BOLYAI SOCIETY
MATHEMATICAL STUDIES, 7

Graph Theory and
Combinatorial Biology
Balatonlelle (Hungary), 1996
Budapest, 1999, pp. 29–85.

THE COLIN DE VERDIÈRE GRAPH PARAMETER

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In 1990, Y. Colin de Verdière introduced a new graph parameter $\mu(G)$, based on spectral properties of matrices associated with G. He showed that $\mu(G)$ is monotone under taking minors and that planarity of G is characterized by the inequality $\mu(G) \leq 3$. Recently Lovász and Schrijver showed that linkless embeddability of G is characterized by the inequality $\mu(G) \leq 4$.

In this paper we give an overview of results on $\mu(G)$ and of techniques to handle it.

Contents

1. Introduction

In 1990, Colin de Verdière [7] (cf. [8]) introduced an interesting spectral parameter $\mu(G)$ for any undirected graph G. The parameter was motivated by the study of the maximum multiplicity of the second eigenvalue of certain Schrödinger operators. These operators are defined on Riemann surfaces. It turned out that in this study one can approximate the surface by a sufficiently densely embedded graph G, in such a way that the maximum multiplicity of the second eigenvalue of such operators is just the parameter $\mu(G)$, and thus can be described fully in terms of spectral properties of matrices related to the adjacency matrix of G.

The interest in Colin de Verdière's graph parameter can be explained not only by its background in differential geometry, but also by the fact that it has surprisingly nice graph-theoretic properties. Among others, it is minor-monotone, so that the Robertson-Seymour graph minor theory applies to it. Moreover, planarity of graphs can be characterized by this invariant: $\mu(G) \leq 3$ if and only if G is planar. More recently it was shown in [20] that $\mu(G) \leq 4$ if and only if G is linklessly embeddable in $\mathbb{R}^3$. So using μ, topological properties of a graph G can be characterized by spectral properties of matrices associated with G.

It turns out that graphs with large values of μ are also quite interesting. For example, for a graph G on n nodes, having no "twin" nodes, and with $\mu(G) \geq n-4$, the complement of G is planar; the converse of this assertion also holds under reasonably general conditions. This result is closely related to a famous construction of Koebe representing planar graphs by touching circles.

In this paper, we give a survey of this new parameter.

Acknowledgement. The authors are grateful to Andrew Kotlov for reading the manuscript carefully and thoroughly, and for suggesting many improvements and correstions.

1.1. Definition

We consider undirected, loopless graphs $G = (V, E)$ without multiple edges. We assuming (without loss of generality) that $V = \{1, \ldots, n\}$. For any subset U of V, let $G|U$ denote the subgraph of G induced by U, and $G - U$ the subgraph of G obtained by deleting U (so $G - U = G|(V \setminus U)$). $N(U)$ is the set of nodes in $V \setminus U$ adjacent to at least one node in U.

Let $\mathbb{R}^{(n)}$ denote the linear space of real symmetric $n \times n$ matrices. This space has dimension $\binom{n+1}{2}$. We will use the inner product $A \cdot B = \sum_{i,j} A_{i,j} B_{i,j} = \mathrm{Tr}(A^T B)$ in this space.

The *corank* $\mathrm{corank}(M)$ of a matrix M is the dimension of its kernel (null space) $\ker(M)$. If S is a set of rows of M and T is a set of columns of M, then $M_{S \times T}$ is the submatrix induced by the rows in S and the columns in T. If $S = T$ then we write M_S for $M_{S \times S}$. Similarly, if x is a vector, then x_S denotes the subvector of x induced by the indices in S. We denote the ith smallest eigenvalue of a symmetric matrix M by $\lambda_i(M)$.

The *Colin de Verdière number* $\mu(G)$ of the graph G is defined as the largest corank of any matrix $M = (M_{i,j}) \in \mathbb{R}^{(n)}$ such that:

1.1.

(M1) *for all* i, j *with* $i \neq j$: $M_{i,j} < 0$ *if* i *and* j *are adjacent, and* $M_{i,j} = 0$ *if* i *and* j *are nonadjacent;*

(M2) M *has exactly one negative eigenvalue, of multiplicity* 1;

(M3) *there is no nonzero matrix* $X = (X_{i,j}) \in \mathbb{R}^{(n)}$ *such that* $MX = 0$ *and such that* $X_{i,j} = 0$ *whenever* $i = j$ *or* $M_{i,j} \neq 0$.

There is no condition on the diagonal entries $M_{i,i}$.

Any symmetric matrix M with properties (M1)–(M3) will be called a *Colin de Verdière matrix* for the graph G. A Colin de Verdière matrix M is *optimal* if $\mathrm{corank}(M) = \mu(G)$.

Note that for each graph $G = (V, E)$, a Colin de Verdière matrix exists. If G is connected, let A be the adjacency matrix of G. Then by the Perron–Frobenius theorem, the largest eigenvalue of A has multiplicity 1, and hence for λ between the two largest eigenvalues of A, the matrix $M = \lambda I - A$ is nonsingular (and hence satisfies (M3) in a trivial way) and has exactly one negative eigenvalue. If G is disconnected, we can choose λ for each component separately and obtain a nonsingular matrix with exactly one negative eigenvalue again.

Let us comment on the conditions (M1)–(M3), which may seem strange or *ad hoc* at the first sight. Condition (M1) means that we are considering the adjacency matrix of G, with the edges weighted with arbitrary negative weights, and arbitrary real values inserted in the diagonal. The negativity of the weights is, of course, just a convention, which is a bit strange now but will turn out more convenient later on.

In the case of connected graphs, (M1) implies, by the Perron–Frobenius theorem, that the smallest eigenvalue of M has multiplicity 1. Since we do not make any assumption about the diagonal, we could consider any matrix M with property (M1), and replace it by $M - \lambda I$, where λ is the second smallest eigenvector of M. So $\mu(G)$ could be defined as the maximum multiplicity of the second smallest eigenvalue λ of a matrix M satisfying (M1) and (M3) (with M replaced by $M - \lambda I$ in (M3)). In this sense, (M2) can be viewed as just a normalization.

Condition (M3) is called the *Strong Arnold Property* (or *Strong Arnold Hypothesis*). There are a number of equivalent formulations of (M3), expressing the fact that M is "generic" in a certain sense. We discuss this issue in Section 2.1.

When arguing that there is a Colin de Verdière matrix for every graph, we used the fact that any nonsingular matrix M trivially satisfies (M3). This remains true if the matrix M has corank 1. Indeed, a nonzero matrix X with $MX = 0$ would then have rank 1, but since X has 0's in the diagonal, this is impossible. We will see that there are other cases when the Strong Arnold Property is automatically satisfied (cf. Section 5.1), while in other cases it will be a crucial assumption.

1.2. Some examples

It is clear that $\mu(K_1) = 0$. We have $\mu(G) > 0$ for every other graph. Indeed, one can put "generic" numbers in the diagonal of the negative of the adjacency matrix to make all the eigenvalues different; then we can subtract the second smallest eigenvalue from all diagonal entries to get one negative and one 0 eigenvalue. The Strong Arnold Property, as remarked at the end of the previous section, holds automatically.

Let $G = K_n$ be a complete graph with $n > 1$ nodes. Then it is easy to guess the all-(-1) matrix $-J$ for M. This trivially satisfies all three constraints, and has corank $n-1$. One cannot beat this, since at least one eigenvalue must be negative by (M2). Thus

$$\mu(K_n) = n - 1.$$

Next, let us consider the graph $\overline{K_n}$ consisting of $n \geq 2$ independent nodes. All entries of a Colin de Verdière matrix M except for the entries in the diagonal must be 0. By (M2), we must have exactly one negative entry in the diagonal. Trying to minimize the rank, we would like to put 0's in the rest of the diagonal, getting corank $n-1$. But it is here where the Strong Arnold Property appears: we can put at most one 0 in the diagonal! In fact, assuming that $M_{1,1} = M_{2,2} = 0$, consider the matrix X with

$$X_{i,j} = \begin{cases} 1, & \text{if } \{i,j\} = \{1,2\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then X violates (M3). So we must put $n-2$ positive numbers in the diagonal, and are left with a single 0. It is easy to check that this matrix will satisfy (M3), and hence

$$\mu(\overline{K_n}) = 1.$$

A similar appeal to the Strong Arnold Property allows us to show:

1.2. *The complete graph is the only graph G on $n \geq 3$ nodes with $\mu(G) = n-1$.*

(For $n = 2$, both K_2 and its complement have this property.) Indeed, the matrix M realizing μ has rank 1, and thus it is of the form $M = -uu^T$ for some vector u. If G is noncomplete, u must have a 0 coordinate, by (M1). Say, $u_n = 0$, so that n is an isolated node in G. Since $n \geq 3$, the matrix M has corank at least 2, and so it has a nonzero vector x in the null space with $x_n = 0$. Now the matrix $X = xe_n^T + e_n x^T$ shows that M does not have the Strong Arnold Property (where e_n is the nth unit basis vector).

As a third example, consider the path P_n on $n \geq 2$ nodes. We may assume that these are labelled $\{1, 2, \ldots, n\}$ in their order on the path. Consider any matrix M satisfying (M1), and delete its first column and last row. The remaining matrix has negative numbers in the diagonal and 0's above the diagonal, and hence it is nonsingular. Thus the corank of M

is at most 1. We have seen that a corank of 1 can always be achieved. Thus $\mu(P_n) = 1$.

Finally, let us consider a complete bipartite graph $K_{p,q}$, where we may assume that $p \le q$ and $q \ge 2$. In analogy with K_n, one can try to guess a matrix with properties (M1) and (M2) with low rank. The natural guess is

$$M = \begin{pmatrix} 0 & -J \\ -J^T & 0 \end{pmatrix},$$

where J is the $p \times q$ all-1 matrix. This clearly satisfies (M1) and (M2) and has corank 2. But it turns out that this matrix violates (M3) unless $p, q \le 3$. In fact, a matrix X as in (M3) has the form

$$X = \begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix},$$

where Y is a $p \times p$ symmetric matrix with 0's in its diagonal and Z is a $q \times q$ symmetric matrix with 0's in its diagonal. The condition $MX = 0$ says that Y and Z have 0 row-sums. Now if, say, $Y \neq 0$, then it is easy to see that we must have $p \ge 4$.

So far we have been able to establish that $\mu(K_{p,q}) \ge p + q - 2$ if $p, q \le 3$; and we know by the discussion above that equality holds here (if $(p, q) \neq (1, 1)$). But if, say, $p \ge 4$, then it is easy to construct a symmetric $p \times p$ matrix with 0 diagonal entries and 0 row sums. This shows that the above guess for the matrix M realizing $\mu(K_{p,q})$ does not work. We will see in Section 2.3 that in this case μ will be smaller (equal to $\min\{p, q\} + 1$, in fact).

(There is a quite surprising fact here, which also underscores some of the difficulties associated with the study of μ. The graph $K_{4,4}$ (say) has a node-transitive and edge-transitive automorphism group, and so one would expect that at least one optimizing matrix in the definition of μ will have the same diagonal entries and the same nonzero off-diagonal entries. But this is not the case: it is easy to see that this would force us to consider the matrix we discarded above. So the optimizing matrix must break the symmetry!)

1.3. Overview

An important property of $\mu(G)$ proved by Colin de Verdière [7] is that it is monotone under taking minors:

1.3. *The graph parameter $\mu(G)$ is minor-monotone; that is, if H is a minor of G then $\mu(H) \leq \mu(G)$.*

(A *minor* of a graph arises by a series of deletions and contractions of edges and deletions of isolated nodes, suppressing any multiple edges and loops that may arise.) Proving 1.3 is surprisingly nontrivial, and the Strong Arnold Property plays a crucial role.

The minor-monotonicity of $\mu(G)$ is especially interesting in the light of the Robertson-Seymour theory of graph minors [24], which has as principal result that if $\mathcal{C}$ is a collection of graphs so that no graph in $\mathcal{C}$ is a minor of another graph in $\mathcal{C}$, then $\mathcal{C}$ is finite. This can be equivalently formulated as follows. For any graph property $\mathcal{P}$ closed under taking minors, call a graph G a *forbidden minor* for $\mathcal{P}$ if G does not have property $\mathcal{P}$, but each proper minor of G does have property $\mathcal{P}$. Note that a minor-closed property $\mathcal{P}$ is completely characterized by the collection of its forbidden minors. Now Robertson and Seymour's theorem states that each minor-closed graph property has only finitely many forbidden minors.

We have seen that $\mu(K_n) = n - 1$ for each n. Let $\eta(G)$ denote the Hadwiger number of G, i.e., the size of the largest clique minor of G. Then by 1.3 we have that

$$\mu(G) \geq \eta(G) - 1$$

for all graphs G. Hence Hadwiger's conjecture would imply that $\chi(G) \leq \mu(G) + 1$ (where $\chi(G)$ denotes the chromatic number of G). This inequality was conjectured by Colin de Verdière [7]. Since Hadwiger's conjecture holds for graphs not containing any K_6-minor (Robertson, Seymour, and Thomas [26]), we know that $\chi(G) \leq \mu(G) + 1$ holds if $\mu(G) \leq 4$. An even weaker conjecture would be that $\vartheta(\overline{G}) \leq \mu(G) + 1$. Here ϑ is the graph invariant introduced in [18] (cf. also [10]). Since ϑ is defined in terms of vector labellings and positive semidefinite matrices, it is quite close in spirit to μ (cf. Sections 3.1, 3.2).

The following results show that with the help of $\mu(G)$, topological properties of a graph can be characterized algebraically:

1.4.

(i) *$\mu(G) \leq 1$ if and only if G is a disjoint union of paths.*

(ii) *$\mu(G) \leq 2$ if and only if G is outerplanar.*

(iii) *$\mu(G) \leq 3$ if and only if G is planar.*

(iv) *$\mu(G) \leq 4$ if and only if G is linklessly embeddable.*

Here (i), (ii), and (iii) are due to Colin de Verdière [7]. In (iv), direction $\Longrightarrow$ is due to Robertson, Seymour, and Thomas [25] (based on the hard theorem of [27] that the Petersen family (Figure 2 in Section 4.3) is the collection of forbidden minors for linkless embeddability), and direction $\Longleftarrow$ to Lovász and Schrijver [20]. In fact, in 1.4 each $\Longrightarrow$ follows from a forbidden minor characterization of the corresponding statement on the right hand side. It would be very interesting to find a direct proof of any of these implications.

In Sections 4.1 and 4.2 we give proofs of (i), (ii), and (iii), and in Section 4.3 we prove (iv), with the help of a certain Borsuk-type theorem on the existence of 'antipodal links'.

The proof by Colin de Verdière [7] of the planarity characterization 1.4 (iii) uses a result of Cheng [5] on the maximum multiplicity of the second eigenvalue of Schrödinger operators defined on the sphere. A short direct proof was given by van der Holst [11], based on a lemma that has other applications and also has motivated other research (see Section 2.5).

Kotlov, Lovász, and Vempala [16] studied graphs for which μ is close to the number of nodes n. They characterized graphs with $\mu(G) \geq n-3$. They also found that the value $n - \mu(G)$ is closely related to the outerplanarity and planarity of the complementary graph. In fact the following was proved.

1.5.
If $\overline{G}$ is a disjoint union of paths then $\mu(G) \geq n - 3$;
if $\overline{G}$ is outerplanar then $\mu(G) \geq n - 4$;
if $\overline{G}$ is planar then $\mu(G) \geq n - 5$.

Conversely, if G does not have 'twin nodes' (two — adjacent or nonadjacent — nodes u, v that have the same neighbors other than u, v), then:

1.6.
If $\mu(G) \geq n - 3$ then $\overline{G}$ is outerplanar;
if $\mu(G) \geq n - 4$ then $\overline{G}$ is planar.

Note that there is a gap of 1 between the necessary and sufficient conditions in terms of μ for, say, planarity. It turns out that in many cases, planarity of the complement implies the stronger inequality $\mu(G) \geq n - 4$. This is the case, for example, if G has a node-transitive automorphism group. Furthermore, at least among maximal planar graphs (triangulations of the sphere), one can characterize the exceptions in terms of small separating cycles. Details of these results are presented in Chapter 5.

The key to these results are two representations of a graph G by vectors in a euclidean space, derived from a Colin de Verdière matrix M. The two representations are in a sense dual to each other. It turns out that both representations have very nice geometric properties. One of these is closely related to the null space of M; the other, to the range. The latter is best formulated in terms of the complementary graph $H = \overline{G}$. We consider vector representations (labellings) of the nodes of H such that adjacent nodes are labelled by vectors with inner product 1, nonadjacent nodes are labelled with vectors with inner product less than 1; we call these *scalar product labellings* (see Chapter 3).

In dimension 3, scalar product labellings give a picture that is related to a classical construction going back to Koebe:

1.7. The Cage Theorem. *Let H be a 3-connected planar graph. Then H can be represented as the skeleton of a 3-dimensional polytope, all whose edge touch the unit sphere.*

A common generalization of scalar product and "cage" representations can be formulated, not so much for its own sake but, rather, to allow us to take the representation in the Cage Theorem and transform it continuously into a representation with the properties we need.

The Cage Theorem is equivalent to a labelling of the nodes of the graph by touching circles. Scalar product labellings with vectors longer than 1 are equivalent to labellings by spheres so that adjacent nodes correspond to orthogonal spheres. One way to look at the method is that we consider labellings where adjacent nodes are labelled by spheres intersecting at a given angle. In dimension 2, such representations were studied by Andre'ev [1] and Thurston [31], generalizing Koebe's theorem. Sphere labellings give rise to a number of interesting geometric questions, which we do not survey in this paper, but refer to [16].

Finally, the definition of μ in terms of vector representations leads to a reformulation of the Strong Arnold Property. It turns out that for $\mu \geq n-4$, this property is automatically fulfilled; the proof of this fact depends on an extension, due to Whiteley, of the classical theorem of Cauchy on the rigidity of 3-polytopes (see Section 5.1).

2. Basic facts

2.1. Transversality and the Strong Arnold Property

Let $M_1, \ldots, M_k$ be smooth open manifolds embedded in $\mathbb{R}^d$, and let x be a common point of them. We say that $M_1, \ldots, M_k$ intersect *transversally* at x if their normal spaces $N_1, \ldots, N_k$ at x are independent (meaning that no N_i shares a non-zero vector with the linear span of the others). In other words, no matter how we select a normal vector n_i of M_i at x for each $1 \le i \le k$, these normal vectors are linearly independent.

Transversal intersections are nice because near them the manifolds behave like affine subspaces. We will need the following simple geometric fact (a version of the Implicit Function Theorem), which we state without proof. A *smooth family of manifolds* $M(t)$ in $\mathbb{R}^d$ is defined by a smooth function $f : U \times (-1, 1) \to \mathbb{R}^d$, where U is an open set in $\mathbb{R}^t$ $(t \le d-1)$, and for each $-1 < t < 1$, the function $f(\,.\,, t)$ is a diffeomorphism between U and the manifold $M(t)$.

Lemma 2.1. *Let $M_1(t), \ldots, M_k(t)$ be smooth families of manifolds in $\mathbb{R}^d$ and assume that $M_1(0), \ldots, M_k(0)$ intersect transversally at x. Then there is a neighborhood $W \subseteq \mathbb{R}^k$ of the origin such that for each $\varepsilon \in W$, the manifolds $M_1(\varepsilon_1), \ldots, M_k(\varepsilon_k)$ intersect transversally at a point $x(\varepsilon)$ so that $x(0) = x$ and $x(\varepsilon)$ depends continuously on ε.*

The following corollary of this lemma will be sometimes easier to apply:

Corollary 2.2. *Assume that $M_1, \ldots, M_k$ intersect transversally at x, and assume that they have a common tangent vector v at x with $\|v\| = 1$. Then for every $\varepsilon > 0$ there exists a point $x' \ne x$ such that $M_1, \ldots, M_k$ intersect transversally at x', and*

$$\left\| \frac{1}{\|x - x'\|}(x - x') - v \right\| < \varepsilon.$$

Now we come to the Strong Arnold Property. For a given matrix $M \in \mathbb{R}^{(n)}$, let $\mathcal{R}_M$ be the set of all matrices $A \in \mathbb{R}^{(n)}$ with the same signature (i.e., the same number of positive, negative and 0 eigenvalues) as M. Let $\mathcal{S}_M$ be the set of all matrices $A \in \mathbb{R}^{(n)}$ such that $A_{i,j}$ and $M_{i,j}$ have the same sign (positive, negative or 0) for every $i \ne j$. (We could consider, without changing this definition, symmetric matrices with the same rank as M, and with the same pattern of 0's outside the diagonal as M.) Then M has the Strong Arnold Property (M3) if and only if

2.3. *$\mathcal{R}_M$ intersects $\mathcal{S}_M$ at M transversally.*

Indeed, it is well known that the tangent space of $\mathcal{R}_M$ at M consists of matrices $N \in \mathbb{R}^{(n)}$ such that $x^T N x = 0$ for each $x \in \ker(M)$. This is the space of all matrices of the form $WM + MW^T$, where W is any $n \times n$ matrix. Thus the normal space of $\mathcal{R}_M$ at M is equal to the space generated by all matrices xx^T with $x \in \ker(M)$. This space is equal to the space of all symmetric $n \times n$ matrices X satisfying $MX = 0$. Trivially, the normal space of $\mathcal{S}_M$ at M consists of all matrices $X = (X_{i,j}) \in \mathbb{R}^{(n)}$ such that $X_{i,j} = 0$ whenever $i = j$ or $M_{i,j} \neq 0$. Therefore, 2.3 is equivalent to (M3).

2.2. Monotonicity and components

We start with proving the very important fact that $\mu(G)$ is minor-monotone (Colin de Verdière [8]). The proof is surprisingly nontrivial!

Theorem 2.4. *If H is a minor of G, then $\mu(H) \leq \mu(G)$.*

Proof. Let M be a Colin de Verdière matrix for the graph H; we construct a Colin de Verdière matrix M' for the graph G with $\operatorname{corank}(M') \geq \operatorname{corank}(M)$.

It suffices to carry out this construction in three cases: when H arises from G by deleting an edge, by deleting an isolated node, or by contracting an edge.

Suppose that H is obtained by deleting an edge $e = uw$. By (M3), the two smooth open manifolds $\mathcal{R}_M$ and $\mathcal{S}_M$ embedded in $\mathbb{R}^{(n)}$ intersect tranversally. Let $\mathcal{S}(\varepsilon)$ be the manifold obtained from $\mathcal{S}$ by replacing, in each matrix in $\mathcal{S}$, the 0's in positions (u, w) and (w, u) by $-\varepsilon$. Then by Lemma 2.1, for a sufficiently small positive ε, $\mathcal{S}(\varepsilon)$ intersects $\mathcal{R}_M$ transversally at some point M'. Now it is trivial that M' is a Colin de Verdière matrix for G trivially.

Next, assume that H arises by deleting an isolated node v. Let M' be the $n \times n$ matrix arising from M by adding 0's, except in position (v, v), where $M'_{v,v} = 1$. Then trivially M' is a Colin de Verdière matrix for G.

Finally, suppose that H arises from G by contracting an edge $e = uw$. It will be convenient to assume that the nodes of H are $\{2, \dots, n\}$, where 2 is the new node. Let P be the matrix obtained from M by adding a first row and column, consisting of 0's except in position $(1, 1)$ where $P_{1,1} = 1$. We may assume that u is adjacent in G to nodes $3, \dots, r$, and that v is not adjacent to these nodes (we may delete the edges between v and these nodes).

Now consider symmetric $n \times n$ matrices A with the following properties:

(a) $A_{i,j} = 0$ for all $i, j > 1$, $i \neq j$, $ij \notin E(H)$, and also $A_{1,j} = 0$ for $j = 2$ and $j > r$;

(b) $\mathrm{rank}(A) = \mathrm{rank}(P)$;

(c) the rank of the submatrix formed by the first two columns and rows $3, \ldots, r$ is 1.

Each of these constraints defines a smooth manifold in $\mathbb{R}^{(n)}$, and each of these manifolds contains P. We claim that they intersect transversally at P. To this end, let us work out their normal spaces at P.

The normal space at P of the manifold $\mathcal{M}_a$ of all matrices satisfying (a) is, trivially, the linear space of matrices X such that $X_{i,j} = 0$ for $ij \in H$ and for $i = 1$ and $j \in \{1, 3, 4, \ldots, r\}$, or vice versa. In other words, these matrices have the shape

$$X = \begin{pmatrix} 0 & x_2 & 0 & \ldots & 0 & x_{r+1} & \ldots & x_n \\ x_2 & & & & & & & \\ 0 & & & & & & & \\ \vdots & & & & & & & \\ 0 & & & & & & & \\ x_{r+1} & & & & & X' & & \\ \vdots & & & & & & & \\ x_n & & & & & & & \end{pmatrix}$$

where X' is as in condition (M3) for H.

The normal space at P of the manifold $\mathcal{M}_b$ of all matrices satisfying (b) consists of all matrices Y such that $PY = 0$. These matrices have the shape

$$Y = \begin{pmatrix} 0 & 0 \\ 0 & Y' \end{pmatrix}$$

where Y' is a symmetric $(n-1) \times (n-1)$ matrix such that $MY' = 0$.

Finally, we show that the normal space at P of the manifold $\mathcal{M}_c$ of all matrices satisfying (c) consists of all symmetric matrices Z with $Z_{i,j} = 0$ unless $i = 1$ and $3 \leq j \leq r$, or vice versa; and, in addition, we have

$$\sum_{j=3}^{r} Z_{1,j} M_{2,j} = 0.$$

Thus Z has the shape

$$Z = \begin{pmatrix} 0 & 0 & z_3 & \dots & z_r & 0 & \dots & 0 \\ 0 & & & & & & & \\ z_3 & & & & & & & \\ \vdots & & & & & & & \\ z_r & & & & & & & \\ 0 & & & & & 0 & & \\ \vdots & & & & & & & \\ 0 & & & & & & & \end{pmatrix}$$

Indeed, let $\mathcal{L}$ denote the set of matrices in $\mathbb{R}^{(n)}$ that are 0 outside the rectangles $1 \le i \le 2$, $3 \le j \le r$ and $3 \le i \le r$, $1 \le j \le 2$. Then M_c is invariant under translation by any matrix orthogonal to $\mathcal{L}$, and hence Z must be in $\mathcal{L}$. Let K denote the projection of $\mathbb{R}^{(n)}$ onto the submatrix formed by rows $1, 2$ and columns $3, \dots, r$. For every matrix $B \in \mathbb{R}^{2\times(r-2)}$ with rank 1, the matrix

$$B' = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

is in M_c, and hence KZ is orthogonal at KP to the manifold of all $2\times(r-2)$ matrices of rank one. It is easy to see that the tangent space at P of this manifold consists of all $2 \times (r-2)$ matrices whose first row is parallel to the second row of KP. This easily implies the description above.

Now assume that nonzero matrices X, Y and Z of the above form are linearly dependent. Since the nonzeros in Z are zeros in X and Y, the coefficient of Z must be 0. But then we can assume that $X' = Y'$, which implies that X' has the pattern of 0's as in (M3) for the graph H, and so we have $X' = 0$. Therefore $Y' = 0$, implying $Y = 0$, and so $X = 0$, which is a contradiction. This proves that $\mathcal{M}_a$, $\mathcal{M}_b$ and $\mathcal{M}_c$ intersect transversally at P.

Also note that the matrix

$$T = \begin{pmatrix} 0 & 0 & M_{2,3} & \dots & M_{2,r} & 0 & \dots & 0 \\ 0 & & & & & & & \\ M_{2,3} & & & & & & & \\ \vdots & & & & & & & \\ M_{2,r} & & & & & & & \\ 0 & & & & & 0 & & \\ \vdots & & & & & & & \\ 0 & & & & & & & \end{pmatrix}$$

is orthogonal to every matrix of the form X, Y or Z, and hence it is a common tangent to each of the manifolds at P. Hence by Corollary 2.2, there is matrix P' in the intersection of the three manifolds such that the three manifolds intersect transversally at P' and $P' - P$ is "almost parallel" to T. It follows that $P'_{1,j} < 0$ for $3 \le j \le r$, and elsewhere P' has the same signs as P. Also, P' has the same signature as P; in particular, P' has exactly one negative eigenvalue and $\operatorname{rank}(P') = \operatorname{rank}(P) = \operatorname{rank}(M) + 1$.

Now (c) implies that we can subtract an appropriate multiple of the first row of P' from the second to get 0's in positions $(2,3), \ldots, (2,r)$. Doing the same with the first column, we get a matrix $M' \in \mathbb{R}^{(n)}$ that satisfies (M1) for the graph G. By Sylvester's Inertia Theorem (cf. Section 5.5), M' also satisfies (M2) and has $\operatorname{rank}(M') = \operatorname{rank}(M) + 1$, i.e., $\operatorname{corank}(M') = \operatorname{corank}(M)$. Finally, M' has the Strong Arnold Property, which follows easily from the fact that M_a, M_b and M_c intersect transversally at P'. ∎

The previous theorem implies, via the Robertson–Seymour Graph Minor Theorem, that, for each t, the property $\mu(G) \le t$ can be characterized by a finite number of excluded minors. One of these is the graph K_{t+2}, which, as we have seen, has $\mu(K_{t+2}) = t + 1$. The fact that K_{t+2} is minor-minimal with respect to this property follows from 1.2. We will be able to give topological characterizations of the property $\mu(G) \le t$ for $t \le 4$, and thereby determine the complete list of excluded minors in these cases.

Using the previous theorem, it will be easy to prove the following:

Theorem 2.5. *If G has at least one edge, then*

$$\mu(G) = \max_K \mu(K),$$

where K extends over the components of G.

(Note that the theorem fails if G has no edge, since we have seen that $\mu(K_1) = 0$ but $\mu(\overline{K_n}) = 1$ for $n \ge 2$.)

Proof. By Theorem 2.4 we know that $\ge$ holds. To see equality, let M be an optimal Colin de Verdière matrix for G. Since G has at least one edge, we know that $\mu(G) > 0$ (since $\mu(K_2) = 1$), and hence $\operatorname{corank}(M) > 0$. Then there is exactly one component K of G with $\operatorname{corank}(M_K) > 0$. For suppose that there are two such components, K and L. Choose nonzero vectors $x \in \ker(M_K)$ and $y \in \ker(M_L)$. Extend x and y by zeros on the positions not in K and L, respectively. Then the matrix $X := xy^T + yx^T$ is nonzero and symmetric, has zeros in positions corresponding to edges of

G and in the diagonal, and satisfies $MX = 0$. This contradicts the Strong Arnold Property.

Let K be the component with $\mathrm{corank}(M_K) > 0$. Then $\mathrm{corank}(M) = \mathrm{corank}(M_K)$. Suppose now that M_K has no negative eigenvalue. Then 0 is the smallest eigenvalue of M_K, and hence, by the connectivity of K and the Perron-Frobenius theorem, $\mathrm{corank}(M_K) = 1$. So $\mu(G) = 1$. Let L be a component of G with at least one edge. Then $\mu(L) \geq 1$, proving the assertion.

So we can assume that M_K has one negative eigenvalue. One easily shows that M_K has the Strong Arnold Property, implying $\mu(G) = \mu(K)$, thus proving the assertion again. ∎

The following remark simplifies some arguments in the sequel.

2.6. *If G has at least two nodes, then we can replace condition* (M2) *by* (M2'): *M has at most one negative eigenvalue.*

Indeed, suppose that the matrix M minimizing the rank, subject to (M1), (M2') and (M3), has no negative eigenvalue, that is, is positive semidefinite. Then by the Perron-Frobenius Theorem, the submatrix corresponding to any component has corank at most 1. By the same argument as in the proof of Theorem 2.5, at most one of these submatrices is singular. Thus M has corank at most 1. But we know that we can do at least this well under (M2) instead of (M2').

Next we prove:

Theorem 2.7. *Let $G = (V, E)$ be a graph and let $v \in V$. Then*

$$\mu(G) \leq \mu(G - v) + 1.$$

If v is connected to all other nodes, and $G - v$ is not $\overline{K_2}$ or empty, then equality holds.

Proof. We may assume that $v = n$. To prove the first assertion, we can assume that G is connected. Let M be an optimal Colin de Verdière matrix for G. Let M' be obtained by deleting the last row and column of M. Clearly, $\mathrm{corank}(M') \geq \mathrm{corank}(M) - 1$, since $\mathrm{rank}(M') \leq \mathrm{rank}(M)$. So it suffices to show that M' is a Colin de Verdière matrix for $G - v$. As the theorem clearly holds if $\mu(G) \leq 2$, we may assume that $\mu(G) \geq 3$.

Trivially, M' satisfies (M1), and it has at most one negative eigenvalue by the theorem on interlacing eigenvalues. By remark, it suffices to show that it satisfies (M3).

As an intermediate step, we show that corank$(M') \leq$ corank(M). If M' has a negative eigenvalue, this is immediate from eigenvalue interlacing. If M' is positive semidefinite, then by the Perron-Frobenius Theorem, the corank of the submatrix corresponding to each component of $G - v$ is at most 1, and hence the corank of M' is the number of such submatrices that are singular.

We claim that there are at most 3 such submatrices. Let $K_1, \ldots, K_4$ be four such components. For $i = 1, \ldots, 4$, let x_i be a nonzero vector with $M_{K_i} x_i = 0$. By the Perron-Frobenius theorem we know that we can assume $x_i > 0$ for each i. Extend x_i to a vector in $\mathbb{R}^V$ by adding components 0.

Let z be a positive eigenvector of M belonging to the negative eigenvalue. By scaling the x_i we can assume that $z^T x_i = 1$ for each i. Now define

$$X := (x_1 - x_2)(x_3 - x_4)^T + (x_3 - x_4)(x_1 - x_2)^T.$$

Then $(x_1 - x_2)^T M (x_1 - x_2) = 0$ and $x_1 - x_2$ is orthogonal to z, hence $M(x_1 - x_2) = 0$. Similarly $M(x_3 - x_4) = 0$, and thus $MX = 0$ This contradicts the fact that M satisfies (M3).

Thus corank$(M') \leq 3 \leq$ corank(M). In other words, rank$(M') \geq$ rank$(M) - 1$. This implies easily that the last row of M is a linear combination of the other rows and the vector e_n^T.

To see that M' has the Strong Arnold Property (M3), let X' be an $(n-1) \times (n-1)$ matrix with 0's in positions (i,j) where $i = j$ or i and j are adjacent, and satisfying $M'X' = 0$. We must show that $X' = 0$. Let X be the $n \times n$ matrix obtained from X' by appending 0's. Then $MX = 0$; this is straightforward except when multiplying by the first row of M, where we can use its representation as a linear combination of the other rows and e_n^T. This proves the first assertion.

Now we show that if v is connected to all other nodes, then $\mu(G) \geq \mu(G - v) + 1$ unless G is a path with 2 or 3 nodes. If $G - v$ has no edge, then this is easily checked, so suppose that it does. Then by Theorem 2.5 we may assume that $G - v$ is connected and has at least 2 nodes. Let M' be an optimal Colin de Verdière matrix for $G - v$. Let z be an eigenvector of M' belonging to the smallest eigenvalue λ_1. We can assume that $z < 0$ and that $\|z\| = 1$. Let M be the matrix

$$M := \begin{pmatrix} M' & z \\ z^T & \lambda_1^{-1} \end{pmatrix}.$$

Since $(x, 0)^T \in \ker(M)$ for each $x \in \ker(M')$ and since $(z, -\lambda_1)^T \in \ker(M)$, we know that corank$(M) \geq$ corank$(M') + 1$. By eigenvalue interlacing it follows that M has exactly one negative eigenvalue. One easily checks that M has the Strong Arnold Property (M3). ■

We end this section with a lemma that gives very useful information about the components of an induced subgraph of G [13].

Lemma 2.8. *Let $G = (V, E)$ be a connected graph and let M be a Colin de Verdière matrix for G. Let $S \subseteq V$ and let $C_1, \dots, C_m$ be the components of $G - S$. Then there are three alternatives:*

(i) *there exists an i with $\lambda_1(M_{C_i}) < 0$, and $\lambda_1(M_{C_j}) > 0$ for all $j \neq i$;*

(ii) *$\lambda_1(M_{C_i}) \geq 0$ for all i, $\operatorname{corank}(M) \leq |S| + 1$, and there are at least $\operatorname{corank}(M) - |S| + 2$ and at most three components C_i with $\lambda_1(M_{C_i}) = 0$;*

(iii) *$\lambda_1(M_{C_i}) > 0$ for all i.*

Proof. Let z be an eigenvector belonging to the smallest eigenvalue of M, and, for $i = 1, \dots, m$, let x_i be an eigenvector belonging to the smallest eigenvalue of M_{C_i}, extended by 0's to obtain a vector in $\mathbb{R}^V$. We can assume that $z > 0$, and $x_i \geq 0$ and $z^T x_i = 1$ for $i = 1, \dots, m$.

By the Interlacing Eigenvalues Theorem, at most one component has a negative eigenvalue. Assume that $\lambda_1(M_{C_1}) < 0$. We claim that (i) holds. Otherwise, we have $\lambda_1(M_{C_2}) \leq 0$ (say). Let $y := x_1 - x_2$. Then $z^T y = z^T x_1 - z^T x_2 = 0$ and $y^T M y = x_1^T M x_1 + x_2^T M x_2 < 0$. But $z^T y = 0$ and $y^T M y < 0$ imply that $\lambda_2(M) < 0$, contradicting (M1)–(M3).

So we may assume that $\lambda_1(M_{C_i}) \geq 0$, that is, M_{C_i} is positive semidefinite for each i. Suppose that (iii) does not hold, say $\lambda_1(M_{C_1}) = 0$. Let D be the vector space of all vectors $y \in \ker(M)$ with $y_s = 0$ for all $s \in S$. Then

2.9. *for each vector $y \in D$ and each component C_i of $G - S$, $y_{C_i} = 0$, $y_{C_i} > 0$ or $y_{C_i} < 0$; if moreover $\lambda_1(M_{C_i}) > 0$ then $y_{C_i} = 0$.*

Indeed, if $y \in D$, then $M_{C_i} y_{C_i} = 0$. Hence if $y_{C_i} \neq 0$, then (as M_{C_i} is positive semidefinite) $\lambda_1(M_{C_i}) = 0$ and y_{C_i} is an eigenvector belonging to $\lambda_1(M_{C_i})$, and hence by the Perron-Frobenius theorem, $y_{C_i} > 0$ or $y_{C_i} < 0$.

Let m' be the number of components C_i with $\lambda_1(M_{C_i}) = 0$. By 2.9, $\dim(D) \leq m' - 1$ (since each nonzero $y \in D$ has both positive and negative components, as it is orthogonal to z).

Since $\lambda_1(M_{C_1}) = 0$, there exists a vector $w > 0$ such that $M_{C_1} w = 0$. Let

$$F := \{x_S \mid x \in \ker(M)\}.$$

Suppose that $\dim(F) = |S|$. Let j be a node in S adjacent to C_1. Then there is a vector $y \in \ker(M)$ with $y_j = -1$ and $y_i = 0$ if $i \in S \setminus \{j\}$. Let u be the jth column of M. Then $My = 0$ implies that $u_{C_1} = M_{C_1}y_{C_1}$. Since $u_{C_1} \leq 0$ and $u_{C_1} \neq 0$, we have $0 > u_{C_1}^T w = y_{C_1}^T M_{C_1} w = 0$, a contradiction.

Hence $\dim(F) \leq |S| - 1$, and so

$$(1) \qquad m' - 1 \geq \dim(D) = \operatorname{corank}(M) - \dim(F) \geq \operatorname{corank}(M) - |S| + 1.$$

Hence there are at least $\operatorname{corank}(M) - |S| + 2$ components C_i with $\lambda_1(M_{C_i}) = 0$. To see that there are at most three such components, assume that $\lambda_1(M_{C_i}) = 0$ for $i = 1, \ldots, 4$. Define $X := (x_1 - x_2)(x_3 - x_4)^T + (x_3 - x_4)(x_1 - x_2)^T$. Then $X_{i,j} \neq 0$ implies $i \in C_1 \cup C_2$ and $j \in C_3 \cup C_4$ or conversely. As $MX = 0$, this contradicts the Strong Arnold Property (M3). ∎

2.3. Clique sums

A graph $G = (V, E)$ is a *clique sum* of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ if $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ and $V_1 \cap V_2$ is a clique both in G_1 and in G_2. Writing a graph as the clique sum of smaller graphs often provides a way to compute its parameters. For example, for the chromatic number χ one has $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$ if G is a clique sum of G_1 and G_2. A similar relation holds for the size of the largest clique minor (the Hadwiger number) of a graph.

We therefore are interested in studying the behaviour of $\mu(G)$ under clique sums. A critical example is the graph $K_{t+3} \setminus \Delta$ (the graph obtained from the complete graph K_{t+3} by deleting the edges of a triangle). One has $\mu(K_{t+3} \setminus \Delta) = t + 1$ (since the star $K_{1,3} = K_4 \setminus \Delta$ has $\mu(K_{1,3}) = 2$, and adding a new node adjacent to all existing nodes increases μ by 1).

However, $K_{t+3} \setminus \Delta$ is a clique sum of K_{t+1} and $K_{t+2} \setminus e$ (the graph obtained from K_{t+2} by deleting an edge), with common clique of size t. Both K_{t+1} and $K_{t+2} \setminus e$ have $\mu = t$. So, generally one does not have that, for fixed t, the property $\mu(G) \leq t$ is maintained under clique sums. Similarly, $K_{t+3} \setminus \Delta$ is a clique sum of two copies of $K_{t+2} \setminus e$, with common clique of size $t + 1$.

These examples where μ increases by taking a clique sum are in a sense the only cases:

Theorem 2.10. *Let $G = (V, E)$ be a clique sum of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, let $S := V_1 \cap V_2$, and $t := \max\{\mu(G_1), \mu(G_2)\}$. If $\mu(G) > t$, then $\mu(G) = t + 1$ and we can contract two or three components of $G - S$ so that the contracted nodes together with S form a $K_{t+3} \setminus \Delta$.*

Proof. We apply induction on $|V| + |S|$. Let M be an optimal Colin de Verdière matrix for G. We first show that $\lambda_1(M_C) \geq 0$ for each component C of $G - S$. Suppose $\lambda_1(M_C) < 0$. Then by Lemma 2.8, $\lambda_1(M_{C'}) > 0$ for each component C' other than C. Let G' be the subgraph of G induced by $C \cup S$; so G' is a subgraph of G_1 or G_2. Let L be the union of the other components; so $\lambda_1(M_L) > 0$. Write

$$M = \begin{pmatrix} M_C & U_C & 0 \\ U_C^T & M_S & U_L \\ 0 & U_L^T & M_L \end{pmatrix}.$$

Let

$$A := \begin{pmatrix} I & 0 & 0 \\ 0 & I & -U_L M_L^{-1} \\ 0 & 0 & I \end{pmatrix}.$$

Then by Sylvester's Inertia Theorem, the matrix

$$AMA^T = \begin{pmatrix} M_C & U_C & 0 \\ U_C^T & M_S - U_L M_L^{-1} U_L^T & 0 \\ 0 & 0 & M_L \end{pmatrix}.$$

has the same signature as M; that is AMA^T has exactly one negative eigenvalue and has the same corank as M. As M_L is positive definite, the matrix

$$M' := \begin{pmatrix} M_C & U_C \\ U_C^T & M_S - U_L M_L^{-1} U_L^T \end{pmatrix}$$

has exactly one negative eigenvalue and has the same corank as M. Since $(M_L)_{i,j} \leq 0$ if $i \neq j$, we know that $(M_L^{-1})_{i,j} \geq 0$ for all i, j. Indeed, for any symmetric positive-definite matrix D, if each off-diagonal entry of D is nonpositive, then each entry of D^{-1} is nonnegative. This can be seen directly, and also follows from the theory of 'M-matrices' (cf. [17] Section 15.2). Without loss of generality, each diagonal entry of D is at most 1. Let $B := I - D$. So $B \geq 0$ and the largest eigenvalue of B is equal to $1 - \lambda_1(D) < 1$. Hence $D^{-1} = I + B + B^2 + B^3 + \cdots \geq 0$ (cf. Theorem 2 in Section 15.2 of [17]).

Hence $(M'_S)_{i,j} \leq (M_S)_{i,j} < 0$ for all $i, j \in S$ with $i \neq j$. Thus M' satisfies (M1) and (M2) with respect to G'.

The matrix M' also has the Strong Arnold Property (M3). To see this, let X' be a symmetric matrix with $M'X' = 0$ and $X'_{i,j} = 0$ if i and j are adjacent or if $i = j$. As S is a clique, we can write

$$X' = \begin{pmatrix} X'_C & Y \\ Y^T & 0 \end{pmatrix}.$$

Let $Z := -YU_LM_L^{-1}$ and

$$X := \begin{pmatrix} X'_C & Y & Z \\ Y^T & 0 & 0 \\ Z^T & 0 & 0 \end{pmatrix}.$$

Then X is a symmetric matrix with $X_{i,j} = 0$ if i and j are adjacent or if $i = j$, and $MX = 0$. So $X = 0$ and hence $X' = 0$.

It follows that $\mu(G') \geq \text{corank}(M') = \text{corank}(M) = \mu(G) > t$, a contradiction, since G' is a subgraph of G_1 or G_2.

So we have that $\lambda_1(M_C) \geq 0$ for each component C of $G - S$. Suppose next that $N(C) \neq S$ for some component C of $G - S$.

Assume that $C \subseteq VG_1$. Let H_1 be the graph induced by $C \cup N(C)$ and let H_2 be the graph induced by the union of all other components and S. So G is also a clique sum of H_1 and H_2, with common clique $S' := N(C)$, and H_2 is a clique sum of $G_1 - C$ and G_2.

If $\mu(G) = \mu(H_2)$, then $\mu(H_2) > t' := \max\{\mu(G_1 - C), \mu(G_2)\}$. As $|VH_2| + |S| < |VG| + |S|$, by induction we know that $\mu(H_2) = t' + 1$, and thus $\mu(G) = \mu(H_2) = t' + 1 \leq t + 1$. Thus $t' = t$ and $\mu(G) = t + 1$. Moreover, either $|S| = t + 1$ and $H_2 - S$ has two components C', C'' with $N(C') = N(C'')$ and $|N(C')| = t$, or $|S| = t$ and $H_2 - S$ has three components C' with $N(C') = S$, and the theorem follows.

If $\mu(G) > \mu(H_2)$, then $\mu(G) > t' := \max\{\mu(H_1), \mu(H_2)\}$. As $|VG| + |S'| < |VG| + |S|$, we know that $\mu(G) = t' + 1$, implying $t' \geq t$, and that either $|S'| = t'+1$ or $|S'| = t'$. However, $|S'| < |S| \leq t+1 \leq t'+1$, so $|S'| = t'$ and $t' = t$. Moreover, $G - S'$ has three components C' with $N(C') = S'$. This implies that $G - S$ has two components C' with $N(C') = S'$, and the theorem follows.

So we may assume that $N(C) = S$ for each component C. If $|S| > t$, then G_1 would contain a K_{t+2} minor, contradicting the fact that $\mu(G_1) \leq t$. So $|S| \leq t$. Since $\text{corank}(M) > |S|$, we have $\lambda_1(M_C) = 0$ for at least one component C of $G - S$. Hence, by (ii) of Lemma 2.8, $G - S$ has at least $\operatorname{corank}(M) - |S| + 2 = \mu(G) - |S| + 2 \geq 3$ components C with $\lambda_1(M_C) = 0$, and by (iii) of Lemma 2.8, $\mu(G) - |S| + 2 \leq 3$, that is $t \geq |S| \geq \mu(G) - 1 \geq t$. ∎

As direct consequence we have:

Corollary 2.11. *Let $G = (V, E)$ be a clique sum of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, and let $S := V_1 \cap V_2$. Then $\mu(G) = \max\{\mu(G_1), \mu(G_2)\}$ unless $\mu(G_1) = \mu(G_2)$ and $|S| \geq \mu(G_1)$.*

As an application of the tools developed in the previous sections, we determine μ for complete bipartite graphs $K_{p,q}$ $(p \leq q)$. We already know that $\mu(K_{p,q}) = p + q - 2$ if $2 \leq q \leq 3$. Now we prove:

$$\mu(K_{p,q}) = \begin{cases} p, & \text{if } q \leq 2, \\ p+1, & \text{if } q \geq 3. \end{cases} \tag{2}$$

The first line is just a reformulation of our findings in Section 1.2, and so is the second line in the case $q = 3$. Assume that $q > 3$. We have $\mu(K_{p,q}) \leq p + 1$ by Theorem 2.11, since $K_{p,q}$ is a subgraph of a clique sum of q copies of K_{p+1}. Since $\mu(K_{1,3}) = 2$, the equality holds for $p = 1$.

Now it is easy to show that equality holds for $p > 1$ as well. Contracting any edge in $K_{p,q}$ we get a graph that arises from $K_{p-1,q-1}$ by adding a new node connected to every old node. Hence by Theorem 2.7, we have

$$\mu(K_{p,q}) \geq \mu(K_{p-1,q-1}) + 1 = p + 1.$$

2.4. Subdivision and ΔY transformation

In this section we show that (except for small values), Colin de Verdière's parameter is invariant under two graph transformations crucially important in topological graph theory: subdivision and ΔY transformation.

In fact, subdivision is easily settled from our results in the previous sections.

Theorem 2.12. *Let G be a graph and let G' arise from G by subdividing an edge. Then*

(a) $\mu(G) \leq \mu(G')$;

(b) *If $\mu(G) \geq 3$ then equality holds.*

Proof. Since G is a minor of G', (a) is trivial by Theorem 2.4. Since G' is a subgraph of the clique sum of G and the triangle along an edge, equality holds if $\mu(G) > \mu(K_3) = 2$ by Corollary 2.11. ■

It should be remarked that the condition $\mu(G) \geq 3$ for equality cannot be dropped. The graph $K_4 - e$ obtained by deleting an edge from K_4 is the clique sum of two triangles, and hence by Corollary 2.11, $\mu(K_4 - e) = 2$. But subdividing the edge opposite to the deleted edge, we get $K_{2,3}$ which, as we know, has $\mu(K_{2,3}) = 3$.

Bacher and Colin de Verdière [2] proved that if μ is large enough, $\mu(G)$ is invariant under the ΔY and YΔ operations. The YΔ-operation works as follows: given a graph G, choose a node v of degree 3, make its three neighbors pairwise adjacent, and delete v and the three edges incident with v. The ΔY-operation is the converse: given a graph G, select a triangle, add a new node, connect it to the nodes of the triangle, and delete the edges of the triangle.

In fact, Corollary 2.11 implies:

Theorem 2.13. *Let G be a graph and let G' arise from G by applying a ΔY transformation to a triangle in G. Then*

(a) $\mu(G) \leq \mu(G')$;

(b) *if $\mu(G) \geq 4$ then equality holds.*

Proof. We start with (a). Let M be an optimal Colin de Verdière matrix for G. Let 1, 2 and 3 be the nodes in the triangle to which we apply the ΔY operation. Let 0 be the new node in G'. Write

$$M = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix},$$

where A has order 3×3.

It is not difficult to see that there exists a positive vector $b \in \mathbb{R}^3$ such that the matrix

$$D := A + bb^T$$

is a diagonal matrix of order 3. Define the $(n+1) \times (n+1)$ matrix L by

$$L := \begin{pmatrix} 1 & 0 & 0 \\ -b & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

where the 0 and I stand for null and identity matrices of appropriate sizes.

Define the $(n+1)\times(n+1)$ matrix M' by

$$M' := L\begin{pmatrix}1 & 0\\ 0 & M\end{pmatrix}L^T = \begin{pmatrix}1 & -b^T & 0\\ -b & D & B^T\\ 0 & B & C\end{pmatrix} \tag{3}$$

We claim that M' is a Colin de Verdière matrix for G' and $\operatorname{corank}(M') = \operatorname{corank}(M)$. Indeed, trivially by (3) (and Sylvester's Inertia Theorem), $\operatorname{corank}(M') = \operatorname{corank}(M)$, and M' has exactly one negative eigenvalue. Moreover, M' satisfies (M1). So it suffices to show that M' has the Strong Arnold Property. Suppose to the contrary that there exists a nonzero symmetric $(n+1)\times(n+1)$ matrix $X = (X_{i,j})$ such that $M'X = 0$ and such that $X_{i,j} = 0$ if $i = j$ or if i and j are adjacent in G'. We can write X as

$$X = \begin{pmatrix}0 & 0 & y^T\\ 0 & Y & Z^T\\ y & Z & W\end{pmatrix},$$

where Y is a 3×3 matrix. So

$$0 = \begin{pmatrix}1 & 0\\ 0 & M\end{pmatrix}L^T X = \begin{pmatrix}0 & -b^TY & y^T - b^TZ^T\\ B^Ty & AY + B^TZ & AZ^T + B^TW\\ Cy & BY + CZ & BZ^T + CW\end{pmatrix}.$$

Then $Y = 0$. Indeed, since $b^TY = 0$, we have

$$Y_{2,3} = -\frac{b_1}{b_3}Y_{1,2} = \frac{b_1}{b_2}Y_{1,3} = -Y_{1,2}.$$

Similarly, $Y_{2,3} = -Y_{1,3}$ and $Y_{1,2} = -Y_{1,3}$. Hence $Y = 0$.

Let

$$X' := \begin{pmatrix}0 & Z^T\\ Z & W\end{pmatrix}.$$

Then

$$MX' = \begin{pmatrix}A & B^T\\ B & C\end{pmatrix}\begin{pmatrix}0 & Z^T\\ Z & W\end{pmatrix} = \begin{pmatrix}B^TZ & AZ^T + B^TW\\ CZ & BZ^T + CW\end{pmatrix} = 0,$$

contradicting the fact that M has the Strong Arnold Property. This proves (a).

As H is a subgraph of a clique sum of G and K_4 along a triangle, (b) follows directly from Corollary 2.11. ■

2.5. The null space of a Colin de Verdière matrix

In this section we study the null space of a Colin de Verdière matrix The main result will be Theorem 2.16 due to van der Holst [11, 12] and its extensions, but some of the preliminary results leading up to it will also be useful.

For any vector x, let $\mathrm{supp}(x)$ denote the support of x (i.e., the set $\{i \mid x_i \neq 0\}$). Furthermore, we denote $\mathrm{supp}^+(x) := \{i \mid x_i > 0\}$ and $\mathrm{supp}^-(x) := \{i \mid x_i < 0\}$.

Let x be a vector in the null space of a Colin de Verdière matrix M for G. We assume that the graph G is connected. Hence an eigenvector z belonging to the unique negative eigenvalue of M is (say) positive. Since $x^T z = 0$, it follows that both $\mathrm{supp}^+(x)$ and $\mathrm{supp}^-(x)$ are nonempty.

Next, $Mx = 0$ implies:

2.14. *if a node $v \notin \mathrm{supp}(x)$ is adjacent to some node in $\mathrm{supp}^+(x)$, then it is also adjacent to some node in $\mathrm{supp}^-(x)$ (and conversely).*

Since we do not assume anything about the diagonal entries of M, the same argument does not give any information about the neighbors of a node in $\mathrm{supp}(x)$. However, the following lemma does give very important information.

Lemma 2.15. *Let G be a connected graph and let M be a Colin de Verdière matrix for G. Let $x \in \ker(M)$ and let J and K be two components of $G|\mathrm{supp}^+(x)$. Then there is a $y \in \ker(M)$ with $\mathrm{supp}^+(y) = J$ and $\mathrm{supp}^-(y) = K$, such that y_J and y_K are scalar multiples of x_J and x_K respectively.*

Proof. Let $L := \mathrm{supp}^-(x)$. Since $M_{j,k} = 0$ if $j \in J$, $k \in K$, we have:

$$(4) \qquad M_{J\times J}\, x_J + M_{J\times L}\, x_L = 0, \qquad M_{K\times K}\, x_K + M_{K\times L}\, x_L = 0.$$

Let z be an eigenvector of M with negative eigenvalue. By the Perron-Frobenius theorem we may assume $z > 0$. Let

$$(5) \qquad \lambda := \frac{z_J^T x_J}{z_K^T x_K}.$$

Define $y \in \mathbb{R}^n$ by: $y_i := x_i$ if $i \in J$, $y_i := -\lambda x_i$ if $i \in K$, and $y_i := 0$ if $i \notin J \cup K$. By 5, $z^T y = z_J^T x_J - \lambda z_K^T x_K = 0$. Moreover, $M_{j,k} = 0$ if $j \in J$ and $k \in K$ and hence

$$\begin{aligned} y^T M y &= y_J^T M_{J\times J}\, y_J + y_K^T M_{K\times K}\, y_K \\ &= x_J^T M_{J\times J}\, x_J + \lambda^2 x_K^T M_{K\times K}\, x_K \\ &= -x_J^T M_{J\times L}\, x_L - \lambda^2 x_K^T M_{K\times L}\, x_L \qquad \text{(using (4))} \\ &\leq 0, \end{aligned}$$

since $M_{J\times L}$ and $M_{K\times L}$ are nonpositive, and $x_J > 0$, $x_K > 0$ and $x_L < 0$.

Now $z^T y = 0$ and $y^T M y \leq 0$ imply that $My = 0$ (as M is symmetric and has exactly one negative eigenvalue, with eigenvector z). Therefore, $y \in \ker(M)$. ∎

We say that a vector $x \in \ker(M)$ *has minimal support* if x is nonzero and for each nonzero vector $y \in \ker(M)$ with $\operatorname{supp}(y) \subseteq \operatorname{supp}(x)$ one has $\operatorname{supp}(y) = \operatorname{supp}(x)$. Then Lemma 2.15 implies immediately the following theorem.

Theorem 2.16. *Let G be a connected graph and let M be a Colin de Verdière matrix for G. Let $x \in \ker(M)$ have minimal support. Then $G|\operatorname{supp}^+(x)$ and $G|\operatorname{supp}^-(x)$ are nonempty and connected.*

Unfortunately, the conclusion of Theorem 2.16 does not remain valid if the assumption that x has minimal support is dropped. For example, if our graph is $K_{1,3}$ and we take the matrix

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(which satisfies (M1)–(M3) and achieves $\mu = 2$), then the vector $x = (0, 1, 1, -2)$ is in the null space but $\operatorname{supp}^+(x)$ is disconnected.

The Petersen graph P provides a more interesting example. We show that $\mu(P) = 5$. It is easy to construct a matrix realizing this value: Let A be the adjacency matrix of the Petersen graph and let $M = I - A$. Clearly, M satisfies (M1). The eigenvalues of P are well known to be 3 (once), 1 (5 times) and -2 (4 times). Hence M satisfies (M2), and has corank 5. We leave it to the reader to verify that it also has the Strong Arnold Property.

It is easy to work out the null space of M. Let e and e' be two edges at distance 2, and define a vector $q_{ee'} \in \mathbb{R}^V$ to be 1 on the endnodes of e, -1

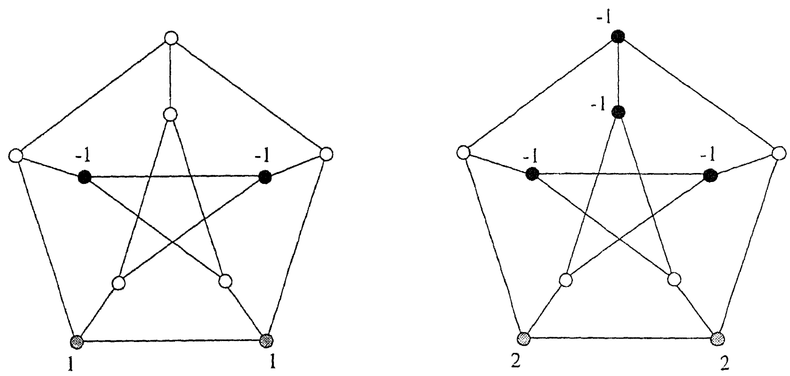

Fig. 1. Two vectors in the null space of M for the Petersen graph

on the endnodes of e', and 0 elsewhere (Figure 1(a)). Then $q_{ee'} \in \ker(M)$, and it is easy to see that $\ker(M)$ is generated by these vectors.

Now if e, e' and e'' are three edges that are mutually at distance 2, then $q_{ee'} + q_{ee''}$ is a vector in $\ker(M)$ with $\mathrm{supp}^-(q)$ having two components (Figure 1(b)).

It is interesting to study linear subspaces with properties similar to those of the null space of a Colin de Verdière matrix M. Given a graph G, what is the maximum dimension of a subspace $L \subseteq \mathbb{R}^V$ such that for every vector $x \in L$ with minimal support, the subgraph spanned by $\mathrm{supp}^+(x)$ is nonempty and connected?

We get a perhaps more interesting graph invariant if we consider the maximum dimension of a subspace $L \subseteq \mathbb{R}^V$ such that $\mathrm{supp}^+(x)$ is nonempty and connected for *every* nonzero vector in L. We denote this maximum dimension by $\lambda(G)$. This invariant was introduced and studied by van der Holst, Laurent and Schrijver [13]. For the interesting properties of these and other related graph invariants, we refer to [13] and to the forthcoming survey by Schrijver [30].

A vector x in the null space of M (not with minimal support) that violates the conclusion of Theorem 2.16 must have rather special properties:

Theorem 2.17. *Let M be a Colin de Verdière matrix for a graph G. Let $x \in \ker(M)$ be such that $G|\mathrm{supp}^+(x)$ is disconnected. Then*

(i) *there is no edge connecting $\mathrm{supp}^+(x)$ and $\mathrm{supp}^-(x)$,*

(ii) *$G|\mathrm{supp}^+(x)$ has exactly two components,*

(iii) *$G|\mathrm{supp}^-(x)$ is connected,*

(iv) $|\mathrm{supp}(x)| + \mathrm{corank}(M) \leq |V(G)| + 1$,

(v) $N(K) = N(\mathrm{supp}(x))$ *for each component* K *of* $G|\mathrm{supp}(x)$.

Proof. Suppose there is an edge connecting component J of $G|\mathrm{supp}^+(x)$ and $G|\mathrm{supp}^-(x)$. Let K be another component of $G|\mathrm{supp}^+(x)$. By Lemma 2.15, there exists a $y \in \ker(M)$ such that $\mathrm{supp}^+(y) = K$ and $\mathrm{supp}^-(y) = J$, and such that y_J and y_K are scalar multiples of x_J and x_K respectively. By adding an appropriate multiple of y to x we obtain a vector $z \in \ker(M)$ with $\mathrm{supp}^-(z) = \mathrm{supp}^-(x)$ and $\mathrm{supp}^+(z) = \mathrm{supp}^+(x) \setminus J$. Since there is no edge connecting J and $\mathrm{supp}^+(x) \setminus J$, this contradicts 2.14, and proves (i).

Let $C_1, \ldots, C_m$ be the components of $G|\mathrm{supp}(x)$. Then $M_{C_i} x_{C_i} = 0$ for each $i = 1, \ldots, m$. Since by (i) each C_i is contained either in $\mathrm{supp}^+(x)$ or in $\mathrm{supp}^-(x)$, we know by the Perron-Frobenius theorem that $\lambda_1(M_{C_i}) = 0$ for $i = 1, \ldots, m$. Hence case (ii) of Lemma 2.8 applies, and we see that $m \leq 3$, implying (ii) and (iii) above. Assertion (iv) also follows immediately from Lemma 2.8.

Similarly to the first part of this proof one shows that, for each two components J, K of $G|\mathrm{supp}(x)$, there is a vector $y \in \ker(M)$ with $\mathrm{supp}^+(y) = J$ and $\mathrm{supp}^-(y) = K$. Hence 2.14 implies that $N(J) = N(K)$. ■

3. Vector labellings

3.1. A semidefinite formulation

We give a reformulation of $\mu(G)$ in terms of positive semidefinite matrices.

Theorem 3.1. *For a graph* $G = (V, E)$ *with at least one edge, the maximum corank of any matrix* $A \in \mathbb{R}^{(n)}$ *with properties* (A1)–(A3) *below is* $\mu(G) + 1$:

(A1) *for all* $i \neq j$, $A_{i,j} < 1$ *if* $ij \in E$, *and* $A_{i,j} = 1$ *if* $ij \notin E$;

(A2) A *is positive semidefinite;*

(A3) *there is no nonzero matrix* $X = (X_{i,j}) \in \mathbb{R}^{(n)}$ *such that* $(A - J)X = 0$ *and* $X_{i,j} = 0$ *whenever* $i = j$ *or* $ij \in E$.

This theorem is quite easy to prove if G is connected, and we give the proof below. The case of disconnected G is somewhat cumbersome, and we refer to [16] for details.

Proof. Assume that G is connected, and let M be an optimal Colin de Verdière matrix for G. We may assume that the unique negative eigenvalue of M is -1. Denote the corresponding unit eigenvector of M by π; we may assume that $\pi > 0$. Consider the diagonal matrix $\Pi = \mathrm{diag}(\pi_1, \dots, \pi_n)$, where the π_i are the components of the vector π. Next; let

$$A := \Pi^{-1} M \Pi^{-1} + J.$$

We claim that A satisfies (A1)–(A3). (A1) is trivial, (A2) follows by standard linear algebra. It is also easy to see that $\mathrm{corank}(A) = 1 + \mathrm{corank}(M)$. Also note that $(\pi_1^2, \dots, \pi_n^2)$ is in the null space of A by construction.

To show that A satisfies (A3), assume that $X \neq 0$ satisfies the conditions in (A3). Then the matrix $X' = \Pi^{-1} X \Pi^{-1} \neq 0$ is symmetric, $X'_{i,j} = 0$ for $ij \in E$ or $i = j$, and

$$MX' = (\Pi(A - J)\Pi)(\Pi^{-1} X \Pi^{-1}) = \Pi(A - J)X\Pi^{-1} = 0.$$

This contradicts the Strong Arnold Property of M.

Thus we have found a matrix A with properties (A1)–(A3) and corank $\mu(G)+1$. To show that no larger corank can be attained, let A be any matrix satisfying (A1)–(A3). Consider the matrix $M = A - J$. Then M satisfies (M1) and (M3) trivially. Moreover, Sylvester's Inertia Theorem implies that M has at most one negative eigenvalue, and $\mathrm{corank}(M) \geq \mathrm{corank}(A) - 1$. By the remark 2.6, this implies that $\mathrm{corank}(M) \leq \mu(G)$. ∎

It follows from the construction that in the case of connected graphs, we could assume that there is a vector $\pi > 0$ in the null space of A, without changing this definition. Perhaps one could also assume that the vector $\mathbf{1}$ is in the null space.

We remark that condition 3.1 (A3) could be replaced by the formally incomparable condition:

(A3') there is no nonzero matrix $X = (X_{i,j}) \in \mathbb{R}^{(n)}$ such that $AX = 0$ and $X_{i,j} = 0$ whenever $i = j$ or $ij \in E$;

and also by the following condition, formally weaker than both (A3) and (A3'):

(A3") there is no nonzero matrix $X = (X_{i,j}) \in \mathbb{R}^{(n)}$ such that $AX = JX = 0$ and $X_{i,j} = 0$ whenever $i = j$ or $ij \in E$.

The fact that (A3") implies (A3') in the presence of (A1) is easy: if $AX = 0$, then in particular $(AX)_{i,i} = 0$; but

$$(AX)_{i,i} = \sum_j A_{i,j} X_{i,j} = \sum_j X_{i,j},$$

since $X_{i,j} = 0$ whenever $A_{i,j} \neq 1$. Thus we get that $\sum_j X_{i,j} = 0$ for all j, i.e., $JX = 0$. But this contradicts (A3").

The fact that (A3") implies (A3) in the presence of (A1) and (A2) (and assuming that $\text{corank}(A) \geq 2$, which excludes only trivial cases), is harder. Again, we prove this for connected graphs G only; the general case can be found in [16].

So let A be a matrix satisfying 3.1 (A1)–(A2) and (A3"), with $\text{corank}(A) \geq 2$. Suppose that there is a matrix X violating 3.1 (A3). If $XJ = 0$ then also $AJ = 0$ and this contradicts (A3"). Suppose that $XJ \neq 0$; then $X\mathbf{1} \neq 0$, and hence we may assume that $\|X\mathbf{1}\|^2 = \mathbf{1}^T X^2 \mathbf{1} = 1$. This implies that $JX^2J = J$. Since A is positive semidefinite, we can write $A = U^2$ for some $U \in \mathbb{R}^{(n)}$. Then we have

$$\begin{aligned}(U - UX^2J)^T(U - UX^2J) &= U^2 - U^2X^2J - JX^2U^2 + JX^2U^2X^2J \\ &= A - AX^2J - JX^2A + JX^2AX^2J \\ &= A - JX^2J - JX^2J + JX^2JX^2J \\ &= A - J.\end{aligned}$$

Thus $A - J$ is positive semidefinite. But by the Perron-Frobenius Theorem, the least eigenvalue of $A - J$ has multiplicity 1 (here is the point where one has to work more for disconnected graphs), and thus $\text{corank}(A - J) \leq 1$. Since J is positive semidefinite, it follows that the corank of A is at most 1, a contradiction.

We note that if G is $\overline{K_n}$, $n \geq 3$, then theorem 3.1 remains valid with (A3'), but not with (A3).

3.2. Gram labellings

Let $G = (V, E)$ be a graph. Considerations in this section will be best applicable when μ is close to n. Graphs with such high values of μ will be very dense, and so it will be convenient to formulate the results in terms of the complementary graph $H = \overline{G} = (V, F)$.

Consider a matrix A satisfying conditions 3.1 (A1)–(A3). Then we can write it as a Gram matrix of vectors in dimension $d = \text{rank}(A)$: $A_{i,j} = u_i^T u_j$, where $u_i \in \mathbb{R}^d$; conversely, such a representation guarantees that A is positive semidefinite. In terms of the u_i, conditions 3.1 (A1) and (A3') can be rephrased as

3.2.

(U1) *for all* $i \neq j$, $u_i^T u_j < 1$ *if* $ij \in E$, *and* $u_i^T u_j = 1$ *if* $ij \in F$.

(U2) *there is no nonzero matrix* $X = (X_{i,j}) \in \mathbb{R}^{(n)}$ *such that* $\sum_j X_{i,j} u_j = 0$ *for all nodes* i, *and* $X_{i,j} = 0$ *whenever* $i = j$ *or* $ij \in E$.

Let $\nu(H)$ be the smallest dimension d in which a vector labelling with these properties exists. We call a mapping $i \mapsto u_i$ with property (U1) a *scalar product labelling* of the graph. A scalar product labelling with property (U2) is *nondegenerate.*

By Theorem 3.1, we have

Theorem 3.3. *For every graph G different from* $\overline{K_2}$,

$$\nu(\overline{G}) = n - \mu(G) - 1.$$

We add that if G has no edges, then $\nu(\overline{G}) = \nu(K_n) = n - 2$. An optimal scalar product labelling is to use 3 copies of e_1 and the vectors $e_1 + e_2, e_1 + e_3, \ldots, e_1 + e_{n-2}$. It is remarkable that, similarly to the best matrix M for $\mu(K_{4,4})$, this optimal labelling breaks the symmetry.

As remarked after 3.1, in the case when G is connected we may assume that there is a positive vector $\pi > 0$ in the null space of A. This implies that the origin is in the convex hull of the u_i. On the other hand, simple linear algebra shows that the origin cannot be contained in the convex hull of any scalar product labelling u_i if G is disconnected.

Condition (U2) has a particularly transparent interpretation. Let $(u_i \in \mathbb{R}^d \mid i \in V)$ be a scalar product labelling of the graph H. This labeling can be described by a point $x \in \mathbb{R}^{dn}$. Condition (U1) then amounts to a number of quadratic equations ($u_k^T u_l = 1$ if kl is an edge) and inequalities ($u_k^T u_l < 1$ if kl is a non-edge). Each such equation defines a surface $U_{k,l}$. Now (U2) is equivalent to saying that the gradients of these surfaces at x are linearly independent; in other words, the surfaces intersect transversally at x.

In the presence of (U1), condition (U2) is equivalent to the following weaker property (this follows from the equivalence of (A3') and (A3")):

(U2') There is no nonzero matrix $X \in \mathbb{R}^{(n)}$ such that $\sum_j X_{i,j} = 0$ and $\sum_j X_{i,j} u_j = 0$ for all nodes i, and $X_{i,j} = 0$ whenever $i = j$ or $ij \in E$.

There is a third, closely related nondegeneracy condition that we shall need. A vector labelling $(u_i \mid i \in V)$ of a graph H is called *stress-free*, if it satisfies the following condition:

(U2") There is no nonzero matrix $X \in \mathbb{R}^{(n)}$ such that $\sum_j X_{i,j}(u_j - u_i) = 0$ for all nodes i, and $X_{i,j} = 0$ whenever $i = j$ or $ij \in E$.

A symmetric matrix X violating (U2") is called a *stress*. If we view the labels of the nodes as their position in d-space, and the edges between them as bars, and interpret $X_{i,j}$ as the "stress" along the edge ij, then $X_{i,j}(u_i - u_j)$ is the force by which the bar ij acts on node i, and so the definition of a stress says that these forces leave every node in equilibrium.

It is clear that (U2") implies (U2') and thus also (U2). In the converse direction we show the following.

Lemma 3.4. *In the presence of* (U1) *and the additional hypothesis that* $\|u_i\| \neq 1$ *for all* i, *conditions* (U2) *and* (U2") *are equivalent.*

Proof. Assume that a vector labelling satisfies (U2') but not (U2"). Then there exists a nonzero symmetric matrix X such that $X_{i,j} = 0$ for $i = j$ and for $ij \in E$, and $\sum_j X_{i,j}(u_j - u_i) = 0$ for every node i. Taking the inner product with u_i and using (U1), we get that

$$\left(\sum_j X_{i,j}\right)(1 - \|u_i\|^2) = 0.$$

Since the second factor is nonzero by the hypothesis, we get that $\sum_j X_{i,j} = 0$ for all i. It follows that also $\sum_j X_{i,j}u_j = 0$ for all i, and hence X violates (U2'). ∎

Scalar product labellings link Colin de Verdière's number with a considerable amount of work done on various geometric representations of graphs, cf. [16].

Scalar product labellings also give a nice geometric picture, some of which we describe now. We discuss this for connected graphs G, and assume that G has no node of degree $n - 1$ (i.e., H has no isolated node). In this case, we know that we may assume that the origin is in the convex hull of the u_i (for the general case we refer to [16]).

Lemma 3.5. *Let* $(u_i \in \mathbb{R}^d \mid i \in V)$ *be a scalar product labelling of a graph* H *with different vectors, and assume that the origin is in the convex hull* P *of the* u_i. *Then for each* $i \in V$, *one of the following alternatives holds:*

(a) u_i *is a vertex of* P.

(b) u_i *is not a vertex of* P, *but a boundary point of* P, *and* $\|u_i\| = 1$. *The neighbors of* i *in* H *are all the vertices of a face* Q_i *of* P, *and* $u_i \in Q_i$.

(c) *u_i is not a vertex of P, but a boundary point of P, and $\|u_i\| < 1$. The neighbors of i in H are the vertices of a face Q_i of P, $u_i \notin Q_i$, but there is a face F_i of P containing u_i and Q_i.*

(d) *u_i is an interior point of P, $\|u_i\| < 1$. Let u_i' be the point where the semiline of u_i intersects the surface of P, and let P_i be the smallest face of P containing u_i'. Then the neighbors of u_i are the vertices of a simplicial face Q_i of P_i (possibly $Q_i = P_i$).*

Proof. For any vector u_i, consider the hyperplane H_i defined by $u_i^T x = 1$. By (U1), all neighbors of i in the graph H are on this hyperplane.

If $\|u_i\| > 1$, then H_i separates u_i from all the other u_j, hence u_i is a vertex of P. Thus (a) holds.

If $\|u_i\| \le 1$ then H_i supports P and so it defines a face Q_i. The neighbors of i are precisely the u_j on this face. Thus every vertex of Q_i is a neighbor of i. But every neighbor u_j of u_i must have $\|u_j\| > 1$ (since $u_i^T u_j = 1$, $\|u_i\| \le 1$ and $u_j \ne u_i$), so u_j is a vertex of P. Thus the neighbors of u_i are precisely the vertices of Q_i.

If $\|u_i\| = 1$ then trivially Q_i contains u_i (u_i could be a vertex of Q_i or not). Thus (a) or (b) holds.

If $\|u_i\| < 1$ then Q_i does not contain u_i. Suppose that u_i is not a vertex, but a boundary point, and let P_i be the smallest face containing u_i. If Q_i is contained in P_i we have (c), so suppose this is not the case.

Let $W = V(Q_i) \setminus V(P_i)$, where $V(Q_i)$ the set of nodes k with $u_k \in Q_i$. If $k \in W$, then we have $u_k^T u_j \le 1$ for all vertices u_j of P_i, and $u_k^T u_i = 1$ for the point u_i in the relative interior of P_i, so we must have $u_k^T u_j = 1$ for all vertices of P_i. Thus all vertices of P_i are neighbors of k.

We show that $F_i = \mathrm{conv}(P_i \cup Q_i)$ is a face. Suppose not; then the affine hull of $P_i \cup Q_i$ intersects the convex hull of vertices not in $P_i \cup Q_i$. This means that there are three collinear points u, v, w such that u is in the convex hull of $U = V(P) \setminus (V(P_i) \cup V(Q_i))$, v is in the affine hull of $V(P_i)$ and w is in the affine hull of of W. Hence it follows that

$$u^T w \le 1, \qquad v^T w = 1.$$

Since $w^T u_i = 1$ and $\|u_i\| < 1$, we also have

$$w^T w > 1.$$

Consider the halfspace $H = \{x \mid w^T x \ge 1\}$. Then w is an interior point of H, v is a boundary point, and u is either on the boundary or outside.

The point v is in the affine hull of the face P_i, and so it cannot be a convex combination of vertices not on this face. In particular, w cannot be in the convex hull of W, and u and v cannot coincide. Thus u is in fact outside the halfspace. Now every vertex $u_k \in W$ has an inner product at most 1 with u, exactly 1 with v, and hence at least 1 with w. Thus $W \subset H$.

Now w must lie on the line through two points x, y in the convex hull of W, where we may assume that x, y are contained in disjoint faces of $\operatorname{conv}(W)$. We may also assume that w, x and y are in this order on the line. $W \subset H$ implies that $y^T w \geq 1$. Since x and y are in the convex hulls of disjoint sets of vertices, we also have $y^T x \leq 1$ and thus $y^T y \leq 1$, which is impossible since then $y^T u_i = 1$, $\|u_i\| < 1$ and $\|y\| \leq 1$. This proves that (c) holds.

Finally, if $\|u_i\| < 1$ and u_i is an interior point, then for any neighbor k of i, the point u_i' defined in (d) will have $(u_i')^T u_k > 1$. This implies that P_i defined in (d) must contain u_k, else u_i' would be a convex combination of vertices u_j with $u_j^T u_k \leq 1$, which would imply that $(u_i')^T u_k \leq 1$. Thus Q_i is a face of P_i. We show that Q_i is simplicial. Suppose not; then we can write u_i' as a convex combination of vertices of P so that at least one vertex k of Q_i is not used:

$$u_i' = \sum_{j \in V(P)} \alpha_j u_j, \qquad \alpha_j \geq 0, \qquad \sum_j \alpha_j = 1, \quad \alpha_k = 0.$$

Then we have

$$1 = u_k^T u_i < u_k^T u_i' = \sum_{j \in V(P) \setminus \{k\}} \alpha_j u_k^T u_j \leq \sum_{j \in V(P)} \alpha_j = 1,$$

a contradiction. So in this case (d) holds. ■

Corollary 3.6. *Under the conditions of Lemma 3.5, the subgraph of H spanned by $V(P)$ is a subgraph of the 1-skeleton of P.*

Proof. We have to show that if $ij \in E(H)$ and both u_i and u_j are vertices of P, then $u_i u_j$ is an edge of P. Since $u_i^T u_j = 1$, we may assume that $|u_i| > 1$. Then the hyperplane $u_i^T x = 1$ separates u_i from all the other vertices of P, and has u_j on its boundary, whence $u_i u_j$ is an edge of P. ■

We can make, for free, the assumption that the vectors u_i are "as generic as possible" in the following sense: if the equations $u_i^T u_j = 1$ $(ij \in E(H))$ do not determine the vector u_i uniquely, then we replace u_i by another solution very near to u_i so that this new u_i does not lie in any affine subspace spanned by other u_j unless all solutions do. Call such a scalar product labelling *generic*. In particular, assume that u_i is an interior point of P. Then u_i is constrained to an affine space A_i defined by the equations $u_k^T x = 1$ $(k \in N(i))$. Genericity means then that a small neighborhood of u_i in A_i is contained in $\text{conv}(P_i \cup \{0\})$. But we also know that $Q_i \subseteq P_i$. It is easy to see that the affine hull of Q_i and A_i together span the whole space. Hence it follows that P_i is a facet.

A similar argument shows that u_i cannot be a point on the surface with $\|u_i\| < 1$. Thus

3.7. *For generic scalar product labellings, we can add in Lemma 3.5 that*

— *in* (b), Q_i *is a facet;*

— (c) *cannot occur;*

— *in* (d), P_i *is a facet.*

Finally, we remark that if all the u_i are vertices of P, then it suffices to check condition (U1) for edges of P only. More exactly,

Lemma 3.8. *Let P be a convex polytope in $\mathbb{R}^d$. If $u^T v \leq 1$ for any two vertices u and v forming an edge of P, then $u^T v < 1$ for every pair u, v of distinct vertices that do not form an edge.*

Proof. Let u and v be two vertices with maximum inner product; it suffices to show that they must form an edge. Suppose not; then there is a point w on the segment uv such that w is in the convex hull of some other vertices $w_1, \ldots, w_k$. We may assume that $\|u\| \geq \|v\|$; then the halfplane $\{x \mid u^T x \geq u^T v\}$ contains both u and v, hence it contains w in its interior, hence it contains at least one w_i in its interior. But then $u^T w_i > u^T v$, a contradiction. ∎

3.3. Null space labellings

Another way of stating some of the theorems above is the following. Let M be the Colin de Verdière matrix of a graph G, and let $x_1, \ldots, x_d$ be a basis of the null space of M $(d = \mu(G))$. Let X be a matrix with column vectors $x_1, \ldots, x_d$, and let $y_1^T, \ldots, y_n^T$ be the row vectors of X. Then $i \mapsto y_i$ is an embedding of the nodes of G in $\mathbb{R}^d$. It is easy to see that, up to a linear transformation, this embedding does not depend on the choice of the basis for the null space. We call (y_i) the *null space labelling* of G (associated with the Colin de Verdière matrix M). Then

3.9. *The origin is in the interior of the convex hull of the y_i. Moreover, for every node i, the linear span of y_i intersects the relative interior of the convex hull of the neighbors of i.*

Theorem 2.16 says the following.

3.10. *For any hyperplane spanned by $d-1$ linearly independent nodes, the subgraph spanned by nodes on one side of this hyperplane is connected.*

Moreover, Theorem 3.17 can be rephrased as follows:

3.11. *For any hyperplane H through the origin, the subgraph spanned by nodes on one side of this hyperplane is connected, except in the following situation: H contains a $(d-2)$-dimensional subspace L such that the set of nodes on L separate the graph into exactly 3 components. Each of these components spans, together with L, a hyperplane that separates the other two components.*

This null space labelling may have some further interesting geometric properties. For example, assume that the graph G is maximal outerplanar, and that M has corank 2 (cf. Section 4). Is it true that the vectors $(1/\|y_i\|)y_i$, with straight edges between them, provide an embedding in the plane with all nodes on the infinite face?

The vector labelling formulation gives us very transparent interpretations (and proofs) of results about the kernel of M. Let G be a connected graph, and M, a Colin de Verdière matrix for G. We may assume that the negative eigenvalue of M is -1, and let π be the positive unit eigenvector belonging to it. Let, as in the proof of 3.1, $\Pi = \mathrm{diag}(\pi)$ and $A = \Pi^{-1}M\Pi^{-1} + J$, and let A be the Gram matrix of vectors $u_i \in \mathbb{R}^d$, where $d = \mathrm{rank}(A)$.

Now clearly $x \in \ker(M)$ if and only if $\pi^T x = 0$ and $\Pi x \in \ker A$. In terms of the vectors u_i, this means that the vector $y = \Pi x$ satisfies $\sum_i y_i = 0$ and $\sum_i y_i u_i = 0$. In other words,

3.12. *$x \in \ker(M)$ if and only if Πx defines an affine dependence between the vectors u_i.*

Next we show the geometric interpretation of the proof of Lemma 2.16 and its extension in these terms. Let y be an affine dependence between the

u_i. Let L be the subgraph of G spanned by $\text{supp}^+(y)$, and let $L_1, \ldots, L_r$ be the connected components of L. Similarly, let $L_{r+1}, \ldots, L_{r+s}$ be the connected components of the subgraph L' spanned by $\text{supp}^-(y)$. Consider the values $\|y_i\|$ as weights associated with the points. Then the fact that y is an affine dependence implies that L and L' have the same weight and the same center of gravity b.

Let c_i be the center of gravity of L_i. Obviously, $c_i^T c_j \leq 1$ if $1 \leq i \leq r < j \leq r+s$ (since $u_i^T u_j \leq 1$ if $i \neq j$), and hence (since b is the center of gravity of $c_{r+1}, \ldots, c_{r+s}$ with appropriate weights) $c_i^T b \leq 1$ for all i. But b is also the center of gravity of $c_1, \ldots, c_r$, and $c_i^T c_j = 1$ for $1 \leq i < j \leq r$ (since there are no edges between L_i and L_j). Hence we must have $c_i^T c_i \leq 1$ for all i.

Now assume that $\text{supp}^+(y)$ spans at least two components, i.e., $r \geq 2$. Then $c_1^T c_2 = 1$ implies that we must have $c_1 = c_2$ and $\|c_1\| = 1$. Thus it follows that $c_1 = c_2 = \cdots = c_r = b$ and b has unit length. Then it follows easily that also $c_{r+1} = \cdots = c_{r+s} = b$.

This implies that $c_i^T c_j = 1$ also for $1 \leq i \leq r < j \leq r+s$, and hence no edge connects L_i to L_j. Thus $L_1, \ldots, L_{r+s}$ are the components of $\text{supp}(y)$.

The Strong Arnold Property can be used to show that $r + s \leq 3$ just like in the proof of Lemma 2.8.

3.4. Cages, projective distance, and sphere labellings

In the case $\nu = 3$, Corollary 3.6 gives a representation of the graph as a subgraph of the 1-skeleton of a convex polytope. This is similar to the representation in the Cage Theorem. Two nodes i and j in the scalar product labelling satisfy

$$u_i^T u_j \leq 1,$$

with equality if and only if ij is an edge of H. In the cage representation, two nodes i and j satisfy

$$u_i^T u_j \leq 1 - \sqrt{(\|u_i\|^2 - 1)(\|u_j\|^2 - 1)},$$

and again equality holds if and only if ij is an edge of H. This suggests that one could consider, more generally, a real parameter a, and labellings of the nodes of H by vectors $u_i \in \mathbb{R}^d$ such that any two nodes satisfy

3.13. $\|u_i\| > 1$ *and* $u_i^T u_j \le 1 - a\sqrt{(\|u_i\|^2 - 1)(\|u_j\|^2 - 1)}$, *with equality if and only if* ij *is an edge of* H.

(The condition $\|u_i\| \ge 1$ is clearly necessary for the definition to be meaningful. Assuming strict inequality will be convenient later on.)

In other words, we could introduce, for any two points u and v outside the unit sphere, a "distance" defined by

$$a(u,v) = \frac{1 - u^T v}{\sqrt{(\|u\|^2 - 1)(\|v\|^2 - 1)}},$$

and then try to represent the graph by points outside the unit sphere so that adjacent points realize the minimum "distance" between all pairs.

While $a(u,v)$ is not a proper distance function, it has nice properties. It is larger than 1 if and only if the line segment S connecting u and v intersects the unit sphere at two points. Let x and y be these two points, and let λ be the cross ratio $(u : v : x : y)$. Then rather straightforward calculations show that

$$a(u,v) = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right).$$

In particular, it follows that $a(u,v)$ is invariant under projective transformations preserving the unit sphere. The formula also shows that if we define an analogous notion of distance for points inside the sphere, we get a function that is closely related to the distance function of the Cayley-Klein model of the hyperbolic plane.

The reason for introducing these generalized scalar product labellings is that we can prove results about ν (or μ) by first constructing representations with another choice of the parameter $a > 0$ (which is sometimes easier), and then use these to obtain scalar product representations (which is the case $a = 0$). Three key properties of scalar product representations can be extended to these generalized scalar product representations without difficulty. We state these without proof. Let $0 \le a \le 1$, and let $(u_i \in \mathbb{R}^d \mid i \in V)$ be a labelling of a graph H with different vectors with property 3.13.

Lemma 3.14. *Let* P *be the convex hull of the* u_i. *Then every* u_i *is a vertex of* P *and every edge of* H *is an edge of* P.

Lemma 3.15. *Let* P *be a convex polytope in* $\mathbb{R}^d$, *and assume that all vertices of* P *are outside the unit sphere. If* $a(u,v) \le a$ *for any two vertices*

u and v forming an edge, then $a(u, v) < a$ for every pair u, v of vertices that do not form an edge.

The third basic fact concerns the Strong Arnold Property. Let $x_i \in \mathbb{R}^d$ ($i \in V$) be unknown vectors, and consider the equations

$$x_i^T x_j = 1 - a\sqrt{(\|x_i\|^2 - 1)(\|x_j\|^2 - 1)} \qquad (ij \in E(H)).$$

These define surfaces in the nd-dimensional space, which intersect at the point $x_1 = u_1, \ldots, x_n = u_n$.

Lemma 3.16. *The surfaces intersect tranversally at $(u_1, \ldots, u_n)$ if and only if the labelling is stress-free.*

We close this section with pointing out that vector labellings with property (U1) are closely related to representations (labellings) of graphs by orthogonal spheres. The advantage of considering sphere labellings is that one can use methods from conformal geometry to study them. The disadvantage is that we do not get an exact reformulation of the definition.

Consider a vector labelling of a graph G satisfying (U1). Also assume that all vectors u_i are longer than 1. Let S^{d-1} be the unit sphere in $\mathbb{R}^d$, and let S_i be the sphere with center u_i and radius $\sqrt{\|u_i\|^2 - 1}$. Then an easy computation gives that

(S1) each S_i is orthogonal to the unit sphere S^{d-1};

(S2) if $ij \in \overline{E}$ then S_i and S_j are orthogonal; if $ij \in E$ then S_i and S_j intersect at an angle larger than $\pi/2$, or have disjoint interiors.

Conversely, every assignment of spheres of the nodes with properties (S1) and (S2) gives rise to a vector labelling with property (U1) such that each vector has length larger than 1.

We can allow vectors of length 1 and spheres degenerating to a single point. (We could even allow vectors shorter than 1 and "imaginary" spheres, but there would not be any real gain in this.)

More generally, vector labellings satisfying 3.13 for some parameter $0 \le a \le 1$ would correspond to sphere labellings where adjacent nodes are labelled with spheres intersecting at a fixed angle. We do not go into the details here, but refer to [16].

4. Small values

4.1. Paths and outerplanar graphs

We now arrive at characterizing the graphs G satisfying $\mu(G) \leq t$, for $t = 1, 2, 3$ and 4, and the corresponding collections of forbidden minors. We already know that $\mu(G) = 0$ if and only if G has at most one node. Next we show that

Theorem 4.1. *$\mu(G) \leq 1$ if and only if G is a node-disjoint union of paths.*

Proof. Since $\mu(K_3) = 2$ and $\mu(K_{1,3}) = 2$, the minor-monotonicity of μ gives the 'only if' part. We already know that a path has $\mu = 1$ and by Theorem 2.5, this is also true for a collection of paths. ∎

Theorem 4.2. *$\mu(G) \leq 2$ if and only if G is outerplanar.*

Proof. Since $\mu(K_4) = 3$ and $\mu(K_{2,3}) = 3$, the minor-monotonicity of μ gives the 'only if' part (using the well known forbidden minor characterization of outerplanarity).

To see the 'if' part, we may assume that G is maximally outerplanar. Then G is a clique sum of a triangle and a smaller outerplanar graph. By Theorem 2.10, it follows by induction that $\mu(G) \leq 2$. ∎

4.2. Planar graphs

Now we get to the main result of Colin de Verdière [7]. The original proof of this theorem appeals to the theory of partial differential equations. While this connection is very interesting, it is also important to see that a purely combinatorial proof is possible. Such a proof was given by van der Holst [11], which we now reproduce.

Theorem 4.3. *$\mu(G) \leq 3$ if and only if G is planar.*

Proof. Since $\mu(K_5) = 4$ and $\mu(K_{3,3}) = 4$, the minor-monotonicity of μ gives the 'only if' part (using Kuratowski's forbidden minor characterization of planarity).

To see the 'if' part, let $\mu(G) > 3$, and assume that G is planar. We may assume that G is maximally planar. Let M be a Colin de Verdière matrix for G. Let uvw be a face of G. Then there exists a nonzero vector $x \in \ker(M)$ with $x_u = x_v = x_w = 0$. We may assume that x has minimal support. By Theorem 2.16, $G|\mathrm{supp}^+(x)$ and $G|\mathrm{supp}^-(x)$ are nonempty and connected.

As G is maximally planar, G is 3-connected. Hence there exist three node-disjoint paths P_1, P_2, P_3 from $\mathrm{supp}(x)$ to $\{u, v, w\}$. Let P_1', P_2', P_3' be the parts outside $\mathrm{supp}(x)$. Then the first nodes of the P_i' belong to $N(\mathrm{supp}(x))$, and hence to both $N(\mathrm{supp}^+(x))$ and $N(\mathrm{supp}^-(x))$. Contracting each of $\mathrm{supp}^+(x)$, $\mathrm{supp}^-(x)$, P_1', P_2', P_3' to single nodes, we obtain an embedded planar graph with uvw forming a face and u, v, and w having two common neighbors. This is impossible. ■

4.3. Linkless embeddable graphs

The next question is to characterize the graphs G with $\mu(G) \leq 4$. To this end, we need some definitions from topology and topological graph theory.

Two disjoint Jordan curves A and B in $\mathbb{R}^3$ are *linked*, if there is no topological 2-sphere in $\mathbb{R}^3$ separating them (in the sense that one curve is in the interior of the sphere, while the other is in the exterior). The curves A and B have an *odd linking number* if there is an embedding of the 2-disk with boundary A that has an odd number of transveral intersections with the image of B (and no other intersections). It is well known that this definition is symmetric in A and B. An odd linking number implies being linked (but not the other way around).

An embedding of a graph G into $\mathbb{R}^3$ is called *linkless* if no two disjoint circuits in G are linked in $\mathbb{R}^3$. A graph G is *linklessly embeddable* if it has a linkless embedding in $\mathbb{R}^3$.

There are a number of equivalent characterizations of linklessly embeddable graphs. Call an embedding of G *flat* if for each circuit C in G there is a disk D (a 'panel') disjoint from (the embedding of) G and having boundary equal to C. Clearly, each flat embedding is linkless, but the reverse does not hold. (For instance, if G is just a circuit C, then any embedding of G is linkless, but only the unknotted embeddings are flat.) However, as was proved by Robertson, Seymour and Thomas [27] if G has a linkless embedding, it also has a flat embedding. So the classes of linklessly embeddable graphs and of flatly embeddable graphs are the same.

We could also consider graphs that are embeddable in $\mathbb{R}^3$ so that no two disjoint circuits have an odd linking number. Again, such a graph has a linkless embedding.

These facts follow from the work of Robertson, Seymour, and Thomas [27], as a byproduct of a proof of a deep forbidden minor characterization of linklessly embeddable graphs.

To understand this forbidden minor characterization, it is important to note that the class of linklessly embeddable graphs is closed under the YΔ- and ΔY-operations. This implies that also the class of forbidden minors for linkless embeddability is closed under applying ΔY and YΔ. Since K_6 is a minimal non-linklessly-embeddable graph, all graphs arising from K_6 by any series of ΔY- and YΔ-operations are forbidden minors. The class of these graphs is called the *Petersen family*, because one of the members is the Petersen graph. The Petersen family consists of seven graphs (see Figure 2).

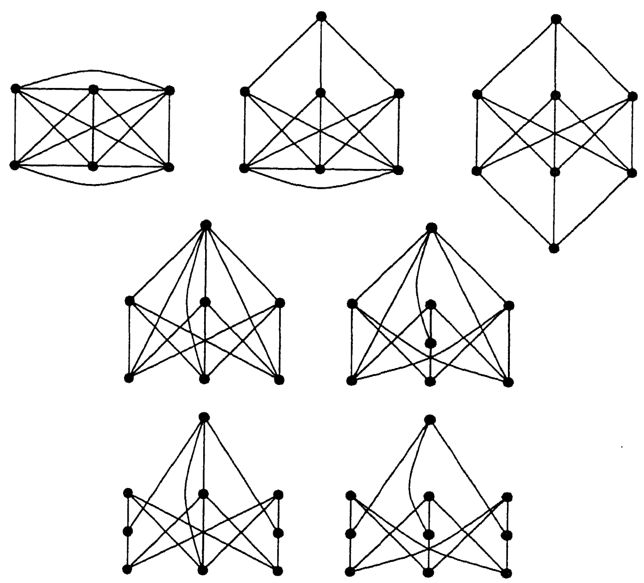

Fig. 2. The Petersen family

Now Robertson, Seymour, and Thomas [27] proved the following.

4.4. *The Petersen family is the collection of forbidden minors for linkless embeddability.*

Since $\mu(K_6) = 5$, it follows from Theorem 2.13 that $\mu(G) = 5$ for each graph G in the Petersen family. Thus it follows that if $\mu(G) \leq 4$ then G is linklessly embeddable. The reverse implication was conjectured by Robertson, Seymour, and Thomas [25] and proved in [20]:

Theorem 4.5. *$\mu(G) \leq 4$ if and only if G is linklessly embeddable.*

The proof uses a topological theorem of independent interest, which we discuss next.

4.3.1. A Borsuk theorem for antipodal links

Let P be a convex polytope in $\mathbb{R}^d$. We say that two faces F and F' are *antipodal* if they are contained in a pair of parallel supporting hyperplanes. So F and F' are antipodal if and only if $F - F'$ is contained in a face of $P - P$.

Call a continuous map ϕ of a cell complex into $\mathbb{R}^m$ *generic* if the images of a k-face and an l-face intersect only if $k + l \geq m$, and for $k + l = m$ they have a finite number of intersection points, and at these points they intersect transversally. (In this section, faces are relatively open.)

For any convex polytope P in $\mathbb{R}^d$, let ∂P denote its boundary.

The following theorem extends a result of Bajmóczy and Bárány [3]. (The difference is that their theorem concludes that $\phi(F) \cap \phi(F')$ is nonempty. Their proof uses Borsuk's theorem. We give an independent proof.)

Theorem 4.6. *Let P be a full-dimensional convex polytope in $\mathbb{R}^d$ and let ϕ be a generic continuous map from ∂P to $\mathbb{R}^{d-1}$. Then there exists a pair of antipodal faces F and F' with $\dim(F) + \dim(F') = d - 1$ such that $|\phi(F) \cap \phi(F')|$ is odd.*

Proof. We prove a more general fact. Call two faces *parallel* if their projective hulls have a nonempty intersection that is contained in the hyperplane at infinity. So faces F and F' are parallel if and only if their affine hulls are disjoint while $F - F$ and $F' - F'$ have a nonzero vector in common. (Note that two antipodal faces are parallel if $\dim(F) + \dim(F') \geq d$.)

Now it suffices to show:

4.7. *Let P be a convex polytope in $\mathbb{R}^d$ having no parallel faces and let ϕ be a generic continuous map from ∂P to $\mathbb{R}^{d-1}$. Then*

$$\sum |\phi(F) \cap \phi(F')|$$

is odd, where the summation extends over all antipodal pairs $\{F, F'\}$ *of faces.*

(It would be enough to sum over all antipodal pairs (F, F') with $\dim(F) + \dim(F') = d - 1$, since the map is generic.)

To see that it suffices to prove 4.7, it is enough to apply a random projective transformation close to the identity. To be more precise, assume that we have a polytope P that has parallel faces. For every pair (E, E') of faces whose affine hulls intersect, choose a (finite) point $p_{EE'}$ in the intersection of the affine hulls. For every pair (E, E') of faces whose projective hulls intersect, choose an infinite point $q_{EE'}$ in the intersection of their projective hulls. Let H be a finite hyperplane having all the points $p_{EE'}$ on one side, and avoiding all the points $q_{FF'}$. Apply a projective transformation that maps H onto the hyperplane at infinity, to get a new polytope P'. It is clear that P' has no parallel faces, and it is easy to argue that every pair of faces that are antipodal in P' correspond to antipodal faces in P. Hence 4.7 implies the theorem.

We now prove 4.7. Let P be a convex polytope in $\mathbb{R}^n$ having no parallel faces. For any two faces F, F', denote $F \leq F'$ if $F \subseteq \overline{F'}$. Then:

4.8.

(i) *if A and B are antipodal faces, then $A - B$ is a face of $P - P$, with* $\dim(A - B) = \dim(A) + \dim(B)$;

(ii) *if F is a face of $P - P$, then there exists a unique pair A, B of antipodal faces with $A - B = F$;*

(iii) *for any two pairs A, B and A', B' of antipodal faces one has: $A - B \leq A' - B'$ if and only if $A \leq A'$ and $B \leq B'$.*

This gives the following observation:

4.9. *For every pair of faces A and B with* $\dim(A) + \dim(B) = d - 2$, *the number of antipodal pairs $\{F, F'\}$ of faces with $A \leq F$ and $B \leq F'$ and* $\dim(F) + \dim(F') = d - 1$ *is* 0 *or* 2.

To see 4.9, it is clear that if A and B are not antipodal, then this number is 0. Suppose that they are antipodal. Then the number is 2, since by 4.8, it is equal to the number of facets of $P - P$ incident with the 3-face $A - B$.

To prove 4.7, we use a "deformation" argument. The statement is true for the following mapping ϕ: pick a point q very near the center of gravity of some facet F (outside P), and project ∂P from q onto the hyperplane H

of F. Then the only nontrivial intersection is that the image of the (unique) vertex of P farthest from H is contained in F.

Now we deform this map to ϕ. We may assume that the images of two faces E and E' with $\dim(E) + \dim(E') \leq d - 3$ never meet; but we have to watch when $\phi(A)$ passes through $\phi(B)$, where A and B are faces with $\dim(A)+\dim(B) = 3$. But then $|\phi(F)\cap\phi(F')|$ changes exactly when $A \subseteq F$ and $B \subseteq F'$. By 4.9, this does not change the parity. This proves 4.7, and hence the theorem. ∎

In what follows, we restrict our attention to dimension 5, which will be the case we need. For the general theorem in an arbitrary dimension, see [20].

For any polytope P, let $(P)_k$ denote its k-skeleton.

Theorem 4.10. *Let P be a convex polytope in $\mathbb{R}^5$ and let ϕ be an embedding of $(P)_1$ into $\mathbb{R}^3$. Then there exists a pair of antipodal 2-faces F and F' of P such that $\phi(\partial F)$ and $\phi(\partial F')$ have an odd linking number.*

Proof. First we extend ϕ with a last coordinate equal to 0, to obtain an embedding ψ of $(P)_1$ into $\mathbb{R}^4$. Next we extend ψ to a generic mapping $\partial P \to \mathbb{R}^4$, in such a way that $\psi(x)$ has last coordinate positive if $x \in \partial P \setminus (P)_1$.

By Theorem 4.6, P has two antipodal faces F and F' such that $\dim(F)+\dim(F') = 4$ and $|\psi(F) \cap \psi(F')|$ is odd. If $\dim(F) \leq 1$, then the last coordinate of each point in $\psi(F)$ is 0, while the last coordinate of each point in $\psi(F')$ is positive (as $\dim(F') \geq 2$). So $\dim(F) \geq 2$ and similarly $\dim(F') \geq 2$. Therefore, $\dim(F) = \dim(F') = 2$.

Then the boundaries of F and F' are 1-spheres S_1 and S_2, mapped disjointly into $\mathbb{R}^3$, and the mappings extend to mappings of the 2-balls into the "upper" halfspace of $\mathbb{R}^4$, so that the images of the balls intersect at an odd number of points. But this implies that the images of the spheres are linked. In fact, if they were not linked, then there exists an extension of the map of ∂F to a continuous mapping ψ' of F into $\mathbb{R}^4$ such that the image of every point in the interior of F has last coordinate equal to 0, and $\psi'(F)$ intersects $\psi(\partial F')$ transversally in an even number of points. We can extend the map of $\partial F'$ to a continuous mapping ψ' of F' into $\mathbb{R}^4$ such that the image of every point in the interior of F has a negative last coordinate 0. Then we get two maps of the 2-sphere into $\mathbb{R}^4$ with an odd number of transversal intersection points, which is impossible. This contradiction completes the proof. ∎

4.3.2. The proof of 4.5

By the results of Robertson, Seymour, and Thomas [27], it suffices to show that if $\mu(G) \geq 5$ then G is not flatly embeddable.

We take a counterexample G with a minimum number of nodes. Then G is 4-connected. For suppose that G has a minimum-size node cut U with $|U| \leq 3$. Consider any component K of $G - U$. Then the graph G' obtained from $G - K$ by adding a clique on U is a linkless embeddable graph again, because, if $|U| \leq 2$, G' is a minor of G, and if $|U| = 3$, G' can be obtained from a minor of G by a YΔ-operation. As G' has fewer nodes than G, we have $\mu(G') \leq 4$. As this is true for each component K, G is a a subgraph of a clique sum of graphs G' with $\mu(G') \leq 4$, along cliques of size at most 3, and hence by Theorem 2.10, $\mu(G) \leq 4$.

Let M be a Colin de Verdière matrix for G. Call two elements x and x' of $\ker(M)$ *equivalent* if $\text{supp}^+(x) = \text{supp}^+(x')$ and $\text{supp}^-(x) = \text{supp}^-(x')$. The equivalence classes decompose $\ker(M)$ into a centrally symmetric complex $\mathcal{P}$ of pointed polyhedral cones. Call a cone f of $\mathcal{P}$ *broken* if $G|\text{supp}^+(x)$ is disconnected for any $x \in f$.

To study broken cones, we first observe:

4.11. *for each $x \in \ker(M)$ with $G|\text{supp}^+(x)$ disconnected, $G|\text{supp}(x)$ has exactly three components, say K_1, K_2, and K_3, with $K_1 \cup K_2 = \text{supp}^+(x)$ and $K_3 = \text{supp}^-(x)$, and with $N(K_i) = V \setminus \text{supp}(x)$ for $i = 1, 2, 3$.*

This follows directly from Theorem 2.17, using the 4-connectivity of G and the fact that G has no $K_{4,4}$-minor ($K_{4,4}$ is not linklessly embeddable, as it contains one of the graphs in the Petersen family).

Now 4.11 gives:

4.12. *Every broken cone f is 2-dimensional.*

Indeed, choose $x \in f$, and let K_1, K_2, and K_3 be as in 4.11. Consider any $y \in f$. As $\text{supp}(y) = \text{supp}(x)$ we have that $M_{K_i} y_{K_i} = 0$ for $i = 1, 2, 3$. As $M_{K_i} x_{K_i} = 0$ and as x_{K_i} is fully positive or fully negative, we know by the Perron-Frobenius theorem that $y_{K_i} = \lambda_i x_{K_i}$ for some $\lambda_i > 0$ ($i = 1, 2, 3$). Moreover, for the positive eigenvector z of M we have that $z^T y = z^T x = 0$. Conversely, any vector $y \in \mathbb{R}^V$ with $z^T y = 0$ and $\text{supp}(y) = \text{supp}(x)$ and for which there exist λ_1, λ_2, $\lambda_3 > 0$ with $y_{K_i} = \lambda_i x_{K_i}$ for $i = 1, 2, 3$, belongs to f, since it belongs to $\ker(M)$. This follows from the fact that $z^T y = 0$ and $y^T M y = 0$. So f is 2-dimensional, proving 4.12.

Now choose a sufficiently dense set of vectors of unit length from every cone in $\mathcal{P}$, in a centrally symmetric fashion, and let P be the convex hull of these vectors. Then P is a 5-dimensional centrally symmetric convex polytope such that every face of P is contained in a cone of $\mathcal{P}$. We choose the vectors densely enough such that every face of P contains at most one edge that is part of a 2-dimensional cone in $\mathcal{P}$. We call an edge of P *broken* if it is contained in a broken cone in $\mathcal{P}$.

We define an embedding ϕ of the 1-skeleton $(P)_1$ of P in $\mathbb{R}^3$. We map each vertex x of P to a point $\phi(x)$ near $\mathrm{supp}^+(x)$, and we map any unbroken edge $e = xy$ of P to a curve connecting $\phi(x)$ and $\phi(y)$ near $G|\mathrm{supp}^+(z)$, where $z \in e$. We do this in such a way that the mapping is one-to-one.

Consider next a broken edge e of P. Choose $x \in e$, let K_1, K_2, and K_3 be as in 4.11, and let $T := N(\mathrm{supp}(x))$.

4.13. *There is a curve C in $\mathbb{R}^3 \setminus G$ connecting K_1 and K_2 such that there is no pair of disjoint linked circuits A in $G|(K_1 \cup K_2 \cup T) \cup C$ and B in $G|(K_3 \cup T)$.*

To see this, let H be the flatly embedded graph obtained from G by contracting K_i to one node v_i $(i = 1, 2, 3)$. It suffices to show that there is a curve C connecting v_1 and v_2 such that the graph $H \cup C$ is linklessly embedded. (Indeed, having C with $H \cup C$ linklessly embedded, we can decontract each K_i slightly, and make C connecting two arbitrary points in K_1 and K_2. Consider a circuit A in $G|(K_1 \cup K_2 \cup T) \cup C$ and a circuit B in $G|(K_3 \cup T)$ disjoint from A. Contracting K_1, K_2, and K_3, we obtain disjoint cycles A' and B' in $H \cup C$. (A *cycle* is an edge-disjoint union of circuits.) As $H \cup C$ is linklessly embedded, A' and B' are unlinked. Hence A and B are unlinked.)

Now $H|T$ is a Hamiltonian circuit on T, or part of it. Otherwise, $H|T$ would contain, as a minor, a graph on four nodes that is either a $K_{1,3}$ or a triangle with an isolated node. In both cases, it implies that H has a minor in the Petersen family, which is not possible since H is linklessly embedded.

So H is isomorphic to the complete bipartite graph $K_{3,|T|}$, with some edges on T added forming part (or whole) of a Hamiltonian circuit on T. As H is flatly embedded, for each edge t_1t_2 of $H|T$ there is an open disk ("panel") with boundary the triangle $t_1t_2v_3$, in such a way that the panels are pairwise disjoint (by Böhme's lemma [4] (cf. [28, 27]). Since the union of $H|(\{v_3\} \cup T)$ with the panels is contractible, there is a curve C from v_1 to v_2 not intersecting any panel. This curve has the required properties, showing 4.13.

We now define ϕ on e close to a curve in $G|(K_1 \cup K_2) \cup C$, again so that it is one-to-one. We do this for each broken edge e, after which the construction of ϕ is finished.

Then by Corollary 4.10, there are two antipodal 2-faces F and F' such that the images of their boundaries are linked. Since P is centrally symmetric, there is a facet D of P such that $F \subseteq \overline{D}$ and $F' \subseteq -\overline{D}$. Let y be a vector in the interior of D. Then ∂F and $\partial F'$ have image in $\text{supp}^+(y)$ and $\text{supp}^-(y)$ respectively. If ∂F and $\partial F'$ do not contain any broken edge, then it would follow that G has two disjoint linked circuits — a contradiction.

So we can assume that ∂F contains a broken edge e. Then it is the only broken edge in ∂F, since by our construction, ∂D contains at most one edge of P that is part of a 2-dimensional cone f in $\mathcal{P}$. So f is broken. Moreover, $\partial F'$ does not contain any broken edge. For suppose that $\partial F'$ contains broken edge e' of P. Then e' is part of a broken 2-dimensional cone f' in $\mathcal{P}$, and hence $f' = -f$ (since D is incident with at most one edge that is part of a 2-dimensional cone of $\mathcal{P}$). However, as f is broken, $-f$ is not broken, since $G|\text{supp}^-(x)$ is connected for any $x \in f$ (by 4.11).

Choose $x \in f$, and consider the partition of V into K_1, K_2, K_3, and T as above, with $\text{supp}^+(x) = K_1 \cup K_2$ and $\text{supp}^-(x) = K_3$. Then $K_1 \cup K_2 \subseteq \text{supp}^+(y)$ and $K_3 \subseteq \text{supp}^-(y)$, and hence $\text{supp}^+(y) \subseteq K_1 \cup K_2 \cup T$ and $\text{supp}^-(y) \subseteq K_3 \cup T$. So the image of ∂F is close to $G|(K_1 \cup K_2 \cup T) \cup C$, where C is the curve constructed for the broken edge e of P, and the image of $\partial F'$ is close to $G|(K_3 \cup T)$. This contradicts 4.13. ∎

5. Large Values

We finally study graphs whose Colin de Verdière parameter is close to its maximum possible value $n-1$ (where n is the number of nodes). It will be convenient to phrase our results in terms of the complementary graph $H = \overline{G}$ and the "complementary parameter" $\nu(H) = n - \mu(G) - 1$ (we may assume that that $G \neq K_2$). We start with showing that we do not have to worry about the Strong Arnold Property. Note that $\nu(H)$ is monotone with repect to edge-deletion, but *not* minor-monotone in general.

5.1. The Strong Arnold Property and rigidity

In this section we show that for small dimensions, scalar product labellings can violate the Strong Arnold Property only in a trivial way. This fact plays an important role later on.

Lemma 5.1. *Every scalar product labelling with different vectors in* $\mathbb{R}^d$, $d \leq 3$, *of a graph is stress-free.*

We give the proof in the case when $d = 3$; the cases $d = 1$ and 2 are much easier. We use a result from the theory of rigidity of bar-and-joint structures. Recall the classical theorem of Cauchy:

5.2. *No nontrivial stress can act along the edges of a convex 3-dimensional polytope.*

We need a generalization of Cauchy's theorem, due to Whiteley [32]. Let P be a convex polyhedron in $\mathbb{R}^3$, and let H be a (planar) graph embedded in the surface of P, with straight edges. A stress on H is called *facial* if there is a facet of P containing all edges with nonzero stress.

5.3. *Every stress on* H *is a sum of facial stresses.*

Now we can prove Lemma 5.1.

Proof. Let $(u_i \,|\, i \in V)$ be a scalar product labelling with different vectors in $\mathbb{R}^3$, and suppose that (U2') is violated. Let X be a matrix violating (U2'):

$$\sum_j X_{i,j} u_j = 0, \qquad \sum_j X_{i,j} = 0 \qquad \text{for all } i,$$

and $X_{i,j} = 0$ unless $ij \in F$. Then X is a nontrivial stress on the edges of H.

If i is a node whose neighbors are affinely independent, then trivially the edges of H incident with this node will have $X_{i,j} = 0$, and so we can delete this node and proceed by induction. Thus by Lemma 3.5, we can delete nodes i that have $\|u_i\| < 1$ and also all nodes i with $\|u_i\| = 1$ unless Q_i has at least four vertices. Clearly $u_i \in Q_i$ in the latter case. Moreover, a given facet contains at most one such additional point. Thus the graph, together with the skeleton of P, satisfies the conditions of Whiteley's theorem. It follows that X is a sum of facial stresses. Let X' be one of the facial stresses occuring in this sum, supported by a facet Q. Clearly Q must be one of the Q_i, and the graph in this facet is a wheel. It is easy to see that a wheel in the plane has only one stress up to scaling, and in this all the "spokes" have the same sign. On the other hand, the other facial stresses do not involve the edges incident with u_i, and hence we must have $\sum_j X'_{i,j} = 0$. This is a contradiction. ∎

5.2. Graphs with $\mu \geq n - 3$

In this section we describe graphs $\mu \geq n - 3$ or, equivalently, with $\nu \leq 2$. We know already that graphs with $\mu = n - 1$ are exactly the cliques.

For $\nu > 0$, we may assume without loss of generality that H does not have isolated nodes (which do not change ν by Theorem 2.7. Using Lemma 3.5 it is not difficult to derive the following results:

Theorem 5.4. *Graphs with $\nu = 1$ and without isolated nodes are exactly those graphs with at most two components, each of which is either a star or a triangle.*

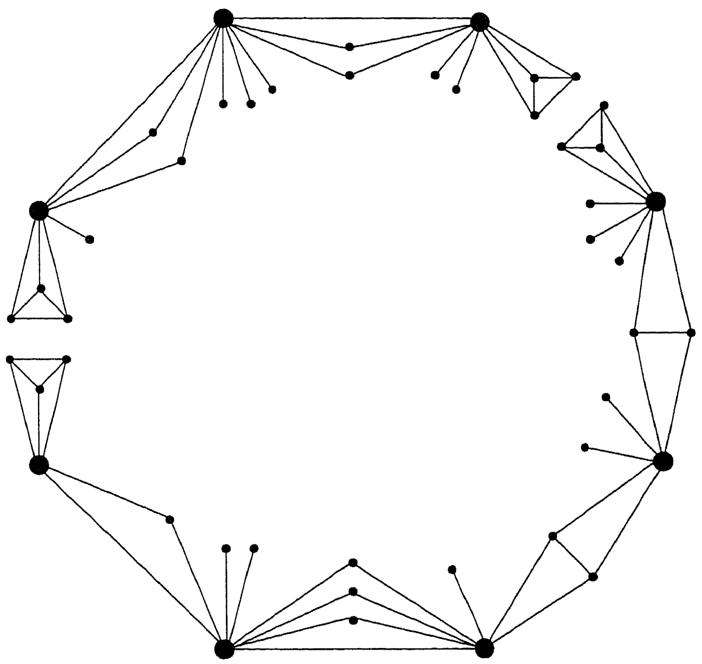

Fig. 3. A typical graph with $\nu = 2$

One can also formulate this result as follows: $\nu(H) \leq 1$ *if and only if* H *does not contain* 3 *disjoint edges or a* 3*-path.*

Theorem 5.5. *A graph H has $\nu \leq 2$ if and only if it is a subgraph of a graph obtained from a k-gon P ($k \geq 3$) as follows. For each node i of P, create a set of new independent nodes and connect them to i. Replace each original edge of P with either*

(a) *an edge and a set of independent nodes connected to its endpoints; or*

(b) *a pair of adjacent nodes connected to its endpoints; or*

(c) *two triangles, connected to its one endpoint each. If $k = 3$ then at least two steps* (b) *or at least one step* (c) *must be used; and if $k = 4$ then step* (b) *or* (c) *must be used at least once (Figure 3).*

To obtain a scalar product labelling of the graph in Figure 3, one should label the fat nodes with vectors of length larger than 1, the small nonadjacent twin nodes with the same vector of length less than 1, and the small adjacent twin nodes with the same unit vector.

Corollary 5.6.

(a) *If $\nu(H) = 2$ then H is planar;*

(b) *If in addition H has no twins then H is outerplanar.*

An alternative way of stating Theorem 5.5 is the following: $\nu(H) \leq 2$ *if and only if H does not contain as a subgraph the disjoint union of a cycle of length at least* 5 *and an edge, nor any of the other four graphs in Figure* 4.

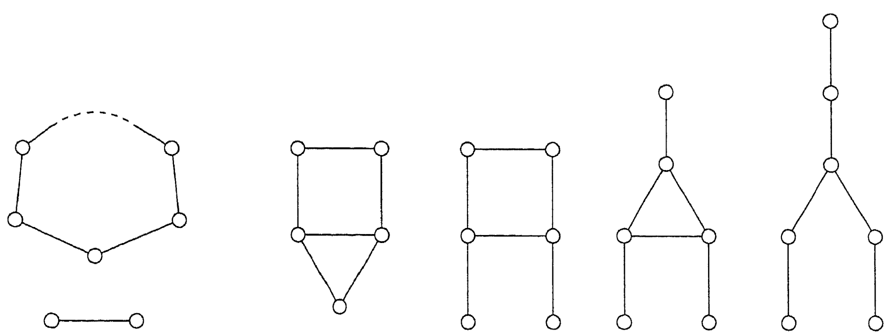

Fig. 4. Minimal graphs with $\nu = 3$

5.3. Planar graphs and $\mu \geq n-4$

In this section we discuss the connections between planar graphs and graphs with $\nu = 3$. Unlike for μ, the correspondence between these two properties is not exact, but we will see that they are quite close.

First, consider a graph H with $\nu(H) = 3$. The next theorem and its corollary show that H must be planar, except possibly for repeated nodes.

Theorem 5.7. *If H is a graph without twin nodes and $\nu(H) = 3$, then H is planar.*

Proof. The theorem is easily derived from Lemma 3.5. The case when $G = \overline{H}$ is not connected is trivial by Lemma 2.5, so suppose that G is connected. Consider a scalar product labelling of H in $\mathbb{R}^3$ so that 0 is in the convex hull P. We may also assume that the labelling is generic. The vectors u_i will be all distinct, since H is twin-free. We claim that projecting the graph on the surface ∂P of P from the origin gives an embedding in ∂P. Supplement 3.7 implies that no two nodes are projected on the same point and that no node is projected on an edge. If the images of two edges ik and jm of H cross, then we would have that an interior point v of the segment $u_i u_k$ is contained in the convex hull of $\{0, u_j, u_m\}$, or vice versa. Without loss of generality, we may assume that $\|u_k\| > 1$. But then $u_k^T u_k > 1$, $u_k^T v \leq 1$, $u_k^T u_i = 1$, which is a contradiction. ∎

Allowing twins in the graph would not lead to any essentially new case, but rather to a discussion of the multiplicities using the Strong Arnold Property. We state the result without proof.

Theorem 5.8. *If $\nu(H) = 3$, then H can be obtained from a planar graph by the following procedure: choose a set A of independent nodes of degrees two and three; replace each node in A of degree 3 by either several independent nodes, or by a pair of adjacent nodes; replace each node in A of degree 2 by a triangle.*

Unfortunately, not all planar graphs have $\nu = 3$. For example, the complement $\overline{P_6}$ of a path on 6 nodes is planar, but has $\nu(\overline{P_6}) = 6 - \mu(P_6) - 1 = 4$. We can add two edges to this graph, to get the octahedron, which is still planar and thus still a counterexample.

The following condition gives an infinite family of planar graphs with $\nu > 3$. A cycle C in a graph H will be called *strongly separating* if $H - V(C)$ has at least two components with more than one node, each being connected by edges to all nodes of C.

Lemma 5.9. *A twin-free graph H with $\nu(H) = 3$ contains no strongly separating 3- or 4-cycle.*

Proof. Suppose that C is a strongly separating 3- or 4-cycle in H. Consider a generic scalar product labelling $(u_i \in \mathbb{R}^3 \mid i \in V)$ of H. It follows easily that the nodes of C give vertices of P. We know that the projection from the origin onto the surface ∂P of P defines an embedding of H. The image of C separates ∂P into two discs. The two non-singleton components of $G - V(C)$ that are attached to all nodes of C must be mapped onto different discs. Moreover, each of these components contains a pair of adjacent nodes, and one of these nodes is labeled by a vector longer than 1. Let u_1 and u_2 be two vectors longer than 1, assigned to nodes in different components of $H - V(C)$.

We consider the case when C is a triangle $v_1v_2v_3$; the other case can be handled in a similar, although more complicated, fashion. It is clear that the origin cannot belong to this triangle, since $v_i^T v_j > 0$ for all i and j. One of u_1 and u_2, say u_1, intersects the triangle at a point v. We may assume that $\|v_1\|^2 > 1$. Then v_1 gives an inner product of 1 with every point on the line v_2v_3, and hence it gives an inner product at least 1 with every point of the triangle. In particular, $1 \leq v_1^T v < v_1^t u_1$ a contradiction. ∎

However, we can show for rather large classes of planar graphs that they have $\nu = 3$. In fact, we believe that a complete characterization should be possible, generalizing Theorem 5.16 below in an appropriate manner. First we give a few classes where $\nu \leq 3$ is relatively easy to prove.

Define the 1-*subdivision* of a graph H as the graph H' obtained by subdividing each edge by one new node.

Theorem 5.10. *H is planar if and only if its 1-subdivision H' satisfies $\nu(H') \leq 3$.*

Proof. The "if" part is trivial by Theorem 5.7, since H' has no twin nodes and since if H' is planar then so is H. To prove the "only if" part, we may assume that H is maximal planar. Consider the cage representation of H, and use the points where the edges of H touch the unit sphere to represent the nodes of $V(H') \setminus V(H)$. It is clear that $u_i^T u_j = 1$ for any edge $ij \in E(H')$, and it is easy to see that $u_i^T u_j < 1$ if i and j are nonadjacent nodes of H'. Thus (U1) is satisfied. By Lemma 5.1, (U2) is also valid. ∎

A variety of planar graphs with $\nu = 3$ can be obtained from the following assertion, trivially implied by Lemmas 3.8 and 5.1.

Theorem 5.11. *Let P be a convex polytope in $\mathbb{R}^3$ such that $u^Tv \le 1$ for every edge uv of P. Let H be the graph on $V(P)$ formed by those edges that give equality here. Then $\nu(H) \le 3$.*

Corollary 5.12. *Let H be a 3-connected planar graph with an edge-transitive automorphism group, different from K_4 and $K_{2,2,2}$. Then $\nu(H) = 3$.*

Proof. By a theorem of Mani [21], H can be represented as the skeleton of a convex polytope P in $\mathbb{R}^3$ so that the group of congruences preserving P acts edge-transitively. We can translate the origin to the center of gravity of P, and scale so that $u^Tv = 1$ if uv is an edge. Then Theorem 5.11 implies the assertion. ■

Theorem 5.13. *If H is an outerplanar graph, then $\nu(H) \le 3$.*

Proof. We may assume that H is maximal outerplanar, i.e., a triangulation of a cycle. First, we show that H can be represented in the plane with property 3.13, for some $a > 1$. We use induction on the number of nodes. The construction is trivial if H is a triangle. Suppose that the graph obtained from H by deleting a node i of degree 1 is already represented. Let j and k be the neighbors of i. We may assume that u_j and u_k have the same length; this can be achieved by a projective transformation preserving the unit circle. But then moving u_i along the bisector of the angle between u_j and u_k, a position with $a(u_i, u_j) = a$ will be found by continuity. By symmetry, this also satisfies $a(u_i, u_k) = a$. It is easy to check that $\|u_i\| > 1$ and the distances between u_i and u_m $(m \ne i, j, k)$ are larger than 1.

Now we append a third coordinate $z_i = \sqrt{a(\|u_i\|^2 - 1)}$ to each u_i. An easy computation shows that we get a scalar product labelling. By Lemma 5.1, this labelling is nondegenerate, which proves the theorem. ■

The following two theorems can be proved by a similar construction; we omit details.

Theorem 5.14. *If H is a planar graph, then $\nu(H) \le 4$.*

Theorem 5.15. *Every graph H has a subdivision H' with $\nu(H') \le 4$.*

Now we come to the main theorem in this section, giving a full description of ν for maximal planar graphs.

Theorem 5.16. *If H is a maximal planar graph, then $\nu(H) \le 3$ if and only if H does not contain any strongly separating 3- or 4-cycle, and is different from the graphs in Figure 5.*

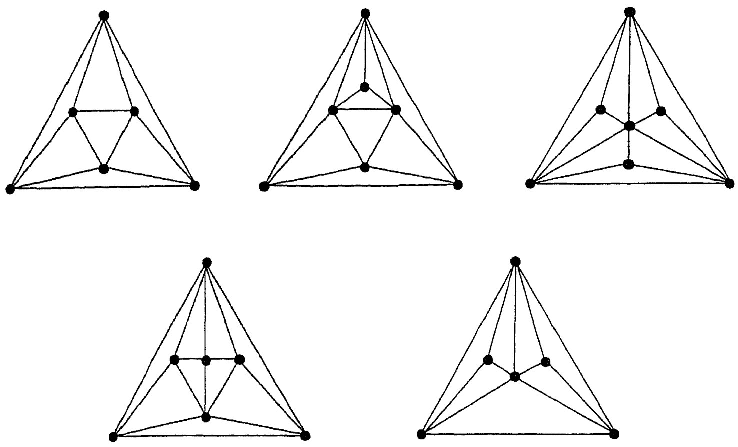

Fig. 5. Exceptional maximal planar graphs with $\mu = n - 5$.

Proof. To check that the graphs in Figure 5 have $\nu = 4$, we can determine their complements, and see that they are disjoint paths (for the 6-node graphs) and outerplanar (for the seven-node graphs). Together with Lemma 5.9, this implies the "only if" part. For the converse, we only sketch the proof; the details are lengthy.

The idea is to start with a cage representation of H; in other words, a labelling in $\mathbb{R}^3$ with property 3.13, with $a = 1$. Consider the set S of all real numbers $0 \leq a \leq 1$ for which H has a stress-free labelling with property 3.13.

First we show that this set is open in $[0, 1]$. In fact, let $0 < a \leq 1$, and assume that $a \in S$. Then H has a labelling by vectors $u_i \in \mathbb{R}^3$ satisfying 3.13. By Lemma 3.14, H is a subgraph of the 1-skeleton of the convex hull of the u_i, and thus by Cauchy's theorem, this labelling is stress-free. Hence by Lemma 3.16 the surfaces in $\mathbb{R}^{3n}$ corresponding to the equality constraints in 3.13 intersect transversally at $(u_1, \ldots, u_n)$. Now by Lemma 2.1, the surfaces intersect transversally at a "nearby" point for every parameter value a' sufficiently close to a, which means that $a' \in S$.

Thus it suffices to show that if S contains the interval $(a, 1]$ then it also contains a. The natural idea is to consider labellings $(u_i^{(t)} \,|\, i \in V)$, $t = 1, 2, \ldots$, satisfying 3.13 for a sequence of parameter values $a_t > a$, $a_t \to a$, take a convergent subsequence, and then take the limit. One problem is that some labels $u_i^{(t)}$ may tend to infinity, so a convergent subsequence. For $a > 0$, it also causes a problem if some $u_i^{(t)}$ tends to a point on the unit sphere, since then the limit does not satisfy 3.13.

The longest part of the proof is to show that if H has no strongly separating 3- or 4-cycles, and is different from the graphs in Figure 5, then such bad occurences can be corrected by applying appropriate unit-sphere-preserving projective transformations. This needs a careful analysis of cases, and is omitted from this survey. For details, we refer to [16]. ∎

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