ASYMPTOTIC ESTIMATES FOR GENERALIZED STIRLING NUMBERS

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Abstract. Uniform asymptotic expansions are given for the Stirling numbers of the first kind for integral arguments and for the second kind as defined for real arguments by Flajolet and Prodinger. The logconcavity of the resulting real valued function of Flajolet and Prodinger is established for a range including the classical integral domain.

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1. Introduction

Recently Flajolet and Prodinger [4] have given a solution to the problem of Graham, Knuth and Patashnik [5] which asks for a good generalization of the Stirling numbers of the second kind, denoted here by S_k^n for complex numbers n and k. They define (equation (2) of [4])

$$S_x^y = \frac{y!}{x!} \frac{1}{2\pi i} \int_C z^{-y-1} \left(e^z - 1\right)^x \, dz$$

Here C is a Hankel contour which starts at $-\infty$, circles the origin and goes back to $-\infty$ subject to $|\operatorname{Im} z| < 2\pi$. Flajolet and Prodinger do not consider this problem for Stirling numbers of the first kind in the same detail although they do establish the identity $S_k^n = s_{-n}^{-k}$,

s_k^n denotes the Stirling number of the first kind using their definition of S_x^y. The definition of s_x^y they give is

$$|s_x^y| = \frac{y!}{x!} \frac{1}{2\pi i} \int_{C_1} \log^x \left(\frac{1}{1-w} \right) \frac{dw}{w^{y+1}},$$

where C_1 is the "raindrop contour", the image of C under $w = e^z - 1$. It is natural to consider the absolute value of the Stirling numbers of the first kind as Moser-Wyman [7] and Temme [11] have done since questions of sign are avoided and $|s_k^n|$ is the number of permutations on n symbols with k cycles. It is natural for us, following these authors, to define for positive real x and y

$$s_x^y = \frac{1}{2\pi i} \int_{C_2} \frac{\Gamma(u+y+1)}{\Gamma(u+1)} \frac{du}{u^{x+1}}.$$

Here C_2 is any Jordan curve which circles the origin in the counterclockwise sense. Note that $\Gamma(u+y+1)/\Gamma(u+1)$ is a polynomial in u for integral y and has no singularities for positive nonintegral y for $|u| < y$. We adopt this last definition, however we are primarily interested in positive integral values where the definitions agree.

Flajolet and Prodinger show that S_x^y is for fixed y an entire function of x and for fixed x a meromorphic function of y with poles at the negative integers.

In this note we derive asymptotic estimates for these generalized Stirling numbers. Our estimates are completely analogous to the estimates stated by Temme [11] for the standard Stirling numbers and are uniform in $\delta < x \leq y$, δ a positive constant as $y \to \infty$ for the Stirling numbers of the second kind and uniform and for the first kind are uniform for integral x and y in the same ranges. One reason for doing this is that certain steps in the proof of Temme's results were omitted (see Odlyzko's comments [8] regarding this), another is to verify that these results hold for Flajolet and Prodinger's generalization. We derive Temme's results using the analysis of Moser and Wyman [6, 7] which turns out to be easily adapted to the generalized numbers. We also, using the method of Merlini-Richmond [10] (based on work of Gardy [3]) show that S_x^y is a logconcave function of x for large y and $\delta < x \leq y$.

We prove three theorems (we let δ denote a positive constant).

Let u_0 be the unique real positive solution of the equation

$$\frac{m}{n} u_0 = 1 - e^{-u_0}.$$

(It is easily seen that $u/(1 - e^{-u})$ is an increasing function). Let

$$t_0 = (y - x)/x, \quad \phi(u) = -y \log u + x \log(e^u - 1),$$

and

$$A = \phi(u_0) - xt_0 + (y - x) \log t_0.$$

Let

$$f(t_0) = (t_0/(1 + t_0)(u_0 - t_0))^{1/2}.$$

(We define t_0, A and $f(t_0)$ as Temme [11] does). Finally let

$$2H_0(u) = e^u(e^u - 1)^{-1} - ue^u(e^u - 1)^{-2}.$$

THEOREM 1. *The relation*

$$S_x^y \sim e^A x^{y-x} f(t_0) \binom{y}{x},$$

where $\binom{y}{x} = y!/x!(y - x)!$ holds uniformly as $y \to \infty$ for $\delta < x \leq y$. Furthermore if $\delta < x \leq y - y^{1/3}$ then

$$S_x^y = \frac{y!}{x!} \frac{(e^{u_0} - 1)^x}{2u_0^y} \frac{1}{[\pi u_0 x H_0(u_0)]^{1/2}} \left[1 + \mathcal{O}\left(y^{-1}\right)\right],$$

where the $\mathcal{O}$-constant depends only on δ. Finally, if $y - y^{1/3} \leq x \leq y$ then

$$S_x^y = \frac{1}{2^{y-x}} \frac{y^{2(y-x)}}{(y - x)!} \left[1 + \mathcal{O}\left(y^{-1/3}\right)\right],$$

where the $\mathcal{O}$-constant is independent of x.

Now let $\psi(u) = \log[(u + 1)(u + 2) \cdots (u + [y])] - x \log u$. Let u_1 be the unique positive solution of $\psi'(u) = 0$ (see Temme [11] for a proof that u_1 is unique). Let $t_1 = x/(y - x)$ and $B = \psi(u_1) - y \log(1 + t_1) + x \log t_1$. Finally, let $g(t_1) = u_1^{-1}[x(y - x)/y\psi''(u_1)]^{1/2}$ and let $H = x - \sum_{h=0}^{[y]-1} u_1^2/(u_1 + h)^2$.

THEOREM 2. *The relation*

$$s_x^y \sim e^B g(t_1) \binom{y}{x}$$

holds uniformly for $0 < x \leq y$ as $y \to \infty$ when x and y are integers. Moreover if $0 < x \leq (\log y)^{1/2}$ then for integral x and y

$$s_x^y = \frac{y!(\log y + \gamma)^{x-1}}{x!} \left\{1 + \mathcal{O}\left[(\log y)^{-1/2}\right]\right\}.$$

If $(\log y)^{1/2} < x \leq y - y^{1/3}$ then for real x and y

$$s_x^y = \frac{\Gamma(y + 1 + u_1)}{\Gamma(u_1)(2\pi H)^{1/2} u_1^{x+1}} \left[1 + \mathcal{O}\left(1/x\right)\right]$$

where the $\mathcal{O}$-constant is independent of x. If $y - y^{1/3} \leq x \leq y$ then independently of x for integral x and y

$$s_x^y = \binom{y+1}{x+1} \left(\frac{x}{2}\right)^{y-x} \left[1 + \mathcal{O}\left(y^{1/3}\right)\right].$$

THEOREM 3. *The function S_x^y is a log-concave function of x for $\delta < x \leq y$ and y large enough.*

REMARK 1. Theorem 3 extends the well-known results that S_k^n is log-concave as a function of the discrete variable k.

REMARK 2. We depend very heavily upon the analysis of Moser- Wyman [6, 7]. We assume the reader has a copy of these papers in hand. The first statement in Theorem 1 and in Theorem 2 is found in Temme [11]. The subsequent statements in each Theorem are derived first following Moser and Wyman. The first statements are then shown to follow. In our approach most of the effort comes from showing that Temme's results are equivalent to the Moser-Wyman results in the extreme values of x with respect to y. Since the Moser-Wyman results are very simply expressed this seems of value in its own right however our main point is that Temme's formula unifies the results of Moser-Wyman. See Section 3 for further discussion.

REMARK 3. It seems very likely that it is not difficult to establish Theorem 2 for real x and y using our definition of s_x^y and a Theorem 3 for s_x^y using the Flajolet-Prodinger definition of s_x^y. It would be more work to establish both results using only one definition.

2. Proofs of the theorems

2.1. Proof of Theorem 1

Let u_0 be defined as in Theorem 1. A significant difference with the Moser-Wyman analysis arises here. The points $k\pi i$ are just zeros of the integrand when x is an integer and a contour can be moved through them with no difficulty. When x is not an integer we avoid doing so. We deform the contour C to the following contour:Let C_1 be the straight line $Im(z) = -2\pi + \delta, 0 < \delta$ however we shall think of δ as small but fixed and $\text{Re}(z) \leq \epsilon$, where ϵ is a small positive number. We let C_2 be the straight line $\text{Re}(z) = \epsilon$, going from

$\epsilon + i(\delta - 2\pi)$ to the circle $|z| = u_0$, C_5 and C_4 be the reflections in the real axis of C_1 and C_2 respectively. We let C_3 be the portion of the circle $|z| = u_0$, meeting C_2 and C_4. The new contour is $C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$ in the counterclockwise sense. We now observe that in the Moser-Wyman analysis [7] n and k can be positive reals. We refer to their analysis leading to their equation (4.3). Note that Stirling's formula gives an asymptotic expansion for $\binom{y}{x}$ provided $x, y - x \to \infty$. We obtain from their equation (5.1) (an instance of (4.3)), letting $x = m$ and $y = n$ the result

$$(1) \qquad \frac{y!}{x!} \frac{1}{2\pi i} \int_{C_3} (e^z - 1)^x z^{-y-1} dz = \frac{y!(e^{u_0} - 1)^x}{x! 2 u_0^y \sqrt{\pi x u_0 H(u_0)}} \left[1 + \mathcal{O}\left(1/y\right)\right],$$

where $2H(u) = e^u(e^u - 1)^{-1} - u e^u(e^u - 1)^{-2}$ and u_0 is denoted by R by Moser and Wyman. This Moser-Wyman result holds uniformly in x if $y - x \geq y^{1/3}$. Let us now check that their analysis applies to the generalized Stirling numbers. Note that $u_0 H(u_0)$ is bounded away from 0 for all u_0, that $x u_0 \geq y - x$ by the comment just after the Moser-Wyman equation (3.4c) in [7], and that $C_3(R)$ and C_4 in their equation (5.1) are bounded for all R. They do not use the fact that their m, n are integral to derive these facts.

The integrals over C_1, C_2, C_4 and C_5 are easily seen to be $\mathcal{O}((2 + \epsilon)^x / (2\pi - \delta)^y)$ since $|e^z| = \exp[\text{Re}(z)] \leq e^\epsilon$ and $|z| \geq 2\pi - \delta$. This error is exponentially small for $\delta < x \leq y$ so our equation (1) holds uniformly in x as $y \to \infty$ provided $y - x \geq y^{1/3}$. We now show that the result stated in Theorem 1 agrees with equation (1). Letting $t_0 = (y - x)/x$ again and using the Stirling approximation we deduce

$$\frac{y!}{x!} = \binom{y}{x} (y - x)^{y-x} e^{-(y-x)} \sqrt{2\pi(y - x)} \left[1 + \mathcal{O}\left(y^{-1}\right)\right].$$

Thus

$$\begin{aligned} e^A &= e^{\phi(u_0) - x t_0 + (y-x)\log t_0} \\ &= \frac{(e^{u_0} - 1)^x e^{-(y-x)} x^{y-x}}{u_0^y (y-x)^{y-x}} \left[1 + \mathcal{O}\left(y^{-1}\right)\right], \end{aligned}$$

so

$$e^A x^{y-x} f(t_0) \binom{y}{x} = \frac{y!(e^{u_0} - 1)^x}{x! u_0^y} \frac{f(t_0)}{\sqrt{2\pi(y-x)}} \left[1 + \mathcal{O}\left(y^{-1}\right)\right].$$

Now

$$f(t_0) = \left[(y - x)/u_0^2 \phi''(u_0)\right]^{1/2}, \quad \phi''(u_0) = y u_0^{-2} - x e^{u_0}(e^{u_0} - 1)^{-2}$$

and using $y/x = e^{u_0} u_0 / (e^{u_0} - 1)$ we find that

$$\begin{aligned} u_0^2 \phi''(u_0) &= x \left[\frac{y}{x} - \frac{u_0 e^{u_0}}{(e^{u_0} - 1)^2}\right] \\ &= x u_0 \left[\frac{e^{u_0}}{e^{u_0} - 1} - \frac{u_0 e^{u_0}}{(e^{u_0} - 1)^2}\right] = 2 x u_0 H(u_0), \end{aligned}$$

so

$$e^A x^{y-x} f(t_0) \binom{y}{x} = \frac{y!(e^x-1)^x}{x!2u_0^y\sqrt{\pi u_0 H(u_0)}} \left[1 + \mathcal{O}\left(y^{-1}\right)\right].$$

Thus if $y - x \geq y^{1/3}$ the first result of Theorem 1 follows from the Moser-Wyman results.

We turn to the case $y - x < y^{1/3}$ and again follow Moser-Wyman starting at their equation (2.1) and continuing to equation (2.11). We write with Flajolet-Prodinger

$$S_{y-x}^y = \frac{y!}{(y-x)!2\pi i} \int_C \left(\frac{e^z-1}{z}\right)^{y-x} z^{-x-1}\, dz.$$

If one uses

$$\log \frac{e^z-1}{z} = \frac{z}{2} + \sum_{k=1}^\infty (-1)^k \frac{\zeta(2k)}{k(2\pi)^{2k}} z^{2k},$$

and the substitutions $z = 2w/(y-x), q = 2/(y-x)$ one obtains, as do Moser and Wyman,

$$\left(\frac{e^z-1}{z}\right)^{y-x} = \exp\left[w + \sum_{k=1}^\infty (-1)^{k+1} \frac{2\zeta(2k)w^{2k}}{k(2\pi)^{2k}} q^{2k-1}\right]$$

$$= e^w \sum_{k=0}^\infty P_k(w) q^k,$$

where the $P_k(w)$ are polynomials in w with degree at least $k+1$ and at most $2k$. Thus agreeing with Moser and Wyman we get

$$S_{y-x}^y = \binom{y}{x} \frac{x!q^{-x}}{2\pi i} \int_C e^w w^{-x-1} \left[1 + \frac{w^2 q}{12} + \frac{w^4 q^2}{288} + \left(\frac{w^6}{10,368} - \frac{w^4}{1440}\right) q^3 + \cdots\right] dw.$$

Now unlike Moser and Wyman, we use the classical identity, due to Hankel

$$\frac{1}{2\pi i} \int_C e^w w^{-t-1}\, dw = \frac{1}{\Gamma(t)}$$

to obtain

$$S_{y-x}^y = \binom{y}{x} q^{-x} x! \left[\frac{1}{x!} + \frac{q}{12(x-2)!} + \frac{q^2}{288(x-4)!} + \cdots\right]$$

$$= \binom{y}{x} q^{-x} \left[1 + \frac{(x)_2}{6(y-x)} + \frac{(x)_4}{72(y-x)^2} + \left(\frac{(x)_6}{1296} - \frac{(x)_4}{180}\right)/(y-x)^3 + \cdots\right].$$

For the reasons that Moser and Wyman give, this is not only an asymptotic series it is a convergent series for $x = o((y-x)^{1/2})$. We therefore conclude that

$$(2) \qquad S_{y-x}^y = \binom{y}{x} \left(\frac{y-x}{2}\right)^x \left[1 + \mathcal{O}\left(y^{-1/3}\right)\right] = \frac{y^{2x}}{2^x x!}\left[1 + \mathcal{O}\left(y^{-1/3}\right)\right]$$

if $y - x = \mathcal{O}(x^{1/3}) = \mathcal{O}(y^{1/3})$.

REMARK 1.   This analysis can be replaced by the general technique known as Watson's Lemma for loop integrals see Olver [9].

We now show that this completes the proof of Theorem 1. We start with

$$\phi(u_0) = -(y-x)\log u_0 + x\log[(e^{u_0}-1)/u_0]$$

and

$$\frac{e^{u_0}-1}{u_0} = \frac{1-e^{-u_0}}{e^{-u_0}u_0} = \frac{x}{y}e^{u_0},$$

so

$$\log\left(\frac{e^{u_0}-1}{u_0}\right) = \log(x/y) + u_0.$$

Thus

$$\phi(u_0) = -(y-x)\log u_0 + xu_0 + x\log[1-(y-x)/y].$$

Now

$$1-(y-x)/y = x/y = \left(1-e^{-u_0}\right)/u_0 = 1 - u_0/2 + u_0^2/6 + \mathcal{O}\left(u_0^3\right)$$

implies

(3)
$$u_0 = \frac{2(y-x)}{y}\left\{1 + \frac{4(y-x)^2}{3y^2} + \mathcal{O}\left[(1-x/y)^3\right]\right\},$$

so

$$\phi(u_0) = -(y-x)\log u_0 + y - x + \mathcal{O}\left[(y-x)^2/y\right].$$

Also

$$A = \phi(u_0) - xt_0 + (y-x)\log t_0 = (y-x)\log(y-x) - (y-x)\log u_0 - (y-x)\log x.$$

From (3)

$$(y-x)\log u_0 = (y-x)\log 2 + (y-x)\log(y-x) - (y-x)\log y + \mathcal{O}\left[(y-x)^3/y^2\right],$$

so

$$\begin{aligned} A &= -(y-x)\log 2 + (y-x)\log(y/x) + \mathcal{O}\left[(y-x)^2/y\right] \\ &= -(y-x)\log 2 + \mathcal{O}\left[(y-x)^2/y\right]. \end{aligned}$$

Thus

$$e^A x^{y-x} f(t_0)\binom{y}{x}\left[1 + \mathcal{O}\left(y^{-1/3}\right)\right] = \frac{y^{2(y-x)}f(t_0)}{2^{y-x}(y-x)!}\left[1 + \mathcal{O}\left(y^{-1/3}\right)\right],$$

if $y - x = \mathcal{O}(y^{1/3})$. Now $f(t_0) = [t_0/(1 + t_0)(u_0 - t_0)]^{1/2}$ and $t_0/(1 + t_0) = (y - x)/y$.
Furthermore $u_0 - t_0 = 2(y - x)y^{-1} + \mathcal{O}[(y - x)^2/y^2] - (y - x)/x = (y - x)/y + \mathcal{O}[(y - x)^2/y]$,
so $f(t_0) = 1 + \mathcal{O}[(y - x)^2/y]$. Thus if $y - x = \mathcal{O}(y^{1/3})$ the first statement of Theorem 1 is
equivalent to

$$\frac{y^{2(y-x)}}{2^{y-x}(y - x)!} \left[1 + \mathcal{O}\left(y^{-1/3}\right)\right],$$

which agrees with equation (2); note that the x of equation (2) is $y - x$ here.

This proves Theorem 1.

2.2. Proof of Theorem 2

If $h(n) \leq m \leq n - n^a$, where $h(n)$ is any function tending to infinity with n, and a is a
positive constant $< 1/2$ then Moser-Wyman [6] show that (they denote u_1 by R)

$$\begin{aligned}
s_{m+1}^{n+1} &= \frac{\Gamma(1 + n + u_1)}{[2\pi H(u_1)]^{1/2} u_1^{n+1} \Gamma(u_1)} \left[1 + \mathcal{O}\left(1/m\right)\right] \\
&= \frac{(n + u_1)!}{[2\pi H(u_1)]^{1/2} u_1^m u_1!} \left[1 + \mathcal{O}\left(1/m\right)\right],
\end{aligned}$$

where

$$H = m - \sum_{k=1}^{n} u_1^2/(u_1 + k)^2.$$

The fact that m and n are integers is not essential in their analysis. We can choose the
contour of integration as they do. We let

$$\begin{aligned}
\psi(u) &= \log\Gamma(u + y + 1) - \log\Gamma(u + 1) - x\log u, \\
\psi'(u) &= \Psi(u + y + 1) - \Psi(u + 1) - x/u, \\
\Psi(u) &= \frac{d}{du}\Gamma(u), (\text{ our } \Psi \text{ is Moser} - \text{Wyman's } \psi),
\end{aligned}$$

etc., and use the asymptotic estimates for these functions given in Chapter 6 of [1]. One
then obtains the same asymptotic estimates as if x and y were integers but the discrete sums
we write should be interpreted in terms of ψ and its derivative. With this interpretation we
get

$$e^{\psi(u_1)} = \frac{(u_1 + y)!}{u_1^x u_1!} \left[1 + \mathcal{O}\left(1/m\right)\right].$$

Also

$$u_1^2 = x - \frac{u_1^2}{(u_1 + 1)^2} - \cdots - \frac{u_1^2}{(u_1 + y)^2} = H.$$

Thus

$$g(t_1) = \frac{1}{u_1\sqrt{H}}\sqrt{x(y - x)/y}.$$

Furthermore

$$(t_1 + 1)^{-y} = \left(\frac{y}{y-x}\right)^{-y} = \frac{(y-x)^y}{y^y}, \quad t_1^x = \frac{x^x}{(y-x)^x}$$

so, using Stirling's approximation,

$$e^B g(t_1) \binom{y}{x} = \frac{\Gamma(y + u_1 + 1)}{\sqrt{2\pi H} u_1^{x+1}} \left[1 + \mathcal{O}\left(1/x\right)\right],$$

which agrees with the Moser-Wyman result.

Suppose now that $x = \mathcal{O}(\log^{1/2} y)$. Then $t_1 = xy^{-1}(1 - x/y)^{-1} = x/y + \mathcal{O}(\log y/y^2)$. To solve for u_1 note that

$$\psi(u_1) = \frac{1}{1 + u_1} + \frac{1}{2 + u_1} + \cdots + \frac{1}{y + u_1} = \log(u_1 + y) + \gamma + \mathcal{O}(1/y)$$

(even when y is not an integer), so

$$u_1 \log(u_1 + y) = \frac{x}{\log y + \gamma} + \mathcal{O}(1/y) = \mathcal{O}\left(1/\log^{1/2} y\right).$$

Now

$$
\begin{aligned}
e^{\psi(u_1)} &= \frac{\Gamma(u_1 + y + 1)}{\Gamma(u_1)} = y! u_1^{-x} \left[1 + \mathcal{O}\left(\log^{-1/2} y\right)\right] \\
&= y!(\log y + \gamma)^x x^{-x} \left[1 + \mathcal{O}\left(\log^{-1/2} y\right)\right].
\end{aligned}
$$

Furthermore

$$(1 + t_1)^{-y} = (1 - x/y)^y = e^{-x} \left[1 + \mathcal{O}\left(y^{-1} \log y\right)\right],$$

$$t_1^x = \frac{x^x}{(y-x)^x} = \frac{x^x e^x}{y^x} \left[1 + \mathcal{O}\left(y^{-1} \log y\right)\right],$$

so, using $y!/((y - x)! \sim y^x[1 + \mathcal{O}(y^{-1} \log y)]$, we have

$$e^B \binom{y}{x} g(t_1) = \frac{y!(\log y + \gamma)^x g(t)}{x!} \left[1 + \mathcal{O}\left(\log^{-1/2} y\right)\right].$$

Finally, if x is bounded away from 0,

$$u_1^2 \psi''(u_1) = x + \mathcal{O}\left(u_1^2\right) = x + \mathcal{O}\left(\log^{-1} y\right),$$

so

$$g(t_1) = \frac{x}{x + \mathcal{O}(\log^{-1} y)} = 1 + \mathcal{O}\left(\log^{-1} y\right),$$

thus

$$e^B g(t_1) \binom{y}{x} = \frac{y!(\log y + \gamma)^x}{x!} \left[1 + \mathcal{O}\left(\log^{-1} y\right)\right],$$

which agrees with Moser and Wyman if $x = \mathcal{O}(\log^{1/2} y)$ when x and y are integral.

We now consider the remaining range $y - x = \mathcal{O}(y^a)$, where $0 < a$ is a constant $< 1/2$. We note first of all that we can follow Moser-Wyman making only very minor changes to get (choosing their $a = 1/3$) to show that when x and y are integral

$$s_{x+1}^{y+1} = \binom{y+1}{x+1} (x/2)^{y-x} \left[1 + \mathcal{O}\left(y^{-1/3}\right)\right].$$

In fact, Moser and Wyman give a convergent series for s_{x+1}^{y+1}, not only the first term, which is a complete asymptotic expansion.

We now show that the first expression in Theorem 2 (which is Temme's) agrees with this formula. We begin by deriving an estimate for u_1. Note that

$$\psi'(u_1) = \frac{1}{1 + u_1} + \frac{1}{2 + u_1} + \cdots + \frac{1}{y + u_1} - x/u_1$$

and that the above sum can be estimated by Euler-Maclaurin summation in a routine way, giving

$$\log(1 + y/u_1) + \mathcal{O}(1/u_1) = x/u_1,$$

or

$$\frac{y}{u_1} - \frac{y^2}{2u_1^2} + \mathcal{O}\left[(y/u_1)^3\right] = \frac{x}{u_1}.$$

Thus

$$\begin{aligned}
u_1 &= \frac{y^2}{2(y-x)} + \mathcal{O}\left[\frac{y^3}{u_1(y-x)}\right] = \frac{y^2}{2(y-x)} + \mathcal{O}(y) \\
&= \frac{y^2}{2(y-x)} \left[1 + \mathcal{O}\left(\frac{y-x}{y}\right)\right].
\end{aligned}$$

Now

$$e^{\psi(u_1)} = (u_1 + y)! e^{-y} u_1^{-x} / u_1!$$

and from Stirling's approximation this is

$$\begin{aligned}
e^{\psi(u_1)} &= (1 + y/u_1)^{u_1} (1 + y/u_1)^y e^{-y} u_1^{y-x} [1 + \mathcal{O}(1/y)] \\
&= e^{y - y^2/2u_1 + \mathcal{O}(y^3/u_1^2)} \frac{y^{2(y-x)}}{2(y-x)^{y-x}} \left\{1 + \mathcal{O}\left[(y-x)^2/y\right]\right\} \\
&= \frac{y^{2y-2x} e^{y-x}}{2^{y-x}(y-x)^{y-x}} \left\{1 + \mathcal{O}\left[(y-x)^2/y\right]\right\}.
\end{aligned}$$

Furthermore

$$(1 + t_1)^y = y^y/(y-x)^y, \quad t_1^x = x^x/(y-x)^x,$$

so

$$e^{\psi(u_1)}(1 + t_1)^{-y} t_1^x = \frac{e^{y-x}}{2^{y-x}} y^{y-x} \left(\frac{x}{y}\right)^x.$$

Since

$$\left(\frac{x}{y}\right)^x = \left(\frac{x}{x+y-x}\right)^x = e^{x-y}\left\{1 + \mathcal{O}\left[(y-x)/y)\right]\right\},$$

we have that

$$
\begin{aligned}
e^{\psi(u_1)}(1+t_1)^{-y}t_1^x &= \left(\frac{y}{2}\right)^{y-x}\left\{1 + \mathcal{O}\left[(y-x)^2/y)\right]\right\} \\
&= \left(\frac{x}{2}\right)^{y-x}\left\{1 + \mathcal{O}\left[(y-x)^2/y)\right]\right\}.
\end{aligned}
$$

Theorem 2 will now be proved if we show that $g(t_1) \sim 1$. Note

$$
\begin{aligned}
u_1^2\psi''u_1) &= -\frac{u_1^2}{(u_1+1)^2} - \cdots - \frac{u_1^2}{(u_1+y)^2} + x \\
&= -\frac{1}{(1+u_1^{-1})^2} - \cdots - \frac{1}{(1+yu_1^{-1})^2} + x \\
&= -y + \frac{y^2}{u_1} + \mathcal{O}\left[(y-x)^2/y^2\right] + x \\
&= -y + 2(y-x) + \mathcal{O}((y-x)^3/y^2) + x = y - x + \mathcal{O}\left[(y-x)^3/y^2\right].
\end{aligned}
$$

Since

$$g(t_1) = \frac{1}{u_1}\sqrt{\frac{x(y-x)}{y\psi''(u_1)}}, \quad y = x\left[1 + \mathcal{O}\left(y^{-1/3}\right)\right]$$

we have $g(t_1) = 1 + \mathcal{O}(y^{-1/3})$ and Theorem 2 is proved.

2.3. Proof of Theorem 3

We follow the approach of Richmond-Merlini [10], however we use the Moser-Wyman results in place of Gardy's results [3]. The notation below is that of Moser- Wyman and Gardy. (Gardy's Theorem 3 was designed to hold under as general a class of functions as possible not only for the Stirling numbers). Clearly $xu_0 < y$ and equation (3.4) of Moser- Wyman [7] shows $xu_0 > y - x$ so xu_0 plays the role of y in Gardy's work and also in that of Richmond-Merlini. Gardy's $\delta h(\rho) = \rho/(1-e^{-\rho}))$ is also the coefficient of θ^2 in Moser-Wyman's $g(\theta, R)$. Moser-Wyman show that the coefficients, c_k, of θ^k satisfy $c_k \le MR$, M an absolute constant. The significant range of integration in Moser-Wyman is $|\theta| \le (mR)^{-3/8} = (mR)^{1/8}(mR)^{-1/2}$, so the factor $\log y$ in Gardy's work is replaced by $(mR)^{1/8}$ or $y^{1/8}$. This does not matter since Gardy's $h(\rho) = \exp(\rho) - 1$ and

$$\theta^k c_k = \theta^k \frac{d^k \log h(\rho e^{i\theta})}{d^k\theta}\bigg|_{\theta=0} = \mathcal{O}\left[(mR)^{1-3k/8}\right]$$

and so the terms with $k \ge 3$ are negligible. The $\mathcal{O}(y^{-1}\log^2 y)$ terms in Merlini-Richmond become $\mathcal{O}(y^{-3/4})$ using the Moser-Wyman analysis. Thus corresponding to Theorem 3.6 of

[10] we have: If $y - x \geq y^{1/3}$, then

$$\frac{dS_x^y}{dx} \sim S_x^y [\log h(\rho) - \log x],$$

$$\frac{d^2 \log S_x^y}{d^2 x} \sim S_x^y \left[-\log^2 x - \frac{1}{x} + \frac{h'(\rho)}{h(\rho)} \frac{d\rho}{dx} \right].$$

Furthermore, Corollary 3.9 of Richmond-Merlini[10] holds; it says that the term involving $d\rho/dx$ is < 0, so S_x^y is log-concave for $y - x \geq y^{1/3}$.

If $y - x = \mathcal{O}(y^{1/3})$ we can use the exact expression for S_x^y obtained in the proof of Theorem 1. It follows that

$$\log S_x^y = \log(y+1)! - \log(x+1)! - \log(y-x)! + (y-x)\log(x/2) + \cdots,$$

and since the convergent series and the Stirling asymptotic series for the factorial function can be differentiated term by term, we get

$$\frac{d^2 \log S_x^y}{d^2 x} = -\frac{1}{y - x} + \mathcal{O}\left(y^{-2/3}\right).$$

So again S_x^y is log-concave.

3. Closing Comments

It is clear that while we have derived Temme's results in the slightly more general context of generalized Stirling numbers, the situation is not satisfactory. It would be much preferable to show how to choose the contour of integration so that Temme's $f(t)$ (or $G(t)$) has no singularities on or in the contour of integration, indeed to see how to do this for integrals other than those representing the Stirling numbers. The Moser-Wyman technique for the extreme ranges can be replaced by Watson's Lemma for loop integrals or other quite general methods.

The log-concavity results show that the generalized Stirling numbers of the second kind have a unique maximum for $\delta < x \leq y$ for large y. The problem of showing this for integral variables seems to be still open. Erdös [2] has shown that the Stirling numbers of the first kind have a unique maximum when the variables are integral.

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