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A FREE BOUNDARY PROBLEM INVOLVING A CUSP
Part II: LOCAL ANALYSIS

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Abstract. We consider a stationary free boundary problem describing the stationary flow of fresh and salt water in a porous medium. The salt water is supposed to be stagnant, while the fresh water on top of it is drawn into wells. In a previous work it has been shown, that for pumping rates $Q < Q_{cr}$ a solution with smooth interface exists. In this part we study the case $Q = Q_{cr}$ in two dimensions. We show that the interface has isolated singularities. At each singularity the free boundary develops a cusp or becomes vertical. By means of local analysis techniques we obtain the asymptotic behaviour of the free boundary at these singularities.

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1. Introduction

In [4] we formulated a free boundary problem which models the stationary flow of fresh and salt groundwater, say, in a reservoir. The fluids are assumed to be separated by an abrupt transition, the interface or free boundary, with salt water below fresh water. The saltwater is supposed to be stagnant, while the fresh water is drawn into wells which are present in the reservoir.

The variables involved in this problem are a reduced potential w and the location u of the interface. Further it contains a parameter $Q > 0$ which is proportional to the pumping rates of the wells. We demonstrated in [4] that a maximal (or critical) value Q_{cr} of Q exists such that for $Q < Q_{cr}$ the free boundary is smooth, i.e. it can be represented by an analytic function u. The proof of this result is based on the local reduction of the problem to the one - phase dam problem. For this it is crucial to have $w > 0$ in an upper neighborhood

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of the free boundary. For $Q = Q_{cr}$, without further investigation, the free boundary is described by a lower semi-continuous function $\underline{u}$ and an upper semi-continuous function $\overline{\overline{u}}$, see Theorem 1.1 below. Further we proved that for $Q = Q_{cr}$ points in the closure of the free boundary exists for which the potential w has points of negativity in any neighborhood.

The aim of this paper is to make precise how the negativity of w leads to loss of smoothness of the free boundary. In particular we show that singularities in the form of cusps occur in the free boundary and we specify the local cusp behaviour of w and u. We prove our results for flow domains of dimension 2.

First we introduce some notation. Let

$$V =]a_1, a_2[\times]0, H[\ , \quad -\infty < a_1 < a_1 < \infty\ ,$$

denote the two dimensional reservoir, where for points $x \in V$ we often write $x = (y, z)$ with $y \in]a_1, a_2[$ representing the horizontal coordinate and $z \in]0, H[$ the vertical coordinate. The N wells are located at the interior points

$$W = \{x_{W(l)} : l = 1, \ldots, N\}\ .$$

In order to compensate for the singularities of w at the wells, we introduced in [4] a truncated fundamental solution h. Along the vertical boundaries of the reservoir, w satisfies the Dirichlet conditions

$$w(a_1, z) = w(a_2, z) = (z - u_0)_+\ ,$$

where u_0, with $0 < u_0 < H$, is the salt water level outside V. At the top of the reservoir w satisfies the Neumann condition

$$\frac{\partial w}{\partial \nu} = 1\ .$$

In [4] we proved, in a more general ($N > 2$ dimensional) context, the following global existence result at $Q = Q_{cr}$.

Theorem 1.1 *There exist functions $\overline{u}, \overline{\overline{u}} : [a_1, a_2] \to [u_0, H]$, satisfying*

$$\begin{aligned} &u_0 \le \overline{u} \le \overline{\overline{u}} \text{ in } [a_1, a_2];\\ &\overline{u} \text{ l.s.c. }, \overline{\overline{u}} \text{ u.s.c. in } [a_1, a_2];\\ &\overline{u} = \overline{\overline{u}} \text{ a.e. in } [a_1, a_2], \end{aligned}$$

and there exists a pair (w, γ), with $w + h \in H^{1,2}(V)$ and $\gamma \in L^\infty(V)$ satisfying

$$(*) \qquad \int_V \nabla\zeta \cdot (\nabla w + \gamma e_z) = 0$$

for all $\zeta \in H^{1,2}(V)$ with $\operatorname{supp}(\zeta) \subset V \setminus W$, *such that*

$$\begin{aligned} &\gamma = \chi_{\{z<\overline{u}(y)\}} \text{ in } V,\\ &w = 0 \text{ in } \{z < \overline{\overline{u}}(y)\}\ ,\\ &w < 0 \text{ in a neighbourhood of } W,\\ &W \text{ lies above } \operatorname{graph}(\overline{\overline{u}}). \end{aligned}$$

From this it follows that

$$\Delta w \geq 0 \text{ in } V \setminus W ,$$
$$\Delta w = 0 \text{ in } \{z > \overline{\overline{u}}(y)\} \setminus W .$$

Throughout this work we assume that the free boundary does not touch the top of the reservoir, i.e., $\overline{\overline{u}} < H$ on $[a_1, a_2]$. For a given configuration of wells, all withdrawing fluid from the reservoir, this assumption seems to be reasonable.

In Section 2 we first prove $\overline{u} = \overline{\overline{u}}$ in $[a_1, a_2]$ and we denote by graph(u),

$$u := \overline{u} = \overline{\overline{u}} \in C([a_1, a_2]) ,$$

the free boundary of the problem, i.e. w is harmonic above and zero below.

In the remaining sections we concentrate on the behaviour of w and u near singular free boundary points $(y^*, u(y^*)) \in \text{int}(V)$ which satisfy Property 4.17 of [4]. This property says that there exists at least one sequence $(y_n, z_n) \to (y^*, u(y^*))$ such that $w(y_n, z_n) < 0$. This can also be characterized by $(y^*, u(y^*)) \in \overline{\{w < 0\}}$.

Using scaling arguments (blow up techniques) we first show in Section 3 that at a singular free boundary point (which we translate to the origin for convenience), the free boundary either forms a cusp ($k = 1$) or becomes vertical ($k = 2$), see [1, Figure2] or Figure 11 of this paper.

In Section 4 and 5 we prove in a number of steps, using blow up arguments, that the scaled function

$$w_r(x) := w(rx)/r^\beta , \quad \gamma_r(x) = \gamma(rx) ,$$

with

$$\beta = \frac{km}{2} ,$$

converge for $r \to 0$ to

$$w_*(x) = c_* \operatorname{Im}(\tilde{x}^m) , \quad \tilde{x} = i^k(-ix)^{k/2}$$

with $c_* > 0$. Moreover, m is odd and $m \geq 3$. It is not clear whether as exceptional case (and probably unstable case) situations with $m \geq 5$ can occur.

It is proven in Section 5 that free boundary points $x = (y, z)$ satisfy $|y| \leq C|z|^\beta$. Further, in Section 6, we show that the branches of the free boundary near the singularity have the form

$$\{\pm z < 0 : y = f(z)\}$$

and that

$$\lim_{z \to 0} \frac{f(z)}{z^\beta} = \pm c_* .$$

This clarifies the asymptotic behaviour of the free boundary near the singularity. For the standard cusp case ($k = 1$, $m = 3$) such an expansion has been expected because special solutions with such a behaviour have been found, see references given in [4]. For the part of the free boundary below the singularty we prove that $f'(z) \to 0$ as $z \nearrow 0$, which shows that indeed the free boundary becomes vertical. In the concluding section we shall pose some conjectures and open questions related to the behaviour of the free boundary. In particular we discuss the occurrence of vertical cusps, the location of cusps in the reservoir and the assumption made that the free boundary does not touch the top of the reservoir.

2. Preliminary remarks and tools

As a first observation we note that the weak differential equation (*) together with the boundary conditions implies that w is Hölder continuous in $\overline{V}\setminus W$. Moreover, w is Lipschitz continuous locally in $V \setminus W$. This can be seen as in Alt & van Duijn [3, Theorem 3.7]. Indeed (*) implies that

$$\Big| \fint_{\partial B_r(x)} (w - w(x)) \Big| \le C \cdot r$$

for all $\overline{B_r(x)} \subset V \setminus W$, and w is harmonic in the set $\{w \neq 0\} \setminus W$.
Next we consider a comparison lemma, that we often shall use to obtain non–oscillation results.

2.1 Comparison Lemma. *Consider a rectangle*

$$R =]a, b[\times]0, c[\subset V \setminus W \ .$$

For $\hat{x} \in R$ and $s_0 \in \mathbb{R}$, consider the unit vector

$$\nu_0 := \frac{1}{\sqrt{s_0^2+1}}(-s_0, 1)$$

and the function $v : \overline{R} \to [0, \infty[$ given by

$$v(x) = \begin{cases} \frac{1}{\sqrt{s_0^2+1}}\nu_0 \cdot (x - \hat{x}) & \text{for } \nu_0 \cdot (x - \hat{x}) > 0 \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

If $\hat{x}$ and s_0 are chosen such that $w \le v$ on ∂R, then

$$\begin{aligned} &(i) \quad w \le v \text{ in } \overline{R}, \\ &(ii) \quad w = 0,\ \gamma = 1 \text{ in } \{v = 0\}. \end{aligned}$$

Remark. The function v is a solution of the dam problem.

Proof. We use the Baiocchi transformation. Let $\zeta \in H^{1,2}(R)$ with $\zeta = 0$ near the vertical walls of R. Then set

$$\overline{\zeta}(y, z) := \int_z^c \zeta(y, s)\, ds.$$

Because $w = 0$ and $\gamma = 1$ in $\{0 \le z \le u_0\}$, the function $\overline{\zeta}$ is an admissible test function in the differential equation for (w,γ). It leads to

$$\int_R \nabla\tilde{\zeta}\cdot(\nabla w-(1-\gamma)e_z)=0.$$

In this equation we substitute

$$\overline{w}(y,z):=\int_0^z w(y,s)\,ds\ ,$$

giving

$$\int_R(\nabla\zeta\cdot\nabla\overline{w}+(1-\gamma)\zeta)-\int_{\{z=c\}}\zeta(\cdot,c)w(\cdot,c)=0\ .$$

As a test function we take $\zeta=(\overline{w}-\overline{v})_+$, where

$$\overline{v}(x,z)=\int_0^z v(x,s)\,ds\ .$$

This gives

$$\int_R|\nabla(\overline{w}-\overline{v})_+|^2+\int_{\{v=0\}}(1-\gamma)(\overline{w}-\overline{v})_+$$
$$-\int_{\{v>0\}}\gamma(\overline{w}-\overline{v})_+ +\int_{\{z=c\}}(\overline{w}-\overline{v})_+(\cdot,c)(v(\cdot,c)-w(\cdot,c))=0\ .$$

The third term only has a contribution when $\overline{w}>\overline{v}$. Suppose there exists $(y_0,z_0)\in R$ such that $\overline{w}(y_0,z_0)>\overline{v}(y_0,z_0)\geq 0$. Then there must also exist $z_1<z_0$ where $w(y_0,z_1)>0$ and hence $w>0$ in $B_\varepsilon((y_0,z_1))$ for some $\varepsilon>0$. This implies that $\gamma=0$ in and above $B_\varepsilon((y_0,z_1))$. In particular $\gamma(y_0,z_0)=0$, which shows that the third term gives no contribution. Since the second and fourth term are nonnegative, the first term implies $\overline{w}\leq\overline{v}$ in $\overline{R}$ and in particular $\overline{w}\leq 0$ in $\{v=0\}$. The equation $\Delta\overline{w}=1-\gamma$ shows that $\overline{w}$ is subharmonic in the set $\{v=0\}$. Then either $\overline{w}<0$ or $\overline{w}\equiv 0$ in $\{v=0\}$. The first possibility contradicts $w=\overline{w}=0$ in $\{0<z<u_0\}$. Hence $\overline{w}=0$, $w=0$ and $\gamma=1$ in $\{v=0\}$. □

We apply the Comparison Lemma to prove that the free boundary is continuous.

2.2 Theorem. $\overline{u}=\overline{\overline{u}}\in C([a_1,a_2])$.

Proof. The continuity and the boundary conditions for w and $\overline{u}\geq u_0$ imply that $\overline{u}=\overline{\overline{u}}$ at the boundary points a_1 and a_2. To show equality for an arbitrary point $y_0\in]a_1,a_2[$, consider sequences $y_n\to y_0$ and $Q_n\nearrow Q_{cr}$ so that

$$u_{Q_n}(y_n)\to\overline{\overline{u}}(y_0)\ ,$$

where u_{Q_n} denotes the free boundary of the solution obtained in [4] with pumping rate $Q=Q_n$. We distinguish two possibilities.

(i) A sequence can be chosen which oscillates around y_0: i.e. y_0 is between y_n and y_{n+1} for all $n \in \mathbb{N}$. We argue as follows. Let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$u_{Q_n}(y_n) > \overline{\overline{u}}(y_0) - \frac{\varepsilon}{2} \ .$$

For $n \geq n_0$ we define

$$z_n := \min\{u_{Q_n}(y_n), u_{Q_{n+1}}(y_{n+1})\}$$

and we consider

$$R_n := \{(y,z) : 0 < z < z_n, \ y \text{ between } y_n \text{ and } y_{n+1}\} \ .$$

Then we have for $n \geq n_0$, n sufficiently large, $w_{Q_{n+1}} = 0$ along the vertical sides of R_n and $w_{Q_{n+1}} < \varepsilon/2$ along the top of R_n (using the monotonicity of w_Q in Q and using the Hölder continuity of w_Q uniformly with respect to $Q \leq Q_{cr}$, see [4; Proposition 4.7]. Using the function $v(y,z) = (z - z_n + \frac{\varepsilon}{2})_+$ and $s_0 = 0$ in the Comparison Lemma we conclude that $w_{Q_{n+1}} = 0$ and $\gamma_{Q_{n+1}} = 1$ in the set

$$\{(y,z) : 0 < z < z_n - \varepsilon/2 \text{ and } y \text{ between } y_n \text{ and } y_{n+1}\} \ ,$$

implying in particular

$$u_{Q_{n+1}}(y_0) \geq z_n - \frac{\varepsilon}{2} > \overline{\overline{u}}(y_0) - \varepsilon.$$

Thus (by definition of $\overline{u}$)

$$\overline{u}(y_0) > \overline{\overline{u}}(y_0) - \varepsilon,$$

giving the desired equality.

(ii) No sequence can be chosen with oscillations around y_0, i.e. all the sequences $(y_n)_n$ come from the same side, say from the right. Then applying the Comparison Lemma similar as in case (i) we are lead to a situation in which we have, see also Figure 1,

$$\overline{u}(y_0) \leq \limsup_{y \uparrow y_0} \overline{\overline{u}}(y) < \overline{\overline{u}}(y_0) \quad \text{ and } \quad \liminf_{y \downarrow y_0} \overline{u}(y) = \overline{\overline{u}}(y_0) \ .$$

Refering to Figure 1 we have

$$w = 0 \ , \ \gamma = 1 \text{ in } B \cap \{y > y_0\}$$

and

$$\Delta w = 0 \ , \ \gamma = 0 \text{ in } B \cap \{y < y_0\}.$$

Moreover, by a global argument, $w \not\equiv 0$ in $B \cap \{y < y_0\}$. Since $-\Delta w = \partial_z \gamma = 0$ in B, we obtain a contradiction with $w = 0$ in $B \cap \{y > y_0\}$. $\square$

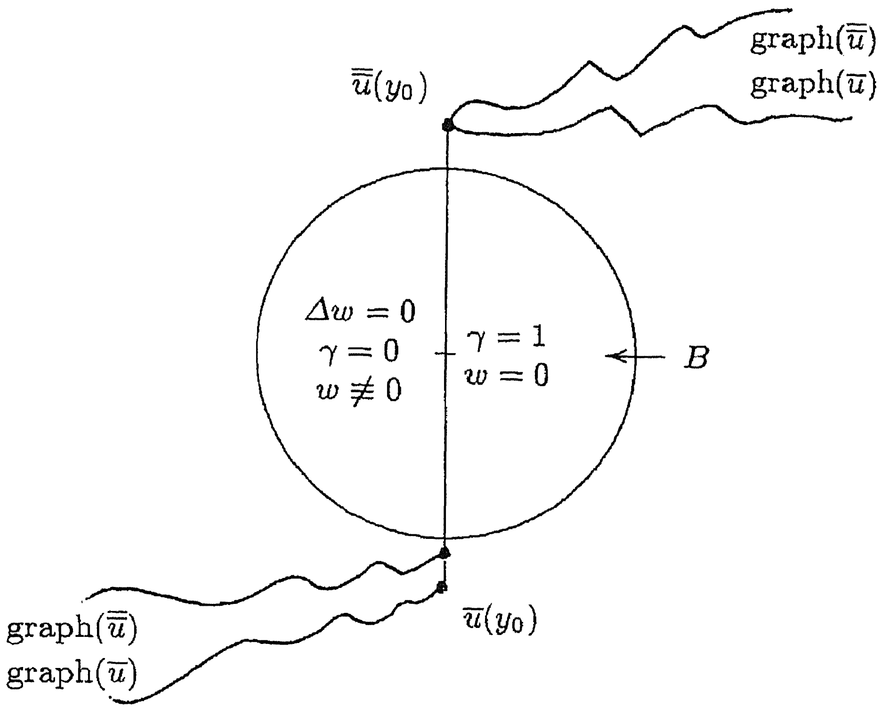


Fig. 1. Possible configuration near discontinuity.

Let $(y^*, u(y^*)) \in \text{int}(V)$ be a free boundary point satisfying Property 4.17 of [4], i.e. a cusp. We translate this point to the origin O, by shifting the coordinates so that $y^* = 0$ and $u(y^*) = 0$. We first define

2.3 Definition. Let $B \subset \mathbb{R}^2$ denote an open ball centered at O. We call $\tilde{w} \in H^{1,2}(B) \cap C^0(B)$ a phase of w at O if $\tilde{w}(w - \tilde{w}) = 0$ in B and if $\{\tilde{w} \neq 0\} \cap B$ is non-empty and connected with O as a boundary point. We have $\nabla w \cdot \nabla \tilde{w} = |\nabla \tilde{w}|^2$ and $\tilde{w}$ has a sign. In section 7 we prove that w has only finitely many ($m \in \mathbb{N}$) phases at O. Moreover, we prove that in some smaller concentric ball $\tilde{B} \subset B$ we have a decomposition $w = \sum_{i=1}^m w_i$, where w_i are the phases of w at O.

Since $\gamma = 1$ and $w = 0$ in a neighborhood of the vertical line below the origin O, any test function from identity $(*)$ can be changed there arbitrarily. We have

2.4 Proposition (Separation Lemma) *Below O test functions from expression $(*)$ can have different values from both sides, i.e.*

$$\int_{B_r} \nabla\zeta \cdot (\nabla w + \gamma e_z) = 0$$

for all $\zeta \in H^{1,2}(B_r \setminus \{(0,z) : -r < z \leq 0\})$ having support in B_r, where B_r denotes the open ball in $\mathbb{R}^2$ with center O and radius $r > 0$.

Proof. For ζ as above and $\varepsilon > 0$ small, consider the expression

$$\frac{\zeta(y,z)+\zeta(-y,z)}{2}+\frac{\zeta(y,z+\varepsilon)-\zeta(-y,z+\varepsilon)}{2}\ .$$

The first term belongs to $H_0^{1,2}(B_r)$. The second term vanishes on $\{(0,z) : z > -\varepsilon\}$ and near ∂B_r. Because $\gamma = 1$, $w = 0$ in a neighborhood of the segment $\{(0,z) : -r < z < -\varepsilon\}$ also the second term is an admissible test function for $(*)$. Hence we may substitute this expression into the equation. Letting $\varepsilon \to 0$ gives the result. □

Next we show that for any phase $\tilde{w}$ of w, the values of $|\nabla \tilde{w}|$ and $\frac{|\tilde{w}|}{r}$ are balanced near O in the following sense:

2.5 Proposition. *There exists a constant $C > 0$ such that for every $r > 0$*

$$\int_{B_{\frac{r}{2}}} |\nabla \tilde{w}|^2 \le C\frac{1}{r^2}\int_{B_r\setminus B_{\frac{r}{2}}} |\tilde{w}|^2 \le C \sup_{B_r\setminus B_{\frac{r}{2}}} |\tilde{w}|^2\ ,$$

and

$$\sup_{B_{\frac{r}{2}}} |\tilde{w}|^2 \le C\Big(\fint_{\partial B_{\frac{3r}{4}}} \tilde{w}\Big)^2 \le C\int_{B_r} |\nabla\tilde{w}|^2\ .$$

Proof. By linear scaling we can take $r = 1$. Set $\zeta = \tilde{w}\eta^2$ with $\eta \in C_0^\infty(B_1)$ in expression $(*)$. Then

$$\int_{B_1} \nabla(\tilde{w}\eta^2)\cdot\nabla w + \int_{B_1}\nabla(\tilde{w}\eta^2)\cdot\gamma e_z = 0\ .$$

Since $\gamma = 0$ in $\{\tilde{w} \neq 0\}$ the second term vanishes. The first term can be written as

$$0 = \int_{B_1}\nabla(\tilde{w}\eta^2)\cdot\nabla\tilde{w} = \int_{B_1}\eta^2|\nabla\tilde{w}|^2 + 2\int_{B_1}\tilde{w}\nabla\eta\cdot\eta\nabla\tilde{w}\ .$$

Hence

$$\int_{B_1}\eta^2|\nabla\tilde{w}|^2 \le 4\int_{B_1}\tilde{w}^2|\nabla\eta|^2\ .$$

When choosing η as cut–off function from $B_{1/2}$ to B_1 we obtain the first pair of estimates. For the second pair we use the fact that $|\tilde{w}|$ is subharmonic in B_1. Then by Poisson's integral for any $\frac{3}{4} \le r \le 1$

$$\begin{aligned}\sup_{B_{1/2}} |\tilde{w}| &\le C \int_0^{2\pi} |\tilde{w}(re^{i\theta})|\, d\theta \\ &\le C \int_0^{2\pi} |\partial_\theta \tilde{w}(re^{i\theta})|\, d\theta \quad (\text{ using } \tilde{w}(re^{-i\pi/2}) = 0) \\ &\le C \int_0^{2\pi} |\nabla \tilde{w}(re^{i\theta})|\, d\theta\ .\end{aligned}$$

Squaring and integrating over r gives the result. □

For several purposes we need that w cannot have long zero curves above and near the free boundary. This is the content of the following two propositions.

2.6 Proposition. *Let (w,γ) be any (sub)solution of the local differential equation $(*)$. Suppose there exists a rectangle $R \subset V$ in which (w,γ) satisfies the properties as listed in Figure 2. Then for some $c > 0$, depending only on the geometry of the rectangle,*

$$\int_R |w|^2 \ge c\ .$$

Proof. In the weak inequality for a subsolution we choose $\zeta \in C_0^\infty(R)$, $\zeta \ge 0$, such that

$$\begin{array}{rclll} \partial_z\zeta \le 0 & \text{in} & \{z_2 - \frac{h}{4} < z < z_2\} \cap R & \text{where} & \gamma = 0\ , \\ \partial_z\zeta \ge 0 & \text{in} & \{z_1 < z < z_2 - \frac{h}{4}\} \cap R & \text{where} & 0 \le \gamma \le 1\ , \\ \partial_z\zeta \ge c > 0 & \text{in} & D \subset\subset \{z_1 < z < z_1 + \frac{h}{4}\} \cap R & \text{where} & \gamma = 1\ . \end{array}$$

This gives (the first inequality arises for subsolutions)

$$-\int_R \nabla w \cdot \nabla\zeta \ge \int_R \gamma\partial_z\zeta \ge \int_D \gamma\partial_z\zeta \ge c\ ,$$

where the constant c also depends on D. Hence

$$\int_{\mathrm{supp}(\zeta)} |\nabla w|^2 \ge c > 0\ .$$

Since $\mathrm{dist}(\mathrm{supp}(\zeta), \partial R) > 0$, we can apply the first part of the proof of Proposition 2.5 with an appropriate test function to w and obtain the inequality. □

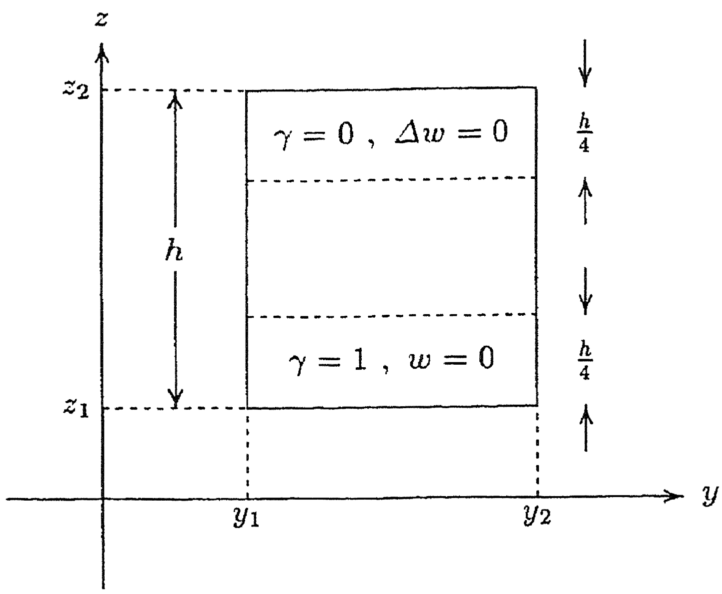


Fig. 2. Properties of (w, γ) in R.

2.7 Proposition. *Suppose there is a continuous Jordan curve (not closed) in the rectangle* $\{z_1 + \frac{h}{4} < z < z_2 - \frac{h}{4}\} \cap R$, *going from the left boundary to the right boundary as in Figure 3, such that*

$$\begin{aligned} &\Gamma \text{ above } \operatorname{graph}(u)\ , \\ &w = 0 \text{ on } \Gamma\ , \\ &w > 0 \text{ in a right neighborhood of } \Gamma, \text{ looking in the direction of } \Gamma\ . \end{aligned}$$

Then for some $c > 0$, depending only on the geometry of the rectangle and on the Lipschitz constant of w,

$$|w(x)| \geq c \qquad \text{for some } x \in R \text{ below } \Gamma\ .$$

Proof. Γ divides the rectangle R into exactly two subdomains R_+ (left of Γ) and R_- (right of Γ). Let

$$w^* := \begin{cases} 0 & \text{in } R_+ \\ w & \text{in } R_- \end{cases}$$

Since $w > 0$ in R_- near Γ, it follows that $\Delta w^* \geq 0$ above the free boundary. Hence (w^*, γ) is a subsolution of equation $(*)$ in R. Applying Proposition 2.6 gives

$$\int_R |w^*|^2 \geq c > 0\ .$$

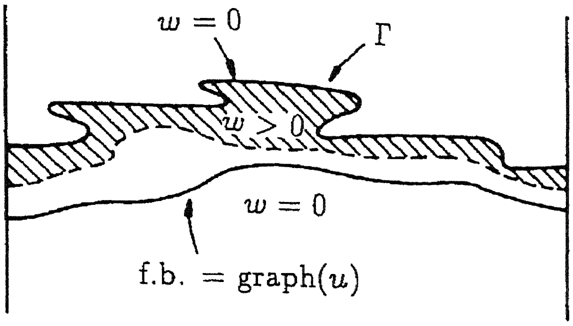


Fig. 3. Situation near Γ.

Hence there must exist points $x \in R_-$ for which $|w(x)| \geq c > 0$, where c only depends on the geometry of the rectangle. □

For future use we also give here the monotonicity formula for the m-phases.

2.8 Monotonicity Formula. Suppose w has $m \in \mathbb{N}$ phases $\{w_i : i = 1, \ldots, m\}$ at O. For each phase w_i we define

$$\varphi_i(r) := \frac{1}{r^{\kappa m}} \int_{B_r} |\nabla w_i|^2 \qquad \text{for } 0 < r < r_0 < \infty \, , \tag{2.1}$$

where $\kappa \geq 1$. Moreover, let

$$\varphi(r) := \prod_{i=1}^{m} \varphi_i(r) \qquad \text{for } 0 < r < r_0 < \infty \, . \tag{2.2}$$

A value $\kappa > 1$ is related to the fact that $\{w = 0\}$ on each sphere might cover a certain sector. To be precise, we assume that there are values $0 \leq \delta(r) < 1 - \frac{1}{\kappa}$ with $\delta(r) \to 0$ as $r \to 0$ such that

$$\frac{1}{2\pi} \mathcal{L}^1(\{\theta \in [0, 2\pi] : w(re^{i\theta}) = 0\}) \geq 1 - \frac{1}{\kappa(1 - \delta(r))} \, , \tag{2.3}$$

where $\mathcal{L}^1$ denotes the one dimensional Lebesgue measure. It then follows, that

$$\frac{d}{dr} \log \varphi(r) \geq -\kappa m^2 \frac{\delta(r)}{r} \tag{2.4}$$

in distributional sense. In particular,

$$\log \varphi(r) \leq \log \varphi(r_0) + \kappa m^2 \int_r^{r_0} \frac{\delta(\tilde{r})}{\tilde{r}}\, d\tilde{r} \; . \tag{2.5}$$

If the function $r \mapsto \delta(r)/r$ is integrable, e.g. if $\delta(r) \leq Cr^\alpha$ for some $\alpha > 0$, then (2.5) implies that φ is bounded. In case that $\delta = 0$ inequality (2.5) gives that φ is monotonically increasing in r. The proof of (2.4) is given in Appendix A. As a special case, see Alt et al. [2], we decompose w into two contributions according to

$$w := w_+ - w_-$$

where $w_\pm := \max\{0, \pm w\}$. Then we consider the functions

$$\varphi_\pm(r) := \fint_{B_r} |\nabla w_\pm|^2 \qquad \text{for } 0 < r < r_0 < \infty \; , \tag{2.6}$$

i.e., $m = 2$ and $\kappa = 1$, consequently $\delta = 0$. It follows that

$$\varphi(r) := \varphi_+(r) \cdot \varphi_-(r) \qquad \text{for } 0 < r < r_0 < \infty \tag{2.7}$$

is monotonically increasing in r.

3. Sublinear decay of solution.

First let us note, that w decays at least linearly at the cusp, here situated at the origin O, i.e.

$$w(x) = \mathcal{O}(|x|) \quad \text{as } x \to O \; .$$

The follows from the Lipschitz continuity. This Lipschitz continuity also implies that the functions $\varphi_\pm$ in 2.8 are bounded and $\varphi_i(r) \leq Cr^{2-\kappa m}$.
The aim of this section is to prove that w decays faster than linearly, i.e.

$$w(x) = o(|x|) \quad \text{as } x \to O \; .$$

For this we apply blow-up techniques to the decomposition $w = w_+ - w_-$.
We first show

3.1 Proposition. *For the function φ in (2.7) we have*

$$\lim_{r \downarrow 0} \varphi(r) = 0 \; .$$

Proof. Suppose that $\lim_{r\downarrow 0} \varphi(r) \geq C > 0$. Then consider the blow-up $(r \downarrow 0)$

$$w_r(x) := \frac{w(rx)}{r} \text{ and } \gamma_r(x) := \gamma(rx) \quad \text{for } x \in B$$

where B denotes any ball in $\mathbb{R}^2$ centered at O. Using the Lipschitz-continuity of w we obtain as in [3] for a subsequence $(r_k)_k$ with $r_k \searrow 0$,

$$w_k := w_{r_k} \to w_0 \text{ uniformly in } B \text{ and strongly in } H^{1,2}(B) \ ,$$
$$\gamma_k := \gamma_{r_k} \to \gamma_0 \text{ weakly star in } L^\infty(B),$$

with $w_0 \in H^{1,2}_{loc}(\mathbb{R}^2)$ and $\gamma_0 \in L^\infty(\mathbb{R}^2)$. Further, because φ is bounded away from zero, the blow–up limit is a linear two–phase solution. Since $w(0,z) = 0$ for $z \leq 0$, we must have $w_0(0,z) = 0$ for all $z \in \mathbb{R}$ with for instance $w_0 > 0, \gamma_0 = 0$ in $\{y > 0\}$ and $w_0 < 0, \gamma_0 = 0$ in $\{y < 0\}$. However by the Separation Lemma we also have $\partial_\nu w_0(0\pm, z) = 0$ for $z < 0$, a contradiction. □

3.2 Proposition. *There is no sequence $r \downarrow 0$ for which*

$$\sup_{B_r} w_+ = o(r) \quad \text{and} \quad \sup_{B_r} w_- \geq cr \quad \text{with } c > 0 \ .$$

Proof. Suppose such a sequence $(r_k)_{k\in\mathbb{N}}$ exists. Then consider the blow up

$$w_k(x) := \frac{w(r_k x)}{r_k} \quad \text{for } x \in B_1 \ .$$

The assumption implies the existence of points x_k in B_1 satisfying

$$-w_k(x_k) \geq c \quad \text{and} \quad x_k \to x_0 \quad \text{in } B_1 \ .$$

By the Lipschitz continuity we have $-w_k \geq c/2$ in $B_\delta(x_0)$ for some $\delta > 0$ and for k large. Therefore the blow up w_0 satisfies $-w_0 \geq c/2$ in $B_\delta(x_0)$, $w_0 \leq 0$ in B_1 and $w_0(0) = 0$. The property $\Delta w_0 \geq 0$ is inherited, hence giving a contradiction. □

3.3 Proposition. $w_-(x) = o(|x|)$ *as* $x \to 0$.

Proof. We argue by contradiction. Suppose there is a sequence $(r_k)_k$ with $r_k \searrow 0$, for which

$$\sup_{B_{r_k}} \frac{w_-}{r_k} \geq c > 0 \ .$$

By Proposition 3.2 also

$$\sup_{B_{r_k}} \frac{w_+}{r_k} \geq c > 0 \ .$$

Applying the second inequality of Proposition 2.5 gives

$$\varphi_\pm(2r_k) = \fint_{B_{2r_k}} |\nabla w_\pm|^2 \geq c \ ,$$

which contradicts the conclusion of Proposition 3.1. □

Therefore we concentrate on the sublinear decay of w_+. We first prove

3.4 Proposition. *Let (w_0, γ_0) be the blow up limit obtained for a sequence $(r_k)_k$ with $r_k \downarrow 0$. If $\tilde{x} \in \mathbb{R}^2$ satisfies*

$$\forall \varepsilon > 0 : \gamma_0 \neq 0 \quad \text{in } L^\infty(B_\varepsilon(\tilde{x})) ,$$

then there exists a sequence $(x_k)_k$ *with* $x_k = (y_k, z_k) \to \tilde{x}$ *such that* $\gamma_k = 1$ *and* $w_k = 0$ *in a neighborhood of the segments*
$\{y_k\} \times]-L_k, z_k[$ *where* L_k *is a suitable big number.*

Proof. The sequence $(x_k)_k$ is constructed as follows. The convergence of γ_k implies that for each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\gamma_k \neq 0$ in $L^\infty(B_\varepsilon(\tilde{x}))$. Since (w_k, γ_k) is obtained from (w, γ) by scaling, we have that

$$\{\gamma_k \neq 0\} = \{\gamma_k = 1\} = \{z < u_k(y)\}$$

is the subgraph of the scaled, continuous free boundary. Hence we can select a point x_k from the open set $\{z < u_k(y)\} \cap B_\varepsilon(\tilde{x})$. Then choose L_k so that $(y_k, -L_k)$ lies on the bottom of the scaled domain V. □

We are now ready to prove the essential part of the section.

3.5 Proposition. $w_+(x) = o(|x|)$ *as* $x \to 0$.

Proof. Again we argue by contradiction. Assume for some $c > 0$, there is a sequence $(x_k)_k$ with $x_k \to 0$ and

$$\frac{w_+(x_k)}{|x_k|} \geq c .$$

Let $r_k := |x_k|$ and consider the corresponding blow up sequence w_k as above. For a subsequence, denoted again by $(r_k)_k$, we have $(w_k, \gamma_k) \to (w_0, \gamma_0)$ as in Proposition 3.1. Moreover we have

$$e_k := \frac{x_k}{r_k} \to e_0 =: (y_0, z_0) .$$

By Proposition 3.3, $w_-(x) = o(|x|)$ as $x \to 0$. Therefore we conclude $w_0 \geq 0$ in $\mathbb{R}^2$. Moreover by the convergence properties of the sequence

$$c \leq \frac{w(x_k)}{r_k} = w_k(\frac{x_k}{r_k}) \to w_0(e_0) ,$$

and the Lipschitz continuity implies

$$(3.1) \qquad w_0 \geq c/2 \quad \text{and } \gamma_0 = 0 \quad \text{in } B_{\delta_0}(e_0) \text{ for some } \delta_0 > 0 .$$

Thus for the blow up limit w_0 we have a situation as show in the figure below.
First we show

$$(3.2) \qquad w_+(0, z) = o(z) \quad \text{for } z \downarrow 0 .$$

If not, we can choose the above sequence such that $x_k = (y_k, z_k)$ with $y_k = 0$ and $z_k > 0$, giving $e_0 = (0, 1)$. Now assume that w_0 is harmonic in the half plane $\{y > 0\}$. Since $w_0 \geq 0$ everywhere and, by (3.1), $w_0 > 0$ in $B_{\delta_0}((0, 1)) \cap \{y > 0\}$ we must have $w_0 > 0$ and therefore also $\gamma_0 = 0$ in $\{y > 0\}$. As in the Separation Lemma we have that the weak differential

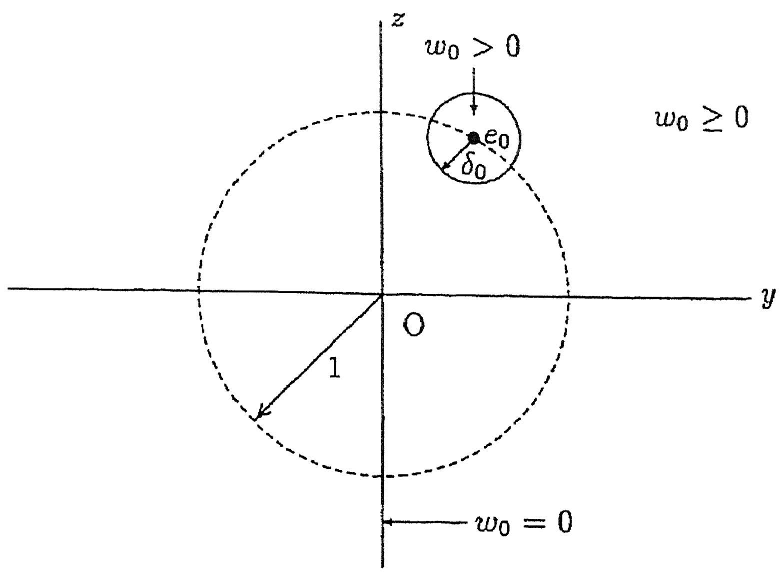


Fig. 4. Situation for w_0, with possible position for e_0.

equation for (w_0, γ_0) also holds for test functions $\zeta \in C_0^\infty(\{y > 0\} \cup \{y = 0, z < 0\})$. Using $\Delta w_0 = 0$ and $\gamma_0 = 0$ in $\{y > 0\}$ this means that

$$\partial_y w_0(0+, z) = 0 \quad \text{for all } z < 0 .$$

But since $w_0(0, z) = 0$ for $z < 0$ (inherited from w) we have a contradiction with the Hopf-principle.

Therefore there exists a point $\tilde{x} = (\tilde{y}, \tilde{z})$ with $\tilde{y} > 0$ so that w_0 is not harmonic in any neighborhood of $\tilde{x}$. Then clearly γ_0 satisfies the assumption of Proposition 3.4 at $\tilde{x}$ (otherwise we would have $\gamma_0 = 0$ and thus $\Delta w_0 = 0$ in some neighborhood of $\tilde{x}$). Let $x_k = (y_k, z_k)$ denote the points from Proposition 3.4 and consider the rectangle

$$R = \{(y, z) : 0 < y < y_k \text{ and } -L_k < z < \min\{0, z_k\}\} ,$$

where again L_k is a suitably chosen large number. By Proposition 3.4 and because $w(0, z) = 0$ for $z < 0$ we have $w_k = 0$ along the vertical boundaries of R. At the top, using the Lipschitz continuity of w, we have $w_k \leq C y_k$ and near the bottom $\gamma_k = 1$ and $w_k = 0$ by the choice of L_k. Then the Comparison Lemma 2.1 with $s_0 = 0$ gives $\gamma_k = 1$ and $w_k = 0$ in

$$\{(y, z) : 0 < y < y_k \text{ and } -L_k < z < \min\{0, z_k\} - C y_k\} .$$

Letting $k \to \infty$ and repeating the same procedure in the half plane $\{y < 0\}$ leads to the situation from Figure 5.

By the regularity theory for the dam problem (see Alt [1]), this implies that the blow-up (w_0, γ_0) has a smooth free boundary, say graph(u_0), passing through the z-axis at a point $(0, z_0)$ with $0 \leq z_0 \leq 1$, such that $w_0 > 0$, $\gamma_0 = 0$ above graph(u_0) and $w_0 = 0$, $\gamma_0 = 1$ below graph(u_0). We show now that this leads to a contradiction.

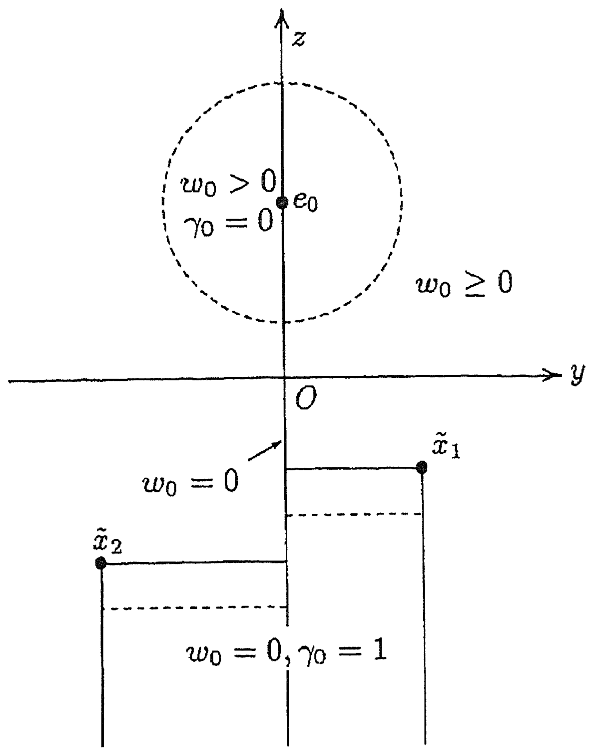


Fig. 5. Situation for blow up limit (w_0, γ_0).

Let $s_0 := u_0'(0)$. For $\delta > 0$ consider the linear solution v_δ from Comparison Lemma 2.1 with $\hat{x} = (0, z_0 - \delta)$.
Now let $0 < \varepsilon < \varepsilon_0$ (small) be given. By the smoothness of u_0 and w_0 we have

$$|u_0(y) - u_0'(0)y| \leq C_1(\varepsilon_0)\varepsilon^2 \quad \text{for } |y| < \varepsilon$$

and

$$|w_0(x) - v_0(x)| \leq C_2(\varepsilon_0)\varepsilon^2 \quad \text{for } x \in B_\varepsilon(0, z_0) \ . \tag{3.3}$$

Then taking $\delta = C\varepsilon^2$, where C is chosen large and independent of ε, we have

$$v_\delta > w_0 \quad \text{in } B_\varepsilon(0, z_0) \cap \{w_0 > 0\}$$

and the free boundary of v_δ is below graph(u_0).
At each free boundary point $(y, u_0(y))$, the function γ_0 satisfies the assumption of Proposition 3.4. Hence for k sufficiently large ($w_k \to w_0$ uniformly) we can select ε_k^- near $-\frac{7}{8}\varepsilon$ and ε_k^+ near $\frac{7}{8}\varepsilon$, and apply the Comparison Lemma 2.1 to the scaled solution (w_k, γ_k) in the rectangle

$$R = \left]-\varepsilon_k^-, \varepsilon_k^+\right[\times \left]-L_k, \sup\{u_0(y) : -\varepsilon_k^- \leq y \leq \varepsilon_k^+\}\right[\ .$$

As a result we find

$$w_k = 0, \gamma_k = 1 \quad \text{below the free boundary of } v_\delta \text{ in } \{\varepsilon_k^- < y < \varepsilon_k^+\}.$$

By (3.3), the positivity of w_0 above graph(u_0) and again the uniform convergence of w_k, it follows that for sufficiently large k

$$w_k > 0 \ , \quad \gamma_k = 0 \qquad \text{above the free boundary of } v_{-\delta} \text{ in } \{\varepsilon_k^- < y < \varepsilon_k^+\}.$$

Thus we are left with the region between the free boundaries of v_δ and $v_{-\delta}$, which is a very flat strip of width $\mathcal{O}(\varepsilon^2)$ and length $\mathcal{O}(\varepsilon)$. Now the origin O is an accumulation point of $\{w_k < 0\}$ because it satisfies Property 4.17 of [4]. First this implies that $z_0 - \delta \leq 0$. Second there must be curves on which $w_k < 0$ coming from outside the strip and approaching O arbitrarily close. These curves must come either from the left or the right. For definiteness consider a curve coming from the right. Define the rectangle

$$R_\varepsilon := \{ \ (y,z) : |y - \tfrac{\varepsilon}{2}| < \tfrac{\varepsilon}{4} \text{ and } |z - z_0| < h \ \}$$

with $h = C\varepsilon$, C large. Then there exists a curve in R_ε going from the left side of R_ε to the right side and lying inside the strip so that $w_k < 0$ on this curve. Moreover, $w_k > 0$ near and above its free boundary. Therefore, after the scaling

$$\tilde{w}_k(x) := \frac{1}{\varepsilon} w_k(\varepsilon x)$$

we obtain a situation as in Proposition 2.7, where a Jordan curve Γ separates $\{\tilde{w}_k < 0\}$ above it from $\{\tilde{w}_k > 0\}$ below it. We deduce that in the flat strip points must exist where $|w_k| \geq c\varepsilon$. Letting $k \to \infty$ we obtain that there exists a point x in the strip with $|w_0(x)| \geq c\varepsilon$. But since $|v_0(x)| \leq C\delta = C\varepsilon^2$ we conclude from (3.3) that $|w_0(x)| \leq C\varepsilon^2$, a contradiction for small ε.
Therefore we conclude that (3.2) holds.
For the blow up w_0 this implies

$$w_0(0,z) = 0 \ , \quad \text{also for } z > 0 \ ,$$

and by (3.1)

$$\operatorname{dist}(e_0, \{y = 0\}) > \delta_0 \ .$$

For definiteness, let $e_0 = (y_0, z_0)$ has $y_0 > 0$.
Next we choose $s_0 > \max\{0, 2\frac{z_0}{y_0}\}$, i.e. the point e_0 is below the line with slope $s_0/2$, passing through the origin. Since $e_k \to e_0$, also e_k is below this line for large k. Choosing such a k (fixed), we consider a second sequence $(e_{kl})_{l \geq k}$ defined by

$$e_{kl} := \frac{r_l}{r_k} e_l \qquad l \in \mathbb{N}, l \geq k \ .$$

It satisfies

$$\begin{aligned} & e_{kl} = \frac{x_l}{r_k} \to 0 \quad \text{for } l \to \infty \ , \\ (3.4) \qquad & \frac{e_{kl}}{|e_{kl}|} = e_l \to e_0 \quad \text{for } l \to \infty \ , \\ & w_k(e_{kl}) = \frac{1}{r_k} w(x_l) > 0 \ . \end{aligned}$$

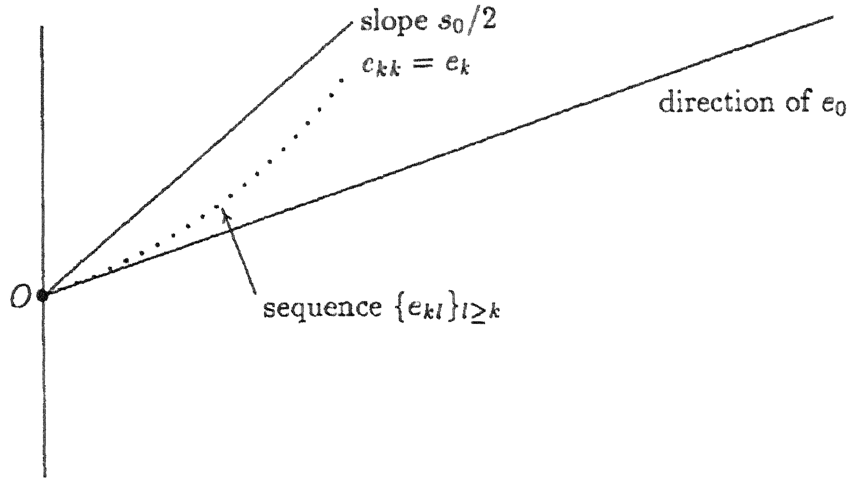


Fig. 6. The sequence $(e_{kl})_{l \geq k} \subset \{w_k > 0\}$ converges to O tangent to the e_0-direction.

Below we shall use the Comparison Lemma 2.1 with a function v defined for $\hat{x} = (0,0)$ and s_0 as above. First fix $h > 2s_0$ and take any L sufficiently large. We have for $0 \leq z \leq h$

$$v(0,z) = \frac{z}{s_0^2+1}$$

and, from (3.2),

$$w_k(0,z) \leq \varepsilon_k z \tag{3.5}$$

where $\varepsilon_k \to 0$ as $k \to \infty$. Therefore if k is large enough (depending on h) we have

$$w_k(0,z) \leq v(0,z) \quad \text{for all } z \in [-L,h] \,.$$

Now assume there is a point $\tilde{x} = (\tilde{y}, \tilde{z}) \in R_\infty :=]0,2[\ \times\ ]h,\infty[$ satisfying the assumption of Proposition 3.4. Then from this proposition it follows that for large k, there is $y_k \in]0,2[$, say, so that (y_k, h) is below the free boundary of w_k. Now consider the rectangle $R_k :=]0, y_k[\ \times\ ]-L, h[$. Then also

$$w_k(y_k, z) = 0 \leq v(y_k, z) \quad \text{for all } z \in [-L,h] \,.$$

Along the top of R_k we have (by the Lipschitz continuity and using (3.5))

$$w_k(y,h) \leq C \quad \text{for } 0 \leq y \leq y_k (< 2) \,,$$

where C, for large k, can be chosen independently of k and h. But

$$v(y,h) = \frac{h - y s_0}{s_0^2+1} \geq \frac{h - 2s_0}{s_0^2+1} \geq C$$

for h large enough. Hence it follows from the Comparison Lemma that $w_k = 0$ in $\{v = 0\}$, i.e. below the line with slope s_0. This contradicts (3.4), see also Figure 6. Therefore $\gamma_0 = 0$

in R_∞. Using the monotonicity of γ_0 ($\partial_z\gamma_0 \le 0$) and refering to (3.1) and Figure 4, we obtain that $\gamma_0 = 0$ in the domain

$$D := R_\infty \cup (]y_0 - \delta_0, y_0 + \delta_0[\times]z_0, \infty[) .$$

Hence

$$\Delta w_0 = 0 \qquad \text{in } D.$$

By (3.1) and the strong maximum principle

$$w_0 > 0 \qquad \text{in } D,$$

while

$$w(0, z) = 0 \qquad \text{for all } z \in \mathbb{R}. \tag{3.6}$$

So far we worked only in the halfspace $\{y > 0\}$. To obtain a contradiction we also have to consider the situation for $y < 0$. There are two possibilities: either

$$w_0(\tilde{e}_0) > 0 \text{ for some } \tilde{e}_0 = (\tilde{y}_0, \tilde{z}_0) \text{ with } \tilde{y}_0 < 0$$

($\tilde{e}_0$ not necessarily a unit vector), or $w_0 = 0$ in $\{y = 0\}$. In the first case there are points $\tilde{x}_k$ with

$$\frac{\tilde{x}_k}{r_k} \to \tilde{e}_0 , \quad \frac{w(\tilde{x}_k)}{r_k} = w_k\Big(\frac{\tilde{x}_k}{r_k}\Big) \to w_0(\tilde{e}_0) > 0 .$$

As in (3.1) we conclude that $\gamma_0 = 0$ in some ball $B_{\tilde{\delta}_0}(\tilde{e}_0)$. Then it follows as above, that for some $\tilde{h}$ the function w_0 is positive and harmonic in $]-2\tilde{y}_0, 0[\times]\tilde{h}, \infty[$, and that $\gamma_0 = 0$ in this rectangle. Therefore $\gamma_0 = 0$ in

$$]-2\tilde{y}_0, 2[\times]\max\{h, \tilde{h}\}, \infty[,$$

so that w_0 has to be harmonic in this region. But then (3.6) contradicts the strong maximum principle. In the second case $\partial_z\gamma_0 = 0$ in

$$]-\infty, 2[\times]h, \infty[,$$

so that again w_0 is harmonic in this region, again a contradiction. This completes the proof of Proposition 3.5. $\square$

As a consequence we have

3.6 Theorem.

$$(i) \qquad w(x) = o(|x|) \qquad \text{as } x \to 0;$$

$$(ii) \qquad \lim_{y\downarrow 0} \frac{u(y)}{y} = +\infty \quad \text{or} \quad -\infty ,$$

$$\lim_{y\downarrow 0} \frac{u(-y)}{y} = +\infty \quad \text{or} \quad -\infty ,$$

where at least one limit is $-\infty$.

Note that

$$\lim_{y\downarrow 0} \frac{u(y)}{y} = -\infty\ , \quad \lim_{y\downarrow 0} \frac{u(-y)}{y} = -\infty$$

refers to the cusp case, and

$$\lim_{y\downarrow 0} \frac{u(y)}{y} = -\infty\ , \quad \lim_{y\downarrow 0} \frac{u(-y)}{y} = +\infty$$

refers to the vertical case with $w = 0$ on the left of the origin (see Figure 2 from [4]).

Proof of the theorem. The first assertion is equivalent to Propositions 3.3 and 3.5. To prove the second part, we first suppose that

$$s := \limsup_{y\searrow 0} \frac{u(y)}{y} > -\infty\ . \tag{3.7}$$

Let $(y_k)_k$ with $y_k \searrow 0$ be a corresponding sequence, choose any $s_0 < s$, and let v be the linear solution in the Comparison Lemma 2.1 with slope s_0 and $\hat{x} = 0$. We consider the rectangle

$$R_k :=]0, y_k[\times] - L, u(y_k)[$$

where the height $-L$ corresponds to the bottom of the translated domain V. By the choice of s_0 and property (i) we have that $w \leq v$ on ∂R_k for k large enough. The Comparison Lemma then gives

$$w = 0\ ,\ \gamma = 1 \text{ in } S_k := \{(y, z); 0 < y < y_k \text{ and } z < ys_0\}\ ,$$

and thus

$$\frac{u(y)}{y} \geq s_0 \qquad \text{for small } 0 < y < y_k.$$

Letting $s_0 \to s$, we therefore obtain from (3.7) that

$$s = \liminf_{y\searrow 0} \frac{u(y)}{y}\ .$$

Next assume that

$$s < \infty \tag{3.8}$$

This means that for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(s-\varepsilon)y < u(y) < (s+\varepsilon)y \qquad \text{for } 0 < y < \delta$$

and consequently

$$\left.\begin{array}{rcl} \Delta w & = & 0 \\ \gamma & = & 0 \end{array}\right\} \quad \text{in } \{(y, z) : z > (s+\varepsilon)y\ ,\ 0 < y < \delta\}$$

and

$$\left.\begin{array}{rcl} w &=& 0 \\ \gamma &=& 1 \end{array}\right\} \quad \text{in } \{(y,z) : z < (s-\varepsilon)y \ , \ 0 < y < \delta\}.$$

Then the same holds for the scaled functions

$$w_k(x) := \frac{1}{r_k} w(r_k x) \text{ and } \gamma_k(x) := \gamma(r_k x)$$

but now with $\delta_k = \frac{\delta}{r_k}$ instead of δ. We obtain for all sufficiently large k the situation from Figure 7. Then we apply Proposition 2.6 and obtain that

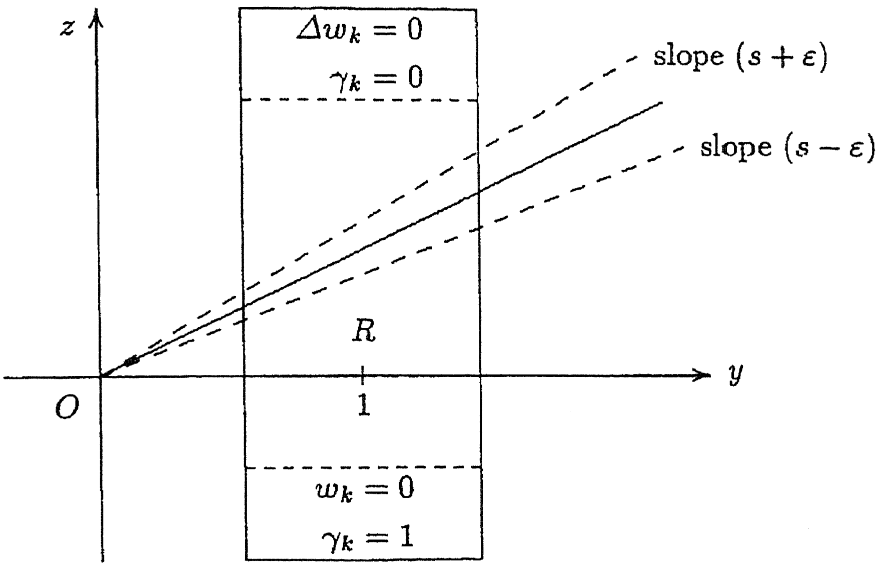


Fig. 7. Situation after scaling for all (w_k, γ_k).

$$\sup_R |w_k| \geq c > 0 \qquad \text{for all } k \text{ sufficiently large.}$$

However this contradicts the o-property of w and rules out the possibility (3.8). The remaining case is

$$\limsup_{y \searrow 0} \frac{u(y)}{y} = -\infty \ .$$

Similar results can be obtained for the left side. Finally, assume that both limits are $+\infty$: i.e.

$$\lim_{y \downarrow 0} \frac{u(y)}{y} = +\infty = \lim_{y \downarrow 0} \frac{u(-y)}{y} \ .$$

Set $y_k = \frac{1}{k}$ and

$$u_k := \min\{u(y_k), u(-y_k)\} .$$

and consider the rectangle

$$R_k :=]-y_k, y_k[\times]-L, u_k[$$

with $L > 0$. Since $|w| \leq \varepsilon_k u_k$ on the top of R_k by Theorem 3.6(i) with $\varepsilon_k \to 0$ as $k \to \infty$, we can apply the Comparison Lemma with $s_0 = 0$ and obtain that

$$w = 0 \ , \ \gamma = 1 \quad \text{in } \{(y,z) : |y| < y_k \text{ and } z < u_k - \varepsilon u_k\}.$$

This means that $\gamma_k = 1$ for large k in a full neighborhood of the origin, contradicting the fact that this is a free boundary point. This completes the proof of the theorem. □

4. Topological properties

In this section we study the properties of local anf global connected components of $\{w \neq 0\}$. Let $x_0 = (y_0, z_0) \in V \setminus W$ with

$$w(x_0) = 0 \quad \text{and} \quad z_0 \geq u(y_0) \ . \tag{4.1}$$

Then x_0 lies on the boundary of $\{w \neq 0\}$. The following statements will be relative to an open set $U \subset V \setminus W$ with $x_0 \in U$. Consider an open set D with

$$D \subset U \cap \{w \neq 0\} \ , \quad w = 0 \quad \text{on} \quad U \cap \partial D \ , \tag{4.2}$$

$$x_0 \in \partial D \ . \tag{4.3}$$

Then the following holds.

4.1 Proposition. *Let D satisfy (4.2) and (4.3). Then the number of sets G satisfying*

(4.4) *G is a connected component of D,*

(4.5) $x_0 \in \partial G$

is positive and finite. Moreover, for each G satisfying (4.4) the closure $\overline{G}$ contains points of $\{w \neq 0\} \cap \partial U$.

Proof. The last statement follows, since otherwise $w = 0$ on ∂G. Since w is harmonic in G, it would follow that $w = 0$ in G.

The assertion follows easily if $z_0 > u(y_0)$ in (4.1). For, in a neighbourhood of x_0

$$w(x) = \mathrm{Re}\, h(x)$$

with a nontrivial holomorphic function h satisfying $h(x_0) = 0$. In other words,

$$h(x) = a(x - x_0)^m (1 + \tilde{h}(x))$$

with $a \in \mathbb{C} \setminus \{0\}$, $m \geq 1$, and a holomorphic function $\tilde{h}$ satisfying $\tilde{h}(x_0) = 0$. Therefore $h(\tau(x)) = a(x - x_0)^m$ for a unique local conformal transformation τ given by

$$\tau(x) = x_0 + (x - x_0)(1 + \tilde{h}(\tau(x)))^{-\frac{1}{m}} .$$

Then near x_0 the set $\{w \circ \tau = 0\}$ consists of $2m$ ray, therefore there are at most $2m$ domains G.

Now let x_0 be on the free boundary. For convenience, let $x_0 = 0$. For $\varepsilon, \delta > 0$ small enough consider the rectangle

$$R :=]-\delta, \delta[\times]-\varepsilon, \varepsilon[,$$

similarly, R' with $\delta' = \frac{\delta}{2}$ and $\varepsilon' = \frac{\varepsilon}{2}$. Since u is continuous we can choose δ so that

$$R \cap \operatorname{graph}(u) \subset \left\{ |z| < \frac{\varepsilon}{4} \right\} . \tag{4.6}$$

Let G be any set satisfying (4.4) and

$$G \cap R' \neq \emptyset . \tag{4.7}$$

Since G touches ∂U there exists a curve $\gamma : [0,1] \to G$ with $\gamma(0) \in \partial R'$ and $\gamma(1) \in \partial R$. Assume there are infinite many domains G_i, $i \in \mathbb{N}$, with corresponding curves γ_i. We claim that

$$\sup_t \operatorname{dist}(\gamma_i(t), \operatorname{graph}(u)) \to 0 \quad \text{as } i \to \infty. \tag{4.8}$$

If not, there are points $\xi_i = \gamma_i(t_i)$ converging, for a subsequence $i \to \infty$, to a point $\xi \in \overline{R}$ above graph(u). Since ξ_i belong to different components G_i we must have $w(\xi) = 0$. But then it follows as in the first part of the proof, that only finitely many domains G_i can enter a small neighbourhood of ξ. This proves (4.8).

Then it follows from (4.6) that $\gamma_i([0,1]) \subset \{|z| < \frac{\varepsilon}{2}\}$ for large i. Therefore $\gamma_i(1) \in \partial R$ has horizontal coordinate $+\delta$ or $-\delta$. For definiteness consider the first case and the rectangle

$$R'' :=]\delta', \delta[\times]-\varepsilon, \varepsilon[.$$

Since γ_i and γ_{i+1} belong to different connected components of D, there must be, at least for a subsequence $i \to \infty$, curves. Γ_i between γ_i and γ_{i+1} going through R'' from left to right and having the property of Proposition 2.7. Consequently there are points $x_i \in R''$ between Γ_i and graph(u) with $|w(x_i)| \geq c > 0$, where c is independent of i. But (4.8) together with the continuity of w gives $w(x_i) \to 0$ as $i \to \infty$. This proves that there are only finitely many domains G_i, $i = 1, \ldots, n$, satisfying (4.4) and (4.7). Since

$$\bigcup_{i=1}^{n} (G_i \cap R') = D \cap R'$$

it follows from (4.3) that some G_{i_0} has to satisfy (4.5). □

4.2 Proposition. *If D satisfies (4.2) and (4.3) then there exists a continuous curve in D with x_0 as continuous limit.*

Proof. Choose a sequence of balls $U_k := B_{r_k}(x_0)$, $k \geq 1$, r_1 sufficiently small, and $r_k \searrow 0$ as $k \to \infty$. Define $D_0 := D$. Using Proposition 4.1 choose inductively D_k, $k \geq 1$, so that

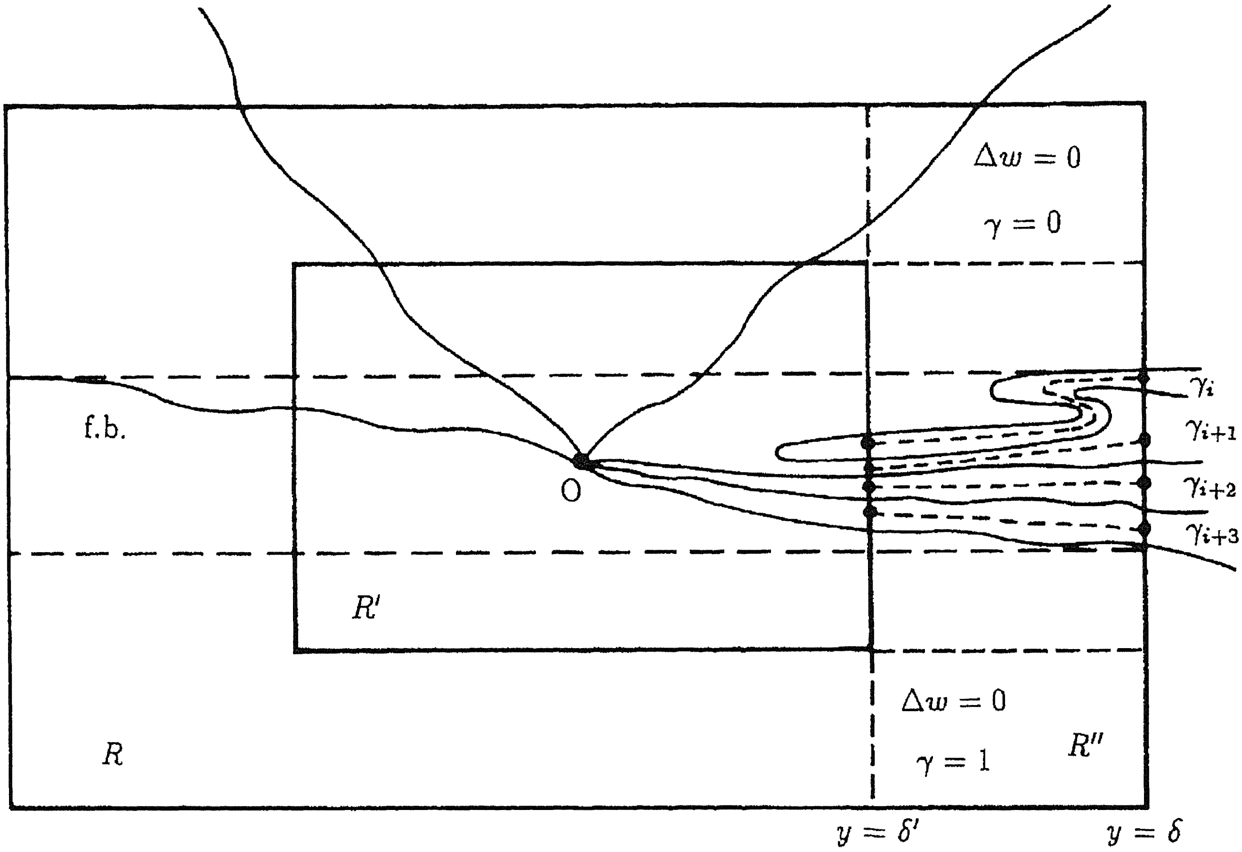


Fig. 8. The curves γ_i.

(4.9) $$D_k \text{ is a connected component of } D_{k-1} \cap U_k$$

with $x_0 \in \partial D_k$. Since D_k touches ∂U_k there are points $x_k \in D_k \cap \partial U_{k+1}$. Fix such a sequence $(x_k)_{k \geq 1}$. By construction $x_{k+1} \in D_{k+1} \subset D_k$. Therefore there are curves

$$\gamma : \left[\frac{1}{k+1}, \frac{1}{k}\right] \to D_k \subset U_k \quad \text{with} \quad \gamma\left(\frac{1}{k+1}\right) = x_{k+1}\,,\ \gamma\left(\frac{1}{k}\right) = x_k\ .$$

Then $\gamma(t) \to x_0$ as $t \to 0$. □

As a consequence we obtain that locally the number phases is well defined.

4.3 Proposition. *Let x_0 as in (4.1) and $U_0 := B_{r_0}(x_0) \subset V \setminus W$. Moreover let U be an open set with $x_0 \in U \subset U_0$. Then the following holds:*

(i) There exists an $m \geq 1$ so that there are exactly m connected components G_i, $i = 1, \ldots, m$, of $\{w \neq 0\} \cap U$ with $x_0 \in \partial G_i$.

(ii) The number m in (i) is independent of U.

(iii) There exists an $r_1 > 0$ with

$$B_{r_1}(x_0) \subset \bigcup_{i=1}^{m} \overline{G}_i\ .$$

Proof. The first assertion is Proposition 4.1 for $D = \{w \neq 0\} \cap U$. To prove (iii) consider the open set

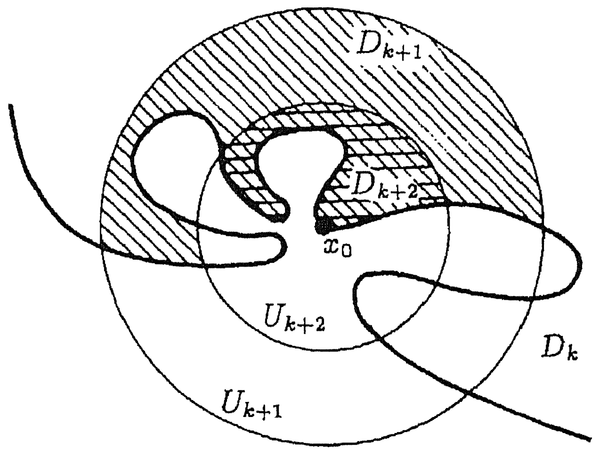


Fig. 9. The domains D_k.

$$D := U \setminus \bigcup_{i=1}^{m} \overline{G}_i \, ,$$

which satisfies (4.2). If D would satisfy (4.3) then by Proposition 4.1 there is a connected component G of D with $x_0 \in \partial G$. This contradicts the definition of m. Therefore $x_0 \notin \overline{D}$, i.e., $B_{r_1}(x_0) \cap \overline{D} = \emptyset$ for some $r_1 > 0$.

To prove (ii) let $x_0 \in \tilde{U} \subset U$ and denote by $\tilde{m}$ the corresponding number from (i). By Proposition 4.2 there are curves $\gamma_i :]0,1] \to G_i$ with $\gamma_i(0) = 0$.

Choose $t_i > 0$ so that $\gamma_i(t) \in \tilde{U}$ for $0 \leq t \leq t_i$ and denote by $\tilde{G}_i$ the connected component of $\{w \neq 0\} \cap \tilde{U}$ containing $\gamma_i(t_i)$. Then $x_0 \in \partial\tilde{G}_i$ and $\tilde{m} \geq m$ is proved. Now assume that $\tilde{m} > m$. Then there are connected components $\tilde{G}_1$, $\tilde{G}_2$ of $\{w \neq 0\} \cap \tilde{U}$ with $x_0 \in \partial\tilde{G}_i$ belonging to the same G_{i_0}. Using Proposition 4.2 there are curves γ_i connecting x_0 within $\tilde{G}_i$ to some point $\tilde{x}_i \in \tilde{G}_i$, and $\tilde{x}_1$ and $\tilde{x}_2$ are connected within G_{i_0} by a curve γ_0. Denote by K the compact set enclosed by γ_0, γ_1, γ_2. By the maximum principle (note that U_0 is a ball not touching W) w has the same sign in K as in G_{i_0}. But then γ_0 can be contracted within $\{w \neq 0\}$ to a curve inside $\tilde{U}$, so that $\tilde{G}_1$ and $\tilde{G}_2$ are connected. □

We now give some consequenses of the above considerations.

4.4 Corollary. *Let m and G_i as in 4.3 (i). Then $w_i := \chi_{G_i} w$ belong to $H^{1,2}(U_0)$ and*

$$w = \sum_{i=1}^{m} w_i \quad \text{in } B_{r_1}(x_0).$$

Therefore m coincides with the number of phases in Definition 2.3.

4.5 Remark. The number of (global) connected components of $\{w \neq 0\}$ is finite.

Proof. Since $\Delta w = 0$ above the free boundary and away from the wells we find, by the maximum principle, that each such component either contains a well, or as part of its boundary a segment $\{a_i\}\times]u_0, H]$ where $w > 0$, or touches the top of V. But there w can have only finitely many sign changes for, the free boundary stays away from the top hence there w is real analytic. □

4.6 Proposition. *Let $D \subset V$ be a connected component of $\{w < 0\}$. Then $\overline{D}$ can contain at most one free boundary point.*

Proof. Suppose $x_0, x_1 \in \partial D \cap V$ are two district free boundary points. By Proposition 4.2, there exists a Jordan arc $\Gamma \subset D$ connecting x_0 and x_1, see Figure 10 (left).

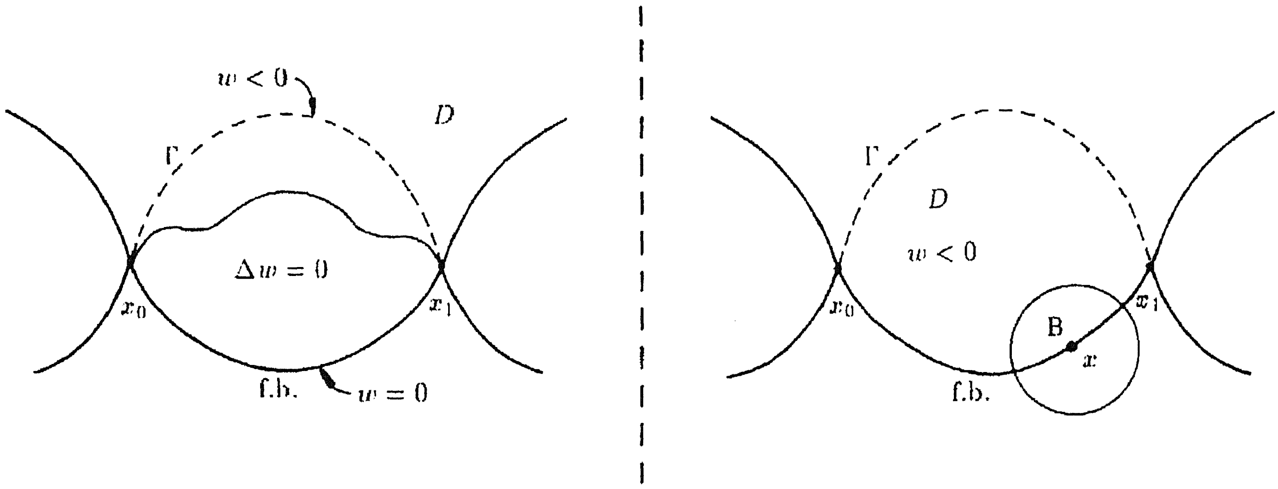


Fig. 10. Consequence of two free boundary points in ∂D.

Applying the maximum principle gives $w < 0$ in the domain bounded by Γ and the free boundary between x_0 and x_1, see Figure 10 (right). Then for a ball B as indicated in the figure, we have $w < 0$ above the free boundary and $w = 0$ below it. This contradicts $\Delta w \geq 0$ in B. □

4.7 Theorem. *The number of cusps is less or equal the number of wells.*

Proof. Each cusp belongs to the closure of $\{w < 0\}$. By Proposition 4.1 the cusp is in the closure of a connected component D of $\{w < 0\}$. But D has to contain a well, since otherwise w is harmonic in D with $w = 0$ on ∂D outside the top on V and $\frac{\partial w}{\partial \nu} = 1$ on the top of V. The maximum principle then gives $w \geq 0$ in D, a contradiction. The assertion then follows using Proposition 4.6. □

4.8 Proposition. *(i) Near a cusp the free boundary is smooth and $w > 0$ in an upper neighbourhood.*

(ii) At a cusp the number m in 4.3 satisfies $m \geq 3$.

Proof. By Theorem 4.7 and the definition of a cusp we know that $w > 0$ in an upper neighborhood of the free boundary near a cusp, except at the cusp. Then (i) follows after applying the regularity theory for the dam problem in suitably chosen left and right neighborhoods of the cusps, and (ii) follows since the cusp lies in the closure of $\{w < 0\}$ as in the proof of the previous theorem. □

Next we consider some local properties of w at a cusp, which again, for convenience, has been translated to the origin O. We first make an assumption about the decay of the free boundary near the cusp. Suppose

$$(A): \text{There exist constants } C, \alpha > 0 \text{ such that for small } |y| \quad |y| \leq C|u(y)|^{1+\alpha}\,.$$

This assumption implies

4.9 Lemma. *Let (A) be satisfied. Then in a neighborhood of O there exists a conformal transformation τ satisfying*

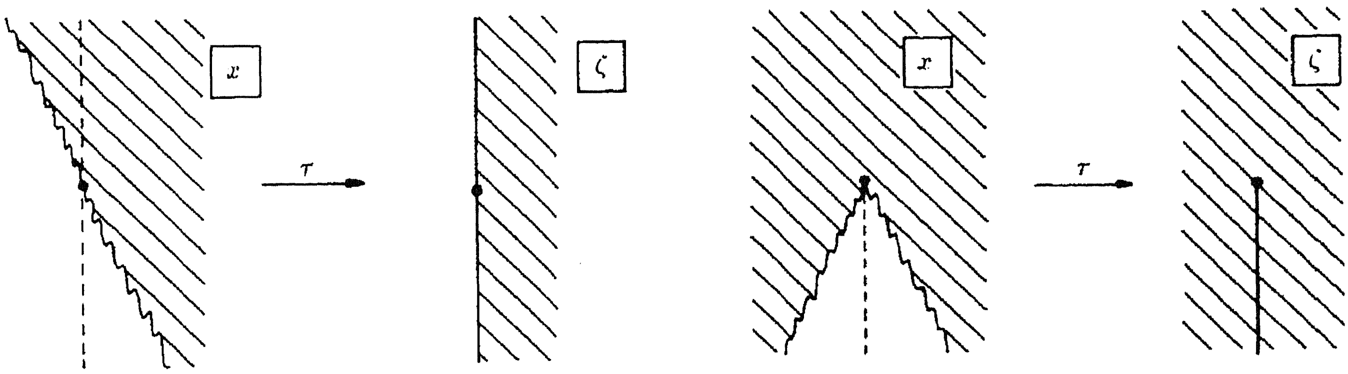
(i) $\tau(0) = 0$,

(ii) τ *and* τ^{-1} *are continuous up to the boundary,*

(iii) on every cone C above the free boundary with vertex at O

$$\frac{1}{|x|}|\tau(x) - x| + |\nabla(\tau(x) - x)| \to 0 \quad \text{for } x \in C,\, |x| \to 0$$

Proof. The proof of this technical lemma is given in Appendix B.



vertical case

cusp case

As a consequence we have the following. The function $w \circ \tau^{-1}$ is harmonic and non-trivial in the transformed (shaded) regions and vanishes along the boundary. This means that for

(4.10) $\qquad k = 1$ in the cusp case, $k = 2$ in the vertical case,

there is a real number $a \neq 0$ and some integer $m \geq 1$ with

(4.11) $$w \circ \tau^{-1}(\zeta) = \mathrm{Re}(-ia\tilde{\zeta}^m(1 + \tilde{h}(\tilde{\zeta}))) \text{ with } \tilde{\zeta} = i^k(-i\zeta)^{k/2} .$$

Here $\tilde{h}$ is a holomorhic function satisfying $\mathrm{Im}\,\tilde{h}(\tilde{\zeta}) = 0$ if $\mathrm{Im}\,\tilde{\zeta} = 0$. Since m is the number of components of $\{w \circ \tau^{-1} \neq 0\}$ near O, it has to coincide with the number m in 4.3. It follows from 4.4 and 4.8 that

$$m \text{ is odd} \quad \text{and} \quad m \geq 3.$$

The properties of τ imply that the m phases are separated by smooth curves which have a tangent at O. For instance, if $m = 3$ the two possibilities are sketched in Figure 11.

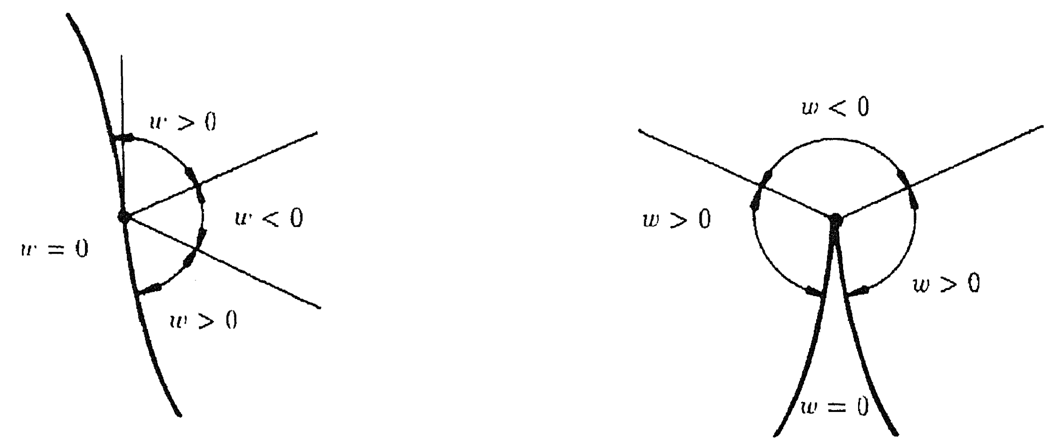


Fig. 11. Distribution of phases with $m = 3$.

Further we obtain for any phase $\tilde{w}$ of w the <u>non-degeneracy</u> result: there exist a constant $c > 0$ such that for small $r > 0$

(4.12) $$\int_{B_r \setminus B_{r/2}} |\nabla \tilde{w}|^2 \geq c r^{km} .$$

5. Blow up.

In this section we investigate the Hölder exponent of the free boundary at a cusp, which again is situated at the origin O. As in Definition 2.3 (see also 4.4) we decompose w according to

$$w = \sum_{i=1}^{m} w_i \quad \text{in } B_{r_0} \subset V \setminus W \ (r_0 \text{ small}),$$

where m denotes the number of phases at O. For each phase w_i we define a corresponding exponent $\alpha_i :]0, r_0] \to \mathbb{R}$ by

$$\left(\fint_{B_r} |\nabla w_i|^2 \right)^{1/2} = r^{\alpha_i(r)} \quad \text{for } 0 < r \le r_0.$$

Using Proposition 2.5 and the sublinear decay of w at O, see Theorem 3.6 (i), we see that for each phase $r^{\alpha_i(r)} \to 0$ along any sequence $r \searrow 0$. Hence all $\alpha_i(r) > 0$ for $0 < r \le r_0$ with r_0 sufficiently small.

5.1 Remark. Let us phrase the Monotonicity Formula 2.8 into terms of $\alpha_i(r)$. It follows from Theorem 3.6 (ii) (for the vertical case) that for $0 < r_1 < r_0$ we can choose $\varepsilon(r_1)$ with $\varepsilon(r_1) \searrow 0$ as $r_1 \searrow 0$ such that (2.4) is satisfied for $0 < r < r_1$ with $\delta(r) = 0$ and

$$\kappa = \kappa(r_1) = \begin{cases} 1 & \text{for } k = 1, \\ 2 - \varepsilon(r_1) & \text{for } k = 2. \end{cases} \tag{5.1}$$

Here k is defined as in (4.10). Then (2.5) becomes

$$\begin{aligned} \prod_{i=1}^{m} r^{\alpha_i(r)} &= \left(r^{-2m} \prod_{i=1}^{m} \int_{B_r} |\nabla w_i|^2 \right)^{1/2} \\ &= \left(r^{\kappa m^2 - 2m} \varphi(r) \right)^{1/2} \le C\, r^{m \frac{\kappa m - 2}{2}} \end{aligned}$$

for $0 < r < r_1$ with $C = \sqrt{\varphi(r_1)}$. Hence

$$\frac{1}{m} \sum_{i=1}^{m} \alpha_i(r) \ge \frac{\kappa m - 2}{2} - \frac{1}{m} \frac{\log C}{\log(1/r)} \quad \text{for } 0 < r < r_1. \tag{5.2}$$

Later we show that this estimate is sharp in the cusp case (i.e. $k = 1$ in (4.10)).

Next define the smallest exponent

$$\alpha(r) := \min_{i=1,\dots,m} \alpha_i(r) \quad \text{for } 0 < r \le r_0$$

and consider the blow-up, for $x \in B_1$ and $0 < \rho \le r_0$,

$$w_\rho(x) := w(\rho x)/\rho^{1+\alpha(\rho)}, \quad \gamma_\rho(x) := \gamma(\rho x).$$

The pair (w_ρ, γ_ρ) satisfies

$$\int_{B_1} \nabla\zeta \cdot (\nabla w_\rho + \frac{\gamma_\rho}{\rho^{\alpha(\rho)}} e_z) = 0 \quad \text{for all } \zeta \in C_0^\infty(B_1) \tag{5.3}$$

and

$$\frac{1}{\pi} \int_{B_1} |\nabla w_\rho|^2 = \rho^{-2\alpha(\rho)} \fint_{B_\rho} |\nabla w|^2 = \sum_{i=1}^{m} \rho^{2(\alpha_i(\rho)-\alpha(\rho))} \in [1, m] \tag{5.4}$$

This means that we have scaled so that w_ρ in B_1 carries the phase with the biggest Dirichlet integral in B_ρ. However other phases might become very small for w_ρ if ρ is small. The reason is that at this point we do not know that the phases, in other words the values α_i, are balanced towards each other. The first result relates the values $\alpha_i(r)$ to Assumption (A) in Section 4.

5.2 Proposition. *There exists a constant $C > 0$ such that for points $(y, z) \in B_r$ on the free boundary of (w, γ) we have*

$$|z| \le \frac{r}{2} \quad \textit{implies} \quad |y| \le C\, r^{1+\alpha(r)} ,$$

for all $r > 0$ sufficiently small.

Proof. To prove this result we first scale and show that for points $(\tilde{y}, \tilde{z}) \in B_1$ on the free boundary, i.e. graph(u_r), of (w_r, γ_r) we have

$$|\tilde{z}| \le \frac{1}{2} \quad \text{implies} \quad |\tilde{y}| \le C\, r^{\alpha(r)} \quad \text{for all } r > 0 \text{ small enough.} \tag{5.5}$$

Let $\eta \in C_0^\infty(B_1)$ be a fixed cut-off function satisfying $0 \le \eta \le 1$ and $\eta = 1$ on $B_{7/8}$. Substitution into (5.3) and using (5.4) yields

$$\frac{1}{r^{\alpha(r)}} \int_{B_1} \gamma_r \partial_z \eta = -\int_{B_1} \nabla\eta \cdot \nabla w_r \le C$$

or

$$\int_{-1}^{+1} \eta(y, u_r(y))\, dy \le C\, r^{\alpha(r)} \quad \text{for } 0 < r < r_0.$$

The integral in this inequality can be bounded from below by

$$\mathcal{L}^1(\{y : |y| \le \tfrac{1}{4} \text{ and } |u_r(y)| \le \tfrac{3}{4}\}).$$

By Theorem 3.6 (ii), the free boundary u is vertical at O, from both sides. Hence for points $(\tilde{y}, \tilde{z}) \in B_1$ on the free boundary of w_r we have that $|\tilde{y}|/|\tilde{z}|$ is small, if r is small. Therefore, for small r, if $|\tilde{z}| \le \frac{1}{2}$ then $|\tilde{y}| \le \frac{1}{4}$.

For definiteness let us consider $0 \le \tilde{y} \le \frac{1}{4}$. If we now can show that

$$|u_r(y)| \leq \frac{3}{4} \quad \text{for all } 0 < y \leq \tilde{y}$$

then assertion (5.5) follows from the above inequalities and the proof of the proposition is complete.

We distinguish two situations.

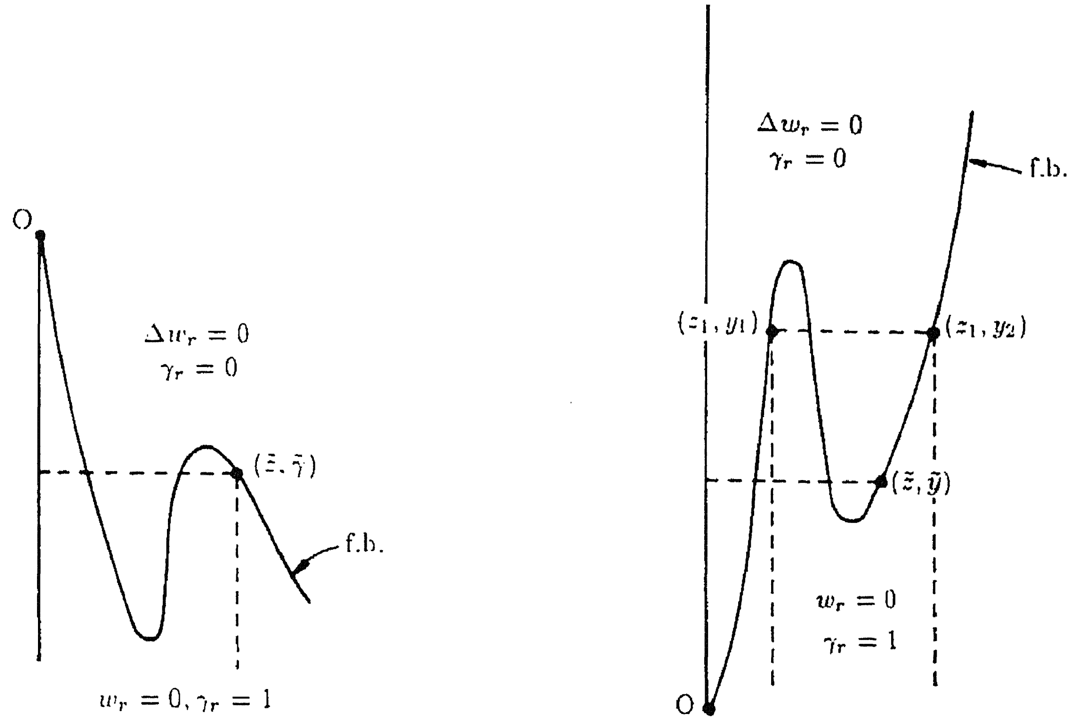


Fig. 12. Possible configurations near O

Case 1: $\tilde{z} < 0$ (see Figure 12 (left)). Then

$$\tilde{y} \leq \varepsilon_r |\tilde{z}| \quad \text{with } \varepsilon_r \to 0 \text{ as } r \to 0,$$

$$u_r(y) \leq 0 \quad \text{for } 0 \leq y \leq \tilde{y} \text{ and small } r,$$

both as a consequence of Theorem 3.6 (ii). Applying the Comparison Lemma 2.1 with $s_0 = 0$ to the scaled equation gives

$$u_r(y) \geq \tilde{z} - C\tilde{y} \quad \text{for } 0 \leq y \leq \tilde{y}$$

or

$$u_r(y) \geq \tilde{z}(1 + C\varepsilon_r) \geq -\frac{3}{4} \quad \text{for } r \text{ small enough.}$$

Case 2: $\tilde{z} > 0$ (see Figure 12 (right)). Then

$$\tilde{y} \leq \frac{\varepsilon_r}{2} \quad \text{with } \varepsilon_r \to 0 \text{ as } r \to 0,$$
$$u_r(y) \geq 0 \quad \text{for } 0 \leq y \leq \tilde{y} \text{ and small } r.$$

Now we argue as follows. Let $z_1 := \frac{3}{4} > \tilde{z}$ and assume that for some $y_1 \in]0, \tilde{y}[$ we have $u_r(y_1) \geq z_1$ (in Figure 12 (right) we have chosen $u_r(y_1) = z_1$). Further, let

$$y_2 := \sup\{y > \tilde{y} : u_r(y) < z_1\} .$$

If y is as in this definition then (ry, rz_1) lies above the free boundary of w with $0 < rz_1 < r$. The vertical shape of the free boundary at O, see Theorem 3.6 (ii), then implies that y_2 exists and

$$y_2 \leq \varepsilon_r z_1 \quad \text{with } \varepsilon_r \to 0 \text{ as } r \to 0.$$

We apply the Comparison Lemma 2.1 with $s_0 = 0$ in the rectangle $]y_1, y_2[\times] - \infty, z_1[$ and obtain that

$$u_r(y) \geq z_1 - C(y_2 - y_1) \quad \text{for } y_1 \leq y \leq y_2.$$

In particular at $y = \tilde{y}$:

$$\tilde{z} \geq z_1 - Cy_2 \geq z_1(1 - C\varepsilon_r)$$

for small r a contradiction to $\tilde{z} \leq \frac{1}{2}$. □

If α would stay strictly positive as $r \to 0$, then by Proposition 5.2 we could apply the conformal transformation of Lemma 4.6 which would tell us that in a sense the phases of w are balanced. If this is not the case then $\alpha(r) \to 0$ for a subsequence $r \to 0$. We then still have the possibility to study the blow-up limit of w_r. For the usual linear blow-up sequence the blow-up limit is globally defined since w is Lipschitz continuous. Here the values of w are stretched more in order to obtain w_r with a Dirichlet integral satisfying (5.4). The purpose of this stretching is to have the chance to pick up a non-trivial blow-up limit. By (5.4) the blow-up limit will exist in B_1, but it needs not to exist outside B_1. Moreover, the problem which could arise is that the blow-up limit might vanish in any ball B_δ with $\delta < 1$, having a gradient concentrated near ∂B_1, despite of property (5.4). On the other hand, such a degeneracy of w_r is in favour of high values of $\alpha(\delta_r)$. The following proposition takes care of this situation in a precise way.

5.3 Proposition. *Let $r_k := 2^{-k} r_0$. Assume that there exist constants $\beta, \gamma > 0$ so that for $r = r_k$*

$$\left(\fint_{B_{r/2}} |\nabla w|^2 \right)^{1/2} \leq 2^{-\gamma} \left(\fint_{B_r} |\nabla w|^2 \right)^{1/2} \quad \textit{or that } \alpha(r) \geq \beta. \tag{5.6}$$

Then

$$\liminf_{s \searrow 0} \alpha(s) \geq \min\{\beta, \gamma\} .$$

Proof. Define for given $M > 0$ the function

$$\Psi(r) := \max\Big\{ \Big(\fint_{B_r} |\nabla w|^2 \Big)^{\frac{1}{2}} \ , \ M r^\beta \Big\} \,, \quad 0 < r < r_0 \ .$$

Let $k \in \mathbb{N}$. If the first inequality in (5.6) is satisfied then it follows from $r_{k+1}^\beta = 2^{-\beta} r_k^\beta$ that

$$\Psi(r_{k+1}) \leq 2^{-\alpha_0} \Psi(r_k) \quad \text{with} \quad \alpha_0 := \min\{\beta, \gamma\} \ .$$

If the second inequality in (5.6) holds, then

$$\fint_{B_{r_k}} |\nabla w|^2 = \sum_{i=1}^m r_k^{2\alpha_i(r_k)} \leq m\, r_k^{2\alpha(r_k)} \leq m\, r_k^{2\beta} \ ,$$

implying

$$\fint_{B_{r_k+1}} |\nabla w|^2 \leq 4 \fint_{B_{k_k}} |\nabla w|^2 \leq 4 m\, r_k^{2\beta} \leq \big(2^{-\gamma} M\, r_k^\beta\big)^2$$

if M was chosen such that $M \geq 2^{\gamma+1}\sqrt{m}$. Hence

$$\Psi(r_{k+1}) \leq \max\big\{ 2^{-\gamma} M\, r_k^\beta \ , \ M\, r_{k+1}^\beta \big\} \leq 2^{-\alpha_0} \Psi(r_k) \ .$$

Thus in either case we have the iterative estimate

$$\Psi(r_{k+1}) \leq \theta \Psi(r_k) \quad \text{with } \theta = 2^{-\alpha_0} < 1,$$

resulting in

$$\Psi(r_k) \leq \theta^k \Psi(r_0) \quad \text{for } k \in \mathbb{N}.$$

Now let $0 < r \leq r_0$ and choose $k \in \mathbb{N}$ such that $r_{k+1} < r \leq r_k$. Then

$$\Big(\fint_{B_r} |\nabla w|^2 \Big)^{\frac{1}{2}} \leq 2 \Big(\fint B_{r_k} |\nabla w|^2 \Big)^{\frac{1}{2}} \leq 2\theta^k \Psi(r_0) \ .$$

Using that

$$r > r_{k+1} = 2^{-k-1} r_0 \quad \Rightarrow \quad k > -\frac{\log \frac{2r}{r_0}}{\log 2} \quad \Rightarrow \quad \theta^k \leq \Big(\frac{2r}{r_0} \Big)^{\alpha_0} \ ,$$

we conclude that

$$\Big(\fint_{B_r} |\nabla w|^2 \Big)^{\frac{1}{2}} \leq C\, r^{\alpha_0}$$

for some $C > 0$. On the other hand

$$\Big(\fint_{B_r} |\nabla w|^2 \Big)^{\frac{1}{2}} = \Big(\sum_{i=1}^m r^{2\alpha_i(r)} \Big)^{\frac{1}{2}} \geq r^{\alpha(r)} \ ,$$

implying the assertion of the proposition. $\square$

If (5.6) holds along the sequence $(r_k)_k$, then according to Proposition 5.2 the free boundary is Hölder continuous at the cusp. This implies Assumption (A) in Section 7 and its consequences. Therefore we consider an arbitrary sequence $\rho \searrow 0$ along which (5.6) does not hold: i.e. for which there exist constants $\delta_0, \beta_0 > 0$ such that

$$\int_{B_{\rho/2}} |\nabla w|^2 \geq \delta_0 \int_{B_\rho} |\nabla w|^2 \quad \text{and} \quad \alpha(\rho) \leq \beta_0 \tag{5.7}$$

as $\rho \searrow 0$. Then for the blow up sequence (w_ρ, γ_ρ), satisfying (5.3), we obtain using (5.4) the nondegeneracy

$$\int_{B_{1/2}} |\nabla w_\rho|^2 \geq \delta_0 \int_{B_1} |\nabla w_\rho|^2 \geq \pi \delta_0 \ , \tag{5.8}$$

i.e. w_ρ will have a nontrivial blow-up limit. To this end we first transform the functions γ_ρ into, see also Figure 13,

$$\tilde{\gamma}_\rho(y,z) := \begin{cases} \gamma_\rho(y,z) & \text{in cusp case,} \\ \gamma_\rho(y,z) & \text{in } \{y > 0\} \\ 1 - \gamma_\rho(y,z) - 1 & \text{in } \{y < 0\} \end{cases} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{in vertical case,} \tag{5.9}$$

where in the vertical case we assume for definiteness the flow domain to be on the right-hand side.

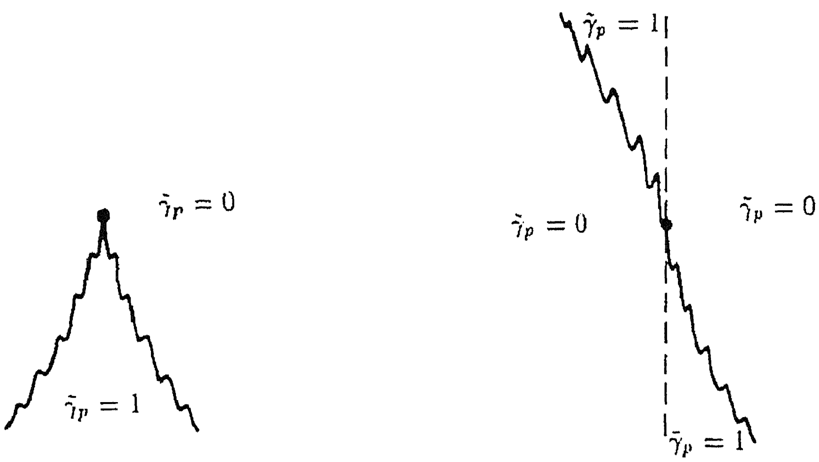


Fig. 13. Definition of $\tilde{\gamma}_\rho$.

Clearly

$$\int_{B_1} \nabla\zeta \cdot \left(\nabla w_\rho + \frac{\tilde{\gamma}_\rho}{\rho^{\alpha(\rho)}} e_z\right) = 0 \quad \text{for all } \zeta \in C_0^\infty(B_1).$$

Moreover, it follows from Proposition 5.2 (see (5.5)) that for $|z| \le \frac{1}{2}$ and ρ sufficiently small

$$\tilde{\gamma}_\rho(x,z) = 0 \quad \text{for } |y| > C \cdot \rho^{\alpha(\rho)}, \tag{5.10}$$

and thus

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{|\tilde{\gamma}_\rho(y,z)|}{\rho^{\alpha(\rho)}}\, dy \le C\,.$$

Further define functions $l_\rho^\pm : [-\frac{1}{2}, +\frac{1}{2}] \to \mathbb{R}$ by

$$l_\rho^\pm(z) := \int_{\{0 \le \pm y \le \frac{1}{2}\}} \frac{\tilde{\gamma}_\rho(y,z)}{\rho^{\alpha(\rho)}}\, dy \quad \text{for } |z| \le \tfrac{1}{2}.$$

They satisfy $0 \le z\, l_\rho^\pm(z) \le C|z|$ and they are monotone non-increasing (since $\partial_z \gamma \le 0$).

It is now possible to choose a subsequence $\rho \searrow 0$ along which

$$\begin{aligned} &w_\rho \to w_* \quad \text{weakly in } H^{1,2}(B_1) \text{ (see (5.2)) and a.e. in } B_1,\\ &l_\rho^\pm \to l_*^\pm \quad \text{weakly star in } L^\infty(]-\tfrac{1}{2}, +\tfrac{1}{2}[),\\ &\alpha(\rho) \to \alpha_* \in [0, \beta_0]\,. \end{aligned}$$

Replacing the test functions in the weak equation for $(w_\rho, \tilde{\gamma}_\rho)$ as was done in the Separation Lemma (thus with $\zeta^\pm(z) := \zeta(0\pm, z)$ having different values on $-\frac{1}{2} < z < 0$, but the same values for $z > 0$) we obtain

$$0 = \int_{B_1} \nabla w_\rho \cdot \nabla\zeta + \int_{B_1 \cap \{y>0\}} \frac{\tilde{\gamma}_\rho}{\rho^{\alpha(\rho)}} \partial_z \zeta + \int_{B_1 \cap \{y<0\}} \frac{\tilde{\gamma}_\rho}{\rho^{\alpha(\rho)}} \partial_z \zeta\,.$$

Then $\delta \to 0$ gives

$$0 = \int_{B_1} \nabla\zeta \cdot \nabla w_* + \int_{\{y=0\}} l_*^+ \partial_z \zeta^+ + \int_{\{y=0\}} l_*^- \partial_z \zeta^- \tag{5.11}$$

for all such test functions ζ. From this limit equation and the convergence properties of w_ρ and the free boundaries we conclude that w_* satisfies the properties from Figure 14. To exclude the possibility of a vanishing blow up limit w_* observe that Proposition 2.5 and inequality (5.8) imply the existence of a positive constant c such that

$$\int_{B_1 \setminus B_{1/2}} |w_\rho|^2 \ge c \int_{B_{1/2}} |\nabla w_\rho|^2 \ge c\delta_0\,.$$

Fig. 14. Properties of blow up limit w_*

Since $w_\rho \to w_*$ strongly in $L^2(B_1)$ we have indeed

$$w_* \not\equiv 0 \text{ in the shaded regions from Figure 14.}$$

As an immediate consequence we have

5.4 Lemma. *There exist* $m_* \geq 1$ *odd and* $c_* > 0$ *such that the following expansion holds:*

$$w_*(x) = c_* \operatorname{Re}(-i\tilde{x}^{m_*}(1 + \tilde{h}(\tilde{x})))$$

for small $|x|$ *with* $\operatorname{Im}\tilde{x} > 0$, *where* $\tilde{x} = i^k(-ix)^{k/2}$ *and* $\tilde{h}$ *is a holomorphic function with* $\operatorname{Im}\tilde{h}(\tilde{x}) = 0$ *for* $\operatorname{Im}\tilde{x} = 0$.

Proof. The asymptotic behaviour of the blow up limit at the orgin, with $m_* \in \mathbb{N}$ and $c_* \in \mathbb{R} \setminus \{0\}$, follows from the properties shown in Figure 14. Moreover, it follows from (5.11) that in the cusp case ($k = 1$)

$$\pm\partial_y w_*(0\pm, z) = -\partial_z l_*^{\pm}(z) \geq 0 \quad \text{for } -1 < z < 0,$$

where the monotonicity of $l_*^{\pm}$ is a consequence of the approximation process. In the vertical case ($k = 2$)

$$\partial_y w_*(0+, z) = -\partial_z l_*^{\pm}(z) \geq 0 \quad -1 < \pm z < 0.$$

Checking the sign of w_* from the above expansion with these inequalities it follows that $c_* > 0$ and that m_* is odd. □

We emphasize once again that the limit function w_* results here from a particular blow up, i.e., for a particular subsequence $\rho \searrow 0$ along which the blow up is non-degenerate. Next we show that the number of phases is conserved in this blow up process. We do this in two steps and show first

5.5 Lemma. $m_* \leq m$.

Proof. Let $\delta > 0$ be fixed and sufficiently small so that in the ball B_δ the distribution of the m_* phases of w_* at 0 over the domains $D_1, \ldots, D_{m_*}$ is as in Figure 15 (a). In the figures we show only the cusp case with $m_* = 3$. We select points $x_i \in D_i$ such that

$$|w_*(x_i)| \geq c\,, \quad \text{for some } c > 0.$$

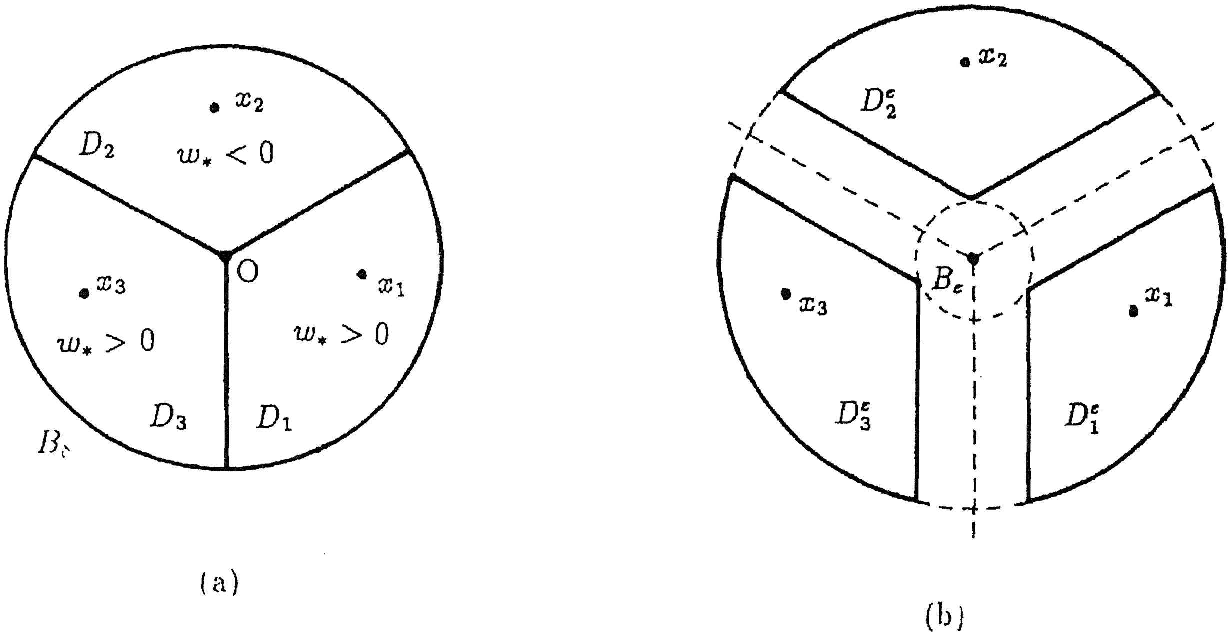


Fig. 15. (a) Distribution of m_* phases of w_* in B_δ; (b) Construction of the subsets D_i^ε

In the arguments below we need that w_ρ becomes uniformly small on circles close to the origin. Using the uniform boundedness of the Dirichlet's integral for w_ρ, we can use Courant [6, Lemma 3.1] to obtain that for any pair $0 < r_1 < r_2 < 1$, there exists $r_\rho \in [r_1, r_2]$ such that

$$\omega^2(r_\rho) \leq \frac{2\pi m}{\log r_2/r_1}\,,$$

where $\omega(r_\rho)$ denotes the oscillation of w_ρ on ∂B_{r_ρ}. Since w_ρ vanishes below O, this implies

$$\sup_{\partial B_{r_\rho}} |w_\rho| \leq \sqrt{\frac{2\pi m}{\log r_2/r_1}} \quad \text{for all } \rho > 0. \tag{5.12}$$

We use this result as follows. Consider a ball B_ε, with ε ($\sqrt{\varepsilon} << \delta$) chosen such that $B_{\sqrt{\varepsilon}} \cap \{x_i\} = \emptyset$ for all $i = 1, \ldots, m_*$. Further we select subsets D_i^ε, satisfying $x_i \in D_i^\varepsilon \subset D_i$, which touch the circle ∂B_ε, see Figure 15 (b). By the convergence of w_ρ we have for ρ sufficiently small,

$$w_\rho(x) \neq 0 \quad \text{for } x \in D_i^\varepsilon \text{ and } |w_\rho(x_i)| > c/2. \tag{5.13}$$

Choose ρ such that (5.13) holds. Then by (5.12) and the choice of ε, there exists $\mu := r_\rho \in [\varepsilon, \sqrt{\varepsilon}]$ such that

$$\sup_{\partial B_\mu} |w_\rho| \leq \sqrt{\frac{4\pi m}{\log \frac{1}{\varepsilon}}} < c/4 \ ,$$

provided ε is chosen small enough. Finally choose points $a_i \in \partial B_\mu \cap D_i^\varepsilon$ for $i = 1, \ldots, m_*$. Now suppose $m^* > m$. Then at least two domains $D_{i_1}^\varepsilon$ and $D_{i_2}^\varepsilon$ must belong to the same component of $\{w_\rho \neq 0\}$ and within this component the sets $D_{i_1}^\varepsilon$ and $D_{i_2}^\varepsilon$ can be connected by a curve σ_ρ on which w_ρ has a fixed sign (for definiteness, say positive). The sets $D_{i_1}^\varepsilon$ and $D_{i_2}^\varepsilon$ are separated by a third set, say $D_{i_0}^\varepsilon$, on which $w_\rho < 0$. We can choose σ_ρ so that it starts at a_{i_1} and stops at a_{i_2}. Now there are two possibilities.

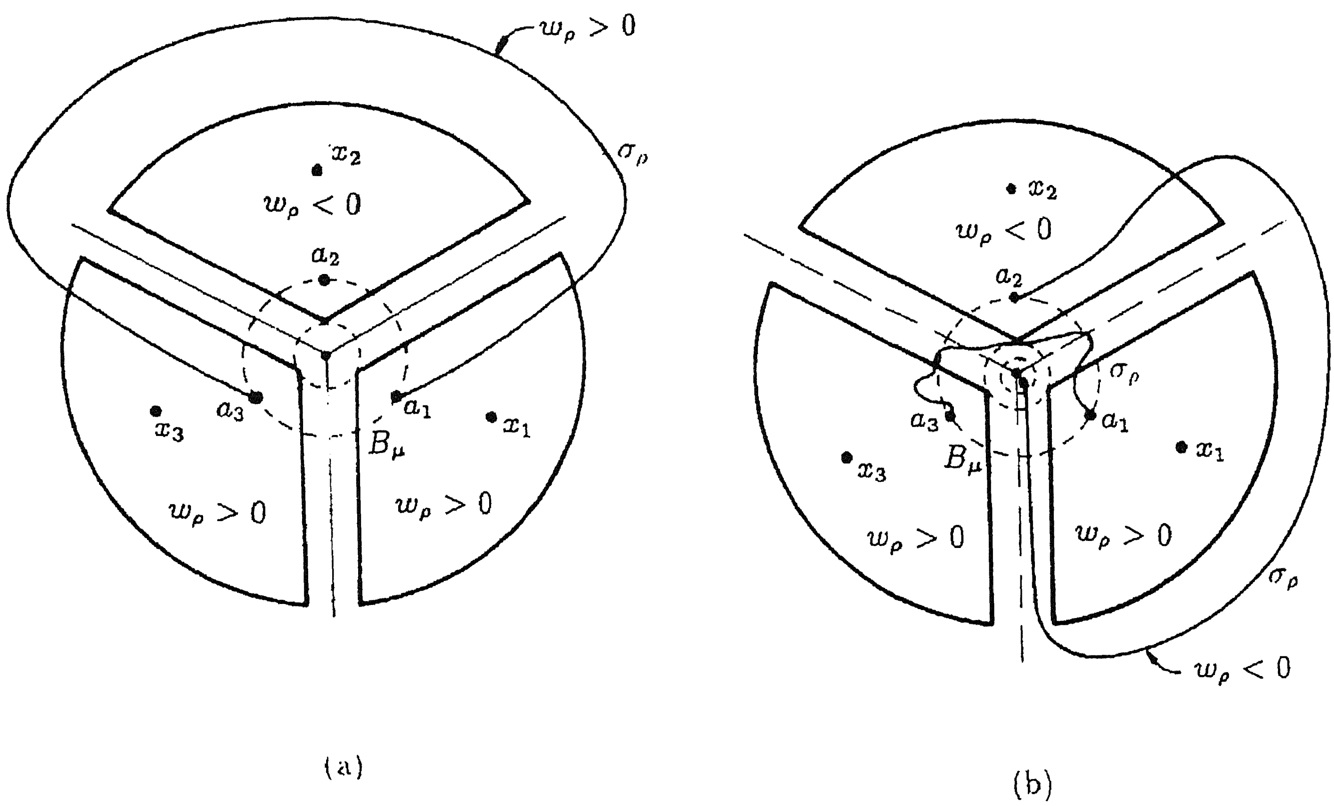


Fig. 16. (a) The curve σ_ρ encloses region where w_ρ has opposite sign; (b) The curve σ_ρ passes throug the small ball B_ε

The curve σ_ρ encloses the set $D_{i_0}^\varepsilon$ where w_ρ has opposite sign, see Figure 16 (a). Since $w_\rho > -c/4$ on ∂B_μ and $w_\rho > 0$ on σ_ρ, the maximum principle gives that $w_\rho > -c/4$ in $D_{i_0}^\varepsilon$ and in particular $w_\rho(x_{i_0}) > -c/4$, a contraction.
The other possibility is that σ_ρ passes through the small ball B_ε when connecting a_{i_1} and a_{i_2}, see Figure 20 b. Then we argue as follows. Choose $0 < \varepsilon^* < \varepsilon$ such that $\overline{B}_{\varepsilon^*} \cap \sigma_\rho = \emptyset$. In the ball B_{ε^*} we select a point b, with $w_\rho(b) < 0$, which belongs to the same component of $\{w_\rho < 0\}$ as the set $D_{i_0}^\varepsilon$. Then the only possible connection between a_{i_0} and b in that

component, is by a curve τ_ρ which encloses either D_{i_1} or D_{i_2}. As before we apply the maximum principle to reach a contradiction. □

Next we show

5.6 Lemma. $m_* \geq m$.

Proof. If $m > m_*$, then between two adjacent domains D_i^ε and D_{i+1}^ε, or between the free boundary and, say, D_1^ε there must be remaining components of $\{w_\rho \neq 0\}$.

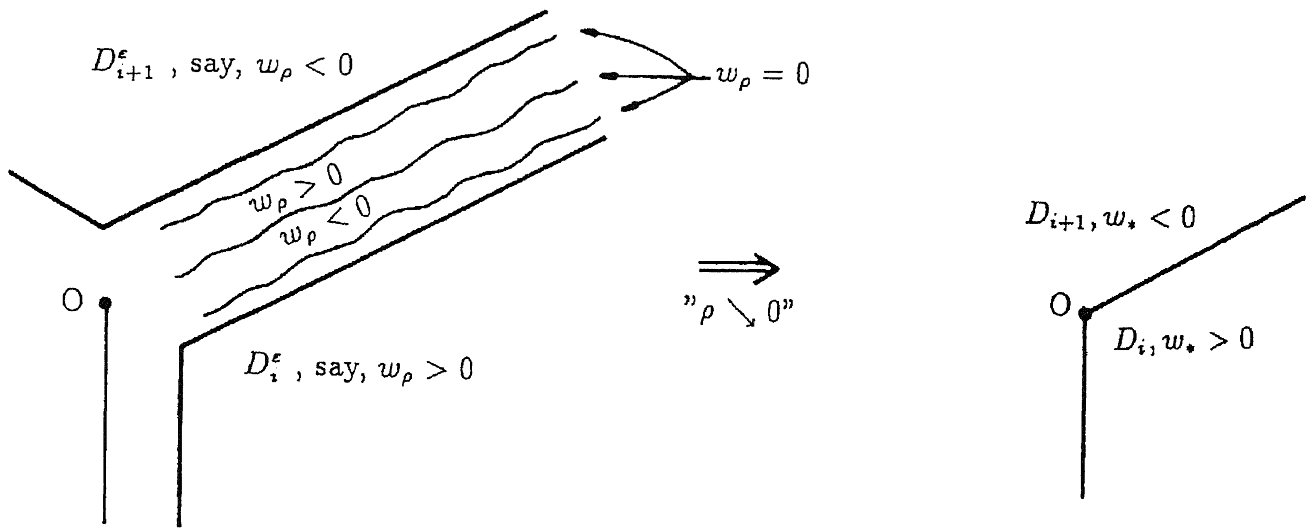


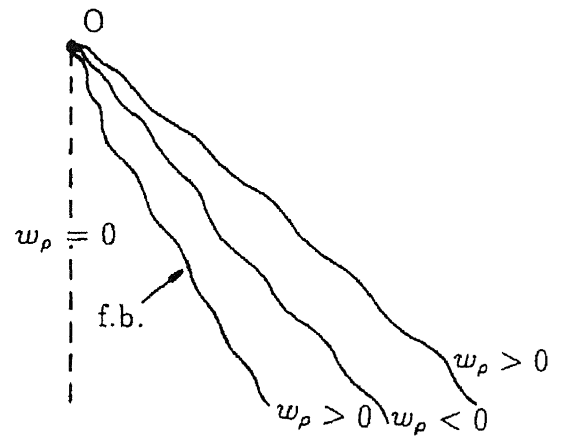
Fig. 17. Additional phases of w_ρ between D_i^ε and D_{i+1}^ε

The first possibility leads to the situation depicted in Figure 17. which holds for all ρ sufficiently small (at least those $\rho \searrow 0$, along which the sequence w_ρ converges). Consequently, in each transversal cross-section along the strip between D_i^ε and D_{i+1}^ε (we selected ε small), there are points at which w_ρ has a zero difference-quotient. By the C^1-convergence of w_ρ we now conclude that $\nabla w_* = 0$ along the curve separating D_i and D_{i+1}, see the picture on the right in Figure 17. This clearly contradicts the behaviour of w_* in Lemma 5.4.

Next we consider the second possibility. Then the additional phases of w_ρ enter along the free boundary. The argument used above does not apply here because of the missing C^1-convergence. We therefore proceed as follows.

Near the free boundary the distribution of components of $\{w_\rho \neq 0\}$ must be similar to the situation shown in Figure 18 (a).

Then for ρ sufficiently small we can choose a domain $D \subset B_1$ and a function $\tilde{w}_\rho : \overline{D} \to \mathbb{R}$, having properties as described in Figure 18 (b). Clearly $\tilde{w}_\rho$ is superharmonic in D. Because it vanishes near $\{y = 0\}$ we have



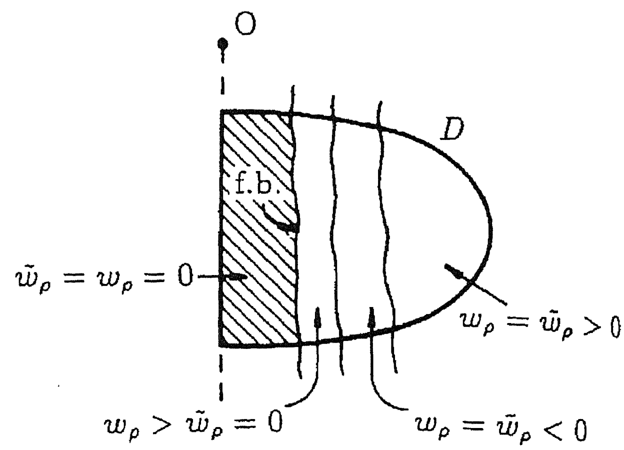


Fig. 18. (a) Sign-changes of w_ρ near free boundary; (b) Definition of the function $\tilde{w}_\rho$ on $D \subset B_1$

$$\int_D \nabla\zeta \cdot \nabla\tilde{w}_\rho \geq 0$$

for all $\zeta \in C_0^\infty(D \cup \{y = 0\})$, $\zeta \geq 0$. Since w_ρ is bounded in $H^{1,2}(B_1)$, see (5.4), we also have $\tilde{w}_\rho$ bounded in $H^{1,2}(D)$. Hence along an appropriate subsequence $\rho \searrow 0$, $\tilde{w}_\rho \to \tilde{w}_*$ as well as $w_\rho \to w_*$ weakly in $H^{1,2}(D)$. Since the domain where $\tilde{w}_\rho \neq w_\rho$ collapses to the vertical line $\{y = 0\}$ as $\rho \searrow 0$ we have $\tilde{w}_* = w_*$ in D. Hence for test functions ζ as above

$$0 \leq \int_D \nabla\zeta \cdot \nabla w_* = \int_{\partial D \cap \{y=0\}} \zeta \partial_\nu w_* \ ,$$

a contraction to Lemma 5.4. □

5.7 Corollary. $m_* \geq 3$.

Proof. By Lemma 5.6 and Proposition 4.8 we have $m_* \geq m \geq 3$. □

Having established that w_* has the same number of phases as w, we prove next that $\alpha_* \geq \frac{km-2}{2}$, independent of the choice of the sequence $\rho \searrow 0$ satisfying (5.7).

5.8 Lemma. $\alpha_* \geq \frac{km-2}{2}$.

Proof. We decompose w_ρ and w_* into their phases at 0:

$$w_\rho = \sum_{i=1}^m w_{\rho i} \quad \text{and} \quad w_* = \sum_{i=1}^m w_{*i} \quad \text{in } B_\rho \ (\rho \text{ small}).$$

Here we used that $m_* = m$ by 5.5 and 5.6, and the numbering is so that $w_{\rho i} \to w_{*i}$ weakly in $H^{1,2}(B_1)$. Therefore

$$\liminf_{\rho \searrow 0} \int_{B_1} |\nabla w_{\rho i}|^2 \geq \int_{B_1} |\nabla w_{*i}|^2 \geq c > 0 .$$

Since, see (5.4),

$$\frac{1}{\pi} \int_{B_1} |\nabla w_{\rho i}|^2 = \rho^{2(\alpha_i(\rho) - \alpha(\rho))} ,$$

we find that

$$\liminf_{\rho \searrow 0} \rho^{2(\alpha_i(\rho) - \alpha(\rho))} \geq \frac{c}{\pi} .$$

Thus for small ρ

$$\alpha_i(\rho) - \alpha(\rho) \leq \frac{C}{\log \frac{1}{\rho}} .$$

Summing over i and using (5.2) we get

$$\alpha(\rho) \geq \frac{\kappa m - 2}{2} - \frac{C}{\log \frac{1}{\rho}}$$

with κ as in (5.1). Letting first $\rho \searrow 0$ and then $r_1 \searrow 0$ we obtain the desired inequality. □

So far we have controlled $\alpha(r)$ from below only for certain subsequences for which (5.7) holds. Using Proposition 5.3 we now show that $\alpha(r)$ remains positive for all small r.

5.9 Lemma. $\liminf_{r \to 0} \alpha(r) \geq \frac{km-2}{2}$.

Proof. Take any $0 < \alpha_0 < \frac{km-2}{2}$ and $\gamma = \alpha_0$. Let us assume that (5.6) does not hold for some small r_0. Then there exists a sequence $\rho \searrow 0$ for which (5.7) holds with

$$\delta_0 = 2^{-2\gamma-2} \quad \text{and} \quad \beta_0 = \alpha_0 .$$

Following the above blow up argument, Lemma 5.8 implies that

$$\beta_0 \geq \liminf_{\rho \searrow 0} \alpha(\rho) = \alpha_* \geq \frac{km-2}{2} ,$$

a contradicition. Hence (5.6) holds for some small r_0. Consequently, by Proposition 5.3,

$$\liminf_{r \searrow 0} \alpha(r) \geq \alpha_0 .$$

Since $\alpha_0 < \frac{km-2}{2}$ was chosen arbitrarily the proof is complete. □

Using this result we are able to prove that on small balls B_r the phases w_i are balanced towards each other.

5.10 Lemma. *There exist constants $c > 0$, $C > 0$ such that for small r and $i = 1, \ldots, m$*

$$(5.14) \qquad c\,r^{km} \le \int_{B_r} |\nabla w_i|^2 \le C\,r^{km} .$$

Proof. By Lemma 5.9 we have $\alpha(r) \ge \alpha_0 > 0$ for small r (α_0 as in the proof of 5.9.). Then Proposition 5.2 implies that the free boundary becomes vertical at 0 in a Hölder sense, that is, Assumption (A) in Section 4 (with $\alpha = \alpha_0$) is satisfied. Thus Lemma 4.9 can be applied and therefore (4.12) holds, i.e.

$$(5.15) \qquad \int_{B_r} |\nabla w_i|^2 \ge c\,r^{km} .$$

Now let us look at the Monotonicity Formula 2.8. It follows from Assumption (A) (in the vertical case) that (2.3) is satisfied with $\kappa = k$ and

$$\delta(r) = \begin{cases} 0 & \text{cusp case,} \\ C\,r^{\alpha_0} & \text{vertical case.} \end{cases}$$

Thus φ is bounded and in Remark 5.1 we obtain instead of (5.2)

$$\frac{1}{m}\sum_{i=1}^{m} \alpha_i(r) \ge \frac{km-2}{2} - \frac{C}{\log\frac{1}{r}} .$$

Using (5.15) we find

$$\alpha_i(r) \le \frac{km-2}{2} + \frac{C}{\log\frac{1}{r}} \quad \text{for } i = 1, \ldots, m.$$

Consequently

$$(5.16) \qquad \left|\alpha_i(r) - \frac{km-2}{2}\right| \le \frac{C}{\log\frac{1}{r}} ,$$

which is equivalent to the assertion. □

Now we are able to consider the blow-up with respect to the exponent $\frac{km}{2} - 2$ instead of $\alpha(r)$:

$$(5.16) \qquad w_r(x) := w(rx)/r^{\beta} , \quad \gamma_r(x) = \gamma(rx) ,$$

where

$$(5.17) \qquad \beta := \frac{km}{2} .$$

Moreover with $\tilde\gamma_r$ as in (5.9) we define now

$$(5.18) \qquad l_r^{\pm}(z) := r^{1-\beta} \int_{\{0 \le \pm y \le \frac{1}{2}\}} \tilde\gamma_r(y, z)\, dy .$$

We now show

5.11 Theorem. *Let w_r and $l_r^\pm$ be as in (5.16), (5.18). Then $w_r \to w_*$ weakly in $H^{1,2}_{loc}(\mathbb{R}^n)$ and $l_r^\pm \to l_*^\pm$ uniformly in $C^0_{loc}(\mathbb{R})$ as $r \to 0$. The limits w_*, $l_*^\pm$ satisfy (5.11) and for some $c_* > 0$ they are given by*

$$w_*(x) = c_* \operatorname{Re}(-i\tilde{x}^m) \quad \textit{with} \quad \tilde{x} = i^k(-ix)^{k/2} \,, \tag{5.19}$$

and for $z \geq 0$

$$\begin{aligned} &l_*^\pm(-z) = c_* z^\beta \textit{ and } l_*^\pm(z) = 0 \textit{ in cusp case,} \\ &l_*^\pm(\mp z) = \pm c_* z^\beta \textit{ and } l_*^\pm(\pm z) = 0 \textit{ in vertical case.} \end{aligned} \tag{5.20}$$

Proof. Let $R > 0$. It follows from (5.14) that the phases w_{ri} of w_r are bounded in $H^{1,2}(B_R)$ for small r. Moreover, by (5.10) and (5.16), the functions $l_r^\pm$ are bounded in $C^0([-R, R])$ for small r. Thus there exist w_*, $l_*^\pm$ such that for certain subsequences $w_r \to w_*$ weakly in $H^{1,2}(B_R)$ and $l_r^\pm \to l_*^\pm$ weakly star in $L^\infty(]-R, R[)$.

Since Assumption (A) in Section 4 is satisfied we can apply Lemma 4.9. It follows from (4.11) that

$$w_r(x) \to a\,Re(-i\tilde{x}^m)$$

uniformly in x, locally in every cone as in 4.9 (iii), which gives (5.19) with $c_* := a$. The identity (5.11) follows as before and repeating the proof of Lemma 5.4 gives $c_* > 0$. Moreover, it follows from (5.19) that with $\tilde{x} = \tilde{y} + i\tilde{z}$

$$\partial_y w_*|_{\tilde{z}=0} = c_*\beta\tilde{y}^{m-\frac{2}{k}} \,.$$

Thus the identities in the proof of Lemma 5.4 give

$$-\partial_z l_*^\pm(z) = c_*\beta|z|^{\beta-1} \tag{5.21}$$

for $-1 < z < 0$ in the cusp case and $-1 < \pm z < 0$ in the vertical case. Now, by (5.16), we have in Proposition 5.2

$$|z| \leq \frac{r}{2} \quad \text{implies} \quad |y| \leq C\,r^\beta$$

for free boundary points $(y, z) \in B_r$, or

$$|y| \leq C\,|z|^\beta \,. \tag{5.22}$$

Then $\tilde{\gamma}_r(y, z) = 0$ for $|y| \geq C\,r^{\beta-1}|z|^\beta$ and we infer that

$$|l_r^\pm(z)| \leq C\,|z|^\beta \,.$$

This also holds for $l_*^\pm$ so that (5.20) follows from (5.21). The uniform convergence of $l_r^\pm$ follows from the monotonicity of these functions and the continuity of the limit $l_*^\pm$.

Finally, since $c_* = a$ is independent of the chosen subsequence it follows that the whole sequence converges. □

5.12 Remark. The free boundary becomes vertical at 0 in the Hölder sense (5.22), where the exponent $\beta \geq \frac{3}{2}$ is given by (5.17). For the standard cusp case ($k = 1$, $m = 3$) we have

$\beta = \frac{3}{2}$. The result (5.22) does not imply that the free boundary is a C^1 curve from the left or right at the cusp. This will be proved in Section 6.

6. Regularity of free boundary at cusp

In a number of steps we show here that at the cusp the free boundary becomes vertical in a C^1-manner. We are able to prove this for the cusp case and partially for the vertical case (that is, for the part of the free boundary which lies below the critical point). For the proof we need to estimate the gradient of a harmonic function, defined in an open, bounded and connected domain, in terms of its value at the boundary. The following proposition gives the precise statement. It is a generalization of a result of Alt & Gilardi [5, Lemma 7.5].

6.1 Proposition. *Let $D \subset \mathbb{R}^2$ be open, bounded and connected, and let $h : D \to \mathbb{R}$ be harmonic. Further let $K \subset \mathbb{R}^2$ be compact such that $\mathbb{R}^2 \setminus K$ is connected. Then*

$$|\nabla h| \le C\ , \quad \textit{for some } C > 0. \tag{6.1}$$

$$\operatorname{dist}(\nabla h(x), K) \to 0 \quad \textit{as} \quad \operatorname{dist}(x, \partial D) \to 0\ . \tag{6.2}$$

implies

$$\nabla h(x) \in K \quad \textit{for all } x \in D.$$

Proof. If ∇h = constant in D then $\nabla h \in K$ by (6.2). If $\nabla h \ne$ constant in D it follows that ∇h is an open mapping (since D is connected and $h : D \to \mathbb{C}$ holomorphic). We argue by contraction. Thus suppose $\nabla h(x_0) \notin K$ for some $x_0 \in D$. Then consider a curve $\sigma : [0, \infty[\to \mathbb{R}^2 \setminus K$ with $\sigma(0) = \nabla h(x_0)$ satisfying

$$|\sigma(s)| \to \infty \quad \text{as } s \to \infty, \tag{6.3}$$

$$\operatorname{dist}(\sigma([0, \infty[), K) \ge d > 0\ . \tag{6.4}$$

Related to σ, consider the interval

$$I = \{t \ge 0 : \sigma(s) \in \{\nabla h(x) : x \in D\} \text{ for } 0 \le s \le t\}\ .$$

I is non empty since $0 \in I$. Because ∇h is an open mapping I is open, and by (6.1), (6.3) it is bounded. Therefore $t_0 := \sup I < \infty$ does not belong to I. Choose $t_m \nearrow t_0$ and $x_m \in D$ with $\sigma(t_m) = \nabla h(x_m)$. Since $t_0 \notin I$ the sequence $(x_m)_m$ has no accumulation point in D, therefore $\operatorname{dist}(x_m, \partial D) \to 0$ as $m \to \infty$. Then $\operatorname{dist}(\sigma(t_m), K) \to 0$ by (6.2), a contradiction to (6.4). $\square$

We consider the free boundary near the origin 0 where the singularity is situated. It sufficies to consider a right neighbourhood. We want to show that u is monotone there. For this let

$$0 \le \varphi < \frac{\pi}{2} \quad \text{and} \quad e = e(\varphi) := \exp(-i\varphi)$$

and consider the ray

$$R := \left\{ r \exp\left(-i\frac{\pi}{2} + i\theta\right) : r > 0 \right\}$$

with $0 < \theta < \frac{\pi}{2}$ (see Figure 19). By Theorem 5.11 we have for $x \in R$

$$\nabla w_*(x) \cdot e = c_* \beta |x|^{\beta - 1} \cos((\beta - 1)\theta + \varphi) > 0$$

provided

$$(\beta - 1)\theta < \frac{\pi}{2} - \varphi \,. \tag{6.5}$$

Since the blow up sequence w_r converges to w_*, with smooth convergence in the set where w_* is harmonic, we also have for a fixed $x_0 \in R$

$$\nabla w_r(x_0) \cdot e > 0 \quad \text{for all small } r,$$

hence

$$\nabla w(x) \cdot e > 0 \quad \text{for } x \in R,\ |x| \text{ small.} \tag{6.6}$$

From now on we assume that (6.5) is satisfied. Let us choose a ball B_ρ around 0 so that (6.6) holds for $x \in R \cap B_\rho$ and so that in B_ρ the free boundary to the right of the cusp lies below R. We then denote by Ω the domain enclosed by R, ∂B_ρ, and the free boundary graph(u).

We show

6.2 Lemma. *There exists a neighbourhood of O in Ω in which*

$$\nabla w \cdot e \geq 0 \,.$$

Proof. Since the free boundary becomes vertical at O (Theorem 3.6 (ii)), there are points $x_i \in \partial\Omega$ on the free boundary with $x_i \to 0$ as $i \to \infty$ so that $\nu(x_i) \cdot e > 0$. Here ν is the normal towards the flow domain. On the free boundary we have

$$(\nabla w - e_z) \cdot \nu = 0 \quad \text{and} \quad w = 0 \,,$$

therefore

$$\nabla w = e_z \cdot \nu \nu \,. \tag{6.7}$$

This implies that

$$\nabla w(x_i) \cdot e = e_z \cdot \nu(x_i) \nu(x_i) \cdot e > 0 \,.$$

Let D_i denote the connected component of $\Omega \cap \{\nabla w \cdot e > 0\}$ containing x_i as boundary point. Let us first make the following assumption:

(6.8) There exists a subsequence, again denoted by $(x_i)_i$, with the property that $\overline{D}_i \cap R \neq \emptyset$.

We shall show that from this assumption the lemma follows. Note that if such a sequence exists, then by (6.6) all the corresponding D_i's coincide and contain part of the ray R up to 0. On R we select points $\tilde{x}_i$ with $\tilde{x}_i \to 0$ as $i \to \infty$, and we consider curves in the connected component, connecting the points x_i and $\tilde{x}_i$ and the points x_j and $\tilde{x}_j$ for a suitable pair

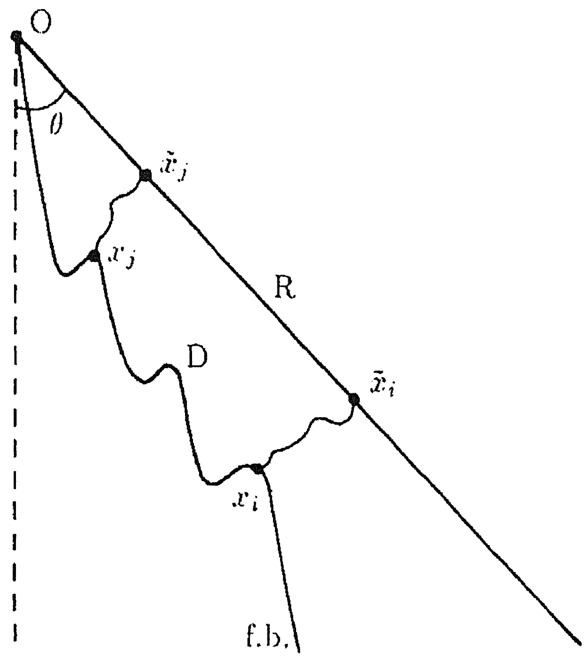


Fig. 19. Construction of the set D.

$j > i$, as in Figure 19. Let D be the region enclosed by the free boundary, R and these two curves.

By construction,

$$\nabla w(x) \cdot e > 0 \quad \text{for all } x \in \partial D \setminus \text{graph}(u).$$

On the free boundary we have by (6.7)

$$|\nabla w - \frac{1}{2} e_z| = \frac{1}{2} .$$

Moreover, since the free boundary does not become vertical on ∂D we have there $\nu \cdot e_z \geq c > 0$, hence

$$\nabla w \cdot e_z \geq c^2 > 0 \quad \text{on } \partial D \cap \text{graph}(u).$$

Consequently ∇w has values on ∂D in the set K from Figure 20.

Then Proposition 6.1 implies

$$\nabla w(\overline{D}) \subset K .$$

Since ∇w is an open mapping, any neighborhood of a free boundary point is mapped into a neighborhood of a point on the circle in Figure 20. Hence, the part of K outside the halfspace $\{z \in \mathbb{C}\, ;\, z \cdot e > 0\}$ cannot be attained. Therefore

$$\nabla w \cdot e \geq 0 \quad \text{in } \overline{D}$$

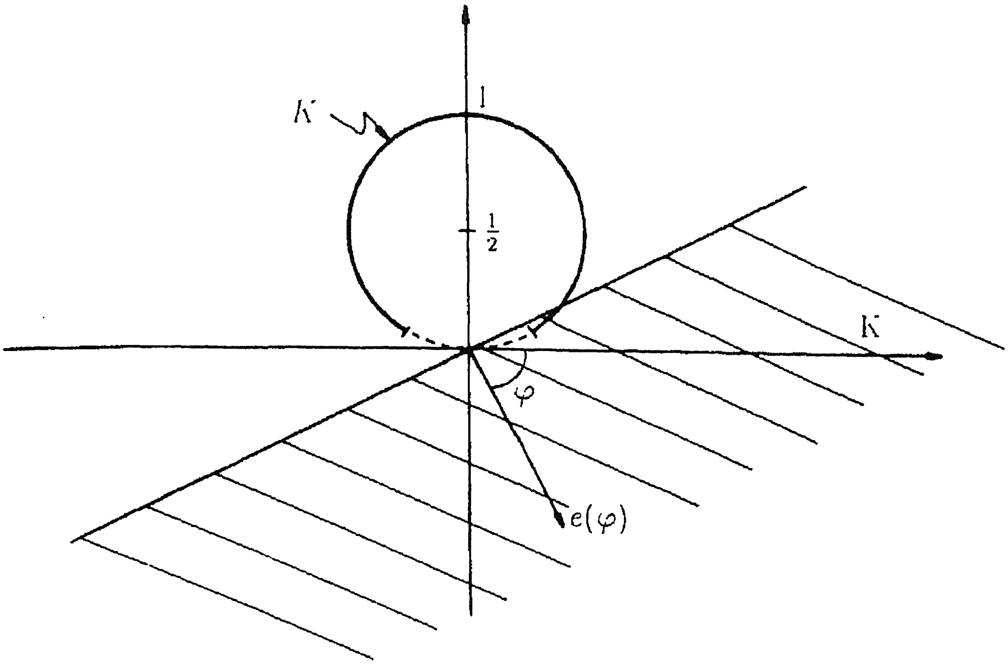


Fig. 20. $\nabla w(\partial D) \subset K$.

from which the lemma follows after letting $i, j \to \infty$. To complete the proof we have to show that assumption (6.8) is the only possibility. We argue by contradicition. If (6.8) does not hold then the following three cases need to be checked.

(i) $\overline{D}_i$ does not touch R, the origin, and ∂B_ρ.
The properties of D_i imply that $\nabla w \cdot e = 0$ on $\partial D \setminus \text{graph}(u)$. Arguing as before with $D := D_i$ we obtain $\nabla w \cdot e = 0$ on D_i contradicting the definition of D_i.

(ii) Infinitely many D_i's reach O.
This implies a situation as in Figure 21. Consider some $\{\nabla w \cdot e < 0\}$ component D enclosed by two of the D_i's and the free boundary. We want to apply the argument used in (i) to the set D. This is straight forward if D does not extend to the origin. If, however, as in Figure 21 the origin belongs to ∂D, we need to estimate $\nabla w(x) \cdot e$, $x \in D$, as $x \to 0$. Since D is contained in the cone bounded by the vertical and R, and since $\nabla w \cdot e$ is harmonic and bounded in D and vanishes on $\partial D \setminus (\text{graph}(u) \cup O)$, we conclude that

$$|\nabla w(x) \cdot e| \to 0 \quad \text{for } x \in D,\, x \to 0.$$

This allows us to apply the argument from (i) to reach a contradiction.

(iii) Infinitely many sets D_i touch ∂B_ρ.
If two domains D_{i_1} and D_{i_2} enclose a set D as in (ii) we proceed as there. Otherwise this leads to a situation as shown in Figure 22, where sign changes of $\nabla w \cdot e$ accumulate in

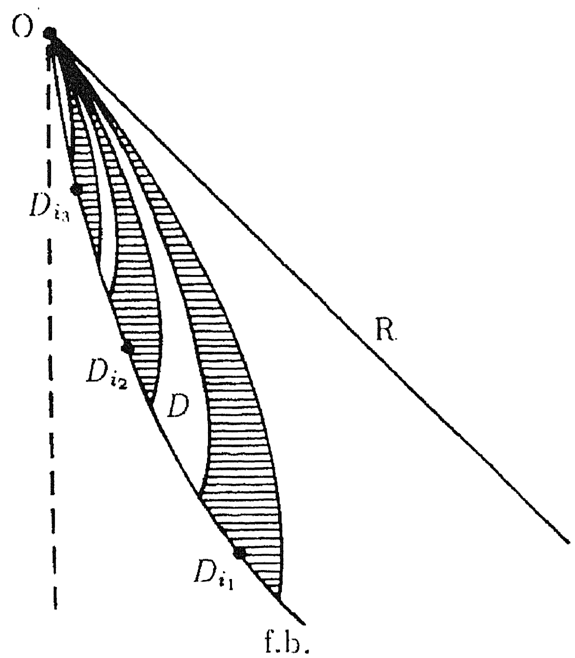


Fig. 21. D_i's reaching 0.

the domain where $\nabla w \cdot e$ is harmonic. This yields a contradiction as in the first part of Proposition 4.1.
We are now in a position to prove

6.3 Theorem. *The free boundary becomes vertical at O in a C^1-manner.*

Proof. Taking $\varphi = 0$ in Lemma 6.2 it follows that $\partial_y w \geq 0$ in a neighborhood of O below R. This implies that the free boundary to the right of the cusp is non-increasing in y, i.e. near O it has the form

$$\{(y,z) : -\delta_0 < z < 0\,,\ y > 0\,,\ y = f(z)\}\ . \tag{6.9}$$

for some $\delta_0 > 0$. Since u is analytic away from the cusp it follows that f is analytic and $f'(z) < 0$. Now choose any $0 \leq \varphi < \frac{\pi}{2}$ in Lemma 6.2. This implies that $\nu \cdot e \geq 0$ on the free boundary in a neighborhood of O below R. Therefore there exists $\delta_\varphi > 0$ so that

$$(1, -f'(z)) \cdot e \geq 0$$

for $-\delta_\varphi < z < 0$, i.e. $|f'(z)| \leq \cot\varphi$. □

Beside this we can show

6.4 Theorem. *The function f in (6.9) satisfies*

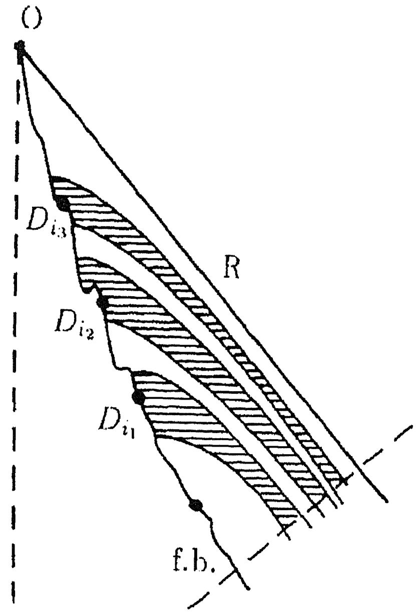


Fig. 22. Accumulation of sign changes of $\nabla w \cdot e$.

$$\lim_{z \nearrow 0} \frac{f(z)}{|z|^\beta} = c_* \, .$$

Proof. Since the free boundary has a representation as in (6.9) it follows that in (5.18)

$$l_r^+(z) = r^{1-\beta} \frac{f(rz)}{r} \, .$$

Set $z = 1$ and use Theorem 5.11. □

6.5 Note. Next we consider, in the vertical case, the part of the free boundary above O. For definiteness we again assume that the flow domain lies to the right. We now take

$$0 \le \varphi < \frac{\pi}{2} \, , \quad e = e(\varphi) = \exp(i\varphi) \, ,$$
$$R := \left\{ r \exp\left(i\frac{\pi}{2} - i\theta \right) : r > 0 \right\}$$

with $0 < \theta < \frac{\pi}{2}$. Then, with θ as in (6.5), we find the same formula for $\nabla w_* \cdot e$ along R. Proceeding as before, we obtain (for $\varphi = 0$) the existence of f (as in (6.9)) with

$$\lim_{z \nearrow 0} \frac{f(z)}{|z|^\beta} = -c_* \, .$$

However, for $\varphi > 0$ we do not get any additional information. Therefore with this method Theorem 6.3 cannot be proven.

7. Concluding remarks.

In this paper we develop the local analysis concerning the behaviour of the reduced potential and the interface near such singular points, provided they belong to the interior of the flow domain and provided $N = 2$. That singular points are in the interior seems to be clear by physical intuition. In fact, for our rectangular domain the interface is expected to be below the position of the highest well, provided all wells withdraw fluid. However we were not able to prove this. The restriction to two space dimensions was imposed to apply typical two dimensional free boundary methods.

As a result of the local analysis we obtain that at a singular free boundary point the free boundary either forms a cusp or becomes vertical. Which of the two will arise is determined by global arguments. For instance, we conjecture that a well configuration as in Figure 23, with one well pumping fluid in and one well pumping fluid out, may lead to vertical interfaces.

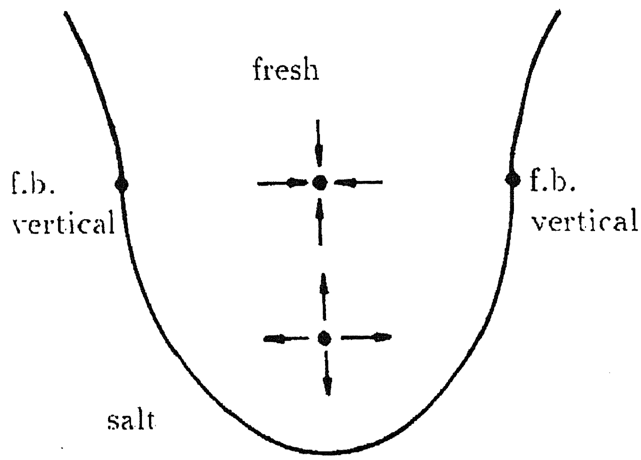


Fig. 23. Well configuration leading to vertical singularities.

With respect to the local behaviour, we observe that we have no regularity results for the function f, see Section 6, related to the branch above the singularity (vertical case).

Also an expansion for the derivative of f, i.e. $f'(z)/z^{\beta-1} \to \pm\beta c_*$ as $z \to 0$, is left as an open problem.

Finally we mention that the proofs in this paper do not carry over to the three dimensional case, in which a different, not polynomial asymptotic expansion, is expected.

Appendix A: Monotonicity formula.

Consider a continous function $w : B_{r_0} \to \mathbb{R}$, $r_0 > 0$, which is harmonic outside its zero set. Assume w has a decomposition

$$w = \sum_{i=1}^{m} w_i$$

where $w_i \in H^{1,2}(B_{r_0}) \cap C^0(B_{r_0})$ are the phases of w at the center O of B_{r_0} (see Definition 2.3). Then φ_i defined as in (2.1) are absolutely continuous positive functions on $]0, r_0[$. We want to show that

$$(A.1) \qquad (\log\varphi)'(r) \geq -\kappa m^2 \frac{\delta(r)}{r}$$

where φ is defined as in (2.2) and the function δ is chosen so that (2.3) holds.

We have for almost all $0 < r < r_0$

$$(A.2) \qquad (\log\varphi)'(r) = \sum_{i=1}^{m} \frac{\varphi_i'(r)}{\varphi_i(r)} = -\frac{\kappa m^2}{r} + \sum_{i=1}^{m} \frac{s_i(r)}{r}$$

with

$$s_i(r) := \frac{r \int\limits_{S_r} |\nabla w_i|^2}{\int\limits_{B_r} |\nabla w_i|^2},$$

where S_r is the sphere ∂B_r. The monotonicity and the harmonicity of w implies that for $\zeta \in C_0^\infty(B_{r_0})$

$$0 = \int\limits_{B_{r_0}} \nabla(\zeta w_i) \cdot \nabla w_i = \int\limits_{B_{r_0}} \zeta |\nabla w_i|^2 + \int\limits_{B_{r_0}} w_i \nabla\zeta \cdot \nabla w_i \,.$$

Therefore for almost all r

$$(A.3) \qquad \int\limits_{B_r} |\nabla w_i|^2 = \int\limits_{S_r} w_i \frac{\partial w_i}{\partial r} \leq \left(\int\limits_{S_r} w_i^2\right)^{1/2} \left(\int\limits_{S_r} \left(\frac{\partial w_i}{\partial r}\right)^2\right)^{1/2} .$$

On the other hand

$$\int_{S_r} |\nabla w_i|^2 = \int_{S_r} \left(\left(\frac{\partial w_i}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial w_i}{\partial \theta} \right)^2 \right) \tag{A.4}$$

$$\geq 2 \left(\int_{S_r} \left(\frac{\partial w_i}{\partial r} \right)^2 \right)^{1/2} \left(\int_{S_r} \left(\frac{1}{r} \frac{\partial w_i}{\partial \theta} \right)^2 \right)^{1/2}$$

Defining $v_i(x) := w_i(rx)$ for $x \in S_1$ it follows from (A.3) and (A.4) that

$$s_i(r) \geq 2 \left(\frac{\int_{S_1} \left(\frac{\partial v_i}{\partial \theta} \right)^2}{\int_{S_1} v_i^2} \right)^{1/2} \geq 2\sqrt{\lambda_i} \, ,$$

if λ_i is the smallest eigenvalue of $\partial^2/\partial\theta^2$ with homogeneous Dirichlet data on $S_1 \cap \{v_i \neq 0\}$. Denoting by l_i the relative length of this set with respect to S_1 we have $\lambda_i \geq (2l_i)^{-2}$ and therefore

$$\sum_{i=1}^{m} s_i(r) \geq \sum_{i=1}^{m} \frac{1}{l_i} \, . \tag{A.5}$$

Moreover, by (2.3),

$$\sum_{i=1}^{m} l_i \leq \frac{1}{\kappa(1 - \delta(r))} \, .$$

With this constraint the right-hand side in (A.5) becomes minimal for $l_i = (m\kappa(1 - \delta(r)))^{-1}$, thus

$$\sum_{i=1}^{m} s_i(r) \geq m^2 \kappa (1 - \delta(r)) \; ,$$

and together with (A.2) the assertion (A.1) follows.

Appendix B: Proof of Lemma 4.9

Let Condition (A) be satisfied. In complex coordinates $\zeta = (-ix)^{k/2} = \zeta_1 + i\zeta_2$ the transformed free boundary Γ lies, near the origin, between the curves $\Gamma_\pm = \{\gamma_\pm(it) : t \in \mathbb{R}\}$. Here $\gamma_\pm(\zeta) := \zeta \cdot (1 \pm M\zeta^\alpha)$ are conformal transformations near the origin, M large.

Let $r_0 > 0$ (small) and D the domain bounded by parts of $\{\zeta_1 = r_0\}$, $\{\zeta_2 = \pm r_0\}$, and Γ. If Γ intersects the lines $\{\zeta_2 = \pm r_0\}$ more than once, we take the points where Γ coming from the origin hits this lines for the first time, see Figure 24. Now consider the harmonic function h on D and continuous in $\overline{D}$ such that $h = r_0$ on the upper boundary, h linear on the sides, and $h = 0$ on the part Γ_0 of ∂D belonging to Γ. Similar define $D_\pm$ with respect to $\Gamma_\pm$ and harmonic functions $h_\pm$. (Note: We do not know that ∂D is a Lipschitz graph near the origin, but the flatness at the origin implies the existence of h.)

Then (extending functions by 0 beyond Γ_0, $\Gamma_\pm$)

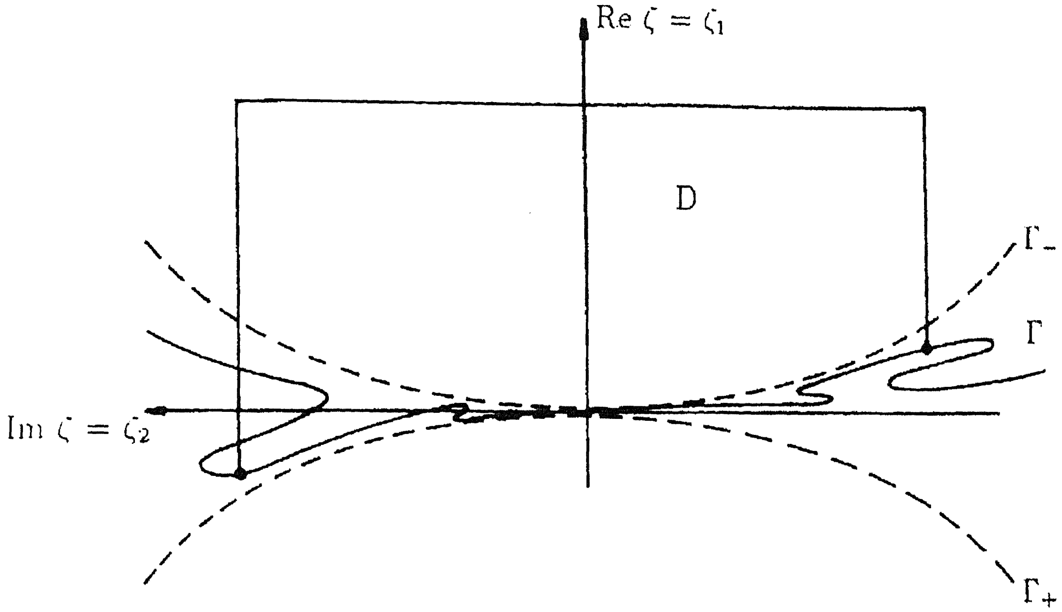


Fig. 24. Construction of domains D, D_+ and D_-.

$$h_- \leq h \leq h_+ \ . \tag{B.1}$$

Moreover, using regularity theory and Hopf principle for the harmonic functions $h_\pm \circ \gamma_\pm$ it follows that $h_\pm$ are $C^{1,\alpha}$ up to the boundaries $\Gamma_\pm$ and that

$$h_-(\zeta) \geq c \operatorname{dist}(\zeta, \Gamma_-) \tag{B.2}$$

for some $c > 0$. Now consider the blow-up sequence

$$h_r(\zeta) := \frac{1}{r} h(r\zeta) \ ,$$

and similarly $h_{\pm r}$. We claim:

B.1 Proposition. *For some constant $c_* > 0$*

$$h_*(\zeta) := \lim_{r \to 0} h_r(\zeta) = c_* \zeta_1$$

locally unifomly in $\{\zeta_1 > 0\}$.

Proof. For small $r > 0$ let s_r be the smallest number such that $h \leq s_r h_+$ in B_r. Clearly s_r decreases when r decreases and by (B.1) and (B.2)

$$s_* := \lim_{r \to 0} s_r > 0 \ .$$

Since

$$0 \le h_r \le s_r h_{+r} \quad \text{in } B_1,$$

h_r are bounded harmonic functions locally in $B_1 \cap \{\zeta_1 > 0\}$. Therefore there exists a harmonic function h_* in $B_1 \cap \{\zeta_1 > 0\}$ so that for a subsequence $r \to 0$

$$h_r \to h_* \quad \text{in } C^0_{loc}(B_1 \cap \{\zeta_1 > 0\}).$$

Since

$$h_{+r}(\zeta) \to \partial_1 h_+(0) \max(\zeta_1, 0) =: \tilde{h}(\zeta)$$

it follows that

$$0 \le h_* \le s_* \tilde{h} .$$

Assume that $h_*(\zeta_0) < s_* \tilde{h}(\zeta_0)$ for some ζ_0. Then

$$h_* \le s_* \tilde{h} - \delta_0 \quad \text{in } \overline{B_{\varepsilon_0}(\zeta_0)}$$

for some $\varepsilon_0 > 0$ and $\delta_0 > 0$. Then for small r

$$f_r := s_r h_{+r} - h_r \ge \frac{\delta_0}{2} \quad \text{in } \overline{B_{\varepsilon_0}(\zeta_0)}.$$

Moreover f_r is superharmonic in $D_{+r} \cap B_1$, non-negative on the boundary. Therefore, by Hopf principle, there is a constant $c_0 > 0$ independent of r such that for $\zeta \in D_{+r} \cap B_{1/2}$

$$f_r(\zeta) \ge c_0 \operatorname{dist}(\zeta, \Gamma_{+r}) \ge c h_{+r}$$

with $c > 0$ independent of r. Thus

$$h_r \le (s_r - c) h_{+r} \quad \text{in } B_{1/2},$$

which says that $s_{r/2} \le s_r - c$. Letting $r \to 0$ this is a contradiction. □

It follows from the Proposition that on each cone $\{re^{i\varphi} : r > 0 \text{ and } |\varphi| \le \frac{\pi}{2} - \delta\}$ and for each multiindex $\beta = (\beta_1, \beta_2) \ge 0$

$$\partial^\beta (h(\zeta) - c_* \zeta_1) = o(|\zeta|^{1-|\beta|}) \quad \text{as } \zeta \to 0. \tag{B.3}$$

Now define the conjugate harmonic function $k : D \to \mathbb{R}$ of h by

$$k(\zeta) := \int_0^1 \nabla h(\sigma_\zeta(t)) \cdot (-i\sigma'_\zeta(t))\, dt$$

where $\sigma_\zeta :]0,1[\to D$ with $\sigma_\zeta(0) = 0$, $\sigma_\zeta(1) = \zeta$, and $\operatorname{Re} \sigma'_\zeta(0) > 0$.

B.2 Proposition. *The holomorphic function*

$$\tau(x) := \frac{1}{c_*}(h(\zeta) + ik(\zeta)) \quad \textit{for } \zeta = (-ix)^{k/2}$$

has the properties stated in Lemma 4.9.

Proof. It follows from (B.3) that k is well defined, and on each cone as above $k(\zeta) = c_*\zeta_2 + o(|\zeta|)$ as $\zeta \to 0$. For $0 < \varepsilon < r_0$ there are exactly to points $\zeta_\varepsilon^\pm \in \partial D$ with $h(\zeta_\varepsilon^\pm) = \varepsilon$. Therefore $D \cap \{h < \varepsilon\}$ and $D \cap \{h > \varepsilon\}$ are connected sets so that $\Gamma_\varepsilon := \partial\{h > \varepsilon\}$ has to be a smooth curve from ζ_ε^- to ζ_ε^+ on which $\nabla h \neq 0$. This implies that k is strictly increasing on Γ_ε. Also k is continuous up to $\Gamma_0 \setminus \{0\}$ and strictly increasing on the two parts of $\Gamma_0 \setminus \{0\}$. Therefore $\tilde{\tau} := h + ik$ is one-to-one if we can show that ∇h is integrable on $\Gamma_0 \setminus \{0\}$ and

$$(B.4) \qquad \int_{\Gamma_\varepsilon} \partial_{-\nu} h \, d\mathcal{H}^1 \to \int_{\Gamma_0} \partial_{-\nu} h \, d\mathcal{H}^1 \quad \text{as } \varepsilon \to 0$$

(ν is chosen so that $\partial_{-\nu} h > 0$). Now, as $\varepsilon \to 0$,

$$\int_{D\cap\{h>\varepsilon\}} |\nabla h|^2 = \int_D \nabla(h-\varepsilon)_+ \nabla h = \int_{\partial D} (h-\varepsilon)_+ \partial_\nu h \, d\mathcal{H}^1$$
$$\to \int_{\partial D \setminus \Gamma_0} h \partial_\nu h \, d\mathcal{H}^1 < \infty \, ,$$

thus $\nabla h \in L^2(D)$. Then with the cut-off function $\eta_r(\zeta) := \min(1, \frac{1}{r}\operatorname{dist}(\zeta, \partial B_r))$

$$\delta_{\varepsilon,r} := \int_{\partial(D\cap\{h<\varepsilon\})} \eta_r \partial_\nu h \, d\mathcal{H}^1 = \int_{D\cap\{h<\varepsilon\}} \nabla\eta_r \nabla h = \mathcal{O}(\|\nabla\zeta\|_{L^2(B_{2r})}) \to 0$$

as $r \to 0$. Since for small r

$$\delta_{\varepsilon,r} = \int_{\Gamma_\varepsilon} \partial_{-\nu} h \, d\mathcal{H}^1 + \int_{\{h<\varepsilon\}\cap\partial D} \partial_\nu h \, d\mathcal{H}^1 - \int_{\Gamma_0\setminus\{0\}} \eta_r \partial_{-\nu} h \, d\mathcal{H}^1$$

(B.4) follows by letting first $r \to 0$ and then $\varepsilon \to 0$. □

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