

A Generic Approach to Coalition Formation

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Abstract

We propose an abstract approach to coalition formation that focuses on simple merge and split rules transforming partitions of a group of players. We identify conditions under which every iteration of these rules yields a unique partition. The main conceptual tool is a specific notion of a stable partition. The results are parametrized by a preference relation between partitions of a group of players and naturally apply to coalitional TU-games, hedonic games and exchange economy games.

1 Introduction

1.1 Approach

Coalition formation has been a research topic of continuing interest in the area of coalitional games. It has been analyzed from several points of view, starting with [3], where the static situation of coalitional games in the presence of a given coalition structure (i.e. a partition) was considered.

In this paper we consider the perennial question ‘how do coalitions form?’ by proposing a simple answer: ‘by means of merges and splits’. This brings us to the study of a natural problem, namely under what assumptions the outcomes of arbitrary sequences of merges and splits are unique.

These considerations yield an abstract approach to coalition formation that focuses on partial preference relations between partitions of a group of

players and simple merge and split rules. These rules transform partitions of a group of players under the condition that the resulting partition is preferred. By identifying conditions under which every iteration of these rules yields a unique partition we are brought to a natural notion of a stable partition.

This approach is parametrized by a generic preference relation. The obtained results depend only on a few simple properties, namely irreflexivity, transitivity and monotonicity, and do not require any specific model of coalitional games.

In the case of coalitional TU-games the preference relations induced by various well-known orders on sequences of reals, such as leximin or Nash order, satisfy the required properties. As a consequence our results apply to the resulting preference relations and coalitional TU-games. We also explain how our results apply to hedonic games (games in which each player has a preference relation on the sets of players that include him) and exchange economy games.

This approach to coalition formation is indirectly inspired by the theory of abstract reduction systems (ARS), see, e.g. [15], one of the aims of which is a study of conditions that guarantee a unique outcome of rule iterations. In an earlier work [1] we exemplified another benefit of relying on ARS by using a specific result, called Newman’s Lemma, to provide uniform proofs of order independence for various strategy elimination procedures for finite strategic games.

1.2 Related work

Because of this different starting point underpinning our approach, it is difficult to compare it to the vast literature on the subject of coalition formation. Still, a number of papers should be mentioned even though their results have no bearing on ours.

In particular, rules that modify coalitions are considered in [16] in the presence of externalities and in [13] in the presence of binding agreements. In both papers two-stage games are analyzed. In the first stage coalitions form and in the second stage the players engage in a non-cooperative game given the emerged coalition structure. In this context the question of stability of the coalition structure is then analyzed.

The question of (appropriately defined) stable coalition structures often focused on hedonic games. [5] considered four forms of stability in such games: core, Nash, individual and contractually individual stability. Each

alternative captures the idea that no player, respectively, no group of players has an incentive to change the existing coalition structure. The problem of existence of (core, Nash, individually and contractually individually) stable coalitions was considered in this and other references, for example [14] and [6].

Recently, [4] compared various notions of stability and equilibria in network formation games. These are games in which the players may be involved in a network relationship that, as a graph, may evolve. Other interaction structures which players can form were considered in [7], in which formation of hierarchies was studied, and [10], in which only bilateral agreements that follow a specific protocol were allowed.

Early research on the subject of coalition formation is discussed in [9]. More recently, various aspects of coalition formation are discussed in the collection of articles [8] and in the survey [11]. Initially, we obtained the corresponding results in [2] in a limited setting of coalitional TU-games and the preference relation induced by the utilitarian order.

1.3 Plan of the paper

The paper is organized as follows. In the next section we set the stage by introducing an abstract comparison relation between partitions of a group of players and the corresponding merge and split rules that act on such partitions. Then in Section 3 we discuss a number of natural comparison relations on partitions within the context of coalitional TU-games and in Section 4 by using arbitrary value functions for such games.

Next, in Section 5, we introduce and study a parametrized concept of a stable partition and in Section 6 relate it to the merge and split rules. Finally, in Section 7 we explain how to apply the obtained results to specific coalitional games, including TU-games, hedonic games and exchange economy games.

2 Comparing and transforming collections

Let $N = \{1, 2, \dots, n\}$ be a fixed set of players called the *grand coalition*. Non-empty subsets of N are called *coalitions*. A *collection* (in the grand coalition N) is any family $C := \{C_1, \dots, C_l\}$ of mutually disjoint coalitions, and l is called its *size*. If additionally $\bigcup_{j=1}^l C_j = N$, the collection C is called a *partition* of N . For $C = \{C_1, \dots, C_k\}$, we define $\bigcup C := \bigcup_{i=1}^k C_i$.

In this article we are interested in comparing collections. In what follows we only compare collections A and B that are partitions of the same set, i.e. such that $\bigcup A = \bigcup B$. Intuitively, assuming a comparison relation \triangleright , $A \triangleright B$ means that the way A partitions K , where $K = \bigcup A = \bigcup B$, is preferable to the way B partitions K .

To keep the presentation uniform we only assume that the relation \triangleright is irreflexive, i.e. for no collection A , $A \triangleright A$ holds, transitive, i.e. for all collections A, B, C with $\bigcup A = \bigcup B = \bigcup C$, $A \triangleright B$ and $B \triangleright C$ imply $A \triangleright C$, and that \triangleright is *monotonic* in the following two senses: for all collections A, B, C, D with $\bigcup A = \bigcup B$, $\bigcup C = \bigcup D$, and $\bigcup A \cap \bigcup C = \emptyset$,

$$A \triangleright B \text{ and } C \triangleright D \text{ imply } A \cup C \triangleright B \cup D, \quad (\text{m1})$$

and for all collections A, B, C with $\bigcup A = \bigcup B$ and $\bigcup A \cap \bigcup C = \emptyset$,

$$A \triangleright B \text{ implies } A \cup C \triangleright B \cup C. \quad (\text{m2})$$

The role of monotonicity will become clear in Section 5, though property (m2) will be already of use in this section.

Definition 2.1. By a *comparison relation* we mean an irreflexive and transitive relation on collections that satisfies the conditions (m1) and (m2). \square

Each comparison relation \triangleright is used only to compare partitions of the *same* set of players. So partitions of different sets of players are incomparable w.r.t. \triangleright , that is no comparison relation is linear. This leads to a more restricted form of linearity, defined as follows. We call a comparison relation \triangleright *semi-linear* if for all collections A, B with $\bigcup A = \bigcup B$, either $A \triangleright B$ or $B \triangleright A$.

In what follows we study coalition formation by focusing on the following two rules that allow us to transform partitions of the grand coalition:

merge: $\{T_1, \dots, T_k\} \cup P \rightarrow \{\bigcup_{j=1}^k T_j\} \cup P$, where $\{\bigcup_{j=1}^k T_j\} \triangleright \{T_1, \dots, T_k\}$

split: $\{\bigcup_{j=1}^k T_j\} \cup P \rightarrow \{T_1, \dots, T_k\} \cup P$, where $\{T_1, \dots, T_k\} \triangleright \{\bigcup_{j=1}^k T_j\}$

Note that both rules use the \triangleright comparison relation ‘locally’, by focusing on the coalitions that take part and result from the merge, respectively split. In this paper we are interested in finding conditions that guarantee that arbitrary sequences of these two rules yield the same outcome. So, once

these conditions hold, a specific *preferred* partition exists such that any initial partition can be transformed into it by applying the merge and split rules in an arbitrary order.

To start with, note that the termination of the iterations of these two rules is guaranteed.

Note 2.2. *Suppose that \triangleright is a comparison relation. Then every iteration of the merge and split rules terminates.*

Proof. Every iteration of these two rules produces by (m2) a sequence of partitions P_1, P_2, \dots with $P_{i+1} \triangleright P_i$ for all $i \geq 1$. But the number of different partitions is finite. So by transitivity and irreflexivity of \triangleright such a sequence has to be finite. \square

The analysis of the conditions guaranteeing the unique outcome of the iterations is now deferred to Section 6.

3 TU-games

To properly motivate the subsequent considerations and to clarify the status of the monotonicity conditions we now introduce some natural comparison relations on collections for coalitional TU-games. A *coalitional TU-game* is a pair (N, v) , where $N := \{1, \dots, n\}$ and v is a function from the powerset of N to the set of non-negative reals¹ such that $v(\emptyset) = 0$.

For a coalitional TU-game (N, v) the comparison relations on collections are induced in a canonic way from the corresponding comparison relations on multisets of reals by stipulating that for the collections A and B

$$A \triangleright B \text{ iff } v(A) \triangleright v(B), \quad (1)$$

where for a collection $A := \{A_1, \dots, A_m\}$, $v(A) := \{v(A_1), \dots, v(A_m)\}$, denoting multisets in dotted braces.

So first we introduce the appropriate relations on multisets of non-negative reals. The corresponding definition of monotonicity for such a relation \triangleright is that for all multisets a, b, c, d of reals

$$a \triangleright b \text{ and } c \triangleright d \text{ imply } a \dot{\cup} c \triangleright b \dot{\cup} d$$

¹The assumption that the values of v are non-negative is non-standard and is needed only to accomodate for the Nash order, defined below.

and

$$a \triangleright b \text{ implies } a \dot{\cup} c \triangleright b \dot{\cup} c,$$

where $\dot{\cup}$ denotes multiset union.

Given two sequences (a_1, \dots, a_m) and (b_1, \dots, b_n) of real numbers we define the (extended) *lexicographic order* on them by putting

$$(a_1, \dots, a_m) >_{lex} (b_1, \dots, b_n)$$

iff

$$\exists i \leq \min(m, n) (a_i > b_i \wedge \forall j < i a_j = b_j)$$

or

$$\forall i \leq \min(m, n) a_i = b_i \wedge m > n.$$

Note that in this order we compare sequences of possibly different length. We have for example $(1, 1, 1, 0) >_{lex} (1, 1, 0)$ and $(1, 1, 0) >_{lex} (1, 1)$. It is straightforward to check that it is a linear order.

We assume below that $a = \{a_1, \dots, a_m\}$ and $b = \{b_1, \dots, b_n\}$ and that a^* is a sequence of the elements of a in decreasing order, and define

- the *utilitarian* order:

$$a \succ_{ut} b \text{ iff } \sum_{i=1}^m a_i > \sum_{j=1}^n b_j,$$

- the *Nash* order:

$$a \succ_{Nash} b \text{ iff } \prod_{i=1}^m a_i > \prod_{j=1}^n b_j,$$

- the *elitist* order:

$$a \succ_{el} b \text{ iff } \max(a) > \max(b),$$

- the *egalitarian* order:

$$a \succ_{eg} b \text{ iff } \min(a) > \min(b),$$

- the *leximin* order:

$$a \succ_{lex} b \text{ iff } a^* >_{lex} b^*.$$

In [12] these relations were considered for sequences of the same length. For such sequences we shall discuss in Section 4 two other natural orders. The intuition behind the Nash order is that when the sum $\sum_{i=1}^m a_i$ is fixed,

the product $\prod_{i=1}^m a_i$ is largest when all a_i s are equal. So in a sense the Nash order favours an equal distribution.

The above relations are clearly irreflexive, transitive and linear. Additionally the following holds.

Note 3.1. *The above relations are all monotonic both in sense (m1) and (m2).*

Proof. The only relation for which the claim is not immediate is \succ_{lex} . We only prove (m1) for \succ_{lex} ; the remaining proof is analogous.

Let arbitrary multisets of non-negative reals a, b, c, d be given. We define, with e denoting any sequence or multiset of non-negative reals,

$$\begin{aligned} len(e) &:= \text{the number of elements in } e, \\ \mu &:= (a \dot{\cup} b \dot{\cup} c \dot{\cup} d)^* \text{ with all duplicates removed,} \\ \nu(x, e) &:= \text{the number of occurrences of } x \text{ in } e, \\ \beta &:= 1 + \max_{k=1}^{len(\mu)} \{\nu(\mu_k, a \dot{\cup} b \dot{\cup} c \dot{\cup} d)\}, \\ \#(e) &:= \sum_{k=1}^{len(\mu)} \nu(\mu_k, e) \cdot \beta^{-k}. \end{aligned}$$

So μ is the sequence of all distinct reals used in $a \dot{\cup} b \dot{\cup} c \dot{\cup} d$, arranged in a decreasing order. The function $\#(\cdot)$ injectively maps a multiset e to a real number y in such a way that in the floating point representation of y with base β , the k th digit after the point equals the number of occurrences of the k th biggest number μ_k in e . The base β is chosen in such a way that even if e is the union of some of the given multisets, the number $\nu(x, e)$ of occurrences of x in e never exceeds $\beta - 1$. Therefore, the following sequence of implications holds:

$$\begin{aligned} a^* >_{lex} b^* \text{ and } c^* >_{lex} d^* &\Rightarrow \#(a) > \#(b) \text{ and } \#(c) > \#(d) \\ &\Rightarrow \#(a) + \#(c) > \#(b) + \#(d) \\ &\Rightarrow \#(a \dot{\cup} c) > \#(b \dot{\cup} d) \\ &\Rightarrow (a \dot{\cup} c)^* >_{lex} (b \dot{\cup} d)^* \end{aligned}$$

□

Consequently, the corresponding five relations on collections induced by (1) are all semi-linear comparison relations.

As a natural example of a transitive relation on multisets of reals that is not monotonic consider \succeq_{av} defined by

$$a \succeq_{av} b \text{ iff } (\sum_{i=1}^m a_i)/m \geq (\sum_{j=1}^n b_j)/n.$$

Note that for

$$a := \{3\}, b := \{2, 2, 2, 2\}, c := \{1, 1, 1, 1\}, d := \{0\}$$

we have both $a \succeq_{av} b$ and $c \succeq_{av} d$ but not $a \dot{\cup} c \succeq_{av} b \dot{\cup} d$ since $\{3, 1, 1, 1, 1\} \succeq_{av} \{2, 2, 2, 2, 0\}$ does not hold.

4 Individual values

In the previous section we defined the comparison relations in the context of TU-games by comparing the values (yielded by the v function) of whole coalitions. Alternatively, we could compare payoffs to individual players. The idea is that in the end, the value secured by a coalition may have to be distributed to its members, and this final payoff to a player may determine his preferences.

To formalize this approach we need the notion of an *individual value function* ϕ that, given the v function of a TU-game and a coalition A , assigns to each player $i \in A$ a real value $\phi_i^v(A)$. We assume that ϕ is *balanced*, i.e. that it exactly distributes the coalition's value to its members:

$$\sum_{i \in A} \phi_i^v(A) = v(A).$$

For a collection $C := \{C_1, \dots, C_k\}$, we put

$$\phi^v(C) := \{\phi_i^v(A) \mid A \in C, i \in A\}.$$

Given two collections $C = \{C_1, \dots, C_k\}$ and $C' = \{C'_1, \dots, C'_l\}$ with $\bigcup C = \bigcup C'$, the comparison relations now compare $\phi^v(C)$ and $\phi^v(C')$, which are multisets of $|\bigcup C|$ real numbers, one number for each player. In this way it is guaranteed that the comparison relations are *anonymous* in the sense that the names of the players do not play a role.

In this section, to distinguish between comparison relations defined only by means of v and those defined using both v and ϕ , we denote the former by \triangleright_v and the latter by \triangleright_ϕ .

We now examine how these two different approaches for defining comparison relations relate. To this end, we will clarify when they coincide, i.e. when given a comparison relation defined in one way, we can also obtain it using the other way, and when they are unrelated. We begin by formalizing the concept of anonymity.

Definition 4.1. Assume a coalitional TU-game (N, v) .

- An individual value function ϕ is *anonymous* if for all v , permutations π of N , $i \in N$, and $A \subseteq N$

$$\phi_i^v(A) = \phi_{\pi(i)}^{v \circ \pi^{-1}}(\pi(A)).$$

- v is *anonymous* if for all permutations π of N and $A \subseteq N$

$$v(A) = v(\pi(A)).$$

Note that for all A we have $(v \circ \pi^{-1})(\pi(A)) = v(A)$. Intuitively, ϕ is anonymous if it does not depend on the names of the players and v is anonymous if it is defined only in terms of the cardinality of the argument coalition.

The following simple observation holds.

Note 4.2. For any v and ϕ , if \triangleright_v and \triangleright_ϕ are the utilitarian order (as defined in Section 3), then for all collections C and C' , we have $\phi^v(C) \triangleright_\phi \phi^v(C')$ iff $v(C) \triangleright_v v(C')$.

Proof. Immediate since

$$\sum_{A \in C} v(A) = \sum_{A \in C} \sum_{i \in A} \phi_i^v(A) = \sum_{A \in C, i \in A} \phi_i^v(A).$$

□

For other orders discussed in Section 3 no relation between \triangleright_v and \triangleright_ϕ holds. In fact, we have the following results.

Theorem 4.3. *Given v and \triangleright_v , it is in general not possible to define an anonymous individual value function ϕ along with \triangleright_ϕ such that for all collections C and C' , we have $\phi^v(C) \triangleright_\phi \phi^v(C')$ iff $v(C) \triangleright_v v(C')$. This holds even if we restrict ourselves to anonymous v .*

Proof. Consider the following game with $N = \{1, 2\}$:

$$v(\{1\}) := 1 \quad v(\{2\}) := 1 \quad v(\{1, 2\}) := 2,$$

and take \triangleright_v to be the Nash order as defined in Section 3. This yields both

$$v(\{\{1, 2\}\}) = \{2\} \triangleright_v \{1, 1\} = v(\{\{1\}, \{2\}\})$$

and

$$v(\{\{1\}, \{2\}\}) \not\triangleright_v v(\{\{1, 2\}\}).$$

However, the symmetry of the game and anonymity of ϕ forces

$$\phi^v(\{\{1, 2\}\}) = \{1, 1\} = \phi^v(\{\{1\}, \{2\}\}),$$

so we have either

$$\phi^v(\{\{1, 2\}\}) \triangleright_\phi \phi^v(\{\{1\}, \{2\}\}) \text{ and } \phi^v(\{\{1\}, \{2\}\}) \triangleright_\phi \phi^v(\{\{1, 2\}\})$$

or

$$\phi^v(\{\{1, 2\}\}) \not\triangleright_\phi \phi^v(\{\{1\}, \{2\}\}) \text{ and } \phi^v(\{\{1\}, \{2\}\}) \not\triangleright_\phi \phi^v(\{\{1, 2\}\}).$$

□

Theorem 4.4. *Given v , ϕ and \triangleright_ϕ , it is in general not possible to define \triangleright_v such that for all collections C and C' , we have $v(C) \triangleright_v v(C')$ iff $\phi^v(C) \triangleright_\phi \phi^v(C')$. This holds even if we restrict ourselves to anonymous v , anonymous ϕ , and a Nash, elitist, egalitarian or leximin order (as defined in Section 3) for \triangleright_ϕ .*

Proof. Consider $N = \{1, \dots, 4\}$ and

$$v(A) := 6 \text{ for all } A \subseteq N$$

$$\phi_i^v(A) := \frac{v(A)}{|A|}.$$

Then we have

$$\begin{aligned}\phi^v(\{\{1\}, \{2, 3, 4\}\}) &= \{6, 2, 2, 2\} \\ \phi^v(\{\{1, 2\}, \{3, 4\}\}) &= \{3, 3, 3, 3\},\end{aligned}$$

which are distinguished by each of the mentioned \triangleright_ϕ , while

$$v(\{\{1\}, \{2, 3, 4\}\}) = v(\{\{1, 2\}, \{3, 4\}\}) = \{6, 6\}.$$

□

These results suggest that the two approaches for defining preference relations are fundamentally different and coincide only for the utilitarian order.

In the case of individual values we can introduce natural orders that have no counterpart for the comparison relations defined only by means of v . The reason is that for each partition P , $\phi^v(P)$ can be alternatively viewed as a sequence (of payoffs) of (the same) length n . Such sequences can then be compared using

- the *majority order*:

$$(k_1, \dots, k_n) \succ_m (l_1, \dots, l_n) \text{ iff } |\{i \mid k_i > l_i\}| > |\{i \mid l_i > k_i\}|,$$

- the *Pareto order*:

$$\begin{aligned}(k_1, \dots, k_n) \succ_p (l_1, \dots, l_n) \text{ iff} \\ \forall i \in \{1, \dots, n\} k_i \geq l_i \text{ and } \exists i \in \{1, \dots, n\} k_i > l_i.\end{aligned}$$

The relation \succ_m is clearly irreflexive and monotonic both in sense (m1) and (m2). Unfortunately, it is not transitive. Indeed, we have both $(2, 3, 0) \succ_m (1, 2, 2)$ and $(1, 2, 2) \succ_m (3, 1, 1)$, but $(2, 3, 0) \succ_m (3, 1, 1)$ does not hold. In contrast, the relation \succ_p is transitive, irreflexive, monotonic both in sense (m1) and (m2), and linear.

5 Stable partitions

We now return to our analysis of partitions. One way to identify conditions guaranteeing the unique outcome of the iterations of the merge and split

rules is through focusing on the properties of such a unique outcome. This brings us to the concept of a stable partition.

We follow here the approach of [2], although now no notion of a game is present. The introduced notion is parametrized by means of a *defection function* \mathbb{D} that assigns to each partition some partitioned subsets of the grand coalition. Intuitively, given a partition P the family $\mathbb{D}(P)$ consists of all the collections $C := \{C_1, \dots, C_l\}$ whose players can leave the partition P by forming a new, separate, group of players $\cup_{j=1}^l C_j$ divided according to the collection C . Two most natural defection functions are \mathbb{D}_p , which allows formation of all partitions of the grand coalition, and \mathbb{D}_c , which allows formation of all collections in the grand coalition.

Next, given a collection C and a partition $P := \{P_1, \dots, P_k\}$ we define

$$C[P] := \{P_1 \cap \bigcup C, \dots, P_k \cap \bigcup C\} \setminus \{\emptyset\}$$

and call $C[P]$ the *collection C in the frame of P* . (By removing the empty set we ensure that $C[P]$ is a collection.) To clarify this concept consider Figure 1. We depict in it a collection C , a partition P and C in the frame of P (together with P). Here C consists of four coalitions, while C in the frame of P consists of three coalitions.

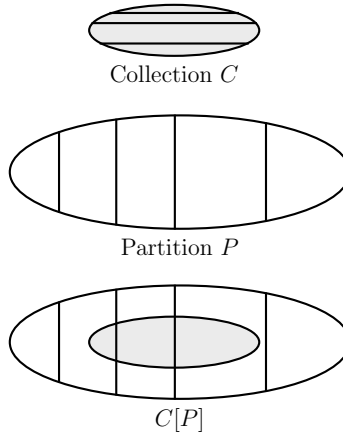


Figure 1: A collection C in the frame of a partition P

Intuitively, given a subset S of N and a partition $C := \{C_1, \dots, C_l\}$ of S , the collection C offers the players from S the ‘benefits’ resulting from the partition of S by C . However, if a partition P of N is ‘in force’, then the

players from S enjoy instead the benefits resulting from the partition of S by $C[P]$, i.e. C in the frame of P .

To get familiar with the $C[P]$ notation note that

- if C is a singleton, say $C = \{T\}$, then $\{T\}[P] = \{P_1 \cap T, \dots, P_k \cap T\} \setminus \{\emptyset\}$, where $P = \{P_1, \dots, P_k\}$,
- if C is a partition of N , then $C[P] = P$,
- if $C \subseteq P$, that is C consists of some coalitions of P , then $C[P] = C$.

In general the following simple observation holds.

Note 5.1. *For a collection C and a partition P , $C[P] = C$ iff each element of C is a subset of a different element of P .* \square

This brings us to the following notion.

Definition 5.2. Assume a defection function \mathbb{D} and a comparison relation \triangleright . We call a partition P \mathbb{D} -stable if $C[P] \triangleright C$ for all $C \in \mathbb{D}(P)$ such that $C[P] \neq C$.

The last qualification, that is $C[P] \neq C$, requires some explanation. Intuitively, this condition indicates that the players only care about the way they are partitioned. Indeed, if $C[P] = C$, then the partitions of $\bigcup C$ by means of P and by means of C coincide and are viewed as equally satisfactory for the players in $\bigcup C$. By disregarding the situations in which $C[P] = C$ we therefore adopt a limited viewpoint of cooperation according to which the players in C do not care about the presence of the players from outside of $\bigcup C$ in their coalitions.

The following observation holds, where we call a partition P of N \triangleright -maximal if for all partitions P' of N different from P , $P \triangleright P'$ holds.

Theorem 5.3. *A partition of N is \mathbb{D}_p -stable iff it is \triangleright -maximal.*

In particular, a \mathbb{D}_p -stable partition of N exists if \triangleright is a semi-linear comparison relation. \square

Proof. Note that if C is a partition of N , then $C[P] \neq C$ is equivalent to the statement $P \neq C$, since then $C[P] = P$. So a partition P of N is \mathbb{D}_p -stable iff for all partitions $P' \neq P$ of N , $P \triangleright P'$ holds. \square

In contrast, \mathbb{D}_c -stable partitions do not need to exist even if the comparison relation \triangleright is semi-linear.

Example 5.4. Consider $N = \{1, 2, 3\}$ and any semi-linear comparison relation \triangleright such that $\{\{1, 2, 3\}\} \triangleright \{\{1\}, \{2\}, \{3\}\}$ and $\{\{a\}, \{b\}\} \triangleright \{\{a, b\}\}$ for all $a, b \in \{1, 2, 3\}$, $a \neq b$.

Then no partition of N is \mathbb{D}_c -stable. Indeed, $P := \{\{1\}, \{2\}, \{3\}\}$ is not \mathbb{D}_c -stable since for $C := \{\{1, 2, 3\}\}$ we have $C[P] = \{\{1\}, \{2\}, \{3\}\} \not\triangleright \{\{1, 2, 3\}\} = C$. Further, any other partition P contains some coalition $\{a, b\}$ and is thus not \mathbb{D}_c -stable either since then for $C := \{\{a\}, \{b\}\}$ we have $C[P] = \{\{a, b\}\} \not\triangleright \{\{a\}, \{b\}\} = C$. \square

In [2] another example is given for the case of TU-games and utilitarian order. More precisely, a TU-game is defined in which no \mathbb{D}_c -stable partition exists, where \triangleright is defined through (1) using the utilitarian order \succ_{ut} .

6 Stable partitions and merge/split rules

We now resume our investigation of the conditions under which every iteration of the merge and split rules yields the same outcome. The following main result of the paper provides an answer in terms of the \mathbb{D}_c -stable partitions.

Theorem 6.1. *Suppose that \triangleright is a comparison relation and P is a \mathbb{D}_c -stable partition. Then*

- (i) *P is the outcome of every iteration of the merge and split rules.*
- (ii) *P is a unique \mathbb{D}_p -stable partition.*
- (iii) *P is a unique \mathbb{D}_c -stable partition.*

To prove it we establish first a sequence of lemmata about \mathbb{D}_c -stable partitions.

Lemma 6.2. *Every \mathbb{D}_c -stable partition is closed under the applications of the merge and split rules.*

Proof. To prove the closure of a \mathbb{D}_c -stable partition P under the merge rule assume that for some $\{T_1, \dots, T_k\} \subseteq P$ we have $\{\bigcup_{j=1}^k T_j\} \triangleright \{T_1, \dots, T_k\}$.

\mathbb{D}_c -stability of P with $C := \{\bigcup_{j=1}^k T_j\}$ yields

$$\{T_1, \dots, T_k\} = \{\bigcup_{j=1}^k T_j\}[P] \triangleright \{\bigcup_{j=1}^k T_j\},$$

which is a contradiction by virtue of the transitivity and irreflexivity of \triangleright .

The closure under the split rule is shown analogously. \square

Next, we provide a characterization of \mathbb{D}_c -stable partitions. Given a partition $P := \{P_1, \dots, P_k\}$ we call here a coalition T *P-compatible* if for some $i \in \{1, \dots, k\}$ we have $T \subseteq P_i$ and *P-incompatible* otherwise.

Lemma 6.3. *A partition $P = \{P_1, \dots, P_k\}$ of N is \mathbb{D}_c -stable iff the following two conditions are satisfied:*

(i) *for each $i \in \{1, \dots, k\}$ and each pair of disjoint coalitions A and B such that $A \cup B \subseteq P_i$*

$$\{A \cup B\} \triangleright \{A, B\}, \quad (2)$$

(ii) *for each P-incompatible coalition $T \subseteq N$*

$$\{T\}[P] \triangleright \{T\}. \quad (3)$$

Proof. (\Rightarrow) Immediate.

(\Leftarrow) Transitivity, monotonicity (m2) and (2) imply by induction that for each $i \in \{1, \dots, k\}$ and each collection $C = \{C_1, \dots, C_l\}$ with $l > 1$ and $\bigcup C \subseteq P_i$,

$$\{\bigcup C\} \triangleright C. \quad (4)$$

Let now C be an arbitrary collection in N such that $C[P] \neq C$. We prove that $C[P] \triangleright C$. Define

$$D^i := \{T \in C \mid T \subseteq P_i\},$$

$$E := C \setminus \bigcup_{i=1}^k D^i,$$

$$E^i := \{P_i \cap T \mid T \in E\} \setminus \{\emptyset\}.$$

Note that D^i is the set of P -compatible elements of C contained in P_i , E is the set of P -incompatible elements of C and E^i consists of the non-empty intersections of P -incompatible elements of C with P_i .

Suppose now that $\bigcup_{i=1}^k E^i \neq \emptyset$. Then $E \neq \emptyset$ and consequently

$$\bigcup_{i=1}^k E^i = \bigcup_{i=1}^k (\{P_i \cap T \mid T \in E\} \setminus \{\emptyset\}) = \bigcup_{T \in E} (\{T\}[P]) \stackrel{(m1), (3)}{\triangleright} E. \quad (5)$$

Consider now the following property:

$$|D^i \cup E^i| > 1. \quad (6)$$

Fix $i \in \{1, \dots, k\}$. If (6) holds, then

$$\left\{P_i \cap \bigcup C\right\} = \left\{\bigcup (D^i \cup E^i)\right\} \stackrel{(4)}{\triangleright} D^i \cup E^i$$

and otherwise

$$\left\{P_i \cap \bigcup C\right\} = \{D^i \cup E^i\}.$$

Recall now that

$$C[P] = \bigcup_{i=1}^k \left\{P_i \cap \bigcup C\right\} \setminus \{\emptyset\}.$$

We distinguish two cases.

Case 1. (6) holds for some $i \in \{1, \dots, k\}$.

Then by (m1) and (m2)

$$C[P] \triangleright \bigcup_{i=1}^k (D^i \cup E^i) = (C \setminus E) \cup \bigcup_{i=1}^k E^i.$$

If $\bigcup_{i=1}^k E^i = \emptyset$, then also $E = \emptyset$ and we get $C[P] \triangleright C$. Otherwise by (5), transitivity and (m2)

$$C[P] \triangleright (C \setminus E) \cup E = C.$$

Case 2. (6) does not hold for any $i \in \{1, \dots, k\}$.

Then

$$C[P] = \bigcup_{i=1}^k (D^i \cup E^i) = (C \setminus E) \cup \bigcup_{i=1}^k E^i.$$

Moreover, because $C[P] \neq C$, by Note 5.1 a P -incompatible element in C exists. So $\bigcup_{i=1}^k E^i \neq \emptyset$ and by (5) and (m2) we get as before

$$C[P] \triangleright (C \setminus E) \cup E = C.$$

□

In [2] the above characterization was proved for the coalitional TU-games and the utilitarian order. We shall now use it in the proof of the following lemma.

Lemma 6.4. *Assume that P is \mathbb{D}_c -stable. Let P' be closed under applications of merge and split rules. Then $P' = P$.*

Proof. Suppose $P = \{P_1, \dots, P_k\}$, $P' = \{T_1, \dots, T_m\}$. Assume $P \neq P'$. Then there is $i_0 \in \{1, \dots, k\}$ such that for all $j \in \{1, \dots, m\}$ we have $P_{i_0} \neq T_j$. Let T_{j_1}, \dots, T_{j_l} be the minimum cover of P_{i_0} . In the following case distinction we use Lemma 6.3.

Case 1. $P_{i_0} = \bigcup_{h=1}^l T_{j_h}$.

Then $\{T_{j_1}, \dots, T_{j_l}\}$ is a proper partition of P_{i_0} . But (2) (through its generalization to (4)) yields $P_{i_0} \triangleright \{T_{j_1}, \dots, T_{j_l}\}$, thus the merge rule is applicable to P' .

Case 2. $P_{i_0} \subsetneq \bigcup_{h=1}^l T_{j_h}$.

Then for some j_h we have $\emptyset \neq P_{i_0} \cap T_{j_h} \subsetneq T_{j_h}$, so T_{j_h} is P -incompatible. By (3) we have $\{T_{j_h}\}[P] \triangleright \{T_{j_h}\}$, thus the split rule is applicable to P' . □

We can now prove Theorem 6.1.

Proof of Theorem 6.1.

(i) By Note 2.2 every iteration of the merge and split rules terminates, so the claim follows by Lemma 6.4.

(ii) Since P is \mathbb{D}_c -stable, it is in particular \mathbb{D}_p -stable. By Theorem 5.3 for all partitions $P' \neq P$, $P \triangleright P'$ holds. So uniqueness follows from the transitivity and irreflexivity of \triangleright .

(iii) Suppose that P' is a \mathbb{D}_c -stable partition. By Lemma 6.2 P' is closed under the applications of the merge and split rules, so by Lemma 6.4 $P' = P$.

□

This theorem generalizes [2], where this result was established for the coalitional TU-games and the utilitarian order. It was also shown there that there exist coalitional TU-games in which all iterations of the merge and split rules have a unique outcome which is not a \mathbb{D}_c -stable partition.

7 Applications

The obtained results do not involve any notion of a game. In this section we show applications to three classes of coalitional games. In each case we define a class of games and a natural comparison relation for which all iterations of the merge and split rules have a unique outcome.

7.1 Coalitional TU-games

To show that the obtained results naturally apply to coalitional TU-games consider first the special case of the utilitarian order, according to which given a coalitional TU-game (N, v) , for two collections $P := \{P_1, \dots, P_k\}$ and $Q = \{Q_1, \dots, Q_l\}$ such that $\bigcup P = \bigcup Q$, we have

$$P \triangleright Q \text{ iff } \sum_{i=1}^k v(P_i) > \sum_{i=1}^l v(Q_i).$$

Recall that (N, v) is called *strictly superadditive* if for each pair of disjoint coalitions A and B

$$v(A) + v(B) < v(A \cup B).$$

Further, recall that given a partition $P := \{P_1, \dots, P_k\}$ of N and coalitional TU-games $(P_1, v_1), \dots, (P_k, v_k)$, their *union* $(N, \oplus_{i=1}^k v_i)$ is defined by

$$(\oplus_{i=1}^k v_i)(A) = \sum_{i=1}^k v_i(P_i \cap A).$$

We now modify this definition and introduce the concept of a *semi-union* of $(P_1, v_1), \dots, (P_k, v_k)$, written as $(N, \overline{\oplus}_{i=1}^k v_i)$, and defined by

$$(\overline{\oplus}_{i=1}^k v_i)(A) := \begin{cases} (\oplus_{i=1}^k v_i)(A) & \text{if } A \subseteq P_i \text{ for some } i \\ (\oplus_{i=1}^k v_i)(A) - \epsilon & \text{otherwise} \end{cases}$$

where $\epsilon > 0$.

So for P -incompatible coalitions the payoff is strictly smaller for the semi-union of TU-games than for their union, while for other coalitions the payoffs are the same. It is then easy to prove using Lemma 6.3 that in the semi-union $(N, \overline{\oplus}_{i=1}^k v_i)$ of strictly superadditive TU-games the partition P is \mathbb{D}_c -stable. Consequently, by Theorem 6.1, in this game P is the outcome of every iteration of the merge and split rules.

The following more general example deals with arbitrary monotonic comparison relations as introduced in Sections 3 and 4.

Example 7.1. Given a partition $P := \{P_1, \dots, P_k\}$ of N , with \triangleright being one of the orders defined in Section 3, we define a TU-game for which P is the outcome of every iteration of the merge and split rules.

Let

$$f(x, y) := \begin{cases} x + y & \text{if } \triangleright \text{ is the utilitarian order} \\ x \cdot y & \text{if } \triangleright \text{ is the Nash order} \\ \max\{x, y\} & \text{if } \triangleright \text{ is the elitist or leximin order} \\ \min\{x, y\} & \text{if } \triangleright \text{ is the egalitarian order} \end{cases}$$

and define

$$v(A) := \begin{cases} 1 & \text{if } |A| = 1 \\ \max_{\substack{B \cup C = A \\ B, C \text{ disjoint} \\ \text{coalitions}}} \{f(v(B), v(C))\} + 1 & \text{if } |A| > 1 \text{ and } A \subseteq P_i \text{ for some } i \\ 0 & \text{otherwise.} \end{cases}$$

Then

(i) for any two disjoint coalitions A, B with $A \cup B \subseteq P_i$ for some i , we have

$$v(A \cup B) > f(v(A), v(B))$$

by construction of v , and thus

- $v(A \cup B) > v(A) + v(B)$ for utilitarian \triangleright ;
- $v(A \cup B) > v(A) \cdot v(B)$ for Nash \triangleright ;
- $v(A \cup B) > \max\{v(A), v(B)\}$ for elitist or leximin \triangleright ;
- $v(A \cup B) > \min\{v(A), v(B)\}$ for egalitarian \triangleright .

Hence in all cases $\{A \cup B\} \triangleright \{A, B\}$.

(ii) for any P -incompatible coalition $T \subseteq N$, we have

$$v(A) > 0 \text{ for all } A \in \{T\}[P], \text{ and } v(T) = 0.$$

Hence $\{T\}[P] \triangleright \{T\}$.

Lemma 6.3 now implies that P is indeed \mathbb{D}_c -stable, so Theorem 6.1 applies.

□

Example 7.2. Given a partition $P := \{P_1, \dots, P_k\}$ of N , with \triangleright being one of the orders defined in Section 3 or Pareto order of Section 4, we define a TU-game and an individual value function for which P is the outcome of every iteration of the merge and split rules.

Let

$$f(x, y) := \begin{cases} |N| \cdot \max\{x, y\} + 1 & \text{if } \triangleright \text{ is elitist, leximin or Pareto} \\ x + y & \text{otherwise,} \end{cases}$$

define v as in Example 7.1, and define

$$\phi_i^v(A) := \frac{v(A)}{|A|}.$$

Then

(i) for any two disjoint coalitions A, B with $A \cup B \subseteq P_i$ for some i , we have

$$v(A \cup B) > f(v(A), v(B))$$

again by construction of v , and thus

- for utilitarian, Nash, or egalitarian \triangleright :
 $v(A \cup B) > v(A) + v(B)$, and since ϕ_i^v distributes the value evenly,
in all cases $\{A \cup B\} \triangleright \{A, B\}$,
- for elitist, leximin, or Pareto \triangleright :
 $v(A \cup B) > |A \cup B| \cdot \max\{v(A), v(B)\}$,
thus $\phi_i^v(A \cup B) > \max\{v(A), v(B)\}$ for all i ,
thus $\{A \cup B\} \triangleright \{A, B\}$ in all cases,

(ii) for any P -incompatible coalition $T \subseteq N$, $\{T\}[P] \triangleright \{T\}$ as before.

Again, Lemma 6.3 implies that P is \mathbb{D}_c -stable, and Theorem 6.1 applies. □

7.2 Hedonic games

Recall that a *hedonic game* $(N, \succeq_1, \dots, \succeq_n)$ consists of a set of players $N = \{1, \dots, n\}$ and a sequence of linear preorders $\succeq_1, \dots, \succeq_n$, where each \succeq_i is the preference of player i over the subsets of N containing i . In what follows we shall not need the assumption that the \succeq_i relations are linear. Denote by \succ_i the associated irreflexive relation.

Given a partition P of N and player i we denote by $P(i)$ the element of P to which i belongs and call it the set of *friends of i in P* .

We now provide an example of a hedonic game in which a \mathbb{D}_c -stable partition w.r.t. to a natural comparison relation \succ exists.

To this end we assume that, given a partition $P := \{P_1, \dots, P_k\}$ of N , each player

- prefers a larger set of his friends in P over a smaller one,
- ‘dislikes’ coalitions that include a player who is not his friend in P .

We formalize this by putting for all sets of players that include i

$$S \succeq_i T \text{ iff } S \cup T \subseteq P(i) \text{ and } S \supseteq T,$$

and by extending this order to the coalitions that include player i and possibly players from outside of $P(i)$ by assuming that such coalitions are the minimal elements in \succeq_i . So $S \succ_i T$ iff either $S \cup T \subseteq P(i)$ and $S \supsetneq T$ or $S \subseteq P(i)$ and not $T \subseteq P(i)$.

We then define

$$Q \triangleright Q' \text{ iff for } i \in \{1, \dots, n\} \text{ } Q(i) \succeq_i Q'(i) \text{ with at least one } \succeq_i \text{ being strict.}$$

(Note the similarity between this relation and the \succ_p relation introduced in Section 4.) It is straightforward to check that \triangleright is indeed a comparison relation and that the partition P satisfies then conditions (2) and (3) of Lemma 6.3. So by virtue of this lemma P is \mathbb{D}_c -stable. Consequently, on the account of Theorem 6.1, the partition P is the outcome of every iteration of the merge and split rules.

7.3 Exchange economy games

Recall that an *exchange economy* consists of

- a market with k goods,
- for each player i an initial endowment of these goods represented by a vector $\vec{\omega}_i \in \mathcal{R}_+^k$,
- for each player i a transitive and linear relation \succeq_i using which he can compare the bundles of the goods, represented as vectors from \mathcal{R}_+^k .

An *exchange economy game* is then defined by first taking as the set of *outcomes* the set of all sequences of bundles,

$$X := \{(\vec{x}_1, \dots, \vec{x}_n) \mid \vec{x}_i \in \mathcal{R}_+^k \text{ for } i \in N\},$$

i.e. $X = (\mathcal{R}_+^k)^n$, and extending each preference relation \succeq_i from the set \mathcal{R}_+^k of all bundles to the set X by putting for $\vec{x}, \vec{y} \in X$

$$\vec{x} \succeq_i \vec{y} \text{ iff } \vec{x}_i \succeq_i \vec{y}_i. \quad (7)$$

This simply means that each player is only interested in his own bundle.

Then we assign to each coalition S the following set of outcomes:

$$V(S) := \{\vec{x} \in X \mid \sum_{i \in S} \vec{x}_i = \sum_{i \in S} \vec{\omega}_i \text{ and } \vec{x}_j = \vec{\omega}_j \text{ for all } j \in N \setminus S\}.$$

So $V(S)$ consists of the set of outcomes that can be achieved by trading between the members of S .

Given a partition $P = \{P_1, \dots, P_k\}$ of $N = \{1, \dots, n\}$ we now define a specific exchange economy game with n goods (one type of good for each player) as follows, where $i \in N$:

$$\begin{aligned} \vec{\omega}_i &:= \text{characteristic vector of } P(i), \\ \vec{x}_i \succeq_i \vec{y}_i &\text{ iff } x_{i,i} \geq y_{i,i} \quad \text{ and } \quad \vec{x}_i \succ_i \vec{y}_i \text{ iff } x_{i,i} > y_{i,i}, \end{aligned}$$

that is, each player's initial endowment consists of exactly one good of the type of each of his friends in P , and he prefers a bundle if he gets more goods of his own type.

Now let $A \triangleright B$ iff

$$\begin{aligned} &\forall A_l \in A \setminus B \exists \vec{x} \in V(A_l) \forall j \in A_l \\ &\left[(\forall \vec{y} \in V(B(j)) \vec{x} \succ_j \vec{y}) \vee (\forall \vec{y} \in V(B(j)) \vec{x} \succeq_j \vec{y} \wedge |A_l| < |B(j)|) \right]. \end{aligned}$$

So a partition A is preferred to a partition B if each coalition A_l of A not present in the partition B can achieve an outcome which each player of A_l strictly prefers to any outcome of his respective coalition in B , or which he likes at least as much as any outcome of his respective coalition in B when that coalition is strictly larger than A_l . The intuition is that the players' preferences over outcomes weigh most, but in case of ties the players prefer smaller coalitions.

It is easy to check that \triangleright is a comparison relation. We now prove that the partition P is \mathbb{D}_c -stable w.r.t. \triangleright . First, note that by the definition of the initial endowments for all $l \in \{1, \dots, k\}$ and coalitions $A \subseteq P_l$ there is an outcome $\vec{z}_A \in V(A)$ which gives exactly $|A|$ units of good j to each player $j \in A$. We have $\vec{z}_A \succeq_i \vec{x}$ for all $i \in A$ and $\vec{x} \in V(A)$. This implies that P is \mathbb{D}_c -stable by Lemma 6.3 since

- (i) for each pair of disjoint coalitions A and B such that $A \cup B \subseteq P_l$ we have $\vec{z}_{A \cup B} \succ_i \vec{z}_A$ for each $i \in A$ and $\vec{z}_{A \cup B} \succ_i \vec{z}_B$ for each $i \in B$ since $|A \cup B| > |A|$ and $|A \cup B| > |B|$, thus $\{A \cup B\} \triangleright \{A, B\}$,
- (ii) for any P -incompatible $T \subseteq N$, $A \in \{T\}[P]$, $i \in A$, and $\vec{x} \in V(T)$, we have $\vec{z}_A \succeq_i \vec{x}$ (since player i can get in T at most all goods of his type from his friends in P , which are exactly the same as in A), and $|A| < |T|$, thus $\{T\}[P] \triangleright \{T\}$.

Consequently, in the above game, by Theorem 6.1 the partition $\{P_1, \dots, P_k\}$ is the outcome of every iteration of the merge and split rules.

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