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1 SUMMARY OF RESULTS

TW

amsterdam 1972

KONINKL. NEDERL. AKADEMIE VAN WETENSCHAPPEN – AMSTERDAM
Reprinted from Proceedings, Series A, 75, No. 2 and Indag. Math., 34, No. 2, 1972

MATHEMATICS

THE ADDITION FORMULA FOR JACOBI POLYNOMIALS

I SUMMARY OF RESULTS *)

BY

T. H. KOORNWINDER

(Communicated by Prof. A. VAN WIJNGAARDEN at the meeting of November 27, 1971)

Recently the author derived a Laplace integral representation, a product formula and an addition formula for Jacobi polynomials $P_n^{(\alpha,\beta)}$. The results are

$$\left\{\begin{aligned} \frac{P_n^{(\alpha,\beta)}(\cos 2\theta)}{P_n^{(\alpha,\beta)}(1)} &= \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}\cdot \\ &\cdot \int_{r=0}^{1}\int_{\phi=0}^{\pi}(\cos^2\theta - r^2\sin^2\theta + ir\cos\phi\sin 2\theta)^n\cdot \\ &\cdot (1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\phi)^{2\beta}\,dr\,d\phi, \end{aligned}\right. \tag{1}$$

$$\left\{\begin{aligned} \frac{P_n^{(\alpha,\beta)}(\cos 2\theta_1)}{P_n^{(\alpha,\beta)}(1)}\frac{P_n^{(\alpha,\beta)}(\cos 2\theta_2)}{P_n^{(\alpha,\beta)}(1)} &= \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}\cdot \\ &\cdot \int_{=0}^{1}\int_{\phi=0}^{\pi}\frac{P_n^{(\alpha,\beta)}(2|\cos\theta_1\cos\theta_2 + re^{i\phi}\sin\theta_1\sin\theta_2|^2-1)}{P_n^{(\alpha,\beta)}(1)}\cdot \\ &\cdot (1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\phi)^{2\beta}\,dr\,d\phi \end{aligned}\right. \tag{2}$$

and

$$\left\{\begin{aligned} &P_n^{(\alpha,\beta)}(2|\cos\theta_1\cos\theta_2 + re^{i\phi}\sin\theta_1\sin\theta_2|^2-1) = \\ &= \sum_{k=0}^{n}\sum_{l=0}^{k} c_{n,k,l}^{(\alpha,\beta)}(\sin\theta_1\sin\theta_2)^{k+l}(\cos\theta_1\cos\theta_2)^{k-l}\cdot \\ &\cdot P_{n-k}^{(\alpha+k+l,\beta+k-l)}(\cos 2\theta_1)\,P_{n-k}^{(\alpha+k+l,\beta+k-l)}(\cos 2\theta_2)\cdot \\ &\cdot P_l^{(\alpha-\beta-1,\beta+k-l)}(2r^2-1)\,r^{k-l}\,\frac{\beta+k-l}{\beta}\,C_{k-l}^{\beta}(\cos\phi). \end{aligned}\right. \tag{3}$$

In formulas (1) and (2) it is supposed that

$$\alpha > \beta > -\tfrac{1}{2}.$$

*) Report TW 131/71 of the Mathematical Center Amsterdam.

In formula (3) the coefficient is

$$c_{n,k,l}^{(\alpha,\beta)} =$$

$$= \frac{(\alpha+k+l)\,\Gamma(\alpha+\beta+n+k+1)\,\Gamma(\alpha+k)\,\Gamma(\beta+1)\,\Gamma(\beta+n+1)\,\Gamma(n-k+1)}{\Gamma(\alpha+\beta+n+1)\,\Gamma(\alpha+n+l+1)\,\Gamma(\beta+k+1)\,\Gamma(\beta+n-l+1)}\,.$$

If $\beta=0$ the factor

$$\frac{\beta+k-l}{\beta}\,C_{k-l}^{\beta}(\cos\phi)$$

in formula (3) is replaced by

$$2\cos(k-l)\phi \quad \text{if } k-l\neq 0$$

and by

$$1 \qquad\qquad \text{if } k-l=0.$$

The analogous formulas in the ultraspherical case $\alpha=\beta$ are due to GEGENBAUER [8] (cf. ERDÉLYI [3], § 3.15, formulas (22), (20) and (19)). They can be obtained as degenerate cases of our formulas (1), (2) and (3) respectively.

It is easy to see that (2) follows from (3) by integration and that (1) follows from (2) by dividing both sides by $P_n^{(\alpha,\beta)}(\cos 2\theta_2)$ and then letting $|\cos 2\theta_2| \to \infty$.

Formula (3) was proved by the author by using group theoretical methods. For integer α the polynomials $P_n^{(\alpha,0)}$ can be interpreted as spherical functions on the complex projective space $SU(\alpha+2)/U(\alpha+1)$ (cf. CARTAN [1], [2]) or as spherical functions on the sphere $U(\alpha+2)/U(\alpha+1)$ (cf. IKEDA [9], IKEDA and KAYAMA [10], IKEDA and SETO [11]). In the last mentioned interpretation the classical notion of spherical harmonics is refined for the case that the sphere is the unit sphere in a complex vector space and the symmetry group is the group of unitary transformations. The usual methods for classical spherical harmonics (cf. for instance ERDÉLYI [4], Ch. 11, MÜLLER [12], VILENKIN [14]) can be extended for this case. The author proved an abstract addition formula and constructed a canonical orthonormal system of generalized spherical harmonics expressed as products of certain Jacobi polynomials. Thus formula (3) was proved for $\alpha=1, 2, \ldots$ and $\beta=0$.

By repeated differentiation with respect to $\cos\phi$ of both sides of (3) this formula can be proved for all integer α and β, such that $\alpha>\beta\geqslant 0$. Analytic continuation with respect to α and β finally gives the general case of formula (3). Detailed proofs will be published in one or more subsequent papers.

By the substitution

$$\cos\theta_3\, e^{i\psi} = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\, re^{i\phi}$$

the product formula (2) is transformed into another product formula which was obtained earlier in a different way by GASPER [7]. Gasper used his result to prove a convolution structure for Jacobi series.

Let for arbitrary complex ν and $1 \leqslant x < \infty$ Jacobi functions be defined as

$$R_\nu^{(\alpha,\beta)}(x) = {}_2F_1\left(-\nu, \nu+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right).$$

Then it follows from (1) and (2) that

$$(4)\quad \left\{\begin{aligned} R_\nu^{(\alpha,\beta)}(\operatorname{ch} 2t) &= \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac{1}{2})} \cdot \\ &\quad \cdot \int_{r=0}^{1} \int_{\phi=0}^{\pi} ((\operatorname{ch} t)^2 + r^2(\operatorname{sh} t)^2 + r\cos\phi \operatorname{sh} 2t)^\nu \cdot \\ &\quad \cdot (1-r^2)^{\alpha-\beta-1} r^{2\beta+1} (\sin\phi)^{2\beta}\, dr\, d\phi \end{aligned}\right.$$

and

$$(5)\quad \left\{\begin{aligned} R_\nu^{(\alpha,\beta)}(\operatorname{ch} 2t_1)\, R_\nu^{(\alpha,\beta)}(\operatorname{ch} 2t_2) &= \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} \cdot \\ &\quad \cdot \int_{r=0}^{1} \int_{\phi=0}^{\pi} R_\nu^{(\alpha,\beta)}(2|\operatorname{ch} t_1 \operatorname{ch} t_2 + \operatorname{sh} t_1 \operatorname{sh} t_2\, re^{i\phi}|^2 - 1) \cdot \\ &\quad \cdot (1-r^2)^{\alpha-\beta-1} r^{2\beta+1} (\sin\phi)^{2\beta}\, dr\, d\phi \end{aligned}\right.$$

for $\alpha > \beta > -\frac{1}{2}$.

This is proved by analytic continuation with respect to ν where a theorem of Carlson is applied (cf. Titchmarsh [13], p. 186). Formulas (4) and (5) generalize well-known results for Legendre and Gegenbauer functions (cf. [3], § 3.15.2 (22) and § 3.11.1 (2)).

Formulas (4) and (5) have applications for the harmonic analysis with respect to the functions $R_\nu^{(\alpha,\beta)}$. Flensted-Jensen who had already obtained some Paley-Wiener type theorems for this case in [5] could simplify his proofs in [6] by using formula (4). In joint work Flensted-Jensen and the author used formula (5) to develop a convolution structure for the harmonic analysis with respect to Jacobi functions. These results will also be published in a near future.

Acknowledgement

The author is due to Professor Richard Askey for suggesting him this problem. The research summarized here was done at the Institute Mittag-Leffler in Djursholm, Sweden. The author is also due to Professor Lennart Carleson for his kind hospitality.

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