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Perturbation Theory for Dual Semigroups

IV. The intertwining formula and the canonical pairing

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The theory of dual semigroups of bounded linear operators on non-reflexive Banach spaces is used to give a natural generalization of the notion of a bounded perturbation of the generator and a new version of the variation-of-constants formula. In order to handle the general (i.e. not necessarily sun-reflexive) case we introduce an intertwining formula to extend the semigroup from the closure of the domain of the generator to the whole space and, in addition, a duality pairing between the second semigroup-dual space $X^{\odot\odot}$ and X^* to describe the duality relations. We quickly review results pertinent to inhomogeneous linear problems, semilinear problems, compactness, positivity and irreducibility.

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1. Introduction

Motivated by the equations of physiologically structured population dynamics (Webb, 1985, Metz & Diekmann, 1986) and by functional differential equations (Hale, 1977, Diekmann, 1987) we have recently begun a systematic study of perturbed dual semigroups. The results so far reported in parts I, II and III of this series all pertain to the so-called sun-reflexive case (see below for the definition). In the present paper we shall deal with the general case which requires two new ideas: the *intertwining formula* as a tool to define a weak * continuous semigroup on a dual Banach space once a strongly continuous semigroup on a maximal closed subspace has been constructed and a *duality pairing* to describe the duality relations when the original space X is not invariant under the perturbed semigroup.

In section 2 we describe the heuristics of our approach. In section 3 we use the variation-of-constants formula to define the perturbed semigroup on the subspace $X^{\odot}$ constructively and then extend the semigroup to X^* by the intertwining formula. The duality pairing and the related sun-topology are introduced in section 4 and then used in section 5 to formulate and derive the relevant duality relations for the perturbed semigroup. A number of results about inhomogeneous linear problems and semilinear problems derived in part III are reformulated in section 6 such that they apply to the general case and the same is done in section 7 for results about compactness, positivity and irreducibility which were, for the sun-reflexive case, proved in section 9.3 of Clément, Heijmans et al., 1987.

Acknowledgement

Already for some years now our work has been stimulated by related work of W. Desch and W. Schappacher (and co-workers). In particular our awareness of the importance of the Favard class was triggered by their lectures and papers.

2. An outline of the basic ideas

This paper is about semigroups of bounded linear operators which are generated by a non-densely defined operator *and* by a densely defined operator. The clue of this paradox lies in the fact that on a dual Banach space one can either use the norm topology or the weak * topology to define the infinitesimal generator.

Let X^* denote the dual of a Banach space X, and let $T^*=\{T^*(t)\}_{t\geqslant 0}$ be the adjoint of a strongly continuous semigroup $T=\{T(t)\}_{t\geqslant 0}$ of bounded linear operators on X with infinitesimal generator A. Then (Butzer & Berens 1967, sections 1.4 and 2.1):

i) $T^*(t)$ is, for any fixed $t\geqslant 0$, a bounded linear operator on X^*.

ii) For any $x\in X$ and $x^*\in X^*$ the function $t\mapsto \langle x, T^*(t)x^*\rangle$ is continuous. In other words, orbits are continuous in the weak * topology.

iii) $\mathfrak{D}(A^*)=\{x^*\in X^*: h^{-1}(T^*(h)x^*-x^*)$ converges as $h\downarrow 0$ in the weak * topology$\}$ and $A^*x^*=w{*}\lim_{h\downarrow 0}h^{-1}(T^*(h)x^*-x^*)$. In other words, A^* is the $w*$ generator of T^*. Moreover $\mathfrak{D}(A^*)$ is weakly * dense in X^*.

iv) Define the Favard class by

$$\mathrm{Fav}(T^*)=\{x^*\in X^*: \limsup_{h\downarrow 0}\frac{1}{h}\|T^*(h)x^*-x^*\|<\infty\}\ .$$

Then $\mathrm{Fav}(T^*)=\mathfrak{D}(A^*)$.

v) Define $X^{\odot}=\overline{\mathfrak{D}(A^*)}$. Then $X^{\odot}$ is the (maximal) subspace of strong continuity of T^*, i.e. $X^{\odot}=\{x^*\in X^*: \|T^*(t)x^*-x^*\|\to 0 \text{ as } t\downarrow 0\}$.

vi) Let $T^{\odot}$ denote the restriction of T^* to $X^{\odot}$. The action of T^* can be reconstructed from the action of $T^{\odot}$ by the *intertwining formula*

$$T^*(t)=(\lambda I-A^*)T^{\odot}(t)(\lambda I-A^*)^{-1} \tag{2.1}$$

(note that the formula makes sense since $\mathfrak{D}(A^*)$ is invariant under $T^{\odot}$).

vii) The generator $A^{\odot}$ of $T^{\odot}$ is the part of A^* in $X^{\odot}$, i.e. $\mathfrak{D}(A^{\odot})=\{x^{\odot}\in\mathfrak{D}(A^*):A^*x^*\in X^{\odot}\}$ and $A^{\odot}x^{\odot}=A^*x^{\odot}$.

The properties listed above show in particular that A^* (which is, in general, not densely defined) generates a strongly continuous semigroup on the closure of its domain which we can extend to the whole space by the intertwining formula (2.1) since $\mathfrak{D}(A^*)$ is invariant. In this paper we shall deal with a class of semigroups for which one can follow the same procedure but for which there is not necessarily a pre-adjoint on the space X. In a forthcoming paper we shall present a Hille-Yosida type theorem in this spirit, but here we restrict ourselves to semigroups which are obtained from a dual semigroup T_0^* by a bounded linear perturbation C of the $w*$ generator A_0^*. Here "bounded" means "mapping $X^{\odot}$ continuously into X^*", and $X^{\odot}:=\overline{\mathfrak{D}(A_0^*)}$.

In a first step we define the "perturbed" semigroup $T^{\odot}$ on $X^{\odot}$ constructively in terms of the "unperturbed" semigroup T_0^* on X^* and the "perturbation" $C:X^{\odot}\to X^*$ by solving the variation-of-constants equation

$$T^{\odot}(t)x^{\odot}=T_0^{\odot}(t)x^{\odot}+\int_0^t T_0^*(t-\tau)CT^{\odot}(\tau)x^{\odot}d\tau\,. \tag{2.2}$$

Note that the integrand takes values in the "big" space X^*. The integral is defined as a weak * Riemann integral on X^* which, as can be proved, takes values in the subspace $X^{\odot}$ (see Lemma 3.1 below).

As to be expected, $h^{-1}(T^{\odot}(h)x^{\odot}-x^{\odot})$ converges in the weak * topology of X^* iff $x^{\odot}\in\mathfrak{D}(A_0^*)$ and in that case the limit equals $A^{\times}x^{\odot}:=A_0^*x^{\odot}+Cx^{\odot}$ (Note the notation: to indicate that an operator acts on X^* but is not necessarily the adjoint of an operator on X we write a $\times$ as superscript instead of a $*$). The invariance of $\mathfrak{D}(A_0^*)$ with respect to $T^{\odot}$ is a consequence of the fact that $\mathfrak{D}(A_0^*)=\text{Fav}\,(T^{\odot})$. Hence

$$T^{\times}(t)=(\lambda I-A^{\times})T^{\odot}(t)(\lambda I-A^{\times})^{-1} \tag{2.3}$$

makes sense and yields a semigroup $T^{\times}$ acting on X^* which has, mutatis mutandis, the properties i-vii listed above.

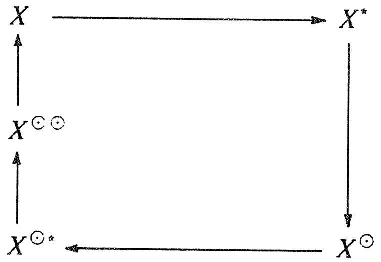
We can get much more information if we extend the duality framework as follows. Playing the game of taking adjoints and restrictions once more we obtain, in self-explaining notation, a semigroup $T_0^{\odot*}$ with $w*$ generator $A_0^{\odot*}$ acting on $X^{\odot*}$ and a strongly continuous semigroup $T_0^{\odot\odot}$ acting on $X^{\odot\odot}$ and generated by the part $A_0^{\odot\odot}$ of $A_0^{\odot*}$ in $X^{\odot\odot}$. The pairing between elements of X and $X^{\odot}$ yields an embedding $j:X\to X^{\odot*}$. Clearly $j(X)\subset X^{\odot\odot}$. Whenever $j(X)=X^{\odot\odot}$ we say that X is $\odot$-reflexive (the symbol $\odot$ denotes the sun; so one should pronounce this as "sun-reflexive") with respect to A_0 (or T_0).

In the $\odot$-reflexive case we can identify X and $X^{\odot\odot}$. So starting from the perturbed semigroup $T^{\odot}$ on $X^{\odot}$ we can, by taking adjoints and restrictions (indeed, $X^{\odot\odot}$ is invariant under $T^{\odot*}$), define a strongly continuous semigroup T on X. It then easily follows that $T^{\times}=T^*$ or, in other words, we actually stay in the realm of dual semigroups. Moreover we may as well describe the perturbation by a bounded linear operator $B:X\to X^{\odot*}$ and the variation-of-constants equation

$$T(t)x=T_0(t)x+\int_0^t T_0^{\odot*}(t-\tau)BT(\tau)x d\tau \tag{2.4}$$

provided we take for B the restriction of $C^*:X^{**}\to X^{\odot*}$ to X (which in turn implies that C is the restriction of $B^*:X^{\odot**}\to X^*$ to $X^{\odot}$). We refer to part I for a detailed analysis of the $\odot$-reflexive case and confine ourselves here to mentioning that de Pagter (this volume) has recently shown that $\odot$-reflexivity is precisely characterized by the weak compactness (and if X is an L^1-space even compactness) of the resolvent $(\lambda I - A_0)^{-1}$ (this improves a result of HILLE and PHILLIPS (1957)).

In the general case we have to deal with the asymmetric diagram



and in general $j(X)$ will not be invariant with respect to the perturbed semigroup $T^{\odot *}$. Hence $T^{\times}$ is in general not a truely dual semigroup. In addition $B: X \to X^{\odot *}$ defined as the restricted adjoint of C may have many continuous extensions $D: X^{\odot\odot} \to X^{\odot *}$ and it is, at this moment, unclear whether or not

$$T^{\odot\odot}(t)x^{\odot\odot} = T_0^{\odot\odot}(t)x^{\odot\odot} + \int_0^t T_0^{\odot *}(t-\tau)DT^{\odot\odot}(\tau)x^{\odot\odot}d\tau \tag{2.5}$$

for some bounded linear operator $D: X^{\odot\odot} \to X^{\odot *}$. It seems we cannot come full circle.

But appearances are deceptive. We shall show that the definitions

$$[x^{\odot\odot}, x^*] = \lim_{t \downarrow 0} \frac{1}{t} \langle x^{\odot\odot}, \int_0^t T_0^*(\tau)x^* d\tau \rangle \tag{2.6a}$$

$$[x^{\odot\odot}, x^*] = \lim_{\lambda \downarrow 0} \lambda \langle x^{\odot\odot}, (\lambda I - A_0^*)^{-1} x^* \rangle \tag{2.6b}$$

make sense, are equivalent and yield a duality pairing between $X^{\odot\odot}$ and X^* which is canonical in the sense that whenever a pairing is already defined (i.e. if either $x^{\odot\odot} \in j(X)$ or $x^* \in X^{\odot}$) the new one is identical to the old one. We then close the circle by showing that the duality relation

$$[T^{\odot\odot}(t)x^{\odot\odot}, x^*] = [x^{\odot\odot}, T^{\times}(t)x^*] \tag{2.7}$$

holds. Moreover, the pairing (2.6) defines an embedding of $X^{\odot\odot}$ into X^{**} and so we can take for $D: X^{\odot\odot} \to X^{\odot *}$ the restriction of $C^*: X^{**} \to X^{\odot *}$ to $X^{\odot\odot}$ and show that (2.5) holds (this amounts to defining D by $\langle Dx^{\odot\odot}, x^{\odot} \rangle = [x^{\odot\odot}, Cx^{\odot}]$; note that now we can define the perturbation either by $B: X \to X^{\odot *}$ or by $C: X^{\odot} \to X^*$ and that we have exploited the behaviour of the unperturbed semigroup T_0 to single out the extension $D: X^{\odot\odot} \to X^{\odot *}$ of B which is relevant for our purposes).

We can now endow X^* with a third topology, the $\sigma(X^*, X^{\odot\odot})$ - topology. Perhaps the appropriate name for this topology is the $\odot\odot$ weak $*$ topology, but we shall abbreviate this to $\odot$-topology. The duality relation (2.7) shows that, for fixed t, $T^{\times}(t)$ maps X^* endowed with the $\odot$-topology continuously into itself and, in addition, that orbits $t \mapsto T^{\times}(t)x^*$ are continuous with respect to the $\odot$-topology. We shall show that $h^{-1}(T^{\times}(h)x^* - x^*)$ converges in the $\odot$-topology iff $x^* \in \mathcal{D}(A_0^*)$ and that the limit equals $A_0^* x^* + Cx^*$.

Note that $T^{\times}(t)$ is *not* continuous as an operator of X^* equipped with the weak $*$ topology into itself unless there is a pre-adjoint on X. This indicates that the $\odot$-topology is more natural than the weak $*$ topology. The original space X is now in a side-line position and one may wonder why we do not leave it out of the picture completely. The reason is that the integrals on X^* are still defined as weak $*$ integrals. We cannot define the integrals on X^* by using the pairing with $X^{\odot\odot}$ since then we may end up in $X^{\odot\odot *}$ (we have no guarantee that the embedding of X^* into $X^{\odot\odot *}$ defined by [. , .] yields a (sequentially) weakly $*$ *closed* subspace). The fact that we use two different spaces X and $X^{\odot\odot}$ to achieve closedness under integration and invariance under the semigroup is one important aspect in which our work differs from the interesting but somewhat neglected paper of FELLER (1953).

3. PERTURBATION THEORY

Let X be a Banach space and let T_0 be a strongly continuous semigroup of bounded linear operators on X, with infinitesimal generator A_0. Let M and ω be such that $\|T_0(t)\| \leq Me^{\omega t}$. The following result is a "dual" reformulation of Lemma 3.1 and Theorem 3.2 in part I. Throughout this paper integrals are weak $*$ Riemann integrals unless explicitly stated otherwise.

LEMMA 3.1.

Let $f : [0, \infty) \to X^*$ be norm-continuous, then $t \mapsto \int_0^t T_0^*(t-\tau)f(\tau)d\tau$ is a norm-continuous function which takes values in the space $X^{\odot} := \overline{\mathfrak{D}(A_0^*)}$ and

$$\|\int_0^t T_0^*(t-\tau)f(\tau)d\tau\| \leq M\frac{e^{\omega t}-1}{\omega} \sup_{0 \leq \tau \leq t} \|f(\tau)\| . \tag{3.1}$$

Let $C: X^{\odot} \to X^*$ be a bounded linear operator. The above lemma is all we need to solve the variation-of-constants equation

$$T^{\odot}(t)x^{\odot} = T_0^{\odot}(t)x^{\odot} + \int_0^t T_0^*(t-\tau)CT^{\odot}(\tau)x^{\odot}d\tau \tag{3.2}$$

by successive approximations in the standard manner. Thus we obtain

THEOREM 3.2.

Equation (3.2) uniquely defines a strongly continuous semigroup $T^{\odot}$ on $X^{\odot}$. The successive approximations converge in the uniform operator topology, uniformly for t in compact sets.

It is convenient to provide the difference between the perturbed and the unperturbed semigroup with a name. So we define

$$U^{\odot}(t) = T^{\odot}(t) - T_0^{\odot}(t) = \int_0^t T_0^*(t-\tau)CT^{\odot}(\tau)d\tau. \tag{3.3}$$

From the estimate (3.1) we deduce

LEMMA 3.3.

$\|U^{\odot}(t)\| = O(t)$ *for* $t \downarrow 0$.

(We refer to Diekmann, Gyllenberg & Heijmans (this volume) for a converse of this result.)

For any semigroup S of bounded linear operators on a Banach space Z we define the Favard class of S by

$$\text{Fav}(S) = \{z \in Z : \limsup_{t \downarrow 0} \frac{1}{t}\|S(t)z - z\| < \infty\} . \tag{3.4}$$

The semigroup property implies at once that Fav (S) is invariant under S. So as a corollary of Lemma 3.3 and a well-known result of dual semigroup theory (BUTZER & BERENS (1967) Thm. 2.1.4) we obtain

THEOREM 3.4.

Fav $(T^{\odot}) = \text{Fav}(T_0^{\odot}) = \mathfrak{D}(A_0^*)$. *As a consequence $\mathfrak{D}(A_0^*)$ is invariant under $T^{\odot}$.*

From the integral representation (3.3) it follows immediately that

LEMMA 3.5.
$\frac{1}{t}<x,\ U^{\odot}(t)x^{\odot}> \to <x,\ Cx^{\odot}>$ *as* $t\downarrow 0$, *for every* $x\in X$ *and* $x^{\odot}\in X^{\odot}$

COROLLARY 3.6.
$t^{-1}(T^{\odot}(t)x^{\odot}-x^{\odot})$ *converges in the weak* $*$ *topology as* $t\downarrow 0$ *iff* $x^{\odot}\in\mathfrak{D}(A_0^*)$ *and the limit equals* $A_0^*x^{\odot}+Cx^{\odot}$.

Motivated by this result we define

$$A^{\times}=A_0^*+C \quad \text{with } \mathfrak{D}(A^{\times})=\mathfrak{D}(A_0^*) \tag{3.5}$$

THEOREM 3.7.
The infinitesimal generator $A^{\odot}$ *of* $T^{\odot}$ *is the part of* $A^{\times}$ *in* $X^{\odot}$.

PROOF.
If $t^{-1}(T^{\odot}(t)x^{\odot}-x^{\odot})$ converges strongly it certainly converges weakly $*$. Corollary 3.6 therefore shows that $x^{\odot}\in\mathfrak{D}(A^{\odot})$ requires that $x^{\odot}\in\mathfrak{D}(A_0^*)$ and $A^{\times}x^{\odot}\in X^{\odot}$ and that $A^{\odot}x^{\odot}=A^{\times}x^{\odot}$. Take any $x^{\odot}\in\mathfrak{D}(A_0^*)$ such that $A^{\times}x^{\odot}\in X^{\odot}$. Then we can rewrite (3.2) in the form

$$T^{\odot}(t)x^{\odot}-x^{\odot}=\int_0^t T_0^*(\tau)(A_0^*x^{\odot}+Cx^{\odot})d\tau+\int_0^t T_0^*(t-\tau)C(T^{\odot}(\tau)x^{\odot}-x^{\odot})d\tau\,.$$

In the first term at the right-hand side we can replace T_0^* by $T_0^{\odot}$ and re-interpret the integral as a strong integral. It follows that t^{-1} times this term converges strongly to $A_0^*x^{\odot}+Cx^{\odot}=A^{\times}x^{\odot}$. A straight-forward estimate shows that t^{-1} times the second term converges strongly to zero as $t\downarrow 0$. □

We emphasize that when we take the part of $A^{\times}$ in $X^{\odot}$ information about the action of C may end up in the description of the domain. In fact in several examples (see part I and DIEKMANN, 1987) *all* information about C is incorporated in the domain in the form of a boundary condition. In GREINER (to appear) such problems are studied directly (i.e. without introducing a larger dual space in which the original space lies embedded).

Using the intertwining formula for T_0^* one can formally rewrite (3.2) as

$$T^{\odot}(t)x^{\odot}=T_0^{\odot}(t)x^{\odot}+(\lambda I-A_0^{\odot})\int_0^t T_0^{\odot}(t-\tau)(\lambda I-A_0^*)^{-1}CT^{\odot}(\tau)x^{\odot}d\tau$$

and forget about weak $*$ integrals (but worry about taking values in $\mathfrak{D}(A_0^{\odot})$ instead). This is the approach of Desch, Schappacher and co-workers.

Our next step is to extend $T^{\odot}$ to the "big" space X^* by the intertwining formula

$$T^{\times}(t)=(\lambda I-A^{\times})T^{\odot}(t)(\lambda I-A^{\times})^{-1}\,. \tag{3.6}$$

Note that this definition makes sense since $\mathfrak{D}(A^{\times})=\mathfrak{D}(A_0^*)$ is invariant under $T^{\odot}$ and that the resolvent identity implies that the definition is independent of the choice of $\lambda\in\rho(A^{\times})$. Theorem 3.7 shows that $T^{\times}$ restricted to $X^{\odot}$ equals $T^{\odot}$, so $T^{\times}$ is indeed an extension of $T^{\odot}$.

It is possible to deduce further properties of $T^{\times}$ from the Hille-Yosida estimates for $(\lambda I-A^{\times})^{-1}$ and a certain closedness property of $A^{\times}$. This is the approach adopted in a paper which we hope to finish very soon. Here we shall follow another road and concentrate on the duality framework.

4. The duality pairing

Starting from $X^{\odot}$ and the strongly continuous semigroup $T_0^{\odot}$ we can introduce the dual space $X^{\odot *}$ and the adjoint semigroup $T_0^{\odot *}$. Let $X^{\odot\odot} = \mathfrak{D}(A_0^{\odot *})$ be the maximal subspace of strong continuity of $T_0^{\odot *}$ and let $T_0^{\odot\odot}$ denote the restriction of $T_0^{\odot *}$ to $X^{\odot\odot}$. Then $T_0^{\odot\odot}$ is generated by $A_0^{\odot\odot}$, the part of $A_0^{\odot *}$ in $X^{\odot\odot}$. By $<jx,x^{\odot}> = <x,x^{\odot}>$ we define a continuous injection $j:X \to X^{\odot *}$. Then $j(X)$ is a closed subspace of $X^{\odot\odot}$. When $j(X) = X^{\odot\odot}$ we have automatically a duality pairing between $X^{\odot\odot}$ and X^* since we have one between X and X^*. We shall show that even when $j(X) \neq X^{\odot\odot}$ we can define a pairing between $X^{\odot\odot}$ and X^* which now, however, involves the behaviour of the semigroup for small t (or, equivalently, the behaviour of the resolvent for large λ).

Theorem 4.1.
For any $x^{\odot\odot} \in X^{\odot\odot}$ and $x^* \in X^*$ the limits

$$\lim_{t\downarrow 0} \frac{1}{t} < x^{\odot\odot}, \int_0^t T_0^*(\tau)x^* d\tau> \tag{4.1}$$

and

$$\lim_{\lambda\to\infty} \lambda < x^{\odot\odot},(\lambda I - A_0^*)^{-1}x^*> \tag{4.2}$$

exist and are equal to each other. Let $[x^{\odot\odot},x^*]$ denote the common limit then

$$|[x^{\odot\odot},x^*]| \leq M\|x^{\odot\odot}\|\|x^*\|. \tag{4.3}$$

Moreover, $[x^{\odot\odot},x^{\odot}] = <x^{\odot\odot},x^{\odot}>$ for all $x^{\odot} \in X^{\odot}$ and $[jx,x^*] = <x,x^*>$ for all $x \in X$.

Proof.
We first observe that

$$X^{\odot} = \{x^* \in X^* : \lim_{t\downarrow 0} \|\frac{1}{t}\int_0^t T_0^*(\tau)x^* d\tau - x^*\| = 0\} \tag{4.4}$$

and

$$X^{\odot} = \{x^* \in X^* : \lim_{\lambda\to\infty} \|\lambda(\lambda I - A_0^*)^{-1}x^* - x^*\| = 0\} \tag{4.5}$$

give alternative equivalent characterizations of $X^{\odot}$. Take any $x^{\odot\odot} \in \mathfrak{D}(A_0^{\odot *})$ and fix $\mu > \omega$ then

$$\frac{1}{t}<x^{\odot\odot},\int_0^t T_0^*(\tau)x^* d\tau> = \frac{1}{t}<(\mu I - A_0^{\odot *})x^{\odot\odot},(\mu I - A_0^{\odot})^{-1}\int_0^t T_0^*(\tau)x^* d\tau> =$$

$$= \frac{1}{t}<(\mu I - A_0^{\odot *})x^{\odot\odot},\int_0^t T_0^*(\tau)(\mu I - A_0^*)^{-1}x^* d\tau>$$

$$\to <(\mu I - A_0^{\odot *})x^{\odot\odot},(\mu I - A_0^*)^{-1}x^*> \qquad \text{as } t\downarrow 0.$$

Similarly

$$\lambda<x^{\odot\odot},(\lambda I - A_0^*)^{-1}x^*> = \lambda<(\mu I - A_0^{\odot *})^{-1}(\mu I - A_0^{\odot *})x^{\odot\odot},(\lambda I - A_0^*)^{-1}x^*> =$$

$$= \lambda<(\mu I - A_0^{\odot *})x^{\odot\odot},(\lambda I - A_0^*)^{-1}(\mu I - A_0^*)^{-1}x^*>$$

$$\to <(\mu I - A_0^{\odot *})x^{\odot\odot},(\mu I - A_0^*)^{-1}x^*> \qquad \text{as } \lambda\to\infty.$$

The estimates

$$\lambda\|(\lambda I - A_0^*)^{-1}\| \leq \frac{M\lambda}{\lambda - \omega}$$

$$\frac{1}{t}\|\int_0^t T_0^*(\tau)d\tau\| \leqslant \frac{M(e^{\omega t}-1)}{\omega t}$$

and the fact that $\mathcal{D}(A_0^{\odot *})$ is dense in $X^{\odot\odot}$ imply at once that the limits exist and equal each other for arbitrary $x^{\odot\odot}\in X^{\odot\odot}$ and that estimate (4.3) holds. Combining either (4.1) and (4.4) or (4.2) and (4.5) we find that $[x^{\odot\odot},x^{\odot}]=<x^{\odot\odot},x^{\odot}>$ for all $x^{\odot}\in X^{\odot}$. Finally, if $x^{\odot\odot}=jx$ we have that

$$\lambda<x^{\odot\odot},(\lambda I-A_0^*)^{-1}x^*>=\lambda<(\lambda I-A_0)^{-1}x,x^*>\rightarrow<x,x^*>$$

for $\lambda\rightarrow\infty$. □

COROLLARY 4.2.
The definition $<kx^{\odot\odot},x^*>=[x^{\odot\odot},x^*]$ *yields an embedding* $k:X^{\odot\odot}\rightarrow X^{**}$ *and* $1\leqslant\|k\|\leqslant M$.

THEOREM 4.3.

(i) $[T_0^{\odot\odot}(t)x^{\odot\odot},\ x^*]=[x^{\odot\odot},\ T_0^*(t)x^*]$

(ii) $<A_0^{\odot *}x^{\odot\odot},x^{\odot}>=[x^{\odot\odot},\ A_0^*x^{\odot}],\qquad x^{\odot\odot}\in\mathcal{D}(A_0^{\odot *}),\ x^{\odot}\in\mathcal{D}(A_0^*)$

(iii) $[x^{\odot\odot},\ \int_0^t T_0^*(t-\tau)f(\tau)d\tau]=\int_0^t[T_0^{\odot\odot}(t-\tau)x^{\odot\odot},\ f(\tau)]d\tau$
for any norm continuous function $f:\mathbb{R}_+\rightarrow X^*$.

PROOF.
Since the proof of (i) and (ii) is rather straightforward we prove only (iii). First consider $x^{\odot\odot}\in\mathcal{D}(A_0^{\odot *})$. Then

$$[x^{\odot\odot},\int_0^t T_0^*(t-\tau)f(\tau)d\tau]=<(\lambda I-A_0^{\odot *})x^{\odot\odot},(\lambda I-A_0^{\odot})^{-1}\int_0^t T_0^*(t-\tau)f(\tau)d\tau>$$

$$=<(\lambda I-A_0^{\odot *})x^{\odot\odot},\int_0^t T_0^{\odot}(t-\tau)(\lambda I-A_0^*)^{-1}f(\tau)d\tau> = \text{(now we have a strong integral)}$$

$$=\int_0^t<(\lambda I-A_0^{\odot *})T_0^{\odot\odot}(t-\tau)x^{\odot\odot},(\lambda I-A_0^*)^{-1}f(\tau)>d\tau = \int_0^t[T_0^{\odot\odot}(t-\tau)x^{\odot\odot},f(\tau)]d\tau.$$

Since $\mathcal{D}(A_0^{\odot *})$ is dense in $X^{\odot\odot}$ and $T_0^{\odot\odot}(t)$ is exponentially bounded the same identity holds for all $x^{\odot\odot}\in X^{\odot\odot}$. □

REMARK.
The definition of the weak $*$ integral is such that

$$<x^{**},\int_0^t T_0^*(t-\tau)f(\tau)d\tau> = \int_0^t<T_0^{**}(t-\tau)x^{**},f(\tau)>d\tau,\qquad x^{**}\in i(X),$$

where i denotes the natural embedding of X into X^{**}. The identity does not extend to general $x^{**}\in X^{**}$. However, Theorem 4.3 (iii) shows that it does extend to $x^{**}\in k(X^{\odot\odot})$!

THEOREM 4.4.

(i) *For fixed* $t\geqslant 0$, *the linear operator* $T_0^*(t)$ *maps* X^* *equipped with the* $\odot$*-topology (i.e. the* $\sigma(X^*,X^{\odot\odot})$ *-topology) continuously into itself.*

(ii) *Orbits are* $\odot$*-continuous, i.e.* $t\mapsto T_0^*(t)x^*$ *is (for any given* x^**) continuous from* $\mathbb{R}_+$ *into* X^* *equipped with the* $\odot$*-topology.*

(iii) $t^{-1}(T_0^*(t)x^* - x^*)$ *converges with respect to the* $\odot$*-topology as* $t\downarrow 0$ *iff* $x^* \in \mathfrak{D}(A_0^*)$ *and in that case the limit equals* $A_0^* x^*$.

PROOF.

(i) and (ii) follow immediately from Theorem 4.3 (i) and the fact that $T_0^{\odot\odot}$ is a strongly continuous semigroup of bounded linear operators on $X^{\odot\odot}$.

If $t^{-1}(T_0^*(t)x^* - x^*)$ converges with respect to the $\odot$-topology as $t\downarrow 0$ it certainly converges with respect to the weak $*$ topology and therefore $x^* \in \mathfrak{D}(A_0^*)$ and the limit equals $A_0^* x^*$. Next take any $x^* \in \mathfrak{D}(A_0^*)$. Then $T_0^*(t)x^* - x^* = \int_0^t T_0^*(\tau)A_0^* x^* d\tau$ and consequently Theorem 4.3 (iii) implies that $t^{-1}[x^{\odot\odot}, T_0^*(t)x^* - x^*] \to [x^{\odot\odot}, A_0^* x^*]$ as $t\downarrow 0$. □

In the next section we shall show that for $T^\times$ the analogue of Theorem 4.4 holds. In other words, the properties (i)-(iii) of this theorem are preserved under bounded perturbations of the generator. In contrast the weak $*$ continuity of the operator $T_0^*(t)$ (for fixed $t>0$) may *not* be preserved under such perturbations.

Note that we cannot use the terminology and results of YOSIDA (1965), Chapter IX, to formulate Theorem 4.4 since we cannot be sure that X^* is sequentially complete in the $\odot$-topology.

5. THE DUALITY RELATIONS PERSIST UNDER BOUNDED PERTURBATIONS

We now exploit the duality pairing defined in terms of the unperturbed semigroup to investigate the perturbed semigroup. Since $\|U^\odot(t)\| = O(t)$ (Lemma 3.3) we know that $t \mapsto T^{\odot *}(t)x^{\odot *}$ is continuous iff $x^{\odot *} \in X^{\odot\odot}$ or, in other words, that $X^{\odot\odot}$ is the maximal subspace of strong continuity for $T^{\odot *}$. As an immediate consequence of Theorem 4.3 (iii) we have

LEMMA 5.1.

$t^{-1} < x^{\odot\odot}, U^\odot(t)x^\odot > \to [x^{\odot\odot}, Cx^\odot]$ *as* $t\downarrow 0$, *for every* $x \in X$ *and* $x^\odot \in X^\odot$.

THEOREM 5.2.

$\mathfrak{D}(A^{\odot *}) = \mathfrak{D}(A_0^{\odot *})$ *and* $A^{\odot *} = A_0^{\odot *} + D$ *with* $D: X^{\odot\odot} \to X^{\odot *}$ *the restricted adjoint of* C *defined by* $<Dx^{\odot\odot}, x^\odot> = [x^{\odot\odot}, Cx^\odot]$. *For D thus defined the variation-of-constants formula (2.5) holds.*

THEOREM 5.3.

(i) $[T^{\odot\odot}(t)x^{\odot\odot}, x^*] = [x^{\odot\odot}, T^\times(t)x^*]$, *for all* $x^{\odot\odot} \in X^{\odot\odot}$ *and* $x^* \in X^*$.

(ii) $<A^{\odot *}x^{\odot\odot}, x^\odot> = [x^{\odot\odot}, A^\times x^\odot]$, *for all* $x^{\odot\odot} \in \mathfrak{D}(A_0^{\odot *})$ *and* $x^\odot \in \mathfrak{D}(A_0^*)$.

(iii) $[x^{\odot\odot}, \int_0^t T^\times(\tau)x^* d\tau] = \int_0^t [x^{\odot\odot}, T^\times(\tau)x^*]d\tau = \int_0^t [T^{\odot\odot}(\tau)x^{\odot\odot}, x^*]d\tau = [\int_0^t T^{\odot\odot}(\tau)x^{\odot\odot}d\tau, x^*]$

(iv) *Let* $f: \mathbb{R}_+ \to X^*$ *be norm continuous. Then* $\int_0^t T^\times(t-\tau)f(\tau)d\tau \in X^\odot$ *and*

$$<x^{\odot\odot}, \int_0^t T^\times(t-\tau)f(\tau)d\tau> = \int_0^t [T^{\odot\odot}(t-\tau)x^{\odot\odot}, f(\tau)]d\tau.$$

PROOF.

(ii) is an immediate consequence of Theorem 5.2 and Theorem 4.3 (ii). To prove (i) we first take $x^{\odot\odot} \in \mathfrak{D}(A_0^{\odot *})$. Then

$$\begin{aligned}
[x^{\odot\odot}, T^\times(t)x^*] &= [x^{\odot\odot}, (\lambda I - A^\times)T^\odot(t)(\lambda I - A^\times)^{-1}x^*] = \\
&= <(\lambda I - A^{\odot *})x^{\odot\odot}, T^\odot(t)(\lambda I - A^\times)^{-1}x^*> \\
&= <(\lambda I - A^{\odot *})T^{\odot\odot}(t)x^{\odot\odot}, (\lambda I - A^\times)^{-1}x^*> = [T^{\odot\odot}(t)x^{\odot\odot}, x^*]
\end{aligned}$$

Since $\overline{\mathfrak{D}(A_0^{\odot *})} = X^{\odot\odot}$ and $T^{\odot\odot}(t)$ is bounded we can extend the identity to arbitrary $x^{\odot\odot} \in X^{\odot\odot}$. Before proving (iii) we first observe that (i) guarantees the weak $*$ continuity of $t \mapsto T^{\times}(t)x^*$. Next we show that

$$\int_0^t T^{\times}(\tau)x^* d\tau = (\lambda I - A^{\odot})\int_0^t T^{\odot}(\tau)(\lambda I - A^{\times})^{-1}x^* d\tau. \tag{5.1}$$

Consider $x \in j^{-1}(\mathfrak{D}(A_0^{\odot *}))$ then

$$<x, \int_0^t T^{\times}(\tau)x^* d\tau> = \int_0^t <x, (\lambda I - A^{\times})T^{\odot}(\tau)(\lambda I - A^{\times})^{-1}x^*> d\tau$$

$$= \int_0^t [(\lambda I - A^{\odot *})jx,\ T^{\odot}(\tau)(\lambda I - A^{\times})^{-1}x^*]d\tau = [(\lambda I - A^{\odot *})jx,\ \int_0^t T^{\odot}(\tau)(\lambda I - A^{\times})^{-1}x^* d\tau]$$

$$= <x, (\lambda I - A^{\odot})\int_0^t T^{\odot}(\tau)(\lambda I - A^{\times})^{-1}x^* d\tau>.$$

Since $j^{-1}(\mathfrak{D}(A_0^{\odot *}))$ is dense in X we have proved (5.1). Next take $x^{\odot\odot} \in \mathfrak{D}(A_0^{\odot *})$. Then, because of (5.1),

$$[x^{\odot\odot}, \int_0^t T^{\times}(\tau)x^* d\tau] = <(\lambda I - A^{\odot *})x^{\odot\odot}, \int_0^t T^{\odot}(\tau)(\lambda I - A^{\times})^{-1}x^* d\tau>$$

$=$ since the integral is a strong integral now $=$

$$= \int_0^t <(\lambda I - A^{\odot *})x^{\odot\odot}, T^{\odot}(\tau)(\lambda I - A^{\times})^{-1}x^*> d\tau$$

$$= \int_0^t [x^{\odot\odot}, (\lambda I - A^{\times})T^{\odot}(t)(\lambda I - A^{\times})^{-1}x^*]d\tau = \int_0^t [x^{\odot\odot}, T^{\times}(\tau)x^*]d\tau$$

$$= \int_0^t [T^{\odot\odot}(\tau)x^{\odot\odot}, x^*]d\tau.$$

Finally we prove (iv). Since f is continuous the integral $\int_0^t T^{\times}(\tau)f(t-\tau)d\tau$ is the *strong* limit as $n \to \infty$ of

$$\sum_{j=0}^{n-1} \int_{j\frac{t}{n}}^{(j+1)\frac{t}{n}} T^{\times}(\tau)f(t-\tau_j)d\tau$$

with $jt/n \leqslant \tau_j \leqslant (j+1)t/n$. Since each term belongs to X (see (5.1)), so does the limit of the sum. Using (iii) we find that

$$<x^{\odot\odot}, \int_0^t T^{\times}(t-\tau)f(\tau)d\tau> = \lim_{n\to\infty} \sum_{j=0}^{n-1} \int_{j\frac{t}{n}}^{(j+1)\frac{t}{n}} [T^{\odot\odot}(\tau)x^{\odot\odot}, f(t-\tau_j)]d\tau$$

$$= \int_0^t [T^{\odot\odot}(\tau)x^{\odot\odot}, f(t-\tau)]d\tau. \qquad \square$$

REMARK 5.4.
The strong continuity of $T^{\odot\odot}$ and Theorem 5.3 (iii) imply that

$$[x^{\odot\odot},x^*] = \lim_{t\downarrow 0}\frac{1}{t}<x^{\odot\odot},\int_0^t T^{\times}(\tau)x^* d\tau> \tag{5.2}$$

which is the analogue of the definition of $[\cdot,\cdot]$ by (4.1). Similarly one proves

$$[x^{\odot\odot},x^*] = \lim_{\lambda\to\infty}\lambda<x^{\odot\odot},(\lambda I-A^{\times})^{-1}x^*> \tag{5.3}$$

by exploiting the Laplace transformed variation-of-constants formula

$$(\lambda I-A^{\times})^{-1} = (\lambda I-A_0^*)^{-1}+(\lambda I-A_0^*)^{-1}C(\lambda I-A^{\times})^{-1}$$

to conclude that $\|(\lambda I-A^{\times})^{-1}-(\lambda I-A_0^*)^{-1}\| = O(\lambda^{-2})$, $\lambda\to\infty$.

THEOREM 5.5.

(i) *For fixed $t\geq 0$, the linear operator $T^{\times}(t)$ maps X^* equipped with the $\odot$-topology continuously into itself*

(ii) The orbit $t\mapsto T^{\times}(t)x^*$ is $\odot$-continuous for all $x^*\in X^*$

(iii) $t^{-1}(T^{\times}(t)x^*-x^*)$ *converges with respect to the $\odot$-topology iff $x^*\in\mathfrak{D}(A_0^*)$ and in that case the limit equals $A_0^*x^*+Cx^*$.*

PROOF.
(i) and (ii) follow directly from Theorem 5.3 (i). Lemma 5.1 and Theorem 4.4 (iii) together imply (iii). □

6. INHOMOGENEOUS LINEAR PROBLEMS AND NONLINEAR PERTURBATIONS

Let us now consider the equations

$$\frac{du}{dt} = A^{\times}u + f \quad\text{and}\quad \frac{du}{dt} = A^{\times}u+F(u)$$

where f is a given X^*-valued function and $F:X^{\odot}\to X^*$ is assumed to be globally (for ease of formulation) Lipschitz continuous. We shall consider $X^{\odot}$ as the state space (so we restrict our attention to initial values in $X^{\odot}$) but conceive of the differential equation as an identity in the wider space X^*. Hence the precise meaning of d/dt depends on the choice of the topology for X^*.

For a given norm continuous function $f:\mathbb{R}_+\to X^*$ we define

$$v(t) = \int_0^t T^{\times}(t-\tau)f(\tau)d\tau. \tag{6.1}$$

As a corollary of Theorem 5.3 (iv) we have

LEMMA 6.1.
For any $x^{\odot\odot}\in\mathfrak{D}(A^{\odot\odot})$ the function $t\mapsto<x^{\odot\odot},v(t)>$ is continuously differentiable and

$$\frac{d}{dt}<x^{\odot\odot},v(t)> = <A^{\odot\odot}x^{\odot\odot},v(t)>+[x^{\odot\odot},f(t)].$$

The next result is meant as a tool in the study of the semilinear problem, but it has some intrinsic interest. Stronger results for the special case in which $A^{\times}$ is densely defined were obtained by BEIRÃO DA VEIGA (1979) (we thank Professor G. da Prato for bringing this to our attention).

THEOREM 6.2.

Let $f:\mathbb{R}_+ \to X^*$ be locally Lipschitz continuous. Then $t \mapsto v(t)$ is continuously differentiable with respect to the $\odot$-topology. Moreover v takes values in $\mathcal{D}(A^\times)$ and

$$\odot - \frac{d}{dt}v(t) = A^\times v(t) + f(t) \tag{6.2}$$

where $\odot - \frac{d}{dt}$ denotes the derivative with respect to the $\odot$-topology.

PROOF.

To show that v takes values in $\mathcal{D}(A^\times)$ we use that $\mathcal{D}(A^\times) = Fav(T^\odot)$. Theorem 5.3 (iv) implies that

$$T^\odot(h)v(t) = \int_0^t T^\times(t+h-\tau)f(\tau)d\tau.$$

Hence

$$h^{-1}\|T^\odot(h)v(t) - v(t)\| \leq h^{-1}\|\int_t^{t+h} T^\times(s)f(t+h-s)ds\|$$

$$+h^{-1}\|\int_h^t T^\times(s)\{f(t+h-s)-f(t-s)\}ds\| + h^{-1}\|\int_0^h T^\times(s)f(t-s)ds\|$$

which stays bounded as $h \downarrow 0$. It also follows from this estimate that $\|A^\times v(t)\|$ is uniformly bounded on compact t-intervals. This observation and the fact that for $x^{\odot\odot}$ in the *dense* subset $\mathcal{D}(A^{\odot\odot})$ the function $t \mapsto [x^{\odot\odot}, A^\times v(t)] = \langle A^{\odot\odot}x^{\odot\odot}, v(t)\rangle$ is continuous, together imply that for *any* $x^{\odot\odot}$ the function $t \mapsto [x^{\odot\odot}, A^\times v(t)]$ is continuous.

For $x^{\odot\odot} \in \mathcal{D}(A^{\odot\odot})$ we know from Lemma 6.1 that

$$\frac{d}{dt}\langle x^{\odot\odot}, v(t)\rangle = [x^{\odot\odot}, A^\times v(t) + f(t)]$$

and hence, using $v(0)=0$, that

$$\langle x^{\odot\odot}, v(t)\rangle = \int_0^t [x^{\odot\odot}, A^\times v(\tau) + f(\tau)]d\tau.$$

Since $\mathcal{D}(A^{\odot\odot})$ is dense in $X^{\odot\odot}$ the last identity actually holds for all $x^{\odot\odot} \in X^{\odot\odot}$. □

We now turn to semilinear problems. With the Cauchy problem

$$\frac{du}{dt} = A^\times u + F(u), \quad u(0) = x^\odot, \tag{6.3}$$

one can associate the integral equation

$$u(t) = T^\odot(t)x^\odot + \int_0^t T^\times(t-\tau)F(u(\tau))d\tau \tag{6.4}$$

which is easily solved by a standard application of the contraction mapping theorem. Exactly as in part III one proves that one obtains a nonlinear semigroup on $X^\odot$ by solving (6.4) and that for initial data in the dense set $\mathcal{D}(A^\times) = \mathcal{D}(A_0^*)$ the orbits satisfy the differential equation in the "sunny" sense. More precisely we have:

THEOREM 6.3.

If $x^\odot \in \mathcal{D}(A^\times)$ then the solution u of the integral equation (6.4) is $\odot$-continuously differentiable and

$$\odot - \frac{d}{dt}u = A^\times u + F(u) \tag{6.5}$$

In general one cannot extend the nonlinear semigroup to $X^{\odot}$, but in section 5 of part III we showed that some special conditions (motivated by concrete applications) guarantee that such an extension does exist.

Next suppose that $F(0)=0$ and that F is continuously Fréchet differentiable. Then $u=0$ is an equilibrium solution. In view of the development of the local stability and bifurcation theory of equilibrium solutions it is important to know that u also satisfies the alternative variation-of-constants equation

$$u(t) = S^{\odot}(t)x^{\odot} + \int_0^t S^{\times}(t-\tau)G(u(\tau))d\tau \tag{6.6}$$

where $S^{\odot}$ is the C_0-semigroup on $X^{\odot}$ generated by the part of $A^{\times}+DF(0)$ in $X^{\odot}$, $S^{\times}$ is the extension of $S^{\odot}$ to X^{*} and $G(u)=F(u)-DF(0)u$. That indeed (6.6) holds can be either shown by formula manipulation (see the proof of Proposition 2.5 in part III) or by specifying a solution concept for which one shows uniqueness (here there are various possibilities: one can take Lemma 6.1 as a starting point or work with integral solutions in the sense of da Prato and Sinestrari) and such that both (6.4) and (6.6) yield solutions. From (6.6) one obtains almost directly that $S^{\odot}(t)$ is the Fréchet derivative of the nonlinear semigroup at $x^{\odot}=0$ and from this the principle of linearized stability follows. We refer once more to part III for a more precise formulation and detailed proofs.

7. Compactness and positivity

In many situations, the study of the large time behaviour of a C_0-semigroup amounts to the investigation of its spectral properties. It is known that (eventual) compactness, positivity and irreducibility of a semigroup can be of considerable help in establishing precise relations between the spectrum of the semigroup and the spectrum of its generator and in locating the element of the latter spectrum with the largest real part. For a number of detailed expositions on the asymptotic behaviour of C_0-semigroups we refer to NAGEL (1986), WEBB (1985), CLÉMENT, HEIJMANS et al. (1987). In this section we shall state a number of results on compactness and positivity for the semigroup $T^{\odot}(t)$ which is obtained via a bounded perturbation of $T_0^{\odot}(t)$ as described above. Similar results can be found in Section 9.3 of CLÉMENT, HEIJMANS et al. (1987) and since the proofs given there carry over to the present situation almost word for word, we shall not repeat them here. We also describe some consequences for the semigroup $T^{\times}$.

Let $T_0^{\odot}(t), T^{\odot}(t)$ and $U^{\odot}(t)$ be as before. Furthermore we define for $\lambda\in\rho(A_0)$

$$K_\lambda := (\lambda I - A_0^{*})^{-1}C \tag{7.1}$$

which is a bounded linear operator mapping $X^{\odot}$ into itself. We give conditions which guarantee that $U^{\odot}(t)$ is eventually compact (i.e., compact after finite time). It is not difficult to show that eventual compactness of $U^{\odot}(t)$ implies eventual norm-continuity. To prove the converse we need an extra assumption.

THEOREM 7.1.
Let $t\mapsto U^{\odot}(t)$ be norm continuous on $[t_0,\infty)$ for some $t_0\geq 0$, and let K_λ be compact for some (and hence for all) $\lambda\in\rho(A_0)$. Then $U^{\odot}(t)$ is compact for every $t\geq t_0$.

The following theorem may be of help in checking the norm continuity of $U^{\odot}(t)$. Let $T_k^{\odot}(t)$ be defined iteratively by

$$T_k^{\odot}(t)x^{\odot} = \int_0^t T_0^{*}(t-\tau)CT_{k-1}^{\odot}(\tau)x^{\odot}d\tau, \qquad k\geq 1 .$$

Then

$$U^{\odot}(t) = \sum_{k=1}^{\infty} T_k^{\odot}(t), \qquad t\geq 0 . \tag{7.2}$$

THEOREM 7.2.
Let $k \geqslant 1$ and $t_0 \geqslant 0$. If $T_1^{\odot}(t) = \cdots = T_{k-1}^{\odot}(t) = 0$ for $t \geqslant t_0$ and $T_k^{\odot}(t)$ is norm continuous for $t \geqslant 0$, then $U^{\odot}(t)$ is norm continuous for $t \geqslant 0$.

Before we start describing the consequences for the semigroup $T^{\times}(t)$ we note that Corollary 4.2 allows to consider X^* as a (strongly) closed subspace of $X^{\odot\odot *}$. By Theorem 4.3 and 5.3, $T_0^{\odot\odot *}(t)$ and $T^{\odot\odot *}(t)$ leave X^* invariant and their restrictions to X^* are identical to $T_0^*(t)$ and $T^*(t)$. This has the following immediate consequences.

THEOREM 7.3.
If $T^{\odot}(t)$ is norm-continuous (compact) at some point t_0, so is $T^{\times}(t)$. The same statement holds for $T_0^{\odot}$ and T_0^* as well as for $U^{\odot}$ and $U^{\times}$ with $U^{\times}(t) := T^{\times}(t) - T_0^*(t)$.

THEOREM 7.4.
If $T^{\odot}(t)$ is norm-continuous for $t \geqslant t_0$, then $T^{\times}(t)$ maps X^* into $X^{\odot}$ for $t \geqslant t_0$. The same holds for $U^{\odot}$ and $U^{\times}$.

PROOF.
For $T^{\odot}$ and $T^{\times}$ the result follows immediately from Theorem 7.3 and the semigroup property. For $U^{\odot}$ and $U^{\times}$ we use the identity:

$$U^{\times}(t+h) = U^{\times}(h)T^{\times}(t) + T_0^*(h)U^{\times}(t).$$

Let $t \geqslant t_0$ and $x^* \in X^*$. Then,

$$T_0^*(h)U^{\times}(t)x^* - U^{\times}(t)x^* = U^{\times}(t+h)x^* - U^{\times}(t)x^* - U^{\times}(h)T^{\times}(t)x^* .$$

According to Lemma 3.3 and Theorem 7.3 $\|U^{\times}(h)\| \to 0$ as $h \downarrow 0$. According to our assumption and Theorem 7.3 $\|U^{\times}(t+h) - U^{\times}(t)\| \to 0$ as $h \downarrow 0$. Hence $\|T_0^*(h)U^{\times}(t)x^* - U^{\times}(t)x^*\| \to 0$ as $h \downarrow 0$. Since $X^{\odot}$ is the *maximal* subspace of strong continuity of T_0^* this implies that $U^{\times}(t)x^* \in X^{\odot}$. □

Since the development of the theory of positive C_0-semigroups by the Functional Analysis group in Tübingen (NAGEL, 1986) positivity arguments have become an important tool in the study of spectral properties of C_0-semigroups. In particular, irreducibility turned out to be a useful notion.

If X is an ordered Banach space then X^* and $X^{\odot}$ are ordered Banach spaces as well. If the positive cone X_+ is normal and generating then the dual cone X_+^* is also normal and generating. This yields that $X_+^{\odot} := X^{\odot} \cap X_+^*$ is normal as well. But it cannot be conduced in general that $X^{+^{\odot}}$ is generating. In particular we mention the following

OPEN PROBLEM.
Let X be a Banach lattice. Under what conditions on the semigroup T_0 and the space X is $X^{\odot}$ a Banach lattice as well?

A rather easy solution to this problem exists in the following (practically important) cases.
Assume that X is a Banach lattice, and let $|\cdot|_X$ denote the absolute value on X. To conclude that $X^{\odot}$ is a Banach lattice it is sufficient to show that $|x^{\odot}|_{X^*} \in X^{\odot}$ if $x^{\odot} \in X^{\odot}$.

THEOREM 7.5.
If $T_0^*(t)$ is a lattice homomorphism for every $t \geqslant 0$ then $X^{\odot}$ is a Banach lattice.

PROOF.
If $T_0^*(t)$ is a lattice homomorphism then $|T_0^*(t)x^*|_{X^*} = T_0^*(t)|x^*|_{X^*}$ for every $x^* \in X^*$. From this we get that

$$\|T_0^*(t)|x^{\odot}|_{X^*} - |x^{\odot}|_{X^*}\| = \||T_0^*(t)x^{\odot}|_{X^*} - |x^{\odot}|_{X^*}\| \leq \|T_0^{\odot}(t)x^{\odot} - x^{\odot}\| \to 0, \quad t \downarrow 0,$$

for every $x^{\odot} \in X^{\odot}$. □

The following criterion is very useful in practical cases.

THEOREM 7.6.
Assume that X is a closed ideal in the Banach lattice Y and that the semigroup T_0 can be extended to a group of positive bounded linear operators on Y. Then X is a Banach lattice.

PROOF.
Let π denote the canonical injection of X into Y and let $T_Y(t)$ be a group of positive bounded linear operators on Y, such that the restriction to X is $T_0(t)$ for $t \geq 0$. So

$$T_Y(t)\pi = \pi T_0(t), \quad t \geq 0.$$

Let $\pi^*: Y^* \to X^*$ be the adjoint. It follows from SCHAEFER (1974, p.86) that π^* is surjective and π^* is a lattice homomorphism, i.e. $|\pi y^*|_{X^*} = \pi^*|y^*|_{Y^*}$.
Let $x^* \in X^*$, and choose $y^* \in Y^*$ such that $\pi^* y^* = x^*$. From the fact that every operator $T_Y^*(t)$ is a lattice homomorphism (SCHAEFER, 1974, p. 60) we get that

$$|T_0^*(t)x^*|_{X^*} = |T_0^*(t)\pi^* y^*|_{X^*} = |\pi^* T_Y^*(t)y^*|_{X^*} = \pi^*|T_Y^*(t)y^*|_{Y^*} =$$
$$\pi^* T_Y^*(t)|y^*|_{Y^*} = T_0^*(t)\pi^*|y^*|_{Y^*} = T_0^*(t)|\pi^* y^*|_{X^*} = T_0^*(t)|x^*|_{X^*},$$

hence $T_0^*(t)$ is a lattice homomorphism on X^*. Now the result follows from Theorem 7.5. □

COROLLARY 7.7.
If $T_0(t)$ can be extended to a group of positive bounded linear operators on X, then $X^{\odot}$ is a Banach lattice.

EXAMPLES.
Let $\phi(t, \omega)$ be a continuous flow on $\mathbb{R}^m$.

a) Let Ω be a compact subset of $\mathbb{R}^m$ which is invariant under the flow ϕ and let $X = C(\Omega)$ or $L^1(\Omega)$. Set $(T_0(t)x)(\omega) = x(\phi(t, \omega))$, $t \in \mathbb{R}$, $\omega \in \Omega$. Then T_0 defines a C_0-group of positive bounded linear operators and therefore $X^{\odot}$ is a Banach lattice.
b) Let Ω be an open subset of $\mathbb{R}^m$ such that the complement of Ω is invariant under the flow $\phi(t,\omega)$ for $t \geq 0$. Let $X = C_0(\Omega)$ be the space of continuous functions on $\mathbb{R}^m$ which are identically zero outside Ω, then X is a closed ideal of $Y = C_0(\mathbb{R}^m)$. Define $(T_Y(t)y)(\omega) = y(\phi(t, \omega))$ for $t \in \mathbb{R}$ and $\omega \in \mathbb{R}^m$. Then $T_Y(t)$ is a group of positive bounded linear operators on Y and, as $\phi(t, \cdot)$ leaves the complement of Ω invariant for $t \geq 0$, it leaves X invariant for $t \geq 0$. If $T_0(t)$ is the restriction to X for $t \geq 0$, then $X^{\odot}$ is a Banach lattice.

Now let X be an ordered Banach space and let $X_+^{\odot}$ be the cone induced on $X^{\odot}$. If $T_0(t)$ is a positive semigroup then $T_0^{\odot}(t)$ is a positive semigroup as well. If, in addition, the perturbation operator $C: X^{\odot} \to X^*$ is positive, then the perturbed semigroup is positive as well: this follows easily if one uses the series expansion (7.2). We now present a criterion for the irreducibility of $T^{\odot}(t)$.

THEOREM 7.8.
Let X be an ordered Banach space and let $X^{\odot}$ with the cone $X_+^{\odot}$ be a Banach lattice. If $T_0^{\odot}(t)$ is a positive semigroup and $C: X^{\odot} \to X^*$ a positive operator, then the perturbed semigroup $T^{\odot}(t)$ is irreducible if and only if $J = \{0\}$ and $J = X^{\odot}$ are the only closed ideals in $X^{\odot}$ satisfying both
(i) $T_0^{\odot}(t)J \subseteq J$, $t \geq 0$
(ii) $K_\lambda J \subseteq J$.

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