

On the countable generator theorem

by

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Abstract. Let T be a finite entropy, aperiodic automorphism of a nonatomic probability space. We give an elementary proof of the existence of a finite entropy, countable generating partition for T .

In this short article we give a simple proof of Rokhlin's countable generator theorem [Ro], originating from considerations in [Se] which use standard techniques in ergodic theory. We hope that these considerations will be useful for elementary expositions in the future. For other proofs see [Pa].

Let (X, \mathcal{A}, μ) be a nonatomic probability space whose σ -algebra \mathcal{A} is generated modulo μ by a countable collection $\{A_1, A_2, \dots\}$ of elements of \mathcal{A} . Let T be an aperiodic automorphism of (X, \mathcal{A}, μ) with finite entropy. For the definitions and properties of entropy and generators used in the sequel, we refer the reader to Billingsley [Bill] and Walters [Wa].

THEOREM. (X, \mathcal{A}, μ, T) has a countable generating partition of finite entropy.

Our proof is based on the following lemma.

LEMMA. Let \mathcal{P} be a finite partition of (X, \mathcal{A}, μ, T) , A an element of \mathcal{A} , and $\varepsilon > 0$. Set

$$\tilde{\mathcal{P}} := \mathcal{P} \vee \{A, A^c\} \quad \text{and} \quad g := \tilde{h} - h,$$

where

$$h := h(T, \mathcal{P}) \quad \text{and} \quad \tilde{h} := h(T, \tilde{\mathcal{P}})$$

denote the respective mean entropies of the partitions \mathcal{P} and $\tilde{\mathcal{P}}$. Then there exists a finite partition \mathcal{Q} of (X, \mathcal{A}, μ, T) such that

$$(1) \quad \mathcal{P} \preceq \mathcal{Q},$$

$$(2) \quad A \in \bigvee_{n=-\infty}^{\infty} T^n \mathcal{Q},$$

$$(3) \quad H(\mathcal{Q}) \leq H(\mathcal{P}) + g + \varepsilon.$$

Assuming the validity of this lemma, here is the proof of the theorem: Using the lemma, we produce inductively a sequence $\mathcal{Q}_0 \preceq \mathcal{Q}_1 \preceq \dots$ of finite partitions as follows. First, set $\mathcal{Q}_0 = \{X\}$. If \mathcal{Q}_k has been defined, then take

$$\varepsilon = \frac{1}{2^{k+1}}, \quad \mathcal{P} = \mathcal{Q}_k, \quad A = A_{k+1}$$

in the lemma to obtain $\mathcal{Q}_{k+1} := \mathcal{Q}$. By (1) and (2), for each $k \geq 0$,

$$A_1, \dots, A_k \in \bigvee_{n=-\infty}^{\infty} T^n \mathcal{Q}_k.$$

Moreover, property (3) yields

$$H(\mathcal{Q}_{k+1}) - H(\mathcal{Q}_k) \leq h(T, \mathcal{Q}_{k+1}) - h(T, \mathcal{Q}_k) + \frac{1}{2^{k+1}}$$

for each $k \geq 0$; summing from zero to k results in

$$H(\mathcal{Q}_k) \leq h(T, \mathcal{Q}_k) + \sum_{j=1}^{k+1} \frac{1}{2^j} \leq (\text{Entropy of } T) + 1.$$

In particular, $\sup_k H(\mathcal{Q}_k) < \infty$. Now set

$$\mathcal{Q} := \bigvee_{k=0}^{\infty} \mathcal{Q}_k;$$

then $H(\mathcal{Q}) = \sup_k H(\mathcal{Q}_k)$ is finite, and $A_k \in \bigvee_{n=-\infty}^{\infty} T^n \mathcal{Q}$ for each k , so that \mathcal{Q} is a countable generating partition of finite entropy. ■

Next, we give a proof of the lemma in the case where T is ergodic. It is clear that we may replace the condition (2) by the condition

$$(4) \quad \text{there exists an } A' \in \bigvee_{n=-\infty}^{\infty} T^n \mathcal{Q} \text{ with } \mu(A \triangle A') < \varepsilon.$$

To see this, suppose that the lemma holds in this modified form, and for $\varepsilon > 0$ choose $\delta > 0$ such that

$$\delta + \{-\delta \log \delta - (1 - \delta) \log(1 - \delta)\} \leq \varepsilon.$$

Apply the modified lemma using δ in place of ε to get a partition \mathcal{Q}' satisfying (1), (4), and (3). Then

$$\mathcal{Q} := \mathcal{Q}' \vee \{A \triangle A', X \setminus (A \triangle A')\}$$

satisfies (1) and (2), and

$$\begin{aligned} H(\mathcal{Q}) &\leq H(\mathcal{Q}') + H(\{A \triangle A', X \setminus (A \triangle A')\}) \\ &\leq H(\mathcal{P}) + g + \delta + \{-\delta \log \delta - (1 - \delta) \log(1 - \delta)\} \\ &\leq H(\mathcal{P}) + g + \varepsilon \end{aligned}$$

as required.

For a fixed positive integer m , which we shall choose in a moment, let

$$\{A_{ij} : 1 \leq i \leq p^m, 1 \leq j \leq 2^m\}$$

be a list of the (possibly empty) atoms of $\bigvee_{n=0}^{m-1} T^n \tilde{\mathcal{P}}$ such that the sets

$$A_i := \bigcup_{j=1}^{2^m} A_{ij}$$

are the atoms (possibly empty) of $\bigvee_{n=0}^{m-1} T^n \mathcal{P}$; here we have assumed that \mathcal{P} has p elements.

By the Shannon–McMillan–Breiman theorem (what we need here is convergence in probability, see [Bill], Thm. 13.2), if $\delta > 0$ and m is large enough, “most” of the A_{ij} have measures in

$$[e^{-(\tilde{h}+\delta)m}, e^{-(\tilde{h}-\delta)m}]$$

and “most” of the A_i have measures in

$$[e^{-(h+\delta)m}, e^{-(h-\delta)m}],$$

“most” meaning, of course, a set with total measure close to 1. For a $\delta > 0$ also to be determined shortly, we now choose m so large that the total measure of the atoms A_i for which

$$(5) \quad \mu(A_i) > e^{-(h-\delta)m}$$

is smaller than δ , and also so that the total measure of the atoms A_{ij} for which

$$(6) \quad \mu(A_{ij}) < e^{-(\tilde{h}+\delta)m}$$

is smaller than δ .

Next, we reorganize the array $\{A_{ij}\}$ as follows. First, delete all the rows i for which (5) holds. Then, in the remaining rows, delete all the A_{ij} for which (6) holds. Finally, renumber the remaining elements to obtain the array

$$\{A'_{ij} : 1 \leq i \leq I, 1 \leq j \leq J_i\},$$

each row of which is a subcollection of a row of the original array. Since now still

$$\sum_{j=1}^{J_i} \mu(A'_{ij}) \leq e^{-(h-\delta)m}$$

for each row i , and since $\mu(A'_{ij}) \geq e^{-(\tilde{h}+\delta)m}$ for each of the remaining elements of a row, it follows that for each $1 \leq i \leq I$,

$$J_i \leq \frac{e^{-(h-\delta)m}}{e^{-(\tilde{h}+\delta)m}} = e^{(g+2\delta)m}.$$

If $\bar{J} := \max_{1 \leq i \leq I} J_i$, and $1 \leq j \leq \bar{J}$, then we set

$$Q'_j := \bigcup_{\{i: j \leq J_i\}} A'_{ij}.$$

Now use Rokhlin's lemma to get a set $M \in \mathcal{A}$ such that $M, TM, \dots, T^{m-1}M$ are pairwise disjoint and

$$\mu\left(X \setminus \bigcup_{n=0}^{m-1} T^n\right) < \delta,$$

and define the partitions

$$\mathcal{Q}' := \left\{ M \cap Q'_1, \dots, M \cap Q'_{\bar{J}}, X \setminus \bigcup_{j=1}^{\bar{J}} M \cap Q'_j \right\}$$

and $\mathcal{Q} := \mathcal{P} \vee \mathcal{Q}'$. Without loss of generality, by choosing m sufficiently large and by replacement of M by one of the $T^n M$ with n small with respect to m ($n < m\sqrt{3\delta}$ will do), we may assume that

$$\frac{\mu(M \cap \bigcup_{j=1}^{\bar{J}} Q'_j)}{\mu(M)} > 1 - \sqrt{3\delta}.$$

Then, by construction, $\bigvee_{n=-m}^m T^{-n} \mathcal{Q}$ contains a set A' with $\mu(A \triangle A') \leq \sqrt{3\delta}$, namely the union of all its atoms contained in A .

As $\mu(M) \leq 1/m$ and $\bar{J} \leq e^{(g+2\delta)m}$, we have

$$H(\mathcal{Q}') \leq -\bar{J} \cdot \frac{1}{m\bar{J}} \cdot \log\left(\frac{1}{m\bar{J}}\right) - \frac{m-1}{m} \log \frac{m-1}{m} \leq g + 2\delta + \frac{\log m}{m} + \frac{1}{m},$$

and hence

$$H(\mathcal{Q}) \leq H(\mathcal{P}) + g + 2\delta + \frac{\log m}{m} + \frac{1}{m}.$$

Thus choosing δ such that

$$\max\left\{ \sqrt{3\delta}, \frac{\log m}{m} + 2\delta + \frac{1}{m} \right\} < \varepsilon$$

finishes the proof. ■

Finally, we give a sketch of how this proof can be modified for the non-ergodic case. Suppose, for instance, that μ has two ergodic components, say μ_1 and μ_2 , with

$$\mu = \alpha\mu_1 + (1 - \alpha)\mu_2.$$

Each μ_i corresponds to entropies \tilde{h}_i, h_i and $g_i = \tilde{h}_i - h_i$ as above, $i = 1$ or 2 . If we produced \mathcal{Q}'_1 and \mathcal{Q}'_2 as above and joined them to \mathcal{P} , the entropy would be too large, and we need to merge the atoms of \mathcal{Q}'_1 and \mathcal{Q}'_2 . For this, the numbers m_1 and m_2 need to be chosen such that $m_1 g_1 \approx m_2 g_2$; all other considerations remain the same. A similar argument applies for arbitrary nonergodic μ by approximation by a finite number of unions of ergodic components with approximately the same \tilde{h} and h values. The details are left to the reader.

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