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ABSTRACT
In this paper we consider a single-server queue with Lévy input, and in particular its workload process (Q(t)), for t > 0, with a focus on the correlation structure. With the correlation function defined as r(t) := Cov(Q(0),Q(t))/Var Q(0) (assuming that the workload process is in stationarity at time 0), we first determine its transform. This expression allows us to prove that r(.) is positive, decreasing, and convex, relying on the machinery of completely monotone functions. We also show that r(.) can be represented as the complementary distribution function of a specific random variable. These results are used to compute the asymptotics of r(t), for t large, for the cases of light-tailed and heavy-tailed Lévy input.

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On the correlation structure of a Lévy-driven queue

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Abstract

In this paper we consider a single-server queue with Lévy input, and in particular its workload process $(Q_t)_{t\geq 0}$, with a focus on the correlation structure. With the correlation function defined as $r(t) := \mathbb{C}\mathrm{ov}(Q_0, Q_t)/\mathbb{V}\mathrm{ar}\, Q_0$ (assuming that the workload process is in stationarity at time 0), we first determine its transform $\int_0^\infty r(t)e^{-\vartheta t}\mathrm{d}t$. This expression allows us to prove that $r(\cdot)$ is positive, decreasing, and convex, relying on the machinery of completely monotone functions. We also show that $r(\cdot)$ can be represented as the complementary distribution function of a specific random variable. These results are used to compute the asymptotics of $r(t)$, for t large, for the cases of light-tailed and heavy-tailed Lévy input.

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1 Introduction

Consider a queueing system, and, more particularly, its workload process $(Q_t)_{t\geq 0}$. Where one usually focuses on the characterization of the (transient or steady-state) workload, another interesting problem relates to the identification of the *correlation function* $r(t) := \mathbb{C}\mathrm{ov}(Q_0, Q_t)/\mathbb{V}\mathrm{ar}\, Q_0$. For several queueing systems this correlation function has been explicitly computed; Morse [16], for instance, analyzes the number of customers in the M/M/1 queue. Often explicit formulae were hard to obtain, but the analysis simplified greatly when looking at the transform

$$\rho(\vartheta) := \int_0^\infty r(t)e^{-\vartheta t}\mathrm{d}t.$$

Beneš [6] managed to compute $\rho(\cdot)$ for the workload in the M/G/1 queue; relying on the concept of complete monotonicity, Ott [17] elegantly proved that, in this case, $r(\cdot)$ is positive, decreasing and convex. We further mention the survey by Reynolds [18], and interesting results by Abate and Whitt [2].

The primary aim of this paper is to extend the results mentioned above to the class of single-server queues fed by *Lévy processes*. Notice that the M/G/1 queue is contained in this class (then the Lévy process under consideration is a compound Poisson process with drift). One could expect that such an extension is possible, as the classical Pollaczek-Khinchine result for the M/G/1 queue, carries over to queues with general Lévy input, see Zolotarev [20] for an early reference. The only condition one usually needs to impose, is that no negative jumps are allowed. In more detail, the setting we consider is the following. We define a 'net input process' $(X_t)_{t\geq 0}$, which is assumed to be a Lévy process with no negative jumps. Then the workload process $(Q_t)_{t\geq 0}$ is defined as the reflected process of $(X_t)_{t\geq 0}$ at 0. Because of the lack of explicit formulae for the probability distributions of the processes considered, we will work most of the time with their Laplace transforms; in our analysis the Laplace exponent $\varphi(\cdot)$ of the process $(X_t)_{t\geq 0}$, as well as its inverse $\psi(\cdot)$, play an important role.
We first obtain an explicit expression, in terms of $\varphi(\cdot)$ and $\psi(\cdot)$, of the transform $\rho(\cdot)$ of the correlation function. Using the concept of complete monotonicity, we use this transform to establish a series of structural properties of the correlation function, viz. we prove that $r(\cdot)$ is positive, decreasing, and convex. These results indeed generalize those obtained by Ott [17] and Abate and Whitt [2] for the M/G/1 queue. We then consider the asymptotic behavior of $r(t)$ for t large. For light-tailed Lévy input these asymptotics are essentially exponential; for the M/G/1 case they resemble those of the busy period. For heavy-tailed input we can use results for regularly varying functions, e.g. Karamata's Tauberian theorem, to obtain the asymptotics of $r(\cdot)$.

This paper is organized as follows. In Section 2 we obtain the Laplace transform of the correlation function, where in Section 3 its structural properties are studied. The cases of light-tailed and heavy-tailed input are treated in Sections 4 and 5, respectively. Concluding remarks are found in Section 6.

2 Laplace transform of the correlation function

In this section we find an expression for the transform $\rho(\cdot)$ of the correlation function. We start this section, however, with a formal introduction of our queueing system.

Lévy processes. Let $(X_t)_{t\geq 0}$ be a Lévy process without negative jumps, with drift $\mathbb{E}X_1 < 0$. Its Laplace exponent is given by the function $\varphi(\cdot) : [0,\infty) \mapsto [0,\infty)$, i.e., $\varphi(\alpha) := \log \mathbb{E}e^{-\alpha X_1}$. It is known that $\varphi(\cdot)$ is increasing and convex on $[0,\infty)$, with slope $\varphi'(0) = -\mathbb{E}X_1$ in the origin. Therefore the inverse $\psi(\cdot)$ of $\varphi(\cdot)$ is well-defined on $[0,\infty)$. In the sequel we also require that X_t is not a *subordinator*, i.e., a monotone process; thus X_1 has probability mass on the positive half-line, which implies that $\lim_{\alpha\to-\infty}\varphi(\alpha) = \infty$.
Important examples of such Lévy processes are the following. (1) *Brownian motion with drift*. We write $X \in \mathbb{B}\mathrm{m}(\mu,\sigma^2)$ when $\varphi(\alpha) = -\alpha\mu + \frac{1}{2}\alpha^2\sigma^2$. (2) *Compound Poisson with drift*. Jobs arrive according to a Poisson process of rate λ; the jobs $B_1, B_2, \ldots$ are i.i.d. samples from a distribution with Laplace transform $\beta(\alpha) := \mathbb{E}e^{-\alpha B}$; the storage system is continuously depleted at a rate -1. We write $X \in \mathbb{C}\mathrm{P}(\lambda, \beta(\cdot))$; it can be verified that $\varphi(\alpha) = \alpha - \lambda + \lambda\beta(\alpha)$.

Reflected Lévy processes; queues. We consider the reflection of $(X_t)_{t\geq 0}$ at 0, which we denote by $(Q_t)_{t\geq 0}$. It is formally introduced as follows, see for instance [4, Ch. IX]. Define the increasing process $(L_t)_{t\geq 0}$ by

$$L_t = -\inf_{0\leq s\leq t} X_s.$$

Then the reflected process (or: workload process, queueing process) $(Q_t)_{t\geq 0}$ is given through

$$Q_t := X_t + \max\{L_t, Q_0\};$$

observe that $Q_t \geq 0$ for all $t \geq 0$. Then the steady-state distribution of $Q := \lim_{t\to\infty} Q_t$ is characterized by [20]:

$$\kappa(\alpha) := \mathbb{E}e^{-\alpha Q} = \frac{\alpha\varphi'(0)}{\varphi(\alpha)}; \tag{1}$$

for the special case of $\mathbb{C}\mathrm{P}$ input this is the celebrated Pollaczek-Khinchine formula. This relation reveals all moments of the steady-state queue Q, and in particular its mean and variance:

$$\mu := \mathbb{E}Q = -\frac{\mathrm{d}}{\mathrm{d}\alpha}\left.\frac{\alpha\varphi'(0)}{\varphi(\alpha)}\right|_{\alpha\downarrow 0} = \frac{\varphi''(0)}{2\varphi'(0)}, \tag{2}$$

and similarly

$$v := \mathbb{V}\mathrm{ar}Q = \frac{1}{4}\left(\frac{\varphi''(0)}{\varphi'(0)}\right)^2 - \frac{1}{3}\frac{\varphi'''(0)}{\varphi'(0)}, \tag{3}$$

which from now on are assumed to be finite.

Correlation structure of the queue. In this paper we are interested in the correlation structure of the queue process $(Q_t)_{t\geq 0}$. Our analysis relies on the following useful relation, see e.g. [4, Section IX.3] and [13]:

$$\mathbb{E}(e^{-\alpha Q_T} \mid Q_0 = q) = \frac{\vartheta}{\vartheta - \varphi(\alpha)} \left(e^{-\alpha q} - \alpha \cdot \frac{e^{-\psi(\vartheta)q}}{\psi(\vartheta)} \right), \tag{4}$$

where T is exponentially distributed with mean ϑ^{-1}, independently of the Lévy process. (As an aside we mention that (4) implies Pollaczek-Khinchine in at least two ways: (a) let $\vartheta \downarrow 0$, so that T corresponds with some epoch infinitely far away, and use elementary calculus ('L'Hôpital'); (b) find $\mathbb{E}e^{-\alpha Q_T}$ by deconditioning, use that in stationarity $\mathbb{E}e^{-\alpha Q_T}$ should coincide with $\mathbb{E}e^{-\alpha Q_0}$, and then solve $\mathbb{E}e^{-\alpha Q_0}$.)

Formula (4) enables us to find explicitly the Laplace transform $\rho(\cdot)$ of

$$r(t) := \mathbb{C}\text{orr}(Q_0, Q_t) = \frac{\mathbb{C}\text{ov}(Q_0, Q_t)}{\sqrt{\mathbb{V}\text{ar}Q_0 \cdot \mathbb{V}\text{ar}Q_t}} = \frac{\mathbb{E}(Q_0 Q_t) - (\mathbb{E}Q_0)^2}{\mathbb{V}\text{ar}Q_0},$$

as we show now. Here it is assumed that the system is in steady-state at time 0, that is, Q_0 obeys the 'generalized Pollaczek-Khinchine' formula (1). First realize that

$$\mathbb{E}(e^{-\alpha Q_T} \mid Q_0 = q) = \int_0^\infty \vartheta e^{-\vartheta t} \mathbb{E}(e^{-\alpha Q_t} \mid Q_0 = q) \mathrm{d}t.$$

By differentiation with respect to α and subsequently letting $\alpha \downarrow 0$, we obtain

$$\int_0^\infty \vartheta e^{-\vartheta t} \mathbb{E}(Q_t \mid Q_0 = q) \mathrm{d}t = -\frac{\varphi'(0)}{\vartheta} + q + \frac{e^{-\psi(\vartheta)q}}{\psi(\vartheta)}. \tag{5}$$

Concentrate on the Laplace transform $\gamma(\vartheta)$ of $\mathbb{C}\text{ov}(Q_0, Q_t)$. Straightforward calculus reveals that

$$\begin{aligned}\gamma(\vartheta) &:= \int_0^\infty \mathbb{C}\text{ov}(Q_0, Q_t) e^{-\vartheta t} \mathrm{d}t = \int_0^\infty (\mathbb{E}(Q_0 Q_t) - \mu^2) e^{-\vartheta t} \mathrm{d}t \\ &= \int_0^\infty \int_0^\infty q \cdot \mathbb{E}(Q_t \mid Q_0 = q) \cdot e^{-\vartheta t} \mathrm{d}\mathbb{P}(Q_0 \leq q) \mathrm{d}t - \frac{\mu^2}{\vartheta};\end{aligned}$$

it is assumed that the queue is in stationarity at time 0 (and hence it is in stationarity at time t as well). By invoking (5) we find that the expression in the previous display equals

$$\begin{aligned}&\int_0^\infty \frac{q}{\vartheta} \left(-\frac{\varphi'(0)}{\vartheta} + q + \frac{e^{-\psi(\vartheta)q}}{\psi(\vartheta)} \right) \mathrm{d}\mathbb{P}(Q_0 \leq q) - \frac{\mu^2}{\vartheta} \\ &= -\frac{\mu\varphi'(0)}{\vartheta^2} + \frac{v}{\vartheta} + \frac{1}{\vartheta\psi(\vartheta)} \mathbb{E}(Q_0 e^{-\psi(\vartheta)Q_0}).\end{aligned} \tag{6}$$

From the generalized Pollaczek-Khinchine formula (1) we obtain by differentiating

$$\mathbb{E}(Q_0 e^{-\alpha Q_0}) = \varphi'(0) \left(-\frac{1}{\varphi(\alpha)} + \alpha \frac{\varphi'(\alpha)}{(\varphi(\alpha))^2} \right).$$

Inserting this relation, in addition to (2) and (3), into (6) we obtain the Laplace transform of $\mathbb{C}\mathrm{ov}(Q_0 Q_t)$:

$$\gamma(\vartheta) = -\frac{\varphi''(0)}{2\vartheta^2} + \frac{v}{\vartheta} + \frac{\varphi'(0)}{\vartheta^2}\left(\frac{1}{\vartheta\psi'(\vartheta)} - \frac{1}{\psi(\vartheta)}\right).$$

This trivially also provides us with the Laplace transform of $\mathbb{C}\mathrm{orr}(Q_0, Q_t)$, as stated in the following theorem. When specializing to $\mathbb{C}$P input, we retrieve Eqn. (6.2) of Beneš [6].

Theorem 2.1. *For any $\vartheta \geq 0$, and v as in* (3)*,*

$$\begin{aligned} \rho(\vartheta) &:= \int_0^\infty r(t)\, e^{-\vartheta t}\mathrm{d}t = \frac{\gamma(\vartheta)}{v} \\ &= \frac{1}{\theta} - \frac{\varphi''(0)}{2v\vartheta^2} + \frac{\varphi'(0)}{v\vartheta^2}\left[\frac{1}{\vartheta\psi'(\vartheta)} - \frac{1}{\psi(\vartheta)}\right]. \end{aligned} \tag{7}$$

Remark 2.2. Using the generalized Pollaczek-Khinchine formula (1), it is readily verified that the result in Thm. 2.1 can be simplified to

$$\rho(\vartheta) = \frac{1}{\theta} - \frac{1}{v}\left(\frac{\varphi''(0)}{2\vartheta^2} + \frac{\kappa'(\psi(\vartheta))}{\vartheta\psi(\vartheta)}\right). \qquad \diamondsuit$$

Example 2.3. Consider the situation that $(X_t)_{t\geq 0}$ corresponds to standard Brownian motion decreased by a linear drift (say of rate 1, so $X \in \mathbb{B}\mathrm{m}(-1,1)$). In other words: the Laplace exponent of the Lévy process is given by $\varphi(\alpha) = \alpha + \frac{1}{2}\alpha^2$, and its inverse is $\psi(\vartheta) = -1 + \sqrt{1+2\vartheta}$. Now consider the workload process $(Q_t)_{t\geq 0}$ and its correlation function. The above theory yields that the Laplace transform of $r(\cdot)$ is given by

$$\rho(\vartheta) = \frac{1}{\vartheta} - \frac{2}{\vartheta^2} + \frac{2}{\vartheta^3}\left(\sqrt{1+2\vartheta} - 1\right).$$

It turns out to be possible to explicitly invert $\rho(\cdot)$:

$$r(t) = 2(1 - 2t - t^2)\left(1 - \Phi_{\mathrm{N}}(\sqrt{t})\right) + 2\sqrt{t}(1+t)\phi_{\mathrm{N}}(\sqrt{t}), \tag{8}$$

with $\Phi_{\mathrm{N}}(\cdot)$ (resp. $\phi_{\mathrm{N}}(\cdot)$) the standard Normal distribution (resp. density). Eqn. (8) is in agreement with the results in [1] and [14, Section 12.1]. $\diamondsuit$

3 Structural properties of the correlation function

This section concentrates on the derivation of a number of key structural properties of the correlation function $r(\cdot)$. More specifically, relying on the concept of completely monotonous functions [7, 17], we prove in Thm. 3.6 that $r(\cdot)$ is a positive, decreasing, and convex function. To this end, we first establish a number of auxiliary results; a key result is Prop. 3.1.

Proposition 3.1. *Define $\xi(\vartheta)$ by*

$$\xi(\vartheta) := \frac{1}{\mu}\left(\frac{1}{\vartheta\psi'(\vartheta)} - \frac{1}{\psi(\vartheta)}\right); \tag{9}$$

then $\xi(\vartheta)$ is the Laplace transform of a (non-negative) random variable Z.

Remark 3.2. The Laplace transform of the stationary-excess distribution Z_e associated with Z is given by [2]

$$\xi_e(\vartheta) = \frac{\xi(\vartheta) - 1}{\vartheta\xi'(0)} = \frac{\varphi''(0)}{2v\vartheta}\Big(1 - \xi(\vartheta)\Big). \tag{10}$$

Hence, the first moment of Z is $2v/\varphi''(0)$. $\diamondsuit$

To prove Prop. 3.1, we need a number of lemmas. These are stated and proved now. They extensively use the concept of *complete monotonicity* [7, 12]. The class $\mathscr{C}$ of completely monotone functions is defined in the Appendix, where also a series of standard properties is given.

Lemma 3.3. $\psi'(\vartheta) \in \mathscr{C}$.

Proof. Consider for $x \geq 0$,

$$T_x := \inf\{t \geq 0 : X_t = -x\};$$

then T_x is a Lévy process with Laplace exponent $-\psi(\vartheta)$, see e.g. [19, Thm. 46.3]. More specifically, T_x is a subordinator. Now apply Lemma A.4. $\square$

Lemma 3.4. *If $f(\alpha) \in \mathscr{C}$, then so does*

$$\frac{f(0) - f(\alpha) + \alpha f'(\alpha)}{\alpha^2}.$$

Proof. This is a consequence of subsequently applying Lemma A.3.(4) and A.3.(5). $\square$

Lemma 3.5. *For $\sigma^2 > 0$ and measure $\Pi_\varphi(\cdot)$ such that $\int_{(0,\infty)} \min\{1, x^2\}\Pi_\varphi(\mathrm{d}x) < \infty$,*

$$\frac{\alpha\varphi'(\alpha) - \varphi(\alpha)}{\alpha^2} = \frac{1}{2}\sigma^2 + \frac{1}{\alpha^2}\int_{(0,\infty)} (1 - e^{-\alpha x} - \alpha x e^{-\alpha x})\Pi_\varphi(\mathrm{d}x) \in \mathscr{C}. \tag{11}$$

Proof. The Laplace exponent $\varphi(\alpha)$ can be written as, with $\sigma^2 > 0$ and measure $\Pi_\varphi(\cdot)$ such that $\int_{(0,\infty)} \min\{1, x^2\}\Pi_\varphi(\mathrm{d}x) < \infty$,

$$\varphi(\alpha) = -\alpha\mu + \frac{1}{2}\alpha^2\sigma^2 + \int_{(0,\infty)} (e^{-\alpha x} - 1 + \alpha x 1_{(0,1)})\Pi_\varphi(\mathrm{d}x),$$

which immediately yields the equality in (11). The claim that this function is in $\mathscr{C}$ follows from the fact that any positive constant is in $\mathscr{C}$, Lemma 3.4, and Lemma A.3.(1). $\square$

Proof of Prop. 3.1. We first decompose

$$\frac{1}{\vartheta\psi'(\vartheta)} - \frac{1}{\psi(\vartheta)} = \eta_1(\vartheta)\eta_2(\vartheta),$$

with

$$\eta_1(\vartheta) := \frac{\psi(\vartheta)}{\vartheta}, \quad \eta_2(\vartheta) := \frac{1}{\psi(\vartheta)\psi'(\vartheta)} - \frac{\vartheta}{(\psi(\vartheta)^2}.$$

Because of (the generalized version of) Pollaczek-Khinchine (1), we have that $\alpha/\varphi(\alpha) \in \mathscr{C}$; now applying Lemma A.3.(3), in conjunction with Lemma 3.3, we obtain that $\eta_1(\vartheta) \in \mathscr{C}$.
To prove that also $\eta_2(\vartheta) \in \mathscr{C}$, we first recall from Lemma 3.5 that $(\alpha\varphi'(\alpha) - \varphi(\alpha))/\alpha^2 \in \mathscr{C}$. Again applying Lemma A.3.(3), in conjunction with Lemma 3.3, it follows that $\eta_2(\vartheta) \in \mathscr{C}$.
As both $\eta_1(\vartheta)$ and $\eta_2(\vartheta)$ are in $\mathscr{C}$, Lemma A.3.(2) yields that $\xi(\vartheta) \in \mathscr{C}$. Applying 'L'Hôpital' twice, and using that $\psi''(0)(\varphi'(0))^3 = -\varphi''(0)$, it is readily verified that

$$\xi(0) = \lim_{\vartheta \downarrow 0} \xi(\vartheta) = 1,$$

Now Thm. A.2 yields the stated. □

Let $\rho^{(1)}(\vartheta)$ and $\rho^{(2)}(\vartheta))$ be the Laplace transforms of $r'(t) := (\mathrm{d}/\mathrm{d}t)r(t)$ and $r''(t) := (\mathrm{d}^2/\mathrm{d}t^2)r(t)$. Their expressions are given respectively as follows

$$\rho^{(1)}(\vartheta) \;:=\; \int_0^\infty r'(t)\, e^{-\vartheta t}\mathrm{d}t = -\frac{\varphi''(0)}{2v\vartheta}\Big(1 - \xi(\vartheta)\Big) = -\xi_e(\vartheta); \tag{12}$$

$$\rho^{(2)}(\vartheta) \;:=\; \int_0^\infty r''(t)\, e^{-\vartheta t}\mathrm{d}t = \frac{\varphi''(0)}{2v}\xi(\vartheta), \tag{13}$$

for $\vartheta \geq 0$. Here the properties that $r(0) = 1$ and

$$r'(0) = \lim_{\varepsilon\downarrow 0} \frac{\mathbb{E}(Q_0 Q_\varepsilon) - (\mathbb{E}Q_0)^2}{\varepsilon \mathbb{V}\mathrm{ar} Q_0} = \lim_{\varepsilon\downarrow 0} \frac{\mathbb{E}(Q_0 X_\varepsilon)}{\varepsilon \mathbb{V}\mathrm{ar} Q_0} = -\frac{\varphi''(0)}{2v},$$

in conjunction with integration by parts, are used.

Theorem 3.6. *$r(t)$ is positive, decreasing and convex. Furthermore $r(t)$ can be written as the tail of the stationary-excess distribution function associated with Z, i.e, $r(t) = \mathbb{P}(Z_e > t)$. If Z has a finite second moment, then $r(t)$ is integrable and*

$$\int_0^\infty r(t)\mathrm{d}t = \frac{1}{4v}\frac{\varphi^{(4)}(0)}{\varphi'(0)^2} - \frac{5}{6v}\frac{\varphi''(0)\varphi^{(3)}(0)}{\varphi'(0)^3} + \frac{1}{2v}\frac{\varphi''(0)^3}{\varphi'(0)^4} \tag{14}$$

Proof. Convexity follows from the expression for $\rho^{(2)}(\vartheta)$ in (13); it is concluded from Prop. 3.1 that $\rho^{(2)}(\vartheta) \in \mathscr{C}$, thus $r''(t)$ is non-negative (for $t \geq 0$). The monotonicity follows from the expression for $\rho^{(1)}(\vartheta)$ in Eqn. (12), by applying Lemma A.3.(4) to $\rho^{(2)}(\vartheta) \in \mathscr{C}$; we find that $-\rho^{(1)}(\vartheta)$ is in $\mathscr{C}$, implying that $r'(t) \leq 0$ (for $t \geq 0$). Then it is easily verified that applying Lemma A.3.(4) to $-\rho^{(1)}(\vartheta) \in \mathscr{C}$, in conjunction with Eqn. (7), implies $\rho(\vartheta) \in \mathscr{C}$, and hence $r(t) \geq 0$ (for $t \geq 0$).

Observe that combining Eqns. (7) and (10) yields

$$\rho(\vartheta)=\frac{1-\xi_e(\vartheta)}{\vartheta}. \tag{15}$$

It is straightforward to verify that the right-hand side of the previous display is just the Laplace transform of $\mathbb{P}(Z_e>t)$. It is concluded that $r(t)=\mathbb{P}(Z_e>t)$ by the uniqueness of the Laplace transform. Eqn. (14) is found, after considerable calculus (i.e., application of 'L'Hôpital' several times, and various series expansions), by evaluating

$$\int_0^\infty r(t)\mathrm{d}t=\rho(0)=\lim_{\vartheta\downarrow 0}\rho(\vartheta);$$

it is noted that $\varphi^{(4)}(0)$ exists if the second moment of Z is finite. □

4 Correlation asymptotics for light-tailed input

When $\varphi(\cdot)$ has an analytic continuation for $\alpha<0$, we are in the regime of light tails, as *a fortiori* then all moments $(-1)^n\varphi^{(n)}(0)$ of X_1 exist. When $(X_t)_{t\geq 0}$ does not correspond to a decreasing subordinator, we also have that $\lim_{\alpha\to-\infty}\varphi(\alpha)=\infty$. Bearing in mind the fact that $\varphi(\cdot)$ has a positive slope at the origin, and that convexity of $\varphi(\cdot)$ implies continuity, there is a unique minimizer $\zeta<0$ such that $\varphi(\zeta)<0$, $\varphi'(\zeta)=0$ and $\varphi''(\zeta)>0$.
In this situation, also $\psi(\cdot)$ is well-defined for negative arguments; more precisely: for all $\vartheta\geq\varphi(\zeta)$ the inverse $\psi(\vartheta)$ has a meaningful interpretation. In fact, $\vartheta^\star:=\varphi(\zeta)$ can be regarded as *branching point.* We thus see that Theorem 2.1 does not only apply for $\vartheta\geq 0$, but also for $\vartheta\in[\vartheta^\star,0)$. Around ζ, we can write $\varphi(\cdot)$ as

$$\varphi(\alpha)=\varphi(\zeta)+\frac{1}{2}(\alpha-\zeta)^2\varphi''(\zeta)+O((\alpha-\zeta)^3),$$

and hence for $\theta\downarrow\vartheta^\star$

$$\psi(\vartheta)-\zeta\sim\sqrt{\frac{2}{\varphi''(\zeta)}}\cdot\sqrt{\vartheta-\varphi(\zeta)}=\zeta+\sqrt{\frac{2}{\varphi''(\zeta)}}\cdot\sqrt{\vartheta-\vartheta^\star}$$

(where '$\sim$' indicates that the ratio of the left-hand side and right-hand side tends to 1). Routine calculations reveal that, for $\theta\downarrow\vartheta^\star$, we have that $\rho(\vartheta)$ looks like

$$\frac{1}{v}\left(-\frac{\varphi''(0)}{2(\vartheta^\star)^2}+\frac{1}{4\vartheta^\star}\left(\frac{\varphi''(0)}{\varphi'(0)}\right)^2-\frac{1}{3\vartheta^\star}\frac{\varphi'''(0)}{\varphi'(0)}-\frac{1}{(\vartheta^\star)^2}\frac{\varphi'(0)}{\psi(\vartheta)}+\frac{1}{(\vartheta^\star)^3}\frac{\varphi'(0)}{\psi'(\vartheta)}\right),$$

or, more precisely,

$$\begin{aligned}\rho(\vartheta)-\frac{1}{v}&\left(-\frac{\varphi''(0)}{2(\vartheta^\star)^2}+\frac{1}{4\vartheta^\star}\left(\frac{\varphi''(0)}{\varphi'(0)}\right)^2-\frac{1}{3\vartheta^\star}\frac{\varphi'''(0)}{\varphi'(0)}-\frac{1}{(\vartheta^\star)^2}\frac{\varphi'(0)}{\zeta}\right)\\&\sim\frac{\sqrt{2}\varphi'(0)}{\sqrt{\varphi''(\zeta)}v(\vartheta^\star)^2}\left(\frac{1}{\zeta^2}+\frac{\varphi''(\zeta)}{\vartheta^\star}\right)\sqrt{\vartheta-\vartheta^\star}.\end{aligned}$$

We now relate the behavior of a transform $\int_0^\infty e^{-\vartheta t} f(t) \mathrm{d}t$ (around a branching point $\vartheta^\star < 0$) to the behavior of the 'transformed' function $f(t)$ (for t large). We obtain the following result, cf. for instance the 'Heaviside approach' of [3, Eqns. (3.21)–(3.23)]; see also [11, pp. 153-154].

Proposition 4.1. *Suppose $\varphi(\alpha) < \infty$ for some $\alpha < 0$. Then*

$$r(t) \sim -\frac{\varphi'(0)}{\sqrt{2\pi\varphi''(\zeta)}v(\vartheta^\star)^2}\left(\frac{1}{\zeta^2}+\frac{\varphi''(\zeta)}{\vartheta^\star}\right)\frac{e^{\vartheta^\star t}}{t\sqrt{t}} \quad \textit{as } t\to\infty.$$

Example 4.2. It can be checked that for Brownian motion with drift, i.e., $X \in \mathbb{B}\mathrm{m}(-1,1)$ as in the setting of Example 2.3,

$$r(t) \sim 8\sqrt{\frac{2}{\pi}}\frac{e^{-t/2}}{t\sqrt{t}};$$

this could be found directly from (8) as well, cf. again [1] and [14, Section 12.1]. ♢

Example 4.3. For the compound Poisson model with exponential jobs (i.e., M/M/1 queue), it can be checked that

$$\psi(\vartheta) = \frac{1}{2}\left(\lambda - \mu + \vartheta + \sqrt{(\lambda-\mu+\vartheta)^2 + 4\vartheta\mu}\right),$$

so that the branching point is $\vartheta^\star = -(\sqrt{\mu}-\sqrt{\lambda})^2$. Also, $\zeta = -\mu + \sqrt{\lambda\mu}$. Prop. 4.1 now yields an explicit expression for the correlation asymptotics:

$$r(t) \sim \frac{1}{2\rho\sqrt{\pi}}\left(\frac{1-\sqrt{\rho}}{1+\sqrt{\rho}}\right)^3 \frac{\exp(-(1-\sqrt{\rho})\sqrt{\mu}t)}{(\sqrt{\mu}t)^{3/2}} \quad \text{as } t\to\infty.$$ ♢

Remark 4.4. For compound Poisson input, that is, $X \in \mathbb{C}\mathrm{P}(\lambda, \beta(\cdot))$, the tail asymptotics of the correlation function are proportional to those of the busy period, at least in this light-tailed regime (where light-tailedness here means that we should require that $\beta(\alpha) < \infty$ for some $\alpha < 0$). This can be seen as follows.

First recall that the Laplace exponent is $\varphi(\alpha) = \alpha - \lambda + \lambda\beta(\alpha)$. With $\pi(\cdot)$ the Laplace transform of the busy period, it is known that it satisfies $\pi(\vartheta) = \beta(\vartheta + \lambda - \lambda\pi(\vartheta))$. Therefore

$$0 = \beta(\vartheta+\lambda-\lambda\pi(\vartheta)) - \pi(\vartheta) = \frac{1}{\lambda}\varphi(\vartheta+\lambda-\lambda\pi(\vartheta)) - \frac{\vartheta}{\lambda},$$

and hence $\varphi(\vartheta+\lambda-\lambda\pi(\vartheta)) = \vartheta$. Apply $\psi(\cdot)$ to both sides, and we obtain

$$\pi(\vartheta) = \frac{\lambda+\vartheta}{\lambda} - \frac{1}{\lambda}\psi(\vartheta).$$

Considering the tail asymptotics of the busy period, first observe that $\pi(\cdot)$ also has a branching point at $\vartheta^\star < 0$ (i.e., it has the same branching point as $\rho(\vartheta)$), such that, for $\vartheta \downarrow \vartheta^\star$,

$$\pi(\vartheta) \sim \frac{\lambda-\vartheta}{\lambda} - \frac{1}{\lambda}\cdot\left(\zeta + \sqrt{\frac{2}{\varphi''(\zeta)}}\cdot\sqrt{\vartheta-\vartheta^\star}\right).$$

Applying 'Heaviside' now yields, with P the busy period,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(P \le t) \sim \frac{1}{\lambda}\sqrt{\frac{2}{\varphi''(\zeta)}} \cdot \frac{1}{2\sqrt{\pi}}\frac{e^{\vartheta^\star t}}{t\sqrt{t}} = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{1}{\beta''(\zeta)}}\frac{e^{\vartheta^\star t}}{\lambda t\sqrt{\lambda t}},$$

in line with the results of Cox and Smith [11, Section 5.6]. These asymptotics are indeed proportional to those of Proposition 4.1. Similarly, applying the relation

$$\mathbb{E}e^{-\vartheta P} = 1 - \vartheta\int_0^\infty \mathbb{P}(P > t)\mathrm{d}t, \tag{16}$$

we obtain

$$\mathbb{P}(P > t) \sim -\sqrt{\frac{2}{\varphi''(\zeta)}} \cdot \frac{1}{2\sqrt{\pi}}\frac{e^{\vartheta^\star t}}{\vartheta^\star\lambda t\sqrt{t}} = -\frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{1}{\beta''(\zeta)}}\frac{e^{\vartheta^\star t}}{\vartheta^\star\lambda t\sqrt{\lambda t}}. \qquad \diamondsuit$$

5 Correlation asymptotics for heavy-tailed input

Where the previous section focused on light-tailed Lévy input, we now consider the heavy-tailed case. We extensively use the concept of slowly (and regularly) varying functions. Prop. 5.4 is the main result of this section; in Corollary 5.5 it is applied to the situation of a queue with $\mathbb{C}$P input with regularly varying jobs.

The following class of functions plays a crucial role in our analysis.

Definition 5.1. We say that $f(x) \in \mathscr{R}_\delta(n,\sigma)$, with $n \in \mathbb{N}$, $\sigma \in \mathbb{R}$, and $\delta \in (n, n+1)$, for $x \downarrow 0$ if

$$f(x) = \sum_{i=0}^{n}\frac{f^{(i)}(0)}{i!}x^i + \sigma x^\delta L(1/x) \quad (x \downarrow 0),$$

for a slowly varying function L (i.e., $L(x)/L(tx) \to 1$ for $x \to \infty$, for any $t > 0$).

Lemma 5.2. *Suppose $\varphi'(\alpha) \in \mathscr{R}_{\delta-1}(n-1,\sigma)$. Then*

$$\varphi(\alpha) \in \mathscr{R}_\delta(n, \sigma/\delta);$$

$$\psi(\vartheta) \in \mathscr{R}_\delta(n,\tau), \quad \textit{with } \tau := -\frac{\sigma}{\delta(\varphi'(0))^{\delta+1}};$$

$$\psi'(\vartheta) \in \mathscr{R}_{\delta-1}(n-1, \tau\delta),$$

for $\alpha \downarrow 0$, resp. $\vartheta \downarrow 0$.

Proof. The first statement is an immediate consequence of Karamata's theorem; the second statement follows from $\psi(\varphi(\alpha)) = \alpha$; the third statement follows in an elementary way by using $\psi'(\vartheta) = 1/\varphi'(\psi(\vartheta))$. □

The following lemma presents the behavior of $\xi_e(\vartheta)$ as $\vartheta \downarrow 0$. We need this type of results, as Karamata's Tauberian theorem then enables us to translate the behavior of transforms around 0 into the behavior of $r(t)$ for t large.

Lemma 5.3. *If $\varphi'(\alpha) \in \mathscr{R}_{\delta-1}(n-1, \sigma)$, with $n \in \{3, 4, \ldots\}$ and $\delta \in (n, n+1)$, then*

$$\xi_e(\vartheta) = 1 - \vartheta\rho(\vartheta) \in \mathscr{R}_{\delta-3}(n-3, \omega), \quad \textit{with}$$

$$\omega := \frac{(\delta - 1)}{v\delta(\varphi'(0))^{\delta-2}}\sigma.$$

Proof. Recall Eqns. (9) and (10). The crucial step is to verify that

$$\frac{\vartheta}{\psi(\vartheta)} \in \mathscr{R}_{\delta-1}(n-1, -\tau(\varphi'(0))^2) \quad \text{and} \quad \frac{1}{\psi'(\vartheta)} \in \mathscr{R}_{\delta-1}(n-1, -\tau\delta(\varphi'(0))^2);$$

use Lemma 5.2. Verification of the claim is now straightforward (though tedious). □

The Tauberian theorem in Bingham, Goldie, and Teugels [10, Thm. 8.1.6] now yields the following result; see also [9].

Proposition 5.4. *If $\varphi'(\alpha) \in \mathscr{R}_{\delta-1}(n-1, \sigma)$, with $n \in \{3, 4, \ldots\}$ and $\delta \in (n, n+1)$, then*

$$r(t) \sim \frac{\omega}{\Gamma(4-\delta)} t^{3-\delta} L(t) \quad \textit{as} \quad t \to \infty.$$

Proof. First recall that $r(t) = \mathbb{P}(Z_e > t)$, and that Z_e has transform $\xi_e(\cdot)$. Lemma 5.3 and Thm. 8.1.6 of [10] yield the stated.

Corollary 5.5. Interestingly, we can now also find a criterion for long-range dependence, cf. the remarks in the introduction of [17].
Suppose $\varphi'(\alpha) \in \mathscr{R}_{\delta-1}(n-1, \eta)$, with $n \in \{3, 4, \ldots\}$ and $\delta \in (n, n+1)$. Then the queueing process is long-range dependent if $n = 3$, as in this case $\int_0^\infty r(t)\mathrm{d}t = \infty$. Consider for instance the case that $X \in \mathbb{CP}(\lambda, \beta(\cdot))$, with $\mathbb{P}(B > t) \sim t^{-\nu}$, for $\nu \in (3, 4)$. Then the first three moments of B exist, and hence also the first two moments of the steady-state queue length, as well as the covariance $\mathbb{C}\text{ov}(Q_0, Q_t)$. The tail of B, however, is so heavy that $\mathbb{C}\text{ov}(Q_0, Q_t)$ decays roughly as $t^{3-\nu}$, giving rise to a long-range dependent queueing process.
Likewise, it follows that the queueing process is short-range dependent if $n \in \{4, 5, \ldots\}$, for instance when considering $\mathbb{CP}$ input with $\mathbb{P}(B > t) \sim t^{-\nu}$, for $\nu \in (4, \infty)$. ♢

6 Concluding remarks

In this paper we studied the correlation function of the workload process of a single-queue fed by a Lévy process (that is, a Lévy process reflected at 0). Relying on the concept of complete monotonicity we have been able to derive a set of structural properties of the correlation function, viz. that it is a positive, decreasing, and convex function. Importantly, we have shown how to represent the correlation function $r(\cdot)$ as the complementary distribution function of a specific random variable. This representation, as well as an explicit characterization of the Laplace transform of

$r(\cdot)$, enabled the analysis of the asymptotic behavior of $r(t)$ for t large; both the light-tailed and heavy-tailed cases were studied.
An alternative way to conclude that the correlation function is positive, decreasing, and convex, may be the following. The Laplace exponent of any Levy process can be approximated arbitrarily closely by that of an appropriately chosen $\mathbb{C}$P process, see e.g. [12, Thm. XVII.1]. As the claim has been proved for $\mathbb{C}$P input [17], a limit argument may lead to an alternative proof of Thm. 3.6. Exploration of this approach is a subject for further research.

Restricting ourselves to the case of $\mathbb{C}$P input, one could say that Section 4 covers the case in which the jobs have a finite moment generating function in a neighborhood of the origin: $\beta(\alpha) < \infty$ for some $\alpha < 0$, and hence all moments are finite. On the other hand, Section 5 addresses the situation in which just a finite set of moments are finite. In between, however, there is a third class of distributions: those for which all moments are finite, but without an analytic continuation for $\alpha < 0$ (that is $\beta(\alpha) = \infty$ for all $\alpha < 0$). Examples of distributions in this class are the Weibull and LogNormal distributions. A subject for further research would be the analysis of the correlation asymptotics for this class of distributions.
As we lack, in most cases, an explicit formula for $r(t)$, one may attempt to estimate it through simulation. This is particularly challenging, as $r(t)$ can be extremely small for large t, and is the difference of two (potentially large) numbers. A way to circumvent this problem is to use *importance sampling* [5, Section V.1], that is, sampling under an alternative measure and correcting the simulation output by likelihood ratios (that keep track of the relative likelihood of the realization under the actual measure, relative to the alternative measure). The resemblance with the busy period asymptotics suggests that, for light-tailed input, the (exponentially-twisted) change of measure proposed in [15] may work well; it is noted that the analysis of [15] indicates that the twisting of the work present at time 0 should be handled with care. An other option could be to rely on the representation of the correlation function $r(\cdot)$ as the complementary distribution function of the random variable Z_e, see Thm. 3.6.

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A Appendix: Complete monotonicity

The concept of complete monotonicity is summarized in the following definition.

Definition A.1. A function $f(\alpha)$ on $[0,\infty)$ is completely monotone if for all $n \in \mathbb{N}$

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\alpha^n} f(\alpha) \geq 0.$$

We write $f(\alpha) \in \mathscr{C}$.

The following deep and powerful result is due to Bernstein [7]. It says that there is equivalence between $f(\alpha)$ being completely monotone, and the possibility of writing $f(\alpha)$ as a Laplace transform. For more background on completely monotone functions, see [12, pp. 439-442].

Theorem A.2. *A function $f(\alpha)$ on $[0,\infty)$ is the Laplace transform of a random variable if and only if* (i) *$f(\alpha) \in \mathscr{C}$, and* (ii) *$f(0) = 1$.*

The concept of complete monotonicity is easy to work with, as one can use a set of practical properties.

Lemma A.3. *The following properties apply:*

(1) *$\mathscr{C}$ is closed under addition: if $f(\alpha) \in \mathscr{C}$ and $g(\alpha) \in \mathscr{C}$, then $f(\alpha) + g(\alpha) \in \mathscr{C}$. This extends to: if $f_x(\alpha) \in \mathscr{C}$ for $x \in \Xi$, then $\int_{x\in\Xi} f_x(\alpha)\mu(\mathrm{d}x) \in \mathscr{C}$ for any measure $\mu(\cdot)$.*

(2) *$\mathscr{C}$ is closed under multiplication: if $f(\alpha) \in \mathscr{C}$ and $g(\alpha) \in \mathscr{C}$, then $f(\alpha)g(\alpha) \in \mathscr{C}$.*

(3) *Properties of composite $\mathscr{C}$ functions: if $f(\alpha) \in \mathscr{C}$ and $g(\alpha) \geq 0$ with $g'(\alpha) \in \mathscr{C}$, then $f(g(\alpha)) \in \mathscr{C}$.*

(4) *Let $U(\alpha)$ non-decreasing on $[0,\infty)$, and $U(0) = 0$, $u := \lim_{\alpha\to\infty} U(\alpha) < \infty$, and*

$$f(\alpha) := \int_{[0,\infty)} e^{-\alpha x} \mathrm{d}U(\alpha);$$

clearly $f(\alpha) \in \mathscr{C}$ and $u = f(0)$. Then also

$$g(\alpha) := \frac{f(0) - f(\alpha)}{\alpha} \in \mathscr{C}.$$

(5) *$\mathscr{C}$ closed under differentiation: if $f(\alpha) \in \mathscr{C}$, then $-f'(\alpha) \in \mathscr{C}$.*

Proof. (1) is trivial from the definition. (2) follows from [12, Criterion 1], and (3) from [12, Criterion 2]. Property (4) can be found in for instance [17, Eqn. (4.2)]. (5) is trivial. □

Lemma A.4. *Let $(Y_t)_{t\geq 0}$ be an increasing subordinator Lévy process, with Laplace exponent $\xi(\alpha)$, then $-\xi'(\alpha) \in \mathscr{C}$.*

Proof. According to Bertoin [8, Ch. III, Eqn. (3)], we can write

$$\xi(\alpha) = -d\alpha + \int_{(0,\infty)} (e^{-\alpha x} - 1)\Pi_\xi(\mathrm{d}x),$$

with $d \geq 0$, and measure $\Pi_\xi(\cdot)$ such that $\int_{(0,\infty)} \min\{1, x^2\}\Pi_\xi(\mathrm{d}x) < \infty$. This implies that

$$-\xi'(\alpha) = d + \int_{(0,\infty)} x e^{-\alpha x}\Pi_\xi(\mathrm{d}x),$$

so that $-\xi'(\alpha) \in \mathscr{C}$; use Lemma A.3.(1). $\square$

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