and epimorphisms.

Now let $F: X \to A$ be a functor. Then the following statements can be proved:

(i) If $F$ is faithful, it reflects monomorphisms and epimorphisms.
   Cf. [Pa], Lemma 1 in Section 2.12.\(^1\)

(ii) If $F$ has a left(right) adjoint, it preserves all limits and monomorphisms (all colimits and all epimorphisms). Cf. [ML], Chap. V, §5, Th. 1.

(iii) Suppose $F$ is left adjoint to a full and faithful functor $G: A \to X$.
    If $D: J \to A$ is a diagram such that the diagram $GD: J \to X$ has a limit (colimit) in $X$, then $D$ has a limit (colimit) in $A$. Cf. [Pa], Prop. 4 in Section 2.14. A straightforward proof can easily be given, using the fact that in the given situation $\epsilon_X: FGX \to X$ is an isomorphism in $A$ ($\epsilon$ is the counit of adjunction).

(iv) If $D$ is a diagram in $X$ such that $FD$ has a limit in $A$, and $F$ creates the limit of $D$, then $F$ preserves the limit of $D$ (indeed, limits are

\(^1\) In [ML], Exercise 9 on p.21, "carries ... to" should be replaced by "reflects".
unique up to isomorphism). In particular, if $F$ creates all limits and $A$ is complete, then $X$ is complete and $F$ preserves all limits.

As an application of (ii) and (iii) we mention the following well-known statement (cf. also [He], p. 88):

If $A$ is a full and reflective subcategory of $X$, closed with respect to isomorphisms in $X$, then limits in $A$ can be calculated in $X$. That is, if a diagram $D$ in $A$ has a limit in $X$, then this limit is completely included in $A$, and it is the limit of $D$ in $A$. Conversely, a limit of $D$ in $A$ is also limit of $D$ in $X$. In particular, if $X$ is complete then so is $A$. Moreover, if $X$ is cocomplete then so is $A$ (use again (iii)). In the latter case, the colimit of a diagram in $A$ is obtained as the reflection in $A$ of the colimit of that diagram in $X$.

0.4.5. A monad in a category $C$ is a triple $(H, \eta, \mu)$, consisting of a functor $H: C \to C$ and two natural transformations,

$$\eta: I_C \to H; \quad \mu: H \circ H \to H,$$

such that the following diagrams commute:

\[
\begin{array}{ccc}
H & \xrightarrow{\eta_H} & H^2 \\
\downarrow H & & \downarrow \mu \\
H^2 & \xrightarrow{\mu} & H
\end{array}
\quad \quad \quad
\begin{array}{ccc}
H^3 & \xrightarrow{H \mu} & H^2 \\
\downarrow & & \downarrow \\
H^2 & \xrightarrow{\mu} & H
\end{array}
\]

If $(F, G, \eta, \epsilon)$ is an adjunction from $C$ to a category $D$, then the monad, defined by this adjunction is the monad $(GF, \eta_G, G\epsilon)$ in $C$ (cf. [ML], p. 134). The following construction shows that, conversely, every monad is defined by an adjunction. For proofs, we refer the reader to [ML], Chap. VI, §2.

Let $(H, \eta, \mu)$ be a monad in a category $C$. Then an $H$-algebra is a pair $(X, \eta)$, consisting of an object $X$ in $C$ and a morphism $\pi: HX \to X$ in $C$ such that the following diagrams commute:
If \((X,\pi)\) and \((Y,\sigma)\) are \(H\)-algebras, then a morphism \(f: X \to Y\) in \(C\) is called a morphism of \(H\)-algebras, from \((X,\pi)\) to \((Y,\sigma)\), whenever the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & HX \\
\downarrow & & \downarrow \\
\pi & & \pi
\end{array}
\]

\[
\begin{array}{ccc}
H^2X & \xrightarrow{\mu_X} & HX \\
\downarrow & & \downarrow \\
HX & \xrightarrow{\pi} & X
\end{array}
\]

Since composites (in \(C\)) of morphisms or \(H\)-algebras are again morphisms of \(H\)-algebras, it is obvious that in this way we obtain a category: the category \(CH\) of all \(H\)-algebras.

There is an obvious "forgetful" functor \(G^H: C^H \to C\), namely

\[
G^H: \left\{ \begin{array}{l}
\{(X,\pi)\} \mapsto X \\
f \mapsto f
\end{array} \right. \quad \text{on objects}
\]

\[
G^H: \left\{ \begin{array}{l}
(X,\pi) \mapsto (HX,\mu_X) \\
f \mapsto Hf
\end{array} \right. \quad \text{on morphisms.}
\]

In the other direction, there is a functor \(F^H: C \to C^H\), defined by

\[
F^H: \left\{ \begin{array}{l}
X \mapsto (HX,\mu_X) \\
f \mapsto Hf
\end{array} \right. \quad \text{on morphisms.}
\]

The \(H\)-algebra \(F^H X := (HX,\mu_X)\) is called the free \(H\)-algebra on \(X\). Notice that \(G^H \circ F^H = H\). The following theorem can be proved (cf. [ML], p.136):

**0.4.6. Theorem.** The functor \(F^H\) is left adjoint to \(G^H\), and the monad defined by this adjunction is just the original monad \((H,\eta,\mu)\). \[\Box\]

The unit for the adjunction of \(F^H\) and \(G^H\) is the natural transformation \(\eta: I_C \to G^H F^H = H\). Its counit \(\xi\) is given by the morphisms
\[ \varepsilon_{(X,\pi)} := \pi: (H_X, \mu_X) \Rightarrow (X, \pi) \]

in \( C^H \), for every \( H \)-algebra \( (X, \pi) \).

0.4.7. It follows from the preceding theorem that \( G^H: C^H \to C \) preserves all limits and monomorphisms, and that \( F^H: C \to C^H \) preserves all colimits and epimorphisms (cf. 0.4.4). For \( G^H \), more can be shown, namely

**THEOREM.** The functor \( G^H: C^H \to C \) creates all limits.

**PROOF.** Cf. [ML], Exercise 2, p.138. For a detailed proof, the reader is referred to E. MANES [1969 b].

If \( C \) is complete, then the preceding theorem implies that \( C^H \) is complete. In that case, the facts that \( G^H \) creates and preserves all limits and that \( G^H \) preserves and reflects all monomorphisms may be expressed by saying that "limits and monomorphisms in \( C^H \) can be calculated in \( C \)."

0.4.8.\(^1\) Let \( (H, \eta, \mu) \) and \( (H', \eta', \mu') \) be monads in \( C \). Then a natural transformation \( \theta: H \to H' \) is called a morphism of monads, from \( (H, \eta, \mu) \) to \( (H', \eta', \mu') \) whenever \( \theta \eta = \eta' \) and \( \theta \mu = \mu' \theta^2 \), that is, whenever for each object \( X \in C \) the following diagrams commute:

\[ \begin{array}{ccc}
X & \xrightarrow{\eta_X} & HX \\
\downarrow{\eta'_X} & & \downarrow{\theta_X} \\
H'X & \xrightarrow{\theta^2_X} & H'X \\
\end{array} \quad \begin{array}{ccc}
H^2X & \xrightarrow{\mu_X} & HX \\
\downarrow{\theta^2_X} & & \downarrow{\theta_X} \\
H^2X & \xrightarrow{\mu'_X} & H'X \\
\end{array} \]

Here \( \theta^2_X \) is defined as the dotted arrow in the commutative diagram

\[ \begin{array}{ccc}
H^2X & \xrightarrow{\theta_{H^2X}} & H'H^2X \\
\downarrow{H\theta_X} & & \downarrow{H'\theta_X} \\
HH'X & \xrightarrow{\theta_{H'H^2X}} & H'H'X \\
\end{array} \]

\(^1\) This is Exercise 3 on p.138 in [ML].
If \( \theta \) is such a morphism of monads from \((H, \eta, \mu)\) to \((H', \eta', \mu')\), then

commutativity of the diagrams

shows that for each \( H' \)-algebra \((X, \pi)\) the morphism \( \pi \theta_X^* \colon HX \to X \) in \( C \) is such that \((X, \pi \theta_X^*)\) is an \( H \)-algebra. Moreover, it is easily seen that for each morphism \( f \colon (X, \pi) \to (Y, \sigma) \) of \( H' \)-algebras, \( f \) may also be interpreted as a morphism of \( H \)-algebras, namely, \( f \colon (X, \pi \theta_X^*) \to (Y, \sigma \theta_Y^*) \). So we have a functor

\( \theta^* \colon C^{H'} \to C^H \), defined by the assignments

\[
\theta^* \colon \begin{cases} 
(X, \pi) & \mapsto (X, \pi \theta_X^*) \text{ on objects} \\
 f & \mapsto f \text{ on morphisms.}
\end{cases}
\]

0.4.9. We shall briefly deal with the dual of 0.4.5 through 0.4.7. A comonad \((H, \delta, \epsilon)\) in a category \( D \) consists of a functor \( H \colon D \to D \), together with natural transformations

\[
\delta \colon H \to I_D; \quad \epsilon \colon H \to H^2
\]

such that the following diagrams commute:

Suppose we are given a pair of functors \( F \colon C \to D \), \( G \colon D \to C \), where \( F \) is left adjoint to \( G \) with unit \( \xi \colon I_C \to GF \) and counit \( \delta \colon FG \to I_D \). Then \((FG, \delta, F\xi)\) is a comonad in \( D \).
Furthermore, any comonad \((H, \delta, \epsilon)\) in \(D\) arises in this way by considering the category \(D_H\) of \(H\)-coalgebras. An object of \(D_H\) is a pair \((X, \sigma)\) with \(X\) an object in \(D\) and \(\sigma: X \rightarrow HX\) a morphism in \(D\) such that the diagrams commute. Moreover, morphisms in \(D_H\) from \((X, \sigma)\) to \((Y, \xi)\) are morphisms \(f: X \rightarrow Y\) in \(D\) such that

\[
\begin{array}{ccc}
X & \xrightarrow{\sigma} & HX \\
\downarrow f & & \downarrow Hf \\
Y & \xrightarrow{\xi} & HY
\end{array}
\]

commutes. Next, one defines functors \(F_H: D_H \rightarrow D\) and \(G_H: D \rightarrow D_H\) by

\[
\begin{align*}
F_H: & \{(X, \sigma)\} & \mapsto X \text{ on objects} \\
f & \mapsto f \text{ on morphisms}
\end{align*}
\]
\[
\begin{align*}
G_H: & \{X \mapsto (HX, \epsilon_X)\} & \mapsto \text{on objects} \\
f & \mapsto Hf \text{ on morphisms}.
\end{align*}
\]

Then we have natural transformations \(\delta: F_H G_H = H \Rightarrow I_D\) and \(\beta: I_{D_H} \Rightarrow G_H F_H\), where

\[\beta(X, \sigma) := \sigma: (X, \sigma) \rightarrow (HX, \epsilon_X)\]

for each \(H\)-coalgebra \((X, \sigma)\). Then \(F_H\) is left adjoint to \(G_H\) with unit \(\beta\) and counit \(\delta\), and the comonad \((F_H G_H, \delta, F_H \delta G_H)\) defined by this adjunction, just coincides with the original comonad \((H, \delta, \epsilon)\).

The functor \(F_H\) preserves colimits and epimorphisms, and, in addition, it reflects epimorphisms and creates colimits (dual of 0.4.7).

0.4.10. **Categorical notions in topology.** All proofs of the following statements can be found in [He] or else they are trivial. First, we describe

\[\text{\textsuperscript{1}}\]

\((HX, \epsilon_X)\) is called the free coalgebra for \(X\).
some categorical notions in the category TOP. Basical is the observation that the forgetful functor \( \text{TOP} \to \text{SET} \) has a left and a right adjoint (providing a set with the discrete and the indiscrete topology, respectively). Hence this forgetful functor preserves all limits, colimits, monomorphisms and epimorphisms. In addition, \( \text{TOP} \) is complete and cocomplete. Now the following descriptions can be given:

Product of \( \{ X_j : j \in J \} \) in \( \text{TOP} \): cartesian product space \( \prod_j X_j \) with projections \( p_i : \prod_j X_j \to X_i \) (\( i \in J \)).

Coproduct of \( \{ X_j : j \in J \} \) in \( \text{TOP} \): disjoint union \( \biguplus_j X_j \) of the spaces \( X_j \) with the topological embeddings \( r_i : X_i \to \biguplus_j X_j \) (\( i \in J \)).

Monomorphisms in \( \text{TOP} \): injective continuous functions.

Epimorphisms in \( \text{TOP} \): surjective continuous functions.

Equalizer of \( f_1, f_2 : X \to Y \) in \( \text{TOP} \): inclusion mapping \( i : Z \to X \), with \( Z := \{ x \in X : f_1(x) = f_2(x) \} \).

Coequalizer of \( f_1, f_2 : X \to Y \) in \( \text{TOP} \): quotient mapping \( q : Y \to Y/R \) with \( R \) the smallest equivalence relation in \( Y \) containing all points \( (f_1(x), f_2(x)) \) \( (x \in X) \).

One particular colimit will be mentioned here: let \( A \) be a subset of a topological space \( X \) and let \( i : A \to X \) be the inclusion mapping. The colimit of the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f_1} & A \\
\downarrow & & \downarrow \\
& X &
\end{array}
\]

in \( \text{TOP} \) shall be denoted \( X \cup_A X \) with coprojections \( f_1, f_2 : X \cup_A X \). Then \( f_1 \) and \( f_2 \) are topological embeddings of \( X \) into \( X \cup_A X \). In fact, \( X \cup_A X \) can be realized as \( (X \times \{1,2\})/R \), where \( \{1,2\} \) is a discrete two-point space (hence \( X \times \{1,2\} \) is the disjoint union of two copies of \( X \)) and \( R \) is the equivalence relation \( \{ (((x,1), (x,1)) : x \in X) \cup \{ ((x,2), (x,2)) : x \in X \} \cup \{ ((x,1), (x,2)) : x \in A \} \) on \( X \times \{1,2\} \).

The most important facts about \( \text{HAUS} \) can be derived from the preceding ones, because the inclusion functor \( \text{HAUS} \to \text{TOP} \) creates, hence preserves, all limits and all coproducts (not coequalizers!). In particular, it preserves and reflects monomorphisms. Notice that the epimorphisms in \( \text{HAUS} \) are just all continuous mappings with dense ranges.

The inclusion functor \( \text{COMP} \to \text{HAUS} \) creates all limits, all finite coproducts and all coequalizers. The epimorphisms in \( \text{COMP} \) are the surjections.

Finally, the inclusion functors \( \text{COMP} \to \text{HAUS} \) and \( \text{HAUS} \to \text{TOP} \) have left adjoints, i.e. \( \text{COMP} \) is (epi-)reflective in \( \text{HAUS} \) and \( \text{HAUS} \) is reflective in
The reflection of an object \( X \in \text{TOP} \) in \( \text{HAUS} \) is always a quotient mapping. The reflection of \( X \in \text{TOP} \) in \( \text{COMP} \) will be denoted \( \beta_X : X \to BX \); if \( X \) is a Tychonov space, it is a dense embedding, and then \( BX \) is called the Stone-Čech compactification of \( X \); in all cases, the mapping \( \beta_X \) has a dense range in \( BX \).

A description of the important category \( \text{KR} \) is postponed until subsection 5.1.

0.4.11. Categorical notions in topological groups. The basic observation is, that the forgetful functor \( \text{TOPGRP} \to \text{GRP} \) has both a left and a right adjoint (assigning to each group the discrete and the indiscrete topology, respectively). So by 0.4.4, the forgetful functor \( \text{TOPGRP} \to \text{GRP} \) preserves all limits, colimits, monomorphisms and epimorphisms. In addition, it can be shown that \( \text{TOPGRP} \) is complete and cocomplete. Limits and colimits in \( \text{TOPGRP} \) can be formed by first computing the corresponding limits and colimits in \( \text{GRP} \) and then the resulting objects have to be provided with a suitable topology, in order to obtain the desired limits and colimits in \( \text{TOPGRP} \). For limits, this suitable topology is easy to find. In fact, the forgetful functor \( \text{TOPGRP} \to \text{TOP} \) plainly creates all limits. For colimits, the situation is more complicated. Cf. for instance S.A. Morris [1971] or E.T. Ordman [1974] and the references given there. Incidentally, it follows from the above remarks that monomorphisms in \( \text{TOPGRP} \) are the injective morphisms and that epimorphisms in \( \text{TOPGRP} \) are the surjective morphisms (indeed, in \( \text{GRP} \) this is well-known).

The full subcategory \( \text{HAUSGRP} \) is reflective in \( \text{TOPGRP} \), for each object \( G \in \text{TOPGRP} \) the reflection being the morphism \( J = G/\text{cl}_0(e) \). So the last paragraphs in 0.4.4 provide the device to compute all limits and colimits in \( \text{HAUSGRP} \). In fact, \( \text{HAUSGRP} \) is closed under the formation of all limits and coproducts (= free products) in \( \text{TOPGRP} \) (not under the formation of coequalizers); for limits this is obvious, for coproducts, cf. for example E.T. Ordman [1974] and the references given there.

\[^{11}\] It follows that \( \text{HAUS} \) and \( \text{COMP} \) are complete and cocomplete (completeness follows also from the earlier remarks, but cocompleteness does not!).

\(^{22}\) Consequently, we cannot characterize epimorphisms in \( \text{HAUSGRP} \). It is an outstanding conjecture that it are the morphisms in \( \text{HAUSGRP} \) with dense ranges.
The category COMPGRP is reflective in HAUSGRP, hence in TOPGRP. The reflection \( \alpha_G: G \to G^c \) of an object \( G \) of TOPGRP in COMPGRP is called the 
Bohr compactification of \( G \). In contradistinction to the reflector of TOP to 
COMP, the reflector of TOPGRP to COMPGRP preserves all products (cf. 
P. HOLM [1964]).

The full subcategory of TOPGRP of all locally compact \( T_2 \)-groups seems 
to be not yet systematically investigated, although a lot of information 
about it is known in the literature. For instance, coproducts in TOPGRP of 
locally compact Hausdorff groups may be not locally compact. The full sub-
category of TOPGRP of all abelian locally compact \( T_2 \)-groups behaves better, 
and a lot is known about it. Although, we shall not use this category, we 
refer the interested reader to D.W. ROEDER [1974] and the references given 
there for a categorical approach of the duality theorem.
1 - GENERALITIES ON TTGS

In this introductory section the basic concepts of this treatise are defined, and some simple properties are derived. First of all, the definition of a topological transformation group (a ttg) is given, and the relation with suitably topologized homeomorphism groups is investigated. Then, in subsection 1.3, more shape is given to the concept of a ttg by introducing orbits, the orbit space and the enveloping semigroup of a ttg. After this superficial glance at the internal structure of a ttg, the possibility of studying relations between ttgs is opened by defining morphisms of ttgs. This happens in subsection 1.4, where also some examples are provided, using the previously defined concepts. In order to facilitate subsequent constructions, this section will be closed by presenting some elementary constructions of new ttgs from given ones. In order to exclude trivialities, all phase spaces of ttgs in this section are supposed to be non-void.

1.1. Definitions and terminology

1.1.1. Let $G$ be a topological group. For any topological space $X \neq \emptyset$, let $\eta_X^G: X \to G \times X$ and $\mu_X^G: G \times (G \times X) \to G \times X$ be defined by

\[
\eta_X^G(x) := (e, x) \quad \mu_X^G(s, (t, x)) := (st, x)
\]

(\(x \in X\) and \(s, t \in G\)). If $G$ is understood we shall often write $\eta_X$ and $\mu_X$ instead of $\eta_X^G$ and $\mu_X^G$.

An *action* of a topological group $G$ on a non-void topological space $X$ is a continuous function $\pi: G \times X \to X$ such that the following diagrams commute:
A topological transformation group (abbr.: a ttg, plural: ttgs) is a triple \(<G,X,\pi>\) with \(G\) a topological group, \(X\) a topological space, \(X \neq \emptyset\), and \(\pi\) an action of \(G\) on \(X\). Here \(G\{X\}\) is called the phase group (the phase space) of the ttg \(<G,X,\pi>\).

In accordance with our notational conventions, let

\[ \pi^t(x) := \pi(t,x) =: \pi_x(t) \]

\((t \in G, x \in X)\). Then the continuous mappings \(\pi^t: X \to X\) \((\pi: G \times X)\) are called the transitions (the motions) of the action \(\pi\).

1.1.2. Let \(G\) be a topological group, \(X\) a topological space, and \(\pi: G \times X \to X\) a continuous function. Then \(\pi\) is an action iff the following conditions are satisfied:

(i) \(\pi^e = 1_X\).
(ii) \(\pi^{st} = \pi^s \pi^t\) for all \(s, t \in G\).

It is convenient to notice that for any ttg \(<G,X,\pi>\) the following diagrams commute for every \((t,x) \in G \times X\) (recall that \(\lambda: G \times G \to G\) denotes the multiplication in \(G\); in addition, \(tx\) stands for \(\pi(t,x))\):

(4)

1.1.3. PROPOSITION. If \(<G,X,\pi>\) is a ttg then each \(\pi^t\) is a homeomorphism of \(X\) onto itself \((t \in G)\). In addition, the mapping \(\pi: t \mapsto \pi^t\) defines a morphism of groups from \(G\) into the full homeomorphism group \(H(X,X)\) of \(X\).
The transition mapping of a ttg \( <G, X, \pi> \) is the mapping \( \bar{\pi}: G \rightarrow X^X \), defined by the rule

\[
\bar{\pi}(t) := \pi^t
\]

for \( t \in G \).\(^1\) Obviously, \( \bar{\pi}(G) \) is a subgroup of the full homeomorphism group \( H(X, X) \) of \( X \). It is called the transition group of \( <G, X, \pi> \).

It follows immediately from 1.1.3 that for every ttg \( <G, X, \pi> \), \( \pi \) may also be considered as an action of the group \( G_d \) on the space \( X \), where \( G_d \) denotes the group \( G \) with its discrete topology. Consequently, if \( <G, X, \pi> \) is a ttg, then we can speak about the ttg \( <G_d, X, \pi> \) as well.

We present now some elementary examples of ttgs. To this end, we fix a ttg \( G \). Several ttgs \( <G, X, \pi> \) will be described by indicating \( X \) and \( \pi \). Most proofs are left to the reader.

(i) The ttg \( <G, G, \lambda> \).

Here \( \lambda(t, s) := ts \). The transitions are the left translations in \( G \), and the motions are the right translations in \( G \).

(ii) The ttg \( <G, G \times X, \mu_X> \), where \( X \) is any topological space.

Here \( \mu_X := \mu^G_X : (s, (t, x)) \mapsto (st, x) : G \times (G \times X) \rightarrow G \times X \). If we take for \( X \) a one-point space, then we may identify \( G \times X \) with \( G \), and we obtain the ttg of example (i).

(iii) Suppose \( G \) is a subgroup of some topological group \( H \). If \( \pi: G \times H \rightarrow H \) is defined by \( \pi(t, u) := tu \), then \( <G, H, \pi> \) is a ttg. The transitions are the left translations in \( H \) over elements of \( G \). If \( G = H \) then we obtain the ttg of example (i) above. If the quotient mapping \( q: H \rightarrow H/G \) admits a continuous cross-section, then the ttg \( <G, H, \pi> \) may be identified with the ttg \( <G, G \times (H/G), \mu_{H/G}> \), defined according to example (ii). Here \( H/G \) denotes the space of all right cosets of \( G \) in \( H \) with its usual quotient topology. (Recall that a continuous cross-section of \( q \) is a continuous mapping \( f: H/G \rightarrow H \) such that \( qf = \operatorname{id}_{H/G} \). If it exists, then the mapping \( u \mapsto (u \cdot (fqu)^{-1}, qu) : H \rightarrow G \times (H/G) \) is a homeomorphism.

\(^1\) Warning: we shall also use the symbol \( \bar{\pi} \) for the corestriction of \( \pi \) to certain subsets of \( X^X \) containing \( \bar{\pi}(G) \); if this occurs, such a corestriction shall also be called a transition mapping.
If we identify $H$ with $G \times (H/G)$ via this homeomorphism, then the action $\pi$ of $G$ on $H$ carries over to the action $\mu_{H/G}$ of $G$ on $G \times (H/G)$.}

(iv) Let $H$ be a subgroup of $G$ and let the space $G\backslash H$ of left cosets of $H$ in $G$ be provided with its usual quotient topology. Then $\pi: G \times (G\backslash H) \to G\backslash H$ may unambiguously be defined by

$$\pi(t,q(s)) := q(ts)$$

for $t,s \in G$. Here $q: G \to G\backslash H$ is the quotient mapping. Then $<G,G\backslash H,\pi> \to G\backslash H$ is a ttg (since $q$ is open, continuity of $\pi$ follows from $0.2.4(i)$).

(v) Let $S$ denote a semigroup with multiplication $(\xi,\eta) \mapsto \xi \eta: S \times S \to S$, not necessarily continuous. Let $\varphi: G \to S$ be a morphism of semigroups, and suppose that the topology on $S$ is such that the mapping

$$\hat{\varphi}: (t,\xi) \mapsto \varphi(t)\xi: G \times S \to S$$

is continuous. If, in addition, $\varphi(e)$ is a left unit in $S$ (i.e. $\varphi(e)\xi = \xi$ for every $\xi \in S$), then $\hat{\varphi}$ is a continuous action of $G$ on $S$, so $<G,S,\hat{\varphi}> \to G\backslash S$ is a ttg. In particular, if $S$ is a topological group and if $\varphi$ is a continuous morphism of groups, then we can define the ttg $<G,S,\hat{\varphi}> \to G\backslash S$ in this way. In the general case, it is useful to notice that $\varphi$ can be recovered from $\hat{\varphi}$ by means of the equality $\varphi = \hat{\varphi}(e)$.

(vi) Let $G$ be a locally compact Hausdorff group, and let $X := G \cup \{\infty\}$ denote its one-point compactification. Then $\pi: G \times X \to X$ may be defined by

$$\pi(t,x) := \begin{cases} \lambda(t,x) = tx & \text{if } x \in G \\ \infty & \text{if } x = \infty \end{cases}$$

for all $t \in G$. Plainly, $\pi$ is continuous, and $<G,X,\pi> \to G\backslash X$ is a ttg.

(vii) Let $X \neq \emptyset$ be any topological space. Then $t_X: (t,x) \mapsto x : G \times X \to X$ defines an action of $G$ on $X$. The transition group of this action consists just of the one-element group $\{1\}$. This action $t_X$ will be referred to as the trivial action of $G$ on $X$.

(viii) Let $X$ be a topological space and let $f$ be an automorphism of $X$. For $n \in \mathbb{N}$, let $f^n := f \circ \ldots \circ f$ (n times), let $f^0 := 1_X$ and $f^k := (f^n)^k$ if $k \in \mathbb{Z}$, $k < 0$. Finally, set $\pi(n,x) := f^n(x)$ for every $n \in \mathbb{Z}$ and $x \in X$. Then $\pi: \mathbb{Z} \times X \to X$ is continuous, and $\pi$ is an action of $\mathbb{Z}$ on $X$. The ttg $<\mathbb{Z},X,\pi>$ is called the discrete ttg, generated by the homeomorphism $f$. Notice, that $\pi^1 = f$, and that, for all $n \in \mathbb{Z}$, $\pi^n = f^n = (\pi^1)^n$. 


Observe that each action \( \sigma \) of \( \mathbb{Z} \) on any space \( X \) is generated in this way by \( \sigma^1 \).

1.1.7. We shall now agree upon some informal terminology and notation. If \( \langle G,X,\pi \rangle \) is a ttg, then \( \pi(t,x) \) is denoted more concisely by \( t \cdot x \) or \( tx \) when there is no risk for ambiguity. The statement "\( \langle G,X,\pi \rangle \) is a ttg" may be rephrased as "\( G \) acts on \( X \) (by \( \pi \))" or "\( X \) is a \( G \)-space (with action \( \pi \))." We shall also use the description "the \( G \)-space \( \langle G,X,\pi \rangle \).

If in a ttg \( \langle G,X,\pi \rangle \) the action \( \pi \) has a certain property, then we may express this also by saying that \( \langle G,X,\pi \rangle \) has that property, and vice versa. Alternatively, we say that \( G \) has the property on \( X \), or that \( X \) has the property under \( G \). In the same spirit we shall use in the sequel sometimes a more or less informal terminology which is not always defined explicitly but which is hoped to be clear from the context (and, of course, from previously given definitions).

1.1.8. In general, the transition mapping \( \pi \) for a ttg \( \langle G,X,\pi \rangle \) is not injective (consider the trivial action of a non-trivial group \( G \) on any non-void space \( X \)). The action \( \pi \) in a ttg \( \langle G,X,\pi \rangle \) is called effective (by 1.1.7, we may call \( \langle G,X,\pi \rangle \) then an effective ttg) if \( \pi : G \to H(X,X) \) is injective. Equivalently, \( \langle G,X,\pi \rangle \) is effective iff

\[
\forall t \in G \left( t \neq e \implies \exists x \in X : \pi(t,x) \neq x \right).
\]

Stated more loosely, in any effective ttg the phase group may be identified with a topologized group of homeomorphisms of the phase space, namely with the transition group. We shall consider the topologies on such homeomorphism groups in more detail in section 1.2.

1.1.9. An action \( \pi : G \times X \to X \) of the topological group \( G \) on the topological space \( X \) is strongly effective provided for every \( x \in X \) the motion \( \pi_X : G \times X \to X \) is injective. Equivalently, \( \pi \) is strongly effective iff

\[
\forall t \in G \left( t \neq e \implies \forall x \in X : \pi(t,x) \neq x \right)
\]

(compare this with (6) above). So a ttg \( \langle G,X,\pi \rangle \) has a strongly effective action provided no transition \( \pi^t \) with \( t \neq e \) has a fixed point \( (t \in G) \).

1.1.10. If \( G \) is a topological group then for any topological space \( X \) the
In [GH], the term "period" is used.
For certain purposes it is, however, undesirable that $\tilde{\pi}: G \to \tilde{\pi}(G)$ may be not continuous if $\tilde{\pi}(G)$ has its discrete topology. However, if we provide $\tilde{\pi}(G)$ with the finest topology making $\tilde{\pi}: G \to \tilde{\pi}(G)$ continuous, then plainly $\tilde{\pi}(G)$ is homeomorphic with $G/\ker \tilde{\pi}$, and the homeomorphism which achieves this is, in addition, an isomorphism of groups. So by [HR], 5.2.6, $\bar{\pi}(G)$ is a topological group, and $\bar{\pi}: G \to \bar{\pi}(G)$ is an open mapping. It follows that $\tilde{\pi} \times_X: G \times_X \to \tilde{\pi}(G) \times_X$ is a quotient mapping. Since $\delta = (\tilde{\pi} \times_X)$ is continuous (it just equals the continuous mapping $\pi: G \times_X \to X$), we obtain that $\delta: \bar{\pi}(G) \times_X \to X$ is continuous. Thus, $\bar{\pi}(G)$ acts on $X$ by means of $\delta$. Since $\delta$ is plainly an effective action, this proves

1.1.15. PROPOSITION. Let $G, X, \pi$ be a ttg and let $\bar{\pi}(G)$ be given the finest topology making $\tilde{\pi}: G \to \bar{\pi}(G)$ continuous. Then $\delta: (\xi, x) \mapsto \xi(x): \bar{\pi}(G) \times_X \to X$ is continuous, and $<\bar{\pi}(G), X, \delta>$ is an effective ttg. □

1.1.16. If $X$ is a $T_0$-space then the topology on $\bar{\pi}(G)$ indicated in 1.1.15 is a Hausdorff topology. Indeed, as mentioned before, $\bar{\pi}(G)$ is topologically isomorphic to $G/\ker \tilde{\pi}$, and if $X$ is a $T_0$-space, then so is $G/\ker \tilde{\pi}$, by 1.1.12 and [HR], 5.2.6. Finally, recall that for topological groups, the $T_0$ and the $T_2$ separation axioms are equivalent.

1.1.17. Intuitively, in effective ttgs the connection between phase group and phase space is stronger than in non-effective ones. This is illustrated by the fact that most theorems relating properties of the phase group to properties of the phase space and of the action apply only to effective ttgs. As an example, we present a relation connecting the local weight of the phase group to the local weight of the phase space and the "measure of effectiveness". First, we have to introduce some terminology in order to be able to give a precise meaning to "measure of effectiveness".

1.1.18. A stabilizing set in an effective ttg $G, X, \pi$ is a subset $A$ of $X$ such that $\pi^t|_A = \pi^e|_A$ implies $t = e (t \in G)$. Equivalently, $A \subset X$ is a stabilizing set if $\cap \{ \pi(x) : x \in A \} = \{ e \}$. If $G, X, \pi$ is an effective ttg, then the cardinal number

$$\varepsilon_{G, X, \pi} := \min\{|A| : A \subset X \text{ and } A \text{ is stabilizing}\}$$

is in a certain sense a measure for the effectiveness of $G, X, \pi$.\]
1.1.19. If $<G,X,\pi>$ is an effective ttg, then $X$ is a stabilizing set, hence $e^{<G,X,\pi>} \leq |X|$. If, in addition, $X$ is a $T_0$-space, then every dense subset of $X$ is stabilizing, so $e^{<G,X,\pi>} \leq d(X)$, the density of $X$. If $<G,X,\pi>$ is strongly effective, then each $\{x\}$ is a stabilizing set ($x \in X$), hence $e^{<G,X,\pi>} = 1$.

1.1.20. **Lemma.** Let $Y$ be any topological space and suppose $y \in Y$ has a compact Hausdorff neighbourhood. Let $B \subseteq V_y$ and $\cap B = \{y\}$. Then $\ell w(Y,y) \leq |B|$.

**Proof.** If $|B|$ is finite, then $y$ is isolated, and $\ell w(Y,y) = 1 \leq |B|$. Suppose $|B| \geq N_0$. Without restriction of generality we may suppose that each $B \subseteq B$ is compact and closed in $Y$. Let $B^*$ denote the family of all intersections of finitely many members of $B$. Then $|B^*| = |B|$ and $B^*$ is easily seen to be a local base at $y$. Hence $\ell w(Y,y) \leq |B^*| = |B|$. \(\square\)

1.1.21. **Proposition.** Let $<G,X,\pi>$ be an effective ttg. If $G$ is a locally compact Hausdorff group and if $X$ is a $T_1$-space, then

\[ \ell w(G) \leq e^{<G,X,\pi>} \cdot \ell w(X). \]

**Proof.** Let $A$ be a stabilizing set such that $|A| = e^{<G,X,\pi>}$. For each $a \in A$, let $B_a$ denote a local base at $a$ such that $|B_a| = \ell w(X,a)$. Observe that $\pi^*_a[V] = V_a$ for every $a \in A$ and $V \in B_a$. Then

$$\bigcap_{a \in A} \bigcap_{V \in B_a} \pi^*_a[V] = \{e\},$$

by the $T_1$-separation property of $X$ and the fact that $A$ is stabilizing. Now apply 1.1.20. \(\square\)

1.1.22. Actually, we proved a little bit more than has been expressed by the inequality (9), namely, that we have $\ell w(G) \leq |A| \cdot \sup_{a \in A} \ell w(X,a)$ for any stabilizing set $A$ in $X$. In particular, if $|A| = 1$, then $\ell w(G) \leq \ell w(X,a)$, where $a$ is the unique point in $A$.

1.1.23. **Corollary.** Let $<G,X,\pi>$ be a ttg with $G$ a locally compact Hausdorff group and $X$ a Hausdorff space. If the action $\pi$ is effective, then $\ell w(G) \leq d(X) \cdot \ell w(X)$. If $\pi$ is strongly effective, then $\ell w(G) \leq \min\{\ell w(x,x) : x \in X\}$. In particular, if $G$ acts effectively (strongly effectively) on a separable first countable (a first countable) Hausdorff space, then $G$ is metrizable.
the multiplication being defined by composition of mappings.

1.2.1. NOTES. We shall not enter into the history and the development of the concept of a ttg. Nor shall we try to convince the reader of the importance of ttgs. For a flavour of it, the reader may read the prefaces to [MZ], [GH] and [El]. See also W.H. GOTTSCALK [1958, 1964, 1968].

Usually, the definition of a ttg is given in the form of 1.1.2. The more "abstract" definition that we have presented in 1.1.1 has been motivated by the needs of §3.

In example 1.1.6(iii), the existence of a continuous cross-section $f: H/G+H$ is equivalent to the existence of a closed subset $S$ in $H$ meeting each right coset of $G$ in $H$ in exactly one point, provided $q$ is a closed mapping. In general, the best one can do is to prove the existence of Borel sets with this property: cf. G. Mackey [1952], or J. FELDMAN & F.P. GREENLEAF [1968]. A sufficient condition for the existence of a continuous cross-section $f: H/G+H$ can be found in E. Michael [1959]: $H$ is metrizable and $G$ is a complete subgroup which is isomorphic to the additive topological group of a Banach space. Another result can be found in P.S. MOSTERT [1956]: if $H$ is any locally compact Hausdorff group and $G$ is a closed subgroup such that $H/G$ is 0-dimensional, then $q$ has a continuous cross-section. For related results, namely the local existence of continuous cross-sections, cf. P.S. MOSTERT [1953; 1956], [MZ], p.221 and [Ch.], p.109.

The question which additional conditions imply that a strongly effective ttg is free (of which the condition in 1.1.6(iii) is an instance) is related to the problem of parallelizability of flows (a flow is nothing but an action of the additive group $\mathbb{R}$). We shall return to this question in the notes to section 1.3.

The statements on the metrizability of $G$ in 1.1.23 are well-known; see for instance [MZ], 2.11. The slight generalization of these statements formulated in 1.1.21 seems to be new, but as such it seems to be of limited interest. As an application we shall show in 2.3.15 that $\omega(G) = \omega(L^0(G))$ for every locally compact Hausdorff group $G$.

1.2. Topological homeomorphism groups

1.2.1. For any topological space $X$, the set $X^X$ of all (not necessarily continuous) mappings of $X$ into itself has a natural semigroup structure, the multiplication being defined by composition of mappings. Obviously,
C(X,X) is a subsemigroup of $X^X$, and the set $H(X,X)$ of all homeomorphisms of X onto itself is a subgroup of $C(X,X)$. This group is called the full homeomorphism group of X. The identity element of $H(X,X)$ is $1_X$, and the inverse of any $\xi$ in the group $H(X,X)$ is $\xi^{-1}$. Thus, $\xi^{-1} = \xi^\ast$.

A homeomorphism group of X (or: on X) is a subgroup of $H(X,X)$. A topological homeomorphism group on X is a subgroup $T$ of $H(X,X)$ with a topology such that $T$ is a topological group and the mapping $\delta:(h,x) \mapsto h(x): T \times X \to X$ is continuous.

1.2.2. If $T$ is a topological homeomorphism group on X and if $\delta: T \times X \to X$ is defined by $\delta(h,x) := h(x)$, then $\delta$ is an effective action of $T$ on X.

Conversely, if $\langle G, X, \pi \rangle$ is an effective topg, then $T := \pi[G]$, endowed with the unique topology making $\pi: G \to T$ a homeomorphism, is a topological homeomorphism group on X. If we identify $G$ with $T$ by means of $\pi$, then $\pi$ corresponds to the mapping $(h,x) \mapsto h(x): T \times X \to X$.

It follows from these remarks, that studying topological homeomorphism groups amounts to the same thing as studying effective topgs. We shall collect now some facts about topologies on homeomorphism groups.

1.2.3. The following statements are well-known. The reader may find proofs e.g. in [Bo], Chapter X. As to the notation, see subsection 0.2.

(i) Let X be a locally compact topological Hausdorff space and let $T \subseteq C(X,X)$. Then the mappings

$$(\xi,\eta) \mapsto \xi \eta: T_c \times T_c \to C_c(X,X); \quad (\xi,x) \mapsto \xi(x): T_c \times X \to X$$

are continuous (cf. also 0.2.7).

(ii) Let X be a uniform space, and let $T \subseteq C(X,X)$. Then the mapping

$$(\xi,x) \mapsto \xi(x): T_u \times X \to X$$

is continuous.

(iii) Let X be a uniform space and let $T$ be an equicontinuous subset of $C(X,X)$. Then the mappings

$$(\xi,\eta) \mapsto \xi \eta: T_p \times C_p(X,X) \to C_p(X,X); \quad (\xi,x) \mapsto \xi(x): T_p \times X \to X$$

are continuous.

Before going into details on the continuity of the mapping $\xi \mapsto \xi^{-1}$ on homeomorphism groups, we wish to stress the fact that the compact-open
topology is in many cases the best candidate for a topology on a homeomorphism group to make it a topological homeomorphism group. In addition, it is important to observe the following: if for a certain topology on a subset $T$ of $C(X,X)$ the evaluation $\delta: (\xi,x) \mapsto \xi(x): T \times X \to X$ is continuous, then this topology is finer than the compact-open topology (this is an immediate consequence of the first statement in 0.2.7(iii)).

1.2.4. Let $X$ be a topological space and let $T$ be a subgroup of $H(X,X)$. The bilateral compact-open topology on $T$ is the weakest topology making the mappings $\xi \mapsto \xi$ and $\xi \mapsto \xi^{-1}: T \to T_c$ continuous. If $T$ is endowed with this topology we shall indicate this by writing $T_{bc}$ instead of $T$.

1.2.5. **Lemma.** Let $T$ be a homeomorphism group on the topological space $X$. Then the bilateral compact-open topology is the weakest topology for which the mapping $\xi \mapsto \xi^{-1}: T \to T$ is continuous and which is finer than the compact-open topology.

**Proof.** A straightforward consequence of the definition in 1.2.4 and the fundamental property of a weak topology (cf. 0.2.3). \[ \Box \]

1.2.6. **Corollary 1.** Suppose $T$ is a topological homeomorphism group on the topological space $X$. Then the topology of $T$ is finer than the bilateral compact-open topology on $T$.

**Proof.** The mapping $(\xi,x) \mapsto \xi(x): T \times X \to X$ is continuous, so the topology on $T$ is finer than the compact-open topology on $T$. Now apply 1.2.5. \[ \Box \]

1.2.7. **Corollary 2.** Let $X$ be a locally compact Hausdorff space. Then for any subgroup $T$ of $H(X,X)$, $T_{bc}$ is a topological homeomorphism group on $X$. In particular, $H_{bc}(X,X)$ is a topological homeomorphism group.

**Proof.** By 1.2.5, the mapping $\xi \mapsto \xi^{-1}: T_{bc} \to T_{bc}$ is continuous. Moreover, the mapping $(\xi,\eta) \mapsto \xi \eta: T_{bc} \times T_{bc} \to T_{bc}$ is continuous because its compositions with $\xi \mapsto \xi$ and $\xi \mapsto \xi^{-1}: T_{bc} \to T_c$ are (use 1.2.3(i)). \[ \Box \]

1.2.8. Let $T$ be a subgroup of $H(X,X)$. A subbase for the bilateral compact-open topology on $T$ is formed by all sets $N(K,U) \cap T$, together with all sets of the type $\{\xi \in T : \xi^* \in N(K,U) \cap T\}$ with $K$ compact and $U$ open in $X$. Since $\xi^* \in T$ iff $\xi \in T$, sets of the latter type are equal to sets of the type $\{\xi \in T : \xi^* \in N(K,U)\}$.

A similar description of a subbase for $H_{bc}(X,X)$ can be given. In parti-
ular, it follows that $T_{bc}$ has just the relative topology of $H_{bc}(X,X)$.

1.2.9. COROLLARY 3. For any $G, X, \pi$ the transition mapping $\pi: G \rightarrow H_{bc}(X,X)$ is continuous. Consequently, $\pi: G \rightarrow H_c(X,X)$ is continuous as well.

PROOF. By the conclusion of 1.2.8, it is sufficient to show that $\pi: G \rightarrow H_{bc}(X,X)$ is continuous. If we give $\pi[G]$ the finest topology making $\pi$ continuous, then $\pi[G]$ is a topological homeomorphism group (cf. 1.1.15). By 1.2.6, this topology is finer than the topology of $\pi[G]_{bc}$. □

1.2.10. Let $X$ be a uniform space with uniformity $U$. In addition, let $T$ be a homeomorphism group on $X$. We shall consider two situations in which $T$ is a topological homeomorphism group in the compact-open topology.

(i) Suppose $T$ is equicontinuous. Then $T_{bc} = T_c = T_p$, and this is a topological homeomorphism group.

(ii) If $X$ is a compact Hausdorff space, then $T_u$ is a topological homeomorphism group. In particular, $H_u(X,X)$ is a topological homeomorphism group.

1.2.11. PROPOSITION. Let $X$ be a locally compact Hausdorff space, $T$ a homeomorphism group on $X$ and $S$ the closure of $T$ in $C_c(X,X)$. If $S_c$ is compact, then $S \subseteq H(X,X)$, $S_p = S_c = S_{bc}$, and this is a compact topological homeomorphism group.

PROOF. Since $S_c$ is a compact space and $S_p$ is a Hausdorff space, it follows that $S_c = S_p$. The proof of Theorem 4 in [Bo], Chap. X, §3.5, implies that $S \subseteq H(X,X)$ and that $\xi \mapsto \xi^*: S \rightarrow H_c(X,X)$ is continuous. Using 1.2.3(i), it follows that $S_c$ is a topological homeomorphism group. Therefore, $S_c = S_{bc}$ by the result of 1.2.6. □

1.2.12. COROLLARY. Let $X$ be a compact Hausdorff space, $T$ an equicontinuous
homeomorphism group on $X$ and $S'$ the closure of $T$ in $X^X$. Then $S' \subseteq H(X,X)$, so that $S'$ equals the closure of $T$ in $H_u(X,X)$. Moreover, $S'_P = S'_u$ and this is a compact topological homeomorphism group.

**PROOF.** Apply 0.2.8 and 1.2.11. □

1.2.13. **NOTES.** The results in this section are well-known. The definition of the bilateral compact-open topology occurs in [GH], 11.44 in a slightly different form, but it follows easily from 1.2.8 that our definition is equivalent to the one in [GH]. Our definition was motivated by [Bo], Chap. X, §3.5, Prop. 12. If $X$ is a locally compact Hausdorff space, then this topology is just the $g$-topology, introduced in R. ARRENS [1946a] (i.e., the relative topology of the given homeomorphism group in $C_u(X^*_\omega,X^*_\omega)$, where $X^*_\omega$ is the one-point compactification of $X$).

There exists a notable generalization of proposition 1.2.11, namely, that for a homeomorphism group $T$ on a locally compact $T_2$-space the following conditions are equivalent:

(i) The closure of $T$ in $C_c(X,X)$ is compact.

(ii) The closure of $T$ in $C_p(X,X)$ is compact and this closure is a subgroup of $H(X,X)$.

Of course, here (i) $\Rightarrow$ (ii) is an immediate consequence of 1.2.11. Crucial in the proof of (ii) $\Rightarrow$ (i) is that $\delta'(\xi,x) \mapsto \xi(x): S \times X \to X$ turns out to be continuous, where $S$ denotes the closure of $T$ in $C_p(X,X)$. This is an immediate consequence of the following famous theorem (cf. R. ELLIS [1957]):

Let $X$ be a locally compact $T_2$-space and let $T$ be a homeomorphism group on $X$. Suppose $T$ is given a locally compact topology which is finer than the point-open topology, such that multiplication is separately continuous. Then $T$ is a topological homeomorphism group.

It is an easy consequence of this theorem that a group with a locally compact $T_2$-topology such that multiplication is separately continuous is a topological group. An alternative proof of this statement for the compact case has been given in K. DE LEEUW & I. GLICKSBERG [1961] (cf. also [Bu], Theorem 1.28).

The following result of J. KEESLING [1971] is related to 1.2.11. In fact, it is an easy consequence of the above mentioned theorem of ELLIS and the fact that the product of a locally compact space and a $k$-space is again a $k$-space:
Let $X$ be a $T_\infty$-space and let $T$ be a homeomorphism group on $X$ such that $T_c$ is locally compact. Then $T$ is a topological group (hence $T_c = T_{bc}$). If, in addition, $X$ is a $k$-space, then $T_c$ is a topological homeomorphism group on $X$.

For more results on topological homeomorphism groups, we refer the reader to R. Arens [1946 a,b], or [GH], Chap. 11.

1.3. Orbit space and enveloping semigroup

1.3.1. In this section $<G,X,\pi>$ always denotes a fixed ttg.

1.3.2. Let $H \subseteq G$ and $Y \subseteq X$. We say that $Y$ is invariant under $H$ whenever $\pi[H \cdot x] \subseteq Y$, or equivalently, whenever $\pi^t y \subseteq Y$ for all $t \in H$. In that case $Y$ is said to be an $H$-invariant subset of $X$. If $x \in X$ and $\{x\}$ is $H$-invariant, then $x$ is called an $H$-invariant point of $X$. The $G$-invariant subsets and points of $X$ will simply be called invariant subsets and points of $X$ (or of $<G,X,\pi>$).

1.3.3. PROPOSITION. Let $H \subseteq G$, and let $Y$ be an $H$-invariant subset of $X$. Then $\text{int}_X Y$ and $\text{cl}_X Y$ are also $H$-invariant, and $\text{cl}_X Y$ is even $\text{cl}_G H$-invariant. If $H = H^{-1}$, then $X \sim Y$ is $H$-invariant.

In addition, intersections and unions of arbitrary classes of $H$-invariant subsets of $X$ are $H$-invariant.

PROOF. Everything except cl$_G H$-invariance of cl$_X Y$ follows trivially from the fact that each $\pi^t$ is a homeomorphism of $X$ ($t \in H$). That cl$_X Y$ is invariant under cl$_G H$ is a consequence of the inclusion $\pi[\text{cl}_G H \cdot x] \subseteq \text{cl}_X \pi[H \cdot y]$. 

1.3.4. If $H$ is a subgroup of $G$ and $Y$ is a non-void $H$-invariant subset of $X$, then $<H,Y,\pi |_{H \cdot Y}>$ is obviously a ttg. The following notational convention will often be employed in this situation: $\pi |_{H \cdot Y}$ will simply be denoted by $\pi$, so that we can speak and write about the ttg $<H,Y,\pi>$.

1.3.5. If $x \in X$, then the orbit $C_\pi [x]$ of $x$ in $X$ (under the action of $G$ by $\pi$) is the set

$$C_\pi [x] := \pi^t [x] = \{\pi^t x : t \in G\}.$$ 

Plainly, $C_\pi [x]$ is the least invariant subspace of $X$ containing the point $x$. More generally, if $A$ is a subset of $X$ then $C_\pi [A] := \cap \{C_\pi [x] : x \in A\} = \pi[A \cdot x]$ is the smallest invariant subset of $X$ including $A$. 


1.3.6. A subset $Y$ of $X$ is invariant iff $C_\pi(y) \subseteq Y$ for every $y \in Y$. In particular, if $x, y \in X$ then either $C_\pi[x] = C_\pi[y]$ or $C_\pi(x) \cap C_\pi(y) = \emptyset$. Consequently, the orbits in $X$ form a partition of $X$. The corresponding equivalence relation in $X$ will be denoted with $C_\pi$. In other words,

$$C_\pi = \{(x,y) \in X \times X : y \in C_\pi(x)\}.$$

1.3.7. The orbit space of the ttg $<G,X,\pi>$ is the quotient space $X/C_\pi$, endowed with its quotient topology. The quotient mapping of $X$ onto $X/C_\pi$ shall consistently be denoted by $c_\pi$.

In discussions where the action $\pi$ is understood we shall often write $C$ and $c$ instead of $C_\pi$ and $c_\pi$.

1.3.8. **EXAMPLES.** We shall indicate here the orbit spaces for some of the ttgs defined in 1.1.6.

(i) The orbit space of $<G,G,\lambda>$ consists of one point.

(ii) The orbit space of $<G,G \times X,\mu_X>$ is homeomorphic to $X$. In fact, the projection $p:(t,x) \mapsto x: G \times X \to X$ establishes a one-to-one correspondence between orbits in $G \times X$ and points of $X$, i.e. there is a bijection $f:(G \times X)/\sim \to X$ such that $f \circ p = \pi$. This bijection is plainly a homeomorphism.

(iii) Let $G$ be a subgroup of the topological group $H$, and let $\pi := \lambda|_{G \times H}$ (cf. 1.1.6(iii)). Then the orbit space of $<G,H,\pi>$ is just the space $H/G$ of right cosets of $G$. It is obvious that the ttg $<G,H,\pi>$ is strongly effective. If it were free, i.e. of the form $<G,G \times X,\mu_X>$, then it would follow from example (ii) that (up to homeomorphism) $X = H/G$, hence $H = G \times (H/G)$. If we take $H = \mathbb{T}$ and $G = \{-1,1\}$ then this is impossible (otherwise $\pi$ would be disconnected). So not each strongly effective ttg is free.

(iv) Let $H$ be a subgroup of the topological group $G$, and consider the action $\pi$ of $G$ on the space $G \setminus H$ of left cosets of $H$ in $G$ defined in 1.1.6(iv). Then the orbit space of $<G,G \setminus H,\pi>$ consists of one point only.

1.3.9. **PROPOSITION.** The quotient mapping $c_\pi: X \to X/C_\pi$ is open.

**PROOF.** Let $U$ be an open subset of $X$. Then $c_\pi^+\{U\} = U(\pi^+U : t \in G)$, hence it is an open subset of $X$. Consequently, $c_\pi[U]$ is open in $X/C_\pi$. $\square$
1.3.10. **PROPOSITION.** The following statements are true:

(i) $X/C_\pi$ is a $T_1$-space iff each orbit in $X$ is closed.

(ii) $X/C_\pi$ is a $T_2$-space iff $C_\pi$ is a closed subset of $X \times X$.

(iii) $X/C_\pi$ is a $T_3$-space iff each orbit in $X$ is closed (i.e. $X/C_\pi$ is $T_1$) and the following "regularity condition" is satisfied (i.e. $X/C_\pi$ is regular): Every invariant neighbourhood of any point in $X$ contains a closed invariant neighbourhood of that point.

**PROOF.** (i) and (iii) are straightforward. For (ii), cf. [Du], Chap. VII, 1.6. Notice that here it is essential that $c_\pi$ is open.

1.3.11. For $x \in X$, the set $K_\pi[x] := \text{cl}_X C_\pi[x]$ is called the orbit-closure of $x$. If $\pi$ is understood, we write $K[x]$ instead of $K_\pi[x]$.

1.3.12. For every $x \in X$, $K_\pi[x]$ is an invariant subset of $X$, by 1.3.3. Obviously, it is the least closed invariant subset of $X$ containing $x$.

Consequently, a closed subset $Y$ of $X$ is invariant iff $K_\pi[y] \subseteq Y$ for all $y \in Y$. In general, the sets $K_\pi[x]$ for $x \in X$ do not form a partition of $X$.

For example, in the ttg of 1.1.6(vi), $K_\pi[x] = X$ for each $x \in X \setminus \{\omega\}$, and $K_\pi[\omega] = \{\omega\}$.

1.3.13. Let $X^X$ be endowed with its usual topology of pointwise convergence. Define a mapping $\pi^* : G \times X^X \to X^X$ by

$$\pi^*(t, \xi) := \pi^t \circ \xi$$

for $t \in G$, $\xi \in X^X$. Since for any $x \in X$ the mapping $(t, \xi) \mapsto \pi(t, \xi(x)) : G \times X^X \to X$ is continuous, it follows that $\pi^*$ is continuous. Moreover, $\pi^*$ is easily seen to be an action, so we have a ttg $<G, X^X, \pi^*>$.

Obviously, $1_X \in X^X$, and the orbit of the element $1_X$ in $X^X$ under the action $\pi^*$ of $G$ is just the transition group $\mathbb{P}[G]$.

1.3.14. The enveloping semigroup $E_{X^X}$ of the ttg $<G, X^X, \pi^*>$ is the closure of the transition group $\mathbb{P}[G]$ in $X^X$. Instead of $E_{X^X}$ we will often write $E_X$ or even $E$. The natural action (sometimes called the obvious action) of $G$ on $E$ is the restriction to $E$ of the action $\pi^*$ of $G$ on $X^X$ (cf. 1.3.4). This action will also be denoted by $\pi^*$.

1.3.15. A few comments are in order about the terminology. The space $X^X$ has a semigroup structure: if $\xi, \eta \in X^X$, then their composite $\xi \eta$ is in $X^X$, and
($\xi, \eta$) $\mapsto \xi \eta$: $X^X \times X^X \to X^X$ is an associative multiplication. Notice, that $1_X$ is the identity of $X^X$ with respect to this multiplication. We shall show that $E_{<G,X,\pi>}$ is a subsemigroup of $X^X$.

1.3.16. **Lemma.** Let $X$ be any topological space. Then the following statements are valid:

(i) For every $\eta \in X$, the mapping $\xi \mapsto \xi \eta: X^X \to X^X$ is continuous.

(ii) If $\xi \in X^X$, then the mapping $\eta \mapsto \xi \eta: X^X \to X^X$ is continuous if $\xi: X \to X$ is continuous.

**Proof.**

(i): For any $x \in X$, the mapping $\xi \mapsto \xi(x): X^X \to X$ is continuous.

(ii): "If": for any $x \in X$, the mapping $\eta \mapsto \eta(x): X^X \to X$ is continuous. Hence $\eta \mapsto \xi \eta$ is continuous, provided $\xi$ is continuous. So $\eta \mapsto \xi \eta$ is continuous in that case. "Only if": we leave this as an exercise for the reader. □

1.3.17. **Proposition.** The enveloping semigroup $E_{<G,X,\pi>}$ of $<G,X,\pi>$ is a subsemigroup of $X^X$, and the mapping $\pi^*: G \to E_{<G,X,\pi>}$ is a continuous morphism of semigroups. In addition, using the notation of 1.1.6(v), the natural action $\pi^*$ of $G$ on $E_{<G,X,\pi>}$ is exactly the action $(\pi)^*$, induced by the morphism $\pi$ of semigroups.

**Proof.** The only non-trivial fact is that $E := E_{<G,X,\pi>}$ is a subsemigroup of $X^X$. The proof is completely standard, but one has to start at the right point, as follows.

First notice that $\mathbb{F}[G]$ is a subgroup of $X^X$, consisting entirely of continuous elements of $X^X$. If $\xi \in \mathbb{F}[G]$, then the mapping $\eta \mapsto \xi \eta: X^X \to X^X$ sends $\mathbb{F}[G]$ into $\mathbb{F}[G]$. By 1.3.16(ii), this mapping is continuous, so it sends $\text{cl}\mathbb{F}[G]$ into $\text{cl}\mathbb{F}[G]$, i.e. its sends $E$ into $E$. Thus $\xi \eta \in E$ for all $\xi \in \mathbb{F}[G]$ and $\eta \in E$. This means that the continuous mapping $\xi \mapsto \xi \eta: X^X \to X^X$ sends $\mathbb{F}[G]$ into $E$. Hence it sends $E (= \text{cl}\mathbb{F}[G])$ into $E (= \text{cl}E)$, that is, $\xi \eta \in E$ for all $\xi, \eta \in E$. □

1.3.18. **Proposition.** If $X$ is a compact Hausdorff space and $<G,X,\pi>$ is an equicontinuous tgg, i.e. $\mathbb{F}[G]$ is an equicontinuous subset of $X^X$, then:

(i) The enveloping semigroup $E$ of $<G,X,\pi>$ is a group of continuous mappings of $X$ into itself, and

\[ \text{Recall that a compact Hausdorff space has a unique uniformity compatible with its topology.} \]
(ii) \( E_p = E_u \), and this is a compact Hausdorff topological homeomorphism group.

**PROOF.** Clearly, (ii) is a direct consequence of 1.2.12. The implication \( (ii) \Rightarrow (i) \) is trivial. □

1.3.19. **REMARKS.**

(i) The converse of 1.3.18 is also valid, i.e. if (i) of 1.3.18 holds, then \( \overline{G} \) is equicontinuous. The proof reads as follows: since \( E \) is compact and \( E \subseteq \mathcal{C}(X,X) \), \( E \) equals the closure of \( \overline{G} \) in \( \mathcal{C}_u(X,X) \). Therefore, by the implication (ii) \( \Rightarrow (i) \) in the theorem mentioned in the notes to section 1.2, the closure of \( \overline{G} \) in \( \mathcal{C}_u(X,X) \) is compact. Now a straightforward compactness argument, viz. 0.2.2(ii), shows that \( \overline{G} \) is equicontinuous (use 1.2.3(iii)).

(ii) The preceding proposition applies also to the ttg \( <G,X,\pi> \). This shows that the topology of \( G \) is irrelevant (this follows also from the proof).

1.3.20. **NOTES.** Orbit spaces are intensively explored in those parts of the theory of ttgs which have to do with bundle theory (in fact, a G-bundle is nothing but the triple \( (X,c,X/C) \) for some G-space \( X \) (cf. [Hu, p.40]). One of the important questions concerning the orbit space of a ttg \( <G,X,\pi> \) is the existence of a cross-section, i.e. a continuous function \( f: X/C \to X \) such that \( c\pi f = \pi \). Clearly, such a cross-section exists, whenever \( <G,X,\pi> \) is isomorphic to \( <G,GX(X/C),\pi_{X/C}> \) as a G-space (for the precise definition of an isomorphism in G-spaces, cf. 1.4). So this problem is related to the following question: **when is a strongly effective ttg free?** (Cf. 1.1.10). For actions of the group \( \mathbb{R} \), this problem is known as the question of when a flow parallelizable? For some pertinent literature, cf. J. DUGUNDJI & H.A. ANTONIEWICZ [1961] and O. HAJEK [1971]. The technique in these papers is to prove first the existence of local cross-sections and then "paste" them together to a global one (cf. also [St], Theorem 12.2 or [Br], Chap. II, 9.2). For the existence of local cross-sections, we have the classical WHITNEY-BERTRON theorem (cf. [Ha], Chap. VI, 2.13), or more generally, the existence theorem for so-called slices (e.g. R.S. PALAIS [1961]). For related results, cf. also Theorem 1.8 in App. II in [HM] and the paper 7.S. MOSTERT [1956]. A related question is, which properties of the phase space of a ttg are inherited by the orbit space. We glanced at this subject already in 1.3.10. For more results in this direction and for some pertinent literature, we refer the
reader to the notes in 4.1.11.

Concerning enveloping semigroups we can be brief here: they play an important role in certain parts of topological dynamics: cf. [El], from which 1.3.17 and 1.3.18 are taken.

1.4. Morphisms and comorphisms

1.4.1. Let \( <G,X,\pi> \) and \( <H,Y,\sigma> \) be ttgs. A morphism of ttgs from \( <G,X,\pi> \) to \( <H,Y,\sigma> \) is a pair \( \psi,f \) with \( \psi:G \rightarrow H \) a continuous morphism of groups and \( f:X \rightarrow Y \) a continuous function such that the following diagram commutes:

\[
\begin{array}{ccc}
G \times X & \xrightarrow{\pi} & X \\
\downarrow \psi \times f & & \downarrow f \\
H \times Y & \xrightarrow{\sigma} & Y
\end{array}
\]

(1)

Notation: \( \psi,f: <G,X,\pi> \rightarrow <H,Y,\sigma> \). If \( \psi,f \) is a morphism of ttgs, then \( \psi \) and \( f \) are called its group component and its space component, respectively.

If \( G \) is a topological group, then a morphism of G-spaces from a G-space \( X \) with action \( \pi \) to a G-space \( Y \) with action \( \sigma \) is a morphism of ttgs of the form \( <1_G,f>: <G,X,\pi> \rightarrow <G,Y,\sigma> \). In this case we shall also say that \( f:X \rightarrow Y \) is a morphism of G-spaces.

A morphism \( \psi,f: <G,X,\pi> \rightarrow <H,Y,\sigma> \) of ttgs is said to be an isomorphism of ttgs whenever \( \psi \) is a topological isomorphism of \( G \) onto \( H \) and \( f \) is a homeomorphism of \( X \) onto \( Y \). A morphism \( <1_G,f>: <G,X,\pi> \rightarrow <G,Y,\sigma> \) of G-spaces is said to be an isomorphism of G-spaces whenever \( f \) is a homeomorphism of \( X \) onto \( Y \) (i.e. \( <1_G,f> \) is an isomorphism of ttgs).

1.4.2. Let \( <G,X,\pi> \) and \( <H,Y,\sigma> \) be ttgs, \( \psi:G \rightarrow H \) a continuous morphism of groups and \( f:X \rightarrow Y \) a continuous function. Then \( \psi,f \) is a morphism of ttgs iff for all \( (t,x) \in G \times X \) one of the following diagrams commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & X \\
\downarrow f & & \downarrow f \\
Y & \xrightarrow{\sigma \psi(t)} & Y
\end{array}
\]  

(2)

\[
\begin{array}{ccc}
G & \xrightarrow{\pi} & X \\
\downarrow \psi & & \downarrow f \\
H & \xrightarrow{\sigma f(x)} & Y
\end{array}
\]
In that event, both diagrams commute for every \((t,x) \in G \times X\).

If \(<G,X,\pi>\) and \(<G,Y,\tau>\) are \(G\)-spaces, then \(f\) is a morphism of \(G\)-spaces iff for all \((t,x) \in G \times X\) one of the following diagrams commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi^t} & X \\
\downarrow{f} & & \downarrow{f} \\
Y & \xrightarrow{t^*} & Y
\end{array}
\quad
\begin{array}{ccc}
G & \xrightarrow{\pi_X} & X \\
\downarrow{T_f(x)} & & \downarrow{f} \\
Y & \xrightarrow{t^*} & Y
\end{array}
\]

In that event, both diagrams commute for every \((t,x) \in G \times X\).

If \((3)\) always commutes, then \(f\) is simply called equivariant. Thus, \(<\psi,f>\) is a morphism of \(G\)-spaces iff \(f\) is \(\psi\)-equivariant and both \(\psi\) and \(f\) are continuous. Similarly, \(f\) is a morphism of \(G\)-spaces iff \(f\) is equivariant and continuous.

1.4.3. Let \(<ljl,f>: <G,X,\pi> \rightarrow <H,Y,\sigma>\) and \(<n,g>: <H,Y,\sigma> \rightarrow <K,Z,\tau>\) be morphisms of \(G\)-spaces. Then clearly \(<\eta, gf>: <G,X,\pi> \rightarrow <K,Z,\tau>\) is a morphism of \(G\)-spaces. We call \(<\eta, gf>\) the composition of the given morphisms \(<\psi, f>\) and \(<\eta, g>\). Notation:

\[<\eta, g \circ \psi, f> := <\eta \circ gf>.\]

In addition, if \(\zeta := \eta \circ \psi\) and \(h := \psi \circ f\), then we shall use diagrams like

\[
\begin{array}{ccc}
<\zeta, h> & \xrightarrow{<\xi, h>} & <\eta, g> \\
<\psi, f> & \xrightarrow{<\zeta, f>} & <\eta, g> \\
<\xi, h> & \xrightarrow{<\eta, h>} & <\eta, g>
\end{array}
\]

(4)

to illustrate the situation.

If in the above situation \(G = H = Z\) and \(\psi = \eta = 1_G\), then, of course, \(\zeta = 1_G\). In other words, the composition of morphisms of \(G\)-spaces is again a morphism of \(G\)-spaces.
1.4.4. **EXAMPLES.** Although we shall consider many examples of morphisms in the subsequent sections, we shall present here some simple examples.

(i) If \( <G_x, \pi> \) is any ttg, then for every \( x \in X \), the motion \( \pi_x : G \to X \) is a morphism of G-spaces from \( G \) (with action \( \lambda \)) to \( X \) (with action \( \pi \)). Cf. the diagrams (4) in 1.1.2 and (3) in 1.4.2.

(ii) If \( G \) and \( H \) are topological groups and \( \psi : G \to H \) is a continuous morphism of groups, then \( \psi : G \to H \) is a morphism of ttgs from \( <G, \lambda(G)> \) to \( <H, \lambda(H)> \) (recall that \( \lambda(G) \) and \( \lambda(H) \) are the actions of \( G \) and \( H \) on themselves by left translations; cf. 1.1.6(i)).

(iii) In the situation of (ii), \( \psi : G \to H \) is a morphism of G-spaces from \( <G, \lambda> \) to \( <G, \psi> \), where \( \psi : G \to H \) is defined by \( \psi(t, u) = \psi(t)u \) for \( (t, u) \in G \times \) (cf. 1.1.6(v)).

(iv) Let \( H \) be a subgroup of the topological group \( G \), \( G \) the space of all left cosets of \( H \) in \( G \) and \( <G, \psi> \) the ttg which is described in 1.1.6(iv). Then the quotient mapping \( q : G \to G \) is a morphism of G-spaces from \( G \) (with action \( \lambda \)) onto \( G \) (with action \( \pi \)).

(v) Let \( <G_x, \pi> \) be a ttg and let \( x \in X \). Let \( G \times _x \) denote the space of all left cosets of \( G_x \) in \( G \) and \( q_x : G \to G \times _x \) the quotient mapping. Since for all \( s, t \in G \) we have \( s \pi_x = t \pi_x \) iff \( s G_x = t G_x \), there exists an injective function \( \varphi_x : G \times _x \to X \) such that \( \pi_x = \varphi_x \pi_x \). It is easily seen that \( \varphi_x \) is continuous and that it is a morphism of G-spaces from \( G \times _x \) (cf. 1.1.6(iv)) into \( X \). Observe, that \( q_x : G \to G \times _x \) and \( \pi_x : G \to X \) are morphisms of G-spaces as well.

It is clear that \( \varphi_x \) maps \( G \times _x \) onto \( C[x] \). So if \( \pi_x \) is the corestriction of \( \varphi_x \) to \( C[x] \) and if \( j_x \) denotes the inclusion mapping of \( C[x] \) into \( X \) then we have the following decomposition of \( \pi_x \) into morphisms of G-spaces (clearly, \( \psi_x \) and \( j_x \) are morphisms of G-spaces when \( G \) acts on \( C[x] \) by \( \pi \))

\[
\begin{array}{c}
G \xrightarrow{q_x} G \times _x \xrightarrow{\psi_x} C[x] \xrightarrow{j_x} X.
\end{array}
\]

Topologically, \( q_x \) is a quotient mapping, \( \psi_x \) is a continuous bijection and \( j_x \) is a topological embedding.

In this context, the following observation is useful, namely, that the statements

(i) \( \psi_x : G \times _x \to C[x] \) is a homeomorphism,

(ii) \( \pi_x : G \to C[x] \) is open,

(iii) \( \pi_x : G \to C[x] \) is open at \( e \),

are all equivalent (the proof is almost trivial).
Let \(<G,X,\pi>\) be a ttg, and consider the ttg \(<G,E,\pi^*>\), where \(E\) is the enveloping semigroup of \(<G,X,\pi>\), and \(\pi^*\) is the natural action of \(G\) on \(E\) (cf. 1.3.14). Then \(\pi^*:G \to E\) is a morphism of \(G\)-spaces from \(G\) (with action \(\lambda\)) into \(E\) (with action \(\pi^*\)). See also (iii) above (notice that \(\pi^* = \pi^*\)).

For every \(x \in X\), let \(\delta_x:E \to X\) be defined by

\[
\delta_x(\xi) := \xi(x)
\]

\((\xi \in E)\). Obviously, \(\delta_x\) is continuous. Moreover, \(\delta_x(\pi^*t) = \pi^*x = \pi^*\xi\), so that \(\delta_x \circ \pi[G] = C_{\pi^*}[x]\). Since \(\pi[G]\) is dense in \(E\), this implies that

\[
\delta_x[E] \subseteq \text{cl} \ C_{\pi^*}[x] = K_{\pi^*}[x].
\]

It is clear that we have equality here iff \(\delta_x[E]\) is closed in \(X\).

In particular, if \(X\) is a compact Hausdorff space, then \(E\) is compact, hence \(\delta_x[E]\) is closed in \(X\), and \(\delta_x[E] = K_{\pi^*}[x]\).

It is clear that, for any \(ttg\) \(<G,X,\pi>\) and \(x \in X\), \(\delta_x\) is a morphism of \(G\)-spaces from \(E\) (with action \(\pi^*\)) into \(X\) (with action \(\pi\)). Indeed, if \(t \in G\) and \(\xi \in E\), then

\[
\delta_x(\pi^*t, \xi) = \delta_x(\pi^* \cdot \xi) = \pi^*\xi(x) = \pi(t, \delta_x \xi).
\]

If \(<G,X,\pi>\) is equicontinuous and, in addition, \(X\) is a compact Hausdorff space, then \(E\) is a compact Hausdorff topological homeomorphism group; cf. 1.3.18. In that case, \(<E,X,\delta>\) is a ttg (here \(\delta\) is defined by \(\delta(\xi,x) := E(x)\), in accordance with the definition of \(\delta_x\) above). In addition, \(<\pi,1_X>:\ <G,X,\pi> \to <E,X,\delta>\) is a morphism of ttgs.

1.4.5. Let \(<\psi,f>: <G,X,\pi> \to <H,Y,\sigma>\) be a morphism of ttgs. If \(A \subseteq X\) is invariant under a subset \(S\) of \(G\), then \(f[A]\) is invariant under the subset \(\psi[S]\) of \(H\). Hence \(\text{cl}_H f[X]\) is invariant under \(\text{cl}_H \psi[S]\) (cf. 1.3.3).

In addition, if \(B \subseteq Y\) is invariant under a subset \(T\) of \(H\), then \(f^*[B]\) is invariant under \(\psi^*[T]\). In particular, for each \(x \in X\), \(f^*[C_{\pi^*}[f(x)]\) is an invariant subset of \(X\). Since it contains \(x\), it includes all of \(C_{\pi^*}[x]\). Hence

\[
f[C_{\pi^*}[x]] \subseteq C_{\sigma}[f(x)].
\]

If \(\psi\) is a surjection, then the inclusion in (5) is easily seen to be an equality. In fact, then the image under \(f\) of any invariant subset of \(X\) is an invariant subset of \(Y\).
1.4.6. **Lemma.** Let \( \langle G, X, \pi \rangle \) and \( \langle H, Y, \sigma \rangle \) be ttgs and \( f : X \to Y \) a continuous function. Then there exists a continuous function \( f' : X/C_\pi \to Y/C_\sigma \) such that \( f' \circ \pi = c_0 \circ f \) iff for all \( x \in X \), the inclusion \( f[C_\pi(x)] \subseteq C_\sigma(f(x)) \) is valid.

**Proof.** Obvious. \( \square \)

1.4.7. **Proposition.** If \( \langle \psi, f \rangle : \langle G, X, \pi \rangle \to \langle H, Y, \sigma \rangle \) is a morphism of ttgs, then there exists a unique continuous function \( f' : X/C_\pi \to Y/C_\sigma \) with the property that \( f' \circ \pi = c_0 \circ f \).

**Proof.** Use 1.4.5 and 1.4.6. \( \square \)

1.4.8. If \( \langle \psi, f \rangle : \langle G, X, \pi \rangle \to \langle H, Y, \sigma \rangle \) is a morphism of ttgs, then the function \( f' : X/C_\pi \to Y/C_\sigma \) for which \( f' \circ \pi = c_0 \circ f \) will be called the continuous mapping of orbit spaces, induced by \( \psi, f \).

In general, \( f \) is not uniquely determined by \( f' \). For example, all \( G \)-endomorphisms of a \( G \)-space which consists of one orbit induce the identity mapping of the (one-point) orbit space onto itself.

1.4.9. **Proposition.** Let \( \langle \psi, f \rangle : \langle G, X, \pi \rangle \to \langle H, Y, \sigma \rangle \) be a morphism of ttgs. If \( f \) is an open mapping then \( f' : X/C_\pi \to Y/C_\sigma \) is open as well. If \( f \) is relatively open and, in addition, \( \psi \) is a surjection of \( G \) onto \( H \), then \( f' \) is relatively open.

**Proof.** The first statement is almost trivial. In order to prove the second one, consider an open subset \( U \) of \( X/C_\pi \). Then \( f_\pi^{-1}(U) = f[X] \cap V \) for some open subset \( V \) of \( Y \). Since \( f_\pi^{-1}(U) \) and \( f[X] \) are \( H \)-invariant subsets of \( Y \), it follows easily that \( f[X] \cap V = f[X] \cap a[H \times V] \). Therefore, we may suppose that \( V \) is \( H \)-invariant. Hence, \( f'[U] = f'[X/C_\pi] \cap c_\sigma[V] \) with \( c_\sigma[V] \) open in \( Y/C_\sigma \). \( \square \)

1.4.10. **Corollary.** If \( A \) is an invariant subset of the ttg \( \langle G, X, \pi \rangle \), then the inclusion mapping \( i : A \to X \) is a morphism of \( G \)-spaces from \( A \) (with action \( \pi \)) into \( X \) (with action \( \pi \)), and the mapping \( i' : A/C_\pi \to X/C_\pi \) induced by \( i \), is a topological embedding. Consequently, if \( \eta \) is any subset of \( X/C_\pi \), then the orbit space of \( \langle i'^{-1}(\eta), \pi \rangle \) may be identified with \( \eta \) in the obvious way. \( \square \)

1.4.11. **Proposition.** Let \( \langle \psi, f \rangle : \langle G, X, \pi \rangle \to \langle H, Y, \sigma \rangle \) be a morphism of ttgs, where \( X \) and \( Y \) are uniform Hausdorff spaces, and \( f \) is uniformly continuous. If \( Y \) is complete and if \( f \) is surjective, then there exists a unique continuous morphism of semigroups \( f'' : E_X \to E_Y \) such that the following diagram commutes for every \( x \in X \):
In particular, it follows that \(<\psi, f"">: \langle G, E_X, \tau^* \rangle \rightarrow \langle H, E_Y, \sigma^* \rangle \) is a morphism of ttgs.

**PROOF.** Let us first observe that the topologies of \(X, Y\) and their subspaces are generated by the weakest uniformities making all evaluations on these spaces uniformly continuous. We shall first define \(f^*: \overline{\pi}[G] \rightarrow \overline{\sigma}[H] \subseteq Y\) in such a way that \(f^*\) is easily seen to be uniformly continuous. Since \(Y\) is a complete uniform space and \(\overline{\pi}[G]\) is dense in \(E_X\), \(f^*\) has a unique uniformly continuous extension denoted by \(f"\), mapping \(E_X\) into \(\overline{\sigma}[H] = E_Y\) (cf. [Bo], Chap. II, §3.6, Theorem 2).

So let us define \(f^*\) on \(\overline{\pi}[G]\) by \(f^*(\pi^t) : = \sigma(\psi(t)) (t \in G)\). This definition is unambiguous, because \(\pi^t = \pi^s\) implies \(\sigma(\psi(t)) = \sigma(\psi(s))\) (\(f\) is surjective!). Now \(f^*: \overline{\pi}[G] \rightarrow Y\) is uniformly continuous, because \(\delta^*_Y \circ f^* = \overline{\sigma}[\pi][G] \rightarrow Y\) is uniformly continuous for every \(y \in Y\). Indeed, if \(y \in Y\), then \(y = f(x)\) for some \(x \in X\), and \(\delta^*_X \circ f^* = f \circ \delta^*_X\) with \(\delta^*_X\) and \(f\) uniformly continuous. So we can extend \(f^*\) to \(f"\) on \(E_X \rightarrow E_Y\) in the way described above.

Finally, the requirements that \(f"\) is a morphism of semigroups, that (6) commutes and that \(<\psi, f"">\) is a morphism of ttgs can be expressed as equations of continuous functions. These equations are easily seen to hold on dense subspaces of the spaces under consideration. Hence they hold everywhere. The details are left to the reader. \(\Box\)

1.4.12. The conclusions of the preceding proposition are in particular valid if \(X\) and \(Y\) are compact Hausdorff spaces and \(f\) is a continuous \(\psi\)-equivariant surjection: then \(f\) is uniformly continuous with respect to the (unique) uniformities for \(X\) and \(Y\).

1.4.13. In this section we have obviously defined a category \(\text{TtG}'\) whose objects are ttgs, and whose morphisms are the ordered pairs \(<\psi, f>\) satisfying diagram (1) in 1.4.1. In addition, 1.4.7 shows that the assignment of the orbit space to a ttg is functorial on all of \(\text{TtG}'\), and 1.4.11 shows that the
same is true for enveloping semigroups on a suitable subcategory of TTG.

However, in considering invariant subsets of a ttg \( <H,Y,\sigma> \) another definition of a "morphism of ttgs" may come to one's mind. If \( X \) is an invariant subset of \( Y \), set \( G := \{ \sigma^t \mid X : t \in H \} \). Then \( G \) is a subgroup of \( H(X,X) \), and \( \psi : t \mapsto \sigma^t \mid X : H \to H \) is a morphism of groups. If we give \( G \) the finest topology making \( \psi \) continuous, then \( \delta : (x,y) \mapsto \xi(y) : G \times X \to X \) is continuous, and \( <G,X,\delta> \) is a ttg (apply 1.1.15 to the ttg \( <H,X,\sigma> \)). If \( f : X \to Y \) denotes the inclusion mapping, then the following diagram commutes for all \( t \in H \):

\[
\begin{array}{cc}
X & \psi \\
\downarrow f & \downarrow f \\
Y & \sigma^t \\
\end{array}
\]

\( (11) \)

This motivates the following definition:

1.4.14. If \( <G,X,\pi> \) and \( <H,Y,\sigma> \) are ttgs, then a pair \( <\psi^G,f> \) is called a comorphism of ttgs from \( <G,X,\pi> \) to \( <H,Y,\sigma> \), if \( \psi : H \to G \) is a continuous morphism of groups, \( f : X \to Y \) a continuous function, and for each \( t \in H \) the diagram (11) commutes (with \( \delta \) replaced by \( \pi \)).

Notation: \( <\psi^G,f> : <G,X,\pi> \to <H,Y,\sigma> \). In this situation, \( \psi \) and \( f \) are called the group component and the space component of the comorphism, respectively.

1.4.15. Notice that the direction of a comorphism is the same as the direction of its space component. This choice is more or less arbitrary, but now we have the advantage that what we would like to call a comorphism of \( G \)-spaces (i.e. \( H = G \) and \( \pi = 1_G \) in definition 1.4.14) is exactly the same as a morphism of \( G \)-spaces.

1.4.16. If \( <\psi^G,f> \) and \( <\eta^G,g> \) are comorphisms of ttgs, and the codomain of \( <\psi^G,f> \) equals the domain of \( <\eta^G,g> \), then \( <(\psi \eta)^G,gf> \) is a comorphism of ttgs.

Notation: \( <(\psi \eta)^G,gf> =: <\eta^G,g> \circ <\psi^G,f> \).

\[ ^{1)} \text{If} \psi : H \to G \text{ is a continuous morphism of groups, we shall express this sometimes by writing } \psi^G : G \to H. \text{ Cf. } 6.1 \text{ for the proper context of this notation.} \]

Obviously, we have defined now another category, denoted $\TTG_\star$. Its objects are just all ordinary ttgs and its morphisms are the comorphisms, defined in 1.4.14. We shall have now a brief look at the behaviour of orbit spaces and enveloping semigroups with respect to comorphisms.

1.4.17. **Proposition.** Let $\langle \psi^{\circ}, f \rangle: \langle G, X, \pi \rangle \to \langle H, Y, \sigma \rangle$ be a comorphism of ttgs. If $\psi: H \to G$ is surjective then there exists a unique continuous function $f': X/\pi \to Y/\sigma$ such that $f' c = c_f$.

**Proof.** Straightforward. □

1.4.18. **Proposition.** Let $\langle \psi^{\circ}, f \rangle: \langle G, X, \pi \rangle \to \langle H, Y, \sigma \rangle$ be a comorphism of ttgs, where $Y$ is a uniform Hausdorff space, $X$ is a subspace of $Y$ and $f: X \to Y$ is the inclusion mapping. If $X$ is complete then there exists a unique continuous morphism of semigroups $f'': E_X \to E_X$ such that the following diagram commutes for every $x \in X$:

\[
\begin{array}{ccc}
E_X & \xrightarrow{f''} & E_Y \\
\downarrow{\delta_x} & & \downarrow{\delta_{fx}} \\
X & \xrightarrow{f} & Y
\end{array}
\]

Moreover, $\langle \psi^{\circ}, f'' \rangle: \langle H, X, \sigma^* \rangle \to \langle G, X, \pi^* \rangle$ is a morphism of ttgs.

**Proof.** Similar to 1.4.11. □

1.4.19. **Notes.** There exist several other definitions of "morphisms of ttgs". Cf. O. HAJEK [1968]. The concept of a comorphism seems to be new, like propositions 1.4.17 and 1.4.18. However, 1.4.17 is an obvious adaptation of the well-known proposition 1.4.7. A similar remark holds with respect to 1.4.18 and 1.4.11. Here it may be noticed that 1.4.11 slightly generalizes [El] 3.8, where only the compact case has been treated.

1.5. **Operations on ttgs**

1.5.1. The operations we have in mind are the usual ones on topological groups and on topological spaces, but now combined in order to obtain operations on ttgs. Most of these operations are exactly what they are expected to be. These will not be treated here; we shall mention them here with a
In this section we shall consider only some questions related to the formation of quotients.

1.5.2. Let \(<G,X,\pi>\) be a ttg and let \(H \subseteq G\). Then an equivalence relation \(R\) on \(X\) is said to be \textit{invariant under} \(H\) or \(H\)-\textit{invariant} whenever \((x,y) \in R\) implies \((tx,ty) \in R\) for all \(t \in H\). If \(R\) is a \(G\)-invariant equivalence relation, then \(R\) will simply be called \textit{invariant}.

1.5.3. Let \(<G,X,\pi>\) be a ttg and let \(R\) be an invariant equivalence relation on \(X\), with quotient map \(q: X \to X/R\). Since \(q(x) = q(y)\) implies \(q(tx) = q(ty)\) for all \(t \in G\) \((x,y \in X)\), it follows that there exists a \textit{unique} function \(\tau: G \times (X/R) \to X/R\) such that the following diagram commutes:

\[
\begin{array}{ccc}
G \times X & \xrightarrow{\pi} & X \\
\downarrow{1_G \times q} & & \downarrow{q} \\
G \times (X/R) & \xrightarrow{\tau} & X/R
\end{array}
\]

(1)

Equivalently, \(\tau\) is the unique mapping such that

\[
\tau^t q(x) = q(t^x)
\]

for all \(t \in G\) and \(x \in X\) (uniqueness: \(q\) is surjective).

1.5.4. In the sequel, up to 1.5.10, we shall use the notation of 1.5.3. In particular, the symbols \(s, t\) will always denote elements of \(G\), and \(x, y\) will denote elements of \(X\). Notice that each point in \(X/R\) is of the form \(q(x)\), because \(q\) is a surjection.

1.5.5. The function \(\tau: G \times (X/R) \to X/R\) is \textit{separately continuous} and, in addition, it is an \textit{action of} \(G_d\) on \(X/R\). It is the unique action of \(G_d\) on \(X/R\) making \(q\) a morphism of \(G_d\)-spaces, from \(X\) \(\text{with action } \pi; \text{ cf. } 1.1.5\) onto \(X/R\).

Continuity of each \(\tau q(x)\) is obvious from the equality \(\tau q(x) = q^{\pi^x}\), and
continuity of any $\tau^t$ follows from the continuity of its composition with the quotient mapping $q$, which equals $q_\pi^t$. The other statements are easily verified.

1.5.6. In 1.5.11 below we present an example which shows that $\tau$ may be not continuous. Notice that $\tau$ is the only candidate for an action of $G$ on $X/R$ for which $q: X \rightarrow X/R$ is a morphism of $G$-spaces from $<G,X,\pi>$ to $<G,X/R,\tau>$.

1.5.7. **PROPOSITION.** If one of the following conditions is fulfilled, then $\tau$ is continuous, i.e. then $\tau$ is the unique action of $G$ on $X/R$ making $q$ a morphism of $G$-spaces:

(i) $R$ is an open equivalence relation, i.e. $q$ is an open mapping.

(ii) $R$ is a closed equivalence relation and each equivalence class $R[x]$ is compact, i.e. $q$ is a perfect mapping.

(iii) $G$ is a locally compact Hausdorff group.

(iv) $G \times (X/R)$ is a $k$-space.

**PROOF.** Apply 0.2.4. □

1.5.8. **COROLLARY 1.** Suppose we are given another action on $X$, say $\sigma: H \times X \rightarrow X$, where $H$ is any topological group, and suppose that $\sigma$ commutes with $\eta$, i.e.

$$(3) \quad \sigma^t \eta = \eta \sigma^t \quad (s \in G, t \in H).$$

Then there exists a unique action $\tau$ of $G$ on $X/C_0$ such that $c_0: X \rightarrow X/C_0$ is a morphism of $G$-spaces (i.e. $\tau$ is the unique action of $G$ on $X/R$ making $q$ a morphism of $G$-spaces).

**PROOF.** Use 1.5.7(i) with $R = C_0$, hence $q = c_0$ (keep in mind that $c_0$ is an open mapping; cf. 1.3.9). □

1.5.9. In 1.5.8, (3) is used in order to prove that $c_0$ is an invariant equivalence relation in $X$. For this, however, it would be sufficient to require

$$(4) \quad \forall (s,t) \in G \times H, \exists t' \in H : \sigma^t \pi^s = \pi^s \sigma^t.$$

This condition will certainly be fulfilled if $H$ is a normal subgroup of $G$ and $\sigma = \pi \big|_{H \times X}$ (i.e. $\sigma^t = \pi^t$ for every $t \in H$). In this case, we obtain an action $\tau$ of $G$ on $X/C_0$, making $c_0$ a morphism of $G$-spaces. Obviously, $H \subseteq \text{Ker} \tilde{\tau}$, hence $\tilde{\tau}: G \rightarrow \tilde{T}[G]$ factorizes over the quotient mapping $\psi: G \rightarrow G/H$, as follows:
Now we can define $\tau' : (G/H) \times (X/C_0) + X/C_0$ by the rule

$$\tau'(u,z) := \alpha(u)(z)$$

for $u \in G/H$ and $z \in X/C_0$. Since $\alpha$ is a morphism of groups (notice that $H$ was assumed to be a normal subgroup of $G$), $\tau'$ is plainly an action of $(G/H)_d$ on $X/C_0$. However, by (5) and (6), $\tau'(\psi(t),c_0(x)) = \overline{\tau}(t)(c_0) = \tau(t,c_0(x)) = c_0(\psi(t,x))$, so the following diagram commutes:

$$
\begin{array}{ccc}
G \times X & \overset{\pi}{\longrightarrow} & X \\
\downarrow{\psi \times c_0} & & \downarrow{c_0} \\
(G/H) \times (X/C_0) & \overset{\tau'}{\longrightarrow} & X/C_0 \\
\end{array}
$$

It follows, that the mapping $\tau' = (\psi \times c_0)$ is continuous. Since $\psi$ and $c_0$ are both open mappings, $\psi \times c_0$ is a quotient mapping. Hence $\tau'$ is continuous.

So we have the ttg $<G/H, X/C_0, \tau'>$ and by (7), $<\psi, c_0>$ is a morphism of ttgs. Moreover, $\tau'$ is the unique action of $G/H$ on $X/C_0$ making $<\psi, c_0>$ a morphism of ttgs.

We shall see later, in 3.3.15, that the preceding construction is a special case of a more general one with nice functorial properties.

1.5.10. COROLLARY 2. Let $<G,Y,\sigma>$ be a ttg and let $A$ be a closed invariant subset of $Y$. Then there exists a unique action $\tau$ of $G$ on $Y \cup_A Y$ such that the canonical injections $\iota_1, \iota_2 : Y + Y \cup_A Y$ are morphisms of $G$-spaces.

PROOF. Recall that the space $Y \cup_A Y$ is obtained in the following way: first, form $X := Y \times \{1,2\}$, the disjoint union of two copies of $Y$, then form the quotient space $Y \cup_A Y := X/R$, where $R$ is the equivalence relation $\{(x,1) : x \in X\} \cup \{(a,1),(a,2) : a \in A\}$ in $X$. Notice that we have canonical embeddings
Define \( \pi : G \times X \times X \) by \( \pi^{-1}(y, i) = (\sigma^i y, i) \) for \( t \in G, y \in X \) and \( i = 1, 2 \). Then \( \pi \) is continuous, \( (G, X, \pi) \) is a tfg, and \( r_1, r_2 : Y \times X \) are morphisms of \( G \)-spaces. Now apply 1.5.7(ii) to the \( \text{tfg} (G, X, \pi) \) and the equivalence relation \( R \) in \( X \), which is obviously invariant. Notice that \( R \) is a closed equivalence relation in \( X \) because \( A \) is closed in \( Y \); each equivalence class \([x] \) is compact, since it consists of at most two points. \( \Box \)

1.5.11. \textbf{Example.} In 0.2.5 we described a locally compact Hausdorff space \( X \) and an equivalence relation \( R \) on \( X \) such that on \( \mathbb{Q} \times (X/R) \) the quotient topology induced by \( 1 \times f : \mathbb{Q} \times X \to \mathbb{Q} \times (X/R) \) (\( f : X \to X/R \) the quotient mapping) is strictly finer than the product topology on \( \mathbb{Q} \times (X/R) \). If \( \mathbb{Q} \times (X/R) \) is endowed with this quotient topology, we shall indicate this by writing \( \mathbb{Q} \star (X/R) \) for this space.

Consider the \( \text{tfg} \langle \mathbb{Q}, \mathbb{Q} \times X, \mu_X \rangle \) (cf. 1.1.6(ii)). Let \( D_{q, R} := \{(t, t) : t \in \mathbb{Q}\} \); then \( D_{q, R} \) is clearly an equivalence relation in \( \mathbb{Q} \times X \). Notice that the quotient mapping \( q : \mathbb{Q} \times X \to \mathbb{Q} \times (X/R) \) is a morphism of \( \mathbb{Q} \)-spaces given by \( \tau(t, (s, f(x))) = (t, s, f(x)) \). We claim that \( \tau : \mathbb{Q} \times (\mathbb{Q} \times (X/R)) \to \mathbb{Q} \times (X/R) \) is not continuous.

Suppose it were. Then in particular the mapping \( (s, 0, y) \mapsto (s, y) : \mathbb{Q} \times A \to \mathbb{Q} \star (X/R) \) would be continuous, where \( A := \{y \in X/R : y \in \mathbb{Q} \star (X/R) \} \). Now \( A \) is a closed subspace of the quotient space \( \mathbb{Q} \star (X/R) \), hence its topology equals the quotient topology when considered as a quotient space of \( \{0\} \times X \); cf. [Du], Chap. VI, 4.2. So \( A \) may be identified with \( X/R \), and the domain of the above mapping may be identified with \( \mathbb{Q} \times (X/R) \) in its product topology. It would follow that this product topology is finer than the quotient topology, which is not true.

\textbf{Remark.} In the above example, \( \mathbb{Q} \) is a \( k \)-space and \( X \) is a locally compact \( T_2 \)-space, hence \( \mathbb{Q} \times X \) is a \( k \)-space. Moreover, \( \mathbb{Q} \star (X/R) \) with the quotient topology induced by \( 1 \times f \) is a \( T_2 \)-space, hence it is a \( k \)-space, by [Du], Chap. XI, Cor. 9.5.

If \( \mathbb{Q} \star (\mathbb{Q} \star (X/R)) \) were a \( k \)-space, then by 1.5.7(iv), the action \( \tau \) of \( D_{q, R} \) on \( \mathbb{Q} \star (X/R) \) would be continuous on \( \mathbb{Q} \times (\mathbb{Q} \star (X/R)) \), which we just proved to be not true. Consequently, the product of the \( k \)-space \( \mathbb{Q} \) and \( \mathbb{Q} \star (X/R) \) is not a \( k \)-space (another example is given in [Da], Ex. 5 on p. 132).
Aim of this section is to provide some examples of tgs which will be needed in the following chapters. In subsection 2.1 we study the action of a topological group $G$ on the space $\mathcal{C}_c(G,Y)$ by means of right translations, where $Y$ is a topological space, fixed throughout the discussion. The most interesting applications are those with $Y = \mathbb{F}$; however, it will only rarely be assumed that $Y = \mathbb{F}$. In general, right translation in $\mathcal{C}_c(G,Y)$ is only separately continuous; if $G$ is locally compact $T_2$, then it is simultaneously continuous, and we have, indeed, an action of $G$ on $\mathcal{C}_c(G,Y)$. If $G$ is not locally compact, but under the assumption that $Y$ is a uniform space, right translation is at least simultaneously continuous on orbit closures of elements of $\mathcal{RUC}(G,Y)$. Moreover, these orbit closures are compact in the compact-open topology. In addition, we shall consider briefly the subspace $\{ \pi, \pi \circ \text{y}\}$ of $\mathcal{C}_c(G,Y)$ when $\pi$ is an action of $G$ on $Y$. This will turn out to be important for the considerations in §7.

If $Y$ is a uniform space, right translations of $G$ on $\mathcal{C}_u(G,Y)$ are the context in which almost periodic functions are to be studied. We shall make some remarks about them in subsection 2.2. Here the original definition of H. Bohr is employed, and the difference between left and right almost periodicity (which does not occur if one uses the Von Neumann definition) is discussed.

Finally, we consider right translations in $L^p(G)$ and so-called weighted translations in $L^2(G)$, where $G$ is a locally compact Hausdorff group. The results on weighted translations in $L^2(G)$ (subsection 2.4) will be needed in §2 for theorems on linearization of actions.

Most material in this section, except perhaps subsection 2.4, is well-known in one form or another, and can be omitted at first reading.
Notation. Throughout this section, the following notation will be used.

- **G**: any topological group (which may be subjected to conditions like local compactness, etc.).
- **Y**: any topological space (which may be specified to be \(\mathbb{R}\) or \(\mathbb{C}\), or which may otherwise be subjected to conditions of being a uniform space, etc.).
- **\(\mathcal{P}\)**: this is a short-hand notation for the mapping \(\mathcal{P}_X^G: G \times Y \to Y_G^G\), defined by \([[(\mathcal{P}_X^G)^t]](t) = f(st)\) for \(f \in Y^G\) and \(s,t \in G\) (right translations in \(Y^G\)).

We shall make an intensive use of the notational convention in 1.3.4: if \(A \subseteq Y^G\) and \(\mathcal{P}[G \times A] \subseteq A\) (i.e. \(A\) is right invariant) then the restriction and corestriction of \(\mathcal{P}\) to the domain \(G \times A\) and the codomain \(A\) will be denoted also by \(\mathcal{P}\).

2.1. Action of a group \(G\) on \(C_c(G,Y)\)

2.1.1. Obviously, \(\mathcal{P}\) is an action of \(G_d\) on the space \(Y^G\) with its discrete topology: in fact, it is easy to see that \(\mathcal{P}_G^e\) is the identity mapping on \(Y^G\), and \(\mathcal{P}_G^s = \mathcal{P}_G^t\) for all \(s,t \in G\). In addition, it is easy to see that \(C(G,Y)\) is a right invariant subspace of \(Y^G\).

2.1.2. Proposition. The mapping \(\mathcal{P}: G \times C_c(G,Y) \to C_c(G,Y)\) is separately continuous. Consequently, \(<G_d C_c(G,Y), \mathcal{P}>\) is a ttg.

Proof. If \(K \subseteq G\) is compact and \(U \subseteq Y\) is open, then for all \(t \in G\) we have (cf. 0.2.6 for notation):

\[
\mathcal{P}^t[\mathcal{N}(Kt,U)] = \mathcal{N}(K,U).
\]

This shows that \(\mathcal{P}^t: C_c(G,Y) \to C_c(G,Y)\) is continuous.

In addition, if \(f \in C_c(K,U)\), then \(K \subseteq f^t[\mathcal{N}]\), where \(f^t[\mathcal{N}]\) is open in \(G\).

By compactness of \(K\), there exists \(V \in \mathcal{U}_c\) such that \(KV \subseteq f^t[\mathcal{N}]\), hence \(\mathcal{P}^s f \in \mathcal{N}(K,U)\) for all \(s \in V\), that is, \(\mathcal{P}^s[V] \subseteq \mathcal{N}(K,U)\). It follows that \(\mathcal{P}\) is continuous at \(e\). By (1), it follows easily that \(\mathcal{P}^t\) is continuous at any point \(t \in G\).

2.1.3. Theorem. If \(G\) is a locally compact Hausdorff group, then \(\mathcal{P}: G \times C_c(G,Y) \to C_c(G,Y)\) is continuous, and, consequently, \(<G, C_c(G,Y), \mathcal{P}>\) is a ttg.
PROOF. By 2.1.2, each $\tilde{\rho}_f$: $G \rightarrow C_c(G,Y)$ is continuous ($f \in C_c(G,Y)$). So in view of 0.2.7(iii) it is sufficient to show that the mapping $f \mapsto \tilde{\rho}_f$: $C_c(G,Y) \rightarrow C_c(G \times C(G,Y))$ is continuous. Now the codomain of this mapping may be identified with $C_c(G \times G,Y)$ according to 0.2.7(iii). In doing so, $\tilde{\rho}_f$ corresponds to $f \circ \rho$, where $\rho(s,t) = ts$ for $s,t \in G$. So we have to prove that the mapping $f \mapsto f \circ \rho$: $C_c(G,Y) \rightarrow C_c(G \times G,Y)$ is continuous. This is easy and well-known (cf. [Du], Chap. XII, 2.1). □

2.1.4. If $G$ is not locally compact, then $\tilde{\rho}$ may be not continuous on $G \times C_c(G,Y)$. To see this, first observe that continuity of $\tilde{\rho}$: $G \times C_c(G,Y) \rightarrow C_c(G,Y)$ implies continuity of the evaluation mapping $\delta: (f,t) \mapsto f(t) = \tilde{\rho}_t f(e) \in C_c(G,Y)$, $G \times G \rightarrow Y$. By the result of ARENS, mentioned in 0.2.7, this is impossible if $G$ is not locally compact (e.g. $G = \mathbb{R}$) and $Y = [0,1]$.

2.1.5. If $G$ is not locally compact, there still are certain useful right invariant subspaces of $C_c(G,Y)$ on which $G$ acts continuously by means of $\tilde{\rho}$. For simplicity, let us assume from now on up to 2.1.11 that $Y$ is a uniform Hausdorff space with uniformity $U$.

2.1.6. LEMMA. The mapping $\tilde{\rho}$: $G \times C_c(G,Y) \rightarrow C_c(G,Y)$ is continuous on each set $G \times A$ with $A \subseteq C(G,Y)$ such that $A$ is equicontinuous at each point of $G$.

PROOF. By equicontinuity $A_c = A_p$ and in addition, the mapping $(f,t) \mapsto f(t)$; $A_c \times G \rightarrow Y$ is continuous (cf. 1.2.3(iii)). Therefore, the mapping $\tilde{\rho}: (f,s,t) \mapsto f(ts)$; $(A_c \times G) \rightarrow Y$ is continuous. So for any compact subset $K$ of $G$, the set $\{\Delta_t : t \in K\}$ of functions from $A_c \times G$ to $Y$ is equicontinuous (cf. 0.2.2(ii)).

Let $f \in A_c$, $s \in G$, and let $M(K,\alpha)$ be a typical element of the uniform base of $C_c(G,Y)$, where $K$ is a compact subset of $G$ and $\alpha \in \mu$. By equicontinuity of $\{\Delta_t : t \in K\}$ on $A_c \times G$ at the point $(f,s)$, there are neighbourhoods of $f$ in $A_c$ and of $s$ in $G$, say $U$ and $V$ respectively, such that $(\Delta_t g, \Delta_t f(s)) \in A_c$ for all $g \in U$ and $v \in V$, and all $t \in K$. That is, $(\rho^y_\mu, \rho^s F) \in M(K,\alpha)$ for all $(v,g) \in V \times U$. This proves continuity of $\tilde{\rho}$ on $G \times A_c$ at the point $(f,s)$. □

2.1.7. Recall that the orbit of $f \in C_c(G,Y)$ under the action $\tilde{\rho}$ of $G_d$ on $C_c(G,Y)$ is denoted by $C_d [f]$. The orbit-closure of $f$, that is, the closure of $C_d [f]$ in $C_c(G,Y)$ is denoted by $K_d [f]$.
2.1.8. **Lemma.** For \( f \in \mathcal{C}_c(G,Y) \) the following conditions are equivalent:

(i) \( f \in \text{RUC}(G,Y) \).
(ii) \( \mathcal{C}_p[f] \) is equicontinuous on \( G \).
(iii) \( \mathcal{K}_p[f] \) is equicontinuous on \( G \).
(iv) \( \mathcal{K}_p[f] \subseteq \text{RUC}(G,Y) \).

If these conditions are fulfilled then \( \tilde{\mathcal{P}}: G \times \mathcal{K}_p[f] \to \mathcal{K}_p[f] \) is continuous. If, in addition, \( f[G] \) is relatively compact in \( Y \), then \( \mathcal{K}_p[f] \) is a compact subspace of \( \text{RUC}(G,Y) \).

In that case, as a set and as a topological space, \( \mathcal{K}_p[f] \) equals the closure of \( \mathcal{C}_p[f] \) in \( Y^G \).

**Proof.**

(i) \( \Rightarrow \) (ii): An immediate consequence of the definitions.

(ii) \( \Rightarrow \) (iii): Cf. 0.2.8(i).

(iii) \( \Rightarrow \) (iv): If \( \mathcal{K}_p[f] \) is equicontinuous on \( G \), then for all \( g \in \mathcal{K}_p[f] \), \( \mathcal{C}_p[g] \subseteq \mathcal{K}_p[f] \), hence \( \mathcal{C}_p[g] \) is equicontinuous on \( G \), and \( g \in \text{RUC}(G,Y) \), by the implication (ii) \( \Rightarrow \) (i) above.

(iv) \( \Rightarrow \) (i): Obvious.

The other statements follow easily from 0.2.8, using the obvious observation that for each \( t \in G \), \( (\mathcal{C}_p[f])(t) = f[t] \).

2.1.9. **Proposition.** \( \text{RUC}^*(G,Y) \) is an invariant subset of the ttg \( <G, \mathcal{C}_c(G,Y), \tilde{\mathcal{P}}> \). Moreover, for each \( f \in \text{RUC}^*(G,Y) \), \( \mathcal{K}_p[f] \) is a compact invariant subset of \( \text{RUC}^*(G,Y) \), and \( \tilde{\mathcal{P}}: G \times \mathcal{K}_p[f] \to \mathcal{K}_p[f] \) is continuous. Hence \( <G, \mathcal{K}_p[f], \tilde{\mathcal{P}}> \) is a ttg with a compact Hausdorff phase space.

**Proof.** Use 2.1.6 and 2.1.8.

2.1.10. In general, \( \tilde{\mathcal{P}}: G \times \text{RUC}^*(G,Y) \to \text{RUC}^*(G,Y) \) is not continuous. In fact, if \( Y = [0,1] \) (consequently, also if \( Y = R \) or \( Y = C \)), then continuity of \( \tilde{\mathcal{P}} \) implies that \( G \) is locally compact. Cf. the remark preceding the final lemma in 0.2.7 and use the method of 2.1.4.

2.1.11. We have shown in 2.1.9 that each point of \( \text{RUC}^*(G,Y) \) has a compact orbit closure in \( \mathcal{C}_c(G,Y) \) under the action of \( G \) by \( \tilde{\mathcal{P}} \). It follows from 0.2.8 and the equivalence of (i) and (ii) in 2.1.8, that the converse is also true if \( G \) is a k-space. Thus, we obtain the following statement:

If \( G \) is a k-space (in particular, if \( G \) is a locally compact \( T_0 \)-space) then an element \( f \in \mathcal{C}_c(G,Y) \) has a compact orbit closure in the \( G \)-space \( \mathcal{C}_c(G,Y) \) (with action \( \tilde{\mathcal{P}} \)) if and only if \( f \in \text{RUC}^*(G,Y) \).

---

1) If \( f \in \text{RUC}(G,Y) \), then \( <G, \mathcal{K}_p[f], \tilde{\mathcal{P}}> \) is a ttg as well, but if \( f \) is not bounded, then \( \mathcal{K}_p[f] \) is not compact.
2.1.12. In the remainder of this section, let \(<G,Y,p>\) be an arbitrary ttg. Then, for every \(y \in Y\) and \(t \in G\),
\[
\rho^t(\pi_y) = \pi_{ty}
\]
(cf. the second diagram in 1.1.2). Consequently, the mapping
\[
\tau : y \mapsto \tau_y : Y \rightarrow C_c(G,Y)
\]
is equivariant with respect to the action \(\pi\) of \(G_d\) on \(Y\) and the action \(\tilde{\rho}\) of \(G_d\) on \(C_c(G,Y)\). In particular, it follows that \(\tau[Y]\) is a right invariant subset of \(C(G,Y)\).

2.1.13. LEMMA. The mapping \(\tau : Y \rightarrow C_c(G,Y)\) defined in (4) above is a topological embedding.

PROOF. For \(y,z \in Y\), \(y \neq z\), we have \(\pi_y(e) = y \neq z = \pi_z(e)\). Hence \(\tau\) is injective. Moreover, \(\tau\) is continuous, by 0.2.7(iii). In order to show that \(\tau\) is a topological embedding, it is sufficient to show that for each \(y \in Y\) and for each \(U \in \mathcal{V}_Y\), there exist a compact subset \(K\) of \(G\) and an open subset \(V\) of \(Y\) such that
\[
\{z \in Y : \tau(z) \in N(K,V)\} \subseteq U.
\]
Obviously, (5) is fulfilled if we take \(K = \{e\}\) and \(V = \text{int} \, U\).

2.1.14. PROPOSITION. The mapping \(\rho : G \times [Y]_c \rightarrow [Y]_c\) is continuous, so \(<G,\rho[Y]_c,\tilde{\rho}>\) is a ttg. Moreover, \(\tau : Y \rightarrow [Y]_c\) is an isomorphism of \(G\)-spaces.

PROOF. A trivial consequence of 2.1.12 and 2.1.13.

2.1.15. NOTES. The reader may have had the feeling that the proof of 2.1.3 as we have given it is somewhat obscure. We have chosen this proof in view of its generalization in 6.2.3 and 6.2.8. (A straightforward formulation of the proof of 2.1.3 is as follows (it is, indeed, exactly the same proof): proceed as in the second half of the proof of 2.1.2; since \(G\) is locally compact \(T_2\), \(V\) may be supposed to be compact, hence \(K\) is compact, and now \(KV \subseteq f^*[U]\) means that \(N(KV,U)\) is a neighbourhood of \(f\) in \(C_c(G,Y)\). By (1), however, \(f(V \times N(KV,U)) \subseteq N(f,U)\), so \(\tilde{\rho}\) is continuous at \((e,f) \in G \times C_c(G,Y)\).
Using (1) again, it follows that \(\tilde{\rho}\) is continuous on all of \(G \times C_c(G,Y)\).
Although 2.1.8 is well-known, the ttgs of the form \(<G,K_\rho[f],\tilde{\rho}>\) with
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\( f \in \text{RUC}^*(G,Y) \), \( Y \) a uniform space, are considered in the literature mainly under the (superfluous) assumption that \( G \) is locally compact \( T_2 \). In this context, the following references are worth to be mentioned: L. AUSLANDER & F. HAHN [1963], J. AUSLANDER & F. HAHN [1967], A.W. KNAPP [1964, 1966, 1967]. In these papers classes of functions on \( G \) are considered which have been defined by means of certain dynamical properties of \( \langle G, K_p \rangle \), mainly for the case that \( G = \mathbb{R} \). Cf. also J.F. KENT [1972], where arbitrary locally compact groups are considered. For related results, cf. J.D. BAUM [1953] and R. ELLIS [1959, 1961].

2.2. Action of a group \( G \) on \( C_u(G,Y) \)

2.2.1. Throughout this subsection we shall assume that \( Y \) is a uniform space with uniformity \( U \). Then \( C(G,Y) \) is a right invariant subset of \( Y^0 \), and we shall consider this space with its topology of uniform convergence on \( G \), i.e. we consider the space \( C_u(G,Y) \). Since for every \( t \in G \) obviously \( G = \{ s \in G \mid s0 \} \) it is clear that for \( a \in U \) we have

\[(1) \quad (p^a_t)_{M(a_o)} = M(a_o).\]

Consequently, \( \langle G, C_u(G,Y), \bar{p} \rangle \) is a ttg, and its transition group \( \{ p^a_t : t \in G \} \) is equi-uniformly continuous.

In general, \( \bar{p} : G \times C_u(G,Y) \to C_u(G,Y) \) is not continuous, not even in the case \( Y = \mathbb{R} \) or \( Y = \mathbb{C} \). This is an immediate consequence of:

2.2.2. PROPOSITION. Let \( f \in C(G,Y) \). The following conditions are mutually equivalent:

(i) \( \bar{p} : G \times C_u(G,Y) \to C_u(G,Y) \) is continuous.

(ii) For every \( t \in G \), \( \bar{p} : G \times C_u(G,Y) \to C_u(G,Y) \) is continuous at the point \( (t,f) \).

(iii) \( f \in \text{LUC}(G,Y) \).

PROOF. The straightforward proofs are left to the reader. \( \square \)

2.2.3. The preceding proposition remains true if we replace \( C(G,Y) \) by \( C^*(G,Y) \) and \( \text{LUC}(G,Y) \) by \( \text{LUC}^*(G,Y) \). In particular, if \( Y = \mathbb{R} \), then for any \( f \in C^*(G) \), the mapping \( \bar{p} : G \times C_u(G,Y) \to C_u(G,Y) \) is continuous iff \( f \in \text{LUC}^*(G) \).

2.2.4. PROPOSITION. The set \( \text{LUC}(G,Y) \) is right invariant and \( \bar{p} : G \times \text{LUC}_u(G,Y) \to \text{LUC}_u(G,Y) \) is continuous. Hence \( \langle G, \text{LUC}_u(G,Y), \bar{p} \rangle \) is a ttg.
PROOF. In view of the preceding proposition it is sufficient to prove that \( \tilde{\beta} u \in \text{LUC}(G,Y) \) for each \( f \in \text{LUC}(G,Y) \) and \( u \in G \). If such \( f \) and \( u \) are fixed, then for every \( \alpha \in U \) there exists \( V \in \mathcal{V} \) such that \( (f(t),f(s)) \in \alpha \) for all \( s,t \in G \) with \( t^{-1}s \in V \). Now there is \( W \in \mathcal{V} \) with \( u^{-1}Wu \subseteq V \). Consequently, if \( t^{-1}s \in W \), then \( (tu)^{-1}su \in V \), and \( (\tilde{\beta} u f(t),\tilde{\beta} u f(s)) \in \alpha \). Therefore, \( \tilde{\beta} u f \in \text{LUC}(G,Y) \). \( \square \)

2.2.5. We shall characterize now the elements in \( C_u(G,Y) \) having a compact orbit closure in \( C_u(G,Y) \) under the action of \( G_d \) by \( \tilde{\beta} \). To this end we introduce some new concepts. Although everything may be done for an arbitrary complete uniform space \( Y \) (completeness is essential in 2.2.13 below), we shall write down the proofs only for the case \( Y = \mathbb{F} \).

2.2.6. A function \( f \in C_u(G) \) is called Von Neumann almost periodic if \( C^u_\tilde{\beta}[f] \) is a relatively compact subset of \( C_u(G) \). The set of all Von Neumann almost periodic functions will be denoted by \( \text{AP}(G) \).

2.2.7. LEMMA. Let \( f \in C(G) \). The following conditions are equivalent:

(i) \( f \in \text{AP}(G) \).

(ii) There exist a compact topological Hausdorff group \( H \) and a continuous morphism of groups \( \psi \colon G \to H \) such that \( f = f' \psi \) for some \( f' \in C(H) \).

PROOF. (i) \( \Rightarrow \) (ii): If \( f \in \text{AP}(G) \), then the closure of \( C^u_\tilde{\beta}[f] \) in \( C_u(G) \) is a compact Hausdorff space. Let this space be denoted by \( X \). Observe that \( X \) is right invariant (indeed, each \( \tilde{\beta}^t \colon C_u(G) \to C_u(G) \) is continuous and leaves \( C^u_\tilde{\beta}[f] \) invariant); so we can consider the tgg \( <G_d,X,\tilde{\beta}> \). By 2.2.1, this tgg is equicontinuous, hence 1.3.16 implies that the enveloping semigroup \( E \) of \( <G_d,X,\tilde{\beta}> \) is a compact topological Hausdorff group. Obviously, \( f' \colon \mathbb{Z} \to \mathbb{Z} f(e) \): \( E \to \mathbb{F} \) is continuous, and \( f = f' \psi \). Since \( \mathbb{F} \) is a continuous morphism of groups, this shows that (i) implies (ii).

(ii) \( \Rightarrow \) (i): Since \( H \) is compact, \( \text{RUC}(H) = \text{LUC}(H) = C(H) \). So for any \( g \in C_u(H) \), \( \tilde{\beta}^g \colon H \to C_u(H) \) is continuous. In particular, \( \tilde{\beta}^g[H] \) is a compact subset of \( C_u(H) \). A straightforward calculation shows that the mapping \( C(\psi) \colon h \mapsto h \psi \colon C^u_\tilde{\beta}(H) \to C_u(G) \) sends \( \tilde{\beta}^g[\psi(g)] \) onto the orbit \( C^u_\tilde{\beta}[g \psi] \) of \( g \psi \) in \( <G_d,C_u(G),\tilde{\beta}> \). Hence \( C^u_\tilde{\beta}[g \psi] \) is included in the image of the compact set \( \tilde{\beta}^g[H] \) under the continuous mapping \( C(\psi) \). It follows that \( g \psi \in \text{AP}(G) \). \( \square \)
2.2.8. Instead of the action $\tilde{\lambda}$ of $G$ on $C_u(G)$ we can consider also the action $\tilde{\lambda}$, defined by $\tilde{\lambda} f(s) = f(t^{-1}s)$ for $f \in C(G)$ and $t, s \in G$. Then, similar to 2.2.7 it can be shown that the following conditions on $f \in C(G)$ are equivalent:

(i) $C_{\tilde{\lambda}}[f]$ is a relatively compact subset of $C_u(G)$.

(ii) There exist a compact topological Hausdorff group $H$ and a continuous morphism $\psi: G \to H$ of groups such that $f = f' \psi$ for some $f' \in C(H)$.

Combining this with 2.2.7 it follows that the "left" and the "right" versions of Von Neumann almost periodicity coincide: if $f \in C(G)$, then $f \in AP(G)$ iff $C_{\tilde{\lambda}}[f]$ has a compact closure in $C_u(G)$, iff $C_{\tilde{\lambda}}[f]$ has a compact closure in $C_u(G)$. For a different proof, see [HR], 18.1.

2.2.9. **Theorem.** The set of Von Neumann almost periodic functions equals the range of the mapping $C(a_G): C(G^c) \to C(G)$, where $a_G: G \to G^c$ denotes the Bohr compactification of $G$.

**Proof.** A straightforward consequence of 2.2.7 and the universal property of the Bohr compactification. \( \square \)

2.2.10. We shall present now another definition of almost periodicity which is, in general, not equivalent to the above defined concept, and for which the left and right versions can be different.

Call a subset $A$ of $G$ relatively dense in $G$ provided there exists a compact subset of $K$ of $G$ such that $G = KA$. Equivalently, $A$ is relatively dense in $G$ provided there exists a compact subset $K_1$ of $G$ such that for each $t \in G$, $A \cap K_1 t \neq \emptyset$ (in this case, $K$ and $K_1$ are related by $K_1 = K^{-1}$).

{In [GH] the term right syndetic is used.}\n
If $f \in C(G)$ and $\varepsilon > 0$, then the set of all $\varepsilon$-almost periods of $f$ is defined as follows:

$$A(f, \varepsilon) := \{ t \in G : \| \rho(f, \varepsilon) - f \| < \varepsilon \}. \tag{2}$$

Notice, that $A(f, \varepsilon)$ is a symmetric subset of $G$, that is, $t \in A(f, \varepsilon)$ iff $t^{-1} \in A(f, \varepsilon)$. Stated otherwise, $A(f, \varepsilon)^{-1} = A(f, \varepsilon)$.

An element $f \in C(G)$ is said to be right almost periodic provided $A(f, \varepsilon)$ is relatively dense in $G$ for every $\varepsilon > 0$. The set of all right almost periodic functions on $G$ will be denoted $RAP(G)$.\n

2.2.11. Let \( f \in C(G) \). Then \( f \in \text{RAP}(G) \) iff for every \( \varepsilon > 0 \) there exists a compact subset \( K(f,\varepsilon) \) of \( G \) such that

\[
(3) \quad \forall t \in G, \exists k \in K(f,\varepsilon) : \| \hat{\rho}^t - \rho_k \| < \varepsilon.
\]

Indeed, if \( f \in \text{RAP}(G) \), then for every \( \varepsilon > 0 \) there is a compact set \( K(f,\varepsilon) \subseteq G \) such that \( G = K(f,\varepsilon)A(f,\varepsilon) \). For each \( t \in G \), take \( k \in K(f,\varepsilon) \) such that \( t^{-1}k \in A(f,\varepsilon)^{-1} = A(f,\varepsilon) \). Then \( \| \hat{\rho}^{-1}k - \rho_k \| < \varepsilon \); since \( \hat{\rho}^t \) is an isometry, this is equivalent to \( \| \hat{\rho}^t - \rho_k \| < \varepsilon \). The proof of the reversed statement is left to the reader.

2.2.12. **Lemma.** \( \text{RAP}(G) \subseteq \text{RUC}^+(G) \).

**Proof.** Consider \( f \in \text{RAP}(G) \), and let \( \varepsilon > 0 \). Fix \( K := K(f,\varepsilon) \) in accordance with (3). For all \( t \in G \), (3) implies that \( |f(t)| \leq \| f \|_K + \varepsilon \), where \( \| f \|_K < \infty \).

It follows that \( f \) is bounded.

Next, apply 0.2.2(ii) to the continuous function \( (u,v) \mapsto f(vu) : G \times G \to \mathbb{F} \). There exists \( V \in V_\varepsilon \) such that \( |f(u)-f(vu)| < \varepsilon \) for all \( u \in K \) and \( v \in V \). If we choose \( k_v \in K \) for each \( t \in G \) in accordance with (3), then we have

\[
|f(vt) - f(t)| \leq |f(vt) - f(vk_v)| + |f(vk_v) - f(k_v)| + |f(k_v) - f(t)| < 3\varepsilon
\]

for every \( v \in V \). It follows that \( |f(s) - f(t)| < 3\varepsilon \) if \( s, t \in G \), \( st^{-1} \in V \). \( \square \)

2.2.13. **Lemma.** Let \( f \in C(G) \), and consider the following statements:

(i) The closure of \( C_\hat{\rho}[f] \) in \( C_u(G) \) is compact, i.e. \( f \in \text{AP}(G) \).

(ii) \( f \in \text{RAP}(G) \).

Then (i) implies (ii). If \( f \in \text{LUC}(G) \), then also (ii) implies (i).

**Proof.** (i) \( \Rightarrow \) (ii): Condition (i) is equivalent with total boundedness of \( C_\hat{\rho}[f] \) in the complete uniform space \( C_u(G) \). This, in turn, is equivalent with the existence, for every \( \varepsilon > 0 \), of a finite set \( K \subseteq G \) such that

\[
\forall t \in G, \exists k \in K : \| \hat{\rho}^t - \rho_k \| < \varepsilon.
\]

Since finite sets are compact, this shows that (i) \( \Rightarrow \) (ii), by 2.2.7.

Conversely, let \( f \in \text{RAP}(G) \cap \text{LUC}(G) \), and let \( \varepsilon > 0 \). Take \( K := K(f,\varepsilon) \) such that (3) holds. By 2.2.2, \( \hat{\rho}_f[K] \) is a compact subset of \( C_u(G) \), hence it is totally bounded. So there is a finite subset \( K_1 \) of \( K \) such that
\[ V_k \in K, \exists l \in K, : |f^k - p^l| < \varepsilon. \]

In view of (3), it follows that
\[ \forall t \in G, \exists k \in K, : |f^k - p^l| < 2\varepsilon. \]

Consequently, \( C_p[f] \) is totally bounded. \( \square \)

2.2.14. Instead of the action \( \hat{\rho} \) of \( G_d \) on \( C_u(G) \) we may also consider the action \( \lambda \), defined in 2.2.8. Then for \( f \in C_u(G) \) and \( \varepsilon > 0 \), set
\[ B(f,\varepsilon) := \{ t \in G : |f^t - f| < \varepsilon \}. \]

Then \( f \) is called \textit{left almost periodic} if \( B(f,\varepsilon) \) is relatively dense in \( G \) for every \( \varepsilon > 0 \). The set of all left almost periodic functions is denoted by \( \text{LAP}(G) \).

2.2.15. By similar methods as before, it may be shown that
(i) \( \text{LAP}(G) \subseteq \text{LUC}^*(G) \) (cf. 2.2.12).
(ii) For any \( f \in C(G) \), relative compactness of \( C_p[f] \) implies that \( f \in \text{LAP}(G) \).
The converse implication is valid whenever \( f \in \text{RUC}(G) \) (cf. 2.2.13).

2.2.16. \textsc{Theorem.} Let \( f \in C(G) \). Then the following conditions are mutually equivalent:
(i) \( C_p[f] \) is a relatively compact subset of \( C_u(G) \).
(i)' \( C_p[f] \) is a relatively compact subset of \( C_u(G) \).
(ii) \( f \in \text{RAP}(G) \cap \text{LUC}^*(G) \).
(iii) \( f \in \text{RUC}(G) \).
(iii)' \( f \in \text{RAP}(G) \cap \text{LUC}^*(G) \).
In particular, it follows that \( \text{AP}(G) = \text{LAP}(G) \cap \text{RAP}(G) \).

\textsc{Proof.} For the equivalence of (i) and (i)', cf. 2.2.8 above.
(i) \( \Rightarrow \) (iii): Use 2.2.13 and 2.2.15(ii) and the equivalence of (i) and (i)'.
(iii) \( \Rightarrow \) (ii) and (iii) \( \Rightarrow \) (ii)': Use 2.2.12 and 2.2.15(i).
(ii) \( \Rightarrow \) (i) and (ii)' \( \Rightarrow \) (i)': Use the converse implications in 2.2.13.
and 2.2.15(ii). \( \square \)

2.2.17. \textsc{Corollary.} If the right and the left uniform structures on \( G \) coincide, then the concepts of left almost periodicity, right almost periodicity and Von Neumann almost periodicity are all equivalent.

\textsc{Proof.} Immediate from 2.2.12, 2.2.15(i) and 2.2.16, since now obviously \( \text{LUC}^*(G) = \text{RUC}^*(G) \). \( \square \)
2.2.18. It may occur that LAP(G) ≠ RAP(G). Notice that for any \( f \in \text{LAP}(G) \) the function \( t \mapsto f(t^{-1}) \) is in RAP(G), and vice versa. So in case of inequality we must have \( \text{LAP}(G) \notin \text{RAP}(G) \) and \( \text{RAP}(G) \notin \text{LAP}(G) \). In addition, in that case the set of von Neumann almost periodic functions cannot coincide with \( \text{LAP}(G) \) nor with \( \text{RAP}(G) \). The following example is due to T.S. Wu [1966].

2.2.19. Let \( G \) be the semidirect product of a compact normal subgroup \( K \) and a subgroup \( H \). This means that \( G = KH \) and \( KnH = \{ e \} \) (cf. [HR], 2.6 and 6.20). Then each \( t \in G \) has a unique representation \( t = k_\tau h_\tau \) with \( k_\tau \in K \) and \( h_\tau \in H \). We shall assume that the mapping \( t \mapsto (k_\tau, h_\tau) : G \to K \times H \) is continuous (hence it is a homeomorphism).

If \( f \in C(K) \), then define \( f^\tau : G \to \mathbb{F} \) by \( f^\tau(t) = f(k_\tau) \). By our assumptions, it is clear that \( f^\tau \in \text{C}(G) \). First we show that \( f^\tau \in \text{RAP}(G) \) for every \( f \in C(K) \). To this end, observe that for every \( \epsilon > 0 \), \( h \in H \) and \( t \in G \) the inequality

\[
| f^\tau(h_{n}^{-1}kh_{n}) - f^\tau(h_{n}) | = 0 < \epsilon
\]

shows that \( \text{L}(f^\tau, \epsilon) \geq H \). Hence \( G = \text{KA}(f^\tau, \epsilon) \), i.e. \( \text{A}(f^\tau, \epsilon) \) is relatively dense in \( G \). So indeed \( f^\tau \in \text{RAP}(G) \).

Finally, let us assume that \( G, K \) and \( H \) satisfy the following condition:

\[
G \text{ is metrizable and there exist } k \in K, k \neq e, \text{ and a sequence } \{ h_{n} : n \in \mathbb{N} \} \text{ in } H \text{ such that } \lim_{n \to \infty} h_{n}^{-1}kh_{n} = e.
\]

Now for any \( f \in C(K) \) with \( f(k) \neq f(e) \) we have

\[
| f^\tau(h_{n}^{-1}kh_{n}) - f^\tau(h_{n}) | = | f^\tau(kh_{n}) - f^\tau(h_{n}) | = | f(k) - f(e) | \neq 0,
\]

hence \( f^\tau \notin \text{LUC}(G) \). So by 2.2.15, \( f^\tau \notin \text{LAP}(G) \).

The only thing that remains is to give an example of a group \( G \) which satisfies all conditions above. To do this, we proceed as follows:

As a topological space, let \( G := \mathbb{T}^{2} \times \mathbb{Z} \). Let \( \psi \) denote any continuous automorphism of \( \mathbb{T}^{2} \), \( \psi \neq \text{id}_{\mathbb{T}^{2}} \), and define a multiplication in \( G \) by

\[
(u, m)(v, n) = (u\psi^{m}(v), m+n).
\]

Then \( G \) is a metrizable topological group, and \( G \) is the semidirect product of the compact normal subgroup \( K := \mathbb{T}^{2} \times \{0\} \) and the closed subgroup
groups,

K. DE LEEUW & I. [1961]

J.S. PYM [1963].

for instance E.M. ALFSEN & P. HOLM [1962],

HOLM [1964],

and for semi­ periodic functions has grown.

Cf.

to define almost periodic functions on arbitrary topological groups. Since then, the literature on almost periodic functions has grown enormously. Cf.

for instance E.M. ALFSEN & P. HOLM [1962], P. HOLM [1964], and for semi­

H := \{e', \mathbb{Z}\}, where e' = (1,1) is the unit of \(T^2\). Now G meets all require­
ments, except possibly (4).

We shall show now that we can choose \(\psi\) is such a way that \(\lim_{t \to \infty} \psi^n(v) = e'\)
for some \(v \in T^2, v \neq e'\), and some sequence \(\{n_i : i \in \mathbb{N}\}\) in \(\mathbb{Z}\). For then,
setting \(k := (v,0)\) and \(h_i := (e',1)^{-n_i} = (e', n_i)\), we obtain

\[h_i^{-1} kh_i = (e', n_i)(v,0)(e',-n_i) = (e', n_i)(v, n_i) = (\psi^n(v), 0)\]

and, consequently, \(\lim_{i \to \infty} h_i^{-1} kh_i = (e',0)\), the identity of \(G\). Thus, condition (4) is also satisfied.

Finally we show that \(\psi\) has the above mentioned property if we take

\(\psi(t_1, t_2) := (t_1^2, t_2, t_1 t_2)\) for \((t_1, t_2) \in T^2\). Obviously, \(\psi\) is a continuous
automorphism of \(T^2\). Then we can apply [GH], 12.28 to the effect that
there exists \(v \in T^2\) and a sequence \(\{n_i : i \in \mathbb{N}\}\) in \(\mathbb{Z}\) such that \(\lim_{i \to \infty} \psi^n(v) = e'\),
as desired. (In order that this theorem can be applied, it must be checked
that \(e' \in T^2\) is invariant under \(\psi\), and that the neighbourhood
\(U := \{(t_1, t_2) \in T^2 : |t_1| = \exp(\alpha)\}
\]
with \(|\alpha| < \pi/3\) of \(e'\) satisfies the condition that
none of its points, except \(e'\), has its orbit under \(\psi\) completely in \(U\).)

2.2.20. NOTES. In view of proposition 2.2.2 it is natural to ask when it
occurs that \(\text{LUC}(G,Y) = C(G,Y)\). Of course, a sufficient condition is that \(G\)
is compact. In the case \(Y = \mathbb{R}\) more can be said: then for any topological
the conditions \(\text{LUC}(G) = C(G)\) and \(\text{LUC}^*(G) = C^*(G)\) are equivalent,
and they imply that either \(G\) is pseudocompact (that is, \(C(G) = C^*(G)\)) or \(G\)
is a P-space (that is, each countable intersection of open sets in \(G\) is open
in \(G\)). Cf. W.K. COMFORT & K.A. ROSS [1966]. In that paper it has also been
shown that pseudocompactness of \(G\) is equivalent with the property that
\(\text{AP}(G) = C(G)\). See also Appendix A.

The definition of almost periodic functions which we have employed in
2.2.10 is a straightforward generalization of the original definition in
H. BOHR [1924]. We borrowed it from [GH], Chap. 4. Notice that 2.2.10
through 2.2.16 are adapted from [GH], 4.58-4.61. Theorem 2.2.16 is a general­
ization of a characterization of almost periodic functions for the case
\(G = \mathbb{R}\), due to S. BOCHNER [1926]. It was used by VON NEUMANN [1934]
to define almost periodic functions on arbitrary topological groups. Since then, the literature on almost periodic functions has grown enormously. Cf.
for instance E.M. ALFSEN & P. HOLM [1962], P. HOLM [1964], and for semi­
groups, K. DE LEEUW & I. GLICKSBERG [1961] and J.S. PYM [1963]. In this
context, also W. P. EBERLEIN [1949] should be mentioned. For almost periodic functions on groups, cf. also [Ma]; [BH] and [Bu] deal with almost periodic functions on semigroups and the relation with compactifications of semi­
groups (i.e. generalizations of 2.2.9).

The proof of 2.2.9 as we have presented it follows roughly the lines of Theorem 16.2.1 in [Di] (cf. also [We], §§33-35). It can also be shown that a compactification \( \psi: G \rightarrow H \) of \( G \) (i.e. \( H \) a compact \( T_2 \)-group, \( \psi \) a con­tinuous morphism of groups with dense range) such that \( C(\psi)[C(H)] = AP(G) \) essentially equals the Bohr-compactification of \( G \). Cf. the papers of ALFSEN and HOLM and cf DE LEEUW and GLICKSBERG mentioned above, or the last chapter in [Lo]. For a very simple proof, applying both to the group and the semi­
group case, cf. J. DE VRIES [1970]. {All these references deal with the Von Neumann definition of almost periodicity: one considers points with compact orbit closures in \( \langle G_d, C(G), \beta \rangle \), where \( C(G) \) is given some suitable topology.}

2.3. Action of a locally compact Hausdorff group \( G \) on \( L^p(G) \) for \( 1 \leq p < \infty \)

2.3.1. In this section, \( G \) is a locally compact Hausdorff group with a fixed right invariant Haar measure \( \mu \). If \( G \) happens to be compact, we take \( \mu \)
normalized, i.e. \( \mu(G) = \int_G 1 \text{d} \mu = 1 \). If \( f \) is an extended real- or complex valued function on \( G \), then we shall often write \( \int_G f(t) \text{d} \mu(t) \) instead of \( \int_G f \text{d} \mu(t) \) whenever this expression has a meaning.

Let \( 1 \leq p < \infty \), and let \( L^p(G) \) be the set of all extended real- or complex valued measurable functions \( f \) on \( G \) such that \( \|f\|_p := \left( \int_G \|f\|^p \text{d} \mu \right)^{1/p} \) is finite. It is well-known that, with the usual pointwise operations, \( L^p(G) \) is a linear space and that \( \| \cdot \|_p \) is a pseudo-norm on it. Given \( f, g \in L^p(G) \), we have \( \| f - g \|_p = 0 \) iff \( f(t) = g(t) \) almost everywhere on \( G \), iff \( f(t) = g(t) \) locally almost everywhere on \( G \). Let \( N := \{ fe \in L^p(G) : \|fe\|_p = 0 \} \). Then \( N \) is a linear subspace of \( L^p(G) \), and \( L^p(G) := L^p(G)/N \) is a Banach space with its usual quotient norm. As is usually done, the elements of \( L^p(G) \) will be denoted by their representatives in \( L^p(G) \). So we will frequently refer to a function \( f \in L^p(G) \), and it will be clear from the context in every case whether we mean the fixed function \( f \) or the equivalence class \( f + N \) containing \( f \).

By right-invariance of \( \mu \) and the fact that each right translation \( s \mapsto
st: $G + G$ is a homeomorphism, it follows that $L^p(G)$ is a right invariant subset of $\mathcal{M}^G$, and that $\tilde{p}^t f_p = f_p$ for each $f \in L^p(G)$. In particular, $N$ is right-invariant, and it follows easily that $\tilde{p}^t [f+N] = \tilde{p}^t f + N$. Therefore, we can define $\tilde{p}^t f$ for $t \in G$ and $f \in L^p(G)$ in an obvious way (cf. also 1.5.3). Thus, we obtain for each $t \in G$ a linear isometry $\tilde{p}^t: L^p(G) \to L^p(G)$. It is clear that $\tilde{p}$ is an action of $G$ on $L^p(G)$, i.e. $<G, L^p(G), \tilde{p}>$. Therefore, we can define $p f$ for $t \in G$ and $f \in L^p(G)$ in an obvious way (cf. also 1.5.3). Thus, we obtain for each $t \in G$ a linear isometry $p_p: L^p(G) \to L^p(G)$. It is clear that $p$ is an action of $G$ on $L^p(G)$, i.e. $<G, L^p(G), p>$ is a tng.

2.3.2. LEMMA. For each $f \in L^p(G)$, the mapping $\tilde{p}_f: G \to L^p(G)$ is continuous.

PROOF. Continuity of each $\tilde{p}_f$ means that the mapping $t \mapsto \tilde{p}^t: G \to GL(L^p(G))$ is continuous when its codomain is given the strong operator topology (i.e. the topology of pointwise convergence). It is well-known that this mapping is continuous: cf. [HR], 20.4. □

2.3.3. PROPOSITION. For each $p \geq 1$, the mapping $\tilde{p}: G \times L^p(G) \to L^p(G)$ is continuous, and $<G, L^p(G), \tilde{p}>$ is an effective tng.

PROOF. The continuity of $\tilde{p}$ follows from the inequality

$$1 \tilde{p}(t,g)-\tilde{p}(s,f)_p \leq 1 \tilde{p}^t(g)-\tilde{p}^s(f)_p + 1 \tilde{p}_f(t)-\tilde{p}_f(s)_p$$

for $s,t \in G$ and $f,g \in L^p(G)$. Indeed, each $\tilde{p}^t$ is an isometry and each $\tilde{p}_f$ is continuous.

In order to show that $<G, L^p(G), \tilde{p}>$ is effective, observe that for every $t \in G$, $t \neq e$, there exists $U \in V_e$ such that $t \notin U U^{-1}$. Since $G$ is a locally compact $T_2$-space there is $f \in C_00(G)$, $f \neq 0$, such that $supp(f) \subseteq U$. Then

$$(1) \quad 1 \tilde{p}^t f-f_p = \int_G |f(st)-f(s)|^p ds = \int_{U t^{-1}} |f(st)|^p ds + \int_U |f(s)|^p ds.$$ 

Therefore, $\tilde{p}^t f \neq f$, hence $\tilde{p}^t \neq p^e$. □

2.3.4. It is not difficult to construct examples showing that in general $<G, L^p(G), \tilde{p}>$ is not strongly effective. On the other hand, there are no $\tilde{p}$-invariant points $\neq 0$ in $L^p(G)$ unless $G$ is compact.

Indeed, if $G$ is compact, then all constant functions are in $L^p(G)$, and constant functions are clearly invariant under the action $\tilde{p}$. If $G$ is not compact, then no constant function $\neq 0$ is in $L^p(G)$, otherwise we would have $\mu(G) < \infty$, contradicting the non-compactness of $G$ (cf. [HR], 15.9).
However, this does not prove our claim that $0$ is the only $\beta^t$-invariant point in $L^p(G)$ for non-compact groups $G$.

The difficulty in proving this claim is the following one: the condition $\beta^t f = f$ for all $t \in G$ means in the context of $L^p(G)$ that for every $t \in G$ there is a local null set $N_t$ such that $f(st) = f(s)$ for all $s \in G \setminus N_t$. We would like to show that there exists $s \in \mathcal{N}(0; N_t : t \in G)$; then $f(st) = f(s)$ for all $t \in G$, and $f$ would be constant. Actually, a little bit less is needed, and that can be shown using FUBINI's theorem: if for some $f \in L^p(G)$ we have $\|\beta^t f - f\|_p = 0$ for all $t \in G$, then

$$\int \int |f(st) - f(s)|^p dt \, ds = \int \int |f(st) - f(s)|^p ds \, dt$$

$$= \int \|f - f\|_p^p dt = 0.$$

It follows, that $\|f(st) - f(s)|_p^p dt = 0$ for almost all $s \in G$. Fix such an $s \in G$: then $|f(st) - f(s)|_p^p = 0$ for almost all $t \in G$. Consequently, $f(u) = f(s)$ for almost all $u \in G$, and $f$ may be assumed to be a constant function in $L^p(G)$. Hence $f = 0$ by the above remarks.

2.3.5. Suppose $GL(L^p(G))$ is given its strong operator topology. Then the mapping $t \mapsto \beta^t: G \to GL(L^p(G))$ is a topological embedding. Consequently, the transition group $\{\beta^t : t \in G\}$ with its point-open topology is a topological group.

Since the point-open topology on $\{\beta^t : t \in G\}$ is just the relative topology of this set in $GL(L^p(G))$, it is sufficient to prove the first statement. Obviously, the mapping $t \mapsto \beta^t: G \to GL(L^p(G))$ is continuous. To prove that it is relatively open, it is sufficient to show that for every $V \in \mathcal{V}_\varepsilon$ there exist $\varepsilon > 0$ and a finite subset $F$ in $L^2(G)$ such that

$$\|f - f\|_p < \varepsilon \quad \text{for all } f \in F \implies t \in V.$$  

This is easy: take $U \in \mathcal{V}_\varepsilon$ such that $UU^{-1} \subseteq V$, take $P = \{f\}$ with $f$ as in the proof of 2.3.3, and take $\varepsilon = \|f\|_p$. Then for all $t \in G$, $t \not\in V$, we have by formula (1) in 2.3.3:
lf(s)

Hence (2) is valid for our choice of ε and F.

2.3.6. For p = 1, 2, the tgs (G, L^p(G), 5), or rather their transition mappings \( \tilde{5}: G \rightarrow GL(L^p(G)) \), play an important role in representation theory of locally compact groups. Cf. [HR], Chap. V. We cannot go into details here, but we wish to make the following remarks.

For p = 1 (and if G is compact, also for p = 2), the tgs \( <G, L^p(G), \tilde{5}> \) has the additional structure of a Banach algebra. Multiplication is provided by convolution: if \( f, g \in L^1(G) \), then \( f \ast g \) is defined by

\[
(f \ast g)(s) := \int \limits_G f(t) g(t^{-1}s) \, dt
\]

(this expression has a meaning for almost every \( s \in G \), and if we take \( f \ast g(s) = 0 \) for all other \( s \))\(^1\), then \( f \ast g \in L^1(G) \). The relation between convolution and right translation is as follows:

\[
\tilde{5}^t(f \ast g) = f \ast \tilde{5}^t g
\]

for all \( t \in G \). The straightforward proof of (4) is left to the reader.

For \( p = 2 \), the space \( L^p(G) \) has the additional structure of a Hilbert space, its inner product being defined by

\[
(f | g) := \int \limits_G f(t) \overline{g(t)} \, dt
\]

for \( f, g \in L^2(G) \) (the horizontal bar denotes complex conjugation). Here each \( \tilde{5}^t \) is a unitary operator.

We shall present now two theorems, one for the Banach algebra \( L^1(G) \) and one for the Hilbert space \( L^2(G) \), which are both based on proposition 1.1.21.

2.3.7. Recall that an approximate unit in \( L^1(G) \) is a set \( \{f_\lambda : \lambda \in I\} \) in \( L^1(G) \) such that \( \lim \|f \ast f_\lambda - f\|_1 = 0 \) for each \( f \in L^1(G) \) (cf. [HR], 20.27).

The following is well-known and easily established by standard methods:

\(\text{If one allows representants of elements of } L^p(G) \text{ which are defined almost everywhere, then (3) suffices as a definition of } f \ast g.\)
\(L^1(G)\) has an approximate unit of the form \(\{f_v : v \in B\}\), where \(B\) is a local base for the neighbourhood system at \(e\) in \(G\), and \(f_v \in C_0(G)\) (in fact, \(\text{supp } f_v \subseteq V\); cf. [HR], 20.2).

2.3.8. **Lemma.** If \(\{f_1 : 1 \in I\}\) is an approximate unit in \(L^1(G)\), then the subset \(\{f_1 : 1 \in I\}\) of \(L^1(G)\) is a stabilizing set in \(\langle G, L^1(G), \rho \rangle\).

**Proof.** Suppose we have \(t \in G\) such that \(\rho^t f_1 = f_1\) for every \(1 \in I\). Then for each \(f \in L^1(G)\) we have
\[
\rho^t f = \lim_{1 \in I} \rho^t(f* f_1) = \lim_{1 \in I} f* \rho^t f_1 = \lim_{1 \in I} f* f_1 = f.
\]
Since \(\langle G, L^1(G), \rho \rangle\) is effective, it follows that \(t = e\). \(\square\)

2.3.9. **Proposition.** For any approximate unit \(\{f_1 : 1 \in I\}\) in \(L^1(G)\) we have \(|I| \geq \omega(G)\). Consequently, the least cardinal number of a directed set for an approximate unit in \(L^1(G)\) equals \(\omega(G)\).

**Proof.** Since \(L^1(G)\) is metrizable, its local weight is \(\aleph_0\). Moreover, if \(\{f_1 : 1 \in I\}\) is an approximate unit in \(L^1(G)\), then \(|I| \geq \omega(G, L^1(G), \rho)\) by 2.3.8. Now 1.1.21 implies that \(|I| \geq \omega(G)\).

The final statement in the proposition is a straightforward consequence of the first one and the observation in 2.3.7. \(\square\)

2.3.10. Recall that the dimension \(\delta(L^2(G))\) of the Hilbert space \(L^2(G)\) is the cardinal number of an orthonormal base for it.

Since all rational combinations of elements in an orthonormal base are dense in \(L^2(G)\) and, conversely, the base elements form a discrete subset in \(L^2(G)\), it is easy to see that
\[
\aleph_0 \cdot \delta(L^2(G)) = d(L^2(G)) = \omega(L^2(G)).
\]

Obviously, each \(\overline{\rho} : L^2(G) \to L^2(G)\), being continuous and linear is completely determined by its values at the elements of an orthogonal base. Consequently, each orthonormal base of \(L^2(G)\) is a stabilizing subset of \(\langle G, L^2(G), \rho \rangle\). In particular, \(\omega(G, L^2(G), \rho) \leq \delta(L^2(G))\).

Yet another well-known inequality regarding the dimension of \(L^2(G)\) is: \(\delta(L^2(G)) \leq \omega(G)\). For a proof, we refer the reader to the proof of Theorem 24.15 in [HR], which gives the desired result with only minor modifications. For compact groups it is known that \(\delta(L^2(G)) = \omega(G)\) (see [HR], 28.2). We
shall prove this equality for an arbitrary locally compact Hausdorff group $G$.

2.3.11. **Lemma.** $\omega(G) \leq \delta w(G) \cdot L(G)$.

**Proof.** Let $B$ denote a local base at $e$ with $|B| = \delta w(G)$. For each $V \in B$, let $F_V$ be a covering of $G$ with $L(G)$ left translates of $V$, then $U(F_V : V \in B)$ is a base for the topology of $G$, and the cardinality of this base is $|B| \cdot L(G) = \delta w(G) \cdot L(G)$. This proves the lemma. $\square$

2.3.12. **Lemma.** $L(G) \leq \delta(L^2(G))$.

**Proof.** If $G$ is finite, then $L(G) = |G| = \delta(L^2(G))$. So we may assume that $G$ is infinite. Then there exists a family $\mathcal{W}$ of pairwise disjoint, non-empty open subsets of $G$ such that $|\mathcal{W}| \geq L(G)$. Indeed, if $G$ is not sigma-compact, then take for $\mathcal{W}$ the family of all left cosets of an open, sigma-compact subgroup of $G$ (more details can be found in the first part of the proof of lemma 7.2.2). And if $G$ is sigma-compact, then let $\mathcal{W} := \{U_n^* : U_n \in G \text{ open} \}$ for some suitable sequence of open subsets $U_n$ in $G$.

For each $W \in \mathcal{W}$, let $f_W \in C_c(G)$, $0 \neq f_W \geq 0$, $\text{supp}(f_W) \subseteq W$. After suitable normalization, $(f_W : W \in \mathcal{W})$ is an orthonormal subset of $L^2(G)$, hence $\delta(L^2(G)) \geq |\mathcal{W}| \geq L(G)$. $\square$

2.3.13. **Corollary 1.** $\delta(L^2(G))$ is finite iff $G$ is finite.

**Proof.** If $G$ is finite, then $\delta(L^2(G)) = |G| < \aleph_0$. Conversely, if $\delta(L^2(G)) < \aleph_0$, then $L(G) < \aleph_0$, by 2.3.12. Now it is easy to see that $G$ is finite iff $L(G) < \aleph_0$. $\square$

2.3.14. **Corollary 2.** $\ell w(G) \leq \delta(L^2(G))$.

**Proof.** If $\delta(L^2(G)) < \aleph_0$ then $G$ is finite, by 2.3.13. Now $\ell w(G) = 1 \leq \delta(L^2(G))$. If $\delta(L^2(G)) = \aleph_0$, then 1.1.21 and 2.3.10 imply the desired inequality. $\square$

2.3.15. **Theorem.** For any locally compact Hausdorff group $G$ the equality $\omega(G) = \delta(L^2(G))$ is valid.

**Proof.** The inequality "$\leq$" was known before (cf. 2.3.10), and "$\geq$" follows from 2.3.11 through 2.3.14. $\square$

2.3.16. **Notes.** The first part of the proof of 2.3.3 is a special case of a much more general theorem, namely the following one: if $E$ is any Banach
space and $\pi: G \times E \to E$ is separately continuous, where each $\pi^t$ is a continuous linear operator on $E$, then the inequality $\|\pi(t,x) - \pi(s,y)\| \leq \|\pi^t - \pi^s\|_y$ shows that $\pi$ is continuous. Indeed, each $s \in G$ has a compact neighbourhood $U$, so that $(\pi^t z : t \in U)$ is compact, hence bounded in $E$, for each $z \in E$. Since $E$ is not a first category space, the principle of uniform boundedness (cf. [Sc], Chap.III, 4.2) implies that $\|\pi^t\| \leq k$ for all $t \in U$, where $k > 0$. Observe, that here local compactness of $G$ is quite essential.

Even more is known. Again, let $E$ be a Banach space, and let $E'_w$ denote $E$ with its weak topology (i.e. the $\sigma(E,E')$-topology). By the principle of uniform boundedness it is easy to see that a linear mapping $t: E \to E$ is continuous with respect to the norm topology iff $t: E'_w \to E'_w$ is continuous. Now the following can be shown: if $\pi: G \times E'_w \to E'_w$ is separately continuous and each $\pi^t: E \to E$ is linear, then $\pi: G \times E \to E$ is continuous. By the preceding remark, it is sufficient to prove that each $\pi^t: G \to E$ is continuous. This can be done, using a certain amount of integration theory; see K. De Leeuw & I. Glicksberg [1965], Theorem 2.8. A proof is also contained in [BH], p.41.

Proposition 2.3.9 is well-known: cf. [HR], 28.70(b); our proof seems to be a little bit simpler then the one suggested there. The inequality in lemma 2.3.11 is a special case of [Ju], 2.27; there it has been shown that we even have equality in 2.3.11. Corollary 2.3.13 is well-known (cf. [HR], 28.1) but our proof differs from that in [HR]. Finally, theorem 2.3.15 for general locally compact groups seems to be new. With only minor modifications, the proof carries over to $L^p(G)$ for $1 < p < \infty$. In that case, $\delta(L^p(G))$ should be defined as the least cardinal number of a discrete subset of $L^p(G)$ spanning a dense subspace of $L^p(G)$.

2.4. Weighted translations in $L^2(G)$

2.4.1. According to an idea of P.C. Baayen (cf. [Ba], Chap.IV; see also P.C. Baayen & J. de Groot [1968]) we wish to modify the $\tau_G < G, L^2(G) >$ by using a "weighted" translation instead of the mapping $\tau$. Notation will be as in the previous section. In particular, $G$ is a locally compact Hausdorff group.

2.4.2. A weight function on $G$ is an element $w \in L^2(G)$ satisfying the following conditions:
(i) For all \( t \in G \), \( w(t) > 0 \).
(ii) The function \( t \mapsto \frac{1}{w(t)} \colon G \to \mathbb{R} \) is bounded on compact subsets of \( G \).
(iii) For all \( s, t \in G \), \( w(st) \geq w(s)w(t) \).

2.4.3. For examples of weight functions and for a proof of the following lemma, we refer to Appendix B, in particular 3.2 through B.7.

2.4.4. **Lemma.** There exists a weight function on \( G \) iff \( G \) is sigma-compact. In that case, we may assume the existence of a lower semicontinuous weight function \( w \) on \( G \) such that \( w(t^{-1}) = w(t) \leq 1 \) for all \( t \in G \). \( \square \)

2.4.5. In the remainder of this section, \( G \) is a sigma-compact, locally compact Hausdorff group. Fix a weight function \( w \) on \( G \). For every \( t \in G \), let a mapping \( \sigma^t \colon L^2(G) \to L^2(G) \) be defined by

\[
(\sigma^t f)(s) = \frac{w(s)}{w(st)} f(st)
\]

if \( s \in G \) and \( f \in L^2(G) \). Observe that \( \sigma^t f \in L^2(G) \) for every \( t \in G \) and \( f \in L^2(G) \). Indeed, \( \sigma^t f \) is a measurable function, and it follows from the inequality

\[
|\sigma^t f(s)|^2 \leq \left( \frac{w(s)}{w(st)} \right)^2 |f(st)|^2 = \frac{|f(st)|^2}{w(t)^2}
\]

that \( |\sigma^t f|^2 \) is dominated by the integrable function \( s \mapsto w(t)^{-2}(\sigma^t f(s))^2 \).

2.4.6. **Lemma.** For each \( t \in G \), \( \sigma^t \) is a bounded invertible linear operator on the Hilbert space \( L^2(G) \), and for its operator norm the inequality \( \|\sigma^t\| \leq w(t)^{-1} \) is valid. If \( w \) is lower semicontinuous, then

\[
\frac{w(e)}{w(t)} \leq \|\sigma^t\| \leq \frac{1}{w(t)}.
\]

**Proof.** We know already that \( \sigma^t \) maps \( L^2(G) \) into itself. Plainly, \( \sigma^t \) is linear, and since formula (2) implies that

\[
\|\sigma^t f\|^2 \leq w(t)^{-2} \int_G |f(st)|^2 \, ds = w(t)^{-2} \|f\|^2,
\]

\( )^1 \)

\( )^1 \) If, in addition, \( w(e) = 1 \), then \( \|\sigma^t\| = w(t)^{-1} \). Notice that in this case \( w \) is continuous, by Appendix B.9.
it follows that $\sigma^t$ is bounded and $\|\sigma^t\| \leq w(t)^{-1}$.

Next, let $U$ be a compact symmetric neighbourhood of $e$ in $G$. In view of condition 2.4.2(ii), there exists a number $k(U) > 0$ such that

$$\forall s \in U : w(s) \geq k(U).$$

Then $w(t) = w(s^{-1}st) \geq w(s^{-1}) w(st)$, whence

$$\forall s \in U : w(st) \leq \frac{w(t)}{k(U)}.$$

Let $f \in C_0(G)$ with $\text{supp}(f) \subseteq U$ and $\|f\|_2 \neq 0$. Then we have by (4) and (5)

$$\|\sigma^tf\|_2^2 = \int_G \left( \frac{w(s)}{w(st)} \right)^2 |f(st)|^2 \, ds \geq \frac{k(U)}{w(t)^2} \|f\|_2^2,$$

and it follows that $\|\sigma^t\| \geq k(U)^2 w(t)^{-1}$. Observe, that this inequality is valid for each compact symmetric neighbourhood $U$ of $e$ in $G$. If $w$ is lower semicontinuous at $e$, then for each $\varepsilon > 0$ we can choose $U$ such that $k(U) \geq (1-\varepsilon)w(e)$. Then we obtain $\|\sigma^t\| \geq (1-\varepsilon)^2 w(e)^2 w(t)^{-1}$. This holds for every $\varepsilon > 0$, so that, indeed, $\|\sigma^t\| \geq w(e)^2 w(t)^{-1}$.

Finally, it is an easy calculation to show that $\sigma^e$ is the identity operator on $L^2(G)$ and that, for all $s,t \in G$, the equality $\sigma^{st} = \sigma^s \sigma^t$ is valid. From this it follows that $\sigma$ is an action of $G_d$ on $L^2(G)$. In particular, it follows that each $\sigma^t$ is invertible, and that $(\sigma^t)^{-1} = \sigma^{t^{-1}}$ is a bounded linear operator on $L^2(G)$ as well. $\square$

2.4.7. **Proposition.** The mapping $\bar{\sigma} : t \mapsto \sigma^t : G \to GL(L^2(G))$ is a morphism of groups and if $GL(L^2(G))$ is given its strong operator topology, it is a topological embedding.

**Proof.** The fact that $\bar{\sigma}$ is a morphism of groups has been indicated at the end of the proof of 2.4.6. So we shall confine ourselves to the proof that $\bar{\sigma}$ is a topological embedding of $G$ into $GL(L^2(G))$.

In order to show that $\bar{\sigma}$ is relatively open, it is sufficient to show that for each neighbourhood $U$ of $e$ in $G$ there exist a finite set $A \subset L^2(G)$ and a real number $\varepsilon > 0$ such that

$$\forall t \in G : \|\sigma^t f - f\| < \varepsilon \text{ for all } f \in A \Rightarrow t \in U.$$
The proof is similar to 2.3.5: let $V$ be a compact symmetric neighbourhood of $e$ in $G$ such that $V^2 \subseteq U$. There is $f \in C^0_\infty(G)$ with $\text{supp}(f) \subseteq V$ and $\|f\|_2 = 1$. Now for all $t \in G$, $t \notin U$ implies $V \cap V^{-1} = \emptyset$, hence

$$\|\sigma^t f - f\|_2^2 = \int\limits_V \left| \frac{w(s)}{w(st)} f(st) - f(s) \right|^2 ds$$

$$= \int\limits_V \left| \frac{w(s)}{w(st)} f(st) \right|^2 ds + \int\limits_V |f(s)|^2 ds$$

$$\geq \|f\|_2^2 = 1.$$
By 2.3.2 (for the case $p = 2$), there exists $V \in V_e$ such that for all $t \in V$

$$\left| \int_{G} |f(st) - f(s)|^2 \, ds \right| < \frac{\varepsilon^2}{4k}$$

and

$$\left| \int_{G} |w(s) - w(st)|^2 |f(s)|^2 \, ds \right| < \frac{\varepsilon^2}{4k}$$

where $\|f\|_G := \sup\{|f(s)| : s \in G\}$ is finite and non-zero. Hence for all $t \in U \cup V$ we obtain

$$\|t f - f\|_G^2 < 2k \left( \frac{\varepsilon^2}{4k} + \frac{\varepsilon^2}{4k} \|f\|_G^2 \right) = \varepsilon^2.$$ 

This shows, that $t \mapsto g^tf: G \to L^2(G)$ is continuous at $e$ for each $f \in C_{00}(G)$.

In the general case, take $f \in L^2(G)$, and let $\varepsilon > 0$. Since $C_{00}(G)$ is dense in $L^2(G)$, there exists $g \in C_{00}(G)$ such that $\|f-g\|_2 < \varepsilon/(2m)$, where $m := 1 + \sup(w(s)^{-1} : s \in U)$ (as before, $U$ is a fixed compact neighbourhood of $e$). Since for each $t \in G$ we have $|t f| \leq w(t)^{-1}$, it follows that for all $t \in U$

$$\|t g - g\|_2^2 \leq \|t f - f\|_2^2 + \|g - g\|_2^2 < \frac{\varepsilon}{2} + \|g - g\|_2^2.$$ 

By our previous result, there exists a neighbourhood $W$ of $e$ such that $\|t g - g\|_2 < \frac{\varepsilon}{2}$ for all $t \in W$. Consequently, $\|t f - f\|_2 < \varepsilon$ for all $t \in U \cup W$. \[\]
2.4.8. COROLLARY. The mapping $\sigma: G \times L^2(G) \to L^2(G)$ is continuous, hence it is an action of $G$ on $L^2(G)$. Moreover, the tupple $(G, L^2(G), \sigma)$ is effective, and the transition mapping $\delta: G \to GL(L^2(G))$ is a topological isomorphism if the transition group $\delta[G]$ is given its point-open topology.

PROOF. The fact that $\delta$ is a topological mapping is equivalent to saying that $\delta$ is a topological embedding of $G$ into $GL(L^2(G))$ if the latter space has its strong operator topology. So by 2.4.7 it remains only to show that $\sigma$ is continuous. To this end, consider for $f, g \in L^2(G)$ and $s, t \in G$ the inequality

$$||\sigma(t, f) - \sigma(s, g)||_2 \leq \|t - s\|_1 \cdot \|f - g\|_2 + \|t - s\|_1 \cdot \|g - \delta s f\|_2.$$  

Since each $s \in G$ has a compact neighbourhood on which the function $t \mapsto w(t)^{-1}$ is bounded, it follows easily, that $\sigma$ is continuous at each point $(s, g) \in G \times L^2(G)$. □

2.4.9. COROLLARY. Any sigma-compact locally compact Hausdorff group $G$ admits an embedding (as a group and as a topological space) $\sigma: G \to GL(L^2(G))$ in such a way that the function $w_0: t \mapsto \|\sigma(t)^{-1}\|$ is an upper semicontinuous weight function on $G$.

PROOF. Take any lower semicontinuous weight function $w$ on $G$ and construct $\delta: G \to GL(L^2(G))$ as before. Then $\delta$ is the desired embedding.

Indeed, it follows from (3) in 2.4.6 that:

$$w(t) \leq \frac{1}{\|t\|_1} \leq \frac{w(t)}{\|e\|^2}.\tag{7}$$

It follows that the function $w_0: t \mapsto \|\delta t^{-1}\|$ on $G$ satisfies the conditions (i) and (ii) of 2.4.2. In addition, for all $s, t \in G$ we have $\|\delta s^t\| = \|\delta s^{-1} t^{-1}\| \leq \|s^{-1}\| \|t^{-1}\|$, so $w_0$ satisfies 2.4.2(iii). It remains to show that $w_0 \in L^2(G)$ (then $w_0$ is a weight function) and that $w_0$ is upper semicontinuous.

To begin with the latter, we shall show that $\|t\|_1$ is lower semicontinuous on $L(L^2(G))$. Obviously this implies that $w_0$ is upper semicontinuous, because $t \mapsto \|t\|$ is a topological embedding of $G$ into $GL(L^2(G)) \subset L(L^2(G))$ (of course, we consider the strong operator topology on $L(L^2(G))$). If we write $\delta_s(t) := \tau(f)$ for any $\tau \in L(L^2(G))$ and any $f \in L^2(G)$ with $\|f\|_2 \leq 1$, then, by the definition of the norm on $L(L^2(G))$, [Details of the proof continue here.]
\[ |f| = \sup\{ |\delta' f(t)|^2 : f \in L^2(G) \} \]

Since each of the \( \delta' f : L^2(G) \to L^2(G) \) is continuous, and a pointwise supremum of continuous functions is lower semicontinuous, it follows that \( |f| \) is lower semicontinuous on \( L^2(G) \).

Finally, note that semicontinuity of \( w_0 \) implies its measurability. As \( w_0 \) is dominated by a scalar multiple of \( w \in L^2(G) \), it follows that \( w_0 \in L^2(G) \).\[2.4.10\]

There is a useful connection between the ttg \( \langle G, C^*_c(G), \beta \rangle \) and the ttg \( \langle G, L^2(G), \sigma \rangle \). If \( w \) is the weight function on \( G \), used in the definition of the action \( \sigma \) according to 2.4.5, let \( F : C^*_c(G) \to L^2(G) \) be defined by

\[ (8) \quad F(f)(t) := w(t) f(t) \]

for all \( f \in C^*_c(G) \) and \( t \in G \). Observe that \( \| F(f) \|_2 \leq \| w \|_2 \| f \|_G \) for all \( f \in C^*_c(G) \).

A straightforward calculation shows that \( F \) is equivariant. Moreover, \( F \) is injective. For if \( f \) and \( g \) are in \( C^*_c(G) \), \( f \neq g \), then there is an open subset of \( G \), i.e., a set with positive Haar measure, on which \( f \) and \( g \) differ from each other. Since \( w(t) \neq 0 \) for all \( t \in G \), it follows that \( F(f) \) cannot equal \( F(g) \) almost everywhere, i.e., \( F(f) \neq F(g) \).

Obviously, \( F \) is linear. Hence the inequality \( \| F(f) \|_2 \leq \| w \|_2 \| f \|_G \) for all \( f \in C^*_c(G) \) and the equality \( \| F(f) \|_2 = \| w \|_2 \| f \|_G \) for \( f = 1_G \) show, that \( F : C^*_c(G) \to L^2(G) \) is a continuous linear operator with operator norm \( \| F \| = \| w \|_2 \).

The following shows that \( F \) is not a topological embedding of \( C^*_c(G) \) into \( L^2(G)^1 \). In fact, \( F \) doesn't even induce a topological embedding of \( C_{00}(G) \) into \( L^2(G) \) unless \( G \) is a finite group. For suppose that \( F^* : F(C^*_c(G)) \to C_{00}(G) \) were a continuous linear operator. Then there would be a number \( c > 0 \) such that \( \| F(f) \|_2 \geq c \| f \|_G \) for all \( f \in C_{00}(G) \), that is,

\[ (9) \quad \int_G w(t)^2 |f(t)|^2 \, dt \geq c^2 \sup_{t \in G} |f(t)|^2. \]

Since \( G \) is sigma-compact, \( G = \bigcup \{ C_n : n \in \mathbb{N} \} \), where \( C_1 \subseteq C_2 \subseteq \ldots \) are compact subsets of \( G \). It follows, that

\[ (10) \quad \int_G w(t)^2 \, dt = \lim_{n \to \infty} \int_{C_n} w(t)^2 \, dt. \]

\[ )\] So by BANACH's homomorphism theorem (cf. [Sc], Chap.III, 2.1), \( F \) has not a closed range in \( L^2(G) \).
Hence there is an index $n$ such that

$$\int_{G-C_n} w(t)^2 \, dt < c^2.$$  

If $G-C_n \neq \emptyset$, there exists $f \in C_0(G)$ such that $\text{supp}(f) \subseteq G-C_n$ and $f \neq 0$. Now (9) and (11) imply

$$c^2 \sup_{t \in G} |f(t)|^2 \leq \int_{G-C_n} w(t)^2 |f(t)|^2 \, dt < c^2 \sup_{t \in G} |f(t)|^2,$$

a contradiction. Hence $G = C_n$, that is, $G$ is compact.

Next, let us assume that $G$ is not discrete. Then for each $m \in \mathbb{N}$ there is $U_m \in \mathcal{V}$ such that $\mu(U_m) < m^{-1}$. Indeed, by [HR] 19.21, $\mu(\{e\}) = 0$, hence $\inf\{\mu(U) : U \in \mathcal{V}\} = 0$ by regularity of $\mu$. Since $w(t) \leq 1$ for all $t \in G$, we can choose $m$ such that (11) is valid with $G-C_n$ replaced by $U_m$. As before, we obtain a contradiction. Thus, the assumption that $F$ is a topological embedding implies that $G$ is compact and discrete, hence finite. The converse is almost trivial, and its proof is omitted.

2.4.11. **Lemma.** If $A$ is a uniformly bounded subset of $C^0_c(G)$, then $F|_A : A \hookrightarrow L^2(G)$ is continuous. In particular, if $A$ is a compact, bounded subset of $C^0_c(G)$, then $F|_A : A \hookrightarrow L^2(G)$ is a topological embedding.

**Proof.** Suppose $\|f\|_G < \varepsilon$ for all $f \in A$. Fix $f \in A$, and let $\varepsilon > 0$. In view of formula (10) in 2.4.10, there exists a compact subset $K$ of $G$ such that

$$\int_{G-K} w(t)^2 \, dt < \frac{\varepsilon^2}{8k}.$$

Now for every $g \in A \cap \mathcal{U}_C(K,\delta)$ with $\delta := \frac{\varepsilon}{4w} \in \mathbb{N}$ we have

$$\|F(g)-F(f)\|_2^2 = \int_K |w(t)^2| |g(t)-f(t)|^2 \, dt + \int_{G-K} w(t)^2 |g(t)-f(t)|^2 \, dt$$

$$< \frac{\varepsilon^2}{4W^2} \|w\|_2^2 + 4k^2 \frac{\varepsilon^2}{8k} < \varepsilon^2.$$  

This shows that $F|_A$ is continuous at $f$.  

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Since a continuous injection of a compact space into a Hausdorff space is a topological embedding, the last statement of the lemma is clear. □

2.4.12. If $G$ is not compact, then compact subsets of $C_c^*(G)$ may be not uniformly bounded. The following example shows that $F|_A$ may be not continuous on $A$ if $A$ is required only to be compact in $C_c(G)$.

Let $G = \mathbb{R}$, and set $w(t) := \exp(-|t|)$ for $t \in \mathbb{R}$. Then $w$ is a weight function on $\mathbb{R}$. Define $A := \{f_n : n \in \mathbb{N}\}$ in $C_c^*(\mathbb{R})$ as follows:

\[
 f_n(t) := \begin{cases} 
 0 & \text{if } t \leq n \\
 (t-n) \exp(2n) & \text{if } n \leq t \leq n+1 \\
 \exp(2n) & \text{if } t \geq n+1
\end{cases}
\]

Then $\{f_n : n \in \mathbb{N}\}$ is pointwise bounded and equicontinuous on $\mathbb{R}$. In fact, the sequence $\{f_n : n \in \mathbb{N}\}$ converges to 0 in $C_c^*(\mathbb{R})$, so $\{f_n : n \in \mathbb{N}\} \cup \{0\}$ is a compact subset of $C_c^*(\mathbb{R})$.

In this situation, $F(f)(t) = f(t) \exp(-|t|)$ for $t \in \mathbb{R}$ and $f \in C_c^*(\mathbb{R})$. In particular,

\[
 F(f_n)(t) = \int_\mathbb{R} \exp(-2|t|) f_n(t)^2 \, dt 
\]

\[
 \geq \int_{[n+1, \infty]} \exp(-2|t|+2n) \, dt \]

\[
 = \frac{1}{2} \exp(2n-1).
\]

Therefore, the sequence $\{F(f_n) : n \in \mathbb{N}\}$ does not converge to 0 in $L^2(G)$. Consequently, $F$ is not continuous on the compact set $\{f_n : n \in \mathbb{N}\} \cup \{0\}$.

2.4.13. **Proposition.** If $A$ is a uniformly bounded, invariant subset of the ttg $\langle G, C_c^*(G), \mathfrak{D} \rangle$, then $F|_A : A \to L^2(G)$ is an injective morphism of $G$-spaces from $A_c$ (with action $\mathfrak{D}$) into $L^2(G)$ (with action $\mathfrak{A}$). If, in addition, $A_c$ is compact, then $F$ is a topological embedding.

**Proof.** Apply 2.4.10 and 2.4.11. □

2.4.14. For our purposes in §8, the ttg $\langle G, L^2(G), \mathfrak{D} \rangle$ is too small. Therefore, we shall consider now the "Hilbert sum" of copies of $\langle G, L^2(G), \mathfrak{D} \rangle$. To this
end, let $\kappa$ be a cardinal number, $A$ a set with $|A| = \kappa$, and let $H(\kappa)$ be the Hilbert sum of $\kappa$ copies of $L^2(G)$. Recall, that the elements of $H(\kappa)$ are just all elements $\xi = (\xi_A)_A \in L^2(G)^A$ for which the expression

$$\|\xi\| := \left( \sum_{a \in A} |\xi_A|^2 \right)^{1/2}$$

is finite. Then (12) defines a norm on $H(\kappa)$, and this norm can be derived from an inner product on $H(\kappa)$ which makes $H(\kappa)$ a Hilbert space.

For each $t \in G$ we have the bounded linear operator $\sigma^t$ on $L^2(G)$. Define $\sigma(\kappa)^t : H(\kappa) + H(\kappa)$ by $\sigma(\kappa)^t \xi := (\sigma^t \xi_A)_A$ for $\xi = (\xi_A)_A \in H(\kappa)$. Then $\sigma(\kappa)^t$ is easily seen to be a bounded linear operator with operator norm $\|\sigma(\kappa)^t\| = \|\sigma^t\|$. In fact, $\sigma(\kappa) : t \mapsto \sigma(\kappa)^t$ is a morphism of groups from $G$ into the group $GL(H(\kappa))$.

2.4.15. **Lemma.** If $GL(H(\kappa))$ is given its strong operator topology, then the mapping $\sigma(\kappa) : t \mapsto \sigma(\kappa)^t : G \to GL(H(\kappa))$ is a topological embedding.

**Proof.** Straightforward (use, among others, 2.4.7).

2.4.16. **Corollary.** For any cardinal number $\kappa$, the mapping $\sigma(\kappa) : G \times H(\kappa) \to H(\kappa)$ is continuous, hence it is an action of $G$ on the Hilbert space $H(\kappa)$.

Moreover, the ttg $<G,H(\kappa),\sigma(\kappa)>$ is effective, and the transition mapping $\delta : G \to [G]$ is a topological isomorphism if the transition group $\delta[G]$ is given its point-open topology.

**Proof.** The proof is similar to the proof of 2.4.8, since we have, again, local boundedness of the mapping $t \mapsto \|\sigma(\kappa)^t\|^{-1}$.

2.4.17. In contradistinction to the ttg $<G,L^2(G),\rho>$, the ttg $<G,L^2(G),\sigma>$ and its "Hilbert sums" $<G,H(\kappa),\sigma(\kappa)>$ do have invariant points $\neq 0$ (for the lack of non-trivial invariant points in $<G,L^2(G),\rho>$, see 2.3.4). The proof is more or less similar to the proof in 2.3.4. Indeed, we have:

2.4.18. **Proposition.** The set of invariant points in $<G,H(\kappa),\sigma(\kappa)>$ is homeomorphic to a Hilbert space of dimension $\kappa$. In fact, $\xi = (\xi_A)_A$ is an invariant point iff $\xi_A = \lambda_A w$ for scalars $\lambda_A \in \mathbb{C}$ such that $\Sigma |\lambda_A|^2 : a \in A < \infty$.

**Proof.** Obviously, it is sufficient to show that $f \in L^2(G)$ is invariant under the action $\sigma$ of $G$ iff $f = \lambda w$ with $\lambda \in \mathbb{C}$. Since a straightforward
calculation shows that $g^tw = w$ for all $t \in G$, it suffices to show that the condition \( \int_0^t f^t f_2 = 0 \) for all $t \in G$ implies that $t \mapsto f(t)w(t)^{-1}$: $G + \mathcal{I}$ is almost everywhere constant on $G$. The proof is an easy application of FUBINI's theorem: if \( \int_0^t f^t f_2 = 0 \) for (almost) all $t \in G$, then

\[
\begin{align*}
\int_G \left( \int_G \left| \frac{f(st)}{w(st)} - \frac{f(s)}{w(s)} \right|^2 dt \right) ds &= \int_G \left( \int_G \left| \frac{w(s)}{w(st)} f(st) - f(s) \right|^2 ds \right) dt \\
&= \int_G \left( \int_G f^t f_2^2 dt \right) ds = 0.
\end{align*}
\]

Consequently, for almost all $s \in G$ we obtain

\[
\int_G \left( \frac{f(st)}{w(st)} - \frac{f(s)}{w(s)} \right)^2 dt = 0.
\]

Fix such an $s \in G$. Then it follows that $f(st)w(st)^{-1} = f(s)w(s)^{-1}$ for almost all $t \in G$, i.e. there exists $c \in \mathcal{I}$ such that $f(u)w(u)^{-1} = c$ for almost all $u \in G$. \( \square \)

2.4.19. By 2.3.15, the dimension of $L^2(G)$ is just $w(G)$, the weight of $G$. Consequently, for any cardinal number $\kappa$, the dimension of $H(\kappa)$ equals $\kappa \cdot w(G)$. If $G$ is infinite, then $w(G) \geq \aleph_0$, hence in this case the dimension of $H(\kappa)$ equals $\max(\kappa, w(G))^1$.

2.4.20. NOTES. The contents of this subsection are needed in section 8. For comments on this material, we refer the reader to the notes in 8.2.17. Most of these results are also contained in J. DE VRIES [1972]. Corollary 2.4.9 forms a partial answer to a problem posed by P.C. BAAYEN in [Ba], p.144.

\[1\] This is equivalent with the statement which appears without proof at the bottom of p.372 in P.C. BAAYEN & J. DE GROOT [1968].
CHAPTER II
CATEGORIES OF TOPOLOGICAL TRANSFORMATION GROUPS

3 - THE CATEGORIES TTG AND TOP^G

In this section we investigate the category TTG. This category is obtained from the category TTG' (cf. 1.4.13) by taking into consideration also "actions" of groups on empty spaces. These additional objects of TTG shall be called ttgs too. In addition, we investigate the category TOP^G of all G-spaces for a fixed topological group G. These categories turn out to be isomorphic to categories of algebras over suitable monads. As a consequence, we find that TTG and TOP^G are complete categories. The limit of a diagram in TTG can be computed by computing the limit of the corresponding diagram of phase groups in TOPGRP and the limit of the corresponding diagram of phase spaces in TOP: then there exists a unique action of this "limit group" on this "limit space" producing the limit in TTG of the given diagram. The situation in TOP^G is similar. For colimits the situation is a little bit more complicated. In order to prove that TTG and TOP^G are cocomplete, we have first to consider "induced actions". This concerns a construction which generalizes the well-known construction of extending the action of a subgroup to an action of the whole group. It turns out that these induced actions have nice functorial and universal properties. Using this, it can be shown that TTG and TOP^G are cocomplete. However, the structure of colimits in TTG is rather complicated. In TOP^G the situation is a little bit simpler: modulo the topology, colimits can be computed in TOP; if G is locally compact, then even the right topology is obtained.

3.1. Limits in TTG

3.1.1. Recall that TTG' denotes the category with all ttgs as its objects and with morphisms as defined in 1.4.1. As composition the operation will
be used that has been defined in 1.4.3. In this subsection, we would like to investigate limits of diagrams in the category TTG'; in particular, we shall consider the categories TOPGRP and TOP as "known", so we shall be satisfied if we are able to express the behaviour of diagrams in TTG' in terms of the behaviour of corresponding diagrams in TOPGRP and TOP. The following convention will be convenient.

We shall consider TOPGRP simply as a subcategory of TOP. Thus, if G is an object in TOPGRP and X is an object in TOP, then the topological (cartesian) product G × X is just the product of G and X in the category TOP. We can express this convention also by saying that we shall suppress the forgetful functor TOPGRP → TOP.

In order to avoid difficulties, we have to extend the object class of TTG' with all objects of the form <G,∅,∅G> (G a topological group), where ∅G denotes the empty mapping from the empty set G × ∅ to the empty set. For each ttg <H,Y,∅> and each morphism ψ: G → H in TOPGRP we can consider the pair <ψ,∅G> as a morphism from <G,∅,∅G> to <H,Y,∅>; similarly, we have <ψ,∅H>: <G,∅,∅G> → <H,Y,∅H> (here ∅H: ∅ → Y is the unique mapping of ∅ into Y). In this way we obtain a category which properly contains TTG'; it will be denoted by TTG. Notice, that there exist no morphisms in TTG from <G,X,∅> to any <H,Y,∅H>, unless X = ∅. In the sequel, all terminology and all notions from §1 and §2, as far as they are meaningful, will be applied to the objects and morphisms in the extended category TTG.

3.1.2. Let the covariant functors G: TTG → TOPGRP and S: TTG → TOP ("forgetful" functors) be defined in the following way:

\[
G: \begin{cases} 
<G,X,∅> & \mapsto G \\
<ψ,∅> & \mapsto ψ 
\end{cases} \text{ on objects,}
\]

\[
S: \begin{cases} 
<G,X,∅> & \mapsto X \\
<ψ,∅> & \mapsto f 
\end{cases} \text{ on morphisms.}
\]

As usual, let TOPGRP × TOP denote the following category: objects are all pairs (G,X) with G and X objects in TOPGRP and TOP, respectively; morphisms are the pairs (ψ,f) with ψ and f morphisms in TOPGRP and TOP, composition of morphisms being defined coordinate-wise. Let G₀: TOPGRP × TOP → TOPGRP and S₀: TOPGRP × TOP → TOP denote the canonical projection functors. Then
there exists a unique covariant functor \( K: \text{TTG} \to \text{TOPGRP} \times \text{TOP} \) such that \( G_0K = G \) and \( S_0K = S \). Obviously, \( K \) is described by:

\[
K: \begin{cases} 
<\theta,X,\pi> \mapsto (\theta,X) & \text{on objects} \\
<\psi,f> \mapsto (\psi,f) & \text{on morphisms.}
\end{cases}
\]

3.1.3. The program suggested in 3.1.1 will partly be carried out by investigation of preservation and reflection properties of the functor \( K \). In this context, the following trivial observations are useful:

(i) A morphism \((\psi,f)\) in \( \text{TOPGRP} \times \text{TOP} \) is monic (epic) if and only if \( \psi \) is monic (epic) in \( \text{TOPGRP} \) and \( f \) is monic (epic) in \( \text{TOP} \). Similar for isomorphisms.

(ii) A diagram \( D: J \to \text{TOPGRP} \times \text{TOP} \) has a limit if and only if the diagrams \( G_0D: J \to \text{TOPGRP} \) and \( S_0D: J \to \text{TOP} \) both have a limit. If so, then \( (\psi,f): (G,X) \to D \) is a limiting cone in \( \text{TOPGRP} \times \text{TOP} \). Here, of course, \((\psi,f)_j := (\psi_j,f_j)\) for each object \( j \) in \( J \).

In particular, it follows that \( \text{TOPGRP} \times \text{TOP} \) is complete, since \( \text{TOPGRP} \) and \( \text{TOP} \) are complete. Similar statements hold with respect to colimits of diagrams, so that \( \text{TOPGRP} \times \text{TOP} \) is cocomplete.

3.1.4. Let \( \langle \psi,f \rangle \) be a morphism in \( \text{TTG} \). Then the following statements are true:

(i) \( \langle \psi,f \rangle \) is an isomorphism in \( \text{TTG} \) if and only if \( \psi \) is an isomorphism in \( \text{TOPGRP} \) and \( f \) is an isomorphism in \( \text{TOP} \). Thus, the isomorphisms of \( \text{ttg} \) defined in 1.4.1 are isomorphisms in \( \text{TTG} \).

(ii) If \( \psi \) and \( f \) are monic (epic) in \( \text{TOPGRP} \) and \( \text{TOP} \), respectively, then \( \langle \psi,f \rangle \) is monic (epic) in \( \text{TTG} \).

The proofs are easy: we leave them to the reader.

3.1.5. We shall see that the converse of 3.1.4 (ii) is also true: the functor \( K \) preserves all monomorphisms and all epimorphisms. Although we can give a direct proof for the case of a monomorphism, we prefer to obtain it as a corollary to 3.1.10 below, where we show that \( K \) has a left adjoint. However, see also 4.1.7 below. Since \( K \) cannot have a right adjoint (cf. 3.4.12 below), we have to proceed in a different way if we want to show that \( K \) preserves epimorphisms. This will be postponed to subsection 3.4, where we shall show first that \( G: \text{TTG} \to \text{TOPGRP} \times \text{TOP} \) has a right adjoint (cf. 3.4.9 and 3.4.10).
3.1.6. In the sequel, we shall denote the category \( \text{TOPGRP} \times \text{TOP} \) simply by \( C \). We shall construct a monad in \( C \) such that the corresponding category of algebras is isomorphic to \( \text{TTG} \). To this end, consider the functor \( H : C \to C \), defined in the following way:

\[
H : \begin{cases}
(G,X) \mapsto (G,G \times X) & \text{on objects} \\
(\psi,f) \mapsto (\psi,\psi \times f) & \text{on morphisms}
\end{cases}
\]

In addition, for each object \((G,X)\) in \( C \), let the morphisms

\[ \eta_{(G,X)} : (G,X) \to (G,G \times X) \]

and

\[ \mu_{(G,X)} : (G,G \times (G \times X)) \to (G,G \times X) \]

in \( C \) be defined by

\[ \eta_{(G,X)} := (1_G, \eta_X^G); \quad \mu_{(G,X)} := (1_G, \mu_X^G). \]

Here \( \eta_X^G \) and \( \mu_X^G \) are the mappings, defined in 1.1.1.

3.1.7. \textbf{Proposition.} The morphisms \( \eta_{(G,X)} \) and \( \mu_{(G,X)} \) in \( C \) for all objects \((G,X)\) in \( C \) form up two natural transformations,

\[ \eta : I_C \to H \quad \text{and} \quad \mu : H^2 \to H, \]

and \((H,\eta,\mu)\) is a monad in \( C \).

\textbf{Proof.} The straightforward verifications that \( \eta \) and \( \mu \) are natural transformations are left to the reader. The proof that \((H,\eta,\mu)\) is a monad now reduces to showing that for each object \((G,X)\) in \( C \) the following diagrams commute:
By definition, an $H$-algebra for the above defined monad $(H, \eta, \mu)$ is an ordered pair $((G, X), (\psi, \pi))$, consisting of an object $(G, X)$ and a morphism $(\psi, \pi): (G, G \times X) \to (G, X)$ in $C$ such that the following diagrams commute:

$$
\begin{align*}
& (G, X) \xrightarrow{(1_G, \mu_X)} (G, G \times X) \\
& (G, G \times X) \xrightarrow{(\psi, \pi)} (G, X)
\end{align*}
\begin{align*}
& (G, G \times X) \xrightarrow{(\psi, \pi)} (G, X) \\
& (G, X) \xrightarrow{(1_G, \mu_X)} (G, G \times X)
\end{align*}

So we need only that $G$ is a semigroup with unit (i.e. $G$ is a monoid).
The first of these diagrams requires that $\psi = 1_G$. Hence the condition that both diagrams commute is equivalent to the condition that the morphism
\[ \pi : G \times X \times X \text{ in } \TOP \] is an action of $G$ on $X$ (cf. diagram (2) in 1.1.1). Stated otherwise: the assignment $K_0 : (G, X, \pi) \mapsto ((G, X), (1_G, \pi))$ defines a bijection of the class of all objects in $\TTG$ onto the class of all $H$-algebras.

A morphism of $H$-algebras, from $((G, X), (1_G, \pi))$ to $((H, Y), (1_H, \sigma))$ is by definition a morphism $(\psi, f) : (G, X) \to (H, Y)$ in $C$ such that the following diagram commutes:

\[
\begin{array}{ccc}
(G, G \times X) & \xrightarrow{(1_G, \pi)} & (G, X) \\
| & \downarrow{(\psi, \psi \times f)} & | \\
(G, H \times Y) & \xrightarrow{(1_H, \sigma)} & (H, Y)
\end{array}
\]

This is equivalent to saying that $<\psi, f> : (G, X) \to (H, Y, \sigma)$ is a morphism in $\TTG$. Stated otherwise: the assignment $K_0 : <\psi, f> \mapsto (\psi, f)$ defines a bijection of each morphism set $\TTG(<G, X, \pi>, <H, Y, \sigma>)$ onto the corresponding set of all morphisms of $H$-algebras with domain $K_0<G, X, \pi>$ and codomain $K_0<H, Y, \sigma>$.

Since $K_0$ is easily seen to preserve compositions of morphisms, this proves most of the following

3.1.9. **Theorem.** There exists an isomorphism $K_0$ of categories from $\TTG$ onto the category $\CH$ of all $H$-algebras. If $G^H : \CH \to C$ denotes the usual forgetful functor, then the following diagram of functors commutes:

\[
\begin{array}{ccc}
\TTG & \xrightarrow{K_0} & \CH \\
\downarrow{G^H} & \downarrow{K} & \\
C & & C
\end{array}
\]

**Proof.** The first statement has been proved in 3.1.8. The second one is trivial, taking into account the definition of $K_0$. \(\square\)
3.1.10. **COROLLARY 1.** The functor $K: \mathbf{TTG} \to \mathbf{C}$ has a left adjoint. In addition, it creates all limits in $\mathbf{TTG}$.

**PROOF.** Apply 0.4.6 and 0.4.7, taking into account 3.1.9. □

3.1.11. **COROLLARY 2.** The category $\mathbf{TTG}$ is complete and the functor $K: \mathbf{TTG} \to \mathbf{C}$ preserves all limits and all monomorphisms.

**PROOF.** Completeness follows immediately from completeness of $\mathbf{C}$ (cf. 3.1.3) and the fact that $K$ creates all limits in $\mathbf{TTG}$. The preservation properties of $K$ are a consequence of its having a left adjoint (cf. 0.4.4(ii)). □

3.1.12. It follows from 3.1.11 and 3.1.4(ii) that a morphism $<\psi, f>$ in $\mathbf{TTG}$ is monic iff $\psi$ is monic in $\mathbf{TOPGRP}$ and $f$ is monic in $\mathbf{TOP}$. We may summarize this by saying that "monomorphisms in $\mathbf{TTG}$ can be calculated in $\mathbf{C}$".

Similarly, the behaviour of $K$ with respect to limits may be expressed by saying that "limits in $\mathbf{TTG}$ can be calculated in $\mathbf{C}$". That is, if $D: J \to \mathbf{TTG}$ is a diagram, then its limiting cone $<\psi, f>: <G_j, X_j, \eta_j> \to D$ is obtained by taking $\psi$ and $G$ such that $\psi: G \to GD$ is the limiting cone of the diagram $GD$ in $\mathbf{TOPGRP}$; in addition, $f: X \to SD$ is the limiting cone of the diagram $SD$ in $\mathbf{TOP}$; finally, $\pi$ is the unique action of $G$ on $X$ such that each $<\psi_j, f_j>: <G_j, X_j, \eta_j> \to D_j (j \in J)$ is a morphism in $\mathbf{TTG}$.

We shall present now a short description of products and equalizers in $\mathbf{TTG}$, using the above characterization:

(i) The product in $\mathbf{TTG}$ of a set $\{<G_j, X_j, \eta_j>: j \in J\}$ of its objects is the triple $<\mathbb{P}G, \mathbb{P}X, \pi>$, together with the projections $<\psi, f>$:

$<\mathbb{P}G_j, \mathbb{P}X_j, \eta_j> \to <G_j, X_j, \eta_j>$. Here $\mathbb{P}G_j$ and $\mathbb{P}X_j$ are the usual products in the categories $\mathbf{TOPGRP}$ and $\mathbf{TOP}$, and $\psi_j, f_j$ denote the usual projections. Moreover, $\pi$ is defined by

$$\pi((t_j, x_j)_j) := (\pi_j(t_j, x_j)_j).$$

(ii) The equalizer of a pair of morphisms $<\psi_1, f_1>, <\psi_2, f_2>: <G, X, \pi> \to <H, Y, \sigma>$ in $\mathbf{TTG}$ is the morphism $<\psi, f>: <K, Z, \pi|_{K \times Z} > \to <G, X, \pi>$ in $\mathbf{TTG}$, where $K := \{t \in G : \psi_1(t) = \psi_2(t)\}$; $Z := \{x \in X : f_1(x) = f_2(x)\}$.

1) We present here just a reformulation of "$K$ creates all limits in $\mathbf{TTG}$".
and \( \psi: K \to G, f: Z \to X \) are inclusion maps (i.e. equalizers of \( \psi_1, \psi_2 \) in TOPGRP and of \( f_1, f_2 \) in TOP, respectively).

The straightforward justifications of (i) and (ii) are left to the reader.

3.1.13. As an application and extension of the preceding results, we prove the following well-known fact:

Let \( \langle G, X, \pi \rangle \) be a ttg and let \( f': Y \to X/C_\pi \) be a continuous function. Then there exists a ttg \( \langle G, Z, \zeta \rangle \) and a morphism of \( G \)-spaces \( f: Z \to X \) such that

(i) \( Y \) is homeomorphic to \( Z/C_\zeta \).
(ii) If we identify \( Y \) with \( Z/C_\zeta \) according to (i), then \( f'c_\zeta = c_\pi f \).

**Proof.** Let \( \sigma \) and \( \tau \) denote the trivial actions of \( G \) on \( Y \), resp. \( X/C_\pi \). Then we have the following diagram in TTG (solid arrows only):

\[
\begin{array}{ccc}
\langle G, Z, \xi \rangle & \to & \langle G, X, \pi \rangle \\
\downarrow \langle G, \sigma \rangle & & \downarrow \langle G, \tau \rangle \\
\langle G, Y, \sigma \rangle & \to & \langle G, X/C_\pi, \tau \rangle
\end{array}
\]

The limit of this diagram is the object \( \langle G, Z, \zeta \rangle \), together with the dotted arrows in the above diagram. Here \( f: Z \to X \) and \( g: Z \to Y \) form the limiting cone of the diagram

\[
Y \xrightarrow{f'} X/C_\pi \xrightarrow{c_\pi} X
\]

in TOP. So we may take \( Z := \{(x, y) \in X \times Y : c_\zeta(x) = f'(y)\} \), and for \( f \) and \( g \) we may take the restrictions to \( Z \) of the projections of \( X \times Y \) onto \( X \) and \( Y \), respectively. The action \( \zeta \) of \( G \) on \( Z \) is given by (still according to 3.1.12) \( \zeta(t, (x, y)) := (\pi X, o^t_x) = (\pi X, y) \). Now some straightforward computations show that \( g: Z \to Y \) is an open mapping, and that \( g \) induces a bijection of \( Z/C_\zeta \) onto \( Y \). Then (i) and (ii) follow readily (cf. also 1.4.9 and notice that \( Y = Y/C \)).

3.1.14. According to 0.4.6 and 3.1.9, the left adjoint \( F: C \to \text{TTG} \) of the functor \( K: \text{TTG} \to C \) is the functor \( K^H \), where \( F_H \) is the left adjoint of \( G^H \).

To be concrete:
For any object \((G,X) \in C\), the free \(H\)-algebra is \((\text{H}(G,X),\mu_{(G,X)})\), which corresponds to the ttg \(<G,G \times X,\mu_X^G>\) (cf. 3.1.8, where \(K_0\) is explicitly described). This ttg will be called the free ttg for \(G\) and \(X\). Combining in a similar way the definitions of \(\mathcal{H}\) and \(K_0\) on morphisms, we obtain the following description of the left adjoint \(F\) of \(K\):

\[
\begin{align*}
F: & \{(G,X) \mapsto <G,G \times X,\mu_X^G> \quad \text{on objects} \\
& \{(\psi,f) \mapsto <\psi,\psi \times f> \quad \text{on morphisms.}
\end{align*}
\]

Translation of the remainder of 0.4.6 to the present situation gives the following results:

The unit of the adjunction of \(F\) to \(K\) is the natural transformation \(\eta: I_C + KF = H\). So for each object \((G,X)\) in \(C\) the arrow

\[
(1^G \eta_X^G): (G,X) \rightarrow (G,G \times X)
\]

is universal from \((G,X)\) to \(K\).

The counit of the adjunction of \(F\) to \(K\) is the natural transformation \(\xi: FK + ITTG\) defined by

\[
\xi_{<G,X,\pi>} := 1^G \pi^G: <G,G \times X,\mu_X^G> \rightarrow \langle G,X,\pi \rangle
\]

for each object \(<G,X,\pi>\) in \(TTG\).

(It may be comforting for a reader who doesn't like monads and algebras that it is easy to show directly that \((F,K,\eta,\xi)\) is an adjunction from \(C\) to \(TTG\); in addition, it can easily be shown that \(K\) creates limits (cf. also 4.1.3.).)

3.1.15. According to 0.4.9, the adjunction of \(F\) and \(K\), with unit \(\eta\) and counit \(\xi\) gives rise to the comonad \((FK,\xi,\eta_FK)\). The coalgebras for this comonad are the pairs \(<<G,X,\pi>,<\psi,f>>\) with \(<\psi,f>: <G,X,\pi> + FK<G,X,\pi> = <G,G \times X,\mu_X^G>\) a morphism in \(TTG\) such that the following diagrams commute:

\[
\begin{array}{c}
\xymatrix{ <G,X,\pi> \ar[r]<G,G \times X,\mu_X^G> \ar[d]<1^G \eta^G_X> & <G,G \times X,\mu_X^G> \\
<\psi,f> \ar[r]<\psi,\psi \times f> & <\psi,\psi \times f> \ar[u]<1^G \eta^G_X> \end{array}
\]

(1)
By (1), we obtain

\[ y(g(x)) = e; \quad g(g(x)) = g(x). \]

In addition, the condition that \( <1_G, f> : <G, X, \eta> \rightarrow <G, G \times X, \mu_X> \) is a morphism in \( TG \) implies that

\[ \gamma(\pi(t, x)) = t . \gamma(x); \quad g(\pi(t, x)) = g(x) \]

for all \( t \in G, x \in X \). If we set \( S := g[X] \) and \( \tau(x) := \gamma(x)^{-1} \), then (3), the first formula in (4) and the first formula in (5) imply that for each \( x \in X \), \( \tau(x) \) is the unique element of \( G \) for which \( \pi(\tau(x), x) \in S \).

Conversely, if we are given an object \( <G, X, \eta> \) in \( TG \), a subset \( S \) of \( X \), and a continuous function \( \tau : X \rightarrow G \) such that for each \( x \in X \), \( \tau(x) \) is the unique element of \( G \) for which \( \pi(\tau(x), x) \in S \), then we can define \( \gamma(x) := \tau(x)^{-1}, g(x) := \pi(\tau(x), x) \) and \( f(x) := (\gamma(x'), g(x)) \). If we do so, then (3), (4) and (5) can be derived (uniqueness of \( \tau(x) \) is essential!), and \( (G, X, \eta, <1_G, f>) \) is an FK-coalgebra.

A pair \( (S, \tau) \) as described above will be called a continuous cross-section of the tte \( (G, X, \eta) \). Now the first statement in the next proposition is just a reformulation of the preceding remarks:

3.1.16. PROPOSITION. The coalgebras for the comonad \( (FK, \xi, \Gamma\eta K) \) in \( TG \) are just the ttegs \( <G, X, \eta> \) admitting a continuous cross-section \( (S, \tau) \). If \( <G, X, \eta> \) (with continuous cross-section \( (S, \tau) \)) and \( <G', X', \eta'> \) (with continuous cross-section \( (S', \tau') \)) are two such coalgebras, then a morphism \( <\psi, f> : <G, X, \eta> \rightarrow <G', X', \eta'> \) can be derived if \( \psi \) is a morphism in \( TG \) and \( f(x) := \psi(\eta(x)) \).

\[^1\] Here terminology is a little bit confusing, because this concept of a cross-section is not the same as the one in 1.3.20. The latter concept is often called a continuous selection.
<G',X',n'> in TTG is a morphism of coalgebras iff f[S] ⊆ S' and, in addition, the following diagram commutes:

\[
\begin{array}{ccc}
G & \xrightarrow{\psi} & G' \\
\downarrow{\tau} & & \downarrow{\tau'} \\
X & \xrightarrow{\tau} & X'
\end{array}
\]

**Proof.** The characterization of the morphisms of coalgebras follows by a straightforward argument from the definitions in 0.4.9 and the remarks preceding our proposition. ⊣

3.1.17. If <G,X,η> is a ttg with continuous cross-section (S,τ), then \( x \mapsto (\tau(x)^{-1}, \#(\tau(x),x)) \) is a homeomorphism of X onto \( G \times S \); in fact, it defines an isomorphism in TTG of \( <G,X,\eta> \) onto \( <G_0 \times S, \eta_0 S> \). In addition, it is easy to see that in this way the category of all FK-coalgebras is isomorphic to the subcategory \( F[C] \) of TTG.)

3.1.18. **Notes.** It is well-known that group actions have a close connection with certain monads \( (H,\eta,\mu) \) and that, in fact, G-spaces can be obtained as H-algebras. This is not surprising, since the abstract definition of H-algebras was constructed on the model of group actions. To be precise, one of the original examples which were of interest to CODEMENT when he introduced the concept of a "standard construction" (which is the same as a monad) in [Go] was the monad, generated in the category of abelian groups by tensoring with a fixed ring \( R \). Moreover, in the paper of S. EILENBERG & J.C. MOORE [1965] where it was proved that each monad arises from a pair of adjoint functors, the construction of H-algebras was motivated by the example of the category of all A-modules. This is just the category of all H-algebras, where \( H \) is the functor in the category of modules over a commutative ring \( K \), defined by tensoring with the fixed K-algebra \( A \). Cf. also the motivation in [ML], p.137. Our functor \( H \), defined in 3.1.6 is just similar to these examples, except that we do not restrict ourselves to a fixed group \( G \); this is the only new point of view in this subsection. In addition, we were unable to find the observations of 3.1.16 in the literature. References to

\[\text{\textsuperscript{1}}\text{ It can be shown that this is not a full subcategory of TTG.}\]

\[\text{\textsuperscript{1}}\text{ It can be shown that this is not a full subcategory of TTG.}\]
the literature about continuous cross-sections in TTGs can be found in the notes to subsection 1.3.

3.2. Limits in TOP\(^G\)

3.2.1. In this subsection, the symbol \(G\) denotes always a given topological group. Then TOP\(^G\) denotes the subcategory of TTG defined by all objects having \(G\) as a phase group and all morphisms with \(1_G\) as a group component. Obviously, the morphisms in TOP\(^G\) are just the morphisms of \(G\)-spaces, defined in 1.4.1.\(^1\) For most groups \(G\), TOP\(^G\) is not a full subcategory of TTG.

Let \(S^G: TOP^G \rightarrow TOP\) be the restriction to TOP\(^G\) of the functor \(S\), defined in 3.1.2. Thus,

\[
\begin{align*}
S^G: \{<G,X,\sigma> \mapsto X \text{ on objects} \\
<1_G, f> \mapsto f \text{ on morphisms.}
\end{align*}
\]

3.2.2. Let \(<1_G, f>\) be a morphism in TOP\(^G\). Then the following statements are true:

(i) \(<1_G, f>\) is an isomorphism in TOP\(^G\) iff \(f\) is an isomorphism in TOP. Thus, the isomorphisms of \(G\)-spaces defined in 1.4.1 are isomorphisms in TOP\(^G\).

(ii) If \(f\) is monic \((\text{epic})\) in TOP, then \(<1_G, f>\) is monic \((\text{epic})\) in TOP\(^G\).

The easy proofs are left for the reader. We shall prove now first of all the converse of (ii) for epimorphisms.

3.2.3. PROPOSITION. A morphism \(<1_G, f>\) in TOP\(^G\) is epic iff \(f\) is epic in TOP; that is, \(S^G\) preserves and reflects epimorphisms.

PROOF. Reflection: cf. 3.2.2(ii).

Preservation \(^2\): Let \(<1_G, f>: <G,X,\sigma> \rightarrow <G,Y,\tau>\) be an epimorphism in TOP\(^G\). Then \(f[X]\) is an invariant subset of \(<G,Y,\tau>\) (cf. 1.4.5). Consequently, \(R := \mathcal{C}_G \cup (f[X] \times f[X])\) is an equivalence relation in \(Y\), which is invariant under the action \(\sigma\) of \(G\). Let \(Z := Y/R\) with its usual quotient topology, and let \(q: Y \rightarrow Z\) be the quotient mapping. In addition, let \(q': Y \rightarrow Z\) be the constant function with \(q'[Y] = q[f[X]]\). Finally, let \(r\) denote the trivial action of

---

\(^1\) According to our agreement in 3.1.1, TOP\(^G\) contains also the object \(<G,0,\sigma>\) and, for every \(G\)-space \(<G,Y,\tau>\), a morphism \(<1_G, \varphi>: <G,0,\varphi> \rightarrow <G,Y,\tau>\).

\(^2\) For alternative proofs, see 3.4.6 and 3.4.8 below.
G on Z. Then q and q' are morphisms of G-spaces, from Y with action σ to Z with action τ. Since \( <1_G,q><1_G,f> = <1_G,q'><1_G,f> \), where \( <1_G,f> \) is an epimorphism in TOP\(^G\), it follows that q' = q, whence \( f(X) = Y \). Therefore, f is an epimorphism in TOP. \( \square \)

### 3.2.4. Theorem

The functor \( S^G : \text{TOP}^G \to \text{TOP} \) has a left adjoint. In addition, it creates all limits in \( \text{TOP}^G \).

**Proof.** Completely similar to the proof of 3.1.10, so we shall present only a brief outline of it. Define a functor \( H^G : \text{TOP} \to \text{TOP} \) by

\[
H^G; \begin{cases} X \mapsto G \times X & \text{on objects} \\ f \mapsto 1_G \times f & \text{on morphisms.} \end{cases}
\]

Then we have natural transformations \( \eta^G_G : I_{\text{TOP}} \to H^G \) and \( \mu^G_G : (H^G)^G \to H^G \), where for each object \( X \in \text{TOP}, \eta_X^G \) and \( \mu_X^G \) are as in 1.1.1 (compare also 3.1.6). Then similar to 3.1.7, one shows that \( (H^G, \eta^G, \mu^G) \) is a monad. The category of all \( H^G \)-algebras may be identified with \( \text{TOP}^G \) in such a way that \( S^G : \text{TOP}^G \to \text{TOP} \) corresponds to the forgetful functor from the category of \( H^G \)-algebras to TOP. Now the theorem follows from 0.4.6 and 0.4.7. \( \square \)

### 3.2.5. Corollary

The category \( \text{TOP}^G \) is complete, and all limits can be calculated in \( \text{TOP} \), i.e. \( S^G \) creates and preserves all limits. In addition, \( S^G \) preserves and reflects all monomorphisms. \( \square \)

### 3.2.6. Theorem

The product in \( \text{TOP}^G \) of a set \( \{<G_j,X_j,\pi_j> : j \in J \} \) of its objects is the G-space \( <G,\prod_{j \in J} X_j,\pi> \), where π is defined by \( \pi(t,(x_j)_j) := (\pi_j(t,x_j))_j \). Moreover, the equalizer in \( \text{TOP}^G \) of a parallel pair of morphisms \( <1_G,f>,<1_G,g> : <G,X,\pi> \to <G,Y,\rho> \) is the morphism \( <1_G,h> : <G,Z,\pi> \to <G,X,\pi> \), where \( Z := \{x \in X : f(x) = g(x)\} \), and \( h : Z \times X \to Z \times X \) is the inclusion mapping.

Notice, that it follows immediately from this description that the inclusion functor of \( \text{TOP}^G \) in \( \text{TTG} \) does not preserve products, but that it creates and preserves all equalizers.

### 3.2.7. For the description of the left adjoint \( F^G \) of the functor \( S^G \) and the unit and counit of adjunction we refer the reader to 3.1.14, where each morphism \( \psi \) in \( \text{TOP}^{G^P} \) has to be replaced by \( 1_G \).

\(^1\) Plainly, this proof fails if \( X = \emptyset \). In that case, however, it is easy to see that \( <1_G,f> \) is not an epimorphism if \( Y \neq \emptyset \). So in that case, \( Y = \emptyset \), and \( f \) is epic in \( \text{TOP} \).
3.2.8. Now we consider another topological group $H$, and the corresponding monad $(H^H, \eta^H, \mu^H)$ in TOP. Again, the category of all $H^H$-algebras may be identified with the category $\text{TOP}^H$. We shall investigate now the morphisms of monads from $(H^H, \eta^H, \mu^H)$ to $(G^H, \eta^G, \mu^G)$.

According to 0.4.8, a natural transformation $\theta: H^H \rightarrow G^H$ is a morphism of monads from $(H^H, \eta^H, \mu^H)$ to $(G^H, \eta^G, \mu^G)$ iff

\[ \theta \circ \eta^H = \eta^G; \quad \theta \circ \mu^H = \mu^G \circ \theta^2. \]

For each object $X \in \text{TOP}$, $\theta_X$ is a continuous mapping.

\[ \theta_X: H \times X \rightarrow G \times X. \]

Identifying the $G^H$- and $H^H$-algebras with $G$- and $H$-spaces, the functor $\theta$ induces the functor $\theta^*: \text{TOP}^G \rightarrow \text{TOP}^H$ according to 0.4.8, as follows:

\[ \theta^*: \begin{cases} <G, X, x> \mapsto <H, X, \theta_X x> \text{ on objects} \\
<1_G, f> \mapsto <1_H, f> \text{ on morphisms}. \end{cases} \]

We shall describe now $\theta$ and $\theta^*$ in terms of morphisms in $\text{TOP}^{\text{GRP}}$ and in $\text{TOP}$.

3.2.9. **Lemma.** There exists a bijection $\theta \mapsto \psi_\theta$ from the set of all natural transformations $\theta: H^H \rightarrow G^H$ onto the set of all continuous functions $\psi_\theta: H \rightarrow G$.

Here $\theta$ and $\psi_\theta$ are related by

\[ \psi_\theta = \theta_X \times 1_X \]

for each object $X \in \text{TOP}$.

**Proof.** Let $(*)$ denote any one-point space. If $\theta: H^H \rightarrow G^H$ is a natural transformation then by (4) there is a continuous function $\psi_\theta: H \rightarrow G$ such that $\theta(*) = \psi_\theta \times 1_\{(*)\}$. Next, fix any non-void object $X \in \text{TOP}$ and any point $x \in X$, and let $f: (\ast) + X$ be defined by $f(*) := x$. Then $f$ is a morphism in $\text{TOP}$, hence naturality of $\theta$ implies that $\theta_X(1_X f) = (1_G f) \ast \theta(*)$. It follows that $\theta_X(s, x) = (\psi_\theta(s), x)$ for all $s \in H$. Since $x \in X$ has been chosen arbitrarily, this proves (6) for each $X \neq \emptyset$. For $X = \emptyset$, (6) is obvious.

Conversely, if $\psi: H \rightarrow G$ is any morphism in $\text{TOP}$, then defining $\theta_X := \psi \times 1_X$ for every object $X \in \text{TOP}$, we obtain a natural transformation $\theta: H^H \rightarrow G^H$ such that $\psi = \theta_\emptyset$. This completes the proof. \[ \square \]
3.2.10. **Lemma.** Let $\theta$ and $\psi_\theta$ be as in 3.2.9. Then $\theta$ satisfies the relations (3) iff $\psi_\theta : H \to G$ is a morphism in $\text{TOPGRP}$.

**Proof.** By a straightforward argument one shows that the natural transformation $\theta^2 : (H^2)_\theta \to (H^2)_\theta$ is related with $\psi_\theta$ as follows: if $X$ is an object in $\text{TOP}$, then $\theta^2_X : H \times (H \times X) \to G \times (G \times X)$ is given by $\theta^2_X = \psi_\theta \times \psi_\theta \times 1_X$. Therefore, the relations (3) are equivalent with

\[
\psi_\theta(e_H) = e_G; \quad \psi_\theta(st) = \psi_\theta(s)\psi_\theta(t)
\]

for $(s,t) \in H$. This proves the lemma. 

3.2.11. **Theorem.** There exists a bijection $\theta \mapsto \psi_\theta$ from the set of all morphisms of monads $\theta : (H^H, \eta^H, \mu^H) \to (H^G, \eta^G, \mu^G)$ onto the set $\text{TOPGRP}(H,G)$. Here $\theta$ and $\psi_\theta$ are related by (6).

**Proof.** Obvious from the preceding lemmas. 

3.2.12. If $\psi : H \to G$ is a morphism in $\text{TOPGRP}$, then let $R_\psi : \text{TOP}^G \to \text{TOP}^H$ be the functor, defined by $R_\psi := \psi^*$, where $\theta$ is the morphism of monads corresponding to $\psi$ (i.e., $\psi = \psi_\theta$). Thus, by (5) and (6),

\[
R_\psi : \begin{cases} (G, X, \pi) & \mapsto (H, X, \pi \circ (\psi \times 1_X)) \\ (1, \tau) & \mapsto (1, \tau) \end{cases}
\]

(7) on objects and morphisms.

3.2.13. **Notes.** Most of the contents of this subsection are classical (cf. the notes in 3.1.18). Only 3.2.3 and 3.2.11 seem to be new.

Another approach to categories of $G$-spaces would be to consider $G$ as a category. Then $G$ is a small strict monoidal category, and as such one can define actions of the category $G$ on the category $\text{TOP}$. For more details, cf. [ML], p. 170, where also some references to pertinent literature are given.

3.3. Induced actions

3.3.1. In this subsection, let $\psi : H \to G$ be a fixed morphism in $\text{TOPGRP}$. For any $G$-space $<G, X, \pi>$, let

\[
\pi_\psi := \pi \circ (\psi \times 1_X).
\]

(1) So the functor $R_\psi : \text{TOP}^G \to \text{TOP}^H$ defined in 3.2.12 can now be described as follows:
For each G-space \( \langle G, X, \pi \rangle \), we have the morphism

(3) \( \langle \psi, 1_X \rangle: \langle H, X, \pi \rangle \to \langle G, X, \pi \rangle \)

in \( \mathcal{T} \mathcal{T} \mathcal{G} \). If \( \langle 1_G, z \rangle: \langle G, X, \pi \rangle \to \langle G, Z, \zeta \rangle \) is a morphism in \( \mathcal{T} \mathcal{O} \mathcal{P} \mathcal{G} \), then plainly the following diagram commutes:

\[
\begin{array}{ccc}
\langle H, X, \pi \rangle & \xrightarrow{\langle \psi, 1_X \rangle} & \langle G, X, \pi \rangle \\
\downarrow{1_H} & & \downarrow{1_G} \\
\langle H, Z, \zeta \rangle & \xrightarrow{\langle \psi, 1_Z \rangle} & \langle G, Z, \zeta \rangle
\end{array}
\]

(The morphisms \( \langle \psi, 1_X \rangle \) form a natural transformation from \( E^H R \psi \) to \( E^G \), where \( E^G \) and \( E^H \) denote the inclusion functors of \( \mathcal{T} \mathcal{O} \mathcal{P} \mathcal{G} \) and \( \mathcal{T} \mathcal{O} \mathcal{P} \mathcal{H} \) in \( \mathcal{T} \mathcal{T} \mathcal{G} \).)

3.3.2. EXAMPLES.

(i) Let \( H \) be a subgroup of \( G \) and let \( \psi: H \to G \) be the inclusion mapping.

Then the functor \( R \psi \) assigns to each \( G \)-space the \( H \)-space which is obtained by restricting the action of \( G \) to \( H \): \( R \psi \langle G, X, \pi \rangle = \langle H, X, \pi \rangle \) where \( \pi \).

(ii) Let \( H = G \) and let \( \psi: G \to G \) be the identical mapping. Then for each \( G \)-space \( \langle G, X, \pi \rangle \), we have \( R \psi \langle G, X, \pi \rangle = \langle G, X, \pi \rangle \) (cf. also 1.1.5).

(iii) Let \( G = \{ e \} \) be a one-point group, and let \( \psi: H \to G \) be the obvious surjection. Identify \( \mathcal{T} \mathcal{O} \mathcal{P}^G \) in the obvious way with \( \mathcal{T} \mathcal{O} \mathcal{P} \). Then \( R \psi \) assigns to each object \( X \in \mathcal{T} \mathcal{O} \mathcal{P} \) the \( H \)-space \( \langle H, X, \pi \rangle \), where \( \pi \) denotes the trivial action of \( H \) on \( X \).

3.3.3. PROPOSITION. Let \( \langle \psi, f \rangle: \langle H, Y, \sigma \rangle \to \langle G, X, \pi \rangle \) be any morphism in \( \mathcal{T} \mathcal{T} \mathcal{G} \).

Then \( \langle 1_H, f \rangle: \langle H, Y, \sigma \rangle \to \langle H, X, \pi \rangle \) is the unique morphism of \( H \)-spaces for which the following diagram in \( \mathcal{T} \mathcal{T} \mathcal{G} \) commutes:
3.3.4. We shall see in 3.3.12 below that this property of the arrow $\langle \psi,1^X \rangle$ is related to the fact that $R_\psi$ has a left adjoint $L_\psi$. We shall prove this according to 0.4.2(i) by constructing for each $H$-space $\langle H,Y,\sigma \rangle$ a $G$-space $\langle G,X,\pi \rangle =: L_\psi \langle H,Y,\sigma \rangle$ and a universal arrow $\gamma_{\langle H,Y,\sigma \rangle}: \langle H,Y,\sigma \rangle \to \langle H,X,\pi \rangle = R_\psi L_\psi \langle H,Y,\sigma \rangle$ from $\langle H,Y,\sigma \rangle$ to $R_\psi$. We shall present first the construction of the object function of $L_\psi$.

3.3.5. Let $\langle H,Y,\sigma \rangle$ be an object in $\text{TOP}^H$. Define an action $\rho$ of $H$ on $G \times Y$ by the rule

\begin{equation}
\rho^H(t,y) := (t\psi(u)^{-1}, \sigma^Y)
\end{equation}

($u \in H$, $(t,y) \in G \times Y$). Obviously, the action $\rho$ of $H$ on $G \times Y$ commutes with the action $\rho^G$ of $G$ on $G \times Y$, so by 1.5.8 there exists a unique action $\pi$ of $G$ on $X := (G \times Y)/\sim^H$ making $\gamma_{\langle G,G \times Y,\sim^G \rangle}: \langle G,G \times Y,\sim^G \rangle \to \langle G,X,\pi \rangle$ a morphism of $G$-spaces. Now set $L_\psi \langle H,Y,\sigma \rangle := \langle G,X,\pi \rangle$.

3.3.6. With notation as in 3.3.5, set $f := c^G \circ \pi^G$. Then $f: Y \to X$ is continuous, and using the fact that $c^G(u,y) = c^G(e,\psi^Y y)$ for all $(u,y) \in H \times Y$, it follows that $\langle \psi,f \rangle: \langle H,Y,\sigma \rangle \to \langle G,X,\pi \rangle$ is a morphism in $\text{TT}^G$. Let

\begin{equation}
\gamma_{\langle H,Y,\sigma \rangle} := \langle \psi,f \rangle: \langle H,Y,\sigma \rangle \to \langle G,X,\pi \rangle.
\end{equation}

Then, indeed, $\langle \psi,f \rangle$ is a morphism of $H$-spaces (cf. 3.3.3). We shall show now that it is a universal arrow from $\langle H,Y,\sigma \rangle$ to $R_\psi$.

3.3.7. Lemma. Let $\langle H,Y,\sigma \rangle$ be an object in $\text{TOP}^H$ and let $\langle G,X,\pi \rangle$ and $f: Y \to X$ be constructed as above. In addition, let $\langle G,Z,\zeta \rangle$ be any object in $\text{TOP}^G$, and let $\langle 1^H,G \rangle: \langle H,Y,\sigma \rangle \to \langle H,Z,\zeta \rangle$ be a morphism in $\text{TOP}^H$. Then there exists a unique morphism $\langle 1^G,G \rangle: \langle G,X,\pi \rangle \to \langle G,Z,\zeta \rangle$ in $\text{TOP}^G$ such that $g = g \circ f$, i.e.
such that the following diagram commutes:

\[
\begin{array}{ccc}
<H,Y,\sigma> & \xrightarrow{<1_{H'},\sigma>^*} & <H,X,\psi> \\
\downarrow & & \downarrow \\
<H,Z,\rho> & \xrightarrow{<1_{H'},\rho>^*} & <G,Z,\zeta> \\
\end{array}
\]

(8)

**PROOF.** Let notation be as in 3.3.5 and 3.3.6. Define a function \( g' : G \times Y \to Z \) by \( g'(t,y) = \xi(t,g(y)) \) for \((t,y) \in G \times Y\). Plainly, \( g' \) is continuous. In addition, for each \( u \in H \) and each \((t,y) \in G \times Y\) we have

\[
g'(t\psi(u)^{-1},\rho^uY) = \xi(t\psi(u)^{-1},\rho^uY) = \xi(t\psi(u)^{-1},\rho^uY,\sigma^uY) = g'(t,y),
\]

that is, \( g'\rho^u(t,y) = g'(t,y) \). Consequently, \( g' \) is constant on the orbits in \( G \times Y \) under the action \( \rho \) of \( H \). Hence there exists a unique continuous function \( \tilde{g} : X = G \times Y / \rho \to Z \) such that \( g' = \tilde{g}\rho \). By the definition of \( g' \), we have \( g = g'\rho_Y \), hence \( g = \tilde{g}\rho_Y \). Moreover, a straightforward calculation shows that \( <1_G,\tilde{g}> : \langle G \times Y, \mu^G \rangle \to \langle G,Z,\zeta \rangle \) is a morphism of \( G \)-spaces. This implies that \( <1_G,\tilde{g}> : \langle G,X,\mu \rangle \to \langle G,Z,\zeta \rangle \) is a morphism of \( G \)-spaces as well.

Finally, suppose that \( <1_G,\tilde{h}> : \langle G,X,\mu \rangle \to \langle G,Z,\zeta \rangle \) is another morphism of \( G \)-spaces such that \( g = \tilde{h} \). Then we have for all \((s,y) \in G \times Y\):

\[
\begin{align*}
\tilde{h}\rho(s,y) &= \tilde{h}\rho_Y(s,(e,y)) = \xi(s,\tilde{h}\rho(e,y)) \\
&= \xi(s,\tilde{h}f(y)) = \xi(s,g(y)) = g'(s,y).
\end{align*}
\]

Hence \( \tilde{h}\rho = g' = \tilde{g}\rho \). Since \( \rho \) is a surjection, it follows that \( \tilde{h} = \tilde{g} \). This proves uniqueness of \( \tilde{g} \). □

3.3.8. **THEOREM.** Let \( \psi : H \to G \) be a morphism in TOPGRP. Then the functor \( R_\psi : \text{TOP}^G \to \text{TOP}^H \) has a left adjoint \( L_\psi : \text{TOP}^H \to \text{TOP}^G \).

**PROOF.** Use 3.3.5 through 3.3.7, and apply 0.4.2(i). □

3.3.9. The unit of the adjunction of \( L_\psi \) and \( R_\psi \) is the natural transformation \( \gamma : I_{\text{TOP}^H} \to R_\psi L_\psi \), indicated in 3.3.6 above (cf. (7)).
We shall describe now the counit $\gamma': \gamma_\psi \circ \eta_{\text{TOP}}$. To this end, consider an arbitrary object $<G,Z,\zeta>$ in $\text{TOP}^G$. According to formula (5) in 0.4.2, $\gamma'_G<,G,Z,\zeta>$ is obtainable as the morphism $<1_G,\gamma>$ in diagram (8) in 3.3.7 by taking there $<H,Y,\sigma> := \gamma_\psi<,G,Z,\zeta>$ and $g = 1_Z$. If we do so, then the tttg $<G,X,\pi>$ occurring in diagram (8) is $\gamma_\psi<,G,Z,\zeta>$, i.e. it is $\gamma_\psi<,G,Z,\zeta>$. According to 3.3.5 and 3.3.6, the phase space of this tttg is the quotient space $(G\times Z)/\sigma$, where $\sigma$ is defined according to (6) (of course, with $\sigma := \zeta\psi$). Obviously, we have for $(t_1,z_1),(t_2,z_2) \in G\times Z$:

\begin{align*}
\sigma(t_1,z_1) = \sigma(t_2,z_2) & \Leftrightarrow \exists u \in \mathcal{H} : t_2^{-1}t_1 = \psi(u) \& \zeta(u)z_1 = z_2 \\
& \Leftrightarrow t_2^{-1}t_1 \in \psi[H] \& \zeta(t_1,z_1) = \zeta(t_2,z_2).
\end{align*}

Let the function $k: G\times Z \to (G\psi[H])\times Z$ be defined by

$$k(t,z) := (q(t),\zeta(t,z))$$

$((t,z) \in G\times Z); \text{here } q: G \to G\psi[H] \text{ is the usual quotient mapping}. \text{It is easy to see that } k \text{ is a surjection. Furthermore, (9) implies that there exists a bijection } h: (G\times Z)/\sigma \to (G\psi[H])\times Z \text{ making the following diagram commutative:}$

\begin{center}
\begin{tikzpicture}
\node (A) at (0,0) {$G\times Z$};
\node (B) at (3,0) {$(G\times Z)/\sigma$};
\node (C) at (3,3) {$(G\psi[H])\times Z$};
\node (D) at (6,0) {$h$};

\draw[->] (A) to node [above] {$c_\sigma$} (B);
\draw[->] (B) to node [right] {$k$} (C);
\draw[->] (A) to node [right] {$h$} (C);
\end{tikzpicture}
\end{center}

Therefore, if we give $(G\psi[H])\times Z \text{ the quotient topology, induced by } k$, then it may be identified with $(G\times Z)/\sigma$, via $h$. In doing so, the action of $G$ on $(G\times Z)/\sigma$ turns out to correspond to the action $\psi$ of $G \to (G\psi[H])\times Z$, defined by

$$\psi^t(q(s),z) := (q(ts),\zeta^t(z))$$

If $p: (G\psi[H])\times Z \to Z$ is the projection, then $p\circ \psi = \zeta$, hence $p$ is continuous with respect to the quotient topology in $(G\psi[H])\times Z$, induced by $k$. In addition, $p$ is equivariant with respect to the actions $\psi$ and $\zeta$. A close examination of the proof of 3.3.7 shows, that under the above mentioned identification $p$ corresponds to the mapping $\tilde{g}$ in diagram (8), provided $g = 1_Z$. 

Thus, up to isomorphism, we have
\[ y'_G, Z, \xi = [1_G, \psi] : G, (G \psi[H]) \times Z, \nu \rightarrow G, Z, \xi. \]

3.3.10. In diagram (8), we can insert arrows \( \psi, 1_X \) and \( \psi, 1_Z \). Then diagram (4) shows, that the resulting diagram is still commutative. Now lemma 3.3.7 can be reformulated as follows:

3.3.11. COROLLARY. Let \( H, Y, \sigma \) be an object in \( \text{TOP}_H \), and let \( f : Y \rightarrow X \) be as in 3.3.6. Then the arrow \( \psi, f : H, Y, \sigma \rightarrow G, X, \pi \) is "universal" in \( \text{TTG} \) for the class of all morphisms in \( \text{TTG} \) having group component \( \psi \) and domain \( H, Y, \sigma \), in the following modified sense: for any morphism \( \psi, g : H, Y, \sigma \rightarrow G, Z, \xi \) in \( \text{TTG} \) there exists a unique morphism \( 1_G, g \) making the following diagram commutative:

\[
\begin{array}{ccc}
H, Y, \sigma & \xrightarrow{\psi, f} & G, X, \pi := L_\psi H, Y, \sigma \\
\downarrow{\psi, g} & & \downarrow{1_G, g} \\
G, Z, \xi & & \\
\end{array}
\]

PROOF. By 3.3.3, any morphism \( \psi, g : H, Y, \sigma \rightarrow G, Z, \xi \) in \( \text{TTG} \) factorizes as \( \psi, g = \psi, 1_H \cdot 1_G, g \). Taking into account the observation made in 3.3.10, the corollary is an immediate consequence of 3.3.7. □

3.3.12. We mention without proof, that a similar reformulation of the universal property of the counit \( \gamma' \) of the adjunction of \( L_\psi \) and \( R_\psi \) shows:
for any object \( G, Z, \xi \) in \( \text{TOP}_G \) the arrow \( \psi, 1_G : G, Z, \xi \rightarrow G, Z, \xi \) is "universal" for all arrows in \( \text{TTG} \) with group component \( \psi \) and codomain \( G, Z, \xi \) in the sense which has been described in 3.3.3.

3.3.13. EXAMPLES.
(i) Suppose \( H \) is a subgroup of \( G \) and \( \psi : H \rightarrow G \) is the inclusion mapping.
Then for each object \( H, Y, \sigma \) the universal arrow \( \psi, f : H, Y, \sigma \rightarrow G, X, \pi \) has the following additional properties. First, \( f : Y \rightarrow X \) is injective. Indeed, \( \circ_0 (e, y_1) = \circ_0 (e, y_2) \iff \text{there exists } u \in H \text{ with } \psi(u) = e \text{ and } \circ_0^u y_1 = y_2, \text{iff } y_1 = y_2 \) (notation: cf. 3.3.5 and 3.3.6).

\[ \text{If there is no morphism } \alpha \neq 1_H \text{ in } \text{TOPGRP} \text{ such that } \psi = \alpha \psi, \text{ then } \psi, 1_G \text{ is unique in } \text{TTG}. \]

\[ \text{In fact, this argument shows that } \psi \text{ injective } \Rightarrow f \text{ injective.} \]
Second, \( f[Y] \) is an \( H \)-invariant subset of \( \langle G, X, \pi \rangle \) and the smallest \( G \)-invariant set containing \( f[Y] \) is just all of \( X \). Indeed, for all \( (t, y) \in G \times Y \) we have \( c_p(t, y) = c_p \mu_Y(t, (e, y)) = \pi^t c_y(e, y) = \pi^t f(y) \), whereas \( c_p(t, y) \notin f[Y] \) iff \( t \in H \). Consequently, \( \pi^t f(y) \in f[Y] \) iff \( t \notin H \). Notice that we proved just a little bit more: \( \pi^t f[Y] \cap f[Y] = \emptyset \) iff \( t \notin H \).

(ii) Let \( H = G_d \) and let \( \psi : G_d \to G \) be the identical mapping. Similar to the argument in 3.3.9 one shows that for each \( G_d \)-space \( \langle G_d, Y, \sigma \rangle \) the universal arrow \( \langle \psi, f \rangle : \langle G_d, Y, \sigma \rangle \to \langle G, X, \pi \rangle \) is as follows: \( X = G \times Y / C_p \) may be identified with \( Y \) (as a set) in such a way that \( c_p : G \times Y \to X \) corresponds to \( \sigma : G \times Y \to Y \); furthermore, the action \( \pi \) of \( G \) on \( X \) corresponds (as a function \( G \times X \to X \)) with the function \( \sigma : G \times Y \to Y \), and, finally, \( f : Y \to X \) corresponds to the identical mapping of \( Y \) onto itself. The only differences between \( \langle G, X, \pi \rangle \) and \( \langle G_d, Y, \sigma \rangle \) are the topologies on \( G \) and \( X \). If \( T \) denotes the original topology on \( Y \), then \( X = (Y, T') \) where \( T' \) is the finest topology making \( \sigma : G \times (Y, T) \to (Y, T') \) continuous.

(iii) Let \( G = \{ e' \} \) be a one point group, and let \( \psi : H \to G \) be the obvious surjection. Identify TOP\(^G\) in the obvious way with TOP (by means of the functor \( S^G \), which is now an isomorphism of categories). According to the construction in 3.3.5 and 3.3.6, for any object \( \langle H, Y, \sigma \rangle \) in TOP\(^H\) we obtain for \( f : Y \to X \) just the quotient mapping \( c_0 : Y / C_0 \to X \). So in identifying TOP\(^G\) with TOP, the functor \( L_\psi : \text{TOP}^H \to \text{TOP}^G \) carries over to the functor \( S^H_1 : \text{TOP}^H \to \text{TOP}^G \), defined as follows:

\[
S^H_1 : \langle H, Y, \sigma \rangle \mapsto Y / C_0, \quad \langle H, g' \rangle \mapsto g' \end{equation}

on objects

where for each morphism \( \langle h, g \rangle : \langle H, Y, \sigma \rangle \to \langle H, Z, \tau \rangle \), \( g' : Y / C_0 \to Z / C_0 \) is the unique continuous function with \( g'c_\sigma = c_\tau g \) (cf. 1.4.7). By 3.3.8, the functor \( S^H_1 \) has a right adjoint \( R^H_1 : \text{TOP}^G \to \text{TOP}^H \) where \( R^H_1 \) is the functor, described in 3.3.2(iii). Moreover, the unit \( \gamma \) of adjunction

\[ \text{In the general case, } \pi^t f[Y] = f[Y] \text{ iff } t \in \psi[H] \text{ and } \pi^t f[Y] \cap f[Y] = \emptyset \text{ iff } t \notin G \cap \psi[H]. \]

\[ \text{In this example we used only that } \psi : H \to G \text{ is bijective, i.e. } G \text{ is just the group } H \text{ with a weaker topology.} \]
is given by

\[ Y_{H,Y,\sigma} = \langle 1_H^*c_\sigma : \langle H,Y,\sigma \rangle \to \langle H,Y/C_\sigma, H/Y/C_\sigma \rangle \rangle \]

where \( T_H^\sigma \) denotes the trivial action of \( H \) on \( Y/C_\sigma \).

3.3.14. We close this subsection with some trivial, though useful remarks about the functors \( R_\psi \) and \( L_\psi \). The easy proofs are left to the reader.

(i) The functor \( R_\psi \) is always faithful. If \( \psi \) is surjective then \( R_\psi \) is full.

(ii) If \( \psi : K \to H \) is a morphism in \( \text{TOPGRP} \), then we have also functors \( R_\psi : \text{TOP}^H \to \text{TOP}^K \) and \( L_\psi : \text{TOP}^K \to \text{TOP}^H \), and \( L_\psi \) is left adjoint to \( R_\psi \). In addition, \( R_{\psi^p} = R_\psi \circ R_\psi \), hence \( L_{\psi^p} = L_\psi \circ L_\psi \).

3.3.15. The reader is invited to calculate the universal arrow \( <\psi, f> : \langle G,X,\pi \rangle \to \langle G/N,Y,\sigma \rangle \), where \( N \) is a normal subgroup of \( G \) and \( \psi : G \to G/N \) is the quotient mapping. Cf. 1.5.9.

3.3.16. NOTES. The central facts in this subsection are the construction of the functor \( L_\psi \) and the proof that \( L_\psi \) is left adjoint to \( R_\psi \). If \( \psi : H \to G \) is an embedding (cf. 3.3.2(i) and 3.3.13(i)), then the construction is well-known, and has as its classical analog the famous FROBENIUS reciprocity theorem. The corresponding construction in ergodic theory can be found in K. LANGE, A. RAMSAY & G.-C. ROTA [1971].

It is interesting to know in example 3.3.13(i) when \( f : Y \to X \) is a topological embedding. It is easy to see that this is so if \( G \) is a discrete group. A second situation in which \( f \) is a topological embedding occurs when \( G \) is a Hausdorff group and \( H \) is a compact subgroup of \( G \): then \( \pi_Y : Y \to GxY \) is a closed embedding and \( c_p : GxY \to GxY/C_\rho \) is now a closed mapping (in any \( T\pi \) with a compact phase group the projection of the phase space onto the orbit space is closed; this is an easy consequence of [GH], 1.18(5); cf. also [Br], Chap. I, Th. 3.1). This case is very important for the study of the structure of transformation groups with compact phase groups. Cf. for instance [Br], Chap. II. Finally, if \( Y \) is a compact Hausdorff space and \( H \) is a closed subgroup of \( G \), then \( C_\rho \) turns out to be a closed subset of \( (GxY) \times (GxY) \), so that \( X = GxY/C_\rho \) is a Hausdorff space, by 1.3.10(ii). Since \( Y \) is compact and \( f : Y \to X \) is injective, \( f \) is a topological embedding. See W.H. GOTTschalk [1973], p.123.

In all cases that \( H \) is a subgroup of \( G \) and \( f : Y \to X \) is a topological
embedding, the construction of $L_{\psi}<H,Y,o>$ can be described as an extension of the action of a subgroup to an action of the whole group by means of an extension of the phase space. Notice that the particular case of $H=\mathbb{Z}$, $G=\mathbb{R}$ fits in this situation with respect to compact spaces $Y$. It is an outstanding problem to give sufficient conditions for an action of $\mathbb{Z}$ on a compact $T_2$-space to be extendable to an action of $\mathbb{R}$ on the same phase space. That is, when can a single homeomorphism on a space $Y$ be described as the transition $\pi^1$ for some $\psi<\mathbb{R},Y,\pi>$? Cf. G.D. JONES [1972], also for further references to this subject and some remarks about its history. For certain spaces, this so-called embedding problem is equivalent to the HILBERT-SMITH conjecture for those spaces. (This conjecture reads as follows: if a compact Hausdorff group acts effectively on a connected manifold, then the group is a Lie group.) Cf. H. CHU [1973].

A close inspection of the construction of $f: Y+X$ in 3.3.6 shows that $f = c_p^G_f$, where $c_p$ is the coequalizer in TOP of the morphisms

$$
G \times H \times Y \twoheadrightarrow (s,u,y) \mapsto (\psi(u),y) \quad G \times Y.
$$

Since $c_p$ is an open mapping, it follows from the first remark in 3.4.4 below, that the functor $S^G$ creates the corresponding coequalizer in $TOP^G$, thus producing the action $\pi$ of $G$ on $G\times Y/C_p^G$. If we look at the construction from this point of view, we see that a basic point in the proof of theorem 3.3.8 is the existence of a certain coequalizer in $TOP^G$. A similar statement, involving arbitrary morphisms of monads, may be found in Corollary 1 in F.E.J. LINTON [1969]. Although the idea of our proof of 3.3.8 is similar to that of LINTON's, our theorem turns out to be not a simple application of his result.

Finally, it is easy to see that 3.3.8 could have been proved by means of the FREYD adjoint functor theorem. Indeed, the solution set condition is obviously fulfilled, whereas $R_\psi$ trivially preserves all limits (use 3.2.5). However, then it would be difficult to describe the functor $L_\psi$ explicitly.

### 3.4. Colimits in $TTG$ and $TOP^G$

#### 3.4.1. We are now in a position to prove that $TOP^G$ and $TTG$ are cocomplete categories. First we shall deal with $TOP^G$. Here all coproducts turn out to
be created by the functor $S^G: \text{TOP}^G \to \text{TOP}$, but $S^G$ does not even preserve all coequalizers. However, if $G$ is locally compact, then $S^G$ creates all coequalizers, and $\text{TOP}^G$ is cocomplete. The case of an arbitrary group $G$ is then reduced to the locally compact case by considering $G_d$, using techniques from the previous subsection.

In $\text{TTG}$ the situation is somewhat more complicated; fortunately, the functor $G: \text{TTG} \to \text{TOPGRP}$ preserves all colimits. Refinements of the arguments used for the proof of cocompleteness of $\text{TOP}^G$ then show that $\text{TTG}$ is cocomplete as well. The bad behaviour of the functor $K: \text{TTG} + \text{C} := \text{TOPGRP} \times \text{TOP}$ with respect to colimits is caused by bad preservation properties of the functor $S: \text{TTG} \to \text{TOP}$. There are several "explanations" for this bad behaviour of $S$. First, $S$ forgets all about actions of groups on spaces. Therefore, it seems quite natural that $S$ has no reasonable preservation properties. However, $S$ behaves nicely with respect to limits, so this explanation is quite unsatisfactory. No doubt, therefore, the difficulties are related to the fact that (unlike for limits) colimits in $\text{TOPGRP}$ are not obtained by giving a suitable group structure to the corresponding colimit in $\text{TOP}$.

On the contrary, colimits in $\text{TOPGRP}$ are calculated in the category of (discrete) groups and afterwards they are provided with a suitable topology (cf. 0.4.11).

There is another functor, $S_1: \text{TTG} \to \text{TOP}$, which behaves better than $S$. It is defined as follows:

\begin{align*}
S_1^G & : \langle G, X, \pi \rangle \mapsto X/C \pi \quad \text{on objects} \\
S_1^G & : \langle \psi, f \rangle \mapsto f' \quad \text{on morphisms}
\end{align*}

where for each morphism $\langle \psi, f \rangle : \langle G, X, \pi \rangle \to \langle H, Y, \sigma \rangle$ in $\text{TTG}$, $S_1^G f := f': X/C \pi \to Y/C \sigma$ is the unique continuous function with $f'c_{\pi} = c_{\sigma}f$ (cf. 1.4.8). The restriction $S_1^G$ of $S_1$ to $\text{TOP}^G$ has already been defined in 3.3.13(iii). The functors $S_1$ and $S_1^G$ will also be considered in this subsection.

All notation will be as in subsections 3.1, 3.2 and 3.3. In particular, $G$ will always denote a topological group.

3.4.2. **Proposition.** The functor $S^G: \text{TOP}^G \to \text{TOP}$ creates all coproducts and, consequently, $S^G$ preserves all coproducts.

**Proof.** It is sufficient to prove that $S^G$ creates all coproducts: then by 0.4.4(iv) $S^G$ preserves them, because $\text{TOP}$ is cocomplete. The proof that $S^G$ creates the coproduct for a given set $\{\langle G, X_j, \pi_j \rangle : j \in J \}$ of objects in $\text{TOP}^G$.
is straightforward. Representing the coproduct $X$ of the set \( \{ X_j : j \in J \} \) in \( \text{TOP} \) as the disjoint union of the spaces $X_j$, the created coproduct of the given set in \( \text{TOP}^G \) is just what it is expected to be: the $G$-space \( \langle G, X, \pi \rangle \) with $\pi^t|_{X_j} = \pi^t_j$ for each $t \in G$ and $j \in J$. Details are left to the reader. \( \square \)

3.4.3. **Theorem.** Suppose $G$ is a locally compact Hausdorff group. Then the functor $S^G : \text{TOP}^G \to \text{TOP}$ creates all colimits. Hence \( \text{TOP}^G \) is cocomplete, and $S^G$ preserves all colimits.

**Proof.** Since \( \text{TOP} \) is cocomplete, it is sufficient to show that $S^G$ creates all colimits. In view of 3.4.2, we can restrict ourselves to coequalizers (cf. [ML], p.109).

Suppose \( \langle 1^G, f_1 \rangle : \langle G, X, \sigma \rangle \to \langle G, Y, a \rangle \) (i=1,2) are morphisms in \( \text{TOP}^G \). Let $g : Y \to Z$ denote the coequalizer of $f_1, f_2 : X \to Y$ in \( \text{TOP} \). Then for each $t \in G$, $g^t : Y \to Z$ is a morphism in \( \text{TOP} \), and

$$g^t f_1 = g f_1^t = g f_2^t = g^t f_2.$$

By the coequalizer property of $g$, it follows that there exists a unique continuous mapping $\zeta^t : Z \to Z$ such that $g^t = \zeta^t g$. Stated otherwise, the quotient mapping $g : Y \to Z$ (cf. 0.4.10) is defined by an equivalence relation in $Y$ which is invariant under the action $\sigma$ of $G$. Then 1.5.7(iii) implies that $\zeta$ is a continuous action of $G$ on $Z$. It is the unique action of $G$ on $Z$ making $g$ a morphism of $G$-spaces (cf. 1.5.5). So the proof will be finished if $\langle 1^G, g \rangle : \langle G, Y, a \rangle \to \langle G, Z, \zeta \rangle$ is shown to be the coequalizer of $\langle 1^G, f_1 \rangle$ and $\langle 1^G, f_2 \rangle$ in \( \text{TOP}^G \). This may be done by a straightforward argument which is left to the reader. \( \square \)

3.4.4. If $G$ is not locally compact, one shows as above that $S^G$ creates coequalizers for those morphisms $\langle 1^G, f_1 \rangle, \langle 1^G, f_2 \rangle : \langle G, X, \sigma \rangle \to \langle G, Y, a \rangle$ in \( \text{TOP}^G \) for which the coequalizer $g : Y \to Z$ in \( \text{TOP} \) of $f_1, f_2 : X \to Y$ is either an open mapping, or a perfect mapping, or for which $G \times Z$ is a k-space. Cf. 1.5.7.

The following example shows that some restriction has to be made in 3.4.3. Let $G = Q$ and let $Y$ be the locally compact Hausdorff space which admits an equivalence relation $R$ such that on $Q \times (Y/R)$ the quotient topology induced by $1_Q \times f$ is strictly finer than the product topology (here $f : Y \to Y/R$ is the quotient mapping). Cf. 0.2.5. In 1.5.11 we pointed out that the equivalence relation $D^Q = \{ \langle Q \times \{ t \} : G \times Y \rangle \}$ is invariant in $Q \times Y / D^Q$, but that there exists no continuous action of $Q$ on $Q \times Y / D^Q \times R$. 


(= Q × (Y/R) with its quotient topology) for which the quotient mapping q: Q×Y → Q×Y/DQ×R is equivariant. Let X := D_q × R, and observe that X is an invariant subset of the product in TOP\(_G\) of \(<\alpha,\xi,\zeta,\mu_\xi^G>\) with itself. Let \(\pi\) denote the action of \(Q\) on X obtained by restriction of the action in this product to X. Then the restrictions \(f_1\) and \(f_2\) to X of the projections of \((Q\times Y)\times(Q\times Y)\) onto \(Q\times Y\) are equivariant, i.e. we have morphisms \(<1'_Q,f'_1>:\langle Q,X,\psi_1^D\rangle + \langle Q,Q\times Y,\mu_\psi^D\rangle\) in TOP\(_G\). It is not difficult to show that the coequalizer in TOP of \(f'_1,f'_2\): \(Q\times Y\rightarrow Q\times Y\) is the quotient mapping q: Q×Y→Q×Y/DQ×R. By what we noticed above, it follows that \(S^G\) cannot create the coequalizer of \(<1'_Q,f'_1>\) and \(<1'_Q,f'_2>\).

We shall see in 3.4.5 below, that the morphisms \(<1'_Q,f'_1>\) and \(<1'_Q,f'_2>\) do have a coequalizer in TOP\(_G\). So the above example shows, in addition, that the functor \(S^G\) does not preserve all colimits in TOP\(_G\).

3.4.5. THEOREM. For any topological group \(G\), the category TOP\(_G\) is cocomplete, but in general the functor \(S^G: TOP^G\rightarrow TOP\) does not preserve all colimits.

PROOF. The bad behaviour of \(S^G\) is already illustrated in 3.4.4. In order to prove that TOP\(_G\) is cocomplete, proceed as follows.

Let \(H := G_\alpha\) and let \(\psi: H \rightarrow G\) be the identical mapping. Observe, that \(\psi\) is a bijection, so that the functor \(R_\psi: TOP^G\rightarrow TOP^H\) is full and faithful (cf. 3.3.14(i)). Since \(H\) is a locally compact Hausdorff group, TOP\(_H\) is cocomplete by 3.4.3. Since \(R_\psi\) has a left adjoint \(L_\psi\) (cf. 3.3.8), an obvious application of 0.4.4(iii) shows that TOP\(_G\) is cocomplete. \(\Box\)

3.4.6. In the preceding proof we can replace the appeal to 0.4.4(iii) by the following argument (which is, in fact, a proof for 0.4.4(iii), adapted to the present situation): if \(D\) is a diagram in TOP\(_G\), then by 0.4.4(ii) the functor \(L_\psi\) preserves the colimit of the diagram \(R_\psi D\) in TOP\(_H\) and thus giving rise to a colimit for the diagram \(L_\psi R_\psi D\). However, it follows immediately from the description of \(L_\psi\) for this particular case in 3.3.13(ii) or from the description of the counit of the adjunction of \(L_\psi\) and \(R_\psi\) in 3.3.9, that \(L_\psi R_\psi\) may be identified with the identity functor on TOP\(_G\) (take into account that for any ttg \(G,Z,T\) the finest topology \(T'\) on \(Z\) making \(\zeta: G\times (Z,T) \rightarrow (Z,T')\) continuous just equals the original topology \(T\) on \(Z\)).

\(^1\) Hence \(S^G\) cannot have a right adjoint, by 0.4.4(ii).
Thus, the image under \( L_\psi \) of the colimit in \( \text{TOP}^H \) of \( R_D \) is just the colimit of \( D \) in \( \text{TOP}^G \).

We may rephrase this loosely by saying that it is only the topologies for which things go wrong. Indeed, colimits in \( \text{TOP}^H \) can be computed in \( \text{TOP} \) (\( H = G_d \) is discrete; cf. 3.4.3), and application of \( L_\psi \) to these colimits means (according to 3.3.13(ii)) that the topologies of their phase spaces have to be altered. Stated otherwise: a colimit in \( \text{TOP}^G \) can be computed in \( \text{TOP} \), but afterwards the topology in the phase space of an obtained "colimit" has to be suitably weakened in order to obtain the colimit in \( \text{TOP}^G \).

It follows immediately from these remarks that the composition of \( S^G \) with the forgetful functor \( P: \text{TOP} \to \text{SET} \) preserves all colimits (recall that \( P \) preserves colimits). In particular, \( P S^G \) preserves all epimorphisms. \textit{Hence \( S^G \) preserves all epimorphisms (\( P \) reflects them).} This yields an alternative proof of 3.2.3.

We close our considerations about \( \text{TOP}^G \) by a brief inspection of the functor \( S^G_1: \text{TOP}^G \to \text{TOP} \), defined in 3.3.13(iii).

3.4.7. \textsc{Proposition.} The functor \( S^G_1 \) preserves all colimits and epimorphisms. In addition, \( S^G_1 \) preserves all equalizers, but it does not preserve all finite products, unless \( G = \{ e \} \).

\textbf{Proof.} By 3.3.13(iii), \( S^G_1 \) has a right adjoint. So 0.4.4(ii) implies that \( S^G_1 \) preserves all colimits and all epimorphisms.

Next, consider morphisms \( \langle G,f_1,' \rangle, \langle G,f_2,' \rangle: \langle G,X,\pi \rangle \to \langle G,Y,\sigma \rangle \) in \( \text{TOP}^G \). Their equalizer in \( \text{TOP}^G \) is the morphism \( \langle G,\zeta,\eta \rangle: \langle G,Z,\xi \rangle \to \langle G,X,\pi \rangle \), where \( Z := \{ x \in X \mid x \in X \land f_1(x) = f_2(x) \} \) is a \( G \)-invariant subset of \( X \), \( g: Z \to X \) is the inclusion mapping, and \( \xi := \pi|_{G \times Z} \) (cf. 3.2.6). By 1.4.10, \( S^G_1 \) is a topological embedding of \( \langle G,Z,\xi \rangle \) into \( \langle G,X,\pi \rangle = X/C \pi \), and its range is easily seen to be the subspace of \( X/C \pi \) on which the mappings \( S^G_1 f_1 \) and \( S^G_1 f_2 \) coincide. This proves that \( S^G_1 \) preserves all equalizers.

Finally, the following observations show that \( S^G_1 \) does not always preserve finite products. Plainly, \( S^G_1 \langle G,0,\lambda \rangle \) is a one-point space. On the other hand, the product of \( \langle G,0,\lambda \rangle \) with itself in \( \text{TOP}^G \) is \( \langle G,G \times G,\pi \rangle \), where \( \pi^*(u,v) := (tu,tv) (t,u,v \in G) \). Hence \( S^G_1 \langle G,G \times G,\pi \rangle \) may be identified with \( G \times G \) (and \( C \pi \) then corresponds to the continuous and open mapping \( (u,v) \mapsto u^{-1}v: G \times G \to G \)). So if \( G \) is not a one-point group, \( S^G_1 \) does not preserve the product of \( \langle G,0,\lambda \rangle \) with itself. \( \square \)
3.4.8. If \( <G,f> : <G,X,\pi> \rightarrow <G,Y,\sigma> \) is an epimorphism in \( \text{TOP}^G \), then \( f' := S^G_f : X/C_\pi \rightarrow X/C_\sigma \) is epic in \( \text{TOP} \), by 3.4.7, hence \( f' \) is a surjection. It follows that for every \( y \in Y \) there exists \( x \in X \) with \( C_\sigma[y] = f'c_\pi x = C_\sigma[fx] \). Since \( fC_\pi[x] = C_\sigma[fx] \), it follows that \( f : X \rightarrow Y \) is a surjection. Consequently, we have proved, again, that \( S^G \) preserves all epimorphisms.

3.4.9. **PROPOSITION.** The functor \( G : \text{TTG} \rightarrow \text{TOPGRP} \) has a right adjoint. Consequently, \( G \) preserves all colimits and epimorphisms.

**PROOF.** Fix a one-point space (*). For any object \( G \in \text{TOPGRP} \), let \( \tau^G \) denote the obvious action of \( G \) on (*). Define the functor \( R : \text{TOPGRP} \rightarrow \text{TTG} \) by

\[
R : \begin{cases}
    G & \mapsto <G,(*),\tau^G> \\
    \psi & \mapsto <\psi,1(*)>
\end{cases}
\]

Then the following diagram shows that \( R \) is right adjoint to \( G \) (apply 0.4.2(ii)):

\[
\begin{array}{c}
    \begin{array}{c}
        <G,(*),\tau^G> \\
        \downarrow \psi f_Y \\
        <H,Y,\sigma>
    \end{array} \\
    \begin{array}{c}
        \downarrow \psi \\
        H
    \end{array}
\end{array}
\]

Then \( f_Y : Y \rightarrow (*) \) is the unique surjection of the object \( Y \) onto (*).

3.4.10. **COROLLARY.** The functor \( K : \text{TTG} \rightarrow C \) preserves and reflects epimorphisms, i.e. a morphism \( \langle \psi,f \rangle \) in \( \text{TTG} \) is epic iff \( \psi \) is epic in \( \text{TOPGRP} \) and \( f \) is epic in \( \text{TOP} \).

**PROOF.** In view of 3.1.4(ii) we need only to prove that \( K \) preserves epimorphisms. So let \( \langle \psi,f \rangle : <G,X,\pi> \rightarrow <H,Y,\sigma> \) be an epimorphism in \( \text{TTG} \). Then 3.4.9 implies that \( \psi : G \rightarrow H \) is epic in \( \text{TOPGRP} \), i.e. \( \psi \) is a surjection. Hence \( f[X] \) is an invariant subset of \( <H,Y,\sigma> \) (cf. 1.4.5). Therefore, the proof of 3.2.3 applies to the present case \([2]\), showing that \( f[X] = Y \).

\( \star \)

\( \star \)

1 Recall that \( C := \text{TOPGRP} \times \text{TOP} \).

2 We have only to replace the trivial action of \( G \) on the space \( Z \) considered in 3.2.3 by the trivial action of \( H \) on \( Z \).
3.4.11. We shall show now that \( \text{TTG} \) is cocomplete. The existence of coequalizers is shown by means of a more or less obvious modification of the proof of the existence of coequalizers in \( \text{TOP}^G \) (cf. 3.4.5). However, the construction of coproducts in \( \text{TTG} \) offers some difficulties. We shall show first that the object that might expected to be the colimit of a given set of ttgs is not the right one.

Let \( \{\langle G_j, X_j, \pi_j \rangle \mid j \in J \} \) be a set of objects in \( \text{TTG} \). If it has a coproduct, the phase group of the colimiting object has to be the coproduct of the set \( \{G_j \mid j \in J \} \) in \( \text{TOPGR}^p \), and the group components of the coprojections in \( \text{TTG} \) have to be the coprojections \( \beta_i : G_i \to G \) of the coproduct in \( \text{TOPGR}^p \) (cf. 3.4.9). There exists an obvious action \( \pi \) of \( G \) on the disjoint union \( \bigcup_j X_j \) of the spaces \( X_j \) (i.e. the coproduct of the set \( \{X_j \mid j \in J \} \) in \( \text{TOP} \)) such that each \( \langle x_j, r_j \rangle \) is a morphism in \( \text{TTG} \); here \( r_j : X_j \to \bigcup_j X_j \) is the canonical embedding (coprojection) of \( X_j \) into \( \bigcup_j X_j \). In order to define this action \( \pi \), first observe that each \( G_i \) admits a canonical embedding \( \alpha_i : G_i \to \bigcup_i G_i \). Since \( \alpha_i \) is a morphism in \( \text{TOPGR}^p \) for each \( i \in J \), there exists a unique morphism \( \alpha : G \to \bigcup_i G_i \) in \( \text{TOPGR}^p \) making the following diagram commutative for every \( i \in J \):

\[
\begin{array}{c}
\alpha
\\ \downarrow
\\ G
\end{array}
\begin{array}{c}
\bigcup_i G_i
\\ \downarrow
\\ G
\end{array}
\begin{array}{c}
\alpha_i
\\ \downarrow
\\ G_i
\end{array}
\begin{array}{c}
\alpha_i
\\ \downarrow
\\ G_i
\end{array}
\begin{array}{c}
\bigcup_j G_j
\\ \downarrow
\\ \bigcup_j G_j
\end{array}
\]

Furthermore, let \( p_i : \bigcup_i G_i \to G_i \) be the canonical projection. Then we can form the object \( \langle G, \bigcup_j X_j, \pi \rangle \) in \( \text{TTG} \). Since \( p_i \alpha_i = 1_{G_i} \), we have \( p_i \alpha \beta_i = 1_{G_i} \), whence

\[
\langle \beta_i \pi, r_j \rangle : \langle G_i, X_j, \pi_i \rangle \to \langle G, \bigcup_j X_j, \pi \rangle
\]

is a morphism in \( \text{TTG} \). Finally, form the coproduct \( \langle G, \bigcup_j X_j, \pi \rangle \) of the set \( \{\langle G, X_j, \pi_i \rangle \mid j \in J \} \) in \( \text{TOP}^G \) (cf. 3.4.2); the coprojections are the morphisms \( \langle 1_{G_i}, r_j \rangle \), and using (5), we see that each \( \langle \beta_i \pi, r_j \rangle : \langle G_i, X_j, \pi_i \rangle \to \langle G, \bigcup_j X_j, \pi \rangle \) is a morphism in \( \text{TTG} \). These morphisms form a cone in \( \text{TTG} \), but we shall show now that it is not a colimiting cone for the given set of objects in \( \text{TTG} \).

To this end, suppose we are given morphisms \( \langle \psi_i, \xi_i \rangle : \langle G_i, X_j, \pi_i \rangle \to \langle H, Y, \sigma \rangle \) in \( \text{TTG} \). Since \( \langle G, \bigcup_j X_j, \pi \rangle \) is the coproduct of the set \( \{\langle G_j, X_j \rangle \mid j \in J \} \) in \( \mathcal{C} \) (where \( \mathcal{C} = \text{TOPGR}^p \times \text{TOP} \)), there exists a unique morphism \( \langle \psi, \xi \rangle : \langle G, \bigcup_j X_j, \pi \rangle \to \langle H, Y \rangle \) in
C such that \((\psi_i \cdot \varepsilon_i^i) = (\psi_i \cdot \varepsilon_i^j)\) for all \(i \in J\). We shall show now that, in general, \((\psi_i \cdot \varepsilon_i)\) is not a morphism in \(\mathbf{TTG}\) from \(\langle G, \Sigma_j X_j, \pi \rangle\) to \(\langle H, \Sigma_j, \sigma \rangle\). This shows that \(\langle G, \Sigma_j X_j, \pi \rangle\) together with the morphisms \(\langle \beta_i, \xi \rangle\) cannot form the desired coproduct in \(\mathbf{TTG}\). To this end, observe first that for \(i, j \in J, \ i \neq j\) implies that \(p_j \alpha_i \cdot t = e_j\) the unit of \(G_j\), for each \(t \in G_i\). Hence for \(t \in G_i\) and \(x \in X_j\), \(\pi_i(\beta_i \cdot t, r_i) \cdot x = r_j \pi^i_j(\beta_i \cdot t, x) = r_j(\pi_i (p_j \alpha_i \cdot t, x) = r_j(x).\) Consequently,

\[g \pi(\beta_i \cdot t, x) = g_i(x),\]

whereas, on the other hand

\[\sigma(\psi_i \cdot t, g_j \cdot x) = \sigma(\psi_i \cdot t, g_j \cdot x).\]

Since there is no guarantee that \(\sigma(\psi_i \cdot t, g_j \cdot x) = g_j(x)\) for all \(i,j \in J, \ i \neq j, \ t \in G_i\) and \(x \in X_j\), it follows that \((\psi_i \cdot \varepsilon_i)\) need not be a morphism in \(\mathbf{TTG}\).

Observe, that the reason for this failure is, that the restriction of the action \(\pi^i_j\) of \(G_i\) to \(r_j[X_j]\) is trivial if \(i \neq j\).

**3.4.12. THEOREM.** The category \(\mathbf{TTG}\) is cocomplete. The functor \(K: \mathbf{TTG} \to \mathbf{C}\) does not preserve all colimits ¹.

**PROOF.** We shall prove separately the existence of coproducts and of coequalizers in \(\mathbf{TTG}\). From the constructions it will be clear that \(K\) does not preserve all coproducts or all coequalizers.

I. Suppose \(\langle G_j, X_j, \pi_j \rangle \upharpoonright j \in J\) is a set of objects in \(\mathbf{TTG}\). Let \(G\) and \(\beta_i: G_i \to G\) be as in 3.4.11, and let for each \(i \in J,\)

\[\beta_i: G_i \to G\]

be the morphism in \(\mathbf{TTG}\) which is universal for the family of all morphisms in \(\mathbf{TTG}\) with domain \(\langle G_i, X_i, \pi_i \rangle\) and group component \(\beta_i\) (cf. 3.3.11). In addition, let \(\langle G, \Sigma_j Y_j, \sigma \rangle\) denote the coproduct of the set \(\{G_j, \Sigma_j, \sigma_j \upharpoonright j \in J\}\) in \(\mathbf{TOP}\), with coprojections \(\langle f_i, \sigma_i \rangle: \langle G, \Sigma_j Y_j, \sigma \rangle \to \langle G_i, \Sigma_j Y_j, \sigma_j \rangle\) (cf. 3.4.2). We claim that \(\langle G, \Sigma_j Y_j, \sigma \rangle\), together with the morphisms \(\langle G, \Sigma_j Y_j, \sigma \rangle \to \langle G_i, \Sigma_j Y_j, \sigma_j \rangle\), form the coproduct of the given set \(\langle G_j, X_j, \pi_j \rangle \upharpoonright j \in J\) in \(\mathbf{TTG}\).

In order to prove this, suppose that we are given morphisms \(\langle \psi_i, \varepsilon_i \rangle:\)

\[\langle G_i, X_i, \pi_i \rangle \to \langle H, \Sigma_j, \sigma \rangle\]

in \(\mathbf{TTG}\) (\(i \in J\)). Then there exists a unique morphism

¹ In view of 3.4.9 this implies that \(S\) does not preserve all colimits. Hence \(S\) cannot have a right adjoint, no more than \(K\) can have.
\(\psi: G \to H\) in \(\text{TOPGRP}\) such that \(\psi_i = \psi_i^G\) for every \(i \in J\). Using 3.3.3, we see that each \(\langle \psi_i, \sigma_i \rangle\) factorizes as \(\langle \psi_i, \sigma_i^G \rangle\langle \beta_i, \sigma_i \rangle\) over the object \(\langle G, Z, \cdot \rangle\) in \(\text{TTG}\).

\[
\begin{array}{c}
\langle G, X, \pi \rangle \\
\downarrow \psi_i \sigma_i \\
\langle H, Z, \cdot \rangle
\end{array}
\xrightarrow{\langle \beta_i, h_i \rangle}
\begin{array}{c}
\langle G, Y, \sigma_i \rangle \\
\downarrow \langle \gamma_i, \gamma_i \rangle
\end{array}
\xrightarrow{\langle \gamma_i, \cdot \rangle}
\begin{array}{c}
\langle G, Z, \cdot \rangle
\end{array}
\]

By the universal property of \(\langle \beta_i, h_i \rangle\), there exists a morphism of \(G\)-spaces
\(\langle \beta_i, h_i \rangle: \langle G, X, \pi \rangle \to \langle H, Z, \cdot \rangle\) in \(\text{TTG}\) such that \(\psi_i \sigma_i = \psi_i \psi_i^G \sigma_i\) for every \(i \in J\). Since \(\langle G, \Sigma Y, \sigma_i \rangle\) is the coproduct of the \(G\)-spaces \(\langle G, Y, \sigma_i \rangle\) in \(\text{TOP}\), this implies the existence of a unique morphism of \(G\)-spaces \(\langle \gamma_i, \cdot \rangle: \langle G, \Sigma Y, \sigma_i \rangle \to \langle G, Z, \cdot \rangle\) such that \(\psi_i^G \sigma_i = \psi_i^G \psi_i^G \sigma_i\) for every \(i \in J\).

Thus, we have obtained a morphism \(\psi_i \psi_i^G: \langle G, Y, \sigma_i \rangle \to \langle H, Z, \cdot \rangle\) in \(\text{TTG}\) such that \(\psi_i \sigma_i = \psi_i \psi_i^G \sigma_i\) for every \(i \in J\). It is easy to see that this is the unique morphism in \(\text{TTG}\) with this property (use the fact that any morphism \(\psi_i \psi_i^G: \langle G, Y, \sigma_i \rangle \to \langle H, Z, \cdot \rangle\) factorizes as \(\psi_i \psi_i^G\langle \gamma_i, \cdot \rangle\) over \(\langle G, Z, \cdot \rangle\)). This proves our claim.

\smallskip

(\textbf{Remark.}) If \(\alpha_i, p_i\) and \(\alpha\) are as in 3.4.11, then set \(\gamma_i := \beta_i p_i \alpha\). Observe that \(\gamma_i \beta_i = \beta_i\). In 3.4.11, we considered the morphisms \(\langle \beta_i, h_i \rangle: \langle G, X, \pi_i \rangle \to \langle G, X, \pi_i \rangle\).

\[
\begin{array}{c}
\langle G, X, \pi_i \rangle
\end{array}
\xrightarrow{p_i \alpha}
\begin{array}{c}
\langle G, X, \pi_i \rangle
\end{array}
\]

Let

\[
(7) \quad \gamma_i \beta_i: \langle G, X, \pi_i \rangle \to \langle G, Y, \sigma_i \rangle
\]

be the universal arrow, according to 3.3.11. Here \(\langle G, Y, \sigma_i \rangle := L \gamma_i \langle G, X, \pi_i \rangle = L \gamma_i \langle G, X, \pi_i \rangle = L \gamma_i \langle G, X, \pi_i \rangle\) (use 3.3.14(ii)).

Since \(p_i \alpha: G \cdot G_1\) is surjective, \(R \gamma_i \beta_i\) is full and faithful (cf. 3.3.14(i)), hence \(L p_i \alpha \langle \gamma_i, \pi_i \rangle\) may be replaced by \(\langle G, X, \pi_i \rangle\), and \(\langle G, Y, \sigma_i \rangle\) may be replaced by \(L \gamma_i \langle G, X, \pi_i \rangle\), hence by \(G, Y, \sigma_i\). In addition, the morphism (6) is just the same (up to isomorphism) as \(\gamma_i \beta_i: \langle G, X, \pi_i \rangle \to \langle G, Y, \sigma_i \rangle\).

It follows that the construction in the present proof is just the construction of 3.4.11, except that we first apply \(L \gamma_i\) to \(\langle G, X, \pi_i \rangle\) for each \(i \in J\).
II. For \( i=1,2 \), let \( \langle \psi_i, f_i \rangle : \langle G, X, \sigma \rangle \to \langle H, Y, \sigma \rangle \) be a morphism in \( T T G \).

Let \( \psi : H \to K \) denote the coequalizer in \( T O P G R P \) of \( \psi_1, \psi_2 : G \to H \), and let \( \sigma_0 : Y \to Z_0 \) be the coequalizer in \( T O P \) of \( f_1, f_2 : X \to Y \). In general, the equivalence relation \( R_0 \) in \( Y \) defined by \( \sigma_0 \) is not invariant under the action \( \sigma \) of \( H \). Let \( R \) be the least invariant equivalence relation in \( Y \) with \( R_0 \subseteq R \), i.e., \( R \) is the intersection of all invariant equivalence relations which include \( R_0 \).

Let \( Z_1 := Y/R \) and let \( \zeta_1 \) be the action of \( H \) on \( Z_1 \) induced by \( \sigma \). It follows easily, that the quotient mapping \( q_1 : Y \to Z_1 \) is universal for all morphisms of \( H \)-spaces \( g : Y \to Z \) with \( g f_1 = g f_2 \).

Next, let \( 1 : H \to H \) denote the identical mapping, and let, according to 3.3.11,

\[
\langle 1, q_2 \rangle : \langle H, Z_1, \zeta_1 \rangle + \langle H, Z_2, \zeta_2 \rangle := L_1 \langle H, Z_1, \zeta_1 \rangle
\]

be the morphism in \( T T G \) which is universal for all morphisms \( \langle 1, h \rangle \) in \( T T G \) with domain \( \langle H, Z_1, \zeta_1 \rangle \). Then obviously

\[
\langle 1, q_2 q_1 \rangle : \langle H, Y, \sigma \rangle + \langle H, Z_1, \zeta_1 \rangle
\]

is a morphism in \( T T G \), and this morphism is easily seen to be universal for all morphisms of \( H \)-spaces \( \langle 1, g \rangle \) with domains \( \langle H, Y, \sigma \rangle \) and satisfying the relation \( f_1 g = f_2 g \).

Finally, let

\[
\langle \phi, q_3 \rangle : \langle H, Z_1, \zeta_1 \rangle + \langle K, Z_2, \zeta_2 \rangle := L_\phi \langle H, Z_2, \zeta_2 \rangle
\]

be the morphism in \( T T G \) which is universal for all morphisms \( \langle \phi, g \rangle \) in \( T T G \) with domain \( \langle H, Z_2, \zeta_2 \rangle \). We claim that \( \langle \phi, q_3 q_2 q_1 \rangle : \langle H, Y, \sigma \rangle + \langle K, Z_3, \zeta_3 \rangle \) is the coequalizer of \( \langle \phi_1, f_1 \rangle \) and \( \langle \psi_2, f_2 \rangle \) in \( T T G \).

To this end, consider the following diagram, where \( \langle \psi, g \rangle : \langle H, Y, \sigma \rangle \to \langle L, Z, \zeta \rangle \) is any morphism in \( T T G \) with \( \langle \psi, g \rangle \langle \psi, f_2 \rangle = \langle \psi, g \rangle \langle \psi_2, f_2 \rangle \). Observe that there exists a unique morphism \( \psi' \) in \( T O P G R P \) such that \( \psi = \psi' \phi \). Now the trick is to factorize \( \psi, g \) a couple of times, using 3.3.3, and then to apply the above mentioned universality properties of \( \langle 1, q_2 q_1 \rangle \) and \( \langle \phi, q_3 \rangle \) in order to obtain the dotted arrows in the diagram.
The proof that \( <\psi',g''^n> \) is the unique morphism in TTG with the property that 
\( <\psi,g> = <\psi',g''^n> \) \( \langle a_3, q, q_2 \rangle \) is left as an exercise for the reader.

3.4.13. In the first part of the preceding proof, each \( \beta_i: G_i \rightarrow G \) is injective (indeed, \( a_3 = a_1 \) and \( a_1 \) is injective). So 3.3.13(i) shows that \( h_i: X_i \rightarrow Y_i \) is an injection (cf. (6)). We can show somewhat more, namely, the functions \( h_i: X_i \rightarrow Y_i \) (i.e.) are topological embeddings, and there exist continuous surjections \( r_i': Y_i \rightarrow X_i \) such that \( r_i'h_i = 1_{X_i} \). Thus, each \( X_i \) is a retract of \( Y_i \).

Indeed, for each \( i \in J \) we have in the first part of the proof of 3.4.12 a morphism \( f_ih_i: X_i \rightarrow \Sigma_j Y_j \) in TOP. Hence there exists a unique morphism \( k: \Sigma_j X_j \rightarrow \Sigma_j Y_j \) in TOP such that \( f_ih_i = kr_i \), where \( r_i: X_i \rightarrow \Sigma_j X_j \) is the canonical embedding. On the other hand, in 3.4.11 we obtained morphisms

\[ \beta_i, r_i: G_i, X_i, \pi \rightarrow G, \Sigma_j X_j, \pi \]

in TTG, and since \( \beta_i, r_i h_i: G_i, X_i, \pi \rightarrow G, \Sigma_j X_j, \pi \) for \( i \in J \) form the coproduct in TTG of the objects \( G_i, X_i, \pi \), it follows that there is a unique morphism \( \gamma, r: G, \Sigma_j Y_j, \pi \rightarrow G, \Sigma_j X_j, \pi \)

in TTG such that \( \beta_i, r_i = \gamma, r \). Then clearly \( \gamma = 1_G \) and \( r_i = r_i h_i \). In particular, the last equality together with the equality \( f_i h_i = kr_i \) shows that \( (rk)r_i = 1_{\Sigma_j X_j} r_i \), hence \( rk = 1_{\Sigma_j X_j} \). Hence \( k \) is a topological embedding of \( \Sigma_j X_j \) into \( \Sigma_j Y_j \), mapping each \( r_i[X_i] \) into \( r_i[Y_i] \). So by the definition of \( k \), each \( h_i \) is a topological embedding of \( X_i \) into \( Y_i \). In addition, if \( r_i \) is defined as the "restriction" and "corestriction" of \( r \) to the domain \( Y_i \) and the codomain \( X_i \), then \( r_i h_i = 1_{X_i} \).

3.4.14. Now we turn our attention to the functor \( S_1: \text{TTG} \rightarrow \text{TOP} \), defined in 3.4.1. We start with a generalization of 3.3.13(iii), where it has been shown that the restriction of \( S_1 \) to \( \text{TOP}^G \) has a right adjoint.
3.4.15. **Proposition.** The functor $S_1: \text{TTG} \to \text{TOP}$ has a right adjoint. Hence $S_1$ preserves all colimits and all epimorphisms.

**Proof.** Fix a one-point topological group $E$, and define the functor $R_1: \text{TOP} \to \text{TTG}$ by

$$ R_1: \left\{ \begin{array}{c} X \mapsto \langle E, X, \tau_X \rangle \\
 f \mapsto \langle f, \tau_f \rangle \end{array} \right. $$

Here $\tau_X$ denotes the trivial action of $E$ on the topological space $X$. Now $S_1$ is easily shown to be left adjoint to $R_1$, with unit $\eta: \text{TTG} \to R_1S_1$, given by

$$ \eta_{\langle G, X, \tau \rangle} = \langle \psi^G, \pi \rangle; \langle G, X, \tau \rangle \mapsto \langle E, X, \tau_X \rangle $$

for every object $\langle G, X, \tau \rangle \in \text{TTG}$; here $\psi^G: G \to E$ is the obvious surjection. \(
\]

3.4.16. If $\langle \psi, \phi \rangle: \langle G, X, \tau \rangle \to \langle H, Y, \sigma \rangle$ is epic in $\text{TTG}$, then $\psi: G \to H$ is surjective, by 3.4.9. Hence the arguments in 3.4.8 can be modified to the effect that we obtain a proof that $f$ is surjective. Thus, we obtain an alternative proof of 3.4.10.

The following proposition should be compared with 3.4.7, where the behaviour of $S_1$ with respect to products and equalizers is considered.

3.4.17. **Proposition.** The functor $S_1: \text{TTG} \to \text{TOP}$ preserves all products, but it does not preserve all equalizers.

**Proof.** Let $\{\langle G_j, X_j, \tau_j \rangle : j \in J \}$ be a set of ttgs. According to 3.1.12(i), their product in $\text{TTG}$ is $\langle \prod_j G_j, \prod_j X_j, \tau \rangle$ with projections $\langle \psi^j \rangle_j$:

$$ \langle \prod_j G_j, \prod_j X_j, \tau \rangle \to \langle G_i, X_i, \tau_i \rangle. $$

For each $(x_j)_j \in \prod_j X_j$ we have plainly

$$ \tau_{(x_j)_j}[x_j] = \tau_j C_j[x_j]. $$

It follows that $C_\pi = \bigcap_j C_j$ (cf. 0.2.1 for products of equivalence relations). Since each $C_j$ is an open mapping there exists a homeomorphism $g: X_j/C_j \to X_j/C_j^{\tau_j}$ such that $g \circ C_j = C_j^{\tau_j}$. If we identify $X_j/C_j$ with $X_j/C_j^{\tau_j}$ via $g$, then $S_1(\psi, f)$ is easily shown to correspond to the projection $\prod_j: \prod_j X_j/C_j^{\tau_j} \to X_j/C_j^{\tau_j}$ in $\text{TOP}$. This proves that the functor $S_1$ preserves all products.

The following example shows that $S_1$ does not preserve equalizers. Let $G$ be any topological group with at least two points, and define actions $\pi$ and $\sigma$ of $G \times G$ on $G \times G$ and on $G$ by
\[ \pi((s,t),(x,y)) := (sx,ty); \quad \sigma((s,t),x) := sx \]

for \((s,t),(x,y) \in G \times G\). Furthermore, let \(f: G \times G \to G\) be the projection (in \(\text{TOP}\)) of \(G \times G\) onto its first coordinate, and let \(\psi_1, \psi_2: G \times G \to G \times G\) in \(\text{TOPGRP}\) be defined by

\[ \psi_1(s,t) := (s,t); \quad \psi_2(s,t) := (s,e) \]

for \((s,t) \in G \times G\). The equalizer of the morphisms \(\langle \psi_1, f \rangle, \langle \psi_2, f \rangle: G \times G, G \times G, \pi \rangle \to G \times G, G, \pi \rangle\) in \(\text{TOP}\) is, by 3.1.12(ii), the morphism \(\langle \psi_1, 1_G \times G \rangle: G \times G, G \times G, \pi \rangle \to G \times G, G, \pi \rangle\), where \(\psi: G \times G \to G \times G\) is the inclusion mapping. Then \(S_1, \psi, 1_G \times G \rangle\) may be seen as the obvious mapping of 1 onto a one-point space, whereas \(S_1, \psi_1, f \rangle\) and \(S_1, \psi_2, f \rangle\) both are mappings of this one-point space onto another one. Since \(G\) has at least two points, \(S_1, \psi, 1_G \times G \rangle\) is not the coequalizer of \(S_1, \psi_1, f \rangle\) and \(S_2, \psi_2, f \rangle\) in \(\text{TOP}\). \(\square\)

3.4.18. NOTES. Most results in this subsection seem to be new. However, it is not unlikely that some of them are straightforward modifications of known facts from category theory concerning categories of algebras over a monad (or, more specifically, of known theorems about the category of \(A\)-modules, say, where \(A\) is some \(K\)-algebra, \(K\) a commutative ring; cf. the notes in 3.1.18). The only result in this direction of which the present author is aware is a theorem in F.E.J. LINTON [1969], stating that the existence of coproducts in an algebra over a monad follows from the existence of certain coequalizers. Although our methods are quite different from LINTON'S, the following similarity is quite striking. As a by-product, LINTON shows that the existence of certain coequalizers in a category \(A\) implies that the induced functor of algebras \(\theta^*: \mathbb{H} \to \mathbb{H}\) has a left adjoint, where \(\theta: \mathbb{H} \to \mathbb{H}\) is a morphism of monads. The analogue of this is our theorem 3.3.8, which played an essential role in the considerations of this subsection.

The attentive reader will have noticed that it is suggested by 1.4.11 that there is a functor from a suitable subcategory of \(\text{TOP}\) to the category of semigroups. Although this "enveloping semigroup functor" plays an important role in topological dynamics (cf. for instance the monograph [El]) it falls outside the scope of the present treatise. We return to it briefly in subsection 4.4.
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4 - SUBCATEGORIES OF TTG

First, in subsection 4.1, we shall analyse the proofs of some of the reflection and preservation properties of the functor $K: \text{TTG} \to \text{TOPGRP} \times \text{TOP}$, given in §3. In addition, some generalizations will be given. We restrict ourselves here to limits, monomorphisms and epimorphisms. This is mainly due to the fact that we are interested in the applicability of the theorem in 0.4.3 to certain subcategories of TTG in order to prove that they are reflective. This will be done in subsection 4.3. We shall consider here only subcategories of TTG of the form $K[A \times B]$, where $A$ is a subcategory of TOPGRP and $B$ is a subcategory of TOP. Consequently, we shall not consider subcategories of TTG which arise by imposing also conditions on the actions of their objects. Nevertheless, some results in subsection 4.4 are related to such subcategories, namely, the full subcategories of TTG, defined by all equicontinuous t.tgs or by all t.tgs on compact spaces having a dense orbit. Therefore we investigate what the reflection of an object of $\text{TOP}^G$ in $\text{COMP}^G$ looks like. This provides us with an example that the functor $S^G: \text{TOP}^G \to \text{TOP}$ does not map reflections of objects of $\text{TOP}^G$ in $\text{COMP}^G$ onto reflections of objects of $\text{TOP}$ in $\text{COMP}$ (i.e. $S^G$ does not "preserve reflections").

4.1. Limits, monomorphisms and epimorphisms

4.1.1. In this section we consider mainly subcategories $X$ of TTG which can be described in the following way. Let $A$ and $B$ denote subcategories of $\text{TOPGRP}$ and $\text{TOP}$, respectively, and set $X := K[A \times B]$; here $K: \text{TTG} \to \text{TOPGRP} \times \text{TOP}$ is the functor defined in 3.1.2. Thus, objects in $X$ are all t.tgs $(G,X,\pi)$ with $(G,X) \epsilon A \times B$; we do not require that $\pi: G \times X \to X$ is a morphism in $B$. 
Morphisms in X are all morphisms \(<\psi,f>\) in \(\text{TTG}\) with \((\psi,f)\) in \(A \times B\). In this section, A, B and X shall always have the above meaning.

If A has only one object \(G\) and one morphism \(1_G\), then \(K^G[A \times B]\) will be denoted \(B^G\). Obviously, \(B^G\) is a subcategory of \(\text{TOP}\), namely, \(B^G = (S^G)^+[B]\).

We shall be a little bit careless with respect to notation. The inclusion functors \(A \to \text{TOPGRP}\), \(B \to \text{TOP}\) and \(X \to \text{TTG}\) are always omitted. In addition, the restriction and corestriction of the functor \(K\) to \(X\) and \(A \times B\) will be denoted simply \(K: X \to A \times B\); similarly, we write \(G: X \to A\), \(S: X \to B\) and \(S^G: B^G \to B\).

4.1.2. At this point, we investigate which conditions have to be imposed upon \(A\) and \(B\) in order that the methods of §3 can be used in order to solve the following questions:

(i) When can limits and monomorphisms in \(X\) be calculated in \(A \times B\) (cf. 3.1.12 and 3.2.5)?

(ii) When can epimorphisms in \(X\) be calculated in \(A \times B\) (cf. 3.4.10 and 3.2.3)?

Ad (i): In order that the monad \((H,\eta,\mu)\) can be defined in \(A \times B\) similar to the definition in 3.1.6, it is necessary and sufficient that the following conditions are fulfilled:

(M1) For each object \((G,X)\) in \(A \times B\), the topological product \(G \times X\) is an object in \(B\).

(M2) For each object \((G,X)\) in \(A \times B\), the continuous functions \(\eta^G_X: x \mapsto (e,x): X \to G \times X\) and \(\mu^G_X: (s,(t,x)) \mapsto (st,x): G \times (G \times X) \to G \times X\) are morphisms in \(B\).

If so, then the category of all \(H\)-algebras may be identified with the full subcategory of \(X\), defined by all its objects \(<G,X,\pi>\) for which \(\pi: G \times X\) is a morphism in \(B\) (cf. 3.1.8); this is all of \(X\) if \(B\) is a full subcategory of \(\text{TOP}\). Resuming: if \(B\) is a full subcategory of \(\text{TOP}\) and if condition (M1) is fulfilled, then limits and monomorphisms in \(X\) can be calculated in \(A \times B\).

Ad (ii): In order to imitate the proof of 3.4.10, one has first to prove the analogue of 3.4.9, i.e. that the functor \(G: X \to A\) has a right adjoint. This can be done if

\[\text{isomorphic}\]
(E1) The category B has a final object.

Then the proof of 3.4.10 works in the present context if

(E2) Epimorphisms in A are surjective.

(E3) For each object <\(H,Y,\sigma>\) in X, the quotient mapping \(c_0: Y \rightarrow Y/C_0\) belongs to B.

(E4) For each object B in B and subset A of B, the quotient mapping \(q: B \rightarrow B/R(A)\) and the constant mapping \(f: B \rightarrow \epsilon/R(A)\) sending B into \(\epsilon[A]\) belong to B; here \(R(A) := (A \times A) \cup \{(b,b) : b \in B\}\).

We might also try to imitate the proofs indicated in 3.4.16 (cf. also 3.4.8). Then we need, among others, again condition (E3). For a discussion of the conditions we refer to the notes in 4.1.11. It appears that (E3) and (E4) are almost never fulfilled. Therefore, we shall now try to develop methods which do not require these conditions.

4.1.3. **Lemma.** Suppose the inclusion functor of B into TOP preserves limits. Then the functor \(K: X \rightarrow A \times B\) creates limits.

**Proof.** Let \(D: J \rightarrow X\) be a diagram, and set \(D_j := (\langle G_j, X_j, \pi_j \rangle, \psi_j, f_j)\) for each object \(j \in J\). Suppose the diagram \(K D: J \rightarrow A \times B\) has a limiting cone \((\psi, f): (G,X) \rightarrow KD\) in \(A \times B\); set \((\psi_j, f_j) := (\psi_j, f_j)\) for \(j \in J\). Note that \(f: X \rightarrow SD\) is a limiting cone for the diagram SD: \(J \rightarrow \text{TOP}\) in TOP. Plainly, the morphisms \(\pi_j \circ (\psi_j, f_j): G \times X \rightarrow X_j\) in TOP form a cone \(G \times X \rightarrow SD\) in TOP. Hence there exists a unique morphism \(\pi: G \times X \rightarrow X\) in TOP such that \(f_j \circ \pi = \pi_j \circ (\psi_j, f_j)\) for each \(j \in J\). It is routine to show that \(\pi\) is an action of \(G\) on \(X\), and that \(\langle \psi, f \rangle: \langle G, X, \pi \rangle \rightarrow D\) is a limiting cone in \(X\) for the diagram \(D\).

4.1.4. **Proposition.** Suppose that A and B are complete, and that the inclusion functor of B into TOP preserves limits. Then the functor \(K: X \rightarrow A \times B\) creates and preserves limits, and X is complete. In addition, \(K: X \rightarrow A \times B\) preserves and reflects monomorphisms.

**Proof.** Use 4.1.3 and 0.4.4.

4.1.5. **Lemma.** Let Y be a subcategory of TTG and let \(\langle \psi, f \rangle: \langle G, X, \pi \rangle \rightarrow \langle H, Y, \sigma \rangle\) be a monomorphism in Y. If either

\[1\] The space \(Z\) constructed in the proof of 3.2.3 can be obtained by identification of the subset \(c_0 f[X]\) of \(Y/C_0\) with a point.
(1) \( \forall x \in X : <1, \pi_x> : <G, G, \lambda, x, f> \) is in \( Y \)
or \( Y \) is a Hausdorff space and \( ^1 \)

(2) \( \forall x \in X : <1, \delta_x> : <G, E, \pi^x> \rightarrow <G, X, \pi> \) is in \( Y \),

then \( f \) is injective.

PROOF. Suppose that (2) is valid and that \( Y \) is a Hausdorff space (the proof under assumption of (1) is similar and is left to the reader). Let \( x, y \in X \) be such that \( f(x) = f(y) \). Then for all \( t \in G, \)

\[
f_{\pi_x}(t) = \sigma(\psi, fx) = \sigma(\psi, fy) = f_{\pi_y}(t)
\]

whence \( f_{\delta_x}(\pi_t^x) = f_{\delta_y}(\pi_t^y) \) for all \( t \in G \). Since \( \pi[G] \) is dense in \( E \), it follows that \( f_{\delta_x} = f_{\delta_y} \). Consequently, the morphisms \( <1, \delta_x> \) and \( <1, \delta_y> \) in \( Y \) have equal compositions with the monomorphism \( <\psi, f> \). Hence \( \delta_x = \delta_y \), and \( x = y \). This shows that \( f \) is injective. \( \Box \)

4.1.6. LEMMA. If \( <\psi, f> : <G, X, \pi> \rightarrow <H, Y, \sigma> \) is a monomorphism in \( X \) and \( f \) is injective, then \( \psi \) is monic in \( A \).

PROOF. Let \( \alpha, \beta : K \rightarrow G \) be morphisms in \( A \) such that \( \psi \alpha = \psi \beta \). Then for all \( s \in K \) and \( x \in X \)

\[
f_{\pi_x}^\alpha(s, x) = \sigma(\psi \alpha s, fx) = \sigma(\psi \beta s, fy) = f_{\pi_y}^\beta(s, x).
\]

Since \( f \) is injective, it follows that \( \pi^\alpha = \pi^\beta \). Let \( \rho := \pi^\alpha = \pi^\beta \). Then \( <\alpha, 1_x> \) and \( <\beta, 1_x> \) are morphisms in \( X \) from \( <K, X, \rho> \) to \( <G, X, \pi> \), and their composites with \( <\psi, f> \) are equal to each other. Since \( <\psi, f> \) is monic in \( X \) it follows that \( \alpha = \beta \). This shows that \( \psi \) is monic in \( A \). \( \Box \)

4.1.7. PROPOSITION. Suppose that \( B \) is a full subcategory of \( \text{TOP} \), and that one of the following conditions is fulfilled:

(i) \( A \subseteq B \).

(ii) \( B \subseteq \text{HAUS} \) and \( B \) is closed under the formation of topological products and closed subspaces.

Then the functor \( K : X \rightarrow A \times B \) preserves and reflects monomorphisms.

PROOF. Reflection is obvious since \( K \) is faithful. Preservation is an easy

\(^1\) Cf. 1.4.4(vi) for the notation.
consequence of the preceding lemmas. Condition (i) implies that (1) in 4.1.5 is fulfilled, and (ii) implies that (2) in 4.1.5 is valid. Hence, the conditions of 4.1.6 are trivially fulfilled. □

4.1.8. The preceding lemmas and propositions, from 4.1.3 up to 4.1.7 may be seen as an effort to save as much as possible if the general method, indicated in 4.1.2 for the computation of limits and monomorphisms cannot be used. For epimorphisms, the method indicated in 4.1.2 is not general at all (condition (E3) is very heavy; cf. the notes in 4.1.11 below). So our next proposition can be seen as an improvement on the above mentioned method.

4.1.9. **Lemma.** Let Y be a subcategory of TTG and let \( \psi, f : \langle G, X, \pi \rangle \rightarrow \langle H, Y, \sigma \rangle \) be an epimorphism in Y. In addition, let A be an H-invariant subset of Y, \( A \supseteq f[X] \), and let there exist an action \( \rho \) of H on \( Y_A \) such that the canonical injections \( f_1, f_2 : Y \rightarrow Y_A \) are morphisms of H-spaces. If the morphisms \( h_i f_i : \langle H, Y, \sigma \rangle \rightarrow \langle H, Y_A, \rho \rangle \) for \( i=1,2 \) belong to Y, then \( A = Y \).

**Proof.** Plainly \( f_1 = f_2 \), hence \( A = Y \). □

4.1.10. **Proposition.** Suppose that A and B satisfy the following conditions:

(i) Epimorphisms in A have a dense range.

(ii) B is a full subcategory of HAUS having a terminal object.

(iii) For any object \( Y \in B \) and closed subset \( A \) of \( Y \) the space \( Y_A \) is an object in B.

Then the functor \( K : X \rightarrow A \times B \) preserves and reflects epimorphisms.

**Proof.** Reflection: \( K \) is faithful.

Preservation: let \( \psi, f : \langle G, X, \pi \rangle \rightarrow \langle H, Y, \sigma \rangle \) be an epimorphism in X.

Since B has a terminal object, we can use the proof of 3.4.9 in order to prove that \( G : X \rightarrow A \) has a right adjoint. In particular, it follows that \( \psi \) is epic in A (cf. the discussion in 4.1.2(ii)). By (i), \( \psi[G] \) is dense in H.

Next, set \( A := \operatorname{cl}_f f[X] \). Then A is H-invariant (cf. 1.4.5). By 1.5.10, there exists an action \( \rho \) of H on \( Y_A \) making \( h_1 f_1 \) and \( h_2 f_2 \) morphisms in TTG from \( \langle H, Y_A, \sigma \rangle \) into \( \langle H, Y_A, \rho \rangle \). Obviously, these morphisms are in X, hence 4.1.9 implies that \( A = Y \). So \( f \) has a dense range in Y. Since B is a subcategory of HAUS, it follows that \( f \) is epic in B. □

\(^1\) For the definition of \( Y_A \) and of \( f_1, f_2 : Y \rightarrow Y_A \), cf. 0.4.10.
4.1.11. *Notes.* If $B$ is a full subcategory of $\text{TOP}$, then $(M1)$ implies $(M2)$, and in $(E3)$ and $(E4)$ we need only to require that the quotient spaces under consideration are objects in $B$ (which is a quite heavy requirement!).

Observe that $(M1)$ is fulfilled whenever $A \subseteq B$ and $B$ is closed with respect to the formation of topological products. Although the condition $A \subseteq B$ seems to be quite natural, it is rather inconvenient. For example, in Topological Dynamics one is interested in actions of discrete groups on compact Hausdorff spaces; here this condition would imply that one could consider only actions of finite discrete groups.\(^1\) Fortunately, the condition $A \subseteq B$ does not occur in 4.1.4, nor in 4.1.7(ii).

A problem, related to the condition $A \subseteq B$, is the following one: if $A$ and $B$ are suitable subcategories of $\text{TOPGRP}$ and $\text{TOP}$, respectively, and if $<G,X,\pi>$ is a ttg, under which additional conditions on the action $\pi$ the assumption $X \in B$ implies $G \in A$? Of course, the condition that $\pi$ is effective seems to be indispensable. As examples of this general problem we mention two particular problems:

(i) When does metrizability of $X$ imply metrizability of $G$ if $<G,X,\pi>$ is an effective ttg?

An answer is included in 1.1.23: $X$ separable and $G$ locally compact Hausdorff.

(ii) When does the condition that $X$ is an $n$-manifold imply that $G$ is a Lie group, if $<G,X,\pi>$ is an effective ttg?

The HILBERT-SMITH conjecture states that compactness of $G$ is a sufficient condition. In its generality, the conjecture is still open. For a survey and for more references to pertinent literature, cf. R.F. WILLIAMS [1968]. See the notes in 3.3.16 for a related problem.

We proceed with a brief discussion of the conditions which are sufficient in order that epimorphisms in $X$ can be calculated in $A \times B$. Let us first observe that the condition on $B$ in 4.1.10 are rather weak. Indeed, many useful full subcategories $B$ of $\text{HAUS}$ contain a one-point space and satisfy 4.1.10(iii); we mention the following ones:

$T_2$-spaces, $T_3$-spaces, Tychonov spaces (easy);
$T_4$-spaces (cf. [Du], Chap. VII, 3.3(1));
paracompact $T_2$-spaces ([Du], Chap. VIII, 2.6);
locally compact $T_2$-spaces (easy);

\(^1\) To avoid misunderstanding, actions of finite groups on compact spaces form an important field of mathematical research. Cf. also [ME], p.222, where the connection with actions of general compact groups is indicated.
k-spaces (easy);
compact $T_2$-spaces (obvious).

Concerning condition 4.1.10(i), observe that each subcategory $A$ of $\text{TOPGRP}$ is admitted in which epimorphisms are surjections (e.g., all discrete groups). However, the question whether the subcategory $\text{HAUSGRP}$ of $\text{TOPGRP}$ satisfies condition 4.1.10(i), seems still to be unsolved (of course, all morphisms in $\text{HAUSGRP}$ with a dense range are epic in $\text{HAUSGRP}$). Although the conditions (E3) and (E4) do not explicitly impose conditions on $A$, they are quite unattractive. First, (E4) works only for nice subcategories of $\text{TOP}$ if we consider closed subsets, and then 4.1.10 seems to be preferable. Second, in practice condition (E3) can only be verified for nice subcategories of $\text{TOP}$ if $A \subseteq \text{COMPGPR}$. Indeed, the question under which additional conditions on a ttg $\langle H, Y, \sigma \rangle$ (either on $H$ or on the action $\sigma$) the orbit space $Y/C_\sigma$ inherits nice properties from the phase space $Y$, has drawn considerable attention in the literature. As a general rule one can state that the orbit space has better properties according as the action looks more like the action of a compact group. In fact, orbit spaces form an important tool in the study of ttgs with a compact phase group. We shall mention now some properties which $Y/C_\sigma$ inherits from $Y$ if the phase group of $\langle H, Y, \sigma \rangle$ is a compact $T_2$-group. First, notice that in any ttg $\langle H, Y, \sigma \rangle$ with $H \in \text{COMPGPR}$ and $Y \in \text{HAUS}$, the function $c_\sigma: Y \rightarrow Y/C_\sigma$ is perfect (its fibers are the orbits, and orbits are compact because they are continuous images of $H$; moreover, $c_\sigma$ is a closed mapping by Theorem 3.1 of [Br], Chap. I, or [GH], 1.18(9)). In that case one can prove that each of the following properties are inherited from $Y$ by $Y/C_\sigma$:

$T_2, T_3$, metrizable, (cf. [Du], Chap. XI, §5);
paracompact Hausdorff (cf. [Du], Chap. VIII, 2.6);
$T_4$ (cf. [Du], Chap. VII, 3.3(1));
Tychonov (cf. [Du], Chap. XI, Problem 5.12 on p.254).

The above references do not use the fact that $Y/C_\sigma$ is the orbit space of a ttg (only the fact that $c_\sigma$ is a perfect mapping is used). Using the peculiar properties of a given ttg $\langle H, Y, \sigma \rangle$ some of the above "inheritance theorems" can be proved easier or in greater generality. For example, using normalized Haar measure on the compact $T_2$-group $H$, it is easy to show that a metrizable phase space may be assumed to have an invariant metric $d$. Then it is easy to see that
\[ \tilde{d}(c_0x, c_0y) := \inf\{d(u,v) : u \in C_0[x] \land v \in C_0[y]\} \]

defines a metric on \( Y/C_0 \). This proof generalizes to arbitrary locally compact \( T_2 \)-groups \( H \), provided \( Y \) is a locally compact metrizable space, \( Y/C_0 \) is given to be paracompact and the action \( \sigma \) of \( H \) on \( Y \) is proper. Cf. [Ks], Chap. I. Here the property "proper" (cf. also [Bo], Chap. III) may be seen as an "approximation" for the action of a compact group. Yet another "approximation" is \textit{uniform equicontinuity}. And indeed, it is easily shown that for any uniformly equicontinuous ttg \( <H,Y,\sigma> \) with \( Y \) metrizable, there exists an invariant metric on \( Y \). Cf. [SK], p. 186.

Other conditions on the action \( \sigma \) of an arbitrary topological group \( H \) on a space \( Y \) implying that \( Y/C_0 \) inherits nice properties of \( Y \) can be found in [Ks], in [Bo], Chap. III, in R.S. PALAIS [1961], and in O. HAJEK [1970; 1971].

4.2. Applications

4.2.1. The notation in this subsection will be as in subsection 4.1, and \( G \) will always denote a fixed topological group. We shall apply now the results of proposition 4.1.4, 4.1.7 and 4.1.10 to some special categories \( A \) and \( B \). Since in all examples the categories \( A \) and \( B \) are complete and \( K : X \rightarrow A \times B \) creates (hence preserves!) limits, the category \( X \) is complete. We shall not repeat this fact in each case separately.

In the case that \( A \) is the category consisting of one object \( G \) and one morphism \( 1_0 \), we shall also consider briefly some coproducts and coequalizers. Cocompleteness for subcategories of TTG will be considered more intensively in subsection 4.3 (cf. in particular 4.3.3).

4.2.2. \( A = \text{TOPGRP}; \ B = \text{HAUS}. \)

The inclusion functor of \( \text{HAUS} \) into \( \text{TOP} \) creates all limits, so by 4.1.4, the functor \( K : X \rightarrow \text{TOPGRP} \times \text{HAUS} \) creates all limits. In addition, it preserves and reflects all monomorphisms (this would also follow from 4.1.7). Finally, 4.1.10 applies in the present situation to the effect that \( K \) preserves and reflects epimorphisms.

\[ \]¹ In topological dynamics such a ttg is often called \textit{stable in the sense of Liapunov}. ¹
4.2.3. $A = \text{HAUSGRP}; B = \text{HAUS}$.
Similar to 4.2.2, except the statement on epimorphisms: we can apply 4.1.10 only if the conjecture that all epimorphisms in $\text{HAUSGRP}$ have dense ranges is assumed to be true.

4.2.4. The category $\text{HAUS}^G$.
The results concerning limits, mono- and epimorphisms are similar to those in 4.2.2 for $K'[\text{TOPGRP} \rightarrow \text{HAUS}]$.
Since coproducts in $\text{HAUS}$ can be computed in $\text{TOP}$, the proof of 3.4.2 can be given entirely within the present context. Thus, the functor $S^G_\text{HAUS} \rightarrow \text{HAUS}$ creates and preserves all coproducts. (Notice that it follows that the inclusion functor $\text{HAUS}^G \rightarrow \text{TOP}^G$ creates and preserves them as well; use 3.4.2 to prove this.)
Finally, the coequalizer $g: Y \rightarrow Z$ in $\text{HAUS}$ of a pair of morphisms $f_1, f_2: X \rightarrow Y$ in $\text{HAUS}$ is always a quotient mapping. Consequently, the proof of 3.4.3 shows that $S^G_\text{HAUS} \rightarrow \text{HAUS}$ creates all coequalizers whenever $G$ is a locally compact Hausdorff group. In this case, $\text{HAUS}^G$ is cocomplete, and $S^G$ creates and preserves all colimits.

4.2.5. $A = \text{TOPGRP}; B = \text{COMP}$.
Similar to 4.2.2.

4.2.6. $A = \text{HAUSGRP}; B = \text{COMP}$.
Similar to 4.2.3.

4.2.7. The category $\text{COMP}^G$.
The results about limits, monomorphisms and epimorphisms are similar to those in 4.2.4 for $\text{HAUS}^G$.

Observe that all finite coproducts in $\text{COMP}$ can be computed in $\text{TOP}$. So similar to 4.2.4 it can be shown that the functor $S^G_\text{COMP} \rightarrow \text{COMP}$ creates and preserves all finite coproducts. In addition, coequalizers in $\text{COMP}$ are always perfect continuous surjections, so in view of the first remark in 3.4.4 we can use the proof of 3.4.3 in order to show that $S^G$ creates all coequalizers. Consequently, $\text{COMP}^G$ is finitely cocomplete, and $S^G_\text{COMP}: \text{COMP}^G \rightarrow \text{COMP}$ creates and preserves all colimits of finite diagrams.

\footnote{We shall see in subsection 4.3 that this category is cocomplete for every topological group $G$. However, $S^G$ may not preserve limits of infinite diagrams.}
4.2.8. The category $\text{COMPG}$ for discrete $G$.

For limits, monomorphisms and epimorphisms the situation is similar to 4.2.7.

We shall indicate now why the functor $S^G: \text{COMPG} \rightarrow \text{COMP}$ creates colimits for all diagrams in $\text{COMPG}$\footnote{In particular, $\text{COMPG}$ is cocomplete. However, we shall show in 4.3.3 that discreteness of $G$ can be omitted as far as it concerns cocompleteness.}. In view of 4.2.7 it will be sufficient to prove that $S^G$ creates all (infinite) coproducts in $\text{COMPG}$. To this end, one has to apply proposition 4.2.9 below to the (created!) coproduct in $\text{HAUS}$\footnote{Epimorphisms in $\text{COMPGRP}$ are surjective; see D. FOGUNIKE [1970].} of a given set of objects in $\text{COMPG}$ (use the fact that coproducts in $\text{COMP}$ are obtained as reflections in $\text{COMP}$ of the corresponding coproducts in $\text{HAUS}$).

4.2.9. **PROPOSITION.** Let $\langle G,X,\pi \rangle$ be a tgg with $G$ a discrete group and let $\beta_X: X \rightarrow \beta X$ denote the reflection of $X$ in $\text{COMP}$. Then there exists a unique action $\sigma$ of $G$ on $\beta X$ making $\beta_X$ a morphism of $G$-spaces\footnote{In particular, $\text{COMPG}$ is cocomplete. However, we shall show in 4.3.3 that discreteness of $G$ can be omitted as far as it concerns cocompleteness.} from $X$ (with action $\pi$) into $\beta X$ (with action $\sigma$).

**PROOF.** For every $t \in G$, let $\sigma^t: \beta X \rightarrow \beta X$ be the unique continuous function satisfying $\sigma^t \beta_X = \beta_X \pi^t$. Since $G$ is discrete, we obtain a continuous mapping $\sigma: G \times \beta X \rightarrow \beta X$, and $G$ is easily seen to meet all requirements. $\square$

4.2.10. There remain several other subcategories of $\text{TTG}$ to be considered, for example the cases

$$ A = \text{COMPGRP}; B = \text{HAUS}.$$  
$$ A = \text{COMPGRP}; B = \text{COMP}. $$

In these cases, limits, mono- and epimorphisms\footnote{Epimorphisms in $\text{COMPGRP}$ are surjective; see D. FOGUNIKE [1970].} in $X$ are created and preserved by $K: X \rightarrow A \times B$ (cf. 4.2.2).

4.2.11. The results of this subsection are summarized in the scheme on p.128. Anticipating the results in 5.3.4 on the category $\text{KRG}$ for locally compact Hausdorff groups $G$, we have also inserted some properties of the functor $S^G: \text{KRG} \rightarrow \text{KR}$.

4.2.13. **NOTES.** In view of proposition 4.2.9 one might ask under what conditions an action $\pi$ of a group $G$ on, say, a Tychonov space $X$ can be extended to an action of $G$ (not merely of $G_d$) on the Stone-$\check{\text{C}}$ech compactification...
SX of X. This, and related questions are dealt with in D.H. CARLSON [1971] for the case G = R. In general, the action of G on X cannot be extended to an action of G on SX (cf. Theorem 4.10 in the above mentioned paper).

We mentioned some cases in which a subcategory $B^G$ of TOP is cocomplete (cf. 4.2.4, 4.2.8). However, if X is any full reflective subcategory of the complete and cocomplete category TTG or TOP (or of any other complete and cocomplete subcategory of TTG), and if X is closed with respect to isomorphisms, then X itself is complete and cocomplete. Therefore, the results in our next subsection show among others that $COMP^G$ and $HAUS^G$ are cocomplete for every topological group G.

**PROPERTIES OF THE FUNCTOR** $K: X \to A \times B$ and the functor $S^G: B^G \to B$

<table>
<thead>
<tr>
<th>A = TOPGRP</th>
<th>HAUS$^G$</th>
<th>A = TOPGRP</th>
<th>COMP$^G$</th>
<th>$KR^G$ (G loc. comp. $T_2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B = HAUS</td>
<td>c,p</td>
<td>c,p</td>
<td>c,p</td>
<td>c,p</td>
</tr>
<tr>
<td></td>
<td>r,p</td>
<td>r,p</td>
<td>r,p</td>
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<td></td>
<td>r,p</td>
<td>r,p</td>
<td>r,p</td>
<td>r,p</td>
</tr>
</tbody>
</table>

1) only for locally compact Hausdorff groups G
2) only creation and preservation of finite coproducts; if G is discrete, then of all coproducts.

$c$ = creates  
$p$ = preserves  
$r$ = reflects

4.3. **Reflective subcategories of TTG**

4.3.1. Notation will be in accordance with the previous subsections. However, we shall consider now subcategories $A_0, A$ of TOP and $B_0, B$ of TOP, subject to the following conditions:
(R1) $A_0 \subseteq A \subseteq \text{TOPGRP}; B_0 \subseteq B \subseteq \text{TOP}$.

(R2) $A_0$ is a full subcategory of $A$, closed with respect to isomorphisms in $A$. Similarly, $B_0$ is a full subcategory of $B$, closed with respect to isomorphisms in $B$.

Let now the subcategories $X_0$ and $X$ of $\text{TTG}$ be given by $X_0 := K^+ [A_0 \times B_0]$ and $X := K^+ [A \times B]$. Obviously, $X_0$ is a full subcategory of $X$, closed with respect to isomorphisms in $X$. Next, let $E (M)$ denote a class of epimorphisms (monomorphisms) in $X$ and suppose that $X$ has the following properties:

(R3) $X$ has the $E$-$M$-factorization property.

(R4) $X$ is co-$E$-small.

(R5) $X$ has all products.

In addition, let $X_0$ satisfy the following conditions:

(R6) $X_0$ is closed under the formation of products in $X$.

(R7) $X_0$ is closed under the formation of $M$-subobjects in $X$.

Under these conditions, $X_0$ is an $E$-reflective subcategory of $X$ (cf. 0.4.3).

We shall consider now classes $E$ and $M$ which are defined in the following way. Let $E_a$ and $E_b$ ($M_a$ and $M_b$) denote classes of epimorphisms (monomorphisms) in $A$ and $B$, respectively, and set $E := K^+ [E_a \times E_b], M := K^+ [M_a \times M_b]$.

Since $K$ is faithful, it follows that $E$ is a class of epimorphisms in $X$ and that $M$ is a class of monomorphisms in $X^1$.

Next, suppose that $A_0$ and $A$ satisfy the conditions (R3) through (R7) above with respect to $E_a$ and $M_a$. In addition, let $B_0$ and $B$ have them with respect to $E_b$ and $M_b$. (Then $A_0$ is $E_a$-reflective in $A$ and $B_0$ is $E_b$-reflective in $B$; however, we shall not use this explicitly.)

Then the categories $X_0$ and $X$ obviously have the properties (R4) and (R7). Moreover, if $K : X + A \times B$ creates all products in $X$, then also conditions (R5) and (R6) are satisfied.

However, in this abstract setting it is not possible to show that $X$ has $E$-$M$-factorization. The difficulty is the following one. Suppose we are given a morphism $\psi f : <G, X, \pi > \rightarrow <H, Y, \sigma >$ in $X$. Let $G \xrightarrow{\psi f} H \xrightarrow{1} H \{x \xrightarrow{f} Y \xrightarrow{1} Y, \text{be the } E_a-M_a \{E_b-M_b \} \text{ factorization of } \psi \text{ in } A \{\text{of } f \text{ in } B\}$. Then

---

1) Here we need only that $K$ reflects all epimorphisms and all monomorphisms, and our efforts in obtaining results on preservation of such morphisms by $K$ seem to be superfluous. Strictly speaking, this is true. However, if $K : X + A \times B$ preserves all monomorphisms and epimorphisms, the above method yields $E$ and $M$ as general as possible.
\( \langle \psi, f \rangle = \langle 1, 1 \rangle \langle \psi', f' \rangle \) is an \( E \)-factorization of \( \langle \psi, f \rangle \) iff \( \langle 1, 1 \rangle \) and \( \langle \psi', f' \rangle \) really are morphisms in \( \text{TTG} \), i.e. iff there exists an action \( \sigma' \) of \( H' \) on \( Y' \) which makes the following diagram commutative:

\[
\begin{array}{ccc}
G \times X & \xrightarrow{\pi} & X \\
\downarrow \psi' \times f' & & \downarrow f' \\
H' \times Y' & \xrightarrow{\sigma'} & Y' \\
\downarrow i \times i & & \downarrow i \\
H \times Y & \xrightarrow{\sigma} & Y \\
\end{array}
\]

Intuitively, this means that \( Y' \) is an \( H' \)-invariant subset of \( Y \). We shall present now a few examples where the above described situation is, indeed, as follows: \( H' \) is a subgroup of \( H \), \( Y' \) is a subspace of \( Y \), and \( 1 \) and \( i \) are embedding mappings. Then such a \( \sigma' \) exists iff \( Y' \) is an \( H' \)-invariant subspace of \( Y \). In that case, \( \sigma' = \sigma \mid_{H' \times Y'} : H' \times Y' \to Y' \). This condition will be fulfilled in all examples below.

4.3.2. EXAMPLES. The following examples are obtained by specification of \( A_0, A, B_0, B \), etc., taking care that the conditions (R1) through (R7) are satisfied for \( E_a \) and \( M_a \) with respect to \( A_0 \) and \( A \), and for \( E_b \) and \( M_b \) with respect to \( B_0 \) and \( B \).

(i) \( K^+[\text{TOPGRP} \times \text{HAUS}] \) is an \( E \)-reflective subcategory of \( \text{TTG} \), where \( E \) denotes the class of all morphisms in \( \text{TTG} \) whose group and space components both are surjective.

{To see this, take (in the notation of 4.3.1):}

- \( A_0 := A := \text{TOPGRP} \); \( B_0 := \text{HAUS} \); \( B := \text{TOP} \);
- \( E_a \) (\( E_b \)): all surjective morphisms in \( \text{TOPGRP} \) (\( \text{TOP} \));
- \( M_a \) (\( M_b \)): all topological embeddings in \( \text{TOPGRP} \) (\( \text{TOP} \)).

In diagram (1), we obtain \( H' = \psi[G] \), \( Y' = f[X] \), and \( 1 \) and \( i \) are embedding mappings. Then by 1.4.5, \( Y' \) is an \( H' \)-invariant subset of \( Y \), and the arguments in 4.3.1 show that not only conditions (R4) through (R7), are fulfilled, but also (R3). For (R5), notice that the functor \( K : \text{TTG} \to \text{TOPGRP} \times \text{TOP} \) creates products.}
(ii) $K^+[\text{HAUSGRP} \times \text{HAUS}]$ is an $E$-reflective subcategory of $\text{TTG}$, where $E$ denotes the class of all morphisms $\langle \psi, f \rangle$ in $\text{TTG}$ with surjective $\psi$ and $f$.

(Similar to (i).)

(iii) $K^+[\text{COMPGRP} \times \text{HAUS}]$ is $E$-reflective in the category $K^+[\text{HAUSGRP} \times \text{HAUS}]$, where $E$ is the class of all morphisms $\langle \psi, f \rangle$ in $K^+[\text{HAUSGRP} \times \text{HAUS}]$ such that $\psi$ and $f$ have dense ranges.

In the notation of 4.3.1, take:

A

$\text{COMPGRP}$, $A := \text{HAUSGRP}$, $B := B := \text{HAUS}$;

$E_a \{E_b\}$: all morphisms in $\text{HAUSGRP} \{\text{HAUS}\}$ with dense ranges;

$M_a \{M_b\}$: all closed embeddings in $\text{HAUSGRP} \{\text{HAUS}\}$.

Then the conditions (R4) through (R7) are fulfilled by $E_a M_a X_0$ and $X$ (for (R5), observe that $K: \text{X+HAUSGRP} \times \text{HAUS}$ creates all products; cf. 4.2.3). Also condition (R3) is fulfilled. Indeed, in the situation of diagram (1), $H' = \text{cl}_H(\psi G)$, $Y' = \text{cl}_f(f X)$. Hence by one of the remarks in 1.4.5, $Y'$ is an $H'$-invariant subset of $Y$. So $\sigma'$ exists in diagram (1).

(iv) $K^+[\text{HAUSGRP} \times \text{COMP}]$ and $K^+[\text{COMPGRP} \times \text{COMP}]$ are $E$-reflective subcategories of the category $K^+[\text{HAUSGRP} \times \text{HAUS}]$, where $E$ is as in (iii).

(Similar to (iii).)

(v) $K^+[\text{COMPGRP} \times \text{COMP}]$ is $E$-reflective in $K^+[\text{HAUSGRP} \times \text{COMP}]$ and in $K^+[\text{COMPGRP} \times \text{HAUS}]$, with $E$ as in (iii).

(Similar to (iii).)

(vi) Fix any topological group $G$. Then $\text{HAUS}^G$ is $\text{epi}^1$-reflective in $\text{TOP}^G$ and $\text{COMP}^G$ is $\text{epi}^2$-reflective in $\text{HAUS}^G$. Consequently, $\text{COMP}^G$ is $E$-reflective in $\text{TOP}^G$, where $E$ denotes the class of all morphisms of $G$-spaces with dense ranges.

(Similar to (i) and (iv).)

4.3.3. We can summarize the above examples by saying that the following inclusion functors have left adjoints (hence all their possible composites have):

\[1\] Recall that epimorphisms in $\text{TOP}^G$ are the surjective morphisms of $G$-spaces.

\[2\] The epimorphisms in $\text{HAUS}^G$ are the morphisms of Hausdorff $G$-spaces with dense ranges.
In particular, it follows that all subcategories of TTG mentioned here are complete and cocomplete (use the last part of 0.4.4 and the fact that TTG and TOP\textsuperscript{G} are both complete and cocomplete).

4.3.4. It is clear how colimits in the above mentioned reflective subcategories of TTG can be computed: for any diagram in the subcategory, first compute the colimit in TTG, and then compute the reflection of the resulting colimiting cone. A similar procedure can be followed for the reflective subcategories of TOP\textsuperscript{G}. Limits can directly be computed in TTG. For the computation of some of the required reflections, cf. 4.3.11 and subsection 4.4.

4.3.5. At this place the reader might expect a theorem like: "if A\textsubscript{0} and B\textsubscript{0} are reflective in A and B, respectively, then X\textsubscript{0} is reflective in X". However, if \langle G,X,T \rangle is an object in X and X\textsubscript{0} is reflective in X, as well as A\textsubscript{0} and B\textsubscript{0} are in A and B, then there is in general not a nice connection between the reflection of \langle G,X,T \rangle in X\textsubscript{0} and the reflection of X in B\textsubscript{0}. See 4.3.13 below. Therefore, it cannot be expected that reflectiveness of A\textsubscript{0} in A and of B\textsubscript{0} in B alone is sufficient for X\textsubscript{0} to be reflective in X. In many cases, however, A\textsubscript{0} and B\textsubscript{0} are known to be reflective in A and B because they satisfy some stronger conditions (e.g. the conditions mentioned in 4.3.1, or, according to the FREYD adjoint functor theorem, completeness together with a "solution set condition"). But then these stronger conditions may be used (as was done in 4.3.2) to prove that X\textsubscript{0} is reflective in X. Consequently, it seems to be not worth troubling about conditions under which reflectiveness of A\textsubscript{0} in A and of B\textsubscript{0} in B imply reflectiveness of X\textsubscript{0} in X. Rather, we shall have a brief look in the converse direction. Notation will be as before, but we shall require only condition (R1) for A\textsubscript{0}, A, B\textsubscript{0} and B.
4.3.6. **PROPOSITION.** Suppose that \( X_0 \) is a reflective subcategory of \( X \). Then:

(i) If \( B_0 \) contains a one-point space, then \( A_0 \) is a reflective subcategory of \( A \).

(ii) If \( A_0 \) contains a one-point group\(^1\), then \( B_0 \) is a reflective subcategory of \( B \).

**PROOF.** We prove only (ii). (The proof of (i) can be given in a similar way.) Let (*) denote a one-point object in \( A_0 \), and for any object \( X \) in \( B \), let \( \tau_X \) denote the obvious action of (*) on \( X \). Then the functor

\[
\begin{align*}
X &\mapsto <(*)_X, \tau_X> \text{ on objects} \\
f &\mapsto <1(*), f> \text{ on morphisms}
\end{align*}
\]

is an embedding of \( B \) into \( X \), carrying \( B_0 \) into \( X_0 \). From this, the result may easily be derived. □

4.3.7. **PROPOSITION.** Suppose that \( X_0 \) is a reflective subcategory of \( X \) and that \( A_0 \) is a reflective subcategory of \( A \). Then the functor \( G : X \rightarrow A \) preserves reflections of objects of \( X_0 \) into \( X \). That is:

If \( <G, X, \pi> \) is an object in \( X \) and \( <\psi, f> : <G, X, \pi> \rightarrow <H, Y, \sigma> \) is its reflection into \( X_0 \), then \( \psi : G \rightarrow H \) is a reflection of \( G \) into \( A_0 \).

**PROOF.** Let \( \varphi : G \rightarrow K \) be a reflection of \( G \) into \( A_0 \). Then \( \psi = \overline{\varphi} \varphi \) for a unique morphism \( \overline{\varphi} : K \rightarrow H \) in \( A_0 \). Hence \( <\psi, f> \) factorizes in \( X \) as follows:

\[
<\psi, f> \Rightarrow <G, X, \pi> \xrightarrow{<\varphi, f>_{*}} <K, Y, \sigma> \xrightarrow{<\overline{\varphi}, \overline{1}_Y>_{*}} <H, Y, \sigma>.
\]

Obviously, \( <K, Y, \sigma>_{\overline{\varphi}} \) is an object in \( X_0 \), so there exists a (unique) morphism \( <\varphi', f'> : <H, Y, \sigma> \rightarrow <K, Y, \sigma>_{\overline{\varphi}} \) in \( X_0 \) such that \( <\psi, f> = <\varphi', f'> <\psi, f> \). Then we have

\[
<\varphi', f'>_{*} <\psi, f> = <\varphi', f'> <\psi, f> = <\psi, f>,
\]

whence \( <\varphi, f'>_{*} <\psi, f> = <\psi, f> \) by universality of \( <\psi, f> \). In particular,

\[ \psi \varphi' = 1_H. \]

On the other hand,

\[ (\varphi' \overline{\varphi}) \psi = \varphi' \psi = \varphi, \]

\(^1\) See also 4.3.12 below for a particular case where \( A_0 \) does not contain a one-point group.
whence \( \psi' \bar{\psi} = 1_X \) by universality of \( \varphi \). It follows that \( \bar{\psi} \) is an isomorphism in \( A_0 \). In particular, we may conclude that \( \psi : G \to H \) is a reflection of \( G \) into \( A_0 \). □

4.3.8. COROLLARY 1. If \( X_0 \) is reflective in \( X \) and \( A_0 \) is reflective in \( A \), then the reflection in \( X_0 \) of an object \( \langle G, X, \pi \rangle \) in \( X \) having \( G \in A_0 \) may assumed to be of the form \( \langle 1_G, f \rangle : \langle G, X, \pi \rangle \to \langle G, Y, \sigma \rangle \).

PROOF. If \( G \in A_0 \), then \( 1_G : G \to G \) is a reflection of \( G \) in \( A_0 \). □

4.3.9. COROLLARY 2. The reflection of an object \( \langle G, X, \pi \rangle \) in \( \text{TTG into} \ K^\text{[COMPGRP\times COMP]} \) has the form \( \langle \alpha, f \rangle : \langle G, X, \pi \rangle + \langle G^C, Y, \sigma \rangle \), where \( \alpha : G \to G^C \) is the Bohr-compactification of \( G \).

4.3.10. PROPOSITION. If \( A_0 = A \), then the following conditions are equivalent:

(i) \( X_0 \) is a reflective subcategory of \( X \).

(ii) For each object \( G \) of \( A \), \( B_0^G \) is a reflective subcategory of \( B^G \).

If these conditions are fulfilled, then for any object \( \langle G, X, \pi \rangle \) of \( X \) the reflection in \( X_0 \) coincides with the reflection in \( B_0^G \).

PROOF. (i) \( \Rightarrow \) (ii): Apply 4.3.8 (plainly, \( A_0 \) is reflective in \( A \)).

(ii) \( \Rightarrow \) (i): Consider an object \( \langle G, X, \pi \rangle \in X \), and let \( \langle 1_G, f \rangle : \langle G, X, \pi \rangle \to \langle G, Y, \sigma \rangle \) be its reflection into \( B_0^G \). If \( \langle \psi, g \rangle : \langle G, X, \pi \rangle \to \langle H, Z, \zeta \rangle \) is a morphism in \( X \) with \( \langle H, Z, \zeta \rangle \in \langle G, X, \pi \rangle \), then \( \langle \psi, g \rangle \) can be factorized as indicated in the following diagram.

Since \( \langle 1_G, f \rangle \) is a universal arrow in \( B^G \) from \( \langle G, X, \pi \rangle \) into \( B_0^G \), there exists a unique morphism \( \langle 1_G, \bar{g} \rangle : \langle G, Y, \sigma \rangle \to \langle G, Z, \psi \rangle \) in \( B_0^G \) such that \( \bar{g} = \bar{g} f \). Now it is easily seen that \( \langle \psi, \bar{g} \rangle : \langle G, Y, \sigma \rangle \to \langle H, Z, \zeta \rangle \) is the unique morphism in \( X_0 \).
such that $\langle \psi, g \rangle = \langle \psi, g \rangle \circ \langle 1_G, f \rangle$ (by 3.3.3, any other morphism in $X_0$ with this property factorizes over $\langle G, Z, \xi \rangle$ with $\langle 1_G, 1_X \rangle$ as a factor). This shows that $\langle 1_G, f \rangle$ is a universal arrow from $\langle G, X, \pi \rangle$ into $X_0$. In particular, it follows that $X_0$ is reflective in $X$. \[\square\]

4.3.11. Next, we consider the question how to "compute" reflections in general. We shall restrict ourselves to the case that $A = \text{TOPGRP}$ and $B = \text{TOP}$, hence $X = \text{TTG}$. In addition, we shall assume that $X_0$ is a reflective subcategory of $\text{TTG}$, hence also of $K[A_0 \times \text{TOP}]$, and that $K[A_0 \times \text{TOP}]$ is a reflective subcategory of $\text{TTG}$. Thus, the following inclusion functors have left adjoints:

$$X_0 = K^+[A_0 \times B_0] \longrightarrow K^+[A_0 \times \text{TOP}] \longrightarrow K^+[\text{TOPGRP} \times \text{TOP}] = \text{TTG}.$$  

Moreover, let us assume that $A_0$ is a reflective subcategory of $A$ (e.g. because $B_0$ contains a one-point object). Now the reflection of an object $\langle H, Y, \sigma \rangle$ of $\text{TTG}$ into $X_0$ can be obtained in two steps (cf. [ML], p.101).

(i) The reflection of $\langle H, Y, \sigma \rangle$ in $K^+[A_0 \times \text{TOP}]$. Let $\psi: H \rightarrow G$ be the reflection of $H$ in $A_0$. According to 4.3.7, the reflection of $\langle H, Y, \sigma \rangle$ in $K^+[A_0 \times \text{TOP}]$ is of the form $\langle \psi, f \rangle: \langle H, Y, \sigma \rangle \rightarrow \langle G, X, \pi \rangle$. Then this arrow is at least universal in $\text{TTG}$ for all arrows $\langle \psi, g \rangle$ with domain $\langle H, Y, \sigma \rangle$. Hence it coincides (up to isomorphism) with the universal arrow which arises from the unit of adjunction of the functors $L_\psi$ and $R_\psi$ (cf. 3.3.11). Consequently, once the reflection $\psi: H \rightarrow G$ of $H$ in $A_0$ is known, the reflection $\langle \psi, f \rangle: \langle H, Y, \sigma \rangle \rightarrow \langle G, X, \pi \rangle$ of $\langle H, Y, \sigma \rangle$ in $K^+[A_0 \times \text{TOP}]$ can be computed by means of the methods of subsection 3.3. In particular, $

\langle G, X, \pi \rangle = \langle 1_G, f \rangle \circ \langle 1_G, g \rangle$:  

$$\langle G, X, \pi \rangle \rightarrow \langle G, Z, \xi \rangle.$$  

Moreover, this morphism is just the reflection of $\langle G, X, \pi \rangle$ in $B_0^G$. Thus, we reduced the more general problem to the following one, where $G$ is a fixed topological group:

Given an object $\langle G, X, \pi \rangle$ in $\text{TOP}^G$, determine the universal arrow $\langle 1_G, k \rangle$: $\langle G, X, \pi \rangle \rightarrow \langle G, Z, \xi \rangle$ in $\text{TOP}^G$ from $\langle G, X, \pi \rangle$ to $B_0^G$ whenever $B_0^G$ is a reflective category.

\[1\] Because $\psi: H \rightarrow G$ is the reflection of $H$ into $A_0$, the condition mentioned in the footnote to 3.3.11 is fulfilled.
**subcategory of **TOP**^G**.

We shall show now that under very weak and quite natural conditions reflectiveness of **B**^G_0 in TOP^G implies that **B**_c is reflective in TOP. However, even in that case, reflections are in general not preserved by the functor **S**^G: TOP^G→TOP. Examples will be indicated in 4.3.13 below.

4.3.12. If **B**^G_0 is a reflective subcategory of **B**^G, then the proof of 4.3.6 cannot be used to show that **B**_0 is a reflective subcategory of **B**. However, if for any object **X** in **B**, **τ**^X denotes the trivial action of **G** on **X**, and if <1_G,f>: <G,**X**,τ^X>→<G,**Y**,**σ**> is the reflection of <G,**X**,**τ**^X> in **B**^G_0, then it is easy to show that f: **X**→**Y** is a universal arrow from **X** to **B**_0, provided f is an epimorphism in **B**. Thus, we proved:

Let **E** be a class of epimorphisms in **B** and let **E**^G be the class of all (epi!) morphisms in **B**^G of the form <1_G,f> with f ∈ **E**. If **B**^G_0 is **E**^G-reflective in **B**^G, then **B**_0 is **E**-reflective in **B**.

4.3.13. NOTES. Although most results in this subsection could not be traced back in the literature (at least in this form), they are not very surprising. The reflection <α_G,f>: <G,**X**,π>→<G,**Y**,**σ**> of a ttg <G,**X**,**π**> in K[COMP**G**×COMP] has been considered earlier by M.B. LANDSTAD [1972].

There it has been shown that f: **X**→**Y** can be obtained as the Hausdorff completion of **X** with respect to a certain uniformity on **X**. (This uniformity is quite similar to the one considered by E.M. ALFSEN & P. HOLM [1962] for topological groups, leading to a construction of the Bohr compactification.) Similar to 2.2.9, there turns out to be a nice relationship between <α_G,f> and a certain subalgebra of **C**_u(**X**).

In contradistinction to the functor **G**: κ→A, the functors **S**: **X**→**B** and **S**^G: **B**^G→**B** behave badly with respect to reflections. For example, COMP^G is reflective in TOP^G, but the functor **S**^G does not preserve reflections of objects of TOP^G in COMP^G. This means, of course, that the space component of the reflection <1_G,k>: <G,**X**,**π**>→<G,**Z**,**ζ**> of <G,**X**,**π**> in COMP^G is in general not the reflection of **X** in COMP. If it were, then the action of **G** on **Z** could be "extended" to an action of **G** on the reflection **BZ** of **Z** in COMP. It has already been indicated in 4.2.13 that this cannot always be done if **G** is not discrete. Other examples will be given in the next subsection.

Another question is, whether the reflection <1_G,k>: <G,**X**,**π**>→<G,**Z**,**ζ**> of an object <G,**X**,**π**> ∈ HAUS^G in COMP^G is such that **k** is a topological
embedding. A necessary condition for this to be so is that $X$ is a Tychonov space, but it is an open problem whether this condition is sufficient. However, if $X$ can equivariantly be embedded in some compact Hausdorff $G$-space $Y$, say by $<1_G, g> : <G, X, \pi> \rightarrow <G, Y, \sigma>$, then $g = \bar{g}_k$ for some equivariant mapping $\bar{g}_k : Z \rightarrow Y$, and it can easily be seen that $k$ has now to be a topological embedding because $g$ is.

This is why we are interested in equivariant embeddings of Tychonov $G$-spaces in compact $G$-spaces. This problem will be considered in subsection 7.3. (To be sure, the compactifications considered there are in general not the reflections into $\text{COMP}^G$).

### 4.4. Some particular reflections

#### 4.4.1. We shall consider in this subsection reflections of a ttg $<G, X, \pi>$ in $\text{COMP}^G$ and in $K^+[\text{COMP}^\text{GRP} \times \text{COMP}]$. As has been pointed out in 4.3.11, the latter reflections can be reduced to the former ones (even to reflections in $\text{COMP}^H$ of ttgs of the type $<H, Y, \sigma>$ with $H = G^2$, an object in $\text{COMP}^\text{GRP}$).

First, we have to consider reflections of objects of $\text{TOP}^G$ into $\text{HAUS}^G$. Essential in the following proposition is that the reflection of any topological space into $\text{HAUS}$ is a quotient mapping.

#### 4.4.2. **Proposition.** Let $<G, X, \pi>$ be an object in $\text{TOP}^G$ and let $f : X \rightarrow Y$ be the reflection of $X$ in $\text{HAUS}$. If one of the following conditions is fulfilled, then there exists a unique action $\sigma$ of $G$ on $Y$ making $f$ equivariant. In that case, $<1_G, f> : <G, X, \pi> \rightarrow <G, Y, \sigma>$ is the reflection of $<G, X, \pi>$ into $\text{HAUS}^G$. The conditions are:

(i) $f$ is an open mapping.

(ii) $f$ is a perfect mapping.

(iii) $G$ is a locally compact Hausdorff group.

(iv) $G \times Y$ is a $k$-space.

**Proof.** Since $f : X \rightarrow Y$ is the reflection of $X$ into $\text{HAUS}$, there exists for each $t \in G$ a unique continuous mapping $\sigma^t : X \rightarrow Y$ such that $\sigma^t f = ft \sigma^t$. It is easily seen that we obtain in this way an action of $G$ on $Y$ such that $f$ is equivariant with respect to the actions $\pi$ and $\sigma$ of $G$ on $X$ and $Y$, respectively. Now $f$ is known to be a quotient mapping. It follows immediately from 1.5.7 that $\sigma : G \times Y \rightarrow Y$ is continuous whenever one of the conditions (i) through (iv) is fulfilled. Therefore, $<G, Y, \sigma>$ is a ttg, and $\sigma$ is the unique action of $G$ on $Y$ making $<1_G, f>$ a morphism in $\text{TOP}^G$. We claim that $<1_G, f> : <G, X, \pi> \rightarrow <G, Y, \sigma>$
is the reflection of $<G, X, \pi>$ in $\text{HAUS}^G$. For if $<1_G, g>: <G, X, \pi> \to <G, Z, \zeta>$ is any morphism in $\text{TOP}^G$ with $Z \in \text{HAUS}$, then $g = \tilde{g}f$ for some (unique) continuous function $\tilde{g}: Y \to Z$. Now the equations

$$(\tilde{g}^{-1} f) = \tilde{g} f t = \tilde{g} t = \zeta^{-1} g = (\zeta^{-1} g) f$$

($t \in G$) and the fact that $f$ is a surjection imply that $\tilde{g}$ is equivariant. So $<1_G, \tilde{g}>$ is the unique morphism in $\text{HAUS}^G$ such that $<1_G, \tilde{g}> = <1_G, g><1_G, f>$. This proves our claim. \[ \square \]

4.4.3. COROLLARY. If $G$ is a locally compact Hausdorff group then the functor $S^G: \text{TOP}^G \to \text{TOP}$ preserves all reflections of objects of $\text{TOP}^G$ into $\text{HAUS}^G$. If $G$ is any topological group, then the functor $S^G$ preserves all reflections into $\text{HAUS}^G$ of objects $<G, X, \pi>$ of $\text{TOP}^G$ with $X$ compact.

PROOF. In both situations, $S^G$ "creates" the reflections \[1\] into $\text{HAUS}^G$ of the objects under consideration (notice that a continuous mapping of a compact space onto a $T_2$-space is perfect). Now the corollary follows from the fact that reflections are unique (up to isomorphism). \[ \square \]

4.4.4. We are now in a position that we can "describe" the reflection of an arbitrary object $<G, X, \pi>$ of $\text{TOP}^G$ in $\text{COMP}^G$.

First, there is the action $\pi'$ of $G_d$ on $\beta X$ making $\beta_X: X \to \beta X$ equivariant with respect to the actions $\pi$ and $\pi'$ of $G_d$ on $X$ and $\beta X$, respectively (cf. proposition 4.2.9).

Next, let $1: G_d \to G$ be the identity, and consider the arrow $<1, g>: <G_d, \beta X, \pi'> \to <G, Z, \zeta>$ which is universal for the class of all morphisms $<1, g'>$ in $\text{TOP}$ with domain $<G_d, \beta X, \pi'>$ (cf. 3.3.11). By 3.3.13(ii), $g: \beta X \to Z$ is a bijection, so that $Z$ is certainly compact (but presumably not Hausdorff). Notice that $<1_G, \tilde{g} X>: <G, X, \pi> \to <G, Z, \zeta>$ is a morphism in $\text{TOP}^G$.

Finally, let $<1_G, \tilde{f}>: <G, Z, \zeta> \to <G, Y, \sigma>$ be the reflection of $<G, Z, \zeta>$ in $\text{HAUS}$. Since $Z$ is compact, it follows from 4.4.3 that $f: Z \to Y$ is the reflection of $Z$ in $\text{HAUS}$ and that $\sigma$ is uniquely determined by the condition that $f$ be equivariant. So $<G, Y, \sigma>$ may be assumed to be known (cf. also the explanation of our policy in 3.1.1). Since $f$ is surjective and $Z$ is

\[1\] "Creation of reflections" has not been defined, neither in [ML], nor by us. What we mean by it is just what has been described in the preceding proposition.
compact, \( Y \) is an object in \( \text{COMP} \). Now a straightforward argument shows, that the arrow
\[
<1, f g \beta_X>: <G, X, \pi> \rightarrow <G, Y, \sigma>
\]
in \( \text{TOP}^G \) is universal from \( <G, X, \pi> \) to \( \text{COMP}^G \), i.e. it is the reflection of \( <G, X, \pi> \) in \( \text{COMP}^G \) (use the several universality properties of \( \beta_X \), \( <1, g> \) and \( <1, f> \), and the fact that \( f g \beta_X \) has a dense range).

If the space \( X \) above is compact (but not Hausdorff), then the preceding construction may be reduced to its last step. So let \( f: X \rightarrow Y \) be the reflection of \( X \) in \( \text{HAUS} \) and let \( \sigma \) be the unique action of \( G \) on \( Y \) making \( f \) equivariant. Then \( Y \in \text{COMP} \), and it is easy to see that \( <1, f>: <G, X, \pi> \rightarrow <G, Y, \sigma> \) is not only the reflection of \( <G, X, \pi> \) into \( \text{HAUS}^G \), but that it is also its reflection into \( \text{COMP}^G \).

If the group \( G \) is compact Hausdorff, then it can be shown that the orbit space \( Y / C_0 \) of the reflection of \( <G, X, \pi> \) in \( \text{COMP}^G \) is just the reflection of \( X / C_0 \) in \( \text{COMP} \). See 4.4.13(v) below. (This case is of particular interest because the computation of the reflection of an arbitrary \( \tau \tau g <H, Z, \xi> \) in \( K^r[\text{COMP}^G \times \text{COMP}] \) requires computation of the reflection of a \( G \)-space \( <G, X, \pi> \) in \( \text{COMP}^G \) with \( G \) a compact Hausdorff group, viz. \( G = \mathbb{H}^2 \); cf. 4.3.11.)

4.4.5. Using 4.3.11, 4.3.9 and 4.4.4, we can give the following description of the reflection of a \( \tau \tau g <G, X, \pi> \) with \( X \in \text{COMP} \) into \( K^r[\text{COMP}^G \times \text{COMP}] \). It is the morphism
\[
<\alpha, g f>: <G, X, \pi> \rightarrow <G^C, X, \sigma>
\]
where \( \alpha \), \( g \), \( f \), \( Y \) and \( \sigma \) are obtained as follows:

\( \alpha: G \rightarrow G^C \) is the Bohr-compactification of \( G \).
\( <\alpha, f>: <G, X, \pi> \rightarrow L_{\alpha}<G, X, \pi> \) is the universal arrow according to 3.3.11; notice that the phase space \( X' \) of \( L_{\alpha}<G, X, \pi> =: <G^C, X', \pi'> \) is a quotient of \( G^C \times X \). In particular, it follows that \( X' \) is compact.
\( g: X' \rightarrow Y \) is the reflection of \( X' \) in \( \text{HAUS} \) (so \( Y \in \text{COMP} \)).
\( \sigma \) is the unique action of \( G^C \) on \( Y \) making \( g \) equivariant.

The reflection of a \( \tau \tau g <G, X, \pi> \) with \( X \in \text{COMP} \) into \( K^r[\text{COMP}^G \times \text{COMP}] \) has obtained considerable attention in the literature. However, there a quite different terminology is used, so that we have to reformulate the matter.
To do so, we introduce a new category, viz. COMPEQ. It is the full subcategory of $K^+[\text{TOPGRP}\times\text{COMP}]$ determined by all objects with an equicontinuous action. Thus, a ttg $<G,X,\pi>$ is in COMPEQ iff $X \in \text{COMP}$ and $\pi[G]$ is equicontinuous on $X$ (with respect to the unique uniformity of $X$).

4.4.6. Obviously, $K^+[\text{COMPGRP}\times\text{COMP}] \subseteq \text{COMPEQ}$. Indeed, an action of a compact group on any uniform space is equicontinuous by a straightforward compactness argument (namely, 0.2.2(ii)).

Moreover, it is not difficult to see that COMPEQ is closed with respect to the formation of products in TTG and with respect to the passage to closed invariant subspaces. Therefore, it can be shown by means of methods similar to the proof of theorem 0.4.3 that COMPEQ is a reflective subcategory of TTG. However, we shall present a proof which relates the reflection of an object of TTG in COMPEQ with its reflection in $K^+[\text{COMPGRP}\times\text{COMP}]$.

4.4.7. **Proposition.** The subcategory COMPEQ is reflective in TTG. For each object $<G,X,\pi>$ in TTG the reflections in COMPEQ and in $K^+[\text{COMPGRP}\times\text{COMP}]$ are related as follows: if $<\alpha,k>: <G,X,\pi> \to <G^C,Y,\sigma>$ is the reflection in $K^+[\text{COMPGRP}\times\text{COMP}]$, then the reflection in COMPEQ is $<1_G,k>: <G,X,\pi> \to <G,Y,\sigma^2>$.

**Proof.** The transition group of $<G,Y,\sigma^2>$ is a subgroup of the transition group of $<G,X,\pi>$. As $<G^C,Y,\sigma>$ is an equicontinuous ttg it follows that $<G,Y,\sigma^2>$ is equicontinuous. Next, we show that $<1_G,k>: <G,X,\pi> \to <G,Y,\sigma^2>$ is the reflection of $<G,X,\pi>$ in COMPEQ.

To this end, consider a morphism $<\psi,\delta>: <G,X,\pi> \to <G,Z,\xi>$ in TTG, where $<G,Z,\xi>$ is an object in COMPEQ. By 1.3.18, the enveloping semigroup $E_Z$ of $<G,Z,\xi>$ is a compact Hausdorff topological homeomorphism group on $Z$. Thus, we obtain a morphism $<\xi,1_Z>: <G,Z,\xi> \to <E_Z,Z,\delta>$ in TTG (cf. 1.4.4(vi)). Now $<\xi,\delta>: <G,X,\pi> \to <E_Z,Z,\delta>$ is a morphism in TTG, where $<E_Z,Z,\delta>$ is an object in $K^+[\text{COMPGRP}\times\text{COMP}]$. Since $<\alpha,k>: <G,X,\pi> \to <G^C,Y,\sigma>$ is the reflection of $<G,X,\pi>$ in the latter category, it follows that there exists a unique morphism $<\varphi,\alpha>: <G^C,Y,\sigma> \to <E_Z,Z,\delta>$ in TTG such that $<\xi,\delta> = <\varphi,\alpha,\alpha>$. 

The following calculation shows that $\langle \psi, h \rangle: \langle G, Y, o^G \rangle \rightarrow \langle H, Z, \xi \rangle$ is a morphism in $\text{TTG}$:

$$h^G(t, y) = h_0(\alpha t, y) = \delta(\delta \alpha t, hy) = \xi(\xi \alpha t, hy)$$

$(t \in G, y \in Y)$. In addition, $\langle \psi, g \rangle = \langle \psi, h \rangle \langle 1_G, k \rangle$, and $\langle \psi, h \rangle$ is the unique morphism in TTG with this property (by 4.3.2 (ii), (iii) and (v), $k$ has a dense range!). This completes the proof. $\square$

4.4.8. As was noticed in the above proof, $k$ has a dense range. Stated otherwise, if $E$ is the class of all morphisms $\langle 1_G, k \rangle$ with $G \in \text{TOPGRP}$ and with $f$ a continuous mapping with a dense range, then COMPEQ is $E$-reflective in $\text{TTG}$.

In the literature the following terminology is often used. If $\langle G, X, \pi \rangle$ is a ttg with $X \in \text{COMP}$, then its reflection $\langle 1_G, k \rangle: \langle G, X, \pi \rangle \rightarrow \langle G, Y, o^G \rangle$ in COMPEQ is called the maximal equicontinuous factor of $\langle G, X, \pi \rangle$. The enveloping semigroup of $\langle G, Y, o^G \rangle$ is an object in $\text{COMPGRP}$ (cf. 1.3.18). It is called the structure group of $\langle G, X, \pi \rangle$.

Notice that in this case $k: X \rightarrow Y$ is a surjection. It can be described following the lines of 4.4.5 (indeed, $k = gf$ with notation as in 4.4.5).

4.4.9. The reader may have noticed that there is a great similarity between the proofs of 4.4.2 and of 3.4.3. The reader might also have asked himself why the functor $S^G: \text{TOP}^G \rightarrow \text{TOP}$ preserves reflections into $\text{HAUS}^G$ if $G$ is locally compact $T_2$, whereas it does not preserve reflections into $\text{COMP}^G$ (not even if $G$ is compact, as we shall see below). The following lemma will provide a partial answer to these, and similar, questions.
4.4.10. **Lemma.** Let \((P, Q, \alpha, \beta)\) be an adjunction from the category \(Y\) to the category \(C\). Let \(Y_0\) be a reflective subcategory of \(Y\), say with reflections \(\rho_Y : Y \to FY\) \((Y \in Y)\), and let \(C_0\) be a subcategory of \(C\) such that the following conditions are fulfilled:

(i) \(P[Y_0] \subseteq C_0\) and for each object \(C\) in \(C_0\), the arrow \(\beta_C : PQC + C\) is in \(C_0\).
(ii) \(Q[C_0] \subseteq Y_0\) and for each object \(Y\) in \(Y_0\), the arrow \(\alpha_Y : Y + QPY\) is in \(Y_0\).

Then for each object \(Y\) in \(Y\) the arrow \(\rho_Y : PY + PFY\) in \(C\) is universal for the class of all arrows \(f : PY + C\) with \(C \in C_0\).

**Proof.** First, notice that for any object \(Y\) in \(Y\), the object \(FY\) is in \(Y_0\), hence \(PFY\) is in \(C_0\). Next, consider a morphism \(f : PY + C\) in \(C\) with \(C \in C_0\).

Then \(Qf \circ \alpha_Y : Y + QC\) is a morphism in \(Y\) with \(Qc \in Y_0\). Hence there exists a morphism \(f' : FY + QC\) in \(Y_0\) such that the first one of the following diagrams commutes:

\[
\begin{array}{ccc}
Y & \xrightarrow{\rho_Y} & FY \\
\downarrow{\alpha_Y} & & \downarrow{f'} \\
QFY & \xrightarrow{Qf} & QC
\end{array}
\quad
\begin{array}{ccc}
PY & \xrightarrow{\rho_Y} & PFY \\
\downarrow{\beta_C} & & \downarrow{Pf'} \\
C & \xleftarrow{PQY} & PQC
\end{array}
\]

We claim that the second diagram commutes as well. To prove this, observe that

\[
\beta_C \circ Pf' \circ \rho_Y = \beta_C \circ P(f' \circ \rho_Y) = \beta_C \circ P(Qf \circ \alpha_Y).
\]

According to 0.4.2 (in particular, diagram (3) and formula (4)), \(Qf \circ \alpha_Y\) is the unique morphism \(h\) in \(Y\) such that \(\beta_C \circ Ph = f\). So, \(f = (\beta_C \circ Pf') \circ \rho_Y\), as was claimed. Observe that \(\beta_C \circ Pf'\) is a morphism in \(C_0\), by condition (i).

Finally, if \(g : PFY + C\) is any other morphism in \(C_0\) such that \(f = g \circ \rho_Y\), then \(g = \beta_C \circ Pg'\) for some unique morphism \(g' : FY + QC\) in \(Y\). Then \(f = \beta_C \circ P(g' \circ \rho_Y)\). However, we have seen above, that \(Qf \circ \alpha_Y\) is the unique morphism in \(Y\) such that \(\beta_C \circ P(Qf \circ \alpha_Y) = f\), so \(g' \circ \rho_Y = Qf \circ \alpha_Y\). Again according to 0.4.2, \(g' = Qg \circ \alpha_{FY}\), so by condition (ii), \(g'\) is a morphism in \(Y_0\). Since \(f'\) was the unique morphism in \(Y_0\) such that \(f' \circ \rho_Y = Qf \circ \alpha_Y\), it follows that \(f' = g'\). Consequently, \(g = \beta_C \circ Pf'\). This shows that \(\beta_C \circ Pf'\) is the unique morphism in \(C_0\) whose composite with \(\rho_Y\) is \(f\). ☐
4.4.11. Notice that the second part of condition (ii) is only used to ensure the uniqueness of the morphism $\beta_C \circ P'f'$ in the above proof. It is clear, that this uniqueness can also be proved if condition (ii) is replaced by the following one:

(ii)' For each object $C$ in $C_0$, $QC \in Y_0$ and, in addition, $\rho_Y$ is an epimorphism in $Y$ for each object $Y \in Y$.

For then the existence of $f'$ in the preceding proof is guaranteed as before, and the uniqueness of $\beta_C \circ P'f'$ follows from the fact that $P\rho_Y$ is epic in $C$ ($P$ preserves epimorphisms because it has a right adjoint).

4.4.12. If in the preceding lemma $C_0$ is given to be a reflective subcategory of $C$, then obviously $P\rho_Y$ is the reflection of $PY$ into $C_0$. Thus, the functor $P$ preserves reflections of objects of $Y$ into $Y_0$.

It is useful to observe that the lemma implies that $C_0$ will be a reflective subcategory of $C$ if, in addition to the conditions (i) and (ii), it is required that $P$ maps the object class of $Y_0$ onto the object class of $C_0$.

4.4.13. APPLICATIONS. We shall describe now briefly some applications of the preceding remarks. Most of the details are left to the reader.

(i) Let $X, X_0, A, A_0, B, B_0$ and $G : X \rightarrow A$ be as in subsection 4.3. Suppose that $B_0$ contains a one-point object $(*)$ and that for each object $X$ in $B$ the obvious function $f : X \rightarrow (*)$ is a morphism in $B$ (so $(*):$ is a final object in $B$). Under these conditions, the assumption that $X_0$ is a reflective subcategory of $X$ implies that $A_0$ is a reflective subcategory of $A$ and, in addition, the functor $G : X \rightarrow A$ preserves the reflections of objects of $X$ in $X_0$. To prove this, take in 4.4.10, $Y := X, Y_0 := X_0, C := A, C_0 := A_0,$ and $P := G$; then the functor $G : X \rightarrow A$ has a right adjoint $Q$, namely the functor

$$Q : \begin{cases} \text{on objects} \\
G \mapsto <G,(*),\gamma^G> \\
\psi \mapsto <\psi,1_(*)> \\
\end{cases} \text{on morphisms}$$

(cf. 4.1.2(ii), or the proof of 3.4.9). Then 4.4.10 and 4.4.12 yield the desired results. (Notice that these results can also be proved by using 4.3.6(i) and 4.3.7.)
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(ii) Let \( G \) be a locally compact Hausdorff group. It will be shown in §6 that the functor \( S^G : \text{TOP}^G \to \text{TOP} \) has a right adjoint \( M^G : \text{TOP} \to \text{TOP}^G \). The functor \( M^G \) is defined by

\[
\begin{align*}
M^G : \{ & \quad X \mapsto <G, C(G, X), p> \quad \text{on objects} \\
& \quad f \mapsto <1^G, f^>_p \quad \text{on morphisms}
\end{align*}
\]

(cf. 6.3.6(iii)). In particular, it maps \( \text{HAUS} \) into \( \text{HAUS}^G \), so all requirements of lemma 4.4.10 are satisfied if we take \( Y := \text{TOP}^G \), \( Y_0 := \text{HAUS}^G \), \( C := \text{TOP} \) and \( C_0 := \text{HAUS} \). Consequently, the functor \( S^G \) preserves the reflections of objects of \( \text{TOP}^G \) in \( \text{HAUS}^G \) (cf. also the first part of 4.4.3). (Notice that this proof fails if we try to replace \( \text{HAUS} \) by \( \text{COMP} \); the functor \( M^G \) does not send \( \text{COMP} \) into \( \text{COMP}^G \)!)"
1.3.10(iii)). Hence the functor $S^G_1: \text{TOP}^G \to \text{TOP}$ (cf. 3.3.13(iii)) maps $	ext{COMP}^G$ into COMP. According to 3.3.13(iii), the functor $S^G_1$ has a right adjoint. Now 4.4.12 can be used to show that $S^G_1$ preserves reflections of arbitrary $G$-spaces in $\text{COMP}^G$. Thus, if $<1_0,f>: <G,X,\pi> \to <G,Y,\sigma>$ is the reflection in $\text{COMP}^G$ of the $G$-space $<G,X,\pi>$, then the induced morphism $f': X/\pi \to Y/\sigma$ is just the reflection of $X/\pi$ in COMP.

4.4.14. Now we can easily provide an example of a $G$-space such that the space component of its reflection in $\text{COMP}^G$ is not the reflection of the phase space in COMP. To this end, consider any compact Hausdorff group $G$ and any Tychonoff space $X$. Then the space component of the reflection of $<G,GxX,\mu>$ in $\text{COMP}^G$ is, according to 4.4.13(iv), the morphism $1GxX: GxX \to GxX$ in TOP. The reflection of $G$ in COMP is $1G: G \to G$, so we can write $1GxX: GxX \to GxX$ for this morphism. By a result of I. GLICKSBERG [1959], this can only be the reflection of $G \times X$ in COMP if $G \times X$ is pseudocompact. So we have our desired counterexample if we take for $X$ a non-pseudocompact space. Another example could be provided by the ttg $<G,G,\lambda>$ for any non-compact, non-discrete locally compact Hausdorff group $G$.

To this end, we shall first describe some properties of the reflection of $<G,G,\lambda>$ in $\text{COMP}^G$, where $G$ is an arbitrary topological group. Let this reflection be denoted by

$$<1_0,g>: <G,G,\lambda> \to <G,U,\nu>.$$ 

In addition, set $u := g(e)$. An easy calculation shows that $g = \nu u$; hence $g[G]$ is the orbit of $u$ in $U$. By 4.3.2(vi), $g$ has a dense range in $U$, that is, the orbit of $u$ is dense in $U$.

4.4.15. **PROPOSITION.** Let $<G,X,\pi>$ be any object in $\text{COMP}^G$ and let $x \in X$. There exists a unique morphism $<1_0,f>: <G,U,\nu> \to <G,X,\pi>$ in $\text{COMP}^G$ such that $x = f(u)$, i.e. $\pi x = f \circ \nu u$.

**PROOF.** For an equivariant mapping $f: U \to X$ the condition $x = f(u)$ is equivalent to the condition $\pi x = f \circ \nu u$. Notice that $<1_0,f>: <G,G,\lambda> \to <G,X,\pi>$ is a morphism in TOP with codomain in $\text{COMP}^G$. So existence and unicity of $<1_0,f>$ as meant in our proposition follow immediately from the universal property of $<1_0,U>$. □

4.4.16. **COROLLARY.** Every compact Hausdorff $G$-space which is the orbit-closure of one of its points is the continuous equivariant image of $U$. □
4.4.17. **Proposition.** The compactification \(^1\) of \(G\) is the unique compactification of \(G\) (up to isomorphism)\(^2\) with the property that \(C^*(u_u): h \mapsto h \circ u_u\) maps \(C(U)\) onto \(RUC^*(G)\).

**Proof.** First, we show that \(C(U)\) is mapped into \(RUC^*(G)\). Let \(h \in C(U)\). By an elementary compactness argument (namely 0.2.2(i), applied to the continuous function \(h \circ u_u\)), it follows that the mapping \(t \mapsto h \circ u_u: G \to C(U)\) is continuous. Since \(C^*(u_u): C(U) \to C(G)\) is continuous, it follows that \(t \mapsto h \circ u_u: G \to C(G)\) is continuous as well. However, \(u_u \circ u_u = \lambda_t\), so the mapping \(t \mapsto (h \circ u_u) \circ \lambda_t\) is continuous from \(G\) into \(C(G)\). Therefore, by the "right" analog of 2.2.2, \(h \circ u_u \in RUC(G)\). Obviously, \(h \circ u_u\) is bounded, hence \(h \circ u_u \in RUC^*(G)\).

Conversely, suppose we are given any \(f \in RUC^*(G)\). By 2.1.9, \(\langle G, K_G[f], \beta \rangle\) is an object in \(\text{COMP}^G\), in which \(f\) has a dense orbit. By 4.4.15, there exists a morphism of \(G\)-spaces \(k: U \to K_G[f]\) such that \(\beta_f = k \circ u_u\). If \(\delta: K_G[f] \to \mathbb{F}\) denotes evaluation-at-\(e\), then \(\delta \circ k \circ u_u \in C(U)\), and \(C^*(u_u)(\delta \circ k) = \delta \circ k \circ u_u = \delta \circ \beta_f = f\). This proves that \(C^*(u_u)\) maps \(C(U)\) onto \(RUC^*(G)\).

Finally, unicity follows from [Se], 7.7.1 and 7.7.2. \(\square\)

4.4.18. **Corollary.** If \(G\) is a Hausdorff group, then \(u_u: G \to U\) is a topological embedding, and \(\langle G, U, u_u \rangle\) is an effective action.

**Proof.** By the lemma in 0.2.7, \(RUC^*(G)\) separates points and closed subsets of \(G\). Now the result that \(u_u: G \to U\) is a topological embedding is an easy consequence of the fact that \(C^*(u_u)\) maps \(C(U)\) onto \(RUC^*(G)\). Finally, if \(u_u(t) = u_u(e)\) for some \(t \in G\), then \(u_u(t) = u_u(e)\), hence \(t = e\). \(\square\)

4.4.19. Since \(\beta_G: G \to BG\) is the unique compactification (up to isomorphism) of \(G\) such that \(C^*(\beta_G): h \mapsto h \circ \beta_G\) maps \(C(BG)\) onto \(C^*(G)\), it is obvious from 4.4.17 that the following statement is true: the forgetful functor \(S^G: \text{TOP}^G \to \text{TOP}\) maps the reflection of \(\langle G, G, \lambda \rangle\) in \(\text{COMP}^G\) onto a reflection of \(G\) in \(\text{COMP}\) iff \(RUC^*(G) = C^*(G)\), that is, iff each bounded continuous function on \(G\) is uniformly continuous.

---

\(^1\) A compactification of \(G\) is just a continuous mapping \(f: G \to X\) with \(X \in \text{COMP}\) and \(\{g \in G \mid f = f\}\) dense in \(X\).

\(^2\) Two compactifications \(g_i: G \to X_i\) \((i=1,2)\) are said to be isomorphic if \(g_2 = f g_1\) for some homeomorphism \(f: X_1 \to X_2\).

\(^3\) Recall that \(K_G[f]\) is the closure of \(\{p^G(t) \mid t \in G\}\) in \(C(G)\).

\(^4\) Cf. [GJ], 6.5 or [He], 2.3.2 and 2.3.3.
By the results mentioned in Appendix A, the equality of $RUC^*(G)$ and $C^*(G)$ implies that $G$ is either pseudocompact or a P-space (i.e. every countable intersection of open sets is open). Now suppose that $G$ is a locally compact Hausdorff group. Then pseudocompactness of $G$ implies its compactness. Moreover, if $G$ is a P-space then it is discrete (by [GJ], Exercise 4K2, compact P-spaces are finite!). Consequently, if $G$ is a non-compact, non-discrete locally compact Hausdorff group, then $RUC^*(G) \subset C^*(G)$, and $S^G$ does not preserve the reflection of $<G,G,\lambda>$ in $\text{COMP}^G$.

We shall mention now some situations in which $RUC^*(G) = C^*(G)$. First, this is of course true if $G$ is compact and if $G$ is discrete. However, if $G$ is pseudocompact, then $RUC^*(G) = C^*(G)$ as well (cf. Appendix A). In that case, $\beta_G: G \rightarrow G^\beta$ is isomorphic to $\alpha_G: G \rightarrow G^\alpha$, hence we may assume that the reflection of $<G,G,\lambda>$ in $\text{COMP}^G$ is $<1_G,\alpha_G>: <G,G,\lambda> + <G,G,\lambda>^\alpha$ (cf. 1.1.6(v) for notation). (Hence each compact $G$-space which is the orbit closure of one of its points is equicontinuous, being the equivariant continuous image of the equicontinuous $\text{ttg} <G,G,\lambda>$.)

4.4.20. NOTES. The concepts of the maximal equicontinuous factor and the structure group of a $\text{ttg} <G,X,\pi>$ with $X \in \text{COMP}$ seem to be introduced in R. ELLIS & W. GOTTSCHALK [1960]. The maximal equicontinuous factor of a $\text{ttg}$ can be trivial, i.e. an action of $G$ on a one-point space. A lot of research has been done in order to find sufficient conditions for non-triviality of the maximal equicontinuous factor. According to 4.4.7, the construction which has been described in 4.4.5 can be used to obtain the maximal equicontinuous factor of an object $<G,X,\pi>$ in $\text{COMP}^G$. This method seems to be new. However, we have not yet explored this alternative description in order to get results about non-triviality of the maximal equicontinuous factor. In the literature, the study of the maximal equicontinuous factor is often related to full subcategories of $\text{COMP}^G$ which are defined by imposing restrictions on the action of $G$ (often $G$ is supposed to be discrete, i.e. in most cases the topology of $G$ plays no role). This falls outside the scope of this treatise, but we cannot resist temptation to mention the following class of compact $G$-spaces: the class of all minimal compact Hausdorff $G$-spaces (a $\text{ttg}$ is said to be minimal if it contains no proper closed invariant subspaces; by ZORN's lemma, each non-void compact Hausdorff $G$-space contains a non-void invariant closed subspace which is minimal under the action of $G$). The classification of compact minimal $G$-spaces forms an important and largely unsolved problem of Topological Dynamics. For an excellent introduction,
cf. [El]. It has been shown in R. PELEG [1972] that a minimal ttg $<G, X, \pi>$ with $X \in \text{COMP}$ has a non-trivial maximal equicontinuous factor iff $<G, X, \pi>$ is weakly mixing (a ttg $<G, X, \pi>$ is said to be weakly mixing if the product $<G, X \times X, \sigma>$ of it with itself in $\text{TOP}^G$ is ergodic; a ttg $<G, Y, \sigma>$ is ergodic whenever every proper closed invariant subset has non-empty interior). More information about the maximal equicontinuous factor of a minimal compact Hausdorff G-space $<G, X, \pi>$ can be found in R. ELLIS & H. KEYNES [1971].

Another class of objects in $\text{COMP}^G$ which has attracted much attention is the class of ambits. An ambit is an object $<G, X, x, \pi>$ such that $<G, X, \pi>$ is an object in $\text{COMP}^G$ and $x$ is a point in $X$ with a dense orbit (in [El] the term "point transitive" is used). In the literature there are several constructions for a universal ambit (or maximal ambit, or greatest ambit), i.e. an ambit $<G, U, u, \nu>$ with the property described in 4.4.15. Our proof of its existence seems to be new. Note that uniqueness (up to isomorphism) of this universal ambit is trivial, because of the requirement that its base point (i.e. the point with a dense orbit) can be mapped onto the base point of any ambit. (In [El], Chap. 7, in particular, on p.63, it is shown that there exists a universal point transitive ttg: an object $<G, X, \pi>$ in $\text{COMP}^G$ such that $K_{\pi}[x] = X$ for some $x \in X$; in addition, if $<G, Y, \sigma>$ is any object in $\text{COMP}^G$ such that $K_{\sigma}[y] = Y$ for some $y \in Y$, then there exists an equivariant mapping of $X$ onto $Y$. Here no uniqueness is required, nor preservation of base points. Yet such a universal point transitive ttg can be shown to be unique up to isomorphism. Using this uniqueness theorem (which is by no means trivial), it follows from 4.4.16 that our ttg $<G, U, u>$ is (isomorphic to) the universal point transitive ttg of ELLIS.)

The property of the universal ambit which we stated in 4.4.17 was used in J. AUSLANDER & F. HAHN [1967] and in R.B. BROOK [1970] as a starting point for their construction of the greatest ambit. Both papers use essentially the theorem that to every suitable left invariant subalgebra $A$ of $\text{RUC}_u^e(G)$ there corresponds a compactification $f: G \rightarrow Y$ of $G$ such that on the space $Y$ an action of $G$ can be defined so as to obtain an ambit. The papers differ from each other with respect to the proof of this theorem (i.e. the construction of a suitable compactification). The former paper invokes Gelfand theory (the space $Y$ is obtained as the maximal ideal space of the algebra $A$, whereas $f: G \rightarrow Y$ assigns to each point $t$ of $G$ the maximal ideal $\{g \in A : g(t) = 0\}$). The proof in R.B. BROOK [1970] uses the following procedure: provide $G$ with the weakest uniformity making each $f \in A$ uniformly
continuous, and let \( f: G \rightarrow Y \) be the completion of the uniform space \( G \) obtained in this way\(^1\) (since each \( f \in A \) is bounded, \( G \) is totally bounded in this uniformity, hence \( Y \) is compact; cf. [En], Example 1 on page 335). Still other methods can be used to prove this theorem. For a quite general method, cf. [Se], 14.2.2.

The paper W.H. GOTTSCALK [1968] only mentions the existence of a greatest ambit. It contains no proof, but the paper strongly suggests that it is constructed completely similar to the proof which we presented of the theorem in 0.4.3 (i.e. form the product of a representative set of ambits and consider the closure of the canonical image of \( G \) in it). In the paper of P. FLOW [1967], this method is used to obtain a maximal semigroup compactification of \( G \), say \( \psi: G \rightarrow S \), such that \( \mathcal{C}^\ast(\psi)[C(S)] = \text{RUC}^\ast(G) \), and \( <G,S,\psi> \) is a tlg (cf. 1.1.6(v) for notation). It follows from the results in this subsection that \( <G,S,\psi(\varepsilon),\emptyset> \) must be isomorphic to the maximal ambit. Notice that this implies that the phase space \( U \) of the greatest ambit \( <G,U,u,1> \) can be given the structure of a semigroup such that \( u: G \rightarrow U \) is a morphism of semigroups. (This can also be proved directly: if \( <G,U,u,\psi> \) is a maximal ambit, e.g. constructed according to the lines of this subsection, then the ambit \( <G,E_u,1_Y,u^\ast> \) has also the properties of a maximal ambit. Hence \( <G,U,u,1> \) and \( <G,E_u,1_Y,u^\ast> \) are isomorphic.)

It is tempting to mention more full subcategories of \( \text{COMP}^G \), defined by means of restrictions on the actions of \( G \). We shall not do so, and we refer the reader to H. CHU [1962], where several subclasses of the object class of \( \text{COMP}^G \) are mentioned admitting universal (or "maximal") objects. Cf. also L. AUSLANDER & F. HAHN [1963].

\(^1\) If \( A \) consists of all uniformly continuous functions on \( G \), then \( f: G \rightarrow Y \) is the Samuel compactification of \( G \).
5 - K-ACTIONS OF K-GROUPS ON K-SPACES

As has been pointed out by N.E. STEENROD [1967], topologists should work mainly with k-spaces. We shall do so in this section, by dealing with k-actions. A k-action of a k-group on a k-space is just an ordinary action in which the requirement of continuity on the usual cartesian product of phase group and phase space has been replaced by the (weaker!) requirement of continuity on their product in the category KR of all k-spaces. The resulting categories k-TTG and k-KRG (G a k-group) behave completely similar to their counterparts TTG and TOP as far as it concerns limits. We cannot say much about colimits in k-TTG, because we don't know anything about colimits in the category KRGRP of all k-groups. Indeed, it is impossible to express colimits in k-TTG in terms of KRGRP and KR without explicit reference to the existence of colimits in KRGRP. On the other hand, k-KRG behaves very nicely with respect to colimits: all its colimits can be computed in KR. The proof of this fact will be postponed to §6. In fact, all material in the present section should be considered only as preliminaries to the considerations in §6 (in particular, to subsection 6.2).

5.1. General remarks on k-spaces and k-groups

5.1.1. We shall review here briefly some facts about k-spaces. All results can be found in N.E. STEENROD [1967] or [ML], p.181-184. Observe that often k-spaces are called compactly generated spaces. Recall that a k-space is a $T_2$-space in which a subset is closed iff its intersection with each compact subset is closed. They can be characterized as $T_2$-spaces which are quotients of locally compact $T_2$-spaces.

5.1.2. The full subcategory of HAUS defined by the class of all k-spaces will be denoted KR. All statements about limits, colimits, epi- and mono-
morphisms in KR can be derived from the following two facts:

(i) KR is a coreflective subcategory of HAUS, i.e. the inclusion functor 
    KR \rightarrow HAUS has a right adjoint. Hence KR is complete.

(ii) The inclusion functor KR \rightarrow HAUS creates all colimits. Hence KR is 
cocomplete.

Ad (i): For any object X \in HAUS, the coreflection of X in KR is the mapping 
1_X: X_1 \rightarrow X, where X_1 is the set X endowed with the finest topology 
making all inclusion mappings of compact subsets of X into X_1 continuous\(^1\). Limits of 
diagrams in KR can be obtained as coreflections in KR of the limits which are 
computed in HAUS. In particular:

If X,Y are objects in KR, then their product X \times Y in the category KR 
consists of the coreflection of the cartesian product space X \times Y into KR, 
together with the "usual" projections.

If f,g: X \rightarrow Y are morphisms in KR, then their equalizer in KR is the 
inclusion mapping i: Z \rightarrow X, where Z := \{x \in X : f(x) = g(x)\} with the usual 
relative topology inherited from X (closed subspaces of k-spaces are again 
k-spaces!).

Notice that the forgetful functor KR \rightarrow SET preserves all products and 
equalizers; so it preserves all limits and all monomorphisms (cf. 0.4.4). It 
follows that monomorphisms in KR are just the injective morphisms.

Ad (ii): All colimits and, consequently, all epimorphisms in KR can be 
computed in HAUS. In particular, epimorphisms in KR are the morphisms with 
dense ranges.

5.1.3. If X and Y are objects in KR, then C_{kc}(X,Y) shall denote the 
k-refinement of the space C(X,Y). For each triple X,Y,Z of objects in KR one has 

\[ C_{kc}(Z \otimes Y,X) \cong C_{kc}(Z, C_{kc}(Y,X)). \]

To be more precise: the mapping

\[ f \mapsto \tilde{f}: C_{kc}(Z \otimes Y,X) \rightarrow C_{kc}(Z, C_{kc}(Y,X)) \]

is a homeomorphism (in particular, it is a bijection); here \( \tilde{f}: Z \rightarrow C_{kc}(Y,X) \)
is defined by

\(^1\) We shall call X_1 the k-refinement of X. Obviously, the topology of X_1 is 
finer than the topology of X.
\( f(z) := f^z : y \mapsto f(z,y) : Y \times X \)

(\( f \in C(Z \otimes Y, X) \)). A proof of this statement can be given by taking k-refinements in 0.2.7(iii). A detailed proof can be found in N.E. STEENROD [1967]; cf. also [ML], p.183,184.

5.1.4. The precise meaning of 5.1.3 is the following one (cf. [ML], p.183; however, justification of the following statements can be given easily in a straightforward way).

Fix an object \( Y \) in \( KR \). Then the functor \( L^Y : KR \to KR \), defined by

\[
L^Y : \begin{cases}
Z \mapsto Z \otimes Y & \text{on objects} \\
 f \mapsto f \otimes 1_Y & \text{on morphisms}
\end{cases}
\]

has a right adjoint, namely \( R^Y : KR \to KR \), where

\[
R^Y : \begin{cases}
X \mapsto C_K(Y,X) & \text{on objects} \\
 f \mapsto f \circ - & \text{on morphisms}.
\end{cases}
\]

The following explanation of notation may be useful. First, if \( f_i : X_i \to Y_i \) \((i=1,2)\) are morphisms in \( KR \), then \( f_1 \times f_2 : (x_1, x_2) \mapsto (f_1(x_1), f_2(x_2)) : X_1 \times X_2 \to Y_1 \times Y_2 \) is continuous. Taking k-refinements, we obtain a continuous mapping \( X_1 \otimes X_2 \to Y_1 \otimes Y_2 \), which will be denoted \( f_1 \otimes f_2 \). Second, if \( f : X \to Z \) is a morphism, then \( f \circ - : C_K(Y,X) \to C_K(Y,Z) \) is defined as the mapping \( g \mapsto f \circ g \).

Unit and counit of the adjunction of \( L^Y \) and \( R^Y \) are given, respectively, by \( \gamma^Y : 1_{KR} \to R^Y L^Y \) and \( \delta^Y : L^Y R^Y \circ 1_{KR} \), where for each object \( Z \) in \( KR \)

\[
\gamma^Y_Z : Z \to C_K(Z, Z \otimes Y); \gamma^Y_Z(z)(y) := (z,y)
\]

and

\[
\delta^Y_Z : C_K(Y,Z) \otimes Y \to Z; \delta^Y_Z(f,y) := f(y).
\]

5.1.5. Well-known examples of k-spaces are locally compact \( T_2 \)-spaces and first countable \( T_2 \)-spaces. In addition, all Hausdorff quotients of k-spaces are again k-spaces. If \( Y \) is a locally compact \( T_2 \)-space and \( X \) is a k-space, then \( X \otimes Y = X \times Y \), i.e. the cartesian product \( X \times Y \) is already a k-space.

In general, for k-spaces \( X \) and \( Y \), the topology of \( X \otimes Y \) is strictly
finer than the topology of $X \times Y$, i.e. $X \times Y$ is not a k-space. Examples can be found in [Du], p.249. Another example is given in 1.5.11.

5.1.6. The following extension of 0.2.4 holds in KR: If $f: X_1 \to Y_1$ and $g: X_2 \to Y_2$ are morphisms in KR, and both $f$ and $g$ are quotient mappings, then also $f \circ g: X_1 @ X_2 \to Y_1 @ Y_2$ is a quotient mapping.

For a proof, cf. N.E. STENEHROD [1967], Theorem 4.4. Alternatively, it is sufficient to prove this when $Y_1 = Y_2 = Y$ and $g = 1_Y$. In that case, observe that quotient mappings in KR are just the coequalizers (just like in HAUS, cf. 5.1.2(ii)), and that the functor $L^1$ of 5.1.4 preserves coequalizers.

5.1.7. Let KRGRP denote the following category. Its objects, the k-groups, are the groups $G$ having a topology such that $G$ is a k-space and such that

$$\lambda: (s,t) \mapsto st: G \times G \to G; s \mapsto s^{-1}: G \to G$$

are continuous. Its morphisms are the continuous morphisms of groups.

If $G$ is a topological group and the underlying topological space of $G$ happens to be a k-space, then plainly $\lambda: G \times G \to G$ is continuous. Hence $G$ is an object in KRGRP. Thus, considering all relevant categories as subcategories of TOP, we can express this symbolically by

$$\text{TOPGRP} \cap \text{KR} \subset \text{KRGRP}.$$  

It can be shown that equality in (9) would imply that the free topological group of a k-space would be a k-space as well. According to a result of B.L.V.S. THOMAS [1974] this need not be true. Hence the inclusion in (9) is strict. Obviously, the category in the left hand side of (9) equals HAUSGRP $\cap$ KRGRP. It is the full subcategory of HAUSGRP defined by all its objects which are k-spaces; alternatively, it is the full subcategory of KRGRP determined by all its objects with simultaneously continuous multiplication.

Thus, in general, if $G$ is a k-space and a group such that the mappings in (8) are continuous, then $\lambda: G \times G \to G$ need not be continuous. However, for any object $G$ in KRGRP the mapping $\lambda: G \times G \to G$ is separately continuous.

Indeed, if $s \in G$, then $t \mapsto (s, t): G \times G$ is continuous; taking k-refinements we see that the mapping $t \mapsto (s, t): G \times G$ is continuous. Hence the composite of this mapping with $\lambda: G \times G \to G$ is continuous, i.e. $t \mapsto st: G \to G$ is continuous. Similarly, $t \mapsto ts: G \to G$ is continuous.
5.1.8. It is easy to see that closed subgroups of objects in KRGRP (with the usual relative topology) are still in KRGRP. In addition, using the fact that the formation of products in KR is associative, it is a straightforward exercise to show that the product in KR of a set of objects in KRGRP, endowed with coordinate-wise multiplication, is an object in KRGRP. These observations show, that the forgetful functor KRGRP → KR creates all limits. In particular, KRGRP is complete, and limits and monomorphisms can be calculated in KR (as topological spaces).

Moreover, the forgetful functor KRGRP → 3RP has a left adjoint (assigning to each group the group itself with the discrete topology), hence it preserves all limits, in accordance with what we found above.

5.1.9. The coreflector of HAUS into KR induces a functor M: HAUSGRP → KRGRP. Indeed, if G is an object in HAUSGRP, then the k-refinement $G_k$ of G equals G as a set, hence it can be given the same group structure as G. Let $M_G$ denote the space $G_k$ with this group structure. Since the mappings $(s,t) → st: G\times G \to G$ and $t → t^{-1}: G \to G$ are continuous, it follows that $(s,t) → st: M_G \times M_G \to M_G$ and $t → t^{-1}: M_G \to M_G$ are continuous, i.e. $M_G$ is an object in KRGRP. Moreover, if $\psi: G \to H$ is a morphism in HAUSGRP, then obviously $\psi: M_G \to M_H$ is continuous, so $\psi$ can be interpreted as a morphism in KRGRP; in doing so, we shall denote it with $M \psi$. In this way a functor $M: HAUSGRP → KRGRP$ is defined. Clearly, $M$ is a faithful functor.

Since the coreflector of HAUS into KR preserves all limits, it follows easily that $M: HAUSGRP → KRGRP$ preserves all limits (use the descriptions of limits in these categories, given in 0.4.11 and 5.1.8). Now we can apply the FREYD adjoint functor theorem (cf. [ML], p.117) to the effect that $M$ has a left adjoint $N: KRGRP → HAUSGRP$. Without any reference to the FREYD adjoint functor theorem, the left adjoint $N$ of $M$ and the corresponding unit of adjunction $u$ can be obtained in the following way. If G is an object in KRGRP, let $\{T_i \mid i \in I\}$ be the set of all topologies on G such that $(G,T_i)$ is a topological group and $T_i$ is weaker than the original topology on G (such topologies do exist; e.g. consider the indiscrete topology on G). Let $T$ be the weakest topology on G which is finer than all topologies $T_i$. Obviously, $(G,T)$ is a topological group (it is the diagonal in $F_i(G,T_i)$). Set $N'G := (G,T)$. Obviously, the underlying groups of G and $N'G$ are identical; only the topologies are different. If $\psi: G \to H$ is a morphism in KRGRP then the weakest topology on G making $\psi: G \to N'H$ continuous makes G a topological group and is weaker than the original topology on G. It follows that $\psi:
N'G → N'H is continuous. In this way, we obtain a functor N': KRGRP → TOPGRP.
Notice that for each object G in KRGRP, \( 1_G: G \to N'G \) is continuous. Now let N be the composition of N' with the reflector of TOPGRP to HAUSGRP. Then N is left adjoint to the functor M. The unit \( u \) of the adjunction is given by the morphism \( u_G: G \to MNG \), where considered as a mapping, \( u_G \) coincides with the reflection of N'G into HAUSGRP. The straightforward proof is left to the reader.

5.1.10. Many questions about the category KRGRP are left undisussed here. For example, using 5.1.8, it can be shown that the forgetful functor KRGRP → KR has a left adjoint (the "free k-group functor"). In addition, it can be shown that the category KRGRP is complete. The existence of coproducts in KRGRP can be shown similar to the existence of coproducts in TOPGRP (see e.g. Theorem 1 in E.T. ORDMAN [1974]). The coequalizer of \( \psi_1, \psi_2: G + H \) in KRGRP can be obtained as follows: let \( q: H \to K_0 \) be the coequalizer of \( \psi_1, \psi_2 \) in GRP, give \( K_0 \) the quotient topology, induced by \( q \) and, finally, let \( q_0: K_0 \to K \) be the reflection of \( K_0 \) in HAUS. Then \( K \) is a k-group, and \( q_0q: H \to K \) is the desired coequalizer. Details will appear elsewhere.

5.2. The category k-TTG

5.2.1. If \((G, X)\) is an object in KRGRP × KR, then the mappings

\[
\begin{align*}
(1) \quad & \eta^G_X: X \to G \otimes X, \\
& \mu^G_X: (G \otimes G) \otimes X \to G \otimes X,
\end{align*}
\]

which are defined according to 1.1.1, are continuous. Taking the coreflections in KR of these mappings, we obtain the following morphisms in KR:

\[
\begin{align*}
(2) \quad & \eta^G_X: X \to G \otimes X, \\
& \mu^G_X: G \otimes (G \otimes X) \to G \otimes X.
\end{align*}
\]

{Note that it is permitted to identify \( G \otimes (G \otimes X) \) with \( (G \otimes G) \otimes X. \})

5.2.2. Let \((G, X)\) be an object in KRGRP × KR. Then a k-action of \( G \) on \( X \) is a morphism \( \pi: G \otimes X \) in KR such that the following diagrams in KR commute:
A \( k \)-topological transformation group (abbreviated: a \( k \)-ttg) is a triple \([G,X,\pi]\) with \((G,X)\) an object in \( \text{KRGRP} \times \text{KR} \) and \( \pi \) a \( k \)-action of \( G \) on \( X \).

All terminology and notation concerning ordinary ttgs will also be used for \( k \)-ttgs, as far as it is meaningful. Thus, we may speak about transitions, motions, orbits, etc., of \( k \)-ttgs. Formally, the definitions are the same as for ordinary ttgs.

5.2.3. PROPOSITION. If \([G,X,\pi]\) is a \( k \)-ttg then \( \pi: G \times X \rightarrow X \) is separately continuous. Consequently, the transition mapping \( \tilde{\pi}: t \mapsto \pi^t \) defines a morphism of groups from \( G \) into \( H(X,X) \).

PROOF. For each \((t,x) \in G \times X\), the mappings \( y \mapsto (t,y): X \rightarrow G \times X \) and \( s \mapsto (s,x): G \rightarrow G \times X \) are continuous. Taking \( k \)-refinements, we obtain the continuous mappings \( y \mapsto (t,y): X \rightarrow G \times X \) and \( s \mapsto (s,x): G \rightarrow G \times X \). The compositions of these mappings with the continuous function \( \pi: G \times X \rightarrow X \) just equal \( \pi^t: X \times X \rightarrow X \) and \( \pi_s: X \times X \rightarrow X \), respectively. Consequently, \( \pi^t \) and \( \pi_s \) are continuous. \( \Box \)

5.2.4. For any \( k \)-ttg \([G,X,\pi]\), \( \pi \) is an action of \( G_d \) on \( X \), so that we can speak about the ttg \( \langle G_d, X, \pi \rangle \); this is immediate from 5.2.3.

On the other hand, if \((G,X) \in (\text{HAUSGRP} \land \text{KRGRP}) \times \text{KR}\) is given, and if \( \pi: G \times X \rightarrow X \) is an action of \( G \) on \( X \), then \( \pi: G \times X \rightarrow X \) is continuous, hence \( \pi \) is a \( k \)-action. So for any ttg \( \langle G,X,\pi \rangle \) with \((G,X) \in \text{KRGRP} \times \text{KR}\), we can speak about the \( k \)-ttg \([G,X,\pi]\).

5.2.5. We shall present now an example which shows that a \( k \)-ttg may not be a ttg, even if the phase group is a topological group. First, however, we make the following useful observation: if \([G,X,\pi]\) is a \( k \)-ttg and if \( G \) is a locally compact \( T_2 \)-space, then \( \pi \) is an action of \( G \) on \( X \), and we have also the ttg \( \langle G,X,\pi \rangle \). The proof of this observation is a trivial consequence of the fact that now \( G \times X = G \times X \) (cf. 5.1.5), whereas \( G \) is a topological group.

Here follows the example of a \( k \)-action of an object in \( \text{HAUSGRP} \land \text{KR} \)
on a k-space which is not an action:

For any k-space Z, the mapping \( \mu_z^Q : (s, t, z) \mapsto (s + t, z) : Q \otimes (Q \otimes Z) \to Q \otimes Z \) is continuous, i.e. we have the k-ttgs \( [Q, Q \otimes Z, \mu_z^Q] \) (notice that Q is metrizable, hence a k-space). Next, take for Z the space \( Q \otimes (X/R) \), considered in 1.5.11. We have seen in the Remark in 1.5.11, that \( Q \otimes Z \) is not a k-space. Similar to the proof in 1.5.11 one shows that \( \mu_z^Q : Q \otimes (Q \otimes Z) \to Q \otimes Z \) is not continuous (otherwise \((s, 0, y) \mapsto (s, y) : Q \times A \to Q \otimes Z \) would be continuous, where \( A := \{ (0, y) : y \in Z \} \) may be identified with Z). So \( \mu_z^Q \) is not an action of \( Q \) on \( Q \otimes Z \).

5.2.6. If \( [G, X, \pi] \) and \( [H, Y, \sigma] \) are k-ttgs, then a morphism of k-ttgs \( [\psi, f] : [G, X, \pi] \to [H, Y, \sigma] \) is a morphism \( (\psi, f) : (G, X) \to (H, Y) \) in \( KRGRP \times KR \) such that the following diagram commutes:

\[
\begin{array}{ccc}
G \otimes X & \xrightarrow{\pi} & X \\
\downarrow{\psi \otimes f} & & \downarrow{f} \\
H \otimes Y & \xrightarrow{\sigma} & Y
\end{array}
\]

So \( [\psi, f] : [G, X, \pi] \to [H, Y, \sigma] \) is a morphism of k-ttgs iff \( <\psi, f> : <G, X, \pi> \to <H, Y, \sigma> \) is a morphism of ttgs and \( \psi : G \to H \) is continuous. If we consider only locally compact \( T_2 \) phase groups, then the concept of a morphism of k-ttgs is equivalent to that of a morphism of ttgs (cf. also the remark in 5.2.5).

5.2.7. Let \( k\text{-TTG} \) denote the category whose objects are the k-ttgs of definition 5.2.2, and whose morphisms are the morphisms of k-ttgs, defined in 5.2.6. As composition of morphisms in \( k\text{-TTG} \) we shall use the operation which is defined similar to the composition in \( TTG \): if \( [\psi, f] : [G, X, \pi] \to [H, Y, \sigma] \) and \( [\varphi, g] : [H, Y, \sigma] \to [K, Z, \xi] \) are morphisms in \( k\text{-TTG} \), then

\[
(5) \quad [\varphi, g] \circ [\psi, f] := [\psi \circ g, f] : [G, X, \pi] \to [K, Z, \xi]
\]
(this is plainly a morphism in k-TTG).

5.2.8. In order to make the notation not too complicated, we shall denote the obvious forgetful functors from k-TTG to KRGRP and KR simply by G and S, respectively (there seems to be no danger of confusion with the notation of 3.1.2). Thus, G and S are defined by

\[
G: \{ [G,X,\pi] \mapsto G \text{ on objects}; \quad [\psi,f] \mapsto \psi \text{ on morphisms}; \quad \}
\]

\[
S: \{ [G,X,\pi] \mapsto X \text{ on objects}; \quad [\psi,f] \mapsto f \text{ on morphisms}. \quad \}
\]

In addition, let \( K: k\text{-TTG} \to \text{KRGRP} \times \text{KR} \) be defined by

\[
K: \{ [G,X,\pi] \mapsto (G,X) \text{ on objects} \quad [\psi,f] \mapsto (\psi,f) \text{ on morphisms}. \quad \}
\]

5.2.9. In the remainder of this subsection, the category \( \text{KRGRP} \times \text{KR} \) will be denoted by \( C \). Let us consider the functor \( H : C \to C \) which is defined in the following way:

\[
H: \{ (G,X) \mapsto (G,\otimes X) \text{ on objects} \quad (\psi,f) \mapsto (\psi,\otimes f) \text{ on morphisms}. \quad \}
\]

Then we have natural transformations \( \eta : I_C \to H \) and \( \mu : H^2 \to H \), where for each object \( (G,X) \) in the category \( C \),

\[
\eta(G,X) := (1_G,\eta^G_X): (G,X) \to (G,\otimes X)
\]

\[
\mu(G,X) := (1_G,\mu^G_X): (G,\otimes (\otimes X)) \to (G,\otimes X).
\]

Then \( (H,\eta,\mu) \) is a monad. As in subsection 3.1 it is easy to determine the \( H \)-algebras: they correspond uniquely to the k-ttgs \( [G,X,\pi] \). In fact, the category of all \( H \)-algebras is isomorphic to the category k-TTG. If we identify these categories, the functor \( K \) coincides with the standard forgetful functor from the category of all \( H \)-algebras to the category \( C \). Consequently, we obtain the following statements as corollaries of the general theory of monads (cf. 0.4.6 and 0.4.7):

First, the functor \( K \) has a left adjoint, namely the functor \( F : C \to k\text{-TTG} \), where

\[
F: \{ (G,X) \mapsto [G,\otimes X,\mu^G_X] \text{ on objects} \quad (\psi,f) \mapsto [\psi,\otimes f] \text{ on morphisms}. \quad \}
\]
Here \([G,G \otimes X, \mu_X^G]\) is called the free \(k\text{-}ttg\) on \(G\) and \(X\). The unit and counit of the adjunction of \(F\) and \(K\) are \(\eta\) and \(\xi\), respectively, where \(\eta\) is as above, and \(\xi\) is the natural transformation \(\xi : FK \to I_k\text{-}TTG\,' defined by

\[
\xi_{[G,X,\pi]} := \{1_{G,\pi}\}: [G,G \otimes X, \mu_X^G] \to [G,X,\pi]
\]

for every object \([G,X,\pi]\) in \(k\text{-}TTG\).

Second, the functor \(K\) creates all limits in \(k\text{-}TTG\). Since \(KRGRP\) and \(KR\) are complete, it follows that \(k\text{-}TTG\) is complete; in addition, \(K\) preserves all limits and \(K\) preserves and reflects all monomorphisms. Shortly, limits and monomorphisms in \(k\text{-}TTG\) can be computed in \(C\).

5.2.10. It is a rather annoying exercise to determine which properties of the category \(TTG\) carry over to the category \(k\text{-}TTG\). There seem to arise no difficulties from the fact that in objects of \(KRGRP\) multiplication is perhaps not simultaneously continuous. Indeed, in §3 simultaneous continuity of multiplication in the phase groups of objects in \(TTG\) has been used only in subsection 3.1 (continuity of \(\mu_X^G\)). However, we have seen above that the results of subsection 3.1 do carry over to \(k\text{-}TTG\).

In our next subsection, we briefly deal with \(k\text{-}KR^G\), the analog of \(TOP^G\). Here the situation turns out to be even better than in \(TOP^G\) (cf. subsection 3.2) as far as it concerns colimits. Consequently, we do not need the analog of subsection 3.3 in order to show that \(k\text{-}KR^G\) behaves nicely with respect to colimits. Yet we shall see that a version of 3.3.11 is valid (cf. 5.3.8). This will be used, similar to the method in subsection 3.4, that the category \(k\text{-}TTG\) is cocomplete. First, however, we want to make a few remarks on epimorphisms.

5.2.11. Similar to 3.4.9 one can show that the functor \(G: k\text{-}TTG \to KRGRP\) has a right adjoint; hence \(G\) preserves all colimits and \(G\) preserves and reflects all epimorphisms. Now the proof of 4.1.10 can be adapted to the present case, showing the following statement:

If epimorphisms in \(KRGRP\) have dense ranges\(^1\) then the functor \(K: k\text{-}TTG \to C\) preserves and reflects epimorphisms.

\(^1\) Recently it has been announced by W.F. LAMARTIN that this conjecture is false. So it is still an open problem whether \(K\) preserves epimorphisms.
In the proof of this statement (which we leave to the reader), one has to use the following version of 1.5.10: if \([G,Y,\sigma]\) is a \(k\)-ttg and \(A\) is a closed invariant subset of \(Y\), then there exists a unique \(k\)-action \(\tau\) of \(G\) on \(Y\) such that \(Y\) is a \(T_2\)-space and it is the quotient of a \(k\)-space, viz. the disjoint union of \(Y\) with itself.\) The problem is that the unique action \(\tau\) of \(G\) on \(Y\) has to be shown to be a \(k\)-action of \(G\) on \(Y\) i.e. \(\tau: G\times_y Y \rightarrow Y\) is continuous. The difficulty in proving this is of course not that \(Y\) is a \(k\)-space (it is a \(T_2\)-space and it is the quotient of a \(k\)-space, viz. the disjoint union of \(Y\) with itself.)

5.2.12. Let \([G,X,\pi]\) be a \(k\)-ttg and let \(R\) be an invariant equivalence relation in \(X\) such that \(X/R\) is a \(T_2\)-space, i.e. the quotient mapping \(q: X \rightarrow X/R\) is a morphism in \(KR\). Then there exists a unique \(k\)-action \(\tau\) of \(G\) on \(X/R\) making \([1_G,q]\) a morphism in \(k\)-TTG.

The proof is as follows: let \(\tau\) be the action of \(G\) on \(X/R\) making \(q\) equivariant. The only thing that has to be shown is that \(\tau: G\times_y X/R \rightarrow X/R\) is continuous. Obviously, \(\tau^*(1_G\circ q) = q\circ \tau: G\times X \rightarrow X\) is continuous. By 5.1.6, \([1_G,q]\) is a quotient mapping. Hence \(\tau: G\times_y X/R \rightarrow X/R\) is a quotient mapping. Hence \(\tau: G\times_y X/R \rightarrow X/R\) is continuous.

5.2.13. We shall see in 5.3.8 below that the following analog of 3.3.11 holds: if \(\psi: H \rightarrow G\) is a morphism in \(KR\), then there exists for each object \([H,Y,\sigma]\) in \(k\)-TTG a unique \((\psi,\tau): [H,Y,\sigma] \rightarrow [G,X,\pi]\) in \(k\)-TTG which is "universal" for all morphisms \([\psi,\sigma]\) in \(k\)-TTG with domain \([H,Y,\sigma]\).

Using this, the proof in 3.4.12 can be modified in such a way that we obtain a proof of the following theorem:

5.2.14. THEOREM. The category \(k\)-TTG is cocomplete. \(\square\)

5.2.15. Although \(k\)-TTG turns out to be cocomplete, the functor \(S: k\)-TTG \rightarrow \(KR\) does not preserve all colimits. This can be shown similar to 3.4.12: the construction of colimits in \(k\)-TTG is completely similar to the construction of colimits in \(TTG\).

\(^1\) Here "universal" has the same modified meaning as in 3.3.11 (viz. uniqueness is only with respect to \(k\)-\(KR\)).
5.3. The category \( k-KR^G \)

5.3.1. Fix an object \( G \) in \( KRGp \). Then \( k-KR^G \) will denote the subcategory of \( k-TTG \), determined by all objects \( [G,X,\pi] \) and all morphisms of the form \( [1_G,f] \). Most facts about the category \( TOP^G \) carry over to \( k-KR^G \). In particular, if the functor \( S^G: k-KR^G \to KR \) is defined by

\[
S^G: \begin{cases}
[G,X,\pi] \mapsto X \text{ on objects} \\
[1_G,f] \mapsto f \text{ on morphisms},
\end{cases}
\]

then we obtain the following proposition:

5.3.2. **PROPOSITION.** The functor \( S^G: k-KR^G \to KR \) creates all limits. In particular, \( k-KR^G \) is a complete category, and \( S^G \) preserves all limits. In addition, \( S^G \) preserves and reflects all monomorphisms.

**PROOF.** Consider a suitable monad in \( KR \). Cf. 5.2.9. \( \square \)

5.3.3. **PROPOSITION.** The functor \( S^G: k-KR^G \to KR \) creates all colimits. In particular, \( k-KR^G \) is a cocomplete category, and \( S^G \) preserves all colimits. In addition, \( S^G \) preserves and reflects all epimorphisms.

**PROOF.** It is sufficient to show that \( S^G \) creates all colimits. This will be done in 6.2.11. We shall show there that \( k-KR^G \) may be identified with the category of all coalgebras for a suitable comonad in \( KR \). \( \square \)

5.3.4. We shall present now two situations in which \( k-ttgs \) are just \( ttgs \).

(i) Let \( G \) be a locally compact \( T_2 \) topological group. Then for each \( k \)-space \( X \), \( G \otimes X = G \times X \), hence the category \( k-KR^G \) just equals the category \( KR^G \), the full subcategory of \( TOP^G \) determined by all \( G \)-spaces with a \( k \)-space as a phase space. So by the previous propositions, all limits, colimits, monomorphisms and epimorphisms in \( KR^G \) can be computed in \( KR \). In particular, since colimits and epimorphisms in \( KR \) can be computed in \( HAUS \), all colimits and epimorphisms in \( KR^G \) can be computed in \( HAUS \). The reader should compare this result with 3.4.3.

---

1) Observe that notating \( KR^G \) would contradict the terminology of subsection 4.1. Indeed, according to 4.1.1, \( KRG \) would denote a subcategory of \( TTG \) (i.e. simultaneously continuous actions!).
(ii) Let $G$ be any object in $\text{HAUSGRP} \cap \text{KRGRP}$. Then for any compact Hausdorff space $X$, $G \otimes X = G \times X$, hence the full subcategory $k\text{-COMPG}$ of $k\text{-KR}^G$, determined by all $k$-ttgs with compact Hausdorff phase spaces coincides with the subcategory $\text{COMPG}$ of $\text{KR}^G \subseteq \text{TOP}^G$. (We have seen in 4.2.7, that all limits, monomorphisms, epimorphisms, and all colimits of finite diagrams in $\text{COMPG}$ can be computed in $\text{COMP}$. Hence a similar statement holds for $k\text{-COMPG}$, thus providing a certain extension of the propositions 5.3.2 and 5.3.3 to a particular subcategory of $k\text{-KR}^G$.)

5.3.5. Let $\psi : H \rightarrow G$ be a morphism in $\text{KRGRP}$. For each $k$-ttg $[G,X,\pi]$, set $\pi^\psi := \pi(\psi_1)_X$. Then $\pi^\psi$ is easily seen to be a $k$-action of $H$ on $X$. Thus, we can define a functor $R_\psi : k\text{-KR}^G \rightarrow k\text{-KR}^H$ by

$$R_\psi : \begin{cases} [G,X,\pi] &\mapsto [H,X,\pi^\psi] \\ [1_G,f] &\mapsto [1_H,f] \end{cases}$$

on objects

$$R_\psi : \begin{cases} [1_G,f] &\mapsto [1_H,f] \end{cases}$$

on morphisms.

(cf. also 3.3.1).

If $[\psi,f] : [H,Y,\sigma] \rightarrow [G,X,\pi]$ is any morphism in $k\text{-TTGs}$, then $[1_H,f]$:

$$[H,1,Y,\sigma^\psi] \rightarrow [H,X,\pi^\psi]$$

and $[\psi,1_X] : [H,1,X,\pi^\psi] \rightarrow [H,1,X,\pi]$ are morphisms in $k\text{-TTGs},$

and $[\psi,f] = [\psi,1_X] \circ [1_H,f]$ (compare this with 3.3.3).

Plainly, $R_\psi$ preserves all limits (apply 5.3.2 to $k\text{-KR}^G$ and to $k\text{-KR}^H$).

Since $\text{KR}$ is a colocally small category (it is a subcategory of $\text{HAUS}$),

$k\text{-KR}^H$ is colocally small, hence the solution set condition in Freyd's adjoint functor theorem is satisfied. Consequently, we obtain the following theorem.

5.3.6. **Theorem.** Let $\psi : H \rightarrow G$ be a morphism in $\text{KRGRP}$. Then the functor $R_\psi : k\text{-KR}^G \rightarrow k\text{-KR}^H$ has a left adjoint $L_\psi$.

**Proof.** Cf. the preceding remark. \(\square\)

5.3.7. We can also repeat the construction of 3.3.5 through 3.3.7, replacing $\times$ by $\otimes$, $\langle \ldots \rangle$ by $[\ldots]$, etc. We obtain in that way a $k$-action $\rho : H(\text{G}Y) \rightarrow \text{G}Y$ commuting with the $k$-action $\mu_G$ of $G$ on $\text{G}Y$. So if $\text{(G}Y)/\rho^C_\rho$ were a $k$-space everything could be proved as in 3.3.5 up to 3.3.7 (using 5.2.12 instead of 1.5.8!). But $\text{(G}Y)/\rho^C_\rho$ may be not a $T_2$-space, hence not a $k$-space. However, it is a quotient space of a $k$-space. Hence, if $g : (\text{G}Y)/\rho^C_\rho \rightarrow Z$ is its Hausdorff reflection, then $Z$ is a $k$-space ($g$ is a quotient mapping). Similar to the proof of 4.4.2 one shows that the equivalence relation on $(\text{G}Y)/\rho^C_\rho$ induced by $g$ is invariant under the $k$-action
of $G$ on $(G\otimes Y)/C$. So 5.2.12 again implies that there is a unique $k$-action $\zeta$ of $G$ on $Z$ making $g$ equivariant. Then $[1_H, gf]: [H, Y, \sigma] \to [G, Z, \zeta]$ is the desired universal morphism ($f$ as in 3.3.6).

5.3.8. **Corollary.** Let $\psi: H \to G$ be a morphism in $\text{KRGRP}$ and let $[H, Y, \sigma]$ be an object in $k$-$TTG$. Then there exists an arrow $[\psi, h]: [H, Y, \sigma] \to [G, Z, \zeta]$ in $k$-$TTG$ which is "universal" for the class of all arrows in $k$-$TTG$ with domain $[H, Y, \sigma]$ and with group component $\psi$.

**Proof.** The arrow $[\psi, h]$ is the composite of $[\psi, 1_Y]: [H, X, \psi] \to [G, Z, \zeta] := L_\psi[H, Y, \sigma]$ and the universal arrow $[1_H, h]: [H, Y, \sigma] \to [H, Z, \psi] = R_\psi L_\psi[H, Y, \sigma]$, arising from the adjunction of $L_\psi$ and $R_\psi$. See also the proof of 3.3.11. \[1\]

\[1\] Cf. the footnote to 5.2.13.
6 - THE CATEGORIES TTG* AND k-TTG*

In subsection 6.1 we consider the category TTG*, defined in 1.4.16. Although the obvious forgetful functor $K*: \text{TTG}^* \to \text{TOPGRP}^{\text{TOP}}$ preserves all colimits, the category TTG* turns out to be not cocomplete. In addition, it is not complete. Then, in subsection 6.2, we consider the category k-TTG* of all k-ttgs (i.e. the objects of k-TTG) and all comorphisms between k-ttgs. The category k-TTG* turns out to be isomorphic to the category of all coalgebras over a suitable comonad. This implies that all colimits can easily be computed. The same methods with similar results can be applied to the categories k-KRG (with $G$ a k-group) and $\text{TOP}^G$ (with $G$ a locally compact Hausdorff group). Incidentally, this provides an explanation for some previously obtained theorems on ttgs with locally compact phase groups (cf. subsection 6.3). Moreover, this forms the basis for some statements about cogenerators (having locally compact phase groups) in TTG*. This, in turn, will place our considerations in the next section in their proper context. In addition, the result on cogenerators is used in the proof that TTG* is not complete.

6.1. The category TTG*

6.1.1. In this section, let $A := \text{TOPGRP}$ and $B := \text{TOP}$. Then to the category $A$ we may associate the opposite category $A^{\text{OP}}$. The objects of $A^{\text{OP}}$ are the objects of $A$, the morphisms in $A^{\text{OP}}$ are arrows $\psi^{\text{OP}}$, in a one-one correspondence $\psi \mapsto \psi^{\text{OP}}$ with the morphisms $\psi$ in $A$. For each morphism $\psi: G \to H$ in $A$, the domain and the codomain of the corresponding $\psi^{\text{OP}}$ are $H$ and $G$, respectively, so that $\psi^{\text{OP}}: H \to G$ (the direction is reversed). The composite $\psi^{\text{OP}} \circ^{\text{OP}} := (\psi \circ)^{\text{OP}}$ is defined in $A^{\text{OP}}$ exactly when the composite $\psi \circ$ is defined in $A$. Moreover, $\psi^{\text{OP}}$ is a monomorphism in $A^{\text{OP}}$ iff $\psi$ is an epimorphism in $A$, $\alpha^{\text{OP}}: G \to D^{\text{OP}}$ is a limiting cone for a diagram $D^{\text{OP}}: J^{\text{OP}} \to A^{\text{OP}}$ iff $\alpha: D \to G$ is a colimiting cone for the diagram $D: J \to A$ (here $J^{\text{OP}}$ is the opposite category of $J$, and $D^{\text{OP}} := D_j$ for every object $j \in J^{\text{OP}}$, $D^{\text{OP}} \xi^{\text{OP}} := (D \xi)^{\text{OP}}$ for every morphism $\xi^{\text{OP}}$ in $J^{\text{OP}}$),
and so on.

6.1.2. If \( <G,X,n> \) and \( <H,Y,o> \) are ttgs, then a comorphism \( \psi_{\text{OP},f>:} <G,X,n> \rightarrow <H,Y,o> \) is a morphism \((\psi_{\text{OP},f}: (G,X) \rightarrow (H,Y) \) in \( A \times B \) for which the following diagram in \( B \) commutes for every \( t \in H^1 \):

\[
\begin{array}{ccc}
X & \xrightarrow{\psi(t)} & X \\
\downarrow f & & \downarrow f \\
Y & \xrightarrow{o^t} & Y
\end{array}
\]

The composite of two comorphisms \( \psi_{\text{OP},f>:} <G,X,n> \rightarrow <H,Y,o> \) and \( \psi_{\text{OP},g>:} <H',Y',o'> \rightarrow <K,Z,c> \) is defined iff \( <H,Y,o> = <H',Y',o'> \); in that case, the composite is the comorphism \( \eta_{\text{OP},g} \circ \psi_{\text{OP},f>:} <G,X,n> \rightarrow <K,Z,c> \).

6.1.3. Let \( \text{TTG}_* \) denote the category whose objects are the ordinary ttgs (i.e. the objects of \( \text{TTG} \)) and whose morphisms are the comorphisms, defined in 6.1.2. In addition, let composition of comorphisms be defined as in 6.1.2.

We have the following "forgetful" functors from \( \text{TTG}_* \) to \( A^\text{OP} \times B, A^\text{OP} \) and \( B \), respectively:

\[
\begin{align*}
K_*: \text{TTG}_* & \rightarrow A^\text{OP} \times B, \\
G_*: \text{TTG}_* & \rightarrow A^\text{OP}, \\
S_*: \text{TTG}_* & \rightarrow B,
\end{align*}
\]

where

\[
\begin{align*}
K_* & \colon \{ <G,X,n> \} \mapsto (G,X) \quad \text{on objects} \\
& \quad \{ \psi_{\text{OP},f}> \} \mapsto (\psi_{\text{OP},f}>) \quad \text{on morphisms} \\
G_* & \colon \{ <G,X,n> \} \mapsto G \quad \text{on objects} \\
& \quad \{ \psi_{\text{OP},f}> \} \mapsto \psi_{\text{OP}}> \quad \text{on morphisms} \\
S_* & \colon \{ <G,X,n> \} \mapsto X \quad \text{on objects} \\
& \quad \{ \psi_{\text{OP},f}> \} \mapsto f \quad \text{on morphisms}.
\end{align*}
\]

\(^1\) Obviously, this definition is equivalent with the one, given in 1.4.14.
6.1.4. Concerning the preservation and reflection properties of the above defined functors, we have the following trivial observations:

(i) \(<\psi^\text{op},f>\) is an isomorphism in \(\text{TTG}^*\) iff \(\psi^\text{op}\) is an isomorphism in \(A^\text{op}\) (i.e. \(\psi\) an isomorphism in \(A\)) and \(f\) is an isomorphism in \(B\) (i.e. \(f\) is a homeomorphism).

(ii) \(K^*\) is faithful, so \(K^*\) reflects monomorphisms and epimorphisms. For example, if \(<\psi^\text{op},f>\) is a morphism in \(\text{TTG}^*\) and we know that \(\psi^\text{op}\) is monic in \(A^\text{op}\) (i.e. \(\psi\) is epic in \(\text{TOPGRP}\)) and that \(f\) is monic in \(B\), then it follows that \(<\psi^\text{op},f>\) is monic in \(\text{TTG}^*\).

6.1.5. Obviously, most of the methods in §3 fail if we want to apply them to \(\text{TTG}^*\). For instance, there is no natural way to associate to a comorphism \(<\psi^\text{op},f>\): \(<G,X,\pi> + <H,Y,\sigma>\) a morphism \(G\times X + H\times Y\) in \(\text{TOP}\) (we only have \(\psi\times f: H\times X + G\times X\), where the phase spaces are multiplied by the wrong groups). There is one proof (viz. the proof of 3.4.9) which can easily be adapted to the present situation:

6.1.6. **PROPOSITION.** The covariant functor \(G^* : \text{TTG} \rightarrow A^\text{op}\) has a right adjoint. Consequently, \(G^*\) preserves colimits and epimorphisms.

**PROOF.** Similar to the proof of 3.4.9. □

6.1.7. **PROPOSITION.** The covariant functor \(S^* : \text{TTG} \rightarrow B\) has a right adjoint. Consequently, \(S^*\) preserves colimits and epimorphisms.

**PROOF.** The idea of proof is the same as in 3.4.15. Let the functor \(R^*_E\): \(B \rightarrow \text{TTG}^*\) be defined as follows. Fix a one-point group \(E\). For any object \(X\) in \(B\), let \(\tau_X^E\) denote the trivial action of \(E\) on \(X\), and set \(R^*_E X := <E,X,\tau_X^E>\). If \(f : X \rightarrow Y\) is a morphism in \(B\), then \(<\psi^\text{op},f>_E : <E,X,\tau_X^E> + <E,Y,\tau_Y^E>\) is a co-morphism, denoted by \(R^*_E f\). Then \(R^*_E\) is a covariant functor, and the following diagram shows that \(R^*_E\) is right adjoint to \(S^*_E\):

\[
\begin{array}{c}
\langle E,X,\tau_X^E \rangle \\
\downarrow \quad \downarrow \quad \downarrow \\
\langle E,Y,\tau_Y^E \rangle \\
\end{array}
\]

\[
\begin{array}{c}
\langle E,X,\tau_X^E \rangle \\
\downarrow \quad \downarrow \quad \downarrow \\
\langle E,Y,\tau_Y^E \rangle \\
\end{array}
\]

In this diagram, \(\psi^\text{op}: H \rightarrow E\) is associated to the morphism \(\iota : E \rightarrow H\) in \(A\), where \(\iota[E] = \{e\}\). □
6.1.8. COROLLARY. The functor \( K_* : \text{TTG}_* \to \text{A}^{\text{OP}} \times \text{B} \) preserves colimits and epimorphisms.

PROOF. Use 6.1.6 and 6.1.7 (cf. also 3.1.3).

6.1.9. We shall show now, that a set of objects in \( \text{TTG}_* \) always has a co-product. Using 6.1.8, we can easily compute its phase group and its phase space. However, we shall show a little bit more:

6.1.10. PROPOSITION. The functor \( K_* : \text{TTG}_* \to \text{A}^{\text{OP}} \times \text{B} \) creates all coproducts. Consequently, \( \text{TTG}_* \) has all coproducts.

PROOF. Let \( \{<G_1,X_1,\pi_1>, i \in J\} \) be a set of objects in \( \text{TTG}_* \). The coproduct in \( \text{A}^{\text{OP}} \times \text{B} \) of the set \( \{(G_1,X_1) : i \in J\} \) is formed by the object \( (G,X) \) and the coprojections \( \{p_1^G, r_1^X : (G_1,X_1) \to (G,X)\} \). Here \( G := \bigcup_{j \in J} G_j \), the product of the groups \( G_j \) in \( A \), with projections \( p_j^G : G \to G_j \), and \( X := \bigoplus_{j \in J} X_j \), the coproduct of the spaces \( X_j \) in \( B \), with coprojections \( r_j^X : X_j \to X \). For each \( i \in J \), we can form the object \( <G_1,X_1,\pi_1^G> \) in \( \text{TOP}^G \) and then we can form the coproduct \( <G,X,\pi> \) of the set \( \{<G_1,X_1,\pi_1^G>, i \in J\} \) in \( \text{TOP}^G \). Now \( \pi \) is easily seen to be the unique action of \( G \) on \( X \) making each \( (p_1^G, r_1^X) \) a comorphism, i.e. a morphism \( <p_1^G, r_1^X> : <G_1,X_1,\pi_1^G> \to <G,X,\pi> \) in \( \text{TTG}_* \). Finally, a straightforward argument shows that \( <G,X,\pi> \) is the desired coproduct in \( \text{TTG}_* \), with coprojections \( <p_1^G, r_1^X> \).

6.1.11. It follows immediately from 6.1.8 that the coequalizer of \( \psi_1^G, \psi_2^G \), \( \psi_1^G, \psi_2^G : <G,X,\pi> \to <H,Y,\sigma> \) in \( \text{TTG}_* \), if it exists, is of the form \( <\psi_1^G, \psi_2^G : <G_1,X_1,\pi_1^G> \to <G_2,X_2,\pi_2^G> \) where \( \psi_1^G \) is the coequalizer of \( \psi_1, \psi_2 : \text{A} \to \text{H} \) and \( \psi_2 : \text{A} \to \text{H} \).

In particular, if the morphisms \( \psi_1^G, \psi_2^G : <G,X,\pi> \to <G,Y,\sigma> \) in \( \text{TTG}_* \) have a coequalizer, then it is of the form \( <\psi_1^G, \psi_2^G : <G_1,X_1,\pi_1^G> \to <G_2,X_2,\pi_2^G> \) where \( \psi_1^G \) is the coequalizer of \( \psi_1, \psi_2 : \text{A} \to \text{H} \) and \( \psi_2 : \text{A} \to \text{H} \). Plainly, \( \tau \) is the unique (cf. 1.5.5) action of \( G \) on \( Z \) making the quotient mapping \( q \) a morphism of \( G \) spaces. Consequently, the example in 3.4.4 shows that not all parallel pairs of morphisms in \( \text{TTG}_* \) have a coequalizer.

6.1.12. PROPOSITION. The functor \( K_* : \text{TTG}_* \to \text{A}^{\text{OP}} \times \text{B} \) creates all coequalizers of parallel pairs \( \psi_{1,i}^G, \psi_{2,i}^G : <G,X,\pi> \to <H,Y,\sigma> \) \((i=1,2)\) of morphisms in \( \text{TTG}_* \) for which \( G \) is a \( T \)-group and \( H \) is locally compact \( T \).

PROOF. The equalizer of \( \psi_1, \psi_2 : \text{A} \to \text{K} \) is \( \{t \in \text{H} : \psi_1(t) = \psi_2(t)\} \) of \( \text{H} \), and \( \tau \) is the inclusion mapping. The coequalizer of \( \psi_1, \psi_2 : \text{A} \to \text{H} \) is a quotient mapping.
say \( q: Y \to Z \).

Consider the \( \text{ttg} <K,Y,o> \) (we use here the convention of 1.3.4). We shall show that there exists an (obviously unique) action \( \zeta \) of \( K_d \) on \( Z \) making \( q \) an equivariant mapping. To this end, observe that for all \( t \in K \),

\[
q_0^t f_1 = q f_1^t = q f_2^t = q_0^t f_2.
\]

Since \( q \) is the coequalizer of \( f_1 \) and \( f_2 \), it follows that there exists a unique continuous mapping \( \zeta^t: Z \to Z \) such that \( \zeta^t q = q_0^t \). It is easy to see that in this way we obtain the desired action \( \zeta \) of \( K_d \) on \( Z \).

Since \( K \) is locally compact, 1.5.7(iv) implies that \( \zeta: K \times Z \to Z \) is continuous. Hence \( \zeta \) is the unique action of \( K \) on \( Z \) for which \( f_{1}^{\text{op}},q> : <H,Y,o> \to <K,Z,\zeta> \) is a morphism in \( \text{ttG}^* \). Finally, a straightforward argument shows that \( f_{1}^{\text{op}},q> \) is the coequalizer of \( f_{1}^{\text{op}},f_{1}>,f_{2}^{\text{op}},f_{2}> : <G,X,n> \to <H,Y,o> \) in \( \text{ttG}^* \).

6.1.13. We shall pay now some attention to limits in \( \text{ttG}^* \). First some remarks on the analogues of subsection 3.3, in particular proposition 3.3.3.

Let \( \psi: G \to H \) be a morphism in \( A \). Then to each object \( <H,X,n> \) in \( \text{ttG} \) there corresponds the object \( <G,X,n,\psi> \) in \( \text{ttG} \) and the morphism \( <G,X,n,\psi>: <G,X,n,\psi> \to <H,X,n> \) in \( \text{ttG} \). Then we have plainly the morphism

\[
<\psi^\text{op},1_X>: <H,X,n> \to <G,X,n,\psi>
\]

in \( \text{ttG}^* \). In addition, a straightforward calculation shows that for each morphism \( <\psi^\text{op},f>: <H,X,n> \to <G,Y,o> \) in \( \text{ttG}^* \), there is the morphism \( f_{1}^{\text{op}},f>: <G,X,n,\psi> \to <G,Y,o> \), so that \( <\psi^\text{op},f> \) can be factorized in \( \text{ttG}^* \) in the following way:

\[
<\psi^\text{op},1_X> \quad <G,X,n,\psi> \\
<\psi^\text{op},f> \quad <G,Y,o>
\]

6.1.14. PROPOSITION. The functor \( G^*: \text{ttG}^* \to A^\text{op} \) preserves all limits. Consequently, it preserves all monomorphisms.
PROOF. We shall show that $G_*$ preserves all products and all equalizers.

I. Suppose the set $\{G_i, X_i, \pi_i : i \in J\}$ of objects in $\mathbf{TTG}_*$ has a product with projections $\psi_i^*: <H, X, \pi> \to <G_i, X_i, \pi_i>$ ($i \in J$). Let $\beta_i^*: G \to G_i$ be the projections of the product of the set $\{G_i : i \in J\}$ in $\mathbf{TOPGRP}$, i.e. $G$ is the coproduct of the set $\{G_i : i \in J\}$ in $\mathbf{TOPGRP}$, with coprojections $\beta_i: G_i \to G$. The morphisms $\psi_i: G \to H$ in $\mathbf{TOPGRP}$ induce a unique morphism $\psi: G \to H$ such that $\psi_i = \psi_\beta_i$ for each $i \in J$. So in view of 6.1.13, each morphism $\psi_i^*, f_i^*$ factorizes over the object $<G, X, \pi>$ in $\mathbf{TTG}_*$ in the following way:

\[
\begin{array}{c}
<\psi_i^*, f_i^*> \\
\downarrow \\
<G, X, \pi>
\end{array}
\]

Since $<\psi_i^*, f_i^*>$ is the product of the objects $<G_i, X_i, \pi_i>$ in $\mathbf{TTG}_*$, the morphisms $\beta_i^*, f_i^*$ induce a unique morphism $\alpha^*: <G, X, \pi> \to <H, X, \pi>$ such that $\beta_i^*, f_i^* = \psi_i^*, f_i^* \alpha^*$. Since morphisms to $<H, X, \pi>$ are uniquely determined by their composites with all morphisms $\psi_i^*, f_i^*$ (property of a product in a category), it follows from the equations

\[
\psi_i^* f_i^* (\alpha^*, g) \psi_i^*, f_i^* = \psi_i^*, f_i^* \psi_i^*, f_i^*,
\]

that $\alpha^*, g \psi_i^*, f_i^* = \psi_i^*, f_i^*, \alpha^*, g \psi_i^*, f_i^*$. In particular, $\psi \alpha = 1_H$. On the other hand, the composites of all morphisms $\beta_i$ in $\mathbf{TOPGRP}$ with a morphism with domain $G$ completely determine that morphism, because the $\beta_i$ are the coprojections of the coproduct of the groups $G_i$ in $\mathbf{TOPGRP}$. Hence it follows from

\[
(\alpha \psi) \beta_i = \alpha \psi_i = \beta_i
\]

that $\psi = 1_G$ (notice that $\alpha \psi_i = \beta_i$ because $\psi_i^*, f_i^*$ was such that $\beta_i^* = \psi_i^*, f_i^*$).

It follows that $\psi$ is an isomorphism in $\mathbf{TOPGRP}$, hence $\psi^*, f^*: <G, X, \pi> \to <G_i, X_i, \pi_i>$ form the projections of the product of the objects $<G_i, X_i, \pi_i>$ in $\mathbf{TTG}_*$. Hence $G_*$ preserves products.

II. $G_*$ preserves all equalizers. The proof is similar to the proof of preservation of products, and we leave it to the reader. □
6.1.15. Although the functor $G^*$ preserves all limits, it cannot have a left adjoint, since it does not satisfy the solution set condition in FREYD's adjoint functor theorem. This is due to the fact that any topological group $G$ admits an action on all topological spaces (namely, at least the trivial action). Similarly, the functor $S^*$ doesn't satisfy the solution set condition, because any topological space admits an action of all topological groups. In addition, the example in 6.1.17 below shows that the functor $S^*$ does not preserve all limits. Yet the functor $K^*$ preserves all monomorphisms:

6.1.16. PROPOSITION. The functor $K^*: \mathcal{T}TG^* \to \text{AOP}\times B$ preserves and reflects all monomorphisms.

PROOF. Reflection: $K^*$ is faithful.
Preservation: The functor $G^*$ preserves all monomorphisms, by 6.1.14. That $S^*: \mathcal{T}TG^* \to \text{AOP}$ preserves all monomorphisms can be shown similar to the proof of the first case in 4.1.5. □

6.1.17. EXAMPLE. Let $G$ be a non-trivial group, let $E$ be a one-point group, let $\tau_G$ denote the trivial action of $E$ on $G$, and consider the morphisms

\[
\begin{array}{c}
\begin{array}{c}
\langle G, G, \lambda \rangle \\
\downarrow \langle 1^\text{OP}, f_2 \rangle
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\langle B, G, \tau_G \rangle
\end{array}
\end{array}
\]

in $\mathcal{T}TG_*$, where $t: E \to G$ is the obvious injection, $f_1 = \tau_G$ and $f_2$ is the constant mapping with $f_2[G] = \{e\}$. If these morphisms have an equalizer in $\mathcal{T}TG_*$, it has to be of the form

\[
\begin{array}{c}
\begin{array}{c}
\langle G, X, \pi \rangle \\
\downarrow \langle 1^\text{OP}, g \rangle
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\langle G, G, \lambda \rangle
\end{array}
\end{array}
\]

since $1^\text{OP}_G: G \to G$ is the equalizer of $1^\text{OP}, 1^\text{OP}: G \to E$ in $\text{AOP}$. Hence $g[X]$ is an invariant subset of $\langle G, G, \lambda \rangle$. On the other hand, the condition $f_1 g = f_2 g$ implies that $g[X] \subseteq \{e\}$, hence $g[X]$ cannot be an invariant subset of $\langle G, G, \lambda \rangle$, unless $g[X] = \emptyset$. Since the injection of $\{e\}$ into $G$ is the equalizer of $f_2$ and $f_2$ in $\text{TOP}$, it follows that $S^*$ does not preserve equalizers.

6.1.18. REMARK. Since the functor $K^*$ does not preserve all limits, it seems to be difficult to solve the question of whether $\mathcal{T}TG_*$ has all limits. It is quite easy to show that $\mathcal{T}TG_*$ is not complete, but for the proof we need a result from subsection 6.4. See 6.4.11 below. We have also seen, that $\mathcal{T}TG_*$
is not cocomplete. But if we restrict ourselves to actions of locally compact $T_2$-groups, then we obtain a finitely cocomplete subcategory of $\text{TTG}_*$. (see 6.1.12 and 6.1.10, and notice that finite products, hence all finite limits in $A_0$ can be computed in $\text{TOPGRP}$). We shall return to this subcategory of $\text{TTG}_*$ in 6.3.2 below. First, we shall deal with $k$-ttgs and comorphisms between them, i.e. the category $k$-$\text{TTG}_*$.

6.2. The category $k$-$\text{TTG}_*$

6.2.1. Let $A := \text{KRGRP}$, $B := \text{KR}$, and let $A^{\text{OP}}$ denote the opposite category of $A$. Notation with respect to $A^{\text{OP}}$ will be as in 6.1.1. In this section, we consider the category $k$-$\text{TTG}_*$, which is related to the category $k$-$\text{TTG}$ just in the same way as $\text{TTG}_*$ is related to $\text{TTG}$. The objects of $k$-$\text{TTG}_*$ are the $k$-ttgs $[G,X,\pi]$, where $(G,X)$ is an object in $A^{\text{OP}} \times B$, and $\pi$ is a $k$-action of $G$ on $X$ (cf. subsection 5.2). The morphisms in $k$-$\text{TTG}_*$ are the comorphisms of $k$-ttgs. Here a comorphism of $k$-ttgs, $[\psi^{\text{OP}}, f] : [G,X,\pi] \rightarrow [H,Y,\sigma]$ is a morphism $(\psi^{\text{OP}}, f) : (G,X) \rightarrow (H,Y)$ in $A^{\text{OP}} \times B$ such that

$$f \pi(\psi(t), x) = \sigma(t, fx)$$

for all $t \in H$ and $x \in X$ (equivalently: $[1_H, f] : [H,X,\pi] \rightarrow [H,Y,\sigma]$ is a morphism of $k$-ttgs).

Similar to 6.1.3, we define forgetful functors, which will also be denoted by $K : k$-$\text{TTG}_* \rightarrow A^{\text{OP}} \times B$, $G : k$-$\text{TTG}_* \rightarrow A^{\text{OP}}$ and $S : k$-$\text{TTG}_* \rightarrow B$. We shall not write down here the definitions: they may be obtained by replacing in 6.1.3 all brackets $\langle , \rangle$ by brackets of the form $[ , ]$.

6.2.2. If $(G,X)$ is an object in $A^{\text{OP}} \times B$, then

$$(1) \quad H_*(G,X) := (G, C_{k^c}(G,X))$$

is also an object in $A^{\text{OP}} \times B$. If $(\psi^{\text{OP}}, f) : (G,X) \rightarrow (H,Y)$ is a morphism in $A^{\text{OP}} \times B$, then $f*\xi*\psi \in C_{k^c}(H,Y)$ for each $\xi \in C_{k^c}(G,X)$. Let $f*\rightarrow \psi$ denote the mapping $\xi \mapsto f*\xi*\psi : C_{k^c}(G,X) \rightarrow C_{k^c}(H,Y)$. This is a morphism in $B$; set...
Clearly, we have defined a functor $H^* : A^O \times B \to A^O \times B$. We shall show now that $H^*$ is part of a comonad $(H^*, \delta, \epsilon)$.

For this end we have to define natural transformations $\delta : H^* \to 1_{A^O \times B}$ and $\epsilon : H^* \to H^*^2$.

6.2.3. **Lemma.** Let $(G, X)$ be an object in $A^O \times B$, and let $e$ be the identity of $G$. Then the following statements hold:

(i) The mapping $\delta^G_X : f \mapsto f(e) : C^G_{kc}(G, X) \to X$ is continuous. Hence it is a morphism in $B$.

(ii) For each $f \in C^G_{kc}(G, X)$, the mapping $\tilde{\delta}^G_f : t \mapsto \tilde{\delta}^G_f t = \tilde{\delta}^G_f f : g + C^G_{kc}(G, X)$ is continuous. In addition, the mapping $\epsilon^G_X : f \mapsto \tilde{\epsilon}^G_f : C^G_{kc}(G, X) \to C^G_{kc}(G, C^G_{kc}(G, X))$ is continuous. Hence it is a morphism in $B$.

**Proof.**

(i): Obvious.

(ii): If $f \in C^G_{kc}(G, X)$, then $\tilde{\delta}^G_f : G \to C^G_{kc}(G, X)$ is continuous, by 2.1.2. Taking into account that $G$ is its own $k$-refinement, it follows that $\tilde{\delta}^G_f : G \to C^G_{kc}(G, X)$ is continuous. Next, recall that we may identify the space $C^G_{kc}(G, C^G_{kc}(G, X))$ with $C^G_{kc}(G \otimes G, X)$, identifying $\tilde{a} \in C^G_{kc}(G, g \otimes G, X)$ with the element $a \in C^G_{kc}(G \otimes G, X)$, where $a(s, t) = (G(s))t(s)$ for $s, t \in G$ (cf. 5.1.3). In doing so, it is clear that we obtain $\epsilon^G_X(f) = f \circ \rho$ for each $f \in C^G_{kc}(G, X)$; here $\rho(s, t) := ts$ for $s, t \in G$. Since the mapping $f \mapsto fp : C^G_{kc}(G, X) \to C^G_{kc}(G \otimes G, X)$ is continuous (notice, that $p : G \otimes G \to G$ is continuous), we see that $f \mapsto f \circ \rho : C^G_{kc}(G, X) \to C^G_{kc}(G \otimes G, X)$ is continuous. Consequently, $\epsilon^G_X$ is continuous. □

6.2.4. For each object $(G, X)$ in $A^O \times B$, let the morphisms $\delta(G, X) : H^*(G, X) = (G, X)$ and $\epsilon(G, X) : H^*^2(G, X) = H^*(G, X)$ in $A^O \times 3$ be defined by

(3) $\delta(G, X) := (\psi^G_{QC})(G, C^G_{kc}(G, X)) : (G, X) \to (G, X)$

(4) $\epsilon(G, X) := (\psi^G_{QC})(G, C^G_{kc}(G, X)) : (G, C^G_{kc}(G, C^G_{kc}(G, X)))$.

As we shall see in 6.2.8 below (in particular, formula (17)), it follows that even $\tilde{\rho} : G \otimes C^G_{kc}(G, X) \to C^G_{kc}(G, X)$ is continuous.
Here $\delta^G_X$ and $\epsilon^G_X$ are as in the preceding lemmas, namely,

$$\delta^G_X(f) := f(e); \quad \epsilon^G_X(f) := \beta_f \quad \text{1) }$$

for each $f \in C_k(G,X)$. It follows from the lemma that, indeed, $\delta(G,X)$ and $\epsilon(G,X)$ are morphisms in $A^{op} \times B$.

6.2.5. PROPOSITION. With the notation of 6.2.4, we have the natural transformations

$$\delta: H_\ast \to I_{A^{op} \times B}; \quad \epsilon: H_\ast \to H_\ast^2,$$

and $(H_\ast, \delta, \epsilon)$ is a comonad.

PROOF. First, we have to show that $\delta$ and $\epsilon$ are natural transformations, i.e. that for each morphism $(\psi^{op}, f): (G,X) \to (H,Y)$ in $A^{op} \times B$ the following diagrams in $B$ commute (plainly, the $A^{op}$-components of the diagrams which we should consider in $A^{op} \times B$ are commutative; the $B$-component of $H_\ast^2(\psi^{op}, f)$ is denoted by $H_\ast^2(\psi^{op}, f)_2$):

$$\begin{array}{ccc}
C_k(G,X) & \xrightarrow{\delta^G_X} & X \\
\downarrow f & & \downarrow f \\
C_k(H,Y) & \xrightarrow{\epsilon^G_Y} & Y
\end{array} 
\quad \quad 
\begin{array}{ccc}
C_k(G,X) & \xrightarrow{\epsilon^G_X} & C_k(G,C_k(G,X)) \\
\downarrow f & & \downarrow f \\
C_k(H,Y) & \xrightarrow{\epsilon^G_Y} & C_k(H,C_k(H,Y))
\end{array} 
\quad \quad 
\begin{array}{ccc}
C_k(G,X) & \xrightarrow{\delta^G_X} & X \\
\downarrow f & & \downarrow f \\
C_k(H,Y) & \xrightarrow{\epsilon^G_Y} & C_k(H,C_k(H,Y))
\end{array}$$

(7)

For the first diagram to commute it is necessary and sufficient that for each $h \in C_k(G,X)$, $f(h(e_G)) = (f \circ h \cdot \psi)(e_H)$. This equality is certainly valid, since $\psi: H \to G$ is a morphism of groups, so that $\psi(e_H) = e_G$.

In the second diagram we have first to determine what $H_\ast^2(\psi^{op}, f)_2$ looks like. To this end we shall use, again, the homeomorphisms $\alpha \mapsto \alpha$:

$$C_k(G, C_k(G,X)) \to C_k(G, C_k(G,X)) \text{ and } \beta \mapsto \beta: C_k(H, C_k(H,Y)) \to C_k(H, C_k(H,Y)).$$

Then it is easily seen that $H_\ast^2(\psi^{op}, f)_2$ corresponds to the continuous mapping $f \circ \cdot (\psi \cdot \psi)$: $C_k(G, C_k(G,X)) \to C_k(H, C_k(H,Y))$. In the proof of 6.2.3 we have seen al-

1) Recall from §2 that we decided to write always simply $\beta$, where we ought to write $\beta^G_X$. Now the $G$ and $X$ occur in $\epsilon^G_X$. 

ready that \( e_h^G = h \circ \rho \) for every \( h \in C_{kc}(G,X) \). Since a similar relation holds for each \( g \in C_{kc}(H,Y) \), we have to check the commutativity of the following diagram:

\[
\begin{array}{ccc}
C_{kc}(G,X) & \xrightarrow{-\circ \rho} & C_{kc}(G \circ G, X) \\
\downarrow \phi & & \downarrow \phi \circ (\psi \circ \phi) \\
C_{kc}(H,Y) & \xrightarrow{-\circ \rho} & C_{kc}(H \circ H, Y)
\end{array}
\]

Commutativity of this diagram is equivalent to the validity of \( f[h(\psi(t)) \psi(s)] = (f \circ h \circ \psi)(ts) \) for each \( h \in C_{kc}(G,X) \) and \( (s,t) \in G \circ G \). This equality is surely valid, because \( \psi \) is a morphism of groups. Consequently, the second diagram in (7) commutes.

Thus, we have shown that \( \delta \) and \( e \) are natural transformations. In order to prove that \((H*, \delta, e)\) is a comonad, it is sufficient to check that the following diagrams commute for each object \((G, X)\) in \( A^{OP} \times B \) (again, we ought to consider diagrams in \( A^{OP} \times B \), but their \( A^{OP} \)-components trivially commute):

\[
\begin{array}{ccc}
C_{kc}(G,X) & \xrightarrow{e^G_X} & C_{kc}(G, C_{kc}(G,X)) \\
\downarrow \delta^G_X & & \downarrow \delta^G_X \circ \delta^G_X \\
C_{kc}(G, C_{kc}(G,X)) & \xrightarrow{e^G_X} & C_{kc}(G, C_{kc}(G,X))
\end{array}
\]

Commutativity of the first diagram amounts to the equalities
\[ \rho_h(e) = h = \delta_X \circ \tilde{\rho}_h \]

for all \( h \in \mathcal{C}_k(G,X) \), i.e. to the equalities \( h(te) = h(t) = h(et) \) for all \( t \in G \). Therefore, (8) commutes (notice, that the reason is exactly the same as the reason for commutativity of the first diagram in the proof of 3.1.7, namely the identity law in \( G \)). As for the diagram (9), here we have to show that

\[ (\rho^h)^*(s) = \rho^h(st) \quad \text{for all} \quad s,t \in G. \]

This equality is valid by the associative law for the multiplication in \( G \), and it follows that diagram (9) commutes (again, the reason is the same as that of commutativity of the second diagram in the proof of 3.1.7).

6.2.6. THEOREM. There exists an isomorphism \( J \) from the category of all coalgebras for the comonad \((H_*, \delta, \varepsilon)\) onto the category \( k\text{-}TTG_* \). Moreover, if \( L \) denotes the forgetful functor from the category of all coalgebras for the comonad \((H_*, \delta, \varepsilon)\) to \( A^{\text{OP}} \times B \), then \( L = K_* \circ J \).

\[ \begin{array}{ccc}
H_*\text{-coalgebras} & \xrightarrow{J} & k\text{-}TTG_* \\
\downarrow{L} & & \downarrow{K_*} \\
A^{\text{OP}} \times B & & 
\end{array} \]

PROOF. By definition, an \( H_* \)-coalgebra is a pair \(((G,X),(\psi^{\text{OP}},\alpha))\) with \((G,X)\) an object in \( A^{\text{OP}} \times B \) and \( (\psi^{\text{OP}},\alpha) : (G,X) \to (G,\mathcal{C}_k(G,X)) = H_*(G,X) \) a morphism in \( A^{\text{OP}} \times B \) such that the following diagrams commute:

\[ \begin{array}{ccc}
(G,X) & \xrightarrow{(\psi^{\text{OP}},\alpha)} & (G,\mathcal{C}_k(G,X)) \\
\downarrow{(1^{\text{OP}},\delta_X)} & & \downarrow{(1_G, \delta^G_X)} \\
(G,X) & & 
\end{array} \]
From the first diagram it follows that $\phi = 1_G$, and that

$$a(x)(e) = x$$

for all $x \in X$. From the second diagram we obtain now the relation $\alpha(x) = a \circ \alpha(x)$ for all $x \in X$. Hence $\alpha(x) = a(a(x)(t))$, that is

$$a(x)(st) = a(a(x)(t))(s)$$

Next, we use again the fact that $C(X, C_{kc}(G, X))$ and $C(X \otimes G, X)$ or $C(G \otimes X, X)$ are in a one-one correspondence, as follows: if for any $\xi \in C(X, C_{kc}(G, X))$ we write $\xi'(t,x) := \xi(x)(t)$, then $\xi' \mapsto \xi'$ is a bijection of $C(X, C_{kc}(G, X))$ onto $C(G \otimes X, X)$. Thus, we may rewrite (14) and (15) as follows:

$$a'(e,x) = x$$
$$a'(st,x) = a'(s, a'(t, x))$$

for all $x \in X$ and $s, t \in G$, or equivalently, $a'$ is a $k$-action of $G$ on $X$. We have shown, that $((G, X), (1^G_0, \alpha))$ is an $H_*$-coalgebra iff $\phi = 1_G$ and $a'$ is a $k$-action of $G$ on $X$. If we write $J((G, X), (1^G_0, \alpha)) := [G, X, \alpha']$, then $J$ defines a bijection of the class of $H_*$-coalgebras onto the class of $k$-ttgs.

We proceed by determining the morphisms of $H_*$-coalgebras. Suppose $(\psi, f): ((G, X), (1^G_0, \alpha)) \rightarrow ((H, Y), (1^H_0, \beta))$ is a morphism of $H_*$-coalgebras. This means that the following diagram commutes:
Equivalently, \( f \circ \alpha(x) \psi = \beta(fx) \) for all \( x \in X \). Using once again the bijections 
\[ \xi \mapsto \xi': \mathcal{C}(x, \mathcal{C}_{KC}(G,X)) \to \mathcal{C}(G\otimes X, X) \]
and 
\[ \eta \mapsto \eta': \mathcal{C}(Y, \mathcal{C}_{KC}(H,Y)) \to \mathcal{C}(H\otimes Y, Y), \]
we obtain 
\[ f(\alpha'(\psi(t),x)) = \beta'(t,fx) \]
for all \( t \in H \) and \( x \in X \). Therefore, \((\psi^\text{OP},f): (G,X),(\gamma^\text{OP},\alpha) \to (H,Y),(\gamma^\text{OP},\beta)\) is a morphism of \( H \)-coalgebras iff \((\psi^\text{OP},f)\) is a comorphism of \( k \)-TTGs, 
\[ [\psi^\text{OP},f]: [G,X,\alpha'] \to [H,Y,\beta']. \]
If we set \( J(\psi^\text{OP},f) := [\psi^\text{OP},f] \) for each morphism \((\psi^\text{OP},f)\) of \( H \)-coalgebras, then clearly we have obtained a functor \( J \) from the category of all \( H \)-coalgebras to the category \( k \)-TTGs. In addition, \( J \) induces not only a bijection of the object classes, but it also induces bijections of morphism sets. So \( J \) is an isomorphism of categories. It is clear from the definitions that we have the relation \( L = K \circ J \).

6.2.7. **Corollary.** The functor \( K: k \text{-} \text{TTGs} \to A^\text{OP} \times B \) has a right adjoint. Hence \( K \) preserves all colimits and epimorphisms (being faithful, \( K \) also reflects epimorphisms). In addition, \( K \) even creates all colimits, and \( k \text{-} \text{TTGs}^* \) is cocomplete.

**Proof.** Immediate from 6.2.6 and 0.4.9. Hence cocompleteness of \( k \text{-} \text{TTGs} \) follows from cocompleteness of \( A^\text{OP} \) and \( B \) (cf. 5.1.8 and 5.1.2(ii)). □

6.2.8. Using the general theory of coalgebras, the right adjoint \( M \) of the functor \( K \) can be determined as follows. First, the right adjoint \( R \) of the functor \( L \) is given by

\[ R: \begin{cases} (G,X) \mapsto ((G,\mathcal{C}_{KC}(G,X)),(\gamma^G_X)) \quad \text{on objects} \\ (\psi^\text{OP},f) \mapsto (\psi^\text{OP}, f \circ \psi) \quad \text{on morphisms} \end{cases} \]

In addition, the unit and counit of adjunction of \( L \) and \( R \) are given, respectively, by the morphisms

\[ ((G,X),\gamma^G_X) \xrightarrow{(\gamma^G_X)^{-1}} ((G,\mathcal{C}_{KC}(G,X)),(\gamma^G_X)) \]

for every \( H \)-coalgebra \((G,X),(\gamma^G_X)\), and

\[ (G,\mathcal{C}_{KC}(G,X)) \xrightarrow{(\gamma^G_X)} (G,X) \]

for every object \((G,X)\) in \( A^\text{OP} \times B \). Cf. 0.4.9 for details.
Since obviously, \( M_* := J \circ R \), we obtain from the definition of \( J \) (cf. the proof of 6.2.6) that \( M_* \) is defined by the assignments

\[
M_* : \begin{cases} 
(G,X) \mapsto [G, C_{K_*} (G,X), \tilde{\mu}_X] \\
(\psi^{op}, f) \mapsto [\psi^{op}, f_{\circ -} \circ \phi]
\end{cases}
\]

on objects

on morphisms.

Here \( \tilde{\mu}_X \) stands for \( (\epsilon^G_X) \), hence for all \( t \in G, \xi \in C_{K_*} (G,X) \):

\[
\tilde{\mu}_X (t, \xi) := \epsilon^G_X (t) = \tilde{\mu}_X (t) = \tilde{\mu}_X (t, \xi).
\]

Thus, \( \tilde{\mu}_X \) just equals the mapping which was abbreviated to \( \tilde{\mu} \) in \( \S \). We shall adopt here the previous usage, and write simply \( \tilde{\mu} \) for \( \tilde{\mu}_X = (\epsilon^G_X) \) if there is no risk of ambiguity.

Similar considerations show that the unit of the adjunction of \( K_* \) and \( M_* \) is given by the morphisms

\[
[G,X,\pi] \xrightarrow{[\psi^{op}, \pi]} [G, C_{K_*} (G,X), \tilde{\mu}]
\]

for every \( k \)-tgg \( [G,X,\pi] \). Here \( \pi : X \to C_{K_*} (G,X) \) is defined by \( \pi (x) := \pi_x \). Recall from 2.1.13 that \( \pi : X \to C_{K_*} (G,X) \) is a topological embedding, so that \( \pi : X \to C_{K_*} (G,X) \) is also a topological embedding. \( \pi \) remains relatively open. Taking \( k \)-tgg in \( \pi : X \to C_{K_*} (G,X) \), we see that \( \pi : X \to C_{K_*} (G,X) \) is continuous, because \( X \) is its own \( k \)-tgg.

The counit of adjunction of \( K_* \) and \( M_* \) is given by the morphisms of the form (16). If \( [H,Y,\sigma] \) is any \( k \)-tgg, and \( (\psi^{op}, f) : (H,Y) \to (G,X) \) is any morphism in \( A^{op} \times B \), then there exists a unique morphism in \( k \)-TTG, namely

\[
M_*(\psi^{op}, f) \circ [H, Y, \sigma] = [\psi^{op}, f_{\circ -} \circ \phi] : [H,Y,\sigma] \to [G, C_{K_*} (G,X), \tilde{\mu}],
\]

such that the following diagram commutes:

\[
\begin{array}{ccc}
[G, C_{K_*} (G,X), \tilde{\mu}] & \xrightarrow{[\psi^{op}, f_{\circ -} \circ \phi]} & [G, C_{K_*} (G,X)] \\
\downarrow & & \downarrow \\
[H,Y,\sigma] & \xrightarrow{[\psi^{op}, f_{\circ -} \circ \phi]} & [H,Y]
\end{array}
\]

6.2.9. The example in 6.1.17 can easily be adapted in order to show that the functor \( K_* : k \)-TTG \( \to A^{op} \times B \) does not preserve equalizers. We leave this to the reader. We do not know whether \( k \)-TTG is complete (the method of 6.4.11 below does not work in this case).
6.2.10. Fix an object $G \in \text{KRGRP}$, and consider the $k$-ttgs $[G,X,\pi]$ and $[G,Y,\sigma]$. If $f: X \to Y$ is a morphism in $\text{KR}$, then obviously $[1^G \circ f, f]: [G,X,\pi] \to [G,Y,\sigma]$ is a morphism in $k\text{-TTG}^*$ if $[1^G, f]: [G,X,\pi] \to [G,Y,\sigma]$ is a morphism in $k\text{-TTG}$, that is, if $[1^G, f]$ is a morphism in $k\text{-KR}^G$. Equivalently, $k\text{-KR}^G$ could have been defined as the subcategory $K'_G[A^G_{\text{op}} \times B]$ of $k\text{-TTG}^*$, where $A^G_G$ is the subcategory of $A = \text{KRGRP}$, consisting of one object $G$ and one morphism $1_G$.

Thus, identifying $k\text{-KR}^G$ with $K'_G[A^G_{\text{op}} \times B]$, we obtain the following results. The proofs are completely similar to 6.2.1 through 6.2.9, replacing $A$ by $A^G_G$.

There exists a functor $H^G_\ast: \text{KR} \to \text{KR}$,

\begin{equation}
H^G_\ast: \begin{cases}
X & \mapsto C_{kG}(G,X) \\
f & \mapsto f^G
\end{cases}
\end{equation}

and there exist natural transformations $\delta^G: H^G_\ast \to 1_{\text{KR}}$ and $\varepsilon^G: H^G_\ast \to (H^G_\ast)^2$,

\begin{equation}
\delta^G_X: f \mapsto f(1_G), \\
\varepsilon^G_X: f \mapsto \delta^G f: C_{kG}(G,X) \to C_{kG}(G,C_{kG}(G,X)),
\end{equation}

such that $(H^G_\ast, \delta^G, \varepsilon^G)$ is a comonad in $\text{KR}$. Moreover, the category of all coalgebras for this comonad is isomorphic to the category $k\text{-KR}^G$.

The functor $S^G: k\text{-KR}^G \to \text{KR}$ has a right adjoint $M^G$, given by the assignments

\begin{equation}
M^G: \begin{cases}
X & \mapsto [G, C_{kG}(G,X), \beta] \\
f & \mapsto [1_G, f^G]
\end{cases}
\end{equation}

The unit of adjunction is given by the morphisms

\begin{equation}
\begin{CD}
[G, X, \pi] @>>> [G, C_{kG}(G,X), \beta] \\
1_G^{G, \pi} @VVV \\
[1^G, \pi] @VVV \\
\end{CD}
\end{equation}

in $k\text{-KR}^G$, for every $k$-ttg $[G,X,\pi]$. The counit of the adjunction is given by the morphisms

\begin{equation}
\begin{CD}
C_{kG}(G,X) @>>> X \\
\delta^G_X @VVV \\
\end{CD}
\end{equation}

in $\text{KR}$, for every $k$-space $X$. Recall that in (24), $\Xi: X \to C_{kG}(G,X)$ is defined by $\Xi(x) := \pi_X(x \in X)$, and that $\Xi$ is a topological embedding of $X$ into $C_{kG}(G,X)$. 


If $G \in \text{KRGRP}$ is fixed, then by the above remarks the category $\kappa\text{-}\text{KR}^G$ can be identified with the category of all coalgebras over a suitable comonad in $\text{KR}$, and, consequently, the functor $S^G: \kappa\text{-}\text{KR}^G \rightarrow \text{KR}$ not only preserves all colimits, but also creates them. In particular, it follows that $\kappa\text{-}\text{KH}^G$ is complete. In addition, the functor $S^G$ preserves and reflects all epimorphisms. These results have already been announced in 5.3.3.

Recall from 5.3.2, that the functor $S^G: \kappa\text{-}\text{KR}^G \rightarrow \text{KR}$ not only has the right adjoint $M^G$, but that it also has a left adjoint $F^G$, where

\[F^G: \begin{cases} \{X \mapsto [G \otimes X, \mu^G_X]\} & \text{on objects} \\ \{f \mapsto [G \otimes f]\} & \text{on morphisms} \end{cases}\]

(cf. also 5.2.9 or 3.2.7). In fact, $\kappa\text{-}\text{KR}^G$ may also be considered as a category of algebras over a suitable monad in $\text{KR}$, viz. $(H^G, \eta^G, \mu^G)$, where

\[H^G: \begin{cases} \{X \mapsto G \otimes X\} & \text{on objects} \\ \{f \mapsto 1_G \otimes f\} & \text{on morphisms} \end{cases}\]

It follows immediately from 5.1.4 that the functor $H^G$ is left adjoint to the functor $H^G_\mu$. It can be shown that the monad $(H^G, \eta^G, \mu^G)$ and the comonad $(H^G_\mu, \epsilon^G, \delta^G)$ are adjoint to each other. For a definition of adjointness of monads, cf. for example S. Eilenberg & J.C. Moore [1965].

### 6.3. Actions of locally compact Hausdorff groups

6.3.1. In preceding sections it sometimes occurred that a construction could be carried out only if one or more of the phase groups under consideration were locally compact and Hausdorff. Cf. for example 3.3.4, 4.4.3 and 6.1.12. Moreover, as was noticed earlier in 5.3.4, if $G$ is a locally compact $T_2$-group, then the category $\kappa\text{-}\text{KR}^G$ equals the category $\text{KR}^G$, which is a subcategory of $\text{TOP}^G$ (ordinary actions of $G$ on $k$-spaces). Then, by 6.2.11, $\text{KR}^G$ is both (isomorphic to) a category of algebras over a monad in $\text{KR}$ and a category of coalgebras over a comonad in $\text{KR}$. Consequently, $S^G: \kappa\text{-}\text{KR}^G \rightarrow \text{KR}$ has both a left and a right adjoint, and all limits, colimits, monomorphisms and epimorphisms in $\text{KR}^G$ can be computed in $\text{KR}$. We shall indicate in 6.3.6, why similar results are valid for all of $\text{TOP}^G$ ($G$ locally compact Hausdorff).

6.3.2. In the proofs of 6.2.3, 6.2.5 and 6.2.6, an essential use has been made of the homeomorphisms
and their inverses ($G \in \text{KRGRP}$, $X \in \text{KR}$). So at first sight the previous methods cannot be applied to the category $\text{TTG}^*_*$, but it follows from 0.2.7(iii) that for any locally compact $T_2$-group $G$ and any topological space $X$ we have the homeomorphisms

$$α \mapsto \tilde{α}: C_{kc}(G \times G, X) \rightarrow C_{kc}(G, X)$$

$$β \mapsto \tilde{β}: C_{kc}(G, X) \rightarrow C_{kc}(G, G \times G)$$

In addition, in that case $f \mapsto f(ε): C _c (G, X) \rightarrow X$ and $\tilde{ε}^*: G \rightarrow C _c (G, X)$ (with $f \in C _c (G, X)$) are continuous. Therefore, the proofs of 6.2.5 and 6.2.6 can be modified into proofs of the following statements:

Let $A_0$ be the full subcategory of $\text{TOPGRP}$, defined by all locally compact $T_2$-groups, let $B_0 := \text{TOP}$, and let $X_* := K_*[(A_0^{op} \times B_0)]$, where $K_*: \text{TTG}^*_* \rightarrow \text{TOPGRP}^{op} \times \text{TOP}$ is the usual forgetful functor. Thus, $X_*$ is a full subcategory of $\text{TTG}^*_*$.

Then we have a comonad $(H_*, δ, ε)$ in $A_0^{op} \times B_0$, where

$$H_*: \begin{cases} (G, X) \mapsto (G, C _c (G, X)) \quad \text{on objects} \\ (ψ^{op} \delta) \mapsto (ψ^{op} δ _c \circ ψ) \quad \text{on morphisms}, \end{cases}$$

and the natural transformations $δ: H_* \rightarrow I_{A_0^{op} \times B_0}$ and $ε: H_* \rightarrow H_*^2$ are defined similarly to 6.2.4 with $C_{kc}(G, X)$ replaced by $C _c (G, X)$. The category of all coalgebras over this comonad $(H_*, δ, ε)$ may be identified with the category $X_*$, defined above, in such a way that the standard forgetful functor from the category of coalgebras to $A_0^{op} \times B_0$ may be identified with $K_*: X_* \rightarrow A_0^{op} \times B_0$.

Consequently, the functor $K_*: X_* \rightarrow A_0^{op} \times B_0$ has a right adjoint, $X_*$ is finitely cocomplete $^{1}$, and all colimits $^{2}$ and epimorphisms in $X_*$ can be computed in $A_0^{op} \times B$.

Notice that these results were already obtained in 6.1.10 and 6.1.12.

There the restriction to locally compact phase groups might seem somewhat

---

1) Observe that the category $A_0$ has at least all finite products, so that $A_0^{op}$ is certainly finitely cocomplete.

2) If some infinite colimit in $X_*$ exists, it is preserved by $K_*$, even created by $K_*$. 

---
In the present context it is clear that this restriction arises as a consequence of the passage from \(k\)-TTG* to TTG*.

6.3.3. Let the notation be as in 6.3.2. The counit of the adjunction of \(K_\ast\) and its right adjoint is given by the arrows

\[
\left(1^\text{OP}_G \circ \delta_X^G\right): (G, C_c(G,X)) \rightarrow (G,X)
\]

in \(A^\text{OP}_0 \times B_0\). \((G,X) \in A^\text{OP}_0 \times B_0\). Cf. also 6.2.8. However, this arrow is universal for a much wider class of arrows than might be expected: the arrows are allowed to have domains outside \(A^\text{OP}_0 \times B_0\). Indeed we have:

6.3.4. Let \((G,X) \in A^\text{OP}_0 \times B_0\). Then for any object \(<H,Y,\sigma> \in \text{TTG}_\ast\) and for any morphism \((\psi^\text{OP}_1, \tau_1): (H,Y) \rightarrow (G,X)\) in \(\text{TOPGRP}^\text{OP} \times \text{TOP}\), there exists a unique morphism \(<\psi^\text{OP}_1, \tau_1>: <H,Y,\sigma> \rightarrow <G,C_c(G,X),\delta>\) in \(\text{TTG}_\ast\) such that the following diagram commutes:

\[
\begin{array}{ccc}
\psi^\text{OP}_1 & \rightarrow & 1^\text{OP}_G \circ \delta_X^G \\
\downarrow \quad \psi^\text{OP}_1 \circ \tau_1 & & \downarrow \quad 1^\text{OP}_G \circ \delta_X^G \\
(H,Y) & \rightarrow & (G,X)
\end{array}
\]

The proof is as follows (cf. also (19) in 6.2.8): If such a \(<\psi^\text{OP}_1, \tau_1>\) exists, then necessarily \(\psi_1 = \psi\) and (after some straightforward computations) \(\tau_1 = f \circ g(-) \circ \psi\): \(y \mapsto f \circ g \circ \psi: Y \rightarrow C_c(G,X)\). This proves uniqueness. Existence is now easy; indeed, set

\[
<\psi^\text{OP}_1, \tau_1> := <\psi^\text{OP}, f \circ g(-) \circ \psi>.
\]

By 2.1.13, \(g: Y \rightarrow C_c(H,Y)\) is a topological embedding, and since \(h \mapsto f \psi: C_c(H,Y) \rightarrow C_c(G,X)\) is continuous, it follows that the above defined mapping \(\tau_1: Y \rightarrow C_c(G,X)\) is continuous. Now it is straightforward to verify that \(<\psi^\text{OP}_1, \tau_1>\) is a morphism in \(\text{TTG}_\ast\) and that it meets all requirements.

Remark. Local compactness of \(G\) is used only to ensure that \(\delta: G \times C_c(G,X) \rightarrow C_c(G,X)\) is continuous, i.e. that \(<G,C_c(G,X),\delta>\) is a ttg (cf. 2.1.3). If \(G\) is not locally compact, then the above statement remains true if we replace \(G\) by \(G_d\) at all places where \(G\) occurs in its role as phase group.
6.3.5. Fix an object \((G,X) \in \mathcal{A}^{\text{OP}} \times \mathcal{B}_0\), i.e. \((G,X) \in \text{TOPGRP}^{\text{OP}} \times \text{TOP}\), \(G\) locally compact \(T_2\). Then by 6.3.4, for any object \(\langle H,Y,\sigma \rangle \in \text{TTG}_*\), there is a bijection\(^1\)

\[
\Psi_{\langle H,Y,\sigma \rangle}: \text{TOPGRP}^{\text{OP}} \times \text{TOP}(\langle H,Y \rangle , \langle G,X \rangle) \to \text{TTG}_*\langle \langle H,Y,\sigma \rangle, \langle G,C_c(G,X),\hat{\varphi} \rangle \rangle.
\]

In fact, it is clear from the proof in 6.3.4, that

\[
(9) \quad \Psi_{\langle H,Y,\sigma \rangle}(\psi^{\text{OP}},f) = \langle \psi^{\text{OP}}, f \circ \varphi(-) \rangle \psi
\]

for any morphism \((\psi^{\text{OP}}, f): (H,Y) \to (G,X)\) in \(\text{TOPGRP}^{\text{OP}} \times \text{TOP}\), and that

\[
(10) \quad (\Psi_{\langle H,Y,\sigma \rangle})^{-1}(\psi^{\text{OP}},f_1) = (\psi_{\text{OP}}^{-1}, \delta^{\text{G}}_{\text{C}_c(G,X)}(f_1))
\]

for any morphism \(\langle \psi_1^{\text{OP}}, f_1 \rangle: \langle H,Y,\sigma \rangle \to \langle C_c(G,X),\hat{\varphi} \rangle\) in \(\text{TTG}_*\). (Recall, that \(\delta^{\text{G}}_{\text{C}_c}(f) = f(e)\) for \(f \in \text{C}_c(G,X)\).) In addition, it is straightforward to show that this bijection \(\Psi_{\langle H,Y,\sigma \rangle}\) is natural in \(\langle H,Y,\sigma \rangle\). (We shall use this in the next subsection in order to prove some facts about cogenerators in \(\text{TTG}_*\).)

6.3.6. Fix a locally compact \(T_2\)-group \(G\). Then \(\text{TOP}^G\) is a subcategory of \(\text{TTG}\), but it may also be considered as a subcategory of \(\text{TTG}_*\); in the latter case, \(\text{TOP}^G\) is a subcategory of the category \(X_*\), considered in 6.3.2. Therefore, we do not only have the results and methods of subsection 3.2 for \(\text{TOP}^G\), but also the methods of subsection 6.2 (in particular, 6.2.10) can be used, replacing \(\Theta\) by \(x\) and \(C_{\text{GC}}\) by \(C_c\). Collecting all these results together, we have:

(i) \(\text{TOP}^G\) is (isomorphic to) the category of all algebras over the monad \((H^G, n^G, \mu^G)\) in \(\text{TOP}\) in such a way that \(S^G: \text{TOP}^G \to \text{TOP}\) coincides with the standard forgetful functor which forgets the structure maps of algebras. Consequently, \(S^G\) creates and preserves all limits and monomorphisms. In particular, \(\text{TOP}^G\) is complete (cf. subsection 3.2).

(ii) \(\text{TOP}^G\) is (isomorphic to) the category of all coalgebras over the comonad \((H^G, \epsilon^G, \delta^G)\) in \(\text{TOP}\)\(^2\). Hence \(S^G: \text{TOP}^G \to \text{TOP}\) creates and preserves all

\(^1\) The adjointness described in 6.3.2 would give this bijection only for locally compact \(T_2\)-groups \(H\); this restriction is shown now to be superfluous.

\(^2\) We shall not write down here the definitions of \(H^G, \epsilon^G\) and \(\delta^G\); the reader may do it himself, using 6.2.10 as a model.
colimits and epimorphisms. In particular, \( \text{TOP}^G \) is cocomplete (cf. 6.2.10 and 6.2.11; also 3.4.3)

(iii) The functor \( S^G : \text{TOP}^G \to \text{TOP} \) has a left adjoint \( F^G \) and a right adjoint \( M^G \), defined by

\[
F^G : \begin{cases} 
X & \mapsto <G, G^X, \mu_X^G > \quad \text{on objects} \\
\tau & \mapsto <1_G, \tau> \quad \text{on morphisms},
\end{cases}
\]

and

\[
M^G : \begin{cases} 
X & \mapsto <G, C_0(G, X), \tilde{\rho}> \quad \text{on objects} \\
\tau & \mapsto <1_G, \tau_{\tilde{\rho}> > \quad \text{on morphisms}.
\end{cases}
\]

6.3.7. **NOTES.** With notation as in 6.3.6, it can be shown that the functor \( H^G : \text{TOP} \to \text{TOP} \) is left adjoint to the functor \( H^G_w : \text{TOP} \to \text{TOP} \) (cf. also 5.1.4 for a similar situation in KR). Moreover, the comonad \( (H^G_w, \delta^G_w, \eta^G_w) \) turns out to be adjoint to the monad \( (H^G, \eta^G, \mu^G) \). As was noticed in 6.2.11, a definition of adjointness of monads can be found in S. EILENBERG & J.C. MOORE [1965]. The results in 6.3.6 seem to be known in one form or another, viz. that \( \text{TOP}^G \) can be seen both as a category of algebras over a monad and as a category of coalgebras over the adjoint comonad. Cf. for instance the first part of C.N. MAXWELL [1966], where this essentially has been shown for locally compact *abelian* \( T_2 \)-groups (MAXWELL doesn't use the language of monads and comonads, but his results can be interpreted in the above sense; this has been remarked earlier by F.E.J. LINTON in his review of the above mentioned paper; cf. Math. Reviews 1967, #3563). All other results in this subsection, as well as in subsection 6.2 may be seen as extensions or generalizations of this result, viz. that \( \text{ttgs} \) can be seen as coalgebras if the phase groups are locally compact: we have weakened local compactness of the phase group to the requirement of being a \( k \)-group, at the cost of replacing actions by \( k \)-actions; moreover, the appropriate morphisms turned out to be the \( \text{comorphisms} \) of \( k \)-\( \text{ttgs} \).

6.4. **Cogenerators in \( \text{TTG} \).**

6.4.1. Recall that a **cogenerator** in a category \( X \) is an object \( A \in X \) for which the contravariant functor \( X(-, A) : X \to \text{SET} \) is faithful. This functor is defined by
where for every morphism \( f: X \to Y \) in \( X \), the function \( f^*: X(Y,A) \to X(X,A) \) is given by \( f^*(g) := g \circ f \) if \( g \in X(Y,A) \) (so \( f^* = -f \) in our previous notation).

It is just a reformulation of the above definition to say that an object \( A \in X \) is a cogenerator in \( X \) iff for every parallel pair of morphisms \( f_1,f_2: X \to Y \) in \( X \), \( f_1 \neq f_2 \), there exists a morphism \( g: Y \to A \) such that \( g f_1 \neq g f_2 \).

6.4.2. **EXAMPLES.** The following examples are standard, and we leave all proofs for the reader. The list is by no means exhaustive: we insert only the examples which we need in the sequel.

(i) The *indiscrete* two-point space \( E_2 \) (the only open sets are \( \emptyset \) and \( E_2 \)) is a cogenerator in \( \text{TOP} \).

(ii) Let \( F_2 \) denote the two-point space \( \{0,1\} \) with the \( T_0 \)-topology \( \{\emptyset,\{0\}, \{0,1\}\} \). Then \( F_2 \) is a cogenerator in the full subcategory of \( \text{TOP} \), determined by the class of all \( T_0 \)-spaces.

(iii) Let \( D_2 \) be the *discrete* two-point space \( \{0,1\} \). Then \( D_2 \) is a cogenerator in the category of all 0-dimensional Hausdorff spaces.

(iv) The closed unit interval \( [0,1] \) is a cogenerator in the category of all Tychonov spaces and also in the category \( \text{COMP} \).

(v) The category \( \text{HAUS} \) doesn't have a cogenerator. This is due to the fact that for each Hausdorff space \( Y \) there exists a Hausdorff space \( Q \) containing two points \( p \) and \( q \) such that each continuous function from \( X \) into \( Q \) has equal values in \( p \) and \( q \). Cf. H. HERRLICH [1965].

(vi) A slight modification of the above mentioned proof in H. HERRLICH [1965] shows that the space \( Q \) constructed there may assumed to be a k-space. Consequently, the category \( \text{KR} \) does not have a cogenerator.

(vii) The dual concept of a cogenerator is a generator. It is easy to see that the group \( \mathbb{Z} \) is a generator in \( \text{GRP} \). Hence the (discrete!) group \( \mathbb{Z} \) is a generator in \( \text{TOPGRP} \) and in any of its subcategories containing \( \mathbb{Z} \). In addition, \( \mathbb{Z} \) is a generator in \( \text{KRGRP} \). Consequently, \( \mathbb{Z} \) is a cogenerator in \( \text{TOPGRP}^{\text{op}} \) and in \( \text{KRGRP}^{\text{op}} \).

It is known that the circle group \( T \) is a cogenerator in the category of all locally compact \( T_2 \) groups (cf. [HR], 22.17). Hence \( T \) is a generator in the opposite category.
6.4.3. Notice that KR can be seen as a subcategory of the category k-TTG*, identifying each k-space Y with the object \([Z,Y,\tau_Y] \in k\text{-}TTG*\), where \(\tau_Y\) is the trivial action of \(Z\) on \(Y\). It follows easily that \(A\) is a cogenerator in KR if \(k\text{-}TTG*\) has a cogenerator with phase space \(A\). In view of 6.4.2(vi) it follows that \(k\text{-}TTG*\) cannot have a cogenerator. This is one of the reasons that we turn now our attention to the category TGG* (another reason is that we are interested in comprehensive objects in TGG*; see §7). First we prove a general lemma about cogenerators.

6.4.4. \textbf{Lemma.} Let \(F: X \to Y\) be a faithful functor, let \(X_0 \in X\), \(Y_0 \in Y\) and suppose that there exists a natural transformation \(\varphi: Y(F\cdot Y_0) \to X(-,X_0)\). If each \(\varphi_z\) is a bijection of \(Y(Fz,Y_0)\) onto \(X(z,X_0)\), then the assumption that \(Y_0\) is a cogenerator in \(Y\) implies that \(X_0\) is a cogenerator in \(X\).

\textbf{Proof.} Let \(f_1, f_2: X \to Y\) be morphisms in \(X\) such that \(f_1 \neq f_2\). For \(i=1,2\), the following diagram commutes:

\[
\begin{array}{ccc}
X(X,X_0) & \xleftarrow{\varphi_X} & Y(FX,Y_0) \\
\downarrow{f_i^*} & & \downarrow{(Ff_i)^*} \\
X(Y,Y_0) & \xleftarrow{\varphi_Y} & Y(FY,Y_0)
\end{array}
\]

If \(Y_0\) is a cogenerator in \(Y\), then \(Y(F\cdot Y_0)\) is the composition of two faithful functors. Hence \((Ff_1)^* \neq (Ff_2)^*\). Since \(\varphi_X\) and \(\varphi_Y\) are bijections, it follows that \(f_1^* \neq f_2^*\). This proves that the functor \(X(-,X_0)\) is faithful. \(\square\)

6.4.5. If \((F,G,\varphi)\) is an adjunction from \(X\) to \(Y\) and \(F\) is faithful, then the preceding lemma can be applied with \(X_0 := GY_0\). Thus, if \(Y_0\) is a cogenerator in \(Y\), then \(GY_0\) is a cogenerator in \(X\). However, in the situation which we want to consider below, there is no adjointness, so that we have to use the lemma as it is formulated in 6.4.4.

6.4.6. We shall use the following notation in the remainder of this subsection. First, we consider the standard forgetful functor \(K_*: TGG* \to \text{TOPGRP}^{\text{OP}} \times \text{TOP}\). Further, let \(A\) and \(B\) denote full subcategories of \(\text{TOPGRP}\) and \(\text{TOP}\), and set \(X_* := K_*[A,B]\). So \(X_*\) is a full subcategory of \(TGG*\). Finally, let the restriction and corestriction of \(K_*\) to the domain \(X_*\) and the codomain \(A^{\text{OP}} \times B\) also be denoted by \(K_*\).
6.4.7. **Proposition.** Let the locally compact Hausdorff group $G$ be a cogenerator in $\mathcal{A}\mathcal{P}$ and let $X$ be a cogenerator in $\mathcal{B}$. If $C_\mathcal{C}(G,X)$ is an object in $\mathcal{B}$, then the ttg $<G,C_\mathcal{C}(G,X),\overline{p}>$ is a cogenerator in $X_\mathcal{C}$.

**Proof.** Observe that $(G,X)$ is a cogenerator in $\mathcal{A}\mathcal{P} \times \mathcal{B}$. In view of 6.3.5, we can apply now lemma 6.4.4 with $F := K_\mathcal{x}$. 

6.4.8. **Corollary.** If $G$ is a locally compact Hausdorff group and if the category $\mathcal{B}$ has a cogenerator $X$, then $<G,C_\mathcal{C}(G,X),\overline{p}>$ is a cogenerator in $\mathcal{B}$, provided $C_\mathcal{C}(G,X) \in \mathcal{B}$. 

6.4.9. Except COMPGRP and its subcategories, each "reasonably nice" subcategory $\mathcal{A}$ of $\mathcal{TOPGRP}$ contains $\mathcal{Z}_\mathcal{Z}$. In such a category $\mathcal{A}$, $\mathcal{Z}$ is a generator, hence $\mathcal{Z}$ is a cogenerator in $\mathcal{A}\mathcal{P}$ (cf. also 6.4.2(vii)). In this situation the condition that $C_\mathcal{C}(\mathcal{Z},X)$ be in $\mathcal{B}$ for the cogenerator $X$ of $\mathcal{B}$ is rather weak: it means nothing else than that the countable cartesian product of copies of $X$, viz. the space $X^{\mathcal{Z}}$, is still in $\mathcal{B}$. This condition is fulfilled in all examples mentioned in 6.4.2. In particular, we obtain the following examples of cogenerators in subcategories of $\mathcal{T}\mathcal{T}\mathcal{G}_\mathcal{x}$ (for the notation, cf. 6.4.2):

(i) $<\mathcal{Z},\mathcal{Z}^{\mathcal{Z}},\overline{p}>$ is a cogenerator in $\mathcal{T}\mathcal{T}\mathcal{G}_\mathcal{x}$. Observe, that here $\mathcal{Z}^{\mathcal{Z}}$ is an indiscrete space.

(ii) $<\mathcal{Z},\mathcal{Z}^{\mathcal{Z}},\overline{p}>$ is a cogenerator for the full subcategory of $\mathcal{T}\mathcal{T}\mathcal{G}_\mathcal{x}$ determined by the class of all ttgs with a $T_0$ phase space.

(iii) $<\mathcal{Z},\mathcal{Z}^{\mathcal{Z}},\overline{p}>$ is a cogenerator for the full subcategory of $\mathcal{T}\mathcal{T}\mathcal{G}_\mathcal{x}$ determined by all ttgs with a 0-dimensional $T_0$ phase space.

(iv) $<\mathcal{Z},[0,1]^{\mathcal{Z}},\overline{p}>$ is a cogenerator for the full subcategories of $\mathcal{T}\mathcal{T}\mathcal{G}_\mathcal{x}$, determined by all ttgs with a Tychonov, resp. with a compact $T_2$, phase space.

6.4.10. Similar applications can be made of the corollary 6.4.8, although here the restriction that $C_\mathcal{C}(G,X)$ has to be an object in $\mathcal{B}$ is more serious. However, it is easy to see that $C_\mathcal{C}(G,F_2)$ is a $T_0$-space and $C_\mathcal{C}(G,D_2)$ is a 0-dimensional Hausdorff space. In addition, it is clear that $C_\mathcal{C}(G,[0,1])$ is a Tychonov space. Thus, 6.4.8 can be used to obtain cogenerators for the categories of all $G$-spaces with arbitrary, or with $T_0$, or with 0-dimensional

*) This statement should be considered as the definition of "reasonably nice" subcategories of $\mathcal{TOPGRP}$, at least in this context.
T₂, or with Tychonov phase spaces. But \( C_c(0, [0, 1]) \) is compact if and only if \( G \) is discrete, so for non-discrete groups \( G \) we cannot apply 6.4.8 to the case \( B = \text{COMP} \). If \( \text{COMP}^G \) has a cogenerator, it seems to be of a quite complicated character. At this moment, it is an open problem whether \( \text{COMP}^G \) has a cogenerator or not.

6.4.11. PROPOSITION. The category \( \text{TTG}^* \) is not complete.

PROOF. According to [Pa], p. 114, a complete, locally small category with a cogenerator is cocomplete. We have seen above that \( \text{TTG}^* \) has a cogenerator. In addition, the category \( \text{TOPGRP}^{\text{TOP}} \times \text{TOP} \) is locally small, hence 6.1.16 implies that \( \text{TTG}^* \) is such. So if \( \text{TTG}^* \) were complete, it would be cocomplete, contradicting the result of 6.1.11.

6.4.12. NOTES. In the literature the elements of the space \( D_2^Z \) are often called bisequences. Notice that this space is homeomorphic to the Cantor discontinuum. The action \( \delta \) of \( \mathbb{Z} \) on \( D_2^Z \) is generated by the homeomorphism \( \delta_1^n \colon \mathbb{Z} \to \mathbb{Z} \), \( a \mapsto a + n \); this homeomorphism is often called the bilateral shift. The ttg \( <Z, D_2^Z, \delta> \) and, more generally, ttgs of the form \( <Z, S^Z, \delta> \) with \( S \) a discrete set, have been investigated intensively in the literature. Cf. [GH], Section 12 and G.A. Hedlund [1969]. In the present context, such ttgs arise as follows: we shall see in the next section that each ttg \( <Z, X, \pi> \) with \( X \) a compact 0-dimensional topological Hausdorff space can be equivariantly embedded in a product of copies of \( <Z, D_2^Z, \delta> \) in \( \text{COMP}^Z \). It is easily seen that a product of \( k \) copies of \( <Z, D_2^Z, \delta> \) in \( \text{COMP}^Z \) is just \( <Z, S^Z, \delta> \) with \( S := D_2^k \).
7 - COMPREHENSIVE OBJECTS IN TOP^G

An object \( X_0 \) in a category \( X \) is said to be comprehensive for a class \( B \) of objects in \( X \) whenever each \( B \in B \) admits an "embedding" into \( X_0 \). Here an "embedding" is to be understood as an element of a distinguished class of monomorphisms in \( X \). The existence of comprehensive objects for certain classes of objects can be based on the existence of cogenerators. In fact, if in a category \( X \) all powers of the object \( A \) exist, then the following statements are easily seen to be equivalent (cf. also [HS], 19.6):

(i) \( A \) is a cogenerator in \( X \).
(ii) For each object \( X \in X \) the unique morphism (induced by the product) \( X \to A^{X(X,A)} \) is monic.
(iii) Each object \( X \in X \) is a subobject of some power \( A^K \) of \( A \).

However, there is a priori no reason why the monomorphism in (ii) above should belong to our distinguished class of monomorphisms. In the concrete situation which we are interested in, viz. \( X \) is a subcategory of \( \text{TOP}^G \) for some given topological group \( G \), the distinguished class of monomorphisms shall be the class of equivariant topological embeddings. Here we feel free to use all known results about \( \text{TOP} \) and its subcategories. In particular, we suppose that \( B \) is a full subcategory of \( \text{TOP} \) having a cogenerator \( X \) such that each object of \( B \) can be topologically embedded in some power of \( X \).

We shall show that for any locally compact Hausdorff group \( G \) each object of \( B^G \) can be equivariantly embedded in some power of \( <G, \text{C}_c(G,X), \beta> \). Notice that if \( \text{C}_c(G,X) \in B \), then \( <G, \text{C}_c(G,X), \beta> \) is a cogenerator in \( B^G \) (cf. subsection 6.4), which is completely in accordance with the equivalence of the statements (i), (ii) and (iii) above. We shall not use explicitly this equivalence nor the concept of a cogenerator. The preceding discussion is only included in order to indicate the relationship of subsections 7.1 and 6.4. In subsection 7.2 we present a highly non-categorical modification.
of the approach of 7.1, which enables us to replace in some cases powers of the space $\mathcal{C}_c(G,X)$ by the simpler space $\mathcal{C}_c(G \times G, X)$. Finally, in subsection 7.3 we apply the results of 7.1 in order to obtain statements about the existence of compactifications of $G$-spaces.

7.1. General remarks

7.1.1. An object $X$ in $\text{TOP}$ will be called comprehensive for a class $B$ of objects in $\text{TOP}$ if for each $Y \in B$ there exists a topological embedding of $Y$ into $X$. Observe that we do not require that $X \in B$.

Similarly, if $G$ is a fixed topological group, then a $G$-space $(G,X,\sigma)$ is said to be comprehensive for a class $C$ of objects in $\text{TOP}^G$ whenever each $(G,Y,\sigma) \in C$ admits an equivariant topological embedding into $(G,X,\sigma)$. Again, we do not require that the comprehensive object $(G,X,\sigma)$ for $C$ is itself in $C$.

7.1.2. If $(G,X,\sigma)$ is comprehensive for a class $C$ of $G$-spaces, then obviously $X$ is comprehensive (in $\text{TOP}$) for the class $S^G[C]$. In this section we shall consider comprehensive objects in $\text{TOP}^G$ of the form $(G,\mathcal{C}_c(G,X),\bar{\rho})$. To do so, we have to assume that $G$ is locally compact Hausdorff. In order to avoid trivialities, we shall also assume that $G$ is infinite. So from now on, $G$ is an infinite locally compact Hausdorff group.

Recall from 6.3.5 that for any $G$-space $(G,Y,\sigma)$ and any topological space $X$ there exists a bijection of $C(Y, X)$ onto $\text{TOP}^G((G,Y,\sigma), (G,\mathcal{C}_c(G,X),\bar{\rho}))$. For simplicity, we shall denote this bijection here by $\Psi_0$. Hence by (9) and (10) in 6.3.5, we have

$$\Psi_0(f)(y) = (\Psi_0^{-1}g)(y)$$

for $f \in C(Y,X)$, $y \in Y$ and $g \in (G,\mathcal{C}_c(G,X),\bar{\rho})$. In particular, if $\Psi_0$ defines a topological embedding of $Y$ into $\mathcal{C}_c(G,X)$, then $f := \Psi_0^{-1}g$: $Y \to X$ is a continuous function such that

$$t \in T \mapsto \Psi_0^{-1}g(t)$$

separates the points of $Y$.

Conversely, if $f: Y \to X$ is any continuous function satisfying (2), then $\Psi_0 f$ induces an equivariant continuous injection of $Y$ into $\mathcal{C}_c(G,X)$.

Therefore, there are two possibilities to solve the question for what
classes the ttg $\langle 0, C, (G, X), \mathcal{D} \rangle$ with given $X$ is comprehensive: a direct
approach, which provides the equivariant embedding $\langle 1, g \rangle$, and an indirect
one, where one looks for continuous functions $f$ satisfying (2), such that
$\mathbb{V}_0 f$ is, in addition, relatively open.

We shall choose the first possibility. In addition, we adopt the same
point of view as in the preceding chapter, viz. that everything about $\text{Top}^\mathbb{B}
should be reduced to facts about $\text{Top}$. To be concrete, we shall consider a
class $\mathcal{B}$ of topological spaces, and a topological space $X_0$ such that $X_0^i$
is comprehensive for $\mathcal{B}$ (with $I$ a given index set). Using this as a starting
point, we shall search for a comprehensive object in $\text{Top}^\mathbb{B}$ for $\langle \mathbb{S}^D \rangle^\mathbb{E}[\mathcal{B}]$.

7.1.3. EXAMPLES. \(^1\) If $\kappa$ is a cardinal number, then $X_0^\kappa$ denotes a product of
$\kappa$ copies of $X_0$. With this notation, we have for any cardinal $\kappa$ (finite or
transfinite):

(i) $[0, 1]^\kappa$ is comprehensive for the class of all Tychonov spaces of
weight $\leq \kappa$. Cf. [En], Chap.2, §3, Theorem 8. Obviously, $\mathbb{H}^\kappa$ is also
comprehensive for this class. In particular, if $\kappa = \aleph_0$, we infer that
$[0, 1]^\aleph_0$ (and similarly, $\mathbb{H}^\aleph_0$) is comprehensive for the class of all
separable metrizable spaces.

(ii) Let $D_2^\kappa := \{0, 1\}$ be the discrete space consisting of two points. Then
$D_2^\kappa$ is comprehensive for the class of all 0-dimensional Hausdorff
spaces of weight $\leq \kappa$. Cf. [En], Chap.6, §2, Theorem 11. In particular,
$D_2^\aleph_0$ is comprehensive for the class of all separable metrizable
0-dimensional spaces.

(iii) Let $F_2^\kappa$ be the space $\{0, 1\}$ with the $T_0$-topology $\{\emptyset, \{0\}, \{0, 1\}\}$. Then
for each cardinal number $\kappa$, $F_2^\kappa$ is comprehensive for the class of all
$T_0$-spaces of weight $\leq \kappa$. Cf. [En], Chap.2, §3, Theorem 9.

(iv) If $\kappa$ is a cardinal number (finite or transfinite), then the space $S(\kappa)$ will
be defined as follows: let $I$ be a set of cardinality $\kappa$, and define an
equivalence relation in the set $I \times [0, 1]$ such that the equivalence
classes are just all singletons $\{(i, x)\}$ with $i \in I$ and $0 < x < 1$
together with the set $\{(i, 0) : i \in I\}$. The resulting quotient set may
be visualized as a "star", i.e. a union of $\kappa$ intervals $[0, 1]$ which
have their left end points 0 in common; the intervals which constitute

\(^1\) As indicated in the introduction, there is a close connection between
these examples and those in 6.4.2.
this star shall be called "rays". Now $S(\kappa)$ will denote this star with the topology which is generated by the following metric $d$:

$$d(x,y) := \begin{cases} 
|x-y| & \text{if } x \text{ and } y \text{ are in the same ray;} \\
|x| + |y| & \text{if } x \text{ and } y \text{ are in different rays;}
\end{cases}$$

for $x, y \in S(\kappa)$. Note that this topology is not the quotient topology inherited from $X[0,1]$, unless $\kappa$ is finite.

A collection $\mathcal{B}$ of subsets of a topological space $X$ is said to be $\alpha$-discrete ($\alpha \geq \aleph_0$) provided $\mathcal{B} = \bigcup\{B_j : j \in J\}$ with $|J| \leq \alpha$ and each $B_j$ a discrete family of subsets of $X$ (i.e. each point in $X$ has a neighbourhood which meets at most one member of $B_j$). For example, for any cardinal $\kappa$, the space $S(\kappa)$ defined above has an $\aleph_0$-discrete base, because $[0,1]$ has a countable base (cf. [Kw], 28.6).

A close examination of the proof of [Kw], 28.7 shows that for any two infinite cardinal numbers $\alpha$ and $\kappa$ with $\alpha \leq \kappa$ the space $[S(\kappa)]^\alpha$ is comprehensive for the class of all $T_\omega$-spaces which have an $\alpha$-discrete base of cardinality $\leq \kappa$ (these spaces have therefore weight $\leq \kappa$). In particular, $[S(\kappa)]^\kappa$ is comprehensive for the class of all $T_\omega$-spaces of weight $\leq \kappa$.

In addition, $[S(\kappa)]^\aleph_0$ is comprehensive for the class of all $T_3$-spaces having an $\aleph_0$-discrete base of cardinality $\kappa$ (cf. [Kw], 28.8). Hence by BING’s metrization theorem, it is comprehensive for the class of all metrizable spaces of weight $\leq \kappa$ (this can also be found in [En], Chap.4, §8, Theorem 7).

7.1.4. Let $\mathcal{B}$ denote a class of topological spaces. Suppose that the space $X_0^\mathcal{B}$ is comprehensive for $\mathcal{B}$ in $\text{TOP}$. We shall construct a comprehensive object in $\text{TOP}^G$ with respect to the class $(G^{SG})^\mathcal{B}$, provided $G$ and $X_0$ fulfill some additional conditions.

The first step is the observation that for any $G$-space $<G,Y,\sigma>$ with $Y \in \mathcal{B}$ we have the equivariant embedding

$$g: Y \to C_c(G,Y)$$

of the $G$-space $Y$ with action $\sigma$ into the $G$-space $C_c(G,Y)$ with action $\tilde{\sigma}$ (cf. 2.1.3).

Let $Y$ admit the embedding $f: Y \to X_0^\mathcal{B}$. Equivalently, there exists a set $\{f_i : i \in I\}$ of continuous mappings of $Y$ into $X_0$ which separates points and
closed sets; then \( f: y \mapsto (f_i(y))_i \) is a topological embedding of \( Y \) into \( X_0^I \).

It is trivial, that the mapping

\[
F: g \mapsto f \circ g: C_c(G,Y) \rightarrow C_c(G,X_0^I)
\]

is a topological embedding. In fact, \( F \) is an equivariant embedding of the \( G \)-space \( C_c(G,Y) \) with action \( \bar{\rho} \) into the \( G \)-space \( C_c(G,X_0^I) \) with action \( \bar{\rho}^I \).

Observe, that we may define \( F \) alternatively by

\[
F(g)(s) := (f_i(g(s)))_i
\]

for \( g \in C_c(G,Y) \) and \( s \in G \), where \( \{f_i \}_{i \in I} \) is an (arbitrary, but fixed) collection of continuous mappings of \( Y \) into \( X_0 \) which separates points and closed sets.

The next step is the observation that the \( G \)-space \( C_c(G,X_0^I) \) is isomorphic with the product of \( |I| \) copies of \( <G, C_c(G,X_0)>, \bar{\rho}^I > \) in \( \text{TOP}^G \). Formally, let \( <G, C_c(G,X_0^I), \bar{\rho}^I > \) denote this product (cf. 3.3.6 for the precise description).

If \( \eta \in C_c(G,X_0^I) \), then for every \( t \in G \), \( \eta(t) = (\eta(t,i))_i \) with \( \eta(t,i) \in X_0 \) for every \( i \in I \). Then each \( \eta_i: t \mapsto \eta(t,i): G \rightarrow X_0 \) is continuous, and in this way we obviously obtain a bijection

\[
\phi: \eta \mapsto (\eta_i)_i : C_c(G,X_0^I) \rightarrow C_c(G,X_0^I).
\]

It is easy to see that \( \phi \) is equivariant with respect to the action \( \bar{\rho} \) of \( G \) on \( C_c(G,X_0^I) \) and the action \( \bar{\rho}^I \) of \( G \) on \( C_c(G,X_0) \). We show now that \( \phi \) is a homeomorphism, so that \( \phi \) is an isomorphism of \( G \)-spaces of \( C_c(G,X_0^I) \) with action \( \bar{\rho}^I \) and \( C_c(G,X_0) \) with action \( \bar{\rho}^I \).

That \( \phi \) is a homeomorphism can be seen as follows. For any space \( Z \), we may write \( Z^I = C_c(I,Z) \), if we give \( I \) the discrete topology. Since both \( G \) and \( I \) are locally compact Hausdorff spaces, we have the homeomorphisms

\[
\begin{align*}
\varphi_1: a &\mapsto \bar{a} : C_c(G \times I, X_0) \rightarrow C_c(G, C_c(I, X_0)), \\
\varphi_2: \beta &\mapsto \bar{\beta} : C_c(G \times I, X_0) \rightarrow C_c(I, C_c(G, X_0))
\end{align*}
\]

(cf. 0.2.7(iii)). Plainly \( \phi = \varphi_2 \varphi_1 \), so \( \phi \) is a homeomorphism.
7.1.5. Resuming, if \( \langle G, Y, \sigma \rangle \) is a tgp with \( Y \in \mathcal{B} \), then we have an equivariant embedding

\[
\Phi \circ F \circ g : Y \to C_c(G, X_0)^I
\]

of the \( G \)-space \( Y \) into the \( G \)-space \( C_c(G, X_0)^I \). Note, that if \( F \) is given by (5), then

\[
(\Phi \circ F \circ g)(y) = (f_i \circ \sigma_y)_i
\]

for all \( y \in Y \). Untill here our construction is "canonical" in the sense that we used only mappings which have some meaning in the categorical context of the preceding sections.

7.1.6. We leave the applications of 7.1.5 to the various classes \( \mathcal{B} \), mentioned in 7.1.3, to the reader. If \( X_0 = \mathbb{R} \), then we have to do with the \( G \)-space \( C_c(G)^I \), which is comprehensive for the class of all \( G \)-spaces \( Y \) such that \( Y \) is a Tychonov space of weight \( w(Y) \leq |I| \). In the next subsection we shall show that under rather weak restrictions, we can obtain a comprehensive object with phase space \( C_c(G \times G) \). Although this space is in a sense simpler than \( C_c(G)^I \), the methods in subsection 7.2 will be non-categorical.

7.1.7. NOTES. In 7.1.2 we mentioned an alternative attack of the problem of comprehensive objects, exploring condition (2). This has been done in S. KAKUTANI [1968] in proving that \( \langle \mathbb{R}, C_c(\mathbb{R}), \mathcal{B} \rangle \) is comprehensive for the class of all compact metrizable \( \mathbb{R} \)-spaces having a fixed point set which is homeomorphic to a subset of \( \mathbb{R} \).

7.2. The comprehensive object \( \langle G, C_c(G \times G, X_0), \overline{\mathcal{P}} \rangle \)

7.2.1. As before, let \( G \) denote a locally compact Hausdorff topological group; to avoid trivialities, we assume that \( G \) is infinite. In addition, we shall consider an object \( X_0 \in \text{TOP} \) such that \( X_0^I \) is comprehensive for a class \( \mathcal{B} \) of topological spaces (where the index set \( I \) and the class \( \mathcal{B} \) are, of course, somehow related to each other; cf. 7.1.3.

We shall show in this subsection that, under certain additional conditions on \( G, \mathcal{B}, X_0 \) and \( I \), the class \( (S^G)^*\mathcal{B} \) admits a comprehensive object with phase space \( C_c(G \times G, X_0) \). See 7.2.9 and 7.2.10 below. The action of \( G \) on \( C_c(G \times G, X_0) \) can be described as follows:
In the usual way, \( C_c(G \times G, X_0) \) can be identified with \( C_c(G, C_c(G, X_0)) \). In doing so, the action \( \tilde{\sigma} \) of \( G \) on \( C_c(G, C_c(G, X_0)) \) corresponds to an action \( \tilde{\pi} \) of \( G \) on \( C_c(G \times G, X_0) \), defined by

\[
\tilde{\pi}^u f(s, t) := f(su, t)
\]

for \( f \in C_c(G \times G, X_0) \), \( u \in G \) and \( (s, t) \in G \times G \). It follows from the above correspondence between \( \tilde{\sigma} \) and \( \tilde{\pi} \), that \( \tilde{\pi} : G \times C_c(G \times G, X_0) \to C_c(G \times G, X_0) \) is continuous, because \( \tilde{\sigma} \) is continuous by 2.1.3 (of course, continuity of \( \tilde{\pi} \) can also be proved directly). In this way we obtain a continuous isomorphic correspondence \( \tilde{\pi}^u \in \tilde{\pi} \).

7.2.2. **Lemma.** Suppose \( G \) is non-compact, and let \( I \) be a set of cardinality \( |I| = L(G) \), the Lindelöf degree of \( G \). Then there exists a locally finite, disjoint \( 1 \) family \( \{ C_i : i \in I \} \) of non-empty open subsets of \( G \). If \( L(G) > \aleph_0 \) (i.e. \( G \) not sigma-compact) or if \( G \) is 0-dimensional, then every \( C_i \) may assumed to be open and closed. In all cases, the family \( \{ C_i : i \in I \} \) may supposed to be disjoint.

**Proof.** First we consider the case that \( G \) is not sigma-compact, i.e. \( L(G) > \aleph_0 \). Let \( U \in V_c \) be compact and symmetric, and set \( H := U \{ U^0 : i \in I \} \).

Then it is well-known (and easy to prove) that \( H \) is a subgroup of \( G \), that \( H \) is open and closed in \( G \) (see [HR], 5.7), and that \( H \) is sigma-compact (each \( U^0 \) is compact). Since the family of all different right cosets of \( H \) is \( G \) forms an open covering which has no proper subcovering, it is clear that \( L(G) \geq |G/H| \). On the other hand, \( |G/H| > \aleph_0 \), otherwise \( G \) would be sigma-compact. Moreover, each of the right cosets of \( H \) is open in \( G \) and its Lindelöf degree equals \( L(H) \) which is \( \leq \aleph_0 \). Now it is not difficult to see that every open covering of \( G \) has a refinement of cardinality \( \aleph_0 \cdot |G/H| = |G/H| \) (take intersections with cosets). Hence it has a subcovering of cardinality \( |G/H| \) and, consequently, \( L(G) = |G/H| \). Hence for the family \( \{ C_i : i \in I \} \) we can take the right cosets of \( H \) in \( G \); this collection satisfies all requirements.

If \( G \) is sigma-compact then the preceding method fails because it may occur that \( H = G \) (e.g. if \( G \) is connected). Now we proceed in the following

---

\(^1\) Disjoint means: \( i \neq j \Rightarrow C_i \cap C_j = \emptyset \). Since each \( C_i \neq \emptyset \), this implies plainly that \( C_i \neq C_j \) for \( i \neq j \).
way. First, note that $L(G) = \aleph_0$, so we may take $I = \mathbb{N}^1$. Again, let $U$ denote a compact neighbourhood of $e$ in $G$. Since $G$ is not compact, there is a sequence $(t_n : n \in \mathbb{N})$ in $G$ such that $t_{n+1} \not\in U(t_n : 1 \leq n)$. Let $V$ be an open neighbourhood of $e$ such that $V = V^{-1}$ and $V_n \subseteq U$. Then the family $(t_n^2 : n \in \mathbb{N})$ is disjoint, hence $(t_n^2 : n \in \mathbb{N})$ is disjoint and locally finite (if $s \in G$, then $sV$ meets at most one of the sets $t_nV$). Hence we can take $C_i = t_iV$ for $i \in \mathbb{N}$. Observe that $\overline{C_i} \subseteq t_i^2$, so that the sets $\overline{C_i}$ are mutually disjoint. If $G$ is $0$-dimensional, then $V$ could have been taken open and closed, so that each $C_i = t_iV$ would be open and closed. \[\square\]

7.2.3. Suppose that $G$ is non-compact (for compact groups, cf. 7.2.13 below). Fix a locally finite family $(C_i : i \in I)$ of non-empty open subsets of $G$, where $I$ is a set of cardinality $L(G)$, and $\overline{C_i} \cap \overline{C_j} = \emptyset$ for $i, j \in I$, $i \neq j$. Next, fix a family of continuous functions $\psi_i : G \times [0, 1] (i \in I)$ as follows:

If each $C_i$ is open and closed, let $\psi_i$ be the characteristic function of $C_i$, that is, $\psi_i(t) = 0$ or $1$, according to $t \in C_i$ or $t \not\in C_i$, respectively. In the other case, fix $t_i \in C_i$ for every $i \in I$. Using complete regularity of $G$, it follows that there exist continuous functions $\psi_i : G \times [0, 1]$ such that $\psi_i(t_i) = 1$ and $\psi_i(t) = 0$ for $t \in G \sim C_i$.

Notice that in both cases we have $\psi_i(t) = 0$ for $t \not\in C_i$; in addition, for each $i \in I$ there exists $t_i \in C_i$ such that $\psi_i(t_i) = 1$.

7.2.4. With $I$ as in 7.2.3, assume that the space $X_0^I$ is comprehensive in TOP with respect to a class $\mathcal{B}$ of topological spaces. Assume that there exists a mapping $m : [0, 1] \times X_0^{I} \rightarrow X_0$ with the following properties:

(U1) $\exists x_0 \in X_0 : m(0, x) = x_0$ for all $x \in X_0$.

(U2) $m(1, x) = x$ for all $x \in X_0$.

(U3) For every $i \in I$, the mapping $(s, x) \mapsto m(\psi_i(s), x) : G \times X_0 \rightarrow X_0$ is continuous.

7.2.5. EXAMPLES.

(i) If $X_0$ is a contractible space, then there exists a continuous mapping $m : [0, 1] \times X_0 \rightarrow X_0$ satisfying (U1) and (U2) of 7.2.4. Condition (U3) is then obviously fulfilled. Notice that for each cardinal number $\kappa$ the space $S(\kappa)$ (cf. 7.1.3) is contractible. Other contractible spaces are

\[\]
If \( x_0 = [0,1] \) or \( x_0 = \mathbb{R} \), we can take for \( m \) the usual multiplication mapping, \( m: (x,y) \mapsto xy \).

(ii) If \( x_0 = D_2 \) the discrete space \( \{0,1\} \), then define \( m: [0,1] \times D_2 \to D_2 \) by

\[
m(a,0) = 0 \quad \text{for all } a \in [0,1]
\]

\[
m(a,1) = \begin{cases} 
0 & \text{if } a = 0 \\
1 & \text{if } 0 < a \leq 1.
\end{cases}
\]

Obviously, \( m \) satisfies (U1) with \( x_0 = 0 \) and (U2). In addition, if all sets \( C_i \) in 7.2.3 could be chosen to be open and closed, then (U3) is also satisfied. (Indeed, then the functions \( \psi_i \) have only the values 0 and 1, so \( m(\psi_i(s),x) = 0 \) for all \( s \in G \sim C_i \) and \( m(\psi_i(s),x) = x \) for all \( s \in C_i \) \((x \in D_2)\). Since both \( C_i \) and \( G \sim C_i \) are open, continuity of \( (s,x) \mapsto m(\psi_i(s),x) \) is obvious.) Thus, if \( G \) is 0-dimensional or if \( L(G) = N_0 \) (cf. 7.2.2) then we may assume that the function \( m \), defined by (2), satisfies (U1), (U2) and (U3) for \( x_0 = D_2 \).

7.2.6. In the remainder of this subsection, we shall write more concisely \( m(s,x) = sx \) for \( (s,x) \in G \times x_0 \), if \( m \) is as in 7.2.4.

With notation as in 7.2.3 and 7.2.4 we can define unambiguously a function \( \Gamma: C_c(G,x_0) \to x_0^{G \times G} \) by

\[
\Gamma(\eta)(s,t) := \begin{cases} 
\psi_i(t)\eta_i(s) & \text{if } t \in cl_i G \setminus C_i \\
x_0 & \text{if } G \sim \bigcup_j C_j
\end{cases}
\]

for \( \eta = (\eta_i) \in C_c(G,x_0)_I \) and \( s,t \in G \). That this is possible follows from the fact that the closures of the sets \( C_i \) are disjoint. Moreover, for \( t \in cl_i G \setminus (G \sim \bigcup_j C_j) \) we have \( t \notin C_i \), so \( \psi_i(t) = 0 \), hence \( \psi_i(t)\eta_i(s) = x_0 \) by (U1). So the definition is unambiguous.

7.2.7. **Lemma.** The mapping \( \Gamma \) defines a topological embedding of \( C_c(G,x_0)_I \) into \( C_c(G \times G,x_0) \).

\( ^{1} \) Stated otherwise, \( \Gamma \) induces a topological embedding of \( C_c[I \times G,x_0] \) into \( C_c(G \times G,x_0) \) or, alternatively, of \( C_c(I,C_c(G,x_0)) \) into \( C_c(G,C_c(G,x_0)) \).
PROOF. Consider \( \eta = (\eta_i) \in C_c(G \times X_0)^I \). First, we have to show that \( \Gamma(\eta) \in C_c(G \times X_0) \). To this end, observe that for every \( i \in I \), the function \( (s, t) \mapsto \psi_i(t) \eta_i(s) : G \times X_0 \to X_0 \) is continuous by (U3). Consequently, the restriction of \( \Gamma(\eta) \) to each set \( G \times \text{cl}_G C_i \) is continuous. Obviously, the restriction of \( \Gamma(\eta) \) to the set \( G \times (G - U_i C_i) \) is continuous. The family \( \{G \times \text{cl}_G C_i : i \in I\} \) is a covering of \( G \times G \) by closed sets, and this covering is easily seen to be locally finite. Indeed, \( \{C_i : i \in I\} \) and hence \( \{\text{cl}_G C_i : i \in I\} \) is a locally finite family in \( G \). Now it is an easy exercise to show that continuity of \( \Gamma(\eta) \) on each member of this locally finite, closed covering of \( G \times G \) implies continuity of \( \Gamma(\eta) \) on \( G \times G \). (Cf. also [Du], Chap.III, Theorem 9.4.)

Next, we show that \( \Gamma : C_c(G \times X_0)^I \to C_c(G \times X_0) \) is continuous. It is sufficient to show that for every subbasical open set of the form \( N(K_1 \times K_2, V) \) with compact \( K_1, K_2 \) in \( G \) and open \( V \) in \( X_0 \), the set \( \Gamma^{-1}[N(K_1 \times K_2, V)] \) is open in \( C_c(G \times X_0)^I \). So consider \( \eta \in \Gamma^{-1}[N(K_1 \times K_2, V)] \), i.e. \( \eta = (\eta_i) \in C_c(G \times X_0)^I \) with \( \Gamma(\eta)(K_1 \times K_2) \subseteq V \). Since \( \{\text{cl}_G C_i : i \in I\} \) is locally finite and \( K_2 \) is compact, there is a finite subset \( I_0 \) of \( I \) such that \( K_1 \cap \text{cl}_G C_i = \emptyset \) iff \( i \in I_0 \).

For every \( i \in I_0 \), set \( U_i := \{y \in X_0 : \psi_i(t) y \in V \text{ for all } t \in A_i\} \). If \( y \in U_i \), then some elementary compactness arguments (namely, an application of 0.2.2(i)) show that \( y \) is an interior point of \( U_i \). This proves that \( U_i \) is an open subset of \( X_0 \). Since \( \Gamma(\eta)(K_1 \times K_2) \subseteq V \), it follows that \( \psi_i(t) \eta_i(s) \in V \) for all \( s \in K_1 \) and \( t \in A_i \). Hence \( \eta_i(s) \in U_i \) for all \( s \in K_1 \), that is, \( \eta_i \in N(K_1, U_i) \). Thus, \( N(K_1, U_i) \) is an open neighbourhood of \( \eta_i \) in \( C_c(G \times X_0)^I \).

Now set \( V_i := N(K_1, U_i) \) if \( i \in I_0 \) and \( V_i := C_c(G \times X_0)^I \) if \( i \in I \setminus I_0 \). Then \( \Gamma(V_i) \) is a neighbourhood of \( \eta_i \) in \( C_c(G \times X_0)^I \). Moreover, if \( \xi \in \Gamma(V_i) \) then we have for all \( (s, t) \in K_1 \times K_2 \):

\[
\Gamma(\xi)(s, t) = \{\psi_i(t) \xi_i(s) \text{ if } t \in A_i, x_0 \text{ if } t \in K_2 - U_j C_j\}
\]

In the first case, \( i \in I_0 \), hence \( \xi_i(s) \in U_i \) and \( \psi_i(t) \xi_i(s) \in V \). If, in the other case, \( s \in K_1 \) and \( t \in K_2 - U_j C_j \) then \( \Gamma(\xi)(s, t) = x_0 \), and consequently, \( \Gamma(\xi)(s, t) = x_0 \in \Gamma(\eta)(K_1 \times K_2) \subseteq V \). In all cases, therefore, we have \( \Gamma(\xi)(s, t) \in \Gamma^{-1}[N(K_1 \times K_2, V)] \). We have proved now, that \( \Gamma(V_i) \subseteq N(K_1 \times K_2, V) \), and the continuity of \( \Gamma \) follows.

That \( \Gamma \) is injective is easy to see: if \( \xi, \eta \in C_c(G \times X_0)^I \), \( \xi \neq \eta \), then for some \( i \in I \) and \( s \in G \) we have \( \xi_i(s) \neq \eta_i(s) \). Take the element \( t_i \in C_i \) with \( \psi_i(t_i) = 1 \). Then, by (U2),
\[
\Gamma(\xi)(s,t) = \psi_1(t)\xi_1(s) = \xi_1(s) \neq \eta_1(s) = \psi_1(t)\eta_1(s) = \Gamma(\eta)(s,t).
\]

Consequently, \( \Gamma(\xi) \neq \Gamma(\eta) \).

Finally, we show that \( \Gamma \) is relatively open. It is sufficient (and, by injectiveness of \( \Gamma \), also necessary) to show the following: given any \( \eta \in C_c(G, X_0^1) \) and any neighbourhood \( V \) of \( \eta \) in \( C_c(G, X_0^1) \), there exists a neighbourhood \( W \) of \( \Gamma(\eta) \) in \( C_c(G \times G, X_0) \) such that

\[
\{ \xi : \xi \in C_c(G, X_0^1) \text{ and } \Gamma(\xi) \in W \} \subseteq V.
\]

We may assume that \( V = \bigcup_{i \in I_1} V_i \), where for some finite subset \( I_1 \) of \( I \), some compact set \( K \) in \( G \) and some open sets \( U_i \) in \( X_0 \) (\( i \in I_1 \)),

\[
V_i = \begin{cases} 
C_c(G, X_0) & \text{if } i \not\in I - I_1 \\
N(K, U_i) & \text{if } i \in I_1.
\end{cases}
\]

Let \( K_i := \{ t_i : i \in I_1 \} \) (recall that each \( t_i \in C_i \) satisfies the condition that \( \psi_i(t_i) = 1 \)). Then \( K_i \) is a finite, hence compact subset of \( G \).

Now for every \( i \in I_1 \) and \( s \in K \) we have

\[
\Gamma(\eta)(s, t_i) = \psi_1(t_i)\eta_1(s) = \eta_1(s) \in U_i
\]

(use (12) and the fact that \( \eta_i \in V_i = N(K, U_i) \)). Consequently, \( \Gamma(\eta) \in N(N(K \times t_i, U_i) : i \in I_1) =: W \). Obviously, it follows that \( W \) is a neighbourhood of \( \Gamma(\eta) \) in \( C_c(G \times G, X_0) \). This \( W \) satisfies condition (4). Indeed, if \( \xi \in C_c(G, X_0^1) \) and \( \Gamma(\xi) \in W \), then for any \( i \in I_1 \) and \( s \in K \) we have

\[
\xi_i(s) = \psi_1(t_i)\xi_1(s) = \Gamma(\xi)(s, t_i) \in U_i
\]

Hence, \( \xi_i \in N(K, U_i) = V_i \) for every \( i \in I_1 \), and, consequently, \( \xi \in V \). \[ Q.E.D. \]

7.2.8. **Lemma.** The mapping \( \Gamma : C_c(G, X_0^1) \to C_c(G \times G, X_0) \) defines a morphism of \( G \)-spaces from \( C_c(G, X_0^1) \) with action \( p^1 \) (af. 7.1.4) into \( C_c(G \times G, X_0) \) with action \( \tilde{\gamma} \).

**Proof.** A straightforward computation. \[ Q.E.D. \]
7.2.9. **PROPOSITION.** The ttg \(<G, C_c(G \times G, X), \tau>\) is comprehensive in \(\text{TOP}^G\) with respect to the class \((S^G)^{[B]}\), provided \(\mathcal{X}_0^L(G)\) is comprehensive in \(\text{TOP}\) with respect to the class \(B\), \(G\) is non-compact, and \(X_0\) satisfies the conditions \((U1), (U2)\) and \((U3)\) of 7.2.4.

**PROOF.** For any \(G\)-space \(<G,Y,\sigma>\) with \(Y \in \mathcal{B}\), we have the equivariant embedding \(\tau \Phi \Phi \Phi \Phi \Phi g\) of \(Y\) into \(C_c(G \times G, X_0)\) (for \(\Phi \Phi \Phi \Phi \Phi g\), cf. 7.1.5). □

7.2.10. **APPLICATIONS.** Suppose \(G\) is a non-compact locally compact Hausdorff topological group. Then:

(i) \(<G, C_c(G \times G, [0,1]), \tau>\) is comprehensive in \(\text{TOP}^G\) with respect to the class of all ttgs \(<G,Y,\sigma>\) with \(Y\) a Tychonov space of weight \(\omega(Y) \leq \ell(G)\).

(ii) \(<G, C_c(G \times G, D_2), \tau>\) is comprehensive in \(\text{TOP}^G\) with respect to the class of all ttgs \(<G,Y,\sigma>\) with \(Y\) a 0-dimensional Hausdorff space of weight \(\omega(Y) \leq \ell(G)\), provided either \(\ell(G) > \aleph_0\) or \(G\) is 0-dimensional.

(iii) \(<G, C_c(G \times G, S(\kappa)), \tau>\) is comprehensive in \(\text{TOP}^G\) with respect to the class of all ttgs \(<G,Y,\sigma>\) with \(Y\) a \(T_\kappa\)-space which has an \(\ell(G)\)-discrete base of cardinality \(\leq \kappa\).

If \(G\) is sigma-compact (i.e. \(\ell(G) = \aleph_0\)), then in (i) and (ii) above, all admitted spaces \(Y\) are separable metrizable, and in (iii), \(Y\) may be any metrizable space of weight \(\leq \kappa\).

**PROOF.** Cf. 7.1.3 and 7.2.5. □

7.2.11. In 7.2.10(i), the space \(C_c(G \times G, [0,1])\) may clearly be replaced by \(C_c(G \times G, B) = C_c(G \times G)\). The ttg \(<G, C_c(G \times G, \tau), \tau>\) seems simple enough to deserve the predicate "nice". On the other hand, this ttg comprises all ttgs \(<G,Y,\sigma>\) with \(Y\) a Tychonov space of weight \(\leq \ell(G)\). This implies that \(<G, C_c(G \times G, \tau), \tau>\) has to have a rather complex structure.

7.2.12. According to 7.1.5, in particular formula (8), and the definition of \(M\) in 7.2.6, the equivariant embedding of a ttg \(<G,Y,\sigma>\) with \(Y \in \mathcal{B}\) into the ttg \(<G, C_c(G \times G, X), \tau>\) mentioned in 7.2.9 may be effected in the following way. Let \(I\) be a set with \(|I| = \ell(G)|\), let \(\{\psi_i \mid i \in I\}\) be a set of functions from \(G\) into \([0,1]\) and let \(m: [0,1]^X \rightarrow X_0\) be as in 7.2.3 and 7.2.4 (these data can be fixed with \(G\) and \(X_0\)). If \(\{f_i \mid i \in I\}\) is a set of continuous functions from \(Y\) into \(X_0\) which separates points and closed sets in \(Y\), then the equivariant embedding \(h\) of \(Y\) into \(C_c(G \times G)\) is given by

\[\]
for $y \in Y$ and $(s,t) \in G \times G$.

(7) \[ h(y)(s,t) = \begin{cases} \psi_1(t) f_1(\sigma(s,y)) & \text{if } t \in \text{cl}_G C_1 \\ x_0 & \text{if } t \notin U_j C_j \end{cases} \]

for $y \in Y$ and $(s,t) \in G \times G$. According to 7.1.2, the mapping $\varphi^*_0 \varphi^* \pi G \times G \times G \to C_0(G_{X_0})$ is such that $\{\varphi^*_0 \varphi^* \pi G \times G \times G \to C_0(G_{X_0})\}$ separates the points of $Y$. Recall, that here

(8) \[ \varphi^*_0 \varphi^* \pi G \times G \times G \to C_0(G_{X_0}) \]

for every $y \in Y$. We might also have started by defining a mapping $Y \to C_0(G_{X_0})$ according to this rule, and then defining $\varphi$ as the $\varphi^*_0$-value of this mapping. The technical difficulties, however, would have been the same (i.e. the several parts of the proof of lemma 7.2.7).

7.2.13. If $G$ is a compact $T_2$ group, then every locally finite family of subsets of $G$ is finite. So the previous method yields only a tte which is comprehensive for tte $<G,Y,\sigma>$ with $Y$ a subset of $X_0^n$, $n \in \mathbb{N}$ such that $G$ admits a locally finite disjoint family consisting of $n$ non-void open subsets. This can be a considerable class of tte: each $k$-dimensional separable metrizable space $Y$ can be embedded in $[0,1]^{2k+1}$. Consequently, if $G$ admits for every $n \in \mathbb{N}$ a disjoint family of $n$ non-void open subsets, then $<G, C_0(G_{X_0}), \pi G \times G \times G \to C_0(G_{X_0})>$ is comprehensive for the class of all tte $<G,Y,\sigma>$ with $Y$ a separable metrizable space of finite dimension.

We shall remove now finite dimensionality from the conditions, i.e. we shall prove that 7.2.10(i) is also valid if $G$ is compact, but not finite.

---

1) Cf. for example [Na], Theorem IV.8.
2) Since $G$ is a compact Hausdorff space, this is equivalent to saying that $G$ is not finite.
First we have to find a substitute for lemma 7.2.7.

7.2.14. **Lemma.** Let $G$ be an infinite compact Hausdorff topological group. Then there exists a sequence $(C_n : n \in \mathbb{N})$ of pairwise disjoint, non-empty open subsets of $G$.

**Proof.** Since $G$ is not finite, $G$ is not discrete. Hence there exists a sequence $(V_n : n \in \mathbb{N})$ of neighbourhoods of $e$ such that $\text{cl}_G V_{n+1} \subseteq V_n$ for every $n \in \mathbb{N}$. Now set $C_n := V_n \setminus \text{cl}_G V_{n+1}$ ($n \in \mathbb{N}$).

7.2.15. Let $G$ be compact and infinite, and fix a sequence $(C_n : n \in \mathbb{N})$ of mutually disjoint, non-empty open subsets of $G$. As in 7.2.3, let $(\psi_n : n \in \mathbb{N})$ be a sequence of continuous functions from $G$ into $[0,1]$ such that $\psi_n(t) = 0$ for $t \in G - C_n$ and $\psi_n(t_n) = 1$ for some $t_n \in C_n$ ($n \in \mathbb{N}$). Define a mapping $\Gamma : C_c(G,[0,1])^{\mathbb{N}} \rightarrow [0,1]^{G \times G}$ by

$$\Gamma(\eta)(s,t) := \sum_{n=1}^{\infty} 2^{-n} \psi_n(t) \eta_n(s)$$

for $\eta = (\eta_n \in C_c(G,[0,1]))^\mathbb{N}$ and $s,t \in G$. Here juxtaposition in the right-hand member of (9) denotes ordinary multiplication in $[0,1]$. Since for every $s,t \in G$ the series in (9) is absolutely dominated by the convergent series $\sum_{n=1}^{\infty} 2^{-n}$ (which has sum 1), it is clear that $\Gamma(\eta)(s,t)$ is well-defined for every $\eta \in C_c(G,[0,1])^{\mathbb{N}}$ and $s,t \in G$, and that $\Gamma(\eta)(s,t) \in [0,1]$.

We can draw even one more conclusion: the convergence of the series in (9) is uniform in $(s,t) \in G \times G$. For fixed $\eta \in C_c(G,[0,1])^{\mathbb{N}}$, the terms in the series are continuous functions of $(s,t)$ on $G \times G$. Consequently, the sum of the series depends continuously on $(s,t)$, i.e. $\Gamma(\eta) \in C(G \times G,[0,1])$.

7.2.16. **Lemma.** The mapping $\Gamma$ defined in (9) is a topological embedding of $C_c(G,[0,1])^{\mathbb{N}}$ into $C_c(G \times G,[0,1])$.

**Proof.** Since $G \times G$ is compact, basic neighbourhoods of $\Gamma(\eta)$ in $C_c(G \times G,[0,1])$ have the form

$$(\xi \in C_c(G \times G,[0,1]) : ||\Gamma(\eta)(s,t)-\xi(s,t)|| < \varepsilon \text{ for all } s,t \in G),$$

with $\varepsilon > 0$. Using this, continuity and relative openness of $\Gamma$ may be proved along the lines of the proof of 7.2.7. In the proof of the continuity of $\Gamma$, the finite subset $I_0$ of $\mathbb{N}$ may be obtained by requiring that $2(2^{-n} : n \in \mathbb{N} - I_0)$ is sufficiently small (i.e. $I_0$ a sufficiently large initial
segment in \(\mathbb{N}\). □

7.2.17. Neither in 7.2.8, nor in 7.2.1, we used the non-compactness of \(G\). Consequently, 7.2.9 remains valid for compact infinite \(G\), provided we substitute \([0,1]\) for \(X_0\). The class \(S\) for which \([0,1]^\mathbb{N}\) is comprehensive is just the class of all separable metrizable spaces, i.e. the class of all Tychonov spaces of weight \(\leq \aleph_0\).

7.2.18. **Theorem.** Let \(G\) be any infinite locally compact Hausdorff topological group. Then the tbr \(<G,\mathcal{C}_C(GxG,[0,1]),\bar{t}\>\) is comprehensive in \(\text{TOP}^G\) with respect to the class of all tbrs \(<G,Y,\sigma>\) with \(Y\) a Tychonov space of weight \(w(Y) \leq L(G)\).

**Proof.** Cf. 7.2.10(i) for the case that \(G\) is non-compact. If \(G\) is compact, we have \(L(G) = \aleph_0\) (in fact, \(L(G) = \aleph_0\), because \(L(G) < \aleph_0\) would imply that \(G\) were finite). Hence \(w(Y) \leq L(G)\) implies that \(Y\) is a separable metrizable space, and we can apply the preceding remark. □

7.2.19. In the above theorem, we may of course replace \(\mathcal{C}_c(GxG,[0,1])\) by the space \(\mathcal{C}_c^*(GxG)\), or by the space \(\mathcal{C}_c(GxG)\). However, it is useful to notice that a uniformly bounded invariant subspace of \(\mathcal{C}_c(GxG)\), namely \(\mathcal{C}_c(GxG,[0,1])\), is comprehensive for the class of \(G\)-spaces described in the theorem.

We mention some particular properties of the tbr \(<G,\mathcal{C}_c(GxG),\bar{t}>,\mathcal{F}\>:

(i) This tbr is effective but not strongly effective. Indeed, if \(t \neq e\), then there is \(f \in \mathcal{C}_c(GxG)\) such that \(f(t,e) \neq f(e,e)\), hence \(\bar{f}\bar{t} \neq \bar{t}\); on the other hand, \(\bar{t}\bar{g} = g\) for any constant function \(g\).

(ii) The set of invariant points in \(<G,\mathcal{C}_c(GxG),\bar{t}>,\mathcal{F}\>\) is homeomorphic with \(\mathcal{C}_c(G)\). This follows immediately from the fact that \(<G,\mathcal{C}_c(GxG),\bar{t}>,\mathcal{F}\>\) is isomorphic to \(<G,\mathcal{C}_c(G),\tilde{\rho}>,\mathcal{F}\>\). (Indeed, for any space \(Y\), the invariant points in \(<G,\mathcal{C}_c(G,Y),\tilde{\rho}>,\mathcal{F}\>\) are the constant functions, and they form a subset of \(\mathcal{C}_c(G,Y)\) which is homeomorphic to \(Y\).) Similarly, the set of invariant points in \(<G,\mathcal{C}_c(GxG,[0,1]),\bar{t}>,\mathcal{F}\>\) is homeomorphic with \(\mathcal{C}_c(G,[0,1])\).

(iii) If \(G\) is compact, then \(\mathcal{C}_c(GxG) = \mathcal{C}_u(GxG)\), and every \(\bar{t}\mathcal{F}\) is an isometrical mapping of the metric space \(\mathcal{C}_u(GxG)\) onto itself (the metric in \(\mathcal{C}_u(GxG)\) is of course, the metric generated by the uniform norm).

7.2.20. **Examples.** We shall describe here three examples concerning the case that \(G = \mathbb{R}, \mathbb{Z}\), or \(T\), respectively.
Suppose $G = \mathbb{R}$. For $i = 1, 2, \ldots$ we can take $C_i := [i-\frac{1}{2}, i+\frac{1}{2}]$, and $\psi_i: \mathbb{R} \to [0,1]$ as follows

$$
\psi_i(t) := \begin{cases} 
\cos^2 2\pi(t-i) & \text{if } i-\frac{1}{2} \leq t \leq i+\frac{1}{2} \\
0 & \text{otherwise.}
\end{cases}
$$

If $<\mathbb{R}, Y, \sigma>$ is any ttg with $Y$ a separable metrizable space and $\{f_i : i \in \mathbb{N}\}$ is any family of continuous functions from $Y$ into $[0,1]$ separating points and closed subsets of $Y$, then an equivariant embedding $h$ of $Y$ into $C_c(\mathbb{R} \times \mathbb{R}, [0,1])$ is given by

$$
h(y)(s,t) = \begin{cases} 
\cos^2 2\pi(t-i) \cdot f_i(\sigma(s,y)) & \text{if } i-\frac{1}{2} \leq t \leq i+\frac{1}{2} \\
0 & \text{if } t \notin \cup_j [j-\frac{1}{2}, j+\frac{1}{2}]
\end{cases}
$$

for $y \in Y$ and $(s,t) \in \mathbb{R} \times \mathbb{R}$. We may interprete this also as an equivariant embedding of $Y$ (with action $\sigma$) into $C_c(\mathbb{R} \times \mathbb{R})$ (with action $\tilde{\sigma}$) or into $C_c(\mathbb{R}, C_c(\mathbb{R}))$ (with action $\tilde{\sigma}$).

Suppose $G = \mathbb{Z}$. For $i \in \mathbb{Z}$, set $C_i := \{i\}$, and define $\psi_i: \mathbb{Z} \to [0,1]$ by

$$
\psi_i(t) := \begin{cases} 
1 & \text{if } t = i \\
0 & \text{if } t \in \mathbb{Z} \sim \{i\}.
\end{cases}
$$

If $Y$ is a separable metrizable space and $\{f_i : i \in \mathbb{Z}\}$ is any family of continuous functions of $Y$ into $[0,1]$ separating points and closed subsets of $Y$, then for any homeomorphism $\sigma^1: Y \to Y$ (equivalently, for any action $\sigma$ of $\mathbb{Z}$ on $Y$; cf. 1.1.6(viii)) we have the following equivariant embedding $h$ of $Y$ into $C_c(\mathbb{Z} \times \mathbb{Z}, [0,1]) = [0,1]^{\mathbb{Z} \times \mathbb{Z}}$:

$$
h(y)(s,t) = f_i(\sigma^1 y)
$$

for $y \in Y$ and $(s,t) \in \mathbb{Z} \times \mathbb{Z}$. Note that the action $\tilde{\sigma}$ of $\mathbb{Z}$ on $[0,1]^{\mathbb{Z} \times \mathbb{Z}}$ is generated by the automorphism

$$
\tilde{\sigma}^1: (\xi(i,j))_{i,j} \mapsto (\xi(i+1,j))_{i,j}
$$

(bilateral shift in the first coordinate.)

If $Y$ is a separable metrizable 0-dimensional space and $\{f_i : i \in \mathbb{Z}\}$ is a family of continuous functions of $Y$ into $\{0,1\}$ separating points and
closed subsets, then for any homeomorphism \( \sigma : Y \to Y \), (13) describes an equivariant embedding of \( Y \) into \( C_c(\mathbb{R} \times \mathbb{Z}, \{0,1\}) = \{0,1\}^{\mathbb{Z}} \). In this space, the action is again described by (14).

Suppose \( G = T \). For \( i = 1, 2, \ldots \), set \( C_k := \{\exp(2\pi it) : t \in D_k\} \), where \( D_k \) is the interval
\[
\left[ \frac{1}{k} - \frac{1}{2k(k+1)}, \frac{1}{k} + \frac{1}{2k(k+1)} \right] = \left[ -\frac{2k+1}{2k(k+1)}, \frac{2k+3}{2k(k+1)} \right].
\]
Define \( \Psi_k \) by
\[
\Psi_k(\exp(2\pi it)) := \begin{cases} 
\cos^2 k(x+1)\pi(t-k^{-1}) & \text{if } t \in D_k \\
0 & \text{if } t \notin D_k.
\end{cases}
\]
If \( <T,Y,G> \) is a ttg with \( Y \) a separable metrizable space and \( \{f_i : i \in \mathbb{N}\} \) is a family of continuous functions of \( Y \) into \([0,1]\) separating points and closed subsets of \( Y \), then an equivariant embedding of \( Y \) into \( C_c(T \times T, \{0,1\}) = C_c(T \times T, [0,1]) \) is obtained by setting
\[
h(y)(u,v) = \begin{cases} 
\cos^2 k(x+1)\pi(t-k^{-1}) \cdot f_k(\sigma(u,y)) & \text{if } v = \exp(2\pi it) \\
0 & \text{otherwise}
\end{cases}
\]
for \( y \in Y \) and \( (u,v) \in T \times T \).

7.2.21. **NOTES.** Fundamental in this subsection is lemma 7.2.2. However, in this lemma it is not essential that \( G \) is a group. In fact, the lemma can be proved for any paracompact locally compact Hausdorff space. Cf. J. DE VRIES [1972b].

7.3. **Compactifications of G-spaces**

7.3.1. For the motivation, or at least, for the connection of the contents of this subsection with the results of Chapter II, we refer to the final remark in 4.3.13. Although all applications will be for locally compact Hausdorff groups \( G \), we shall not make any particular assumption about \( G \) up to 7.3.7 (except that it has to be a topological group).

7.3.2. Let \( <G,X,\pi> \) be a ttg with \( X \) a completely regular space (i.e. the topology of \( X \) can be generated by a uniformity). Then \( <G,X,\pi> \) is said to be
bounded with respect to the uniformity \( U \) provided
(i) \( U \) generates the topology of \( X \).
(ii) \( \forall x \in U, \exists U \in V_x \ni (\pi^t x, x) \in \alpha \) for all \( t \in U, x \in X \).

The ttg \( <G,X,\pi> \) is said to be bounded if it is bounded with respect to some uniformity \( U \). If \( X \) is metrizable and \( <G,X,\pi> \) is bounded with respect to a metrical uniformity (i.e. a uniformity with a countable base), then \( <G,X,\pi> \) is called metrically bounded.

7.3.3. LEMMA. Let \( <G,X,\pi> \) be a ttg, \( X \) a uniform space with uniformity \( U \). The following conditions are equivalent:
(i) \( <G,X,\pi> \) is bounded with respect to \( U \).
(ii) The family \( \{\pi^t : x \in X\} \) of functions from \( G \) into \( X \) is equicontinuous
(iii) The family \( \{\pi^t : x \in X\} \) is equi-uniformly continuous on \( G \), if \( G \) is endowed with its right uniformity.

PROOF. (i) \( \Rightarrow \) (ii): Obvious from the definitions.
(ii) \( \Rightarrow \) (iii): Let \( \alpha \in U \). Take \( U \in V_x \) such that \( (\pi^t y, y) \in \alpha \) for all \( t \in U \) and \( y \in X \). In particular, for every \( s \in G \) and \( x \in X \), setting \( y := \pi^s x \), we obtain \( (\pi^t(s) \pi^t(x)) \in \alpha \) for all \( x \in X \), \( s \in G \) and \( t \in V \).
(iii) \( \Rightarrow \) (ii): Obvious. \( \square \)

7.3.4. In contradistinction to 7.3.14 below, we present now an example of a ttg \( <G,X,\pi> \) with \( G \) a sigma-compact locally compact Hausdorff topological group and \( X \) a (non-separable!) metrizable space, such that \( <G,X,\pi> \) is bounded but not metrically bounded.

7.3.5. EXAMPLE. Let \( I \) be an uncountable set, and let, for every \( i \in I \),
\( <H_i,Y_i,\rho_i> \) be a ttg with the following properties:
(i) \( Y_i \) is a compact metric space, say with metric \( d_i \) and metrical uniformity \( U_i \).
(ii) \( <H_i,Y_i,\rho_i> \) is transitive, i.e. for every \( x,y \in Y_i \) there exists \( t \in H_i \) such that \( \rho_i^t(x) = y \).
(iii) \( H_i \) is a sigma-compact locally compact Hausdorff topological group.

In addition, for a fixed finite, non-void subset \( I_0 \) of \( I \) we require

\[1 \text{ In } [GH], \text{ a ttg with this property is called motion equicontinuous.} \]
that \( H_i \) is compact if \( i \in I - \{ i_0 \} \) and \( H_i \) is non-compact if \( i \in I_0 \).

(Observe that such collections \( \{H_i, Y_i, \rho_i\} \mid i \in I \} \) of tgs exist. For example, let \( Y_i = T \) for every \( i \in I \), fix \( i_0 \in I \), and set \( H_{i_0} = \mathbb{R} \), \( \rho_{i_0} = \text{rotation of } T \) over \( 2\pi \) radians. For \( i \in I \), \( i \neq i_0 \), let \( H_i = T \) (as a topological group) and \( \rho_i = \lambda \) (= ordinary multiplication in \( T \)).)

Since each \( Y_i \) is compact, \( U_i \) is the unique uniformity in \( Y_i \) which is compatible with the topology of \( Y_i \). We shall use this fact without further reference. In addition, by 7.3.6 below, each \( <H_i, Y_i, \rho_i> \) is (metrically!) bounded.)

Let \( <G, X, \pi> \) denote the coproduct of the set \( \{<H_i, Y_i, \rho_i> \mid i \in I \} \) in \( \text{TOPGRP} \) (cf. 6.1.10), with coprojections \( \psi^i_P, f_i^*: <G, X, \pi> \to <H_i, Y_i, \rho_i> \). Then \( G \) is the product of the set \( \{H_i \mid i \in I \} \) in \( \text{TOPGRP} \), with projections \( \psi_i: G \to H_i \). So by condition (iii), \( G \) is a sigma-compact, locally compact Hausdorff group, but \( G \) is not compact. Moreover, \( X \) is the disjoint union of the spaces \( Y_i \).

Suppressing the canonical injections \( f_i: Y_i \to X \), set for \( i \in I \) and \( n \in \mathbb{N} \)

\[
U(i, n) := \{(x, y) \in Y_i \times Y_i \mid d_i(x, y) < n^{-1}\}
\]

and for every finite subset \( J \) of \( I \)

\[
V(J, n) := U(U(i, n) \mid i \in J) \cup U(Y_i \times Y_i \mid i \in I - J).
\]

Then \( \mathcal{B} := \{V(J, n) \mid n \in \mathbb{N} \text{ and } J \text{ a finite subset of } I \} \) is a base for a uniformity \( U \) in \( X \) which is compatible with the topology of \( X \). Since for any \( \alpha \in U \) we can have \( \alpha \cap (Y_i \times Y_i) \subseteq Y_i \times Y_i \) for only finitely many \( i \in I \) and since \( <H_i, Y_i, \rho_i> \) is bounded for those \( i \), it is easy to see that \( \{\pi_x \mid x \in X\} \) is equicontinuous with respect to the uniformity \( U \) in \( X \). So by 7.3.3, \( <G, X, \pi> \) is bounded.

However, it is not difficult to show that \( U \) cannot have a countable base, because \( I \) is uncountable. On the other hand, there exist uniformities for \( X \), generating its topology, which have a countable base, because \( X \) is obviously metrizable. Let \( V \) denote any such a uniformity. Then for some \( \beta \in V \) the set

\[
\]

\(1\) If we restrict ourselves to the concrete example with each \( Y_i = T \), etc., then this can be seen directly, without any reference to 7.3.6.
\[ J(\beta) := \{ i \in I : \beta \cap (Y_1 \times Y_1) \neq Y_1 \times Y_1 \} \]

is infinite. Otherwise, \( V \) would be equal to the uniformity \( U \) which has not a countable base. Fix such a \( \beta \). Let \( V = \bigcup V_i \) be a neighbourhood of \( e \) in \( G \), with \( V_i \neq V_j \) for only a finite number of indices \( i \in I \). Since \( J(\beta) \) is infinite, there exists \( j \in J(\beta) \) such that \( V_j = H_j \). In view of condition (ii) and the choice of \( \beta \), there exists \( t \in H_j = V_j \) such that \( (p_j(t,y), y) \notin \beta \cap (Y_j \times Y_j) \) for some \( y \in Y_j \). Considering \( y \) as an element of \( X \), this means that \( (\pi \cdot t(y), y) \notin \beta \) for \( t = (t_i)_i \in V \), where \( t_i \) is the unit of \( G_i \) for \( i \neq j \) and \( t_j \in V_j \) is as above. We have proved now, that there exists \( \beta \in V \) such that every neighbourhood \( V \) of \( e \) in \( G \) contains an element \( t \) such that \( (\pi \cdot t(y), y) \notin \beta \) for some \( y \in X \). Thus, \( <G, X, \pi> \) is not bounded with respect to \( V \). This shows that \( <G, X, \pi> \) is not metrically bounded.

7.3.6. PROPOSITION. If the phase space \( X \) of a ttg \( <G, X, \pi> \) is a compact Hausdorff space, then \( <G, X, \pi> \) is bounded. If \( X \) is compact and metrizable, then \( <G, X, \pi> \) is metrically bounded.

PROOF. Suppose \( X \) is a compact Hausdorff space. By an elementary compactness argument, namely 0.2.2(ii), \( <G, X, \pi> \) is bounded with respect to the unique uniformity \( U \) of \( X \). The second statement in the proposition is now trivial. \( \square \)

7.3.7. From a different point of view, we can formulate the proof of 7.3.6 as follows. If \( X \) is compact, then \( \mathbb{F}(X) \) is a compact subset of \( C_c(G, X) \), by 2.1.3. Moreover, the restriction of the evaluation mapping \( (\xi, t) \mapsto \xi(t) : C_c(G, X) \times G \to \mathbb{F}(X) \times G \) is continuous on \( \mathbb{F}(X) \times G \); indeed, \( \mathbb{F}: X \to \mathbb{F}(X) \) is a homeomorphism, and \( (x, t) \mapsto \pi \cdot t(x) : X \times G \to X \) is continuous. So by 0.2.8 (in particular, the converse to (iii)), \( \mathbb{F}(X) \) is equicontinuous at every point of \( G \), hence \( <G, X, \pi> \) is bounded by 7.3.3. This proof suggests that it may be useful to have a look at ttgs of the form \( <G, C_c(G, Y), \beta> \) with \( Y \) a uniform space. Although in general local compactness of \( G \) is needed to ensure that this is a ttg (cf. 2.1.3), we can dispense with this condition for \( G \) in the following proposition.

7.3.8. PROPOSITION. Let \( G \) be a topological group, let \( X \) be a uniform space, and let \( X \) be an invariant subset of the ttg \( <G, C_c(G, Y), \beta> \). Consider the following statements:

(i) \( X \) is equicontinuous on \( G \).

(ii) The mapping \( \beta : G \times X \to X \) is continuous, and the ttg \( <G, X, \beta> \) is bounded with respect to the relative uniformity of \( X \) in \( C_c(G, Y) \).
(iii) \(<G,X,\bar{\beta}>\) admits an equivariant embedding into a ttg \(<G,X,\bar{\beta}>\) with \(\bar{X}\) a compact Hausdorff space.

(iv) \(<G,X,\bar{\beta}>\) is bounded.

Then (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iv) and (iii) \(\Rightarrow\) (iv). In addition, if \(X \subseteq C^*(G,Y)\), then also the implication (i) \(\Rightarrow\) (iii) is valid.

**Proof.** (i) \(\Rightarrow\) (ii): By 2.1.6, \(\bar{\beta}: G \times X \to X\) is continuous, so \(<G,X,\bar{\beta}>\) is a ttg. Next, consider a compact subset \(K\) of \(G\) and an element \(a \in U\). Since \(X\) is equi-uniformly continuous on \(K\), there exists \(V \in \mathcal{V}_e\) such that \((f(s), f(t)) \in a\) for all \(s \in K\) and \(t \in G\) such that \(t^{-1}s \in V\), and for all \(f \in X\). Hence \((f, \bar{\beta}^t f) \in M(K,a)\) for all \(f \in X\) and \(u \in V^{-1}\).

(ii) \(\Rightarrow\) (i): Let \(U\) denote the uniformity of \(Y\), and take \(s \in G, a \in U\). Boundedness of \(<G,X,\bar{\beta}>\) implies that there exists \(U \in \mathcal{V}_e\) such that \((\bar{\beta}^tf, f) \in M([s], a)\) for all \(f \in X\) and \(t \in U\). This proves that \(X\) is equicontinuous.

(ii) \(\Rightarrow\) (iv) is trivial, and (iii) \(\Rightarrow\) (iv) is an obvious consequence of 7.3.6. Moreover, if \(X \subseteq C^*_c(G,Y)\), then equicontinuity of \(X\) implies that the closure \(\bar{X}\) of \(X\) in \(C_c(G,Y)\) is compact, by the Ascoli theorem. Therefore, \(\bar{X}\) is a compact invariant subset of \(C_c(G,Y)\). Moreover, \(\bar{X}\) is equicontinuous as well (cf. 0.2.8), so \(\bar{\beta}\) is continuous on \(G \times \bar{X}\), by the implication (i) \(\Rightarrow\) (ii) for \(\bar{X}\). Thus, (i) \(\Rightarrow\) (iii) with \(\bar{\eta} := \bar{\beta}|_{G \times X}\).

**7.3.9.** If in 7.3.8 the group \(G\) is locally compact and sigma-compact, and \(Y\) is metrizable, then in (iii), \(\bar{X}\) may required to be metrizable, and in (iv) we may demand: \(<G,X,\bar{\beta}>\) is metrically bounded.

Indeed, this follows immediately from the above proof and the fact that in the given situation \(C_c(G,Y)\) is metrizable (cf. Appendix C.4, or alternatively, [Dv], Chap. XII, 8.5).

**7.3.10. Examples.**

(i) Any ttg \(<G,X,\pi>\) with \(X\) locally compact Hausdorff is bounded. If in addition, \(X\) is a separable metrizable space, then \(<G,X,\pi>\) is metrically bounded. Indeed, \(X\) admits an equivariant embedding in the compact Hausdorff G-space \(\tilde{X} = X \cup \{\pi\}\), the one-point compactification of \(X\), with action \(\tilde{\pi}\) defined by \(\tilde{\pi}^t|_{\tilde{X}} = \tau^t\) and \(\tilde{\pi}^t(\pi) = \infty (t \in G)\). (See J. DE VRIES [1975c].) Now apply 7.3.6. Notice that \(\tilde{X}\) is metrizable if \(X\) is separable and metrizable.

(ii) Any ttg \(<G,X,\pi>\) with \(X\) a Tychonov space and \(G\) a discrete group is bounded. In fact, \(<G,X,\pi>\) is bounded with respect to every uniformity...
which generates the topology of $X$. This is a trivial consequence of (ii) $\Rightarrow$ (i) in 7.3.3.

We have not been able to find an example of a ttg $<G,X,n>$ with $X$ a Tychonov space which is not bounded. Observe that such an example would provide an instance of a Tychonov $G$-space which cannot be embedded in a compact $G$-space.

7.3.11. The behaviour of boundedness under the application of (co)morphisms of ttgs is very similar to the behaviour of ttgs with a compact phase space.

For example:

(i) If $\langle \psi, f \rangle: <G,X,n> \to <H,Y,a>$ is a morphism in TTG, where $f: X \to Y$ is a topological embedding of $X$ in the Tychonov space $Y$, then boundedness of $<H,Y,a>$ implies boundedness of $<G,X,n>$. We leave the straightforward proof to the reader. Notice that we have applied this statement several times in the preceding proofs with $Y$ compact and $\psi = 1_G$.

(ii) If $\langle \psi, f \rangle: <G,X,n> \to <H,Y,a>$ is a morphism in TTG, $X$ and $Y$ Tychonov spaces, $f$ a surjection and $\psi$ an open mapping, then boundedness of $<G,X,n>$ with respect to a uniformity $U$ implies boundedness of $<H,Y,a>$ with respect to any uniformity $V$ for which $(f \times f)^{-1}(V) \subseteq U$ (i.e. $f$ uniformly continuous). Straightforward.

(iii) If $\langle \psi, f \rangle: <G,X,n> \to <H,Y,a>$ is a morphism in TTG, $X$ and $Y$ Tychonov spaces, $f$ a surjection, then boundedness of $<G,X,n>$ with respect to a uniformity $U$ implies boundedness of $<H,Y,a>$ with respect to any uniformity $V$ in $Y$ which makes $f$ uniformly continuous (no additional conditions on $\langle \psi, f \rangle$). Straightforward.

(iv) Arbitrary products in TTG (cf. 3.1.12 for what they look like) of bounded ttgs are again bounded. Similarly, coproducts in TTG, of bounded ttgs are bounded (cf. 7.3.5); there the proof that the coproduct $<G,X,n>$ of the given set of objects in TTG is bounded makes only use of boundedness of each of those objects). We leave the details to the reader.

The preceding statements show that bounded ttgs behave like ttgs with a compact phase space. A link between the two classes of ttgs is provided by our next proposition. First, recall that a Tychonov space $X$ of weight $\omega(X)$ admits for every compatible uniformity $U$ a topological embedding $f: x \mapsto (f_i(x))_i: X \to [0,1]^I$ such that $|I| = \omega(X)$ with the additional property that each $f_i: X \to [0,1]$ is uniformly continuous with respect to the uniformity $U$ in $X$. Indeed, the usual proof that $X$ can be embedded in $[0,1]^I$ with $|I| = \omega(X)$ (e.g. the proof of [En], Theorem 2.3.8) can easily be modified to a
proof of the previous statement, using the lemma in 0.2.7. Then $f: X \rightarrow [0, 1]^I$ is uniformly continuous, hence the induced topological embedding $F: \xi \mapsto f \circ \xi: C_c(G, X) \rightarrow C_c(G, [0, 1]^I)$ sends equicontinuous subsets of $C_c(G, X)$ into equicontinuous subsets of $C_c(G, [0, 1]^I)$. This will be used in our next result.

7.3.12. THEOREM. Let $<G, X, \pi>$ be a ttg with $G$ an arbitrary topological group and $X$ a Tychonov space. The following conditions are equivalent:

(i) $<G, X, \pi>$ is bounded.

(ii) There exists an equivariant embedding of $X$ into a compact Hausdorff $G$-space with

$$w(\tilde{X}) \leq \max\{w(G), w(X)\}.$$ 

PROOF. $(ii) \Rightarrow (i)$: Apply 7.3.6 and 7.3.11(i).

$(i) \Rightarrow (ii)$: Suppose $<G, X, \pi>$ is bounded with respect to the uniformity $U$ in $X$. If $X$ is finite, take $\tilde{X} = X$ and $\pi = \pi$, and there remains nothing to be proved. So we may assume that $w(X) \geq \kappa_0$. By 7.3.3, $\pi[X]$ is an equicontinuous subset of $C(G, X)$ with respect to the uniformity $U$ in $X$. By the preceding remark, the topological embedding $f \circ \pi$ of $X$ into $C_c([0, 1]^I)$ maps $X$ equivariantly onto an equicontinuous, invariant subset of the $G$-space $C_c([0, 1]^I)$ (with action $\bar{\delta}$); here $|I| = w(X)$. By 7.3.8 there exists a compact Hausdorff $G$-space $\tilde{X}$ in which $X$ can equivariantly be embedded. Recall, that $\tilde{X}$ is the closure of $X$ in $C_c([0, 1]^I)$. Hence the inequality $w(\tilde{X}) \leq w(X) \cdot w(G)$ can be proved as follows. First, notice that $w(C_c(G, [0, 1]^I)) = w(G) \cdot \kappa_0$ (cf. Appendix C). In addition, it is well-known that for any space $Z$ and any infinite cardinal number $\kappa$, $w(2^\kappa) = \kappa \cdot w(Z)$. Combining these results with the fact that $C_c([0, 1]^I)$ is homeomorphic to $C_c(G, [0, 1]^I)$ (cf. for instance 7.1.4), we see that

$$w(C_c([0, 1]^I)) = |I| \cdot w(C_c([0, 1]^I)) = w(X) \cdot w(G)$$

(here we use that $w(X) \cdot \kappa_0 = w(X)$ because of the assumption $w(X) \geq \kappa_0$). Now $\tilde{X}$ is a subspace of $C_c([0, 1]^I)$, so clearly $w(\tilde{X}) \leq w(C_c([0, 1]^I)) = w(X) \cdot w(G) = \max\{w(X), w(G)\}$. This completes the proof. □

7.3.13. PROPOSITION. If in 7.3.12 $G$ is locally compact and sigma-compact and $X$ is a separable metricizable space, then $\tilde{X}$ may also assumed to be separable and metricizable.
PROOF. Repeat the proof of 7.3.12 with \( |I| = N_0 \); so now \( \mathcal{C}_c(G,[0,1]^I) \) is metrizable (cf. Appendix C). Hence \( \tilde{X} \) is metrizable and \( w(\tilde{X}) = w(X) = N_0 \) (use the final remark in 0.2.10).

7.3.14. COROLLARY 1. Let \( G \) be a sigma-compact locally compact Hausdorff topological group and let \( X \) be a separable metrizable space. For any action \( \pi \) of \( G \) on \( X \) the following are equivalent:

(i) \( <G,X,\pi> \) is bounded.

(ii) \( <G,X,\pi> \) is metrically bounded.

PROOF. (i) \( \Rightarrow \) (ii): Apply 7.3.13.

(ii) \( \Rightarrow \) (i): Obvious.

7.3.15. COROLLARY 2. Let \( <G,X,\pi> \) be a ttg with \( G \) a countable discrete group and \( X \) a separable metrizable space. Then \( X \) admits an equivariant dense embedding in a compact metrizable \( G \)-space \( \tilde{X} \).

PROOF. \( <G,X,\pi> \) is bounded because \( G \) is discrete. In addition, \( G \) is sigma-compact. Now apply 7.3.13.

7.3.16. NOTES. The term bounded has been borrowed from D.H. CARLSON [1972]. However, what is called there "bounded" is what we call "metrically bounded". The close connection between boundedness and embeddability in compact \( G \)-spaces seems to be not earlier recognized. In particular, theorem 7.3.12 seems to be new.

Essential in 7.3.15 is the metrizability of the compactification \( \tilde{X} \). Indeed, if \( G \) is discrete, the action \( \pi \) of \( G \) on \( X \) extends in a natural way to an action \( \bar{\pi} \) of \( G \) on \( \beta X \), the Stone-\v{C}ech compactification of \( X \) (cf. \S 2.9).

Then \( <G,\beta X,\bar{\pi}> \) is a ttg in which \( <G,X,\pi> \) can be embedded, but \( \beta X \) is not metrizable (cf. [GJ], 9.6), unless \( X \) itself is already compact and metrizable. Originally, corollary 7.3.15 is due to J. DE GROOT & R.H. MC DOWELL [1960]. Another proof has been given in [Ba], 3.4.11. The case \( G = Z \) is also handled in R.D. ANDERSON [1968].

In R.B. BROOK [1970] one may find a general compactification theorem for ttgs. Roughly speaking, it is our theorem 7.3.12, except that the actions are not only required to be motion equicontinuous (= bounded), but in addition, each transition has to be a unimorphism of the phase space. By our theorem, this latter condition is superfluous.
A linearization of a ttg \( \langle G, X, \pi \rangle \) may roughly be described as an embedding of \( X \) into the phase space \( Y \) of a ttg \( \langle H, Y, \sigma \rangle \), where \( Y \) is a topological linear space and \( \sigma \) an effective action of \( H \) on \( Y \) such that each \( \sigma^t \) is a linear operator on \( Y \). In addition, each \( \pi^t \) has to be the restriction to \( X \) of some \( \sigma^t \). Therefore, a linearization should be a morphism in \( \text{TTG}_* \). See also the motivation for the introduction of comorphisms in 1.4.13. Now such linearizations turn out to exist as soon as \( X \) can topologically be embedded in some topological vector space, i.e. \( X \) is a Tychonov space. Therefore some restrictions on the admitted linearizations are considered. First, the linearization has to be strict, i.e. it should be a morphism in \( \text{TOP}_G \). Second, in constructing a strict linearization \( \langle \Gamma_0, t \rangle: \langle G, X, \pi \rangle \to \langle G, Y, \sigma \rangle \), one should try to meet the following conditions:

(i) The topological vector space \( Y \) should be "nice".

(ii) A large class of other ttgs \( \langle G, X', \pi' \rangle \) can also be strictly linearized in \( \langle G, Y, \sigma \rangle \).

Of course, these conditions are more or less contradictory. As to condition (i), we shall interpret it in the following sense: topologically, \( Y \) should belong to the same distinguished class of spaces as \( X \) does (e.g. if \( X \) is metrizable, then so should be \( Y \); moreover, it would be nice that \( Y \) were a Fréchet space or even a Hilbert space). Condition (ii) obviously relates the problem of linearization to the existence of comprehensive objects, considered in §7.

In subsection 8.1 we shall make some general remarks about linearizations. Then, in subsection 8.2, we consider strict linearizations of metric \( G \)-spaces in Fréchet spaces. Using the main result from subsection 7.2, it follows readily that for every infinite locally compact Hausdorff group \( G \), each action of \( G \) on any Tychonov space \( X \) with \( \omega(X) \leq l(G) \) can be strictly
linearized in \(<G, C_c(G \times G), \tilde{p}^r>\). If \(G\) is sigma-compact, then \(C_c(G \times G)\) is a Fréchet space, and each action of \(G\) on a separable metric space can be strictly linearized in it. Finally, we shall consider strict linearizations in Hilbert spaces. The main result is, that each action of a sigma-compact locally compact Hausdorff group on a metric space \(X\) can be strictly linearized in the linear ttg \(<G, \mathcal{H}(\kappa), o(\kappa)\>\) (cf. subsection 2.4 for its definition), with \(\kappa = \omega(X)\). In the notes to this subsection we mention some earlier results, which motivated our investigations.

8.1. General remarks on linearization

8.1.1. The action \(\pi\) in a ttg \(<G, X, \pi>\) is said to be linear, and \(<G, X, \pi>\) is called a linear ttg provided

(i) \(X\) is a topological vector space.\(^1\)
(ii) \(\tilde{p}(G) \subseteq GL(X)\), the group of invertible continuous linear operators on \(X\).
(iii) \(<G, X, \pi>\) is effective.

8.1.2. Obviously, linear ttgs are in a one-to-one correspondence with subgroups of general linear groups of topological vector spaces, endowed with a topology such that it is a topological homeomorphism group. Classical examples are matrix groups, acting on finite dimensional spaces. Other examples can be found in §2. Indeed, if \(G\) is a topological Hausdorff group and \(Y\) is a topological vector space, then \(C_c(G, Y)\) and \(C_u(G, Y)\) are topological vector spaces (note that a topological vector space \(Y\) has a uniformity compatible with its topology, viz. the left (= right) uniformity of the underlying additive group of \(Y\)). Moreover, each \(\tilde{p}^r\) is a linear operator, and \(p^t \neq p^s\) for \(t \neq s\).\(^2\)

Consequently, \(<G, C_c(G, Y), \tilde{p}>\), \(<G, C_u(G, Y), \tilde{p}>\) and \(<G, LUC_u(G, Y), \tilde{p}>\) are linear ttgs for any topological vector space \(Y\) (cf. 2.1.2, 2.2.1 and 2.2.2). Moreover, if \(G\) is locally compact, then \(<G, C_c(G, Y), \tilde{p}>\) is a linear ttg (2.1.3). In that case, we have also the linear ttg \(<G, L_p(G), \tilde{p}>\) for \(1 \leq p < \infty\) (2.3.3) and if \(G\) is, in addition, sigma-compact, the linear ttg \(<G, L_1^p(G), o>\) defined in 2.4.9.

\(^1\) All topological vector spaces are assumed to have a Hausdorff topology.

\(^2\) Immediate from the fact that \(C(G, Y)\) separates the points of \(Y\) (note, that \(\mathbb{R}\) is topologically embedded in \(Y\), and \(G\) is a Tychonov space).
8.1.3. A linearization of a ttg \( <G,X,TI> \) is a comorphism of ttgs, i.e. a morphism \( \psi^G_{OP},f: <G,X,\sigma> \to <H,Y,\sigma> \) in \( TTG \) such that

(i) \( <H,Y,\sigma> \) is a linear ttg;

(ii) \( \psi: H \to G \) is a surjection and \( f: X \to Y \) is a topological embedding.

In that case we say that \( <G,X,TI> \) is linearized in \( <H,Y,\sigma> \), or that \( <G,X,\pi> \) has a linearization, viz. \( \psi, f \). A strict linearization of \( <G,X,\pi> \) is a linearization of the form \( \psi^G_{OP},f: <G,X,\pi> \to <G,Y,\sigma> \), which we may and shall denote in the sequel by \( <1_G,f> \).

8.1.4. PROPOSITION. Let \( <G,X,\pi> \) be a ttg. The following conditions are equivalent:

(i) \( <G,X,\pi> \) has a linearization.

(ii) \( X \) is embeddable in a topological vector space.

(iii) \( X \) is a Tychonov space.

In that case, \( <G,X,\pi> \) has a linearization of the form \( \psi^G_{OP},f: <G,X,\pi> \to <G,Y,\sigma> \). Moreover, if \( G \) is a locally compact Hausdorff group then the equivalent conditions (i), (ii) and (iii) imply that \( <G,X,\pi> \) has a strict linearization.

PROOF. (i) \( \Rightarrow \) (ii): Obvious.

(ii) \( \Rightarrow \) (i): Any topological vector space is a Tychonov space, and subspaces of Tychonov spaces are still Tychonov spaces.

(iii) \( \Rightarrow \) (i): If \( X \) is a Tychonov space, it can be topologically embedded in \( [0,1]^I \) hence in \( \mathbb{R}^I \), where \( I \) is some index set (in fact, we may assume that \( |I| = \omega(X) \), but this is irrelevant here). Let \( E := \mathbb{R}^I \). Then \( E \) is a topological vector space (ordinary product topology and coordinate wise linear operations), and we may assume that \( X \subseteq E \). Then \( G \cdot C_c(G,E),\beta > \) is a linear ttg by 8.1.2, and using 2.1.13 it is easy to see that \( \psi: X \to C_c(G,X) \subseteq C_c(G,E) \) is an equivariant embedding. Thus we obtain the linearization \( \psi^G_{OP}: <G,X,\pi> \to <G \cdot C_c(G,E),\beta> \). If \( G \) is locally compact, then \( \beta: G \cdot C_c(G,E) \to C_c(G,E) \) is continuous, and \( \psi^G_{OP}: <G,X,\pi> \to <G_c(G,E),\beta> \) is a strict linearization of \( <G,X,\pi> \). This proves (i) and the remaining statements in our proposition.

8.1.5. The topological vector space \( C_c(G,E) \) in the preceding proof is independent of the particular choice of the space \( X \), except that the index set \( I \) used in its definition is such that \( X \) can be embedded in \( [0,1]^I \). Thus, the only requirement is that \( \omega(X) \leq |I| \). It follows (cf. also 7.1.4) that we have, in fact, also a result about comprehensive objects. Stated other-
wise, the above proposition meets condition (ii) of the introduction to this section. In this context, one might also ask if there is a linear ttg in which all ttgs from a certain class can be linearized, where the ttgs of that class do not have identical phase groups. See 8.2.12 and 8.2.14 below.

8.1.6. The preceding proposition shows that an action can always be linearized as soon as its phase space can be embedded in a topological vector space E. However, the space $C_c(G,E)$ in the above proof is isomorphic to $C_c(G)^I$ (I as above), hence it is not always metrizable if X is. Stated otherwise, the space $C_c(G,E)$ seems to be too complicated. This is why we shall consider other methods in the following subsection. Incidentally, it should be observed that the above proof of (iii) $\Rightarrow$ (i) is similar to the first part of 7.1.4. In the next subsection, we shall replace this by the results of subsection 7.2. A second motivation for the next subsection is that, by the preceding proposition, linearizations are not very interesting: they do always exist if X is Tychonov. Hence strict linearizations shall deserve our attention.

8.2. Strict linearizations in Fréchet spaces and in Hilbert spaces

8.2.1. In this subsection, G shall always be an infinite locally compact Hausdorff topological group. Recall that a Fréchet space is a locally convex topological vector space which is metrizable in such a way that it becomes a complete metric space.

8.2.2. The space $C_c(G\times G)$ is a complete locally convex topological vector space. Indeed, a local base at 0 in $C_c(G\times G)$ is formed by the collection of all sets $\{ f : |f(s,t)| < \varepsilon \text{ for } (s,t) \in K \}$ with $\varepsilon > 0$ and $K \subseteq G\times G$ compact. These sets are easily seen to be convex. So $C_c(G\times G)$ is locally convex. Completeness follows from [Bo], Chap. X, §1.5, Theorem 1, taking into account that the uniformity of $C_c(G\times G)$ induced by its topological vector space structure coincides with the uniformity of convergence on compact sets (the theorem in [Bo] deals with the latter uniformity).

If G is sigma-compact, then $C_c(G\times G)$ is a Fréchet space. For $G\times G$ is sigma-compact and locally compact, so we can apply results from Appendix C.

\[1\] Unless, of course, $|I| \leq \aleph_0$ and G is sigma-compact; see Appendix C.
8.2.3. **Theorem.** Let \( G \) be an infinite locally compact Hausdorff topological group. Then any tgg \( \langle G, X, \pi \rangle \) with \( X \) a Tychonov space of weight \( w(X) \leq L(G) \), can be strictly linearized in the linear tgg \( \langle G, C_c(GG), \pi \rangle \).

**Proof.** Apply 7.2.18 and observe, that \( C_c(GG, [0, 1]) \) may be replaced there by \( C_c(GG) \). Moreover, \( \langle G, C_c(GG), \pi \rangle \) is plainly a linear tgg (for effectiveness, cf. 7.2.19(i)). \( \square \)

8.2.4. **Corollary.** Any action of an infinite locally compact, sigma-compact Hausdorff group \( G \) on a separable metric space can be strictly linearized in a Fréchet space, viz. in \( C_c(GG) \) with action \( \pi \) of \( G \).

**Proof.** Apply 8.2.2 and 8.2.3. \( \square \)

8.2.5. In the above corollary we have obtained linearization of an important class of \( G \)-spaces in a quite simple Fréchet space. Now we shall consider topological linearization in Hilbert spaces.

Recall from subsection 2.4 that for any sigma-compact locally compact Hausdorff group \( H \) and any weight function \( w_0 \) on \( H \) we have the tgg \( \langle H, L^2(H), \pi \rangle \). Moreover, for any uniformly bounded compact invariant subset \( A \) of \( C_c(H) \) the mapping \( F_A : f \mapsto w_0 f : A \to L^2(H) \) is an equivariant embedding of the \( H \)-space \( A \) (with action \( \pi \)) into the \( H \)-space \( L^2(H) \) (with action \( \pi \)).

We shall apply this with \( H = GG \). However, the action \( \sigma \) of \( GG \) on \( L^2(GG) \) will be replaced by the action \( \tilde{\sigma} := \psi^\psi \) of \( G \) on \( L^2(GG) \), where \( \psi : GG \to G \) is the morphism in \( \text{TOPGRP} \) defined by \( \psi(s) := (s, e) \). The weight function \( w_0 \) on \( GG \) will be defined by \( w_0(s, t) := w(s) w(t) \) for \( s, t \in G \), where \( w \) is a weight function on \( G \) (cf. Appendix B.2). Then \( \tilde{\sigma} \) is defined by

\[
(1) \quad \tilde{\sigma}^\psi f(u, v) = \frac{w(u)}{w(us)} f(us, v)
\]

for \( f \in L^2(GG) \), \( s \in G \) and \( (u, v) \in GG \). Moreover, it is not difficult to see that the above mentioned mapping \( F_A \) is also equivariant if we consider the action \( \tilde{\pi} \) of \( G \) on \( A \subseteq C_c(GG) \) and the action \( \tilde{\pi} \) of \( G \) on \( L^2(GG) \). Indeed, \( \tilde{\pi} = \psi^\psi \), where \( \psi \) is the usual action of \( GG \) on \( C_c(GG) \), and \( \psi \) is as above.

Using these preparatory remarks, we can prove:

8.2.6. **Proposition.** Let \( G \) be an infinite locally compact sigma-compact Hausdorff topological group. Then every tgg \( \langle G, X, \pi \rangle \) with \( X \) a compact metric space can be strictly linearized in the linear tgg \( \langle G, L^2(GG), \tilde{\pi} \rangle \).
PROOF. By 7.2.18, $X$ can be equivariantly embedded in $C_c(G \times G)$ (with action $\mathcal{F}$) as a uniformly bounded subset $(C_c(G \times [0,1]))$ is, indeed, a uniformly bounded subset of $C_c(G \times G)$. Therefore, we may assume that $X$ is a compact, uniformly bounded, invariant subset of the ttg $<G,C_c(G \times G),\mathcal{F}>$, and that $\pi$ is the restriction to $X$ of the action $\mathcal{F}$. So we can apply the preceding remark. \(\square\)

8.2.7. The Hilbert space $L^2(G \times G)$ occurring in 8.2.6 has dimension $\omega(G \times G) = \omega(G)$ (cf. 2.3.15). The same is true for $L^2(G)$. Consequently, there exists an isomorphism of Hilbert spaces (i.e. a linear inner product preserving bijection) between $L^2(G \times G)$ and $L^2(G)$. Via this isomorphism the action $\mathcal{D}$ defined above induces plainly a linear action $\tau$ of $G$ on $L^2(G)$ such that $<G,L^2(G),\tau>$ and $<G,L^2(G),\mathcal{D}>$ are isomorphic as $G$-spaces.\(^1\) So we proved:

If $G$ is a locally compact sigma-compact Hausdorff topological group, then there exists a linear action $\tau$ of $G$ on $L^2(G)$ such that each ttg $<G,X,\pi>$ with $X$ a compact metric space can be strictly linearized in $<G,L^2(G),\tau>$.\(^2\)

By 7.3.13, compactness of $X$ may be replaced by the conditions that $X$ is an arbitrary separable metric space and that $\pi$ is a bounded action of $G$ on $X$. We shall show now that there exists a linear action $\tau'$ of $G$ on $L^2(G)$ such that $<G,L^2(G),\tau'>$ is comprehensive for the class of all ttgs $<G,X,\pi>$ with $X$ a metrizable space of weight $\omega(X) \leq \omega(G)$. (Since $\omega(G) = \omega(L^2(G))$, the condition $\omega(X) \leq \omega(G)$ is obviously necessary for $X$ to be embeddable in $L^2(G)$). Cf. 8.2.13 below.

As a motivation for the proof, recall that the basic step in the proof of 8.2.6 is the application of 7.2.18, and that in the proof of 7.2.18 it is used that a separable metric space can be embedded in $[0,1]^\mathbb{N}_0$ or $\mathbb{R}^\mathbb{N}_0$ by means of a suitable sequence of continuous functions. For metric spaces, however, we can take this sequence subject to certain additional conditions, and this enables a more direct approach. In this approach, the mapping $F$ used in 8.2.5 (hence 8.2.6) is used in the construction from the beginning.

8.2.8. **Lemma.** Let $X$ be a metrizable space of weight $\kappa$. Then there exist a set $I$ with $|I| = \kappa$ and a set $\{f_i : i \in I\} \subseteq (X,[0,1])$ such that for every

\(^1\) Observe that we only noticed the existence of the linear action $\tau$; we cannot describe it more explicitly. For a proof, that two Hilbert spaces of the same dimension are isomorphic, see for instance p.30 in P.R. Halmos, *Introduction to Hilbert Space*, 2nd ed., Chelsea Publishing Company, New York, 1957.
8.2.3. **Theorem.** Let $G$ be an infinite locally compact Hausdorff topological group. Then any $\text{ttg} <G,X,\nu>$ with $X$ a Tychonov space of weight $w(X) \leq L(G)$, can be strictly linearized in the linear $\text{ttg} <G,L^2_c(G\times G),\tilde{r}>$.

**Proof.** Apply 7.2.18 and observe, that $L^2_c(G\times G,[0,1])$ may be replaced there by $C_c(G\times G)$. Moreover, $<G,L^2_c(G\times G),\tilde{r}>$ is plainly a linear $\text{ttg}$ (for effectiveness, cf. 7.2.19(i)). \[ \square \]

8.2.4. **Corollary.** Any action of an infinite locally compact, sigma-compact Hausdorff group $G$ on a separable metric space can be strictly linearized in a Fréchet space, viz. in $C_c(G\times G)$ with action $\bar{r}$ of $G$.

**Proof.** Apply 8.2.2 and 8.2.3. \[ \square \]

8.2.5. In the above corollary we have obtained linearization of an important class of $G$-spaces in a quite simple Fréchet space. Now we shall consider topological linearization in Hilbert spaces.

Recall from subsection 2.4 that for any sigma-compact locally compact Hausdorff group $H$ and any weight function $v_0$ on $H$ we have the $\text{ttg} <H,L^2(H),\sigma>$. Moreover, for any uniformly bounded compact invariant subset $A$ of $C_c(H)$ the mapping $F|_A: r \mapsto v_0 f: A \hookrightarrow L^2(H)$ is an equivariant embedding of the $H$-space $A$ (with action $\tilde{g}$) into the $H$-space $L^2(H)$ (with action $\sigma$).

We shall apply this with $H = G\times G$. However, the action $\sigma$ of $G\times G$ on $L^2(G\times G)$ will be replaced by the action $\tilde{g} := \psi^0$ of $G$ on $L^2(G\times G)$, where $\psi: G \to G\times G$ is the morphism in $\text{TOPGRP}$ defined by $\psi(s) := (s,e)$. The weight function $v_0$ on $G\times G$ will be defined by $v_0(s,t) := w(s)w(t)$ for $s,t \in G$, where $w$ is a weight function on $G$ (cf. Appendix B.2). Then $\tilde{g}$ is defined by

\begin{equation} \tilde{g} f(u,v) = \frac{v(u)}{w(us)} f(us,v) \end{equation}

for $f \in L^2(G\times G)$, $s \in G$ and $(u,v) \in G\times G$. Moreover, it is not difficult to see that the above mentioned mapping $F|_A$ is also equivariant if we consider the action $\tilde{r}$ of $G$ on $A \subseteq C_c(G\times G)$ and the action $\tilde{g}$ of $G$ on $L^2(G\times G)$. Indeed, $\tilde{r} = \tilde{g}^0$, where $\tilde{g}$ is the usual action of $G\times G$ on $C_c(G\times G)$, and $\psi$ is as above.

Using these preparatory remarks, we can prove:

8.2.6. **Proposition.** Let $G$ be an infinite locally compact sigma-compact Hausdorff topological group. Then every $\text{ttg} <G,X,\nu>$ with $X$ a compact metric space can be strictly linearized in the linear $\text{ttg} <G,L^2_c(G\times G),\tilde{r}>$. 

PROOF. By 7.2.18, X can be equivariantly embedded in $C_c(G\times G)$ (with action $\mathbb{F}$) as a uniformly bounded subset $(C_c(G\times [0,1]))$ is, indeed, a uniformly bounded subset of $C_c(G\times G)$. Therefore, we may assume that X is a compact, uniformly bounded, invariant subset of the set $<G, C_c(G\times G), \mathbb{F}>$, and that $\pi$ is the restriction to X of the action $\mathbb{F}$. So we can apply the preceding remark. \( \square \)

8.2.7. The Hilbert space $L^2(G\times G)$ occurring in 8.2.6 has dimension $\omega(G\times G) = \omega(G)$ (cf. 2.3.15). The same is true for $L^2(G)$. Consequently, there exists an isomorphism of Hilbert spaces (i.e., a linear inner product preserving bijection) between $L^2(G\times G)$ and $L^2(G)$. Via this isomorphism the action $\mathcal{O}$ defined above induces plainly a linear action $\tau$ of G on $L^2(G)$ such that $<G, L^2(G), \tau>$ and $<G, L^2(G), 3>$ are isomorphic as G-spaces.\(^1\) So we proved:

If G is a locally compact sigma-compact Hausdorff topological group, then there exists a linear action $\tau$ of G on $L^2(G)$ such that each tgg $<G, X, \pi>$ with X a compact metric space can be strictly linearised in $<G, L^2(G), \tau>$.\( \quad \)

By 7.3.13, compactness of X may be replaced by the conditions that X is an arbitrary separable metric space and that $\pi$ is a bounded action of G on X. We shall show now that there exists a linear action $\tau'$ of G on $L^2(G)$ such that $<G, L^2(G), \tau'>$ is comprehensive for the class of all ttg $<G, X, \pi>$ with X a metrizable space of weight $\omega(X) = \omega(G)$. (Since $\omega(G) = \omega(L^2(G))$, the condition $\omega(X) = \omega(G)$ is obviously necessary for X to be embeddable in $L^2(G)$). Cf. 8.2.13 below.

As a motivation for the proof, recall that the basical step in the proof of 8.2.6 is the application of 7.2.18, and that in the proof of 7.2.18 it is used that a separable metric space can be embedded in $[0,1]^0$ or $\mathbb{R}^0$ by means of a suitable sequence of continuous functions. For metric spaces, however, we can take this sequence subject to certain additional conditions, and this enables a more direct approach. In this approach, the mapping $\mathbb{F}$ used in 8.2.5 (hence 8.2.6) is used in the construction from the beginning.

8.2.8. **Lemma.** Let X be a metrizable space of weight $\kappa$. Then there exist a set I with $|I| = \kappa$ and a set $\{f_i : i \in I\} \subseteq C(X, [0,1])$ such that for every \( \quad \)

\(^1\) Observe that we only noticed the existence of the linear action $\tau$; we cannot describe it more explicitly. For a proof, that two Hilbert spaces of the same dimension are isomorphic, see for instance p.30 in P.R. Halmos, *Introduction to Hilbert Space*, 2nd ed., Chelsea Publishing Company, New York, 1957.
x \in X, f_i(x) \neq 0 \text{ for at most countably many } i \in I, \text{ and } \sum_i f_i(x)^2 \leq 1.

Moreover, by

\[(2) \quad r(x,y) := \left( \sum_i |f_i(x) - f_i(y)|^2 \right)^{\frac{1}{2}}\]

\((x,y \in X)\) a metric \(r\) is defined in \(X\), and \(r\) generates the topology of \(X\).

In this metric, \(X\) has a finite diameter (in fact, \(r(x,y) \leq 2\) for all points \(x,y \in X\)).

PROOF. It is well-known that \(X\) can be topologically embedded in the unit sphere of a Hilbert space \(H\) of dimension \(\kappa\) (cf. the proof of the Nagata-Smirnov metrization theorem as given in [Du], p. 194). Let \(\{\xi_i : i \in I\}\) be an orthonormal base of \(H\). Then we have \(|I| = \kappa\). For \(i \in I\), set \(f_i(x) := (x|\xi_i)\), where \(\langle \cdot \rangle\) denotes the inner product in \(H\). So \(f_i(s)\) is the \(j\)-th Fourier coefficient of \(s\) with respect to the base \(\{\xi_i : i \in I\}\).

Hence by elementary Hilbert space theory, \(f_i(x) \neq 0\) for at most countably many \(i \in I\), \(\sum_i f_i(x)^2 = \|x\|^2 \leq 1\), and the metric which \(X\) inherits from \(H\) is given by \(r(x,y)^2 = \|x-y\|^2 = \sum_i |(x-y|\xi_i)|^2 = \sum_i |(x|\xi_i) - (y|\xi_i)|^2 = \sum_i |f_i(x) - f_i(y)|^2\).

This proves our lemma. \(\square\)

8.2.9. Recall from 2.4.14 and 2.4.16 that, given a locally compact \(\sigma\)-compact Hausdorff topological group \(G\) and a weight function \(\omega\) on it, we can construct a \(\text{ttg} <G,H(\omega),o(\omega)>\) for every cardinal number \(\kappa\). This \(\text{ttg} <G,H(\omega),o(\omega)>\) is obtained as a Hilbert sum of \(\kappa\) copies of \(<G,L^2(G),o>\), with \(o\) defined according to 2.4.5. Therefore, it is a linear \(\text{ttg}\).

8.2.10. THEOREM. Let \(<G,x,\pi>\) be a \(\text{ttg}\) with \(G\) a \(\sigma\)-compact, locally compact Hausdorff group, and \(X\) a metrizable space. Then the action \(\pi\) can be strictly linearized in the linear \(\text{ttg} <G,H(\omega),o(\omega)>\) with \(\omega = \omega(x)\).

PROOF. Fix a set \(\{f_i : i \in I\}\) of continuous functions of \(X\) into \([0,1]\) according to 8.2.8. For each \(i \in I\), we have the equivariant, continuous mapping \(f_i \circ \pi : X \to C_c^*(G,L^2(G)) \subseteq C_c(g)\) (\(X\) with action \(\pi\) and \(C_c(g)\) with action \(\hat{\pi}\)).

Recall that the mapping \(F\) introduced in 2.4.10 is equivariant with respect to the actions \(\hat{\pi}\) on \(C_c(g)\) and \(\sigma\) in \(L^2(G)\). It follows that \(F_i := F \circ f_i \circ \pi : X \to L^2(G)\) is an equivariant mapping of \(G\)-spaces, from \(X\) with action \(\pi\) into \(L^2(G)\) with action \(\sigma\). In this way, we obtain the mapping \(F' : X \mapsto (F_i(x))_i : X \to (L^2(G))^I\).

First, we show that \(F'[X] \subseteq H(\omega)\). To this end, we have to show that
\[ \sum_{i} \| F_i(x) \|_2^2 < \infty \] for every \( x \in X \). Since \( F_i(x)(t) = w(t)f_i(\pi(t,x)) \), we obtain

\[ \sum_{i} \| F_i(x) \|_2^2 = \int_{G} w(t)^2 \sum_{i} f_i(\pi(t,x))^2 \, dt \]

(2)

\[ \leq \int_{G} w(t)^2 \, dt < \infty. \]

(Here the exchange of integration and summation can be justified by the Lebesgue theorem. Moreover, we have used the fact that \( \sum f_i(y)^2 \leq 1 \) for all \( y \in X \).) So we have \( F'[X] \subseteq H(K) \).

Next, \( F' : X \to H(K) \) is continuous. Indeed, similar to the above computation, we have for every \( x, y \in X \), using formula (2):

\[ \| F'(x) - F'(y) \|_2^2 = \sum_{i} \| F_i'(x) - F_i'(y) \|_2^2 \]

(3)

\[ = \int_{G} w(t)^2 \sum_{i} \| f_i'(\pi(t,x)) - f_i'(\pi(t,y)) \|_2^2 \, dt \]

\[ = \int_{G} w(t)^2 r(\pi^t x, \pi^t y)^2 \, dt. \]

Let \( \varepsilon > 0 \). In view of formula (10) in 2.4.10, there exists a compact subset \( K \) of \( G \) such that

\[ \int_{G-K} w(t)^2 \, dt < \varepsilon^2. \]

Fix \( x \in X \). Then 0.2.2(i) implies that \( r(\pi^t x, \pi^t y) < \varepsilon^2 \) for all \( y \) in some neighbourhood \( U \) of \( x \) and for all \( t \in K \). Bearing in mind that \( X \) has diameter \( \leq 2 \), we see that

\[ \| F'(x) - F'(y) \|_2^2 \leq 4 \int_{G-K} w(t)^2 \, dt + \varepsilon^2 \int_{K} w(t)^2 \, dt \]

\[ \leq 4\varepsilon^2 + \varepsilon^2 \int_{G} w(t)^2 \, dt = (4\| w \|_2^2)\varepsilon^2. \]
for all \( y \in U \). This proves continuity of \( F' \).

In order to show that \( F' \) is relatively open and injective, it is sufficient to show that for any \( x \in X \) and \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\forall y \in X : \|F'(x) - F'(y)\| < \delta \Rightarrow r(x,y) \leq \varepsilon.
\]

Suppose the contrary. Then for some \( x \in X \) and \( \varepsilon > 0 \) there exists a sequence \( \{y_n : n \in \mathbb{N}\} \) in \( X \) such that \( \|F'(x) - F'(y_n)\| < 1/n \) and \( r(x,y_n) > \varepsilon \) for all \( n \). Then (3) implies for every \( n \in \mathbb{N} \)

\[
\int_G w(t)^2 r(p^tx,p^ty_n)^2 \, dt = \|F'(x) - F'(y_n)\|^2 \leq n^{-2}.
\]

If we set \( f(t) := \inf \{r(p^tx,p^ty_n)^2 : n \in \mathbb{N}\} \), then obviously

\[
0 \leq \int_G w(t)^2 f(t) \, dt \leq \inf_{n \in \mathbb{N}} n^{-2} = 0.
\]

Hence \( f(t) = 0 \) for almost all \( t \in G \) (recall that \( w(t) > 0 \) for all \( t \in G \)). However, \( f(t) = 0 \) for some \( t \in G \) implies that there is a subsequence \( \{n_i\} \) of \( \mathbb{N} \) such that \( \lim_{i} p^tx_{n_i} = x \), whence \( \lim_{i} y_{n_i} = x \). This contradicts the choice of the points \( y_n \) subject to the condition \( r(x,y_n) > \varepsilon \). Consequently, for any \( x \in X \) and \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that (4) holds.

Thus, \( F' : X \to H(K) \) is a topological embedding. Notice that \( F' \) is equivariant with respect to the given action \( \pi \) on \( X \) and the action \( \sigma(k) \) on \( H(k) \), because each \( F_i : X \to L^2(G) \) is equivariant with respect to \( \pi \) and \( G \).

8.2.11. COROLLARY 1. Let \( G \) be a sigma-compact locally compact Hausdorff topological group, and let \( \kappa \) denote any cardinal number. Then the linear ttg \( <G,H(K),\sigma(k)> \) is comprehensive for the class of all ttgs \( <G,X,\pi> \) with \( X \) a metrizable space of weight \( \leq \kappa \).

8.2.12. COROLLARY 2. Let \( G \) and \( \kappa \) be as above, and let \( <G',X,\pi> \) be any ttg satisfying the conditions

(i) there exists a surjective morphism \( \psi : G \to G' \) in \( \text{TOPGRP} \);
(ii) \( X \) is a metric space of weight \( \leq \kappa \).

Then \( <G',X,\pi> \) can be linearized in \( <G,H(k),\sigma(k)> \).
PROOF. If \(<G',X,n>\) satisfies the conditions (i) and (ii), then \(<G,X,n>\)
can be equivariantly embedded in the ttg \(<G,H(\kappa),\sigma(\kappa)>\), i.e. there exists
an equivariant topological embedding \(f: X \rightarrow H(\kappa)\). It is easy to see that
\(<\psi^G,f>: <G',X,n> \rightarrow <G,H(\kappa),\sigma(\kappa)>\) is a morphism in \(TTG_\kappa\). Since \(\psi\) is given
to be a surjection, \(<\psi^G,f>\) is the desired linearization. \(\square\)

8.2.13. COROLLARY 3. Let \(G\) be an infinite \(\sigma\)-compact locally compact
Hausdorff topological group. Then there exists a linear action \(\sigma^*\) of \(G\) on
\(L^2(G)\) such that \(<G,L^2(G),\sigma^*>\) is comprehensive in \(TOP^G\) with respect to the
closest to all tgg \(<G,X,n>\) with \(X\) a metrizable space of weight \(\leq w(G)\).

PROOF. Since \(G\) is infinite, \(w(G) \geq \aleph_0\). It follows from 2.3.15 that \(L^2(G)\)
has Hilbert dimension \(\omega(G)\). Consequently, for any cardinal number \(\kappa\), \(H(\kappa)\)
has Hilbert dimension \(\kappa \cdot \omega(G) = \max(\kappa, \omega(G))\). In particular, if \(\kappa = w(G),\)
\(H(\kappa)\) and \(L^2(G)\) have the same Hilbert dimension. In that case, there exists
an isomorphism of Hilbert spaces \(\gamma: H(\kappa) \rightarrow L^2(G)\). Let \(\sigma^*\) be the unique
action of \(G\) on \(L^2(G)\) such that \(g, L^2(G), \sigma^*\) is equivalent to \(\kappa\) with respect to the action
\(\sigma(\kappa)\) of \(G\) on \(H(\kappa)\) and \(\sigma^*\) of \(G\) on \(L^2(G)\). Thus, \(\sigma^* t = \gamma \circ \sigma(\kappa) \circ \gamma^t\)
for all \(t \in G\). Then obviously \(\sigma^*\) is a linear action, and \(<G,L^2(G),\sigma^*>\) is isomorphic
to \(<G,H(\kappa),\sigma(\kappa)>\), where \(\kappa = w(G)\). Now apply 6.2.11 with \(\kappa = w(G)\). \(\square\)

8.2.14. COROLLARY 4. Let \(G\) be an infinite \(\sigma\)-compact locally compact
Hausdorff topological group, and let \(<G,L^2(G),\sigma^*>\) be as in 8.2.13.
Then any tgg \(<G',X,n>\) satisfying the conditions
(i) there exists a surjective morphism \(\psi: G \rightarrow G'\) in \(TOP^{G'}\),
(ii) \(X\) is a metric space of weight \(\leq w(G)\),
\(\sigma^*\) can be linearized in \(<G,L^2(G),\sigma^*>\).

PROOF. Similar to 8.2.12.

8.2.15. If \(G\) is compact, then we may take as a weight function on \(G\) the
function \(w: t \mapsto 1: G \rightarrow \mathbb{R}\). In that case, we have \(\sigma^t f(s) = f(st)\), for \(f \in L^2(G)\)
and \(s,t \in G\), so that \(\sigma = \delta\) on \(L^2(G)\). So for any cardinal number \(\kappa\),
the action \(\sigma(\kappa)\) of \(G\) on \(H(\kappa)\) is by means of unitary operators (orthogonal
operators, if we consider only \(\mathbb{R}\)-valued functions as elements of \(L^2(G)\),
in which case \(H(\kappa)\) is a real Hilbert space).

A close examination of the proof of 8.2.13 shows that in the case that
\(G\) is an infinite compact Hausdorff topological group we may assume that
the action \(\sigma^*\) of \(G\) on \(L^2(G)\) with the properties mentioned in 8.2.13 is by
means of unitary operators as well.
8.2.16. If $G$ is discrete and sigma-compact, then $G$ is countable. In that case, $L^2(G)$, is isomorphic to the (separable) Hilbert space $l^2(N_0)$ (cf. [Dua], p. 191 for the notation). In particular, if $G = \mathbb{Z}$, then for any cardinal number $\kappa$, $H(\kappa)$ is isomorphic to the Hilbert sum of $\kappa$ copies of the space $l^2(N_0)$. A weight function on $\mathbb{Z}$ is defined by $w(n) := 2^{-|n|}$ for $n \in \mathbb{Z}$. So the action $\sigma$ of $\mathbb{Z}$ on $l^2(N_0) = L^2(\mathbb{Z})$ is the action generated by the homeomorphism $\sigma^1$, where

$$
(\sigma^1)_n = \begin{cases} 
2^n x_n + 1 & \text{if } n \geq 0 \\
\frac{1}{2} x_n + 1 & \text{if } n \leq -1 
\end{cases}
$$

for $x = (x_n)_{n \in \mathbb{Z}} \in L^2(\mathbb{Z})$. Now let $I$ be a set with $|I| = \kappa$. Plainly, an element $((x_n)_{n \in \mathbb{Z}})_{i \in I}$ in $H(\kappa)$, may be identified with the element $((x_n)_{i \in I})_{n \in \mathbb{Z}}$ in the Hilbert sum of $|I|$ copies of the Hilbert space $l^2(\kappa)$ of dimension $\kappa$. In this way we obtain an isomorphism of the Hilbert space $H(\kappa)$ onto the Hilbert sum $K$ of $|I|$ copies of $l^2(\kappa)$. Under this isomorphism, the action $\sigma(\kappa)$ of $\mathbb{Z}$ on $H(\kappa)$ carries over to an action of $\mathbb{Z}$ on $K$ which may also be described by (5), now interpreting the $x_i$ as elements of $l^2(\kappa)$ for $n \in \mathbb{Z}$. (This action was described for the first time in J.H. COPELAND & J. DE GROOT [1961]).

8.2.17. NOTES. The question whether certain ttgs can be embedded in a ttg whose phase space is a topological vector space and whose action is by means of a linear representation of $G$ is almost as old as the theory of ttgs itself. In connection with the existence of comprehensive objects for certain classes of ttgs, one of the most notable early results is BEBUTOV's theorem (the literature gives conflicting references to the original; see for instance V.V. NEMYCII [1949]). In our terminology, it reads as follows:

The ttg $<\mathbb{R}, \mathcal{L}, \beta>$ is comprehensive for the class of all ttgs $<\mathbb{R}, X, \pi>$ with $X$ a compact metric space and with an action $\pi$ such that $X$ contains at most one invariant point.

In S. KAKUTANI [1968] this theorem has been strengthened in the sense that the condition that $<\mathbb{R}, X, \pi>$ has at most one critical point may be replaced by the condition that the set of invariant points in $X$ is homeomorphic with a subset of $\mathbb{R}$. In O. HAJEK [1971] a further modification was presented:

The ttg $<\mathbb{R}, \mathcal{L}, \beta>$ is comprehensive for the class of all ttgs $<\mathbb{R}, X, \pi>$ with $X$ a locally compact separable metric space, such that the set
of invariant points in $X$ is homeomorphic to a closed subset of $\mathbb{R}^n$.

It should be noticed that in these theorems certain restrictions are imposed on the actions in order to describe the class of tgs for which the above mentioned objects are comprehensive. However, D.H. CARLSON [1972] described a linear tg $<\mathbb{R}, C_u(\mathbb{R}^2), \tau>$ which is comprehensive for the class of all tgs $<\mathbb{R}, X, \pi>$ with $X$ a separable metrizable space. The action $\tau$ of $\mathbb{R}$ on $C_u(\mathbb{R}^2)$ in this comprehensive object are "weighted" translations; in fact,

$$\tau_t f(u,v) = e^{(u+v)t + t^2} f(u+t,v+t).$$

In this context, our theorem 7.2.18 (\textsuperscript{1} proposition 8.2.6) is on the one hand a generalization of the results of BERNOU, KAKUTANI and HAJEK, and on the other hand it is a simplification and generalization of the result of CARLSON.\textsuperscript{1}

Euclidean spaces and Hilbert spaces appear for instance in work of L. ZIPPIN, D. MONTGOMERY, R.H. BING and others. Most of these results are special cases of results of G.B. MOSTOV [1957]. We quote one of MOSTOV's theorems:

If $G$ is a compact Lie group and $X$ is a separable metrizable $G$-space of finite dimension and with a finite number of orbit types, then any action of $G$ on $X$ can be strictly linearized in a Euclidean $G$-space where the action is by means of orthogonal linear transformations.

For a nice proof, cf. R.S. PALAIS [1960]. In R.S. PALAIS [1961], these results are generalized to non-compact Lie groups: if $G$ is a matrix group and $X$ a separable finite dimensional metrizable $G$-space with a proper action, having only finitely many orbit types, then $X$ admits an equivariant embedding in a linear $G$-space of finite dimension. In the same paper, PALAIS shows that if $G$ is any Lie group and $X$ is a separable metrizable $G$-space with a proper action, then $X$ admits an equivariant embedding in a real Hilbert $G$-space where the action is by means of orthogonal linear transformations.\textsuperscript{2}

\textsuperscript{1} To be honest, although the action $\tau$ of $\mathbb{R}$ on $C_u(\mathbb{R}^2)$ in the CARLSON system is not as simple as the action $\tilde{F}$, his system is related to the solution space of a first order partial differential equation.

\textsuperscript{2} The paper of PALAIS does not contain statements about comprehensive objects, nor seem such statements to be obtainable from it.
Meanwhile, theorems on linearization in Hilbert spaces were also obtained by J.H. COPELAND & J. DE GROOT [1961] for cyclic groups and in J. DE GROOT [1962] for compact groups and discrete countable groups. These results were extended to more general locally compact groups by P.C. BAAYEN. Cf. Chap. 4 in [Ba], and also P.C. BAAYEN & J. DE GROOT [1968]. These "more general locally compact groups" were described as locally compact Hausdorff groups admitting weight functions ("W-groups"). However, they did not incorporate 2.4.2(ii) in the definition of a weight function, and consequently, they obtained only linearizations, no strict linearizations. Their main result was that for any such a W-group $G$ and any cardinal number $\kappa$ there exists a Hilbert space $H$ such that every ttg $<G,X,\pi>$ with $X$ a metrizable space of weight $\leq \kappa$ admits a morphism $<\psi,f>:<G,W,X,\pi>\rightarrow<GL(H)_d,H,\delta>$ in TTG with $\psi$ injective and $f$ a topological embedding. Here $\delta$ is the obvious action of $GL(H)_d$ on $H$. (Our methods for e.g. the proof of 8.2.10 are similar to those of BAAYEN). In a subsequent note (P.C. BAAYEN [1967]) it was shown that in the above mentioned theorem, $\psi:G\rightarrow GL(H)$ is a topological embedding if $GL(H)$ is given its strong operator topology, provided $G$ admits a continuous weight function. In that case, however, it was not yet clear that a strict linearization in the sense of 8.1.3 had been obtained. Indeed, it was not yet shown that in this case the subgroup $\psi[G]$ of $GL(H)$ with the strong operator topology is a topological homeomorphism group on $H$ (this is our corollary 2.4.16). The results in the present section became possible by the paper of A.B. PAALMAN - DE MIRANDA [1971], who proved that the locally compact Hausdorff groups admitting weight functions are exactly the sigma-compact ones. Some of our results in this section have been published earlier in J. DE VRIES [1972a; 1975a].

Finally, it should be noticed, that [Ba] contains many results on comprehensive objects in TTG; however, most classes of ttgs considered there have discrete phase groups, this in contradistinction with our results in subsection 7.3 and in §8. More information about the history of this subject can be found in the paper P.C. BAAYEN & M.A. MAURICE, Johannes de Groot 1914-1972, General Topology and Appl. 3 (1973), 3-32. Cf. also section 6 in "The topological works of J. DE GROOT", a lecture by P.C. BAAYEN, contained in Topological Structures (Proceedings of a Symposium, organized by the Wiskundig Genootschap of the Netherlands on November 7, 1973, in honour of J. de Groot (1914-1972)), Mathematical Centre Tracts 52, Mathematicisch Centrum, Amsterdam, 1974.
Pseudocompactness for topological groups

A.1. In this appendix, \( G \) shall always denote a topological Hausdorff group. Recall that \( G \) is totally bounded whenever for every \( U \in \mathcal{V} \), \( G \) can be covered by finitely many left translates of \( U \). This means that \( G \) is precompact in its left uniformity. If \( G \) is totally bounded, then the left and right uniformities on \( G \) coincide. The following is well-known:

A.2. **Lemma.** The following statements are equivalent:

1. \( \alpha_G : G \to \mathcal{C} \) is a topological embedding.
2. \( G \) is totally bounded.
3. \( G \) is a subgroup of a compact Hausdorff group \( H \).

In this case, \( \alpha_G : G \to \mathcal{C} \) may be identified with the inclusion mapping of \( G \) into \( \text{cl}_H G \) for any compact Hausdorff group \( H \) in which \( G \) is topologically embedded as a subgroup. In addition, \( \text{AP}(G) = \text{LUC}^*(G) \).

**Proof.** That (i) \( \iff \) (ii) is trivial. In order to prove that (ii) \( \implies \) (iii), consider the completion \( H \) of \( G \) with respect to its left uniformity, and apply [Bo], Chap. IV, §3.4. Next, assume (iii). Then every continuous morphism of groups from \( G \) into a compact Hausdorff group \( K \), being a uniformly continuous function into a complete uniform Hausdorff space, can be extended to \( \text{cl}_H G \). This extension is obviously a morphism of groups. So \( \text{cl}_H G \) may be identified with the reflection of \( G \) in \( \text{COMPGRP} \), i.e. the Bohr compactification of \( G \). Similarly, each \( f \in \text{LUC}^*(G) \) can be extended to a continuous function \( f' \) on \( \text{cl}_H G \). Since \( \text{cl}_H G \) is a compact group, it follows that \( f \in \text{AP}(G) \), by 2.2.7. Thus, \( \text{LUC}^*(G) = \text{AP}(G) \) (cf. 2.2.16). This shows that (iii) \( \implies \) (i) and that the final statement is true. \( \square \)

A.3. Since \( G \) is a Tychonov space, the reflection \( \beta_G : G \to \mathcal{B} \) of \( G \) in \( \text{COMP} \) is a topological embedding. Obviously, there is a unique continuous mapping \( \tilde{\alpha} : \mathcal{B} \to \mathcal{C} \) such that \( \alpha_G = \tilde{\alpha} \circ \beta_G \). The following lemma describes groups \( G \) for
which \( \tilde{\alpha} \) is a homeomorphism.

A.4. **LEMA.** The following conditions are equivalent:

(i) \( \beta G \) can be given the structure of a group in such a way that it becomes a topological group and \( \beta_G : G \to \beta G \) a morphism of groups.

(ii) \( \text{AP}(G) = \mathcal{C}^*(G) \).

(iii) There exists a homeomorphism \( \tilde{\alpha} : \beta G \to G^c \) such that \( \alpha_G = \tilde{\alpha} \circ \beta_G \).

(iv) \( G \) is totally bounded and \( \text{LUC}^*(G) = \mathcal{C}^*(G) \).

**PROOF.**

(i) \( \Rightarrow \) (ii): Since \( \mathcal{C}^*(\beta_G) : f \mapsto f \circ \beta_G \) maps \( \mathcal{C}(\beta G) \) onto \( \mathcal{C}^*(G) \), this is an immediate application of lemma 2.2.7.

(ii) \( \Rightarrow \) (iii): If (ii) is valid, then \( \mathcal{C}^*(\alpha_G) : f \mapsto f \circ \alpha_G \) maps \( \mathcal{C}(G^c) \) onto \( \mathcal{C}^*(G) \), by 2.2.18. Now use the fact that \( \beta_G \) is uniquely determined by the property that \( \mathcal{C}^*(\beta_G) \) is a surjection (cf. [GJ], 6.5).

(iii) \( \Rightarrow \) (i): Obvious.

(i) \( \Leftrightarrow \) (iv): Clear from A.2 and the equivalence of (i) and (ii). \( \square \)

A.5. Recall that \( G \) is pseudocompact whenever \( \mathcal{C}(G) = \mathcal{C}^*(G) \). It is well-known that \( G \) is pseudocompact iff the following condition is fulfilled (cf. [GJ], 6.11):

(i) Any non-void closed \( G_0 \)-set in \( \beta G \) meets \( G \).

Notice, that this characterization is valid for any Tychonov space. For topological groups \( G \), one can prove that \( G \) is pseudocompact iff

(ii) \( G \) is a dense subgroup of a compact group \( H \) and every non-void \( G_0 \)-set in \( H \) meets \( G \).

This result is due to W.W. COMFORT & K.A. ROSS [1966]. For an elementary proof, cf. J. DE VRIES [1975b]. Using this characterization it is easy to prove the following theorem (which is also contained in the above mentioned paper).

A.6. **THEOREM.** An arbitrary product of pseudocompact Hausdorff groups is again pseudocompact. \( \square \)

A.7. **THEOREM.** \( G \) is pseudocompact iff one of the conditions in A.4 is fulfilled.

**PROOF.** Cf. W.W. COMFORT & K.A. ROSS [1966]. \( \square \)

A.8. Another question, which was considered in the paper of COMFORT and ROSS was, under which conditions on \( G \) one has \( \text{LUC}^*(G) = \mathcal{C}^*(G) \) (i.e. condition A.4(iv) without total boundedness). It turned out that the condition
LUC$^+(G) = C^*(G)$ is equivalent to the condition LUC$(G) = C(G)$. If this condition is fulfilled, then either $G$ is pseudocompact, or $G$ is a P-space (i.e. each $G_δ$-set in $G$ is open). (For related results, cf. O.T. ALAS [1971], and also a forthcoming paper by W.W. COMFORT & A.W. HAGER.)

A.9. There exists an abundance of non-compact pseudocompact groups. Cf. the above mentioned paper by COMFORT and ROSS. See also H.J. WILCOX [1966; 1971]. For additional facts about pseudocompact groups, cf. W. MORAN [1970]: barring the existence of measurable cardinals, all groups which admit invariant means on $C(G)$ are pseudocompact (the converse is almost trivial).
APPENDIX B

Weight functions on sigma-compact locally compact Hausdorff groups

B.1. Throughout this appendix, let $G$ denote a locally compact Hausdorff topological group. In addition, from B.4 on up to the end, $G$ will be assumed to be sigma-compact, i.e. $G = \bigcup \{ C_n : n \in \mathbb{N} \}$, where each $C_n$ is a compact subset of $G$. Recall that a weight function on $G$ is an element $w \in L^2(G)$ such that

(i) $\forall t \in G: w(t) > 0$.
(ii) $\forall s, t \in G: w(st) \leq w(s)w(t)$.
(iii) The function $t \mapsto w(t)^{-1}: G \to \mathbb{R}$ is bounded on compact subsets of $G$.

B.2. EXAMPLES. The following examples are taken from P.C. BAAYEN & J. DE GROOT [1968]; cf. also [Ba], section 4.2, where all proofs can be found.

(i) If $G$ is compact, then the constant function $t \mapsto 1: G \to \mathbb{R}$ is a weight function on $G$.
(ii) The function $t \mapsto \exp(-|t|): \mathbb{R} \to \mathbb{R}$ is a weight function on the additive group $\mathbb{R}$.
(iii) The function $t \mapsto 2^{-|t|}: \mathbb{Z} \to \mathbb{R}$ is a weight function on the group $\mathbb{Z}$.

More generally, let $G$ denote the free group generated by the countable set $\{ t_1, t_2, \ldots \}$. Every $t \in G$, $t \neq e$ can be written uniquely as a reduced word

$$ t = t_{n_1}^{k_1} t_{n_2}^{k_2} \cdots t_{n_m}^{k_m} $$

then we put

$$ w(t) := 2^{-\sum_{i=1}^{m} |k_i| n_i}. $$

If, in addition, we define $w(e) := 1$, then $f$ is a weight function on the group $G$. 

(iv) If \( G = G_1 \times G_2 \times \cdots \times G_n \) where each \( G_i \) admits a weight function \( w_i \), then
\[
w: (t_1, t_2, \ldots, t_n) \mapsto w_1(t_1)w_2(t_2)\cdots w_n(t_n): G_1 \times G_2 \times \cdots \times G_n \to \mathbb{R}
\]
is a weight function on \( G_1 \times G_2 \times \cdots \times G_n \).

B.3. **THEOREM.** The following statements are equivalent:

(i) \( G \) is sigma-compact.

(ii) \( G \) admits a weight function.

**PROOF.** This theorem and its proof are due to A.B. PAALMAN-DE MIRANDA [1971]. We shall confine ourselves here to the following remarks.

The proof of (ii) \( \Rightarrow \) (i) is based on the observation that in a locally compact group the closure of a sigma-compact subset is again sigma-compact, and on the well-known property that for any \( f \in L^1(G) \), \( f \geq 0 \) (i.e. \( f = w^2 \), if \( w \) is the weight function on \( G \)), there exists an \( f' \in L^1(G) \) such that
\[
0 \leq f' \leq f \text{ on } G, \int_G f'(t)dt = \int_G f(t)dt, \text{ and the set } \{ t : t \in G \text{ and } f'(t) > 0 \}
\]
is sigma-compact.

The proof of (i) \( \Rightarrow \) (ii) is much more complicated. Actually, in A.B. PAALMAN-DE MIRANDA [1971] the existence of an element \( f \in L^2(G) \) has been proved, satisfying conditions (i) and (ii) of B.1. However, it follows immediately from the construction that there exist \( V \in \mathcal{C} \) and \( n_1 \in \mathbb{N} \) such that \( f(t) \geq n_1^{-1} \) for all \( t \in V \). Now any compact subset \( K \) of \( G \) can be covered by finitely many left translates of \( V \). Hence the fact that \( \sup \{ f(t)^{-1} : t \in K \} < \infty \) follows from the observation that for each \( s \in G \)
\[
\sup \{ f(t)^{-1} : t \in sV \} = \sup \{ f(st)^{-1} : t \in V \} \leq \sup \{ f(s)^{-1}f(t)^{-1} : t \in V \} \leq n_1f(s)^{-1}. \quad \Box
\]

B.4. From now on we shall assume that \( G \) is a sigma-compact, locally compact Hausdorff group. Then \( G \) admits a weight function \( w \). The question may be raised, if \( G \) admits a continuous weight function. The construction in A.B. PAALMAN-DE MIRANDA [1971] does not necessarily produce a continuous weight function: if we apply that construction to \( G = \mathbb{R} \), then we obtain the function \( w: \mathbb{R} \to \mathbb{R} \) defined as follows:
\[
w(t) = \begin{cases} 
1 & \text{if } t = 0, \\
3^{-k} & \text{if } k-1 < |t| < k \ (k=1,2,\ldots). 
\end{cases}
\]

On the other hand, \( \mathbb{R} \) admits a continuous weight function (cf. B.2(ii)). More generally, it follows from the proof of Theorem 3.3 in P.C. BAAYEN &
J. DE GROOT [1968], that every locally compact abelian Hausdorff group which is either separable or compactly generated admits a continuous weight function.

We have not been able to improve on this result. In the following sequence of lemmas we provide some material showing how "nice" a weight function can always be chosen.

B.5. Lemma. There exists a weight function \( w \) on \( G \) satisfying the following additional conditions:

(i) \( \forall t \in G : w(t) \leq 1 \).

(ii) \( \forall t \in G : w(t) = w(t^{-1}) \).

Proof. (i): In fact, we show that (iv) is always implied by the requirements that \( w \in L^2(G) \) satisfies (i) and (ii) of B.1. If \( w \) is a weight function on \( G \), then \( \|w\|_2 > 0 \) because of B.1(i). Then by B.2(ii) and right invariance of Haar measure, we obtain

\[
\|w\|_2^2 = \int_G w(st)^2 ds \geq \int_G w(t)^2 ds = \|w\|_2^2 \int_G w(t)^2 ds.
\]

Consequently, \( w(t) \leq 1 \) for all \( t \in G \).

(v): Let \( w_0 \) be any weight function on \( G \), set \( w(t) := w_0(t)w_0(t^{-1}) \) for all \( t \in G \). Then \( w \) is obviously measurable. Since by (iv), \( w_0(t^{-1}) \leq 1 \) for all \( t \in G \), it follows that \( 0 \leq w(t) \leq w_0(t) \), so that \( w \in L^2(G) \). Now conditions (i), (ii) and (iii) of B.1 are easily verified for \( w \). □

B.6. Lemma. There exist a lower semicontinuous weight function \( \omega' \) on \( G \) and an upper semicontinuous weight function \( \omega'' \) such that \( \omega'(t) \leq \omega''(t) \) for all \( t \in G \).

Proof. Let \( w \) be a weight function on \( G \). Define \( \omega' \) and \( \omega'' \) by

\[
\omega'(t) := \lim \inf_{s \to t} \omega(s); \quad \omega''(t) := \lim \sup_{s \to t} \omega'(s)
\]

for each \( t \in G \). Then the following inequalities are valid:

\[
(1) \quad \omega'(e) \leq \omega'(t) \leq \omega(t); \quad \omega'(t) \leq \omega''(t) \leq \omega(t).
\]

Indeed, since \( V_t = \{ Ut : U \in \omega \} \), we have for each \( t \):
\[
\begin{align*}
   w'(t) &= \sup_{U \in \mathcal{V}_t} \left( \inf_{s \in U} w(s) \right) = \sup_{U \in \mathcal{V}_e} \left( \inf_{s \in U} w(st) \right) \\
   &= \sup_{U \in \mathcal{V}_e} \left( \inf_{s \in U} w(s)w(t) \right) = w'(e)w(t).
\end{align*}
\]

On the other hand, it is trivial that for each \( U \in \mathcal{V}_t \), \( \inf_{s \in U} w(s) \leq w(t) \), whence \( w'(t) \leq w(t) \). This proves half of (1); the remaining part of (1) is trivial.

It is routine to check that \( w' \) is lower semicontinuous and that \( w'' \) is upper semicontinuous. In particular, \( w' \) and \( w'' \) are measurable, so that the inequalities \( 0 \leq w'(t) \leq w''(t) \leq w(t) \) \((t \in G)\) imply that \( w', w'' \in L^2(G) \).

Next, observe that \( w'(e) > 0 \), because, by condition B.1(iii), the function \( w \) is bounded away from zero in a (compact) neighbourhood of \( e \) in \( G \). Hence (1) implies that \( w' \) and \( w'' \) satisfy condition B.1(i). That they satisfy condition B.1(ii) follows from a straightforward computation, and B.1(iii) for \( w' \) and \( w'' \) is, again, an easy consequence of (1).

B.7. As was remarked in the proof of B.5(iv), the functions \( w' \) and \( w'' \) satisfy \( w'(t) \leq w''(t) \leq 1 \) for each \( t \in G \). If \( w \) in the proof of B.6 has property B.6(v), then so do \( w' \) and \( w'' \).

The process of "regularization" described in the proof of B.6 does in general not produce a continuous weight function. Indeed, if we take \( w : \mathbb{R} \to \mathbb{R} \) as in B.4, then

\[
   w'(t) = \begin{cases} 
   3^{-1} & \text{if } t = 0, \\
   3^{-k} & \text{if } k-1 \leq |t| < k \ (k=1,2,\ldots) 
\end{cases}
\]

and

\[
   w''(t) = \begin{cases} 
   3^{-1} & \text{if } t = 0, \\
   3^{-k} & \text{if } k-1 \leq |t| \leq k \ (k=1,2,\ldots) .
\end{cases}
\]

So we gained only one point of continuity, namely, the point \( t = 0 \).

B.8. **Lemma.** Let \( w \) be a lower semicontinuous weight function on \( G \). Then for every \( \varepsilon > 0 \) there exists \( U \in \mathcal{V}_e \) such that

\[
   (2) \quad (1-\varepsilon)w(e)w(t) \leq w(s) \leq \frac{1}{1-\varepsilon} \frac{w(t)}{w(e)}
\]

for all \( t \in G \) and \( s \in Ut \).
PROOF. Let $\varepsilon > 0$. By lower semicontinuity of $w$ there exists $U \subseteq V$ such that $w(u) \geq (1-\varepsilon)w(e)$ for all $u \in U$. Thus, if $s \in Ut$ for some $t \in G$, say $s = ut$ with $u \in U$, then $w(s) \geq w(u)w(t) > (1-\varepsilon)w(e)w(t)$.

In addition, we may and shall assume that $U$ is symmetric, i.e. $U = U^{-1}$. Then for $s \in Ut$ we have $ts^{-1} \in U$, hence $w(t) = w(ts^{-1}s) \geq w(ts^{-1})w(s) \geq (1-\varepsilon)w(e)w(s)$.

B.9. COROLLARY. A lower semicontinuous weight function $w$ on $G$ such that $w(e) = 1$ is right uniformly continuous.

PROOF. Since $w(t) \leq 1$ for all $t \in G$, there exists for each $\delta > 0$ a real number $\varepsilon > 0$ such that $w(t) - \delta < (1-\varepsilon)w(t) < (1-\varepsilon)^{-1}w(t) < w(t) + \delta$. Now apply B.8.

B.10. In 2.4.10 we gave another proof of the existence of an upper semicontinuous weight function on $G$. We do not know whether the weight function constructed there is actually continuous. For this, it would be sufficient to show the continuity of the norm, i.e. the mapping $t \mapsto \|t\|$, on the image of $G$ in $GL(L^2(G))$. (Observe, that for a continuous weight function $w$ with $w(e) = 1$ we have $\|e\|^w = w(t)^{-1}$ for all $t \in G$ (cf. 2.4.10). In that case, the norm is actually continuous on the image of $G$ in $GL(L^2(G))$.}


APPENDIX C

The weight of $C_c(X)$

C.1. In this appendix, $X$ shall denote an infinite\(^1\) locally compact Hausdorff space; so $\omega(X) \geq N_0$, and $L(X) \geq N_0$. We state our results only for $C_c(X)$. However, similar results (with the same proofs) are valid for $C^*_c(X)$ and for $C_c(X,[0,1])$. Most proofs are straightforward; they can be found in J. DE VRIES \([1972\, b]\).

The basic observation which enables us to determine the local weight of $C_c(X)$ is the following

C.2. LEMMA. For any transfinite cardinal number $\kappa$ the following conditions are equivalent:
   (i) $L(X) \leq \kappa$
   (ii) $X$ can be covered by $\kappa$ relatively compact, open subsets.
   (iii) $X$ can be covered by $\kappa$ compact sets. \(\square\)

C.3. LEMMA. $\omega(C_c(X)) = L(X)$. \(\square\)

C.4. COROLLARY. $C_c(X)$ is metrizable iff $X$ is sigma-compact. In that case, $C_c(X)$ is a Fréchet space\(^2\).

PROOF. If $C_c(X)$ is metrizable, then $L(X) = N_0$, by C.3. Hence $X$ is sigma-compact by C.2. Conversely, if $X$ is sigma-compact, then it follows in a similar way that $\omega(C_c(X)) = N_0$. Since $C_c(X)$ is a locally convex topological vector space, it follows from [Sc], 6.1, that this implies that $C_c(X)$ is metrizable by means of an invariant metric $d$ (i.e. a metric $d$ such that

\(\square\)

\(\)\(^1\) If $X$ is finite, then all results remain true if we add a factor $N_0$ at the appropriate places.

\(\)\(^2\) Of course, here $C_c(X)$ cannot be replaced by $C^*_c(X)$ or $C_c(X,[0,1])$: the former space is not complete, and the latter one is not a vector space.
d(f*g,h*g) = d(f,h) for all f,g,h ∈ \( C_c(X) \). In addition, the usual uniformity for \( C_c(X) \) derived from this metric \( d \) coincides with the uniformity of uniform convergence on compact sets in \( X \). With the latter, \( C_c(X) \) is complete (cf. [Bo], Chap. X, §1.5, Theorem 1). Consequently, \( C_c(X) \) admits a metric making it a locally convex vector space which is complete in this metric.

An alternative proof is as follows: according to [Du], Chap. XII, 8.5, sigma-compactness of \( X \) implies that \( C_c(X) \) is metrizable. It is quite easily checked that the metric indicated there, generates exactly the uniformity of uniform convergence on compact sets in \( X \). Then proceed as above. This proof works also for \( C_c([0,1]) \).

C.5. **Lemma.** \( w(X) \geq d(C_c(X)) \).

**Proof.** For the compact case, cf. [Se], 7.6.5. Using this, the general case can be proved quite easily. □

C.6. **Lemma.** \( w(X) = L(X).d(C_c(X)) \). □

C.7. **Proposition.** \( w(X) = w(C_c(X)) \).

**Proof.** First, observe that

\[
(1) \quad w(C_c(X)) = d(C_c(X)).\omega(C_c(X)).
\]

This is due to the fact that \( C_c(X) \) is a uniform space. In any uniform space \( Y \) it can be shown that \( w(Y) \leq d(Y).u(Y) \), where \( u(Y) \) is the uniform weight of \( Y \) (that is, the least cardinal number of a uniform base of \( Y \)). Since the uniform weight of \( C_c(X) \) is equal to \( \omega(C_c(X)) \), this proves that \( w(C_c(X)) \leq d(C_c(X)).\omega(C_c(X)) \). In order to prove equality, use (3) in 0.2.10. Using (1), the desired equality follows easily from the preceding lemmas. □

C.8. **Remark.** It follows from [Du], Chap. XII, Theorem 5.2, that \( w(C_c(X)) \leq w(X) \).\( W_0 = w(X) \). Conversely, C.6, C.3 and (1) (or rather the obvious "≤" in it; cf (3) in 0.2.10) imply that \( w(X) \leq w(C_c(X)) \). Thus, the use of "≤" in (1) can be avoided by this appeal to the theorem in [Du]. □
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LIST OF SYMBOLS

The following list includes only those non-standard notations which are of a more than local application in this book (i.e. which are used not only immediately after the definition). For notational conventions concerning set-theory and topology, see pp. 1-3 and 9.

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A list of the categories which are considered as "known" can be found on p. 12. In this book, we have defined the following categories and functors:

\begin{align*}
\text{COMPEQ, } & 140 \\
\text{COMP}^G, & 119 \\
\text{HAUS}^G, & 119 \\
G, & 84, 119, 158 \\
G_*, & 165, 171 \\
K, & 85, 119, 158 \\
K_*, & 165, 171 \\
k-KR^G, & 161 \\
k-TTG, & 157 \\
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S, & 84, 119, 158 \\
S_*, & 165, 171 \\
S^G, & 94, 119, 161 \\
TTG, & 84 \\
TTG_*, & 165 \\
TOP^G, & 94
\end{align*}
A leaflet containing an order-form and abstracts of all publications mentioned below is available at the Mathematical Centre, 2e Boerhaavestraat 49, Amsterdam-1005, The Netherlands. Orders should be sent to the same address.


**MCT 7** W.R. van Zwet, Convex transformations of random variables, 1964. ISBN 90 6196 007 X.


**MCT 11** A.B. Paalman-de Miranda, Topological semigroups, 1964. ISBN 90 6196 011 B.


**MCT 13** H.A. Laanvrees, Asymptotic expansions, 1966, out of print; replaced by MCT 54.


**MCT 18** R.P. van de Riet, Formula manipulation in ALGOL 60, part II, 1968. ISBN 90 6196 030 X.


**MCT 22** T.J. Dekker, ALGOL 60 procedures in numerical algebra, part I, 1968. ISBN 90 6196 029 0.


**MCT 25** E.R. Fabel, Representations of the Lorentz group and projective geometry, 1969. ISBN 90 6196 039 B.


**MCT 28** J. Oosterhoff, Combination of one-sided statistical tests, 1969. ISBN 90 6196 041 X.


An asterisk before the number means "to appear."