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mathematisch
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MC

DEPARTMENT OF APPLIED MATHEMATICS TW 135/76 JUNE

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THE ADDITION FORMULA FOR JACOBI POLYNOMIALS
III: COMPLETION OF THE PROOF

First printing December 1972
Second printing June 1976

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 33A65, 33A75, 43A85, 43A90

Abstract

This report concludes the proof of the addition formula for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, which was initiated in the previous paper (Math. Centrum Amsterdam Report TW 133 (1972)). The results are obtained by interpreting Jacobi polynomials $P_n^{(q-2,0)}(x)$ as spherical functions on the homogeneous space $U(q)/U(q-1)$.

KEY WORDS & PHRASES: *the sphere as homogeneous space of the unitary group; decomposition of harmonics with respect to the stationary subgroup; addition formula for disk polynomials; addition formula for Jacobi polynomials*

5. Decomposition of surface harmonics with respect to the subgroup U(q-1).(*)

At the end of section 3 we obtained the second stage of the addition theorem (theorem 3.8), which was formulated in terms of an arbitrary orthonormal base of harm(m,n). In this section we will decompose the class harm (m,n) with respect to the subgroup U(q-1). Thus a canonical orthonormal base can be chosen for harm (m,n) and the addition formula (3.17) can be written in a more explicit way. The result is an addition formula for the polynomials $R_{m,n}^{(\alpha)}(x+iy)$, defined in (3.15), which are orthogonal in the circle, and which include the Jacobi polynomials $P_n^{(\alpha,0)}$. We call this result the third stage of the addition theorem.

Like in formula (2.16) we will write a vector $\xi \in \Omega_{2q}$ as

$$\xi = \cos\theta\, e^{i\phi}\, e_1 + \sin\theta\, \xi' \ , \ \xi' \in \Omega_{2q-2} \ .$$

(*)In this report we continue the numbering of sections, formulas and theorems as started in our previous paper "The addition formula for Jacobi polynomials, II, The Laplace type integral representation and the product formula" (Math. Centrum Amsterdam Report TW 133 (1972)). Whenever in the present paper references are given to section 1,2,3 or 4 these have to be understood as references to the corresponding place in report TW 133. Numbers between square brackets in the present paper refer to the list of references at the end of report TW 133. Reference [11] appeared in Nederl. Akad. Wetensch. Proc. Ser. A 75 and Indag. Math. 34 (1972), pp. 188-191. See also the note added in proof at p. 11 of the present report.

Recall that by theorem 3.3 and formula (3.15) the zonal functions in harm (m,n) are constant multiples of

$$R^{(q-2)}_{m,n}(\cos\theta\, e^{i\phi}) = R^{(q-2,|m-n|)}_{m\wedge n}(\cos 2\theta)(\cos\theta)^{|m-n|}e^{i(m-n)\phi}.$$

Theorem 5.1. Let k, l be integers such that $0 \le k \le m$ and $0 \le l \le n$. Let the class harm (m,n;k,l) consist of the functions

$$S(\xi) = (\sin\theta)^{k+l}\, R^{(q-2+k+l)}_{m-k,n-l}(\cos\theta\, e^{i\phi})\, S'_{k,l}(\xi'), \tag{5.1}$$

where $S'_{k,l}$ is a surface harmonic of type (k,l) on Ω_{2q-2}. Then the classes harm (m,n;k,l) are contained in harm (m,n), they are invariant and irreducible under U(q-1), and there is the orthogonal decomposition

$$\mathrm{harm}(m,n) = \sum_{k=0}^{m}\sum_{l=0}^{n}{}^{\oplus}\ \mathrm{harm}\,(m,n;k,l).$$

Proof. The orthogonality follows from formula (2.18) and proposition 2.1. The invariance and irreducibility follow from theorem 3.4. By formula (3.14) we have

$$\dim.(\mathrm{harm}(m,n)) = \sum_{k=0}^{m}\sum_{l=0}^{n} \dim.(\mathrm{harm}(m,n;k,l)).$$

It is only left to prove that the functions $S(\xi)$ given by (5.1) belong to harm(m,n).

First observe that the polynomial $H^{(\alpha)}_{m,n}(z,\bar z) = H^{(\alpha)}_{m,n}(z_1,\dots,z_q,\bar z_1,\dots,\bar z_q)$ defined by

$$H^{(\alpha)}_{m,n}(z,\bar z) = \begin{cases} z_1^{m-n}\,(z_1\bar z_1+\dots+z_q\bar z_q)^n\, R^{(\alpha,m-n)}_n\left(\dfrac{z_1\bar z_1 - z_2\bar z_2 - \dots - z_q\bar z_q}{z_1\bar z_1+z_2\bar z_2+\dots+z_q\bar z_q}\right) & \text{for } m \ge n,\\[2ex] \bar z_1^{n-m}\,(z_1\bar z_1+\dots+z_q\bar z_q)^m\, R^{(\alpha,n-m)}_m\left(\dfrac{z_1\bar z_1 - z_2\bar z_2 - \dots - z_q\bar z_q}{z_1\bar z_1+z_2\bar z_2+\dots+z_q\bar z_q}\right) & \text{for } n \ge m \end{cases} \tag{5.2}$$

is homogeneous od degree m in z and homogeneous of degree n in $\bar z$. For

$\alpha = q-2$ it is a solid harmonic of type (m,n) by theorem 3.3. By putting $w = z_2\bar{z}_2 + \ldots + z_q\bar{z}_q$ and by using [5], §10.8 (16) it follows that for all integers $m, n \geq 0$

$$H^{(\alpha)}_{m,n}(z,\bar{z}) = z_1^m \bar{z}_1^n \, {}_2F_1(-m,-n;\alpha+1;-\frac{w}{z_1\bar{z}_1}) =$$

$$= \sum_{j=0}^{m\wedge n} \frac{(-m)_j \, (-n)_j}{(\alpha+1)_j \; j!} (-1)^j \, z_1^{m-j} \, \bar{z}_1^{n-j} \, w^j .$$

From this expansion we derive that

$$\frac{\partial}{\partial w} H^{(\alpha)}_{m,n}(z,\bar{z}) = -\frac{mn}{\alpha+1} H^{(\alpha+1)}_{m-1,n-1}(z,\bar{z}).$$

Let $a_2, a_3, \ldots, a_q, b_2, b_3, \ldots, b_q$ be complex numbers such that $a_2b_2 + \ldots + a_qb_q = 0$. Let $0 \leq k \leq m$ and $0 \leq l \leq n$. Then it follows easily that the polynomial

$$(b_2 \frac{\partial}{\partial z_2} + \ldots + b_q \frac{\partial}{\partial z_q})^l \, (a_2 \frac{\partial}{\partial \bar{z}_2} + \ldots + a_q \frac{\partial}{\partial \bar{z}_q})^k \, H^{(q-2)}_{m+l,n+k}(z,\bar{z})$$

is a solid harmonic of type (m,n) and equals

$$(a_2z_2 + \ldots + a_qz_q)^k \, (b_2\bar{z}_2 + \ldots + b_q\bar{z}_q)^l \, (\frac{\partial}{\partial w})^{k+l} \, H^{(q-2)}_{m+l,n+k}(z,\bar{z}) =$$

$$= \text{const.} \; (a_2z_2 + \ldots + a_qz_q)^k \, (b_2\bar{z}_2 + \ldots + b_q\bar{z}_q)^l \, H^{(q-2+k+l)}_{m-k,n-l}(z,\bar{z}),$$

where the constant is non-zero.

The polynomials $(a_2z_2 + \ldots + a_qz_q)^k \, (b_2\bar{z}_2 + \ldots + b_q\bar{z}_q)^l$ for which $a_2b_2 + \ldots + a_qb_q = 0$ are solid harmonics of type (k,l) on C^{q-1} and they span this class of solid harmonics because the class is irreducible under $U(q-1)$. Hence the polynomials $F(z_2,\ldots,z_q,\bar{z}_2,\ldots,\bar{z}_q) \, H^{(q-2+k+l)}_{m-k,n-l}(z,\bar{z})$ is a solid harmonic of type (m,n) on C^q if F is a solid harmonic of type (k,l) on C^{q-1}. By restricting this polynomial to Ω_{2q} and by using (5.2), (2.16) and (3.15) it follows that the functions given by (5.1) are in the class harm (m,n). This proves the theorem.

By repeated application of theorem 5.1 to classes of surface harmonics on Ω_{2q-2}, $\Omega_{2q-4}, \ldots, \Omega_2$, respectively, we can obtain an explicit base for harm (m,n). Such an explicit base is given by Ikeda and Kayama [9]. They obtained the result by solving the partial differential equation (2.10) for solid harmonics, introducing suitable coordinates and applying the method of separation of variables. Our theorem 5.1 can also be proved as a corollary of their results.

We will need the L^2-norm of the functions S in (5.1). By formulas (2.18), (3.15), (2.21) and by [5], § 10.8 (4) it follows that

$$(5.3)\qquad \int_{\Omega_{2q}} |S(\xi)|^2 \, d\omega_{2q}(\xi) =$$

$$\int_0^{\frac{1}{2}\pi} \int_0^{2\pi} |R^{(q-2+k+l)}_{m-k,n-l} (\cos\theta \, e^{i\phi})|^2 (\sin\theta)^{2q+2k+2l-3} \cos\theta \, d\theta \, d\phi .$$

$$\cdot \int_{\Omega_{2q-2}} |S'_{k,l}(\xi')|^2 \, d\omega_{2q-2}(\xi') =$$

$$\frac{\pi(m-k)!\,(n-l)!}{(q-1+m+n)\,(q-1+k+l)_{m-k}\,(q-1+k+l)_{n-l}} \cdot \int_{\Omega_{2q-2}} |S'_{k,l}(\xi')|^2 \, d\omega_{2q-2}(\xi').$$

Let the functions $S'_j(\xi')$ ($j = 1,\ldots, N(q-1;k,l)$) form an orthonormal base for the surface harmonics of type (k,l) on Ω_{2q-2}. Then, by theorem 5.1 and formula (5.3), the functions

$$\left(\frac{(q-1+m+n)\,(q-1+k+l)_{m-k}\,(q-1+k+l)_{n-l}}{\pi(m-k)!\,(n-l)!}\right)^{\frac{1}{2}} (\sin\theta)^{k+l} .$$

$$\cdot\; R^{(q-2+k+l)}_{m-k,n-l} (\cos\theta \, e^{i\phi}) \, S'_j(\xi'), \; j = 1,\ldots, N(q-1;k,l),$$

form an orthonormal base for harm (m,n;k,l).By using theorem 3.8 we can conclude the following.

Corollary 5.2. Let the functions $S_j(\xi)$ $(j = 1,\ldots, N(q-1;k,l))$ form an orthonormal base of harm $(m,n;k,l)$. Let for $\xi,\ \eta \in \Omega_{2q}$

$$\xi = \cos\theta_1\, e^{i\phi_1}\, e_1 + \sin\theta_1\, \xi',\ \xi' \in \Omega_{2q-2} \text{ and}$$

$$\eta = \cos\theta_2\, e^{-i\phi_2}\, e_1 + \sin\theta_2\, \eta',\ \eta' \in \Omega_{2q-2}.$$

Then

$$\sum_{j=1}^{N(q-1;k,l)} S_j(\xi)\, \overline{S_j(\eta)} = \text{const.}\ .$$

$$.\ (\sin\theta_1)^{k+l}\, R^{(q-2+k+l)}_{m-k,n-l}(\cos\theta_1\, e^{i\phi_1})\ .$$

$$(\sin\theta_2)^{k+l}\, R^{(q-2+k+l)}_{m-k,n-l}(\cos\theta_2\, e^{i\phi_2}).$$

$$.\ R^{(q-3)}_{k,l}((\xi',\eta')).$$

Here the constant equals

$$\frac{N(q-1;k,l)\ (q-1+m+n)\ (q-1+k+l)_{m-k}\ (q-1+k+l)_{n-l}}{w_{2q-2}\ \pi\ (m-k)!\ (n-l)!}\ .$$

Combination of theorem 3.8, theorem 5.1 and corollary 5.2 gives the formula

$$R^{(q-2)}_{m,n}((\xi,\eta)) =$$

$$= R^{(q-2)}_{m,n}(\cos\theta_1\, e^{i\phi_1}\cos\theta_2\, e^{i\phi_2} + \sin\theta_1\sin\theta_2\, (\xi',\eta')) =$$

$$= \sum_{k=0}^{m}\sum_{l=0}^{n} c_{k,l}\, (\sin\theta_1)^{k+l}\, R^{(q-2+k+l)}_{m-k,n-l}(\cos\theta_1\, e^{i\phi_1}).$$

$$.\ (\sin\theta_2)^{k+l}\, R^{(q-2+k+l)}_{m-k,n-l}(\cos\theta_2\, e^{i\phi_2})\, R^{(q-3)}_{k,l}((\xi',\eta')),$$

$$\text{where } c_{k,l} = \frac{\omega_{2q}}{\pi\omega_{2q-2}}\, \frac{N(q-1;k,l)}{N(q;m,n)}\, \frac{(q-1+k+l)_{m-k}\ (q-1+k+l)_{n-l}\ (q-1+m+n)}{(m-k)!(n-l)!}\ .$$

By putting $(\xi',\eta') = r\,e^{i\psi}$, $\alpha = q-2$ and by using (2.13) and (3.12) this formula is reduced to

(5.4) $$R_{m,n}^{(\alpha)}(\cos\theta_1\, e^{i\phi_1}\cos\theta_2\, e^{i\phi_2} + \sin\theta_1 \sin\theta_2\, r\, e^{i\psi}) =$$

$$= \sum_{k=0}^{m}\sum_{l=0}^{n} c_{m,n,k,l}^{(\alpha)}\, (\sin\theta_1)^{k+l}\, R_{m-k,n-l}^{(\alpha+k+l)}(\cos\theta_1\, e^{i\phi_1}).$$

$$.\ (\sin\theta_2)^{k+l}\, R_{m-k,n-l}^{(\alpha+k+l)}(\cos\theta_2\, e^{i\phi_2})\, R_{k,l}^{(\alpha-1)}(r\, e^{i\psi}),$$

$\alpha = 1,2,\ldots$, where

$$c_{m,n,k,l}^{(\alpha)} = \frac{\alpha}{\alpha+k+l}\binom{m}{k}\binom{n}{l}\frac{(\alpha+n+1)_k\ (\alpha+m+1)_l}{(\alpha+1)_k\ \ (\alpha+k)_l}.$$

We call this formula the third stage of the addition theorem.(*)

6. The addition formula for Jacobi polynomials in the general case.

The addition formula (5.4) was proved by interpreting the functions in (5.4) as functions on the homogeneous space $U(q)/U(q-1)$. Next we can obtain some corollaries of (5.4) by considering the formula just as an analytic formula and by using such simple methods as differentiation and analytic continuation. Thus we will prove the fourth (final) stage of the addition theorem , i.e. the result for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with α and β arbitrary. It will turn out that this general result already follows from the special case of (5.4) for which $\alpha = 1$ and $m = n$. In other words, the addition formula for general Jacobi polynomials can be obtained by doing analysis on the homogeneous space $SU(3)/U(2)$, which is the simplest non-trivial complex

(*) See note added in proof at p. 11.

projective space.

We proved in the previous section that formula (5.4) is true for positive integer values of α. We will show that the formula holds for arbitrary complex values of α. Using (2.20) and the identity $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$ we have for $m \geq n$

$$(6.1) \qquad R_{m,n}^{(\alpha)}(z) = R_n^{(\alpha,m-n)}(2z\bar{z}-1)\, z^{m-n} =$$

$$= \frac{(m-n+1)_n}{(\alpha+1)_n} (-1)^n R_n^{(m-n,\alpha)}(1-2z\bar{z})\, z^{m-n} =$$

$$= \frac{(m-n+1)_n}{(\alpha+1)_n} (-1)^n \sum_{k=0}^{n} \frac{(-n)_k\,(m+\alpha+1)_k}{(m-n+1)_k\, k!} z^{k+m-n} \bar{z}^k .$$

Similarly, for $m \leq n$

$$(6.2) \qquad R_{m,n}^{(\alpha)}(z) = \frac{(n-m+1)_m (-1)^m}{(\alpha+1)_m} \sum_{k=1}^{m} \frac{(-m)_k\,(n+\alpha+1)_k}{(n-m+1)_k\, k!} z^k \bar{z}^{k+n-m} .$$

It follows that $R_{m,n}^{(\alpha)}(z)$ is a rational function of α. Hence, both sides of (5.4) are rational functions of α. After multiplication of both sides of (5.4) with a suitable factor we obtain a polynomial identity in α, which holds for $\alpha = 1,2,3,\ldots$. Therefore formula (5.4) is true for all complex values of α.

We want to apply (6.1) and (6.2) in another way. It follows easily from these formulas that for all integers $m,n \geq 0$

$$(6.3) \qquad \frac{\partial}{\partial z} R_{m,n}^{(\alpha)}(z) = \frac{m(n+\alpha+1)}{\alpha+1} R_{m-1,n}^{(\alpha+1)}(z),$$

$$(6.4) \qquad \frac{\partial}{\partial \bar{z}} R_{m,n}^{(\alpha)}(z) = \frac{n(m+\alpha+1)}{\alpha+1} R_{m,n-1}^{(\alpha+1)}(z).$$

After putting $z = r\,e^{i\psi}$ in (5.4) it turns out that differentiation of both sides of (5.4) with respect to z gives the same formula with α replaced by $\alpha+1$ and m by $m-1$. Similarly α is raised and n is lowered in (5.4) by differentiation with respect to $\bar{z}$. Hence, if formula (5.4) is known for one value of α (for instance for $\alpha = 1$) and for all values of m,n , then repeated differentiation of (5.4) with respect to z and $\bar{z}$ gives the addition formula for $\alpha+1$, $\alpha+2$, $\alpha+3,\ldots$ and the case of general α follows again by analytic continuation.

Differentiation of both sides of (5.4) with respect to $z = r\,e^{i\psi}$ or $\bar{z} = r\,e^{-i\psi}$ gives the recursion formulas

$$\frac{(\alpha+1)(\alpha+l)\ (k+1)}{\alpha\ (\alpha+n+1)\ m}\, c^{(\alpha)}_{m,n,k+1,l} = c^{(\alpha+1)}_{m-1,n,k,l} \quad \text{and}$$

$$\frac{(\alpha+1)(\alpha+k)\ (l+1)}{\alpha\ (\alpha+m+1)\ m}\, c^{(\alpha)}_{m,n,k,l+1} = c^{(\alpha+1)}_{m,n-1,k,l}\ ,$$

respectively

By putting $\theta_2 = 0$, $\phi_2 = 0$ in (5.4) it is evident that $c^{(\alpha)}_{m,n,0,0} = 1$.

This provides another method to compute the coefficients $c^{(\alpha)}_{m,n,k,l}$ in (5.4).

The right hand side of the addition formula (5.4) can be considered as an orthogonal expansion of the left hand side with respect to the measure $(1-r^2)^{\alpha-1}\ r\,dr\,d\psi$, $\alpha > 0$. By integration both sides of (5.4) with respect to this measure the product formula (4.11) can be obtained.

Putting $m = n$, $\phi_1 = \phi_2 = 0$ and taking the terms (k,l) and (l,k) together in (5.4) we can reduce this formula to

$$R_n^{(\alpha,0)}(2\cos^2\theta_1\cos^2\theta_2 + 2\sin^2\theta_1\sin^2\theta_2 + \sin 2\theta_1 \sin 2\theta_2\, r\cos\psi - 1) =$$
$$= \sum_{k=0}^{n}\sum_{l=0}^{k} \frac{\alpha}{\alpha+k+l}\binom{n}{k}\binom{n}{l}\frac{(\alpha+n+1)_k}{(\alpha+1)_k}\frac{(\alpha+n+1)_l}{(\alpha+k)_l}\,\cdot$$
$$\cdot\,(\sin\theta_1)^{k+l}(\cos\theta_1)^{k-l} R_{n-k}^{(\alpha+k+l,k-l)}(\cos 2\theta_1)\,\cdot$$
$$\cdot\,(\sin\theta_2)^{k+l}(\cos\theta_2)^{k-l} R_{n-k}^{(\alpha+k+l,k-l)}(\cos 2\theta_2)\,\cdot$$
$$\cdot\, R_l^{(\alpha-1,k-l)}(2r^2-1)\, r^{k-l}\,\varepsilon(k-l)\cos(k-l)\psi, \tag{6.5}$$

where $\varepsilon(k-l)=1$ for $k=l$,
$=2$ for $k\neq l$.

Let $C_n^\lambda(x) = \frac{(2\lambda)_n}{n!} R_n^{(\lambda-\frac12,\lambda-\frac12)}(x)$ and $T_n(x) = R_n^{(-\frac12,-\frac12)}(x) = \cos(n \arccos x)$ denote a Gegenbauer polynomial and a Chebyshev polynomial, respectively.

By [5], §10.9(6) and §10.11(5) we have the limit formula $\lim\limits_{\lambda\to 0}\frac{\lambda+n}{n} C_n^\lambda(x) = \varepsilon(n)T_n(x)$. We conclude with the final stage of the addition theorem.

Theorem 6.1. For all complex α and β ($\beta\neq 0$) the formula

(6.6)

$$P_n^{(\alpha,\beta)}(2\cos^2\theta_1\ \cos^2\theta_2 + 2\sin^2\theta_1\ \sin^2\theta_2\ r^2 +$$

$$+ \sin 2\theta_1\ \sin 2\theta_2\ r\cos\psi - 1) =$$

$$= \sum_{k=0}^{n} \sum_{l=0}^{k} c_{n,k,l}^{(\alpha,\beta)} (\sin\theta_1\ \sin\theta_2)^{k+l} (\cos\theta_1\ \cos\theta_2)^{k-l}.$$

$$. P_{n-k}^{(\alpha+k+l,\beta+k-l)}(\cos 2\theta_1)\ P_{n-k}^{(\alpha+k+l,\beta+k-l)}(\cos 2\theta_2).$$

$$. P_l^{(\alpha-\beta-1,\beta+k-l)}(2r^2-1)\ r^{k-l}\ \frac{\beta+k-l}{\beta}\ C_{k-l}^{\beta}(\cos\psi)$$

is valid, where $C_{n,k,l}^{(\alpha,\beta)} =$

$$= \frac{(\alpha+k+l)\Gamma(\alpha+\beta+n+k+1)\Gamma(\alpha+k)\Gamma(\beta+1)\Gamma(\beta+n+1)\Gamma(n-k+1)}{\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+n+l+1)\Gamma(\beta+k+1)\Gamma(\beta+n-l+1)}.$$

For $\beta=0$ the same formula holds with $\frac{\beta+k-l}{\beta}C_{k-l}^{\beta}(\cos\psi)$ replaced by $\varepsilon(k-l)T_{k-l}(\cos\psi)$.

<u>Proof</u>. For $\beta=0$ the formula follows from (6.5), by using (2.20) and (2.21). For $\beta=1,2,3,\ldots$ the result follows by differentiating both sides of (6.6) with respect to $\cos\psi$.
Here we use the formulas

$$\left(\frac{d}{dx}\right)^m P_n^{(\alpha,\beta)}(x) = 2^{-m}\ \frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+\beta+n+1)}\ P_{n-m}^{(\alpha+m,\beta+m)}(x)$$

$$\left(\frac{d}{dx}\right)^m T_n(x) = 2^{m-1}(m-1)!\ n\ C_{n-m}^{m}(x)$$

(see [5], §10.8(17) and §10.11(26)).

For general β the formula is proved by analytic continuation with respect to β. This is permitted because both sides of (6.6) are rational functions in β.

q.e.d.

Note added in proof. A recent review in Zbl. Math. called the author's attention to the following two papers.

[17] Vilenkin, N.Ja. and R.L.Šapiro,
Irreducible representations of the group SU(n) of class one with respect to SU(n-1), Izv. Vysš. Učebn. Zaved. Matematika 62(1967), 9-20 (Russian).

[18] Šapiro, R.L.,
The special functions connected with the representations of the group SU(n) of class one with respect to SU(n-1) (n ≥ 3), ibidem 71(1968), 97-107 (Russian).

These two papers anticipate the author's work and derive some of our results along the same lines. In particular, our formula (5.4), i.e. the third stage of the addition theorem, is the same as formula (1.20) in [18]. It seems that these two papers were completely unknown outside the Soviet Union until the present day. Probably our final addition theorem 6.1 is new.