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THE GRAPHS WITH ALL SUBGRAPHS T-PERFECT*

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Abstract. The richest class of t-perfect graphs known so far consists of the graphs with no so-called odd-K_4. Clearly, these graphs have the special property that they are *hereditary t-perfect* in the sense that every subgraph is also t-perfect, but they are not the only ones. In this paper we characterize hereditary t-perfect graphs by showing that any non–t-perfect graph contains a non–t-perfect subdivision of K_4, called a *bad-K_4*. To prove the result we show which "weakly 3-connected" graphs contain no bad-K_4; as a side-product of this we get a polynomial time recognition algorithm.

It should be noted that our result does not characterize t-perfection, as that is not maintained when taking subgraphs but only when taking induced subgraphs.

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1. Introduction.

A graph $G = (V, E)$ is *t-perfect* if the polyhedron

$$\mathcal{P}(G) := \{x \in \mathbb{R}^V \mid \begin{array}{rcll} x_v & \geq & 0 & (v \in V), \\ x_u + x_v & \leq & 1 & (uv \in E), \\ \displaystyle\sum_{v \in V(C)} x_v & \leq & \frac{|V(C)|-1}{2} & (C \text{ is odd circuit in } G) \end{array} \tag{1}$$

has integral vertices only, i.e., when $\mathcal{P}(G)$ is the stable set polytope of G. T-perfection was introduced by Chvátal [4], and a characterization of it has proved elusive. The first two classes of graphs known to be t-perfect are series-parallel graphs (conjectured by Chvátal [4] and proved by Boulala and Uhry [2]) and *almost bipartite graphs*, i.e., graphs with a node that is contained in every odd circuit [5]. A common extension of these two classes is the class of graphs that do not contain an odd-K_4 as a subgraph. Here *odd-K_4* means a subdivision of K_4, the complete graph on four nodes, in which all triangles have become odd circuits (cf. Figure 1a). Graphs containing no odd-K_4 are t-perfect [9]. However, there are odd-K_4's that are t-perfect, namely, the good-K_4's: a *good-K_4* is a subdivision of K_4, in which two nonadjacent edges are not subdivided and the other four edges have become even paths (cf. Figure 1b). An odd-K_4 that is not good is called a *bad-K_4*; bad-K_4's are not t-perfect (Lemma 11). The main result of this paper is the following theorem.

THEOREM 1. *If G contains no bad-K_4 as a subgraph, then it is t-perfect.*

We prove this in section 3. One of the main tools is the following decomposition result.

THEOREM 2. *If G is* weakly 3-connected, *i.e., a subdivision of a 3-node-connected simple graph, then it contains no bad-K_4 if and only if one of the following holds:*

- *G contains no odd-K_4;*
- *G is an odd-P_9;*

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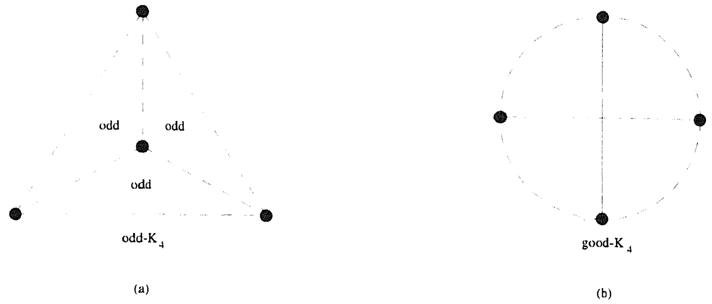

FIG. 1. *Dashed curves indicate internally node disjoint paths of positive length, which in* (b) *all have even length.*

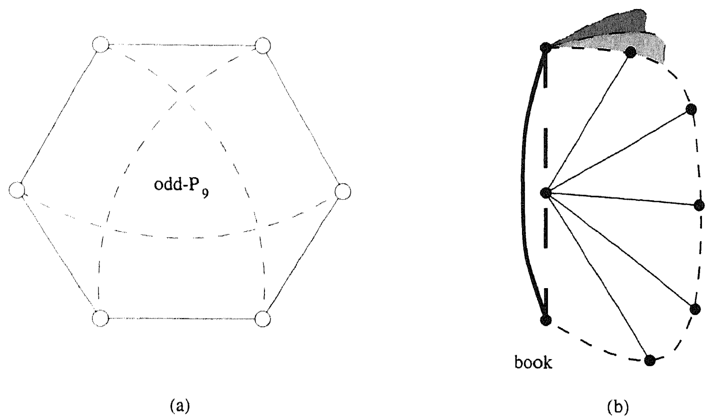

FIG. 2. *Dashed curves indicate internally node disjoint paths of positive even length. The shaded regions in* (b) *indicate the second and third leaf of the book.*

- *G is a clean pad;*
- *G is a book.*

An *odd-P_9* is a graph obtained from a six circuit $u_1u_2, \ldots, u_5u_6, u_6u_1$ by adding three node disjoint even u_iu_{i+3}-paths ($i = 1, 2, 3$); see Figure 2a. Note that the smallest odd-P_9 is the Petersen graph with a node removed.

A *pad* is a graph G with a Hamiltonian circuit $w_1, u_1, w_2, u_2, \ldots, w_k, u_k$ such that an edge not on the Hamiltonian circuit has both end nodes in $U(G) := \{u_1, u_2, \ldots, u_k\}$. (We also define $W(G) := \{w_1, w_2, \ldots, w_k\}$.)

Clearly, a pad has exactly one Hamiltonian circuit, which we denote by $R(G)$ and call the *rim* of the pad. The set of edges not on the rim, called *chords*, will be denoted by $K(G)$. A pad G is *clean* if neither of the two pads in Figure 3 can be derived from G by deleting chords and contracting edges on the rim.

A *book* is any graph that can be constructed as follows:

- Take two nodes h_1 and h_2 (the *hinges* of the book), and join them by an edge.
- Take a third node c, the *center* of the book, and add two internally node disjoint even paths, one from c to h_1 and one from c to h_2 (together with h_1h_2 these paths form the *spine* of the book).
- Add n internally node disjoint even h_1h_2-paths $P_1, \ldots, P_n$, and select on each P_i a nonempty collection T_i of nodes that are an even distance from h_1 on P_i.

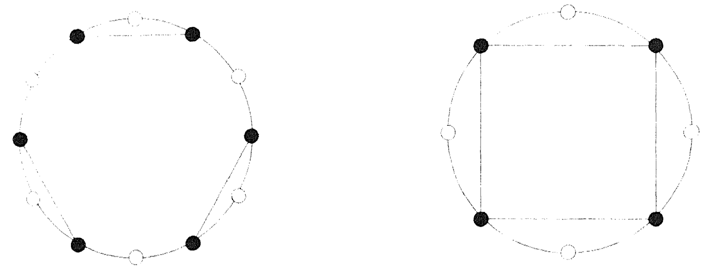

FIG. 3.

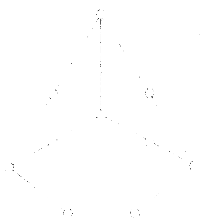

FIG. 4.

- Finally, add all edges in $R_i := \{cr \mid r \in T_i\}$, $i = 1, \dots, n$.

Note that the union of each $P_i \cup R_i$ with the spine forms a pad. We call these pads the *leaves* of the book. The path P_i is called the *trim* of the leaf. Figure 2b indicates a book with 3 leaves.

As side-product we obtain the following result (we shall give the easy proof in section 2.3).

THEOREM 3. *There exists a polynomial time algorithm that decides whether or not a given graph G contains a bad-K_4.*

Another easy side-product, of which we skip the proof, is that graphs with no bad-K_4 are 3-colorable. This generalizes a result of Catlin [3] that graphs with no odd-K_4 are 3-colorable. Toft [12] conjectures that a graph is 3-colorable if it does not contain a subgraph isomorphic to a graph obtained from K_4 by replacing all six edges with odd paths.

Characterizations around t-perfection. Shepherd [11] characterized which near-bipartite graphs are t-perfect. (A graph is *near-bipartite* if for each node v and each odd circuit C there is a neighbor of v on C. In fact, Shepherd [11] characterized the stable set polytopes of all near-bipartite graphs.) However, the characterization of t-perfection among all graphs is still open.

The graph in Figure 4 is t-perfect—as is easily proved—but contains a bad-K_4, which is not t-perfect. Thus t-perfection is not closed under taking subgraphs. T-perfection is however closed under taking induced subgraphs, i.e., under the deletion of nodes, but a complete list of minimally induced non-t-perfect graphs is not yet known.

However, combining Theorem 1 and Lemma 11, we do have the following:

(2) A graph contains no bad-K_4 if and only if all its subgraphs are t-perfect.

The result of Gerards and Schrijver shows that graphs with no odd-K_4 are t-perfect. In fact, there it is proved that a graph $G = (V, E)$ has no odd-K_4 if and only if for all $a, b \in \mathbb{Z}^V$ and all $c, d \in \mathbb{Z}^E$ the polyhedron

$$\{x \in \mathbb{R}^V \mid a_v \le x_v \le b_v \ (v \in V); c_{uv} \le x_u + x_v \le d_{uv} \ (uv \in E)\} \tag{3}$$

has Chvátal-rank 1, which means that the convex hull of the integral vectors in that polyhedron is obtained by adding all rank-1 Chvátal–Gomory cuts. From Theorem 1 it is not hard to see that a similar result holds for graphs with no bad-K_4.

COROLLARY 4. *$G = (V, E)$ contains no bad-K_4 if and only if for all $a, b \in \mathbb{Z}^V$ and all $c \in \mathbb{Z}^E$ the polyhedron*

$$\{x \in \mathbb{R}^V \mid a_v \le x_v \le b_v \ (v \in V); x_u + x_v \le c_{uv} \ (uv \in E)\} \tag{4}$$

has Chvátal-rank 1.

The rank-1 Chvátal–Gomory cuts needed here are

$$\sum_{v \in V(C)} x_v \le \frac{1}{2} \left\lfloor \sum_{uv \in E(C)} c_{uv} \right\rfloor \ (C \text{ is an odd circuit in } G). \tag{5}$$

One of the main open questions about t-perfection is whether the system of linear inequalities given in (1) is totally dual integral. This property holds for graphs with no odd-K_4 [6], but we have not yet been able to verify this for graphs with no bad-K_4. By the decomposition results used in Gerards [6], it follows that to check for which graphs the system in (1) is totally dual integral for all subgraphs, we may confine ourselves to clean pads and books.

Preliminaries. If G is a graph and u and v are nodes in G of degree at least 3, then a *uv-leg* of G is a uv-path P in G such that all nodes of P, except u and v, have degree 2 in G.

If P is a path in G and $u, v \in V(P)$ we denote the uv-path in P by P_{uv}. If $e = uv \in E(G)$, $P_e := P_{uv}$.

2. Structure of graphs with no bad-K_4. We first prove that if a weakly 3-connected graph with no bad-K_4 contains an odd-K_4, then it is either an odd-P_9, a book, or a pad (Lemma 5). Next we prove that a weakly 3-connected pad with no bad-K_4 is clean (Lemma 6). Together these two lemmas prove the only-if direction of the equivalence in Theorem 2. As odd-P_9's clearly have no bad-K_4, the if direction follows by proving that clean pads (Lemma 7) and books (Lemma 8) have no bad-K_4. We conclude this section with a recognition algorithm for graphs with no bad-K_4.

2.1. Books and pads. Let G be a pad. If H is a subgraph of G and not a pad itself, we denote by $K(H)$ the edges in $K(G)$ with both end nodes in $V(H)$.

If P is a path on $R(G)$, we say that chords e and f are *nested on P*, written as $e \succ_P f$, if $e, f \in K(P)$ and P_f is a subpath of P_e. Chords e, f of $K(G)$ are *nested* if they are nested on some path on $R(G)$; if not, e and f *cross* (notation: $e \times f$).

LEMMA 5. *Let G be a weakly 3-connected graph with no bad-K_4. If G contains an odd-K_4, then G is an odd-P_9, a book or a pad.*

Proof. We first give some definitions: Let H be a subgraph of a graph G. A *route* of H or an *H-route* is a uv-path P in G such that $V(P) \cap V(H) = \{u, v\}$ and such that no leg of H contains both u and v. We say that nodes u_1, u_2, and u_3 *induce an extended triangle* in H if each pair is connected by a leg of H. A collection of three

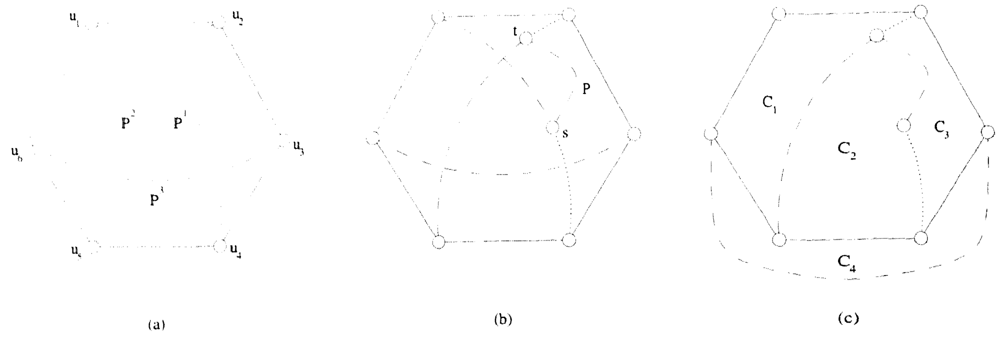

FIG. 5. *Dotted and dashed curves indicate internally node disjoint paths; dashed curves have positive length, whereas dotted curves may have length zero. In* (a), *dashed curves have an even number of edges.*

internally node disjoint vu_i-paths P_i $(i = 1, 2, 3)$ that are internally node disjoint from H is called an *H-tripod* if $v \notin V(H)$ and u_1, u_2, u_3 induce an extended triangle in H.

It is an easy graph theoretical fact that if H is a weakly 3-connected proper subgraph of a weakly 3-connected graph G, then G contains an H-route, or each leg of H is a leg of G and G contains an H-tripod. Moreover, adding an H-route to a weakly 3-connected graph H yields a weakly 3-connected graph.

Assume that G is a counterexample to the lemma with a minimum number of edges.

CLAIM 1. *G contains no odd-P_9.*

Proof of Claim 1. Suppose the claim is false and that H is an odd-P_9 in G. Let $u_1u_2, u_2u_3, \ldots, u_6u_1$ be the six length-1 legs of H; and, for $i = 1, 2, 3$, let P^i be the even u_iu_{i+3}-leg of H (see Figure 5a). By assumption $G \neq H$. As H is weakly 3-connected and has no extended triangle, there exists an H-route P in G. Let s and t be the end nodes of P. One argues that without loss of generality, $s \in P^1 \setminus u_1$ and $t \in P^2 \setminus u_5$ (see Figure 5b). Let $G' := (H \setminus P^1_{u_1s}) \cup P$ and C_1, C_2, C_3, and C_4 be circuits as indicated in Figure 5c. Clearly, C_1 and C_4 are odd circuits. Moreover, C_2 is even, as otherwise the union of C_1, C_2, and C_4 is a bad-K_4. Hence, C_3 is odd, so the union of C_4, C_3, and the symmetric difference of C_1 and C_2 forms a bad-K_4. □

CLAIM 2. *If H is a good-K_4 and P an H-route, then P is an edge and $H \cup P$ is a pad with $R(H \cup P) = R(H)$.*

Proof of Claim 2. H is a pad. Let u_1u_3 and u_2u_4 be the two chords of H and Q^1, Q^2, Q^3 and Q^4 be the four legs of H on $R(H)$ (see Figure 6a). Let s and t be the two end nodes of P. We may assume that $s \in V(Q^1) \setminus \{u_1, u_2\}$ and $t \in V(Q^2) \cup V(Q^3) \setminus \{u_2, u_4\}$. Let C be the unique circuit in $(R(H) \cup P) \setminus Q^4$ (see Figure 6b, c).

First suppose that C is even. If t were in $V(Q^2) \setminus \{u_2\}$ (Figure 6b), then $(H \setminus Q^2_{u_2t}) \cup P$ would be an odd-K_4, with $R_1 := u_1u_3$ and $R_2 := u_4u_2 \cup Q^1_{u_2s}$ as a pair of node disjoint legs. As R_1 has length 1 and R_2 does not, this odd-K_4 would be bad, so $t \in V(Q^3) \setminus \{u_3, u_4\}$ (see Figure 6c). As $H \cup P$ is not an odd-P_9, one of $Q^1_{u_1s}$, $Q^1_{u_2s}$, $Q^3_{u_3t}$, and $Q^3_{u_4t}$ has more than one edge. By symmetry we may assume that this is the case for $Q^1_{u_1s}$. But then all the legs of the odd-K_4 $(H \setminus Q^2) \cup P$, except maybe P or $Q^3_{u_4t}$, have more than one edge. Hence this odd-K_4 is bad.

Therefore, C is odd and thus $H^* := R(H) \cup P \cup \{u_4u_2\}$ is an odd-K_4. Therefore, P has length 1 and H^* is a pad with $R(H^*) = R(H)$. From this it trivially follows

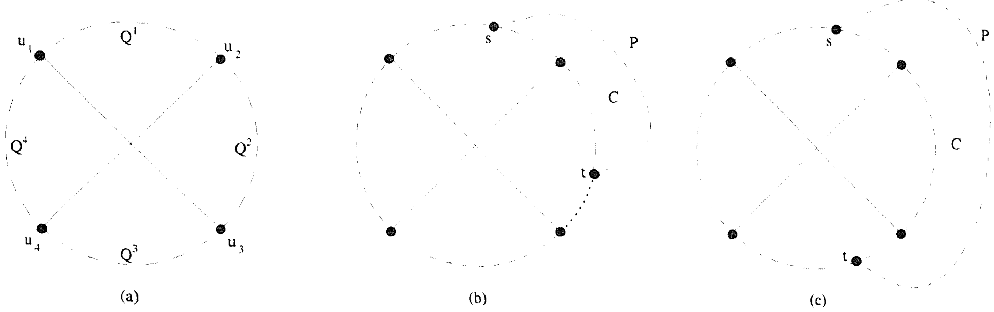

FIG. 6. *Dotted and dashed curves indicate internally node disjoint paths; dashed curves have positive length, whereas dotted curves may have length zero. In* (a), *dashed curves have an even number of edges.*

that also $H \cup P$ is a pad with $R(H \cup P) = R(H)$. $\square$

A pad is called *maximal* if there is no larger pad with the same rim. A subgraph H of G is *induced* if all edges of G with both end nodes in $V(H)$ are in H.

CLAIM 3. *No weakly* 3*-connected maximal pad has a route; hence each one is an induced subgraph of* G *and has a tripod.*

Proof of Claim 3. Let H be a weakly 3-connected pad, and let P be an H-route with end nodes s and t. Let Q_1 and Q_2 be the two st-paths on $R(H)$. As $H' := H \cup P$ is weakly 3-connected, there exists a chord $e = u_1u_2$ of H with $u_i \in V(Q_i) \setminus \{s,t\}$ for $i = 1, 2$. Moreover, as H is weakly 3-connected, there exists a chord f of H crossing e. Now, $H^* := R(H) \cup \{e, f\}$ is a good-K_4. As P is an H-route, $f \neq st$. Thus, s and t lie in different legs of H^*. Hence, by Claim 2, P is an edge—e^*, say—and $H^* \cup e^*$ is a pad with $R(H^* \cup e^*) = R(H^*) = R(H)$. It is trivial to see from this that $H' = H \cup e^*$ is a pad as well. Hence H is not maximal. Therefore, weakly 3-connected maximal pads have no routes.

Now, let H be a weakly 3-connected maximal pad. As it is not equal to G, it must have a tripod. Moreover, if it were not induced, one of its legs would not be a leg of G, but then there would be an H-route. As we have seen, this is not the case, so H is an induced subgraph of G. $\square$

If H is a pad, $u \in V(H)$ is called a *center* of H if the following hold: H has a chord vw such that all other chords cross it and have u as end node, and H has a tripod such that (i) one of its three paths has end node u and this path is of length 1 and (ii) the other two paths end in v and w and are even. We call such a tripod *fitting* H *at* u.

CLAIM 4. *Each weakly* 3*-connected maximal pad has at least one center, and each of its tripods fits at some center of the pad.*

Proof of Claim 4. Let H be a weakly 3-connected maximal pad; let P_1, P_2, and P_3 be the legs of any H-tripod. Denote the end node of P_i on H by u_i. Let Q^{ij} be the u_iu_j-path on $R(H)$ that does not contain the third node in $\{u_1, u_2, u_3\}$. As H is weakly 3-connected, one of Q^{12}, Q^{23}, and Q^{31} is not a leg of H; i.e., one of the legs of the extended triangle induced by u_1, u_2, u_3 is an edge of $K(G)$. Suppose that Q^{13} is not a leg and, consequently, $u_1u_3 \in K(H)$.

(6) If $u_iu_j \in$ K(H), then $P_i \cup P_j$ is an even path.

Indeed, if not then $R(H), P_i \cup P_j$ and one of the chords of H crossing u_iu_j form a

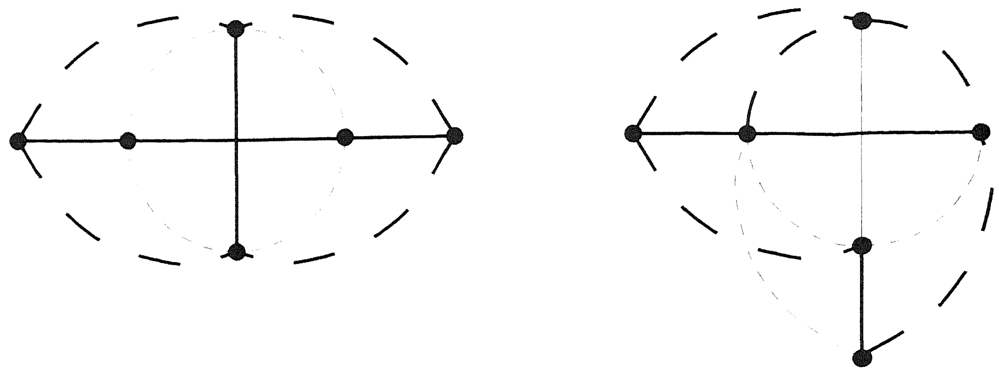

FIG. 7. *Dotted curves indicate internally node disjoint even paths. The bold edges and curves form a bad-K_4.*

bad-K_4.

(7) $P_1 \cup P_2$ and $P_2 \cup P_3$ are odd paths, and so, by (6), Q^{12} and Q^{23} are legs of H.

To see this, let xy be a chord of H crossing u_1u_3; assume $x \in Q^{23}$. It follows from (6) that if (7) were false, then $P_1, P_2, P_3, Q^{23}_{u_2x}, xy, Q^{13}$, and u_1u_3 would constitute a bad-K_4. By (6) and (7), $P_1, P_2, P_3, Q^{12}, Q^{23}$, and u_1u_3 form an odd-K_4, which implies that

$$(8) \qquad P_2 \text{ consists of a single edge.}$$

It remains to prove that u_2 is a center of H. Suppose that this is not the case; then there exists a chord e of H with both end nodes in Q^{13} (recall that Q^{12} and Q^{23} are legs of H). But then $P_1, P_2, P_3, Q^{12}, Q^{23}$, and $(Q^{13} \setminus Q^{13}_e) \cup \{e\}$ form a bad-K_4. □

CLAIM 5. *G contains a book with at least two leaves.*

Proof of Claim 5. There exists a weakly 3-connected pad (namely, each good-K_4 is one). As G is not a pad, by Claim 3 there exists a weakly 3-connected pad with no route and hence has a tripod. This pad and that tripod together form a book with two leaves. □

Let $\widetilde{H}$ be a book with center c and hinges v and w, maximum number of leaves $L_1, \ldots, L_n$, and maximum number of edges. Note that for any $i \neq j$, L_i contains an L_j-tripod centered at c. As in the proof of Claim 4, this implies that each chord of L_j has one end in c and the other on the trim of L_j. Moreover, each $V(L_j)$ induces a maximal pad, so by maximality of $\widetilde{H}$, each L_j is a maximal pad.

CLAIM 6. *There exists no $\widetilde{H}$-tripod.*

Proof of Claim 6. Let T be a tripod of $\widetilde{H}$. As all extended triangles are contained in leaves, we may assume that T is a tripod of leaf L_1. If T fits L_1 at the center of the book, $\widetilde{H} \cup T$ would be a larger book. Hence T fits L_1 at a node different from c. However, then L_1 has two tripods (namely, T and one in L_2) that fit at different nodes of L_1, so L_1 has at least two centers, which implies that it is a good-K_4. There are two possibilities for how the tripods fit at different nodes (see Figure 7). It is not hard to see that in either case, $L_1 \cup L_2 \cup T$ contains a bad-K_4. □

As G itself is not a book, $\widetilde{H}$ has a route—P, say. Let x and y be the end nodes of P. As the leaves of $\widetilde{H}$ are maximal pads, no one contains both x and y, so we may assume that $x \in V(L_1) \setminus V(L_2)$ and $y \in V(L_2) \setminus V(L_1)$.

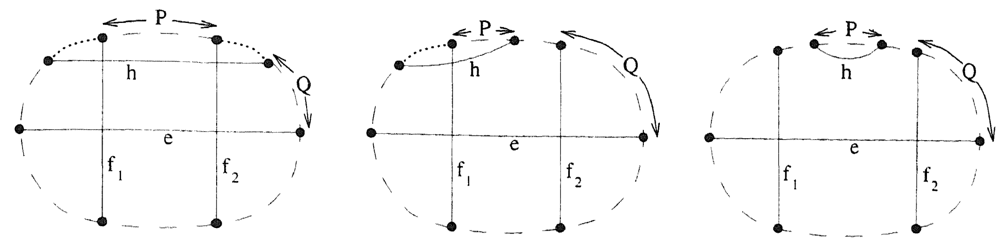

FIG. 8. *Dotted and dashed curves indicate internally node disjoint even paths; dashed curves have positive length, whereas dotted curves may have length zero. The closed curve on the outside is the rim.*

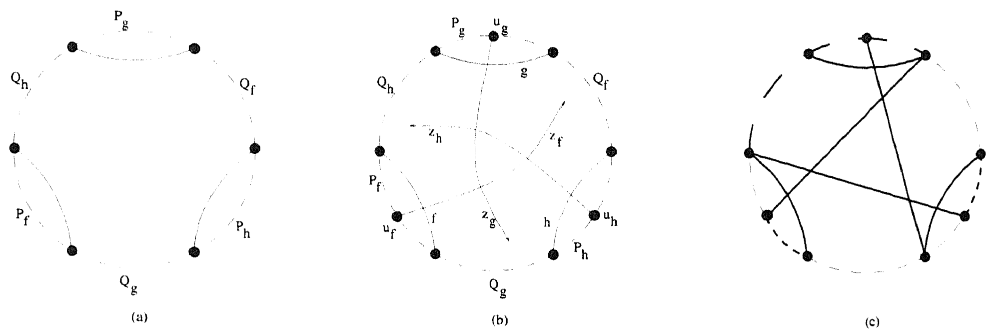

FIG. 9. *Dashed curves indicate internally node disjoint even paths of positive length. The closed curve on the outside is the rim.*

Let Q be the trim of L_2. First, if Q and P do not form a tripod of L_1, then the trim of L_1 contains at least three legs, so L_1 has a route, contradicting Claim 3. Thus Q and P form an L_1-tripod, which—as Q is even—fits at x (by (7)), so P consists of a single edge and L_1 has exactly one chord other than vw, namely, xc. By symmetry, the only chords of L_2 are vw and yc. However, now, xc, yc, xy, and the three even paths in $L_1 \cup L_2$ from v to x, y, and c form a bad-K_4. This yields a final contradiction. □

2.2. Clean pads. Before we can state and prove the next lemma, we need some further definitions. Let G be a pad. Chords e and f *touch*, written as $e \vee f$, if they share an end node. Chords e and f are *parallel* ($e \| f$) if they are nested but do not touch.

A *mesh* is a collection of four chords e, f_1, f_2, h with the following properties:

- $e \times f_1$, $e \times f_2$, $f_1 \| f_2$, and $h \| e$;
- h is not a chord of any of the four legs on $R(G)$ of the pad $R(G) \cup \{e, f_1, f_2\}$ that are adjacent with e.

There are several possibilities for four chords to form a mesh. They are listed in Figure 8. If we delete the paths P and Q on $R(G)$ indicated in Figure 8, we obtain a bad-K_4. Hence, a pad with no bad-K_4 contains no mesh.

A 3-*chain* is a triple $e, f, g \in K(G)$ such that $e \succ_P f \succ_P g$ for some path P on $R(G)$. A *dirty triple* is a collection of three pairwise parallel edges that do not form a 3-chain (see Figure 9a). A path P on $R(G)$ is *nesting* if each pair of chords on $K(P)$ is nested. G is *nesting* if, for each pair of nodes $s, t \in V(G)$, one of the two st-paths on $R(G)$ is nesting.

It is straightforward to prove that

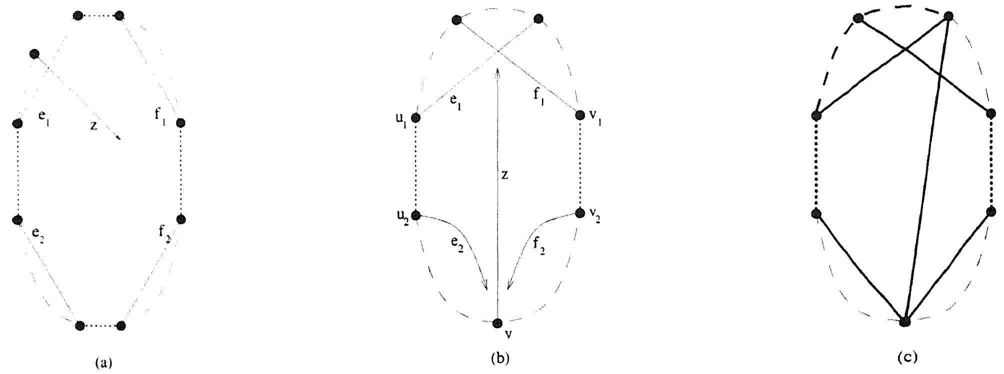

FIG. 10. *Dotted and dashed curves indicate internally node disjoint even paths; dashed curves have positive length, whereas dotted curves may have length zero. The closed curve on the outside is the rim. The bold edges and curves in* (c) *form a test.*

(9) a pad is clean if and only if it is nesting and contains no mesh and no dirty triple.

LEMMA 6. *Each weakly* 3*-connected pad with no bad-*K_4 *is clean.*

Proof. Let G be a weakly 3-connected pad with no bad-K_4. We have already seen that G contains no mesh. Assume that G is not clean.

CLAIM 7. *G is nonnesting.*

Proof of Claim 7. Suppose that G is nesting. Hence, it contains a dirty triple $T := \{f, g, h\}$. Let P_e, Q_e ($e \in T$) be as in Figure 9a. As G is weakly 3-connected, for each $e \in T$ there exists an edge $z_e := u_e v_e$ crossing e. Assume $u_e \in P_e$ for each $e \in T$. Then, for each $e \in T$, $v_e \in Q_e$, because if v_f, say, were not in Q_f, then z_f, g, f, h would form a mesh or G would be nonnesting.

By symmetry, we may assume that $z_f \| h$. As z_f, z_h, f, h is no mesh, $z_h \vee f$, so $z_h \| g$. Repeating this argument we get that $z_g \vee h$ and $z_f \vee g$. However, now G contains a bad-K_4 (namely, the bold lines in Figure 9c)—a contradiction! □

CLAIM 8. *There exist two edge disjoint paths P_1 and P_2 on $R(G)$ and edges $e_1, f_1 \in K(P_1)$ and $e_2, f_2 \in K(P_2)$ such that*

(i) *e_i and f_i are not nested on P_i ($i = 1, 2$),*

(ii) *both e_i and f_i share an end node with P_i ($i = 1, 2$),*

(iii) *$e_1 \times f_1$.*

Proof of Claim 8. By the previous claim, there exist two edge disjoint paths P_1 and P_2 on $R(G)$ and chords e_1, e_2, f_1, and f_2 satisfying (i). It is not hard to see that these paths and chords can be chosen to satisfy (ii) as well. If neither e_1 and f_1 nor e_2 and f_2 are crossing, choose z crossing e_1 (G is weakly 3-connected). With the aid of z, it is straightforward to see that either we can find edge disjoint paths P_1 and P_2 satisfying (i), (ii), and (iii) or we find a mesh (see Figure 9a). As the latter is impossible, the claim follows. □

Choose P_1, P_2, e_1, f_1, e_2, and f_2 as in the previous claim, with $|E(P_1)| + |E(P_2)|$ maximal. Let u_i, v_i be the end nodes of P_i ($i = 1, 2$). As G is weakly 3-connected, there exists a chord $z = uv$ with $v \in P_2 \setminus \{u_2, v_2\}$ and $u \notin P_2$. By the maximality of $|E(P_1)| + |E(P_2)|$, $u \in P_1 \setminus \{u_1, v_1\}$ (see Figure 10b).

First, consider the special case in Figure 10c. It contains a bad-K_4, indicated by the bold edges. However, the general case, as in Figure 10b, can be transformed to

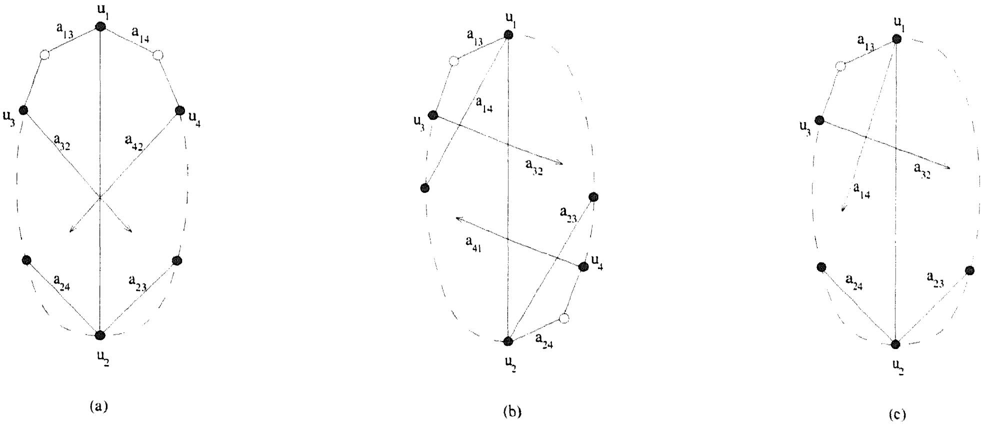

FIG. 11. *Dashed curves indicate internally node disjoint even paths of positive length. The closed curve on the outside is the rim.*

that special case by contracting legs on $R(G)$. As legs are even paths, this contraction could not have created a bad-K_4 if one in G did not already exist. Hence we have a contradiction, so G is clean. □

A chord of a pad is called *universal* if it is not parallel with any other chord.

LEMMA 7. *No clean pad contains a bad-K_4.*

Proof. Let G be a clean pad containing a bad-K_4 H such that $|E(G)|$ is minimal. Let u_1, u_2, u_3, and u_4 be the four nodes of H that have degree three in H. For $i, j = 1, \ldots, 4$, let P^{ij} be the u_iu_j-leg of H.

CLAIM 9. *The following hold:*

(i) $K(G) \subseteq E(H)$.

(ii) *All legs of G on $R(G)$ have length* 2.

(iii) *If P is a leg of H, then $|P \cap R(G)| \leq 2$. If $|P \cap R(G)| = 2$, then P is a leg of G on $R(G)$ or P has length* 3. *In the latter case, the four legs of H meeting P are even, and the sixth leg consists of a single edge.*

(iv) *If $u, v \in U(G)$ form a 2-node cutset of G, then there exists a uv-path P on $R(G)$ with* 2 *or* 4 *edges. If P has* 4 *edges, it has one chord, which meets exactly one of u and v.*

Proof of Claim 9. If (i) were false, deleting an edge from $K(G) \setminus E(H)$ would contradict the minimality of G, as would contracting legs of G into legs of length 2 if (ii) were false.

To prove (iii), suppose P is a leg of H that contains edges of $R(G)$. Let e_1 and e_2 be consecutive edges on $P \cap R(G)$. By the minimality of G, contracting e_1 and e_2 in H does not yield a bad-K_4. This means that the leg P of H containing e_1 and e_2, has length 2 or 3. Moreover, in the latter case the four legs of H meeting P are even and the sixth one has length one. Hence (iii) follows.

To see (iv), note that if G has a two node cutset, then H lies mainly on one "side" of that cutset in the sense that one side of the cutset contains at least five legs of H and the other side contains at most (part of) the sixth leg. □

CLAIM 10. *G has no universal chord.*

Proof of Claim 10. Let uv be a universal chord. This means that $G \setminus \{u, v\}$ is bipartite, so uv is a leg of H. Assume $u = u_1$ and $v = u_2$. Let Q^1 and Q^2 be the two uv-paths in $R(G)$. We call $Q^1 \cup K(Q^2)$ and $Q^2 \cup K(Q^1)$ the two *sides* of G. For

$i, j = 1, \ldots, 4$, let a_{ij} be the first edge on P^{ij} going from u_i to u_j. (Thus, $a_{ij} = a_{ji}$ if and only if $|P^{ij}| = 1$.)

As $|P^{12}| = 1$, it follows by Claim 9 that for $i = 1, 2$ and $j = 3, 4$, $a_{ij} \in R(G)$ if and only if P^{ij} is a leg of G in $R(G)$. Moreover, as the circuit $P^{1i} \cup P^{i2} \cup \{u_1u_2\}$ is odd for $i = 3, 4$, we have the following:

(10) If $i = 3, 4$, then a_{1i} and a_{2i} lie on the same side of G. Moreover, P^{1i} and P^{2i} are both even or both odd.

Also, as the circuit $P^{i3} \cup P^{34} \cup P^{4i}$ is odd for $i = 1, 2$, we have that

(11) if $i = 1, 2$, then a_{i3} and a_{i4} lie on different sides of G.

Next we rule out the different cases one by one:

(12) At least one of a_{13}, a_{14}, a_{23}, and a_{24} is in $R(G)$.

Suppose that this is not the case and that $a_{13} \in K(Q^1)$. Then from (10) and (11) it follows that $a_{23} \in K(Q^1)$ and $a_{14}, a_{24} \in K(Q^2)$. Thus both Q^1 and Q^2 are nonnesting, which is a contradiction.

(13) For $i = 1, 2$, either a_{i3} or a_{i4} is in $K(G)$.

To see this, assume that $a_{13} \in Q^1$ and $a_{14} \in Q^2$ (see Figure 10a). By (10), all legs of H adjacent to u_1u_2 are even. Hence P^{34} is odd, and as H is bad, it has at least three edges. By symmetry, we may assume that $Q^2_{u_2u_4}$ is not internally node disjoint with P^{34}. Hence $P^{24} \neq Q^2_{u_2u_4}$. Therefore, by (10), $a_{24} \in K(Q^1)$ and $a_{42} \times u_1u_2$ (by Claim 9(iii) and since u_1u_2 is universal). However, this implies that $P^{23} \neq Q^1_{u_2u_3}$, so, by (10), $a_{23} \in K(Q^2)$ and $a_{32} \times u_1u_2$. If $a_{32} \times a_{23}$, then a_{32}, a_{23}, a_{42}, and a_{24} form a mesh, so a_{32} and a_{23} do not cross. Similarly, a_{42} and a_{24} do not cross. However, this implies that G is nonnesting, a contradiction. Hence (13) follows.

From the above we may assume that $a_{13} \in Q^1$ and $a_{14} \in K(Q^1)$, so P^{23} cannot be $Q^1_{u_2u_3}$. Hence $a_{23} \in K(Q^2)$ and $a_{32} \times u_1u_2$. First assume that $a_{24} \in Q^2$ and consequently $a_{41} \times u_1u_2$ (see Figure 10b). As G is nesting, by symmetry we may assume that $a_{32} \times a_{23}$, but this implies that a_{32}, a_{41}, a_{23}, and a_{14} form a mesh. As G is clean, this is a contradiction, so $a_{24} \notin Q^2$. Hence, $a_{24} \in K(Q^1)$ (see Figure 10c). As a_{32}, a_{41}, a_{23}, and a_{24} is not a mesh, a_{32} does not cross a_{23}. Similarly, a_{24} does not cross a_{14}. But this means that G is nonnesting—a contradiction! □

A chord is *crossed* if it is crossed by at least one other chord. We call chords e_1 and e_2 *distant* if $e_1 \| e_2$ and, for $i = 1, 2$, the path P_i on $R(G)$ with the same end nodes as e_i but node disjoint from e_{3-i} satisfies $K(P_i) = \{e_i\}$.

CLAIM 11. *Each pair of distant chords contains a noncrossed chord.*

Proof of Claim 11. Let e_1 and e_2 be a pair of distant chords. Suppose that e_1 is crossed by z_1 and e_2 by z_2. For $i = 1, 2$, z_i does not cross e_{3-i}, as otherwise, z_i would be universal, or there would be a mesh, or e_1 and e_2 would not be distant. As G is nesting $z_1 \times z_2$. Let x_1 be the end node of z_1 and x_2 be the end node of z_2 such that there exists an x_1x_2-path on $R(G)$, called Q, that is internally node disjoint with e_1 and e_2. Assume z_1 and z_2 are selected such that Q is as short as possible. As G contains no mesh, either $z_1 \vee e_2$ or $z_2 \vee e_1$; assume the latter is the case (see Figure 11).

For $i = 1, 2$, let y_i be a chord parallel with z_i (z_1 and z_2 are not universal). From the fact that G is clean, that e_1 and e_2 are distant, and that Q is minimal, one is able

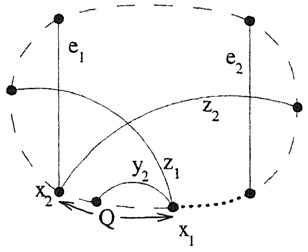

FIG. 12. *Dotted and dashed curves indicate internally node disjoint even paths; dashed curves have positive length, whereas dotted curves may have length zero. The closed curve on the outside is the rim.*

to deduce that $y_2 \in K(Q \setminus x_2)$. As e_1, e_2, y_2 cannot form a dirty triple, y_2 is adjacent to e_2, so x_1 is an end node of e_2. Hence we have symmetry between $i = 1$ and $i = 2$. Therefore, $y_1 \in K(Q \setminus x_1)$ and is adjacent to e_1, but now the edges e_1, e_2, y_1, and y_2 show that G is not nesting—a contradiction! □

If $e = uv$ is a noncrossed chord, then u and v share a common neighbor in $R(G)$ (by Claim 9(iv)), which we denote by u_e. As $e \in E(H)$, the node u_e will not be in $V(H)$.

CLAIM 12. *Each pair of distant chords contains a crossed chord. Moreover, the noncrossed chords in G are pairwise adjacent and there are at most two of them.*

Proof of Claim 12. To prove the first statement, suppose that it is false. Let e and f be two parallel nonadjacent noncrossed chords. Let Q^1 and Q^2 be the two paths on $R(G)$ joining an end of e with an end of f. As G is nesting, we may assume that $K(Q^2) = \emptyset$ and that Q^1 is nesting. As H is contained in $G' = G \setminus \{u_e, u_f\}$, G' is nonbipartite. Hence $K(Q^1) \neq \emptyset$. Let $h \in K(Q^1)$ with Q^1_h minimal. As G has no dirty triple, h is adjacent to e or to f. Thus, let us assume that h and e share an end node—v, say. As Q^1 is nesting all chords in $K(Q^1)$ are in $\delta(v)$. But that means that all odd circuits in G' contain v. This is impossible since not all odd circuits in H can go through a single node.

The second statement easily follows from the first. Indeed, two parallel noncrossed chords are clearly distant by Claim 9(iv), so by the first statement of this claim they cannot exist. Suppose there are three pairwise adjacent noncrossed chords e_1, e_2, and e_3. They cannot meet at a single node, as this would contradict Claim 9(iv), so they form a triangle. Hence $K(G) = \{e_1, e_2, e_3\}$ and $R(G)$ is a circuit of length 6, but that graph has no bad-K_4. □

CLAIM 13. *There is exactly one noncrossed chord.*

Proof of Claim 13. Suppose that this claim is false. Let $e = xy$ and $f = yz$ be two noncrossed chords. Let Q be the xz-path on $R(G)$ not containing y. As e is not universal, $K(Q) \neq \emptyset$. Let $g \in K(Q)$ with Q_g minimal. Let $h \times g$ (by the previous claim, g is crossed). As Q is nesting, $h \in \delta(y)$ and each chord in $K(Q)$ crosses h. Hence, h is universal—a contradiction! □

As there are no universal edges, there exists a pair of distant chords. By Claims 11 and 12 one of the two—$e = uv$, say—is crossed, and the other, f, is not. Let P be the uv-path on $R(G)$ not containing u_f. Let Q^1 and Q^2 be the two paths constituting $R(G) \setminus (P \cup \{u_f\})$. For $i = 1, 2$, let K_i be the collection of edges crossing e with end node in Q^i.

CLAIM 14. $K(Q^1) = K(Q^2) = \emptyset$, *$K_1 \neq \emptyset$, and $K_2 \neq \emptyset$.*

Proof of Claim 14. As G is nesting, (i) $K(Q^1) = \emptyset$ or $K(Q^2) = \emptyset$, (ii) $K(Q^1) = \emptyset$

or $K^2 = \emptyset$, and (iii) $K(Q^2) = \emptyset$ or $K^1 = \emptyset$. From this it is easy to check that if the claim is false, then either $K^1 = \emptyset$ and $K(Q^1) = \emptyset$ or $K^2 = \emptyset$ and $K(Q^2) = \emptyset$. Assume that the latter is the case. Let w be the common end node of P and Q^1. There exists an odd circuit in H not containing w. As $u_f \notin V(H)$, this means that $G \setminus \{u_f, w\}$ is nonbipartite. It is straightforward to check that this implies that $K(Q^1)$ contains a chord parallel with e. Let h be such a chord with Q^1_h minimal. Then e and h are distant, so by Claim 11, h is noncrossed, but this contradicts Claim 13. □

Let Q be the path $R(G) \setminus u_f$. For $i = 1, 2$, let $e_i \in K^i$ with $Q_{e_i} \cap P$ maximal. As e, e_1, e_2, f do not from a mesh, $e_1 \times e_2$ or $e_1 \vee e_2$, so there exists a node—w, say—that lies on $Q_{e_1} \cap P$ and on $Q_{e_2} \cap P$. By Claim 14 this means that w lies on Q_g for each chord $g \in K(Q) = K(G)$. Hence $G \setminus \{w, u_f\}$ is bipartite. As H does not contain u_f, this is a final contradiction. □

LEMMA 8. *No book contains a bad-K_4.*

Proof. Suppose that G is a book, and let H be any odd-K_4 in G. Let C be the spine, h_1 and h_2 be the hinges, and c be the center of G. It is easy to see that for every $e \in E(C)$ there is a node $v \in \{h_1, h_2, c\}$ such that each odd circuit in $G \setminus e$ contains v. Hence H must contain C. Consequently, H should be entirely contained in one of the leaves of G. As all leaves are clean pads, H must be a good-K_4. □

2.3. Recognizing graphs with no bad-K_4. In this section we prove Theorem 3, which says that one can check—in polynomial time—whether or not a given graph G contains a bad-K_4.

First of all, note that odd-P_9's, books and clean pads are easily recognized. Second, a polynomial-time recognition algorithm for the containment of an odd-K_4 is given by Gerards et. al. [8] (cf. Gerards [7]). Hence, by Theorem 2, it suffices to prove that we can find for each graph G in polynomial time a polynomial-length list $\mathcal{L}$ of weakly 3-connected graphs smaller than G such that G contains a bad-K_4 if and only if at least one member of $\mathcal{L}$ contains a bad-K_4. The following two easy-to-prove lemmas show that this is indeed the case.

We need some definitions and notations. If G is a graph, then $[G_1, G_2]_{u,v}$ is called a *split* if G_1 and G_2 are subgraphs of G such that $V(G_1) \cap V(G_2) = \{u, v\}$; $E(G_1)$ and $E(G_2)$ partition $E(G)$, $|E(G_1)|, |E(G_2)| \geq 4$; and neither G_1 nor G_2 is an odd circuit. If G_2 is bipartite and contains an odd uv-path, we call the split *odd*. If G_2 is bipartite and contains an even uv-path, we call the split *even*. If both G_1 and G_2 are nonbipartite, we call the split *strong*.

If u and v are two nodes of a graph H and $\ell \in \mathbb{N}$, then $[H]^{\ell}_{u,v}$ denotes the graph obtained from H by adding a path from u to v with ℓ edges; we abbreviate this as $[H]^{k,\ell}_{u,v} := [[H]^k_{u,v}]^{\ell}_{u,v}$.

LEMMA 9. *Let $[G_1, G_2]_{u,v}$ be a split of a 2-connected graph G. Then the following hold:*

- *If $[G_1, G_2]_{u,v}$ is odd, then G contains a bad-K_4 if and only if $[G_1]^3_{u,v}$ contains a bad-K_4.*
- *If $[G_1, G_2]_{u,v}$ is even, then G contains a bad-K_4 if and only if $[G_1]^2_{u,v}$ contains a bad-K_4.*
- *If $[G_1, G_2]_{u,v}$ is strong and G has no odd or even split, then G contains a bad-K_4 if and only if at least one of $[G_1]^{2,3}_{u,v}$ and $[G_2]^{2,3}_{u,v}$ contains a bad-K_4.*

It follows from this lemma that given a graph G we can construct a polynomial-sized list $\mathcal{L}'(G)$ of graphs with no splits such that G has a bad-K_4 if and only if at least one member of the list has a bad-K_4. Therefore, we may restrict ourselves to graphs with no split. It is easy to see that a graph with no split can be obtained from a 3-connected graph H by replacing some edges in H by a path of length 2 or 3 or by a circuit of length 3 or 5. More precisely, a graph G has no split if and only if there exists a 3-connected graph H and five sets P_1, P_2, P_3, C_3, and C_5 partitioning $E(H)$, such that G can be obtained from H as follows: for each edge $uv \in P_2 \cup C_3 \cup C_5$ add a path from u to v with 2 edges; moreover, for each edge $uv \in P_3 \cup C_5$ add a path from u to v with 3 edges; finally, remove all the edges in $P_2 \cup P_3 \cup C_5$. We denote G by $H(P_1, P_2, P_3, C_3, C_5)$. Note that, given G, it is easy to find H and the proper partition of its edge set.

So we see that a graph with no split only fails to be weakly 3-connected because it may have pairs of "parallel" legs. Clearly, from each such pair of legs a bad-K_4 can use at most one leg. So if we would consider the list of graphs obtainable by dropping a leg from each pair of parallel ones, we do not gain or lose bad-K_4's. The nice thing about the graphs on this list is that they are weakly 3-connected; the bad thing is that there may be exponentially many of them. Fortunately there is an easy way out of this; we do not need to create the whole list.

LEMMA 10. *Let $G = H(P_1, P_2, P_3, C_3, C_5)$ be a graph with no split. Then G contains a bad-K_4 if and only if there exists a $T_3 \subseteq C_3$ and a $T_5 \subseteq C_5$ with $|T_3|+|T_5| \leq 6$, such that the graph $H(P_1 \cup T_3, P_2 \cup (C_3 \setminus T_3) \cup (C_5 \setminus T_5), P_3 \cup T_5, \emptyset, \emptyset)$ contains a bad-K_4.*

(In fact, this lemma remains correct if we replace $|T_3| + |T_5| \leq 6$ with $|T_3| + |T_5| \leq 3$.)

3. T-perfection. The main goal of this section is to prove Theorem 1, but we first show that bad-K_4's are not t-perfect. In the remainder of the paper, for a subset $S \subseteq V(G)$, we use χ_S to denote the incidence vector of S in $\mathbb{R}^{V(G)}$.

LEMMA 11. *No bad-K_4 is t-perfect.*

Proof. First, note that K_4 is not t-perfect, as the vector $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ is in $\mathcal{P}(K_4)$, but obviously not the convex combination of characteristic vectors of stable sets in K_4. Next, note that each bad-K_4 can be reduced to K_4 by repeated application of the following operation: take a node u and contract all the edges incident with u. However, this operation preserves t-perfection, which we easily obtain from the following:

(14) Let G be a graph, $u \in V(G)$, and $x \in \mathbb{R}^{V(G)}$ such that $x_v = 1 - x_u$ for each neighbor v of u. Moreover, let $\widetilde{G}$ be obtained from G by contracting all the edges in $\delta(u)$ into a new node $\widetilde{u}$, and let $\widetilde{x} \in \mathbb{R}^{V(\widetilde{G})}$ be defined by $\widetilde{x}_v := x_v$ if $v \in V(\widetilde{G}) \setminus \widetilde{u}$ and $\widetilde{x}_{\widetilde{u}} := 1 - x_u$. Then x is a vertex of $\mathcal{P}(G)$ if and only if $\widetilde{x}$ is a vertex of $\mathcal{P}(\widetilde{G})$.

Hence no bad-K_4 is t-perfect. □

The proof of Theorem 1 uses the following lemma (the graphs $[G_i]^{\ell}_{u,v}$ are defined in section 2.3).

LEMMA 12. *Let G be a graph with induced subgraphs G_1 and G_2 such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.*

(a) *If $V(G_1) \cap V(G_2)$ induces a clique in G, then G is t-perfect if and only if G_1 and G_2 are t-perfect (Chvátal [4]).*

(b) *If G is 2-connected, G_2 is bipartite, and $V(G_1) \cap V(G_2) = \{u, v\}$ with $uv \notin E(G)$, then if u and v are on the same side of the bipartition of G_2, G is t-perfect if and only if $[G_1]^2_{u,v}$ is t-perfect; otherwise, G is t-perfect if and only if $[G_1]^3_{u,v}$ is t-perfect (Sbihi and Uhry [10]).*

(c) *If G is 2-connected, both G_1 and G_2 are nonbipartite, and $V(G_1) \cap V(G_2) = \{u, v\}$ with $uv \notin E(G)$, then G is t-perfect if and only if $[G_1]^2_{u,v}$, $[G_1]^3_{u,v}$, $[G_2]^2_{u,v}$, and $[G_2]^3_{u,v}$ are t-perfect (Boulala and Uhry [2], Gerards [6]).*

In fact, the lemma above can be generalized beyond t-perfection: It has been proved by Chvátal [4]—for case (a)—and Barahona and Mahjoub [1]—for cases (b) and (c)—that one can obtain linear descriptions for the stable set polyhedron recursively through decompositions as in Lemma 12.

Proof of Theorem 1. Let $\widetilde{G}$ be a counterexample to the theorem with $|E(\widetilde{G})|$ minimal. By Lemma 12

(15) $\widetilde{G}$ is weakly 3-connected and each of its legs has at most 3 edges.

Let $\widetilde{x}$ be a fractional vertex of $\mathcal{P}(\widetilde{G})$. An edge $uv \in E(\widetilde{G})$ is *tight* if $\widetilde{x}_u + \widetilde{x}_v = 1$; an odd circuit C is *tight* if $\sum_{v \in V(C)} \widetilde{x}_v = \frac{1}{2}(|V(C)| - 1)$. We denote the collection of tight edges by $\mathcal{T}$ and the collection of tight odd circuits by $\mathcal{C}$.

(16) $$0 < \widetilde{x}_v < 1 \text{ for each } v \in V(\widetilde{G}).$$

Indeed, if $\widetilde{x}_u = 0$, then $\widetilde{G} \setminus \{u\}$ would be a smaller counterexample, and if $\widetilde{x}_u = 1$, u has a neighbor v with $\widetilde{x}_v = 0$.

(17) $\widetilde{x}$ is the unique solution of the system

$$\begin{array}{rcll} x_u + x_v & = & 1 & (uv \in \mathcal{T}), \\ \displaystyle\sum_{u \in V(C)} x_u & = & \frac{1}{2}(|V(C)| - 1) & (C \in \mathcal{C}), \end{array}$$

as otherwise $\widetilde{x}$ would not be a vertex of $\mathcal{P}(\widetilde{G})$. For $V_0 \subseteq V(\widetilde{G})$, we define $\mathcal{T}(V_0) := \{uv \in \mathcal{T} | u \in V_0\}$ and $\mathcal{C}(V_0) := \{C \in \mathcal{C} | V(C) \cap V_0 \neq \emptyset\}$.

(18) $$\text{For each } V_0 \subsetneq V(\widetilde{G})\text{: } |\mathcal{T}(V_0)| + |\mathcal{C}(V_0)| > |V_0|.$$

If this were not true, the restriction of $\widetilde{x}$ to $V(\widetilde{G}) \setminus V_0$ would be a unique solution of the system

$$\begin{array}{rcll} x_u + x_v & = & 1 & (uv \in \mathcal{T} \setminus \mathcal{T}(V_0)), \\ \displaystyle\sum_{u \in V(C)} x_u & = & \frac{1}{2}(|V(C)| - 1) & (C \in \mathcal{C} \setminus \mathcal{C}(V_0)). \end{array}$$

So $\widetilde{G} \setminus V_0$ would be a smaller counterexample to Theorem 1. From (14), it also follows that

(19) $$\delta(v) \not\subseteq \mathcal{T} \text{ for each } v \in V.$$

CLAIM 15. *If C is an odd circuit, then $E(C) \cap \mathcal{T}$ contains no matching of size $\frac{1}{2}(|V(C)| - 1)$. If C is an even circuit and $E(C) \cap \mathcal{T}$ contains a perfect matching, then $E(C) \subseteq \mathcal{T}$.*

Proof of Claim 15. Let $M \subseteq E(C) \cap \mathcal{T}$ be a matching with at least $\frac{1}{2}(|V(C)| - 1)$ edges. If C is even, then $\frac{1}{2}|V(C)| = \sum_{uv \in M}(\widetilde{x}_u + \widetilde{x}_v) = \sum_{uv \in E(C) \setminus M}(\widetilde{x}_u + \widetilde{x}_v) \leq \frac{1}{2}|V(C)|$; thus, we have equality throughout,which implies that also edges in $E(C) \setminus M$ are in $\mathcal{T}$. If C is odd, then there is exactly one node $u' \in V(C)$ that is incident with none of the edges in M, so we have $\widetilde{x}_{u'} = \sum_{v\in V(C)} x_v - \sum_{uv\in M}(x_u + x_v) \leq \frac{1}{2}(|V(C)|-1) - \frac{1}{2}(|V(C)|-1) = 0$, which contradicts (16). □

CLAIM 16. *Let u and v be two nodes on a circuit $C \in \mathcal{C}$ and P be a uv-path that is internally node disjoint from C. If $\mathcal{T} \cap E(P)$ contains a matching M covering each node in $V(P) \setminus \{u,v\}$, then the unique odd circuit in $C \cup P$ using P is tight.*

Proof of Claim 16. Let Q_1 and Q_2 be the two uv-paths in C, and assume that $P \cup Q_1$ is an odd circuit—C', say. Let N be the largest matching in $E(Q_2)$ with $V(N)\cap\{u,v\} = V(M)\cap\{u,v\}$. Then $\sum_{r\in V(C')}\widetilde{x}_r = \sum_{r\in V(C')\setminus V(M)}\widetilde{x}_r + \sum_{rs\in M}(\widetilde{x}_r + \widetilde{x}_s) = \sum_{r\in V(C)\setminus V(N)}\widetilde{x}_r + |M| \geq \sum_{r\in V(C)\setminus V(N)}\widetilde{x}_r + \sum_{rs\in N}(\widetilde{x}_r+\widetilde{x}_s) - |N| + |M| = \sum_{r\in V(C)}\widetilde{x}_r - |N| + |M| = \frac{1}{2}(|V(C)|-1) - |N| + |M| = \frac{1}{2}(|V(C')|-1)$. Thus $C' \in \mathcal{C}$. □

A circuit C in a graph G is called *separating* if G has subgraphs G_1 and G_2, each properly containing C, such that $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$, $V(C) = V(G_1) \cap V(G_2)$, and $E(C) = E(G_1) \cap E(G_2)$.

CLAIM 17. *No circuit in $\mathcal{C}$ is separating.*

Proof of Claim 17. Let $C \in \mathcal{C}$ be separating, and let G_1 and G_2 be as indicated just above this claim (with $G = \widetilde{G}$). For $i = 1,2$, let $\widetilde{x}^i$ be the restriction of $\widetilde{x}$ to $V(G_i)$. As both G_1 and G_2 have no bad-$K_4$and fewer edges than $\widetilde{G}$, they are t-perfect. Therefore, there exists a $K \in \mathbb{N}$, stable sets $S_1^1,\ldots,S_K^1$ in G_1, and stable sets $S_1^2,\ldots,S_K^2$ in G_2 (with possible repetitions) such that

$$(20)\qquad \widetilde{x}^1 = \frac{1}{K}(\chi_{S_1^1} + \cdots + \chi_{S_K^1})\text{and}\widetilde{x}^2 = \frac{1}{K}(\chi_{S_1^2} + \cdots + \chi_{S_K^2}).$$

Consequently,

$$(21)\qquad |S_j^i \cap V(C)| = \frac{1}{2}(|V(C)|-1)\text{for}i = 1,2\text{and}j = 1,\ldots,K.$$

For $i = 1,2$ and $uv \in E(C)$, we denote the number of stable sets S_j^i with $u,v \notin S_j^i$ by $\sigma_i(uv)$. As, $\sigma_i(uv) = \sum_{j=1}^K(1-|S_j^i\cap\{u,v\}|) = K - \sum_{j=1}^K \chi_{\{u,v\}}^\top \chi_{S_j^i} = K - K\chi_{\{u,v\}}^\top \widetilde{x}^i = K(1-\widetilde{x}_u^i - \widetilde{x}_v^i) = K(1-\widetilde{x}_u - \widetilde{x}_v)$, we have that

$$(22)\qquad \sigma_1(uv) = \sigma_2(uv) \text{ for each } uv \in E(C).$$

By (21) and (22), we can renumber the sets $S_1^2,\ldots,S_K^2$, such that

$$(23)\qquad \text{for all } j = 1,\ldots,K,\ S_j^1 \cap V(C) = S_j^2 \cap V(C).$$

Hence, each $S_j^1 \cup S_j^2$ is a stable set in $\widetilde{G}$ and

$$(24)\qquad \widetilde{x} = \frac{1}{K}(\chi_{S_1^1\cup S_1^2} + \cdots + \chi_{S_K^1\cup S_K^2}),$$

but this contradicts that $\widetilde{x}$ is a fractional vertex of $\mathcal{P}(\widetilde{G})$. □

As $\widetilde{G}$ is not t-perfect, it contains an odd-K_4. So, by (15) and Theorem 2, $\widetilde{G}$ is an odd-P_9, a book or a clean pad. We will deal with these cases separately.

CASE 1. *$\widetilde{G}$ is an odd-P_9.*

By (15), $\widetilde{G}$ is in fact the Petersen graph with a node removed; see Figure 13. Let $S_{3,6} = \{u_3,u_6,u_{14},u_{25}\}$. By (17), there exists an edge $uv \in \mathcal{T}$ with $S_{3,6} \cap \{u,v\} = \emptyset$ or a $C \in \mathcal{C}$ such that $|S_{3,6} \cap V(C)| < (|V(C)|-1)/2$. It is easy to check that the only

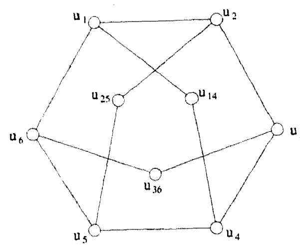

FIG. 13.

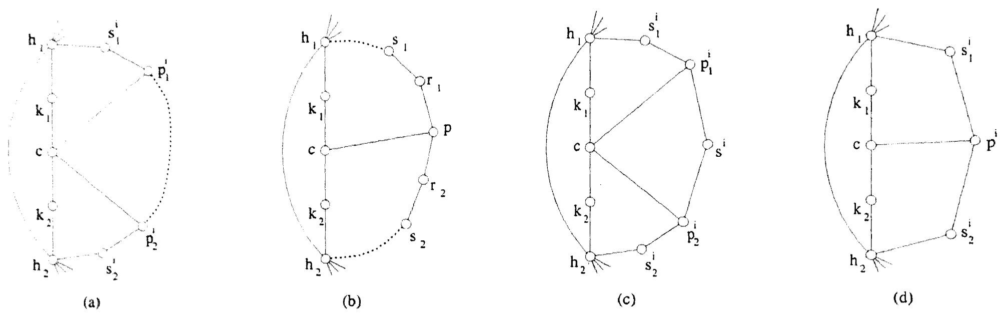

FIG. 14. *Dotted curves indicated internally node disjoint even paths; they may have length zero.*

possibility for this to hold is that either $u_1u_2 \in \mathcal{T}$ or $u_5u_4 \in \mathcal{T}$. By symmetry, we also have $u_2u_3 \in \mathcal{T}$ or $u_6u_5 \in \mathcal{T}$ and $u_3u_4 \in \mathcal{T}$ or $u_6u_1 \in \mathcal{T}$. Again by symmetry, we may assume that $u_1u_2 \in \mathcal{T}$. Hence by Claim 15, $u_6u_5 \notin \mathcal{T}$ and $u_3u_4 \notin \mathcal{T}$. So $u_6u_1 \in \mathcal{T}$ and $u_2u_3 \in \mathcal{T}$. However, that contradicts Claim 15.

CASE 2. $\widetilde{G}$ *is a book.*

Let h_1, h_2 be the hinges of the book and c be the center. Let $L_1, \ldots, L_n$ be the trims of the book. By (15), the spine of $\widetilde{G}$ is a circuit of length 5—$h_1k_1ck_2h_2$, say—and the legs of each L_i have length two. Let $h_1s_1^ip_1^i$ be the first leg of L_i and $p_2^is_2^ih_2$ be the last leg of L_i (going from h_1 to h_2; see Figure 14a). It is straightforward to check that

(25) each nonseparating odd circuit in $\widetilde{G}$ is one of $h_1h_2 \cup L_i$; $h_1s_1^ip_1^ick_1$ or $h_2s_2^ip_2^ick_2$ for some $i = 1, \ldots, n$.

CLAIM 18. *If* $p \in L_i$ *and* $cp \in E(\widetilde{G})$, *then* $|\mathcal{C}(p)| \geq 2$. *Hence* $p \in \{p_1^i, p_2^i\}$.

Proof of Claim 18. Assume $|\mathcal{C}(p)| \leq 1$. Let s_1r_1p and pr_2s_2 be the two legs of L_i adjacent to p; see Figure 14b. By (18), $|\mathcal{T}(r_1, p, r_2)| \geq 4 - |\mathcal{C}(r_1, p, r_2)| \geq 3$. By (18) and (19), $|\mathcal{T}(r_1)| = |\mathcal{T}(r_2)| = 1$. Hence $cp \in \mathcal{T}$. By (19), we may assume that $pr_1 \notin \mathcal{T}$; hence $r_1s_1 \in \mathcal{T}$. But now the circuit cpr_1s_1 or, if $s_1 = h_1$, the circuit $cpr_1s_1k_1$ violates Claim 15. $\square$

CLAIM 19. *For each* $i = 1, \ldots, n$, $p_1^i = p_2^i =: p^i$ (*see Figure* 14d).

Proof of Claim 19. If not, L_i has three legs; see Figure 14c. By (19), (18), and (25), $|\mathcal{T}(s^i)| = 1$, so, by symmetry, we may assume that $s^ip_2^i \in \mathcal{T}$. By Claim 18, the circuit $ck_1h_1s_1^ip_1^i$ is tight, so by Claim 16, $ck_1h_1s_1^ip_1^is^ip_2^i$ is tight as well. But it has a chord, contradicting Claim 17. $\square$

CLAIM 20. $ck_1, ck_2 \in \mathcal{T}$.

Proof of Claim 20. Suppose $ck_2 \notin \mathcal{T}$. Let $S := \{k_1, h_2\} \cup \{p^i | 1 = i, \ldots, n\}$.

By (17), there exists an edge $uv \in \mathcal{T}$ with $S \cap \{u,v\} = \emptyset$ or an odd circuit $C \in \mathcal{C}$ with $|S \cap V(C)| < \frac{1}{2}(|V(C)| - 1)$. Using (25), it is easy to check that this implies that $h_1 s_1^i \in \mathcal{T}$ for some $i = 1, \ldots, n$. Fix such an i. By (19) and Claim 15, none of $s_1^i p^i, p^i c, p^i s_2^i$, and $s_2^i h_2$ is in $\mathcal{T}$. By (18), $|\mathcal{C}(s_1^i, p^i, s_2^i)| \geq 4 - |\mathcal{T}(s_1^i, p^i, s_2^i)| = 3$, so, $ck_2h_2s_2^ip^i \in \mathcal{C}$ and $s_1^ih_1h_2s_2^ip^i \in \mathcal{C}$. Hence $\widetilde{x}_c + \widetilde{x}_{k_2} = \widetilde{x}_{s_1^i} + \widetilde{x}_{h_1}$, contradicting that $s_1^ih_1$ is tight and ck_2 is not. □

Now, by (19), we may assume that $cp^1 \notin \mathcal{T}$. By Claims 20 and 15, $\mathcal{T}(s_1^1) = \mathcal{T}(s_1^2) = \emptyset$. Hence $|\mathcal{T}(s_1^1, p^1, s_2^1)| + |\mathcal{C}(s_1^1, p^1, s_2^1)| \leq 3$, contradicting (18).

CASE 3. *$\widetilde{G}$ is a clean pad.*

A priori, the tight odd circuits might run quite wildly through $\widetilde{G}$. However, this is not the case, as is shown by the following lemma, which can be understood independently of the present proof.

LEMMA 13. *Let C be a nonseparating odd circuit in a clean pad G. Then $|E(C) \cap K(G)| = 1$.*

Proof. Let G be a counterexample with $|E(G)|$ minimal. Let C be a nonseparating odd circuit in G with $|E(C) \cap K(G)| \neq 1$. As contracting all edges on $E(C) \cap R(G)$ yields another counterexample, $E(C) \subseteq K(G)$. Moreover, if $e \in K(G) \setminus E(C)$, then its end nodes lie in different components of $R(G) \setminus V(C)$, as otherwise, $G \setminus \{e\}$ would be a smaller counterexample. We first prove that

(26) $\qquad E(C)$ contains no pair of parallel chords.

Indeed, suppose that it is false. Choose parallel chords $f, g \in E(C)$ that are distant in the pad $G \setminus (K(G) \setminus E(C))$. As C is nonseparating, there exist edges $e_f, e_g \in K(G) \setminus E(C)$ with no end node in $V(C)$ such that $e_f \times f$ and $e_g \times g$. If $e_f \| g$ and $e_g \| f$, then G is not clean. Thus, we may assume $e_f \times g$. As C is odd, not all edges on C can cross e_f, so there exists an $h \| e_f$, but then, as f and g are distant in the pad $G \setminus (K(G) \setminus E(C))$, the chords e_f, f, g, and h form a mesh.

Let $c_0, \ldots, c_{2k}$ be the nodes of C, numbered in the order in which they lie around $R(G)$. From (26) it then follows that the edges of C are $c_i c_{i+k}$ (indices modulo $2k+1$). Let P_i be the $c_i c_{i+1}$-path on $R(G)$ that contains no nodes of C other than c_i and c_{i+1}, see Figure 14a. Let $K_i := \{uv \in K(G) \setminus E(C) | u \in V(P_i)\}$; note that for each $i = 0, \ldots, 2k$, $K_i \neq \emptyset$. For each $e \in K(G) \setminus E(C)$ let C_e be the odd circuit in $R(G) \cup \{e\}$ that uses the fewest nodes of $V(C)$.

(27) $\qquad$ If $e, f \in K_i$, then $V(C_e) \cap \{c_i, c_{i+1}\} = V(C_f) \cap \{c_i, c_{i+1}\}$.

Indeed, if not, $c_ic_{i+k+1}, c_{i+1}c_{i+k+1}, e$, and f show that G is nonnesting or has a mesh, and hence is not clean.

From (27), it is easy to see that there exists an $i = 0, \ldots, 2k$, such that $V(C_e) \ni c_i$ for all $e \in K_i \cup K_{i-1}$. By circular symmetry, we may assume that $k+1$ is such an i. Let $f \in K_0$. By the symmetry $i \leftrightarrow 2k+2-i \pmod{2k+1}$, we may assume that $c_1 \in V(C_f)$; hence $f \| c_0c_{k+1}$ and $f \times c_1c_{k+2}$. Let $e \in K_{k+1}$. Then $e \| c_1c_{k+2}$ and $e \times c_0c_{k+1}$. Hence $c_0c_{k+1}, c_1c_{k+2}, e$, and f form a mesh (see Figure 14b). □

For each $C \in \mathcal{C}$, we denote the unique edge in $E(C) \cap K(\widetilde{G})$ by $k[C]$. Our next task is to study the structure of the collection of tight edges and odd circuits as a whole. The outcome will be summarized in (35), (36), and (37); for proving those we need to derive some claims. We define, for each $\ell = 0, 1, \ldots$, $K_\ell := \{e \in K(\widetilde{G}) | e \text{ is in } \ell \text{ tight odd circuits}\}$, $K_\ell^{\text{tight}} := K_\ell \cap \mathcal{T}$, and $K_\ell^{\text{free}} := K_\ell \setminus \mathcal{T}$. By Lemma 13, $K_\ell = \emptyset$ for $\ell \geq 3$. Moreover,

(28) $\qquad K_0^{\text{free}} = \emptyset,$

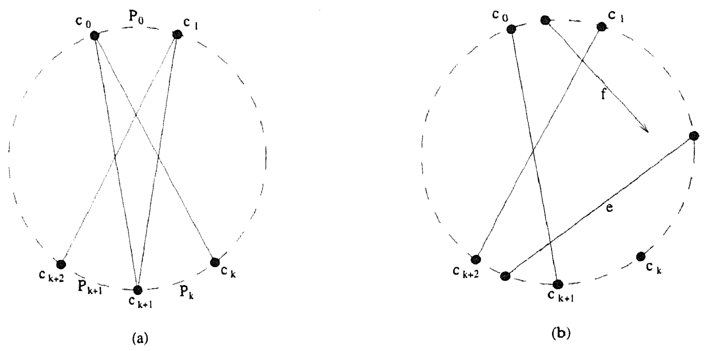

FIG. 15. *Dashed curves indicate internally node disjoint even paths of positive length. The closed curve on the outside is the rim.*

as deleting an edge in K_0^{free} form $\widetilde{G}$ would yield a smaller non–t-perfect graph.

CLAIM 21. *If $uv \in K(\widetilde{G}) \cap \mathcal{T}$, then uv is the only chord with end node u.*

Proof of Claim 21. Let uw be a second chord. Let $P := vv' \ldots w'w$ be the vw-path on $R(\widetilde{G})$ not containing u. If there exists a tight odd circuit using both v and w, then by Claim 16, there exists a tight odd circuit using vu and uw, but this contradicts Claim 17 or Lemma 13. Let $C_w \in \mathcal{C}(w')$ and $C_v \in \mathcal{C}(v')$. By Claim 17, $u \notin V(C_w) \cup V(C_v)$, so $k[C_w]$ crosses uw and $k[C_v]$ crosses uv. Hence, $uw, uv, k[C_w]$, and $k[C_v]$ show that $\widetilde{G}$ has a mesh or is nonnesting—a contradiction! □

CLAIM 22. *If $uv \in K(\widetilde{G}) \cap \mathcal{T}$, then uv is not a universal chord of $\widetilde{G}$.*

Proof of Claim 22. Suppose that the claim is false. We construct a new graph G from $\widetilde{G}$ as follows. For each neighbor w of u, we introduce a new node w^* and two new edges uw^* and w^*w and remove the original edge uw. Moreover, we define $x \in \mathbb{R}^{V(G)}$ by $x_w := \widetilde{x}_w$ if $w \in V(\widetilde{G}) \setminus \{u\}$, $x_{w^*} := \widetilde{x}_u$ if w is a neighbor of u in $\widetilde{G}$, and $x_u := 1 - \widetilde{x}_u$. Then, by (14), x is a vertex of $P(G)$.

Let G' be obtained from G by contracting uv^* and v^*v into one new node, called v again. As $x_u + x_{v^*} = 1 = x_{v^*} + x_v$, we get from (14) that G' is not t-perfect. On the other hand, as uv is universal in $\widetilde{G}$, each odd circuit in G' goes through v. However, Fonlupt and Uhry [5] have proved that graphs containing a node that lies on each odd circuit are t-perfect—a contradiction. □

As tight odd circuits have no chords, we have by Claim 21 and (28) that

$$|\delta(u) \cap K(\widetilde{G})| \leq 2 \text{ for all } u \in V(\widetilde{G}) \tag{29}$$

and

$$\text{if } e \in K_2, \text{ then all other chords cross } e. \tag{30}$$

By (30) and Claim 22,

$$K_2^{\text{tight}} = \emptyset. \tag{31}$$

For each $e \in K(\widetilde{G})$ define y_e to be the total number of tight odd circuits and edges containing e. From Claims 21 and 22 and by (29) and (30), we see that

$$\sum_{e \in \delta(u) \cap K(\widetilde{G})} y_e \leq 2 \quad \text{for each } u \in U(\widetilde{G}). \tag{32}$$

Moreover, by (19),

$$|\mathcal{T}(u)| \leq 1 \text{ for each } u \in W(\widetilde{G}), \tag{33}$$

and thus, by (17),

$$\begin{aligned} |V(\widetilde{G})| &= |U(\widetilde{G})| + |W(\widetilde{G})| \\ &\geq \frac{1}{2}\sum_{u\in U(\widetilde{G})}\ \sum_{e\in\delta(u)\cap K(\widetilde{G})} y_e + \sum_{u\in W(\widetilde{G})} |\mathcal{T}(u)| \\ &= \sum_{e\in K(\widetilde{G})} y_e + \sum_{u\in W(\widetilde{G})} |\mathcal{T}(u)| \\ &= |\mathcal{C}| + |\mathcal{T}| \\ &\geq |V(\widetilde{G})|. \end{aligned} \tag{34}$$

Thus, we have equality throughout, which implies that we have equality in (32) and (33). So we get

$$|\mathcal{T}(u)| = 1 \text{ for each } u \in W(\widetilde{G}); \tag{35}$$

(36) each chord in $K_1^{\text{tight}} \cup K_2^{\text{free}}$ is node disjoint from all other chords; moreover, the edges in K_1^{free} form node disjoint circuits;

and, by Claim 22,

$$K_0^{\text{tight}} = \emptyset. \tag{37}$$

As $W(\widetilde{G})$ is a stable set, by (17), there exists an equation in (17) that does not hold for $\chi_{W(\widetilde{G})}$. Case checking yields that this means that

$$K_1^{\text{tight}} \neq \emptyset. \tag{38}$$

CLAIM 23. $K_1^{\text{free}} \neq \emptyset$.

Proof of Claim 23. Suppose that $K_1^{\text{free}} = \emptyset$. Then, by (36), no two chords touch. By (38), there exists at least one tight chord, so, by Claim 22, there exists a pair of parallel chords. Choose $e, f \in K(\widetilde{G})$ parallel, such that the shortest path—Q, say—on $R(\widetilde{G})$ that connects an end node of e with an end node of f is as short as possible. Let $C_e \in \mathcal{C}(e)$ and $C_f \in \mathcal{C}(f)$ (as e and f are parallel, these two odd circuits are unique). Let $u \in W(G) \cap V(Q)$ and $C \in \mathcal{C}(u)$. (C exists by (18) and (35).) As $u \notin V(C_e) \cup V(C_f)$, $e, f \neq k[C]$, and by the choice of e and f, $k[C]$ is not parallel to e nor to f. Therefore, as there are no touching chords, $k[C]$ crosses both e and f. As $k[C]$ is tight, there exists an edge $h \| k[C]$. As h is not a chord of C_e nor of C_f, $k[C], e, f$, and h form a mesh—a contradiction! □

For each $e \in K_1$ let $C[e]$ be the unique tight odd circuit using e.

CLAIM 24. *K_1^{free} contains no pair of parallel chords.*

Proof of Claim 24. Let $f_1 = u_1u_2$ and $f_2 = v_1v_2$ be two parallel chords in K_1^{free}; see Figure 15. Let P be the u_2v_1-path on $R(\widetilde{G})$ containing v_2. By symmetry we may assume that P is nesting. Let u_2w be the second edge in K_1^{free} incident with u_2. As

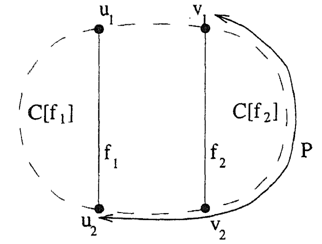

FIG. 16.

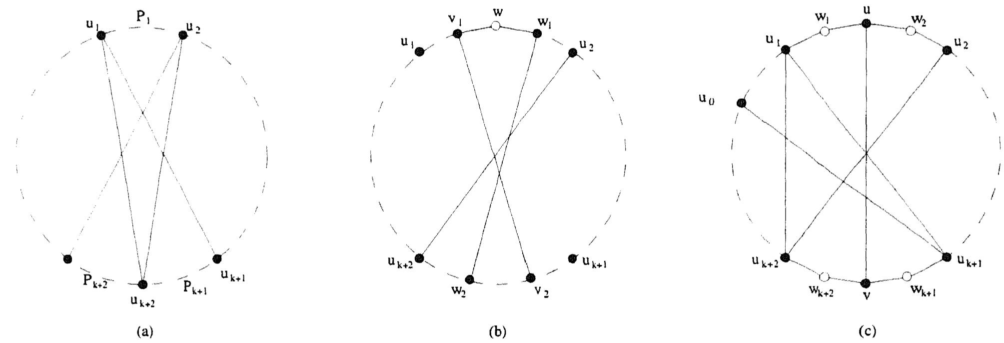

FIG. 17. *Dashed curves indicate internally node disjoint even paths of positive length. The closed curve on the outside is the rim.*

P is nesting, $w \notin P \setminus \{v_1\}$, but then either u_1u_2 or v_1v_2 is a chord of $C[u_2w]$, or u_2w is a chord of $C[u_1u_2]$—a contradiction! □

Let Γ be a circuit in K_1^{free}. Let $u_0, \dots, u_N$ be the nodes of Γ in the order in which they lie around $R(\widetilde{G})$. From Claim 24, it follows that N is even ($2k$, say) and that the edges in Γ are of the form u_iu_{i+k+1} (indices modulo $2k+1$); see Figure 17a. All chords not in Γ are parallel with at least one edge in Γ. Thus, by (28), (30), and Claim 24, we have that

(39) $\quad K_2 = \emptyset$ and $K_1^{\text{free}} = E(\Gamma)$.

For $i = 0, \dots, 2k$, let P_i be the u_iu_{i+1}-path on $R(\widetilde{G})$ that is internally node disjoint from Γ. By (38), there exists an edge uv in K_1^{tight}. By symmetry we may assume that $u \in P_1$ and $v \in P_1 \cup \dots \cup P_{k+1}$. As $C[u_1u_{k+1}]$ has no chords, we have that

(40) $$v \in P_{k+1}.$$

CLAIM 25. *Each chord in K_1^{tight} has one end node in P_1 and one in P_{k+1}.*

Proof of Claim 25. Let $xy \in K_1^{\text{tight}} \setminus \{uv\}$. As we proved for uv, we may assume that $x \in P_i$ and $y \in P_{i+k}$. Hence, $uv \| u_1u_{k+2}$ and $xy \| u_iu_{k+i+1}$. If i were different from 1, then xy, uv,u_1u_{k+2} and u_iu_{k+i+1} would form a mesh or show that $\widetilde{G}$ is nonnesting. Hence $i = 1$ and the claim follows. □

CLAIM 26. $|K_1^{\text{tight}}| = 1$.

Proof of Claim 26. Suppose not; then there are chords v_1v_2 and w_1w_2 in K_1^{tight}, such that u_1 and w_1 are both on P_1 and share a common neighbor w on P_1, see Figure 17b. From (35) we may assume that $v_1w \in \mathcal{T}$, but now the path $v_2v_1ww_1w_2$

and the circuit $C[u_2u_{k+2}]$ satisfy the assumptions in Claim 16. Hence there exists a tight odd circuit using both v_1v_2 and w_1w_2, contradicting Claim 17 or Lemma 13. $\square$

Hence P_1 and P_{k+1} are paths of length 4. Let $P_1 = u_1w_1uw_2u_2$ and $P_2 = u_{k+1}w_{k+1}vw_{k+2}u_{k+2}$; see Figure 17c.

We have that $w_2u_2 \notin \mathcal{T}$, as otherwise the path $vuw_2u_2u_{k+2}$ and the circuit $C[u_0u_{k+1}]$ would satisfy the assumptions of Claim 16 and thus yield a tight odd circuit using three chords of $\widetilde{G}$. By symmetry also $u_1w_1 \notin \mathcal{T}$. Hence, by (35), $uw_1, uw_2 \in \mathcal{T}$. But as $uv \in \mathcal{T}$ this contradicts (19). This completes the proof of Case 3 and thus of Theorem 1. $\square$

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