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Interpolation on Sparse Grids and Tensor Products of Nikol'skij–Besov Spaces[1]

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We investigate the order of convergence of periodic interpolation on sparse grids (blending interpolation) in the framework of tensor products of Nikol'skij–Besov spaces. To this end, we make use of the uniformity of the considered tensor norms and provide a unified approach to error estimates for the interpolation of univariate periodic functions from Nikol'skij–Besov spaces.

KEY WORDS: Periodic interpolation; sparse grids; Nikol'skij–Besov spaces; spaces of dominating mixed smoothness.

1. INTRODUCTION

In this paper, we deal with the error of approximation of periodic functions obtained by interpolation on sparse grids. The method itself is well known, see Baszenski and Delvos [3], Delvos and Schempp [6], Pöplau and Sprengel [18] and Sprengel [25,26]. What is new here is the choice of the underlying function spaces. Baszenski and Delvos [3], Delvos and Schempp [6], Pöplau and Sprengel [18] dealt with tensor products of spaces defined by certain decay properties of the Fourier coefficients (Korobov spaces or potential spaces built on L_2, respectively). Based on [24], we are able to investigate the problem in more appropriate spaces, namely, on Nikol'skij–Besov spaces and its tensor products.

Interpolation on sparse grids can be reduced to error estimates of corresponding interpolation processes in the one-dimensional situation. As a result, we obtain estimates like

$$\|f - B_j f \,|\, L_p(\mathbb{T}^2)\| \le Cj2^{-js}, \tag{1}$$

where B_j denotes the interpolation operator with respect to the sparse grid (having $j2^j$ knots approximately). Comparing this with interpolation on the

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full grid (having 2^{2j} knots approximately), we would end with an estimate

$$\|f - I_j f | L_p(\mathbb{T}^2)\| \le C 2^{-js}. \tag{2}$$

Thus, except for a logarithmic term, the error is of the same order. The price we have to pay consists in the following: whereas (2) is true for all functions taken from the Nikol'skij–Besov space $B^s_{p,\infty}(\mathbb{T}^2)$ inequality (1) holds true for the functions with dominating mixed smoothness properties from $B^s_{p,\infty}(\mathbb{T}) \otimes_{\alpha_p} B^s_{p,\infty}(\mathbb{T})$ (the tensor product with respect to the p-nuclear norm). These classes of functions are close to the usual Besov spaces of dominating mixed smoothness, cf. Subsection 2.2.

We restrict ourselves to the bivariate situation mostly for transparency. In addition, all can be done in the higher dimensional situation.

Investigations of approximation procedures in spaces of dominating mixed smoothness attracted much attention—see the monograph of Temlyakov [27] or the recent papers by DeVore, Petrushev, and Temlyakov [8] and DeVore, Konyagin, and Temlyakov [7].

The paper is organized as follows. In Section 2, we collect information about the underlying spaces. Further, to prepare those estimates as in (1), we investigate the approximation power of certain interpolation processes in the univariate situation. This can be found in Section 3. Finally, Section 4 deals with interpolation on sparse grids.

2. PERIODIC BESOV SPACES

2.1. Besov Spaces on the Torus

Recall some definitions. For our purpose, it will be sufficient to deal with the one-dimensional situation. Let $1 < p < \infty$. As usual, for a natural number m and $f \in L_p(\mathbb{T})$, we put

$$\omega_m(t, f)_p = \sup_{|h| < t} \|\Delta_h^m f | L_p(\mathbb{T})\|, \qquad t > 0,$$

where Δ_h^m denotes an mth order difference with step length h. Let $m > s > 0$ and $1 \le q < \infty$. Then the Besov space $B^s_{p,q}(\mathbb{T})$ is defined as the set of all 2π-periodic functions in $L_p(\mathbb{T})$ such that

$$\|f | B^s_{p,q}(\mathbb{T})\| = \|f | L_p(\mathbb{T})\| + \left(\int_0^1 [t^{-s} \omega_m(t, f)_p]^q \frac{dt}{t} \right)^{1/q} < \infty.$$

Similarly, the Nikol'skij–Besov space $B^s_{p,\infty}(\mathbb{T})$ is the set of all 2π-periodic functions in $L_p(\mathbb{T})$ such that

$$\|f | B^s_{p,\infty}(\mathbb{T})\| = \|f | L_p(\mathbb{T})\| + \sup_{t < 1} t^{-s} \omega_m(t, f)_p < \infty.$$

In both cases, different m will lead to equivalent norms. We refer to De Vore and Lorentz [9], Nikol'skij [15], and Triebel [30] for the basics of theory of Besov spaces.

2.2. Besov and Sobolev Spaces of Dominating Mixed Smoothness

We restrict ourselves here to the two-dimensional situation. To introduce mixed differences, we need the following modification. For natural numbers m_1, m_2, real numbers h_1, h_2, and $f \in L_p(\mathbb{T})$, let $\Delta_{h_1,1}^{m_1} f(x_1, x_2)$ be the difference of order m_1 taken in the first variable. Analogously, $\Delta_{h_2,2}^{m_2} f(x_1, x_2)$ is defined. Finally, we put

$$\Delta_{h_1,h_2}^{(m_1,m_2)} f(x_1, x_2) = \Delta_{h_2,2}^{m_2}(\Delta_{h_1,1}^{m_1} f)(x_1, x_2).$$

Further, we use $c_{k_1,k_2}(f)$ as an abbreviation of the Fourier coefficients of f, more precisely

$$c_{k_1,k_2}(f) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x_1, x_2)\, e^{-i(k_1x_1 + k_2x_2)}\, dx_1\, dx_2.$$

Definition 1. Let $1 < p < \infty$, $1 \le q \le \infty$ and $s_1, s_2 > 0$.

(i) Let m_1 and m_2 denote natural numbers such that $s_1 < m_1$ and $s_2 < m_2$. Then the Besov space of dominating mixed smoothness $S_{p,q}^{s_1,s_2}B(\mathbb{T}^2)$ is the set of all functions $f \in L_p(\mathbb{T}^2)$ such that

$$\begin{aligned} \|f \,|\, S_{p,q}^{s_1,s_2}B(\mathbb{T}^2)\| = {} & \left(\int_0^1 [h_1^{-s_1} \|\Delta_{h_1,1}^{m_1} f \,|\, L_p(\mathbb{T}^2)\|]^q \frac{dh_1}{h_1}\right)^{1/q} \\ & + \left(\int_0^1 [h_2^{-s_2} \|\Delta_{h_2,2}^{m_2} f \,|\, L_p(\mathbb{T}^2)\|]^q \frac{dh_2}{h_2}\right)^{1/q} \\ & + \left(\int_0^1 \int_0^1 [h_1^{-s_1} h_2^{-s_2} \|\Delta_{h_1,h_2}^{(m_1,m_2)} f \,|\, L_p(\mathbb{T}^2)\|]^q \frac{dh_1}{h_1} \frac{dh_2}{h_2}\right)^{1/q} < \infty, \end{aligned}$$

(usual modification if $q = \infty$).

(ii) The Sobolev space (of fractional order) of dominating mixed smoothness $S_p^{s_1,s_2}H(\mathbb{T}^2)$ is the set of all functions $f \in L_p(\mathbb{T}^2)$, such that

$$\begin{aligned} & \|f \,|\, S_p^{s_1,s_2}H(\mathbb{T}^2)\| \\ & \quad = \left\| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} c_{k_1,k_2}(f)(1 + |k_1|)^{s_1}(1 + |k_2|)^{s_2} e^{i(k_1x_1 + k_2x_2)} \,|\, L_p(\mathbb{T}^2) \right\| < \infty, \end{aligned}$$

For a detailed investigation of these classes of functions, we refer to Amanov [1] and Schmeisser and Triebel [23], cf. also Schmeisser [22]. In addition, we refer to the recent monograph by Temlyakov [27], where similar spaces (but not the same) are investigated.

The mixed smoothness spaces can be seen as tensor products of univariate function spaces

$$S^{s_1,s_2}_{p,q}B(\mathbb{T}^2) = B^{s_1}_{p,q}(\mathbb{T}) \otimes_b B^{s_2}_{p,q}(\mathbb{T})$$

for $q < \infty$, where the index b indicates that we took the completion of the algebraic tensor product $B^{s_1}_{p,q}(\mathbb{T}) \otimes B^{s_2}_{p,q}(\mathbb{T})$ with respect to the usual Besov norm, see Definition 1. For our error estimates, we will need another tensor product space

$$B^{s_1}_{p,\infty}(\mathbb{T}) \otimes_{\alpha_p} B^{s_2}_{p,\infty}(\mathbb{T}),$$

where the completion was taken with respect to the p-nuclear norm α_p, cf. [13,24] for details. In what follows, we need the properties:

$$S^{s_1,s_2}_{p,q_0}B(\mathbb{T}^2) \hookrightarrow S^{s_1,s_2}_{p,q_1}B(\mathbb{T}^2) \hookrightarrow S^{s_1,s_2}_{p,\infty}B(\mathbb{T}^2) \qquad \text{if } q_0 \le q_1,$$

$$S^{s_1,s_2}_{p,q}B(\mathbb{T}^2) \hookrightarrow C(\mathbb{T}^2) \qquad \text{if } \min(s_1, s_2) < \frac{1}{p},$$

cf. [23, Chap. 2.4] and

$$S^{s_1,s_2}_{p}H(\mathbb{T}^2), \qquad S^{s_1,s_2}_{p,p}B(\mathbb{T}^2) \hookrightarrow B^{s_1}_{p,\infty}(\mathbb{T}) \otimes_{\alpha_p} B^{s_2}_{p,\infty}(\mathbb{T}) \hookrightarrow S^{s_1,s_2}_{p,\infty}B(\mathbb{T}^2). \tag{3}$$

Further, we need

$$L_p(\mathbb{T}) \otimes_{\alpha_p} L_p(\mathbb{T}) = L_p(\mathbb{T}^2),$$

cf. [13, Cor. 1.5.2]. The advantage of the tensor product approach using the p-nuclear norms consists in the following. Suppose $P, Q \in \mathscr{L}(C(\mathbb{T}))$ and define

$$(P \otimes Q)(f(x_1) \otimes g(x_2)) = P(f)(x_1) \otimes Q(g)(x_2)$$

for all $f, g \in C(\mathbb{T})$. Moreover, we assume $s_1, s_2 > 1/p$, $P \in \mathscr{L}(B^{s_1}_{p,\infty}(\mathbb{T}), L_p(\mathbb{T}))$ and $Q \in \mathscr{L}(B^{s_2}_{p,\infty}(\mathbb{T}), L_p(\mathbb{T}))$. Then, due to the uniformity of the p-nuclear norms, the restriction of $P \otimes Q$ to $B^{s_1}_{p,\infty}(\mathbb{T}) \otimes_{\alpha_p} B^{s_2}_{p,\infty}(\mathbb{T})$ belongs to

$$\mathscr{L}(B^{s_1}_{p,\infty}(\mathbb{T}) \otimes_{\alpha_p} B^{s_2}_{p,\infty}(\mathbb{T}),\ L_p(\mathbb{T}^2))$$

and

$$\begin{aligned} \|(P \otimes Q)h \,|\, L_p(\mathbb{T}^2)\| \le C \|P \,|\, \mathscr{L}(B^{s_1}_{p,\infty}(\mathbb{T}), L_p(\mathbb{T}))\| \\ \times \|Q \,|\, \mathscr{L}(B^{s_2}_{p,\infty}(\mathbb{T}), L_p(\mathbb{T}))\| \, \|h \,|\, B^{s_1}_{p,\infty}(\mathbb{T}) \otimes_{\alpha_p} B^{s_2}_{p,\infty}(\mathbb{T})\| \end{aligned} \tag{4}$$

with some constant C independent of P, Q and $h \in B^{s_1}_{p,\infty}(\mathbb{T}) \otimes_{\alpha_p} B^{s_2}_{p,\infty}(\mathbb{T})$, cf. [13,24].

3. INTERPOLATION ON EQUIDISTANT GRIDS

We are interested in chains of interpolation operators. For technical reasons, we will be restricted to those that are induced by interpolation processes on the real line. We shall start with recalling some well known results of Ries and Stens [20] for approximation by sampling sums in the supremum norm. Then we investigate the case $1<p<\infty$. In a final step, we use complex interpolation of Besov spaces to weaken the restrictions under which we have derived our estimates for $1<p<\infty$.

3.1. Preliminaries

Let N be a natural number and denote by $J_N=\{k\in\mathbb{Z}: -N/2\leq k< N/2\}$ a related set of indices. Further

$$T_N=\left\{\sum_{k\in J_N}\eta_k\, e^{ikx}: \eta_k\in\mathbb{C}\right\}$$

denotes a corresponding set of trigonometric polynomials. The discrete Fourier coefficients of a continuous function f are given by

$$c_k^N(f)=\frac{1}{N}\sum_{l\in J_N} f\left(\frac{2\pi l}{N}\right)e^{i2\pi kl/N}, \qquad k\in J_N.$$

Discrete Fourier coefficients $c_k^N(f)$ and Fourier coefficients $c_k(f)$ of a function f are connected by aliasing

$$c_k^N(f)=\sum_{l\in\mathbb{Z}} c_{k+lN}(f),$$

as long as f belongs to the Wiener algebra $\mathscr{A}(\mathbb{T})$ of functions having an absolutely summable Fourier series. We shall consider interpolation on equidistant grids of type $\mathscr{T}_N=\{2\pi k/N: k\in J_N\}$. The continuous and 2π-periodic function Λ_N^π is called a fundamental interpolant for $\mathscr{T}_N$ if

$$\Lambda_N^\pi\left(\frac{2\pi k}{N}\right)=\begin{cases}1 & \text{if } k=0,\\ 0 & \text{if } k\neq 0,\end{cases} \qquad k\in J_N.$$

The associated interpolation operator I_N is defined as

$$I_N f(x)=\sum_{k\in J_N} f\left(\frac{2\pi k}{N}\right)\Lambda_N^\pi\left(x-\frac{2\pi k}{N}\right). \tag{5}$$

The Fourier coefficients of $I_N f$ can be easily computed as

$$c_k(I_N f) = N c_k^N(f) c_k(\Lambda_N^\pi) = N c_k(\Lambda_N^\pi) \sum_{l \in \mathbb{Z}} c_{k+lN}(f) \tag{6}$$

as long as $f \in \mathscr{A}(\mathbb{T})$. Finally, we denote the Nth Fourier partial sum by

$$S_N f(x) = \sum_{k \in J_N} c_k(f)\, e^{ikx}, \qquad N \in \mathbb{N}, \qquad f \in L_1(\mathbb{T}).$$

3.2. The Error of Approximation in the Supremum Norm

On the real line, there is a well-developed theory of so-called quasi-interpolation. Then, given some basic function ϕ, the rate of convergence of

$$\left\| f(x) - \sum_{k \in \mathbb{Z}} \alpha_{h,k}(f) \phi(hx - k) \middle| L_p(\mathbb{R}) \right\|, \qquad h \to 0, \qquad h > 0,$$

can be determined in dependence of the local reproduction of polynomials by $\sum_{k \in \mathbb{Z}} \alpha(1,k)\phi(x-k)$. Here $\alpha_{h,k}$ are appropriate functionals (cf. DeVore and Lorentz [9, Chap. 13.7] and Jia and Lei [12] for details). Until now, there seems to be no complete periodic counterpart. However, if we consider approximation in $\|\cdot | C(\mathbb{T})\|$, then simple arguments allow a transformation of estimates obtained in the nonperiodic situation to the periodic one.

Let $\Lambda: \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\Lambda(2\pi k) = \delta_{0,k}, \qquad k \in \mathbb{Z}, \tag{7}$$

$$\sum_{k \in \mathbb{Z}} |\Lambda(x - 2\pi k)| \qquad \text{converges uniformly on } [0, 2\pi]. \tag{8}$$

The second assumption guarantees $\Lambda \in L_1(\mathbb{R})$ and hence, the Fourier transform of Λ

$$(\mathscr{F}\Lambda)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, \Lambda(x)\, dx$$

is a continuous function on $\mathbb{R}$. Hence, the following definition makes sense

$$\Lambda_N^\pi(x) = \sum_{l=-\infty}^{\infty} \Lambda(Nx - 2l\pi N), \qquad N \in \mathbb{N}, \qquad x \in \mathbb{T}. \tag{9}$$

Some simple properties are collected in the following lemma.

Lemma 1. Let Λ be a continuous function satisfying (7) and (8).

(i) The function Λ_N^π defines a 2π-periodic continuous fundamental interpolant, i.e.,

$$\Lambda_N^\pi\left(\frac{2k\pi}{N}\right) = \delta_{0,k}, \qquad k \in J_N.$$

Its Fourier series is given by

$$\frac{1}{N\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\mathscr{F}\Lambda\left(\frac{k}{N}\right)e^{ikx}.$$

(ii) We have the equivalence of

$$\sum_{l=-\infty}^{\infty}\Lambda(x-2\pi l)=1,\qquad x\in\mathbb{T},\tag{10}$$

and

$$\mathscr{F}\Lambda(k)=\sqrt{2\pi}\,\delta_{0,k},\qquad k\in\mathbb{Z}.\tag{11}$$

(iii) If Λ satisfies a refinement equation

$$\frac{1}{2}\Lambda\left(\frac{x}{2}\right)=\sum_{k\in\mathbf{Z}}h_k\Lambda(x+2\pi k),\qquad x\in\mathbb{R},\tag{12}$$

with the mask $\{h_k\}_{k\in\mathbf{Z}}\in l_1$, then

$$\Lambda_N^{\pi}(x)=2\sum_{k=0}^{2N-1}\left(\sum_{r=-\infty}^{\infty}h_{k+2rN}\right)\Lambda_{2N}^{\pi}\left(x+\frac{2\pi k}{2N}\right).$$

Proof. Continuity of Λ_N^{π} becomes a consequence of (8). Also, the interpolation conditions can be derived from this requirement taking into account (7). The calculation of the Fourier series is elementary. Part (ii) is taken from Ries and Stens [20]. Finally, (iii) is again elementary. □

The absolute moment of order $\alpha>0$ of Λ is given by

$$m_\alpha(\Lambda)=\sup_{0\le x\le 2\pi}\sum_{k\in\mathbf{Z}}|x-2\pi k|^{\alpha}|\Lambda(x-2\pi k)|.$$

Proposition 1. Let $\Lambda:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying (7), (8) and (11). Let Λ_N^{π} and I_N be the associated fundamental interpolant and interpolation operator, respectively.

(i) We assume $m_\alpha(\Lambda)<\infty$ for some $0<\alpha<1$. Let $0<s\le\alpha$. Then there exists a constant C (independent of N) such that

$$\|f-I_Nf\,|\,C(\mathbb{T})\|\le CN^{-s}\|f\,|\,B^s_{\infty,\infty}(\mathbb{T})\|\tag{13}$$

holds for all $f\in B^s_{\infty,\infty}(\mathbb{T})$.

(ii) We assume $m_r(\Lambda)<\infty$ and $(\mathscr{F}\Lambda)^{(j)}(k)=0$, $j=1,2,\ldots,r-1$, $k\in\mathbb{Z}$, for some $r\in\mathbb{N}$. Let $0<s<r$. Then (13) holds.

Proof. Employing periodicity of the interpolated function and Lemma 1, we obtain

$$\begin{aligned} f(x) - I_N f(x) &= \sum_{k=0}^{N-1} \left(f(x) - f\left(\frac{2\pi k}{N}\right)\right)\Lambda_N\left(x - \frac{2\pi k}{N}\right) \\ &= \sum_{k=0}^{N-1} \left(f(x) - f\left(\frac{2\pi k}{N}\right)\right) \sum_{l=-\infty}^{\infty} \Lambda(Nx - 2\pi k - 2\pi l N) \\ &= \sum_{m=-\infty}^{\infty} \left(f(x) - f\left(\frac{2\pi m}{N}\right)\right) \Lambda(Nx - 2\pi m). \end{aligned}$$

Hence, the periodic situation can be traced back to the nonperiodic one; for this, see Ries and Stens [20]. □

Remark 1. As we stated before, Proposition 1 has been proved by Ries and Stens [20] in the nonperiodic case.

3.3. The Error of Approximation in the L_p Norm

We consider functions $f \in B^s_{p,\infty}(\mathbb{T}) \cap \mathscr{A}(\mathbb{T})$. Our approach is based on Fourier multiplier assertions and this requires additional restrictions compared with the preceding subsection.

In what follows, we shall need, from time to time, a smooth cut-off function. To this end, let ψ be a C^∞-function defined on the real line such that $\psi(x) = 1$ if $|x| \le 1$ and $\psi(x) = 0$ if $|x| \ge 2$.

Proposition 2. Let $\Lambda: \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (7) and (8). Moreover, we assume

$$\int_{-\infty}^{\infty} \left| \mathscr{F}^{-1}\left\{\psi(\xi)|\xi|^{-\alpha}\left(1 - \frac{\mathscr{F}\Lambda(\xi)}{\sqrt{2\pi}}\right)\right\}(w)\right| dw < \infty, \tag{14}$$

$$\sum_{l \ne 0} \int_{-\infty}^{\infty} \left| \mathscr{F}^{-1}\left\{\psi(\xi)|\xi|^{-\alpha} \mathscr{F}\Lambda(\xi + l)\right\}(w)\right| dw < \infty, \tag{15}$$

$$\sum_{l \in \mathbb{Z}} \int_{-\infty}^{\infty} \left| \mathscr{F}^{-1}\left\{(1 - \psi(\xi))|\xi|^{-\beta} \mathscr{F}\Lambda(\xi - l)\right\}(w)\right| dw < \infty \tag{16}$$

for some $\alpha > 0$ and $\beta \ge 0$.

Let Λ_N^π (cf. (9)) and I_N (cf. (5)) be the associated fundamental interpolants and interpolation operator, respectively. Let $0 < s < \alpha$, $\beta < s$ and let

$1<p<\infty$. Then there exists a constant C (independent of N) such that

$$\|f-I_N f\,|\,L_p(\mathbb{T})\| \le CN^{-s}\|f\,|\,B^s_{p,\infty}(\mathbb{T})\| \tag{17}$$

holds for all $f\in B^s_{p,\infty}(\mathbb{T})\cap\mathscr{A}(\mathbb{T})$.

Remark 2. In case $p=2$, one can greatly simplify the conditions on Λ. One can even avoid taking the long way around the nonperiodic case. All conditions can be formulated in terms of the Fourier coefficients of the periodic fundamental interpolant Λ^π_N (cf. Brumme [4] or Pöplau and Sprengel [18]).

We subdivide the proof of (17) into two lemmata. In the first one, we investigate the error for trigonometric polynomials taken from T_N.

We have to consider sums of the type $\sum_{m\in\mathbb{Z},m\neq 0} a_m$ fairly often. In such situation, we simply write $\sum_{m\neq 0} a_m$.

Lemma 2. Let $\Lambda:\mathbb{R}\to\mathbb{R}$ be a function satisfying (7), (8), (14) and (15). Let $0<s<\alpha$ and $1<p<\infty$. Then there exists a constant C (independent of N) such that

$$\|f-I_N f\,|\,L_p(\mathbb{T})\| \le CN^{-s}\|f\,|\,B^s_{p,\infty}(\mathbb{T})\| \tag{18}$$

holds for all $f\in T_N$.

Proof. We shall employ the following Fourier analytical characterization of Nikol'skij–Besov spaces. We put

$$\|f\,|\,B^s_{p,\infty}(\mathbb{T})\|^* = |c_0(f)| + \sup_{j=0,1,\ldots} 2^{js}\left\| \sum_{2^j\le|k|<2^{j+1}} c_k(f)\,e^{ikx}\,\middle|\,L_p(\mathbb{T})\right\|. \tag{19}$$

Then $f\in B^s_{p,\infty}(\mathbb{T})$ if, and only if, $f\in L_p(\mathbb{T})$ and $\|f|B^s_{p,\infty}(\mathbb{T})\|^*<\infty$. Moreover, $\|\cdot|B^s_{p,\infty}(\mathbb{T})\|^*$ yields an equivalent norm (cf. e.g. [23, Chap. 3.7.2]).

Our assumption $f\in T_N$ together with Lemma 1 and (6) yield

$$\begin{aligned} f(x)-I_N f(x) &= \sum_{m=-\infty}^{\infty}\left[c_m(f) - N\sum_{l\in\mathbb{Z}} c_{m+lN}(f)c_m(\Lambda^\pi_N)\right]e^{imx} \\ &= \sum_{m\neq 0}\left[c_m(f) - Nc_m(f)\frac{1}{N\sqrt{2\pi}}\mathscr{F}\Lambda\left(\frac{m}{N}\right)\right]\psi\left(\frac{m}{N}\right)e^{imx} \\ &\quad - \sum_{m\in\mathbb{Z}}\sum_{l\neq 0}\frac{1}{\sqrt{2\pi}}\mathscr{F}\Lambda\left(\frac{m}{N}\right)c_{m+lN}(f)\,e^{imx}. \end{aligned}$$

The two parts of the sum we denote by

$$F_1(x) = \sum_{m\in\mathbb{Z}} \left[c_m(f) - N c_m(f) \frac{1}{N\sqrt{2\pi}} \mathscr{F}\Lambda\left(\frac{m}{N}\right)\right] \psi\left(\frac{m}{N}\right) e^{imx}$$

$$F_2(x) = \sum_{m\neq 0} \sum_{l\neq 0} \frac{1}{\sqrt{2\pi}} \mathscr{F}\Lambda\left(\frac{m}{N}\right) c_{m+lN}(f)\, e^{imx}.$$

Next, we employ that the function

$$M(\xi) = \psi(\xi/N)|\xi|^{-\alpha}\left(1 - \frac{\mathscr{F}\Lambda(\xi/N)}{\sqrt{2\pi}}\right)$$

is continuous in $\mathbb{R}\setminus\{0\}$ and has an absolutely integrable Fourier transform. Hence, interpreting F_1 as a convolution (cf. [23, Chap. 3.3.4, formula (2)]), we find

$$\begin{aligned} \|F_1|L_p(\mathbb{T})\| &= \left\| \sum_{m\neq 0} c_m(f)\psi\left(\frac{m}{N}\right)\left(1 - N\frac{1}{N\sqrt{2\pi}}\mathscr{F}\Lambda\left(\frac{m}{N}\right)\right) e^{imx} \middle| L_p(\mathbb{T}) \right\| \\ &= \frac{1}{\sqrt{2\pi}} \left\| \int_{-\infty}^{\infty} (\mathscr{F}^{-1}M(y))\left(\sum_{m\neq 0} c_m(f)|m|^{\alpha} e^{im(x-y)}\right) dy \middle| L_p(\mathbb{T}) \right\| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\mathscr{F}^{-1}M(y)|\, dy \left\| \sum_{m\neq 0} c_m(f)|m|^{\alpha} e^{imx} \middle| L_p(\mathbb{T}) \right\|. \end{aligned} \tag{20}$$

Observe, that the homogeneity of the Fourier transform implies

$$\begin{aligned} \int_{-\infty}^{\infty} |\mathscr{F}^{-1}M(y)|\, dy &= \int_{-\infty}^{\infty} \left| \mathscr{F}^{-1}\left\{\psi\left(\frac{\xi}{N}\right)|\xi|^{-\alpha}\left(1 - \frac{\mathscr{F}\Lambda(\xi/N)}{\sqrt{2\pi}}\right)\right\}(y)\right| dy \\ &= N^{-\alpha} \int_{-\infty}^{\infty} \left| \mathscr{F}^{-1}\left\{\psi(\xi)|\xi|^{-\alpha}\left(1 - \frac{\mathscr{F}\Lambda(\xi)}{\sqrt{2\pi}}\right)\right\}(y)\right| dy \\ &= N^{-\alpha} C, \end{aligned} \tag{21}$$

where C is a positive number independent of N. It remains to estimate $\|\sum_{m\neq 0} c_m(f)|m|^{\alpha} e^{imx}|L_p(\mathbb{T})\|$. Suppose $2^r \leq N < 2^{r+1}$. We find

$$\begin{aligned} N^{-\alpha}\left\| \sum_{m\neq 0} c_m(f)|m|^{\alpha} e^{imx} \middle| L_p(\mathbb{T}) \right\| &\leq N^{-\alpha} \sum_{l=0}^{r} \left\| \sum_{2^l \leq |m| < 2^{l+1}} c_m(f)|m|^{\alpha} e^{imx} \middle| L_p(\mathbb{T}) \right\| \\ &\leq C_1 N^{-\alpha} \sum_{l=0}^{r} 2^{l\alpha} \left\| \sum_{2^l \leq |m| < 2^{l+1}} c_m(f)\, e^{imx} \middle| L_p(\mathbb{T}) \right\| \\ &\leq C_1 \|f|B^s_{p,\infty}(\mathbb{T})\| N^{-\alpha} \sum_{l=0}^{r} 2^{l(\alpha - s)} \\ &\leq C_2 \|f|B^s_{p,\infty}(\mathbb{T})\| N^{-s}, \end{aligned} \tag{22}$$

where we used the Fourier multiplier assertion

$$\left\| \sum_{2^l \le |m| < 2^{l+1}} c_m(f) 2^{-l\alpha} |m|^\alpha e^{imx} \,\middle|\, L_p(\mathbb{T}) \right\| \le C_1 \left\| \sum_{2^l \le |m| < 2^{l+1}} c_m(f)\, e^{imx} \,\middle|\, L_p(\mathbb{T}) \right\|.$$

Here C_1 can be estimated as follows

$$C_1 \le c \| \, |\xi|^\alpha \chi(\xi) | W_2^1(\mathbb{R}) \|,$$

where $\chi: \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $\chi(\xi) = 1$ if $1 \le \xi \le 2$ and $\chi(\xi) = 0$ if $|\xi| < \frac{1}{2}$ (cf. Schmeisser and Triebel [23, Chap. 3.3.4]). Hence, C_1 is bounded independent of l. Consequently, C_2 becomes a constant independent of N. Plugging this into (20) and taking into account (21), then (18) follows with F_1 instead of $f - I_N f$.

We shall derive an estimate of F_2. We put

$$M_l(\xi) = \psi\left(l + \frac{\xi}{N}\right) |\xi + lN|^{-\alpha} \mathscr{F}\Lambda\left(\frac{\xi}{N}\right).$$

In a similar way as above, we obtain

$$\begin{aligned} &\|F_2 | L_p(\mathbb{T})\| \\ &\le \left\| \sum_{m \in \mathbb{Z}} \sum_{l \ne 0} c_{m+lN}(f) \mathscr{F}\Lambda\left(\frac{m}{N}\right) e^{imx} \,\middle|\, L_p(\mathbb{T}) \right\| \\ &\le \sum_{l \ne 0} \left\| \int_{-\infty}^{\infty} (\mathscr{F}^{-1} M_l(y)) \left(\sum_{m \in \mathbb{Z}} c_{m+lN}(f) |m + lN|^\alpha e^{i(m+lN)(x-y)} \right) dy \,\middle|\, L_p(\mathbb{T}) \right\| \\ &\le \sum_{l \ne 0} \int_{-\infty}^{\infty} |\mathscr{F}^{-1} M_l(y)|\, dy \left\| \sum_{m \in \mathbb{Z}} c_m(f) |m|^\alpha e^{imx} \,\middle|\, L_p(\mathbb{T}) \right\|. \end{aligned}$$

The same arguments as used in case of F_1 can be applied to finish the estimate of F_2 (here one has to apply (15)) ending up with (18) but with F_2 instead of $f - I_N f$. □

Remark 3. With some additional effort [one has to use a different equivalent norm in $B^s_{p,\infty}(\mathbb{T})$], one can extend the validity of above lemma to $p = 1$ and $p = \infty$.

Lemma 3. Let $\Lambda: \mathbb{R} \to \mathbb{R}$ satisfy (7), (8) and (16). Let $\beta < s$ and let $1 < p < \infty$. Then there exists a constant C (independent of N) such that

$$\|I_N(f - S_N f) | L_p(\mathbb{T})\| \le C N^{-s} \|f | B^s_{p,\infty}(\mathbb{T})\| \tag{23}$$

holds for all $f \in B^s_{p,\infty}(\mathbb{T}) \cap \mathscr{A}(\mathbb{T})$.

In the proof of this lemma, we will also need Sobolev spaces $H_p^s(\mathbb{T})$ of fractional order on the torus equipped with the norm

$$\|f|H_p^s(\mathbb{T})\| = \left\| \sum_{k\in\mathbb{Z}} (1+|k|)^s c_k(f)\, e^{ikx} \right| L_p(\mathbb{T}) \Bigg\|.$$

Proof. By assumption, f is continuous and has absolutely summing Fourier coefficients. Thus, we may apply (6), as well. Similar to the proof of the preceding lemma, we derive

$$\begin{aligned} &I_N(f-S_Nf)(x) \\ &\quad = \sum_{l\in\mathbb{Z}} \sum_{m=-\infty}^{\infty} \left[\left(1-\psi\left(2\frac{m+lN}{N}\right)\right) N c_{m+lN}(f-S_Nf) \frac{1}{N\sqrt{2\pi}} \mathcal{F}\Lambda\left(\frac{m}{N}\right) \right] e^{imx}. \end{aligned}$$

This yields

$$\begin{aligned} &|I_N(f-S_Nf)(x)| \\ &\quad \le \sum_{l\in\mathbb{Z}} \left| \sum_{m=-\infty}^{\infty} \left[\left(1-\psi\left(2\frac{m}{N}+2l\right)\right) c_{m+lN}(f-S_Nf) \mathcal{F}\Lambda\left(\frac{m}{N}\right) \right] e^{i(m+lN)x} \right|. \end{aligned}$$

Employing that the functions

$$M_l(\xi) = \left(1-\psi\left(2\frac{\xi}{N}+2l\right)\right) |\xi+lN|^{-\beta} \mathcal{F}\Lambda\left(\frac{\xi}{N}\right)$$

are continuous in $\mathbb{R}$ and have integrable Fourier transforms, we obtain

$$\begin{aligned} &\|I_N(f-S_Nf)|L_p(\mathbb{T})\| \\ &\quad \le \sum_{l\in\mathbb{Z}} \left\| \sum_{m\in\mathbb{Z}} c_{m+lN}(f-S_Nf) \left(1-\psi\left(2\frac{m}{N}+2l\right)\right) \mathcal{F}\Lambda\left(\frac{m}{N}\right) e^{i(m+lN)x} \right| L_p(\mathbb{T}) \Bigg\| \\ &\quad \le \sum_{l\in\mathbb{Z}} \left\| \int_{-\infty}^{\infty} \mathcal{F}^{-1}M_l(y) \left(\sum_{m\in\mathbb{Z}} c_{m+lN}(f-S_Nf) |m+lN|^{\beta} e^{i(m+lN)(x-y)} \right) dy \right| L_p(\mathbb{T}) \Bigg\| \\ &\quad \le \sum_{l\in\mathbb{Z}} \int_{-\infty}^{\infty} |\mathcal{F}^{-1}M_l(y)|\, dy \left\| \sum_{m\in\mathbb{Z}} c_{m+lN}(f-S_Nf) |m+lN|^{\beta} e^{i(m+lN)x} \right| L_p(\mathbb{T}) \Bigg\| \\ &\quad \le \sum_{l\in\mathbb{Z}} \int_{-\infty}^{\infty} |\mathcal{F}^{-1}M_l(y)|\, dy \|f-S_Nf|H_p^{\beta}(\mathbb{T})\|. \end{aligned} \tag{24}$$

The homogeneity of the Fourier transform leads to

$$
\begin{aligned}
&\int_{-\infty}^{\infty} |\mathscr{F}^{-1}M_l(y)|\,dy \\
&\quad = \int_{-\infty}^{\infty} \left|\mathscr{F}^{-1}\left\{\left(1-\psi\left(2\frac{\xi}{N}+2l\right)\right)|\xi+lN|^{-\beta}\mathscr{F}\Lambda\left(\frac{\xi}{N}\right)\right\}(y)\right|\,dy \\
&\quad = N^{-\beta}\int_{-\infty}^{\infty} |\mathscr{F}^{-1}\{(1-\psi(2\xi+2l))|\xi+l|^{-\alpha}\mathscr{F}\Lambda(\xi)\}(y)|\,dy \\
&\quad = N^{-\beta}\int_{-\infty}^{\infty} |\mathscr{F}^{-1}\{(1-\psi(2\xi))|\xi|^{-\alpha}\mathscr{F}\Lambda(\xi-l)\}(y)|\,dy. \qquad (25)
\end{aligned}
$$

In case $s>\beta$ and q arbitrary, one knows

$$\|f-S_Nf\,|\,B^{\beta}_{p,q}(\mathbb{T})\| \le CN^{\beta-s}\|f\,|\,B^{s}_{p,\infty}(\mathbb{T})\|,$$

where $B^{\beta}_{p,q}(\mathbb{T})$ denotes a Besov space (cf. e.g. Pietsch [17]). Because of $B^{\beta}_{p,q}(\mathbb{T}) \hookrightarrow H^{\beta}_{p}(\mathbb{T})$, $q\le\min(p,2)$ (cf. e.g. [23, Chap. 3.5.4]), this yields

$$\|f-S_Nf\,|\,H^{\beta}_{p}(\mathbb{T})\| \le CN^{\beta-s}\|f\,|\,B^{s}_{p,\infty}(\mathbb{T})\|, \qquad (26)$$

where again the constant C does not depend on N and f. By inserting this inequality into (24), taking into acount (25), then (23) follows. □

Remark 4. Again, one could try to extend the lemma to values $p=1$ and $p=\infty$. However, then the approximation properties of the partial sums S_Nf are no longer sufficient. Replacing S_Nf by the following de la Vallée Poussin type means

$$V_Nf(x)=\sum_{k\in\mathbb{Z}}\psi(k/N)c_k(f)\,e^{ikx}, \qquad N\in\mathbb{N}, \qquad f\in D'(\mathbb{T}),$$

one can prove

$$\|I_N(f-V_Nf)\,|\,L_p(\mathbb{T})\| \le CN^{-s}\|f\,|\,B^{s}_{p,\infty}(\mathbb{T})\|$$

also if $p=1$ or $p=\infty$.

Proof of Proposition 2. This can be done in one line now. Lemmata 2 and 3 and (26) (applied with $\beta=0$) yield

$$
\begin{aligned}
\|f-I_Nf\,|\,L_p(\mathbb{T})\| &\le \|f-S_Nf\,|\,L_p(\mathbb{T})\| + \|S_Nf-I_N(S_Nf)\,|\,L_p(\mathbb{T})\| \\
&\quad + \|I_N(f-S_Nf)\,|\,L_p(\mathbb{T})\| \\
&\le CN^{-s}\|f\,|\,B^{s}_{p,\infty}(\mathbb{T})\|.
\end{aligned}
$$

The proof is complete. □

Remark 5. We compare the conditions of Propositions 1 and 2. Because of

$$\left|\psi(\xi)|\xi|^{-\alpha}\left(1-\frac{\mathscr{F}\Lambda(\xi)}{\sqrt{2\pi}}\right)\right| \le \int_{-\infty}^{\infty}\left|\mathscr{F}^{-1}\left\{\psi(\xi)|\xi|^{-\alpha}\left(1-\frac{\mathscr{F}\Lambda(\xi)}{\sqrt{2\pi}}\right)\right\}(w)\right| dw,$$

$$|\psi(\xi)|\xi|^{-\alpha}\mathscr{F}\Lambda(\xi+l)| \le \int_{-\infty}^{\infty}|\mathscr{F}^{-1}\{\psi(\xi)|\xi|^{-\alpha}\mathscr{F}\Lambda(\xi+l)\}(w)|\, dw,$$

we see that in case $\alpha > 1$, the conditions in Proposition 1 are satisfied if in addition $m_\alpha(\Lambda) < \infty$. In particular, if (14) and (15) are satisfied for some $\alpha > 0$, then (11) holds. Further, if $\mathscr{F}\Lambda$ is $(r-1)$-times differentiable in a neighborhood of the integers, then we may apply Proposition 1(ii) as long as $\alpha > r-1$ and $m_r(\Lambda) < \infty$.

Remark 6. We add a second observation. The inequality

$$\begin{aligned}&|(1-\psi(\xi+l))|\xi+l|^{-\beta}\mathscr{F}\Lambda(\xi)|\\ &\quad \le \int_{-\infty}^{\infty}|\mathscr{F}^{-1}\{(1-\psi(\xi+l)|\xi+l|^{-\beta}\mathscr{F}\Lambda(\xi))\}(w)|\, dw,\end{aligned}$$

evaluated in point $\xi = 0$ and summed up over l, shows that β must be strictly larger than 1 in any case.

Remark 7. Recall, if X is a Banach space and W a subspace of X, then the linear N-width is defined as

$$\lambda_N(W, X) = \inf_{\substack{L_N \in Lin_N(X)\\ \Lambda \in \mathscr{L}(X, L_N)}} \sup_{f \in W} \|f - \Lambda f | X\|,$$

where the infimum is taken over all subspaces L_N of X of finite dimension $\le N$ and all linear operators Λ from X to L_N. Here we are interested in $X = L_p(\mathbb{T})$ and W the unit ball in the Nikol'skij–Besov space $B^s_{p,\infty}(\mathbb{T})$, denoted by $B^s_p(\mathbb{T})$. If $s > 0$, then

$$\lambda_N(B^s_p(\mathbb{T}), L_p(\mathbb{T})) \sim N^{-s} \tag{27}$$

(cf. Lorentz, von Golitschek and Makovoz [14, Theorem 14.3.8]). In this sense, approximation with those interpolation operators I_N is nearly optimal (nearly optimal means the order of approximation is correct but may be not the constants). More details about widths may be found in Lorentz, von Golitschek, and Makovoz [14] and Tichomirov [28].

Remark 8. The conditions of Proposition 2 can be simplified by means of Szasz' theorem (cf. e.g. Jawerth [10] or [23, Proposition 1.7.5]). It holds that

$$\int_{-\infty}^{\infty} |\mathcal{F}^{-1}f(w)|\, dw \leq C\|(1+|\xi|^2)^{1/4}\mathcal{F}f(\xi)|L_2(\mathbb{R})\|.$$

Here the right-hand side represents the norm of f in the fractional order Sobolev space $H_2^{1/2}(\mathbb{R})$, which coincides with the Besov space $B_{2,1}^{1/2}(\mathbb{R})$ (equivalent norms).

As it turns out, if $s > 1$, then Proposition 2 will be sufficient for our purposes. In case $s < 1$, we add the following observation. In the proof of Proposition 2, we applied the most simple convolution inequality. Thus our philosophy is as follows. If p is approaching 1 or ∞, then the quality of the estimates stated in Proposition 2 becomes better. Hence, interpolating between spaces with p close to 1 (here we apply Proposition 2) and the spaces $C(\mathbb{T})$ and $B^s_{\infty,\infty}(\mathbb{T})$, respectively (here we apply Proposition 1), we can improve the restrictions on s. We do not formulate a general result. It will be described in detail during the investigations of some examples which follow.

3.4. Examples

The conditions in Proposition 2 look rather technical. We shall show by example that they are not very restrictive.

3.4.1. Periodized B-Splines

The cardinal centralized B-spline $\mathcal{M}_r$ of order r is defined as

$$\mathcal{M}_r(x) = (\underbrace{\mathcal{M}_1 * \cdots * \mathcal{M}_1}_{m\text{-fold}})(x), \qquad x \in \mathbb{R}, \qquad r \in \mathbb{N}.$$

Here $\mathcal{M}_1$ denotes the characteristic function of the interval $[-1/2, 1/2]$. Elementary calculations give

$$\mathcal{F}\mathcal{M}_r(\xi) = \frac{1}{\sqrt{2\pi}}\left(\frac{\sin\frac{1}{2}\xi}{\frac{1}{2}\xi}\right)^r.$$

To construct a fundamental interpolant on $\mathbb{R}$, we follow a standard procedure (cf. e.g. Jetter [11]). The symbol corresponding to $\mathcal{M}_r$ is given by

$$\mathcal{M}_r^{\sim}(\xi) = \sum_{m\in\mathbb{Z}} \mathcal{F}\mathcal{M}_r(2\pi\xi + 2\pi m).$$

We define

$$\mathscr{F}\Lambda_{2r}(\xi)=\sqrt{2\pi}\,\frac{\mathscr{F}\mathscr{M}_{2r}(2\pi\xi)}{\tilde{\mathscr{M}}_{2r}(\xi)},\qquad \xi\in\mathbb{R},\qquad r\in\mathbb{N}.$$

The decay properties of $\mathscr{M}_{2r}$ and the well-known positivity of the symbol imply that the function Λ_{2r} is a fundamental interpolant on $\mathbb{R}$.

Because of the well-known exponential decay of Λ_{2r}, periodization makes sense and we may investigate the periodic fundamental interpolants $\Lambda_{2r,N}^{\pi}$ [cf. (9)]. The function $\Lambda_{2r,N}^{\pi}$ is a periodic spline of order $2r$ and with nodes $2\pi k/N$.

Lemma 4. Let $r\in\mathbb{N}$.

(i) The function Λ_{2r} satisfies (14) with $\alpha<2r$.
(ii) The function Λ_{2r} satisfies (15) with $\alpha<2r$.
(iii) The function Λ_{2r} satisfies (16) if $\beta>1$.

Proof.

Step 1. We have

$$\begin{aligned}1-\frac{\mathscr{F}\Lambda_{2r}\xi}{\sqrt{2\pi}}&=\frac{\sum_{m\neq0}\mathscr{F}\mathscr{M}_{2r}(2\pi\xi+2\pi m)}{\tilde{\mathscr{M}}_{2r}(\xi)}\\&=\pi^{-2r}\sin^{2r}\pi\xi\,\frac{\sum_{m\neq0}(1/|\xi+m|^{2r})}{\tilde{\mathscr{M}}_{2r}(\xi)}.\end{aligned}$$

The functions

$$F_1(\xi)=\psi(4\xi)\,\frac{\sum_{m\neq0}(1/|\xi+m|^{2r})}{\tilde{\mathscr{M}}_{2r}(\xi)}$$

and

$$F_2(\xi)=(1-\psi(4\xi))\psi(\xi)\,\frac{\sum_{m\neq0}\mathscr{F}\mathscr{M}_{2r}(2\pi\xi+2\pi m)}{\tilde{\mathscr{M}}_{2r}(\xi)}$$

belong to $C^1(\mathbb{R})$. Making use of Szazs' Theorem (cf. Remark 8), and of assertions on pointwise multipliers in Besov spaces (cf. [30, Chap. 2.8] or

[21, Chap. 4.7]), it follows that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left|\mathcal{F}^{-1}\left\{\psi(\xi)|\xi|^{-\alpha}\left(1-\frac{\mathcal{F}\Lambda_{2r}(\xi)}{\sqrt{2\pi}}\right)\right\}(w)\right|\,dw \\
&\le C_1\left\|\psi(\xi)|\xi|^{-\alpha}\left(1-\frac{\mathcal{F}\Lambda_{2r}(\xi)}{\sqrt{2\pi}}\right)\middle|B_{2,1}^{1/2}(\mathbb{R})\right\| \\
&\le C_2(\|F_1(\xi)\sin^{2r}\pi\xi|\xi|^{-\alpha}|B_{2,1}^{1/2}(\mathbb{R})\|+\|F_2(\xi)|\xi|^{-\alpha}|B_{2,1}^{1/2}(\mathbb{R})\|) \\
&\le C_3(\|F_1|C^1(\mathbb{R})\|\,\|\psi^2(\xi/2)|\xi|^{-\alpha}\sin^{2r}\pi\xi|B_{2,1}^{1/2}(\mathbb{R})\| \\
&\qquad+\|F_2|C^1(\mathbb{R})\|\,\|(1-\psi(8\xi))\psi(\xi/2)|\xi|^{-\alpha}|B_{2,1}^{1/2}(\mathbb{R})\|) \\
&\le C_4\left(\|\psi(\xi/2)|\xi|^{2r-\alpha}|B_{2,1}^{1/2}(\mathbb{R})\|\left\|\psi(\xi/2)\left(\frac{\sin\pi\xi}{\pi\xi}\right)^{2r}\middle|B_{2,1}^{1/2}(\mathbb{R})\right\|\right. \\
&\qquad\left.+\|(1-\psi(8\xi))\psi(\xi/2)|\xi|^{-\alpha}|B_{2,1}^{1/2}(\mathbb{R})\|\right).
\end{aligned}\tag{28}$$

The membership of power-type functions in Besov spaces is well known (cf. e.g. [21, Chap. 2.3.1]). As a consequence, the right-hand side in (28) is finite as long as $\alpha < 2r$.

Step 2. To prove (ii), we shall employ

$$\left\|\psi(\xi/2)\frac{1}{|\xi+l|^{2r}}\middle|B_{2,1}^{1/2}(\mathbb{R})\right\| \le C_1|l|^{-2r}$$

with C_1 independent of l, $|l|\ge 5$. Hence, as above we conclude

$$\begin{aligned}
&\sum_{|l|\ge 5}\int_{-\infty}^{\infty}|\mathcal{F}^{-1}\{\psi(\xi)|\xi|^{-\alpha}\mathcal{F}\Lambda_{2r}(\xi+l)\}(w)|\,dw \\
&\le C_2\sum_{|l|\ge 5}\left\|\frac{\psi(\xi/2)}{|\xi+l|^{2r}}\middle|B_{2,1}^{1/2}(\mathbb{R})\right\|\left\|\psi(\xi)|\xi|^{2r-\alpha}\left(\frac{\sin\pi\xi}{\pi\xi}\right)^{2r}\middle|B_{2,1}^{1/2}(\mathbb{R})\right\| \\
&\le C_3\|\psi(\xi/2)|\xi|^{2r-\alpha}|B_{2,1}^{1/2}(\mathbb{R})\|\left\|\left(\frac{\sin\pi\xi}{\pi\xi}\right)^{2r}\middle|C^1(\mathbb{R})\right\|.
\end{aligned}\tag{29}$$

It remains to observe the finiteness of the summands in (15) with $|l|<5$. That can be done, as in Step 1, by isolating the singularities of $|\xi|^{-\alpha}$ and $|\xi+l|^{-2r}$, respectively.

Step 3. In a similar way as above, we derive

$$\sum_{l\in\mathbb{Z}}\int_{-\infty}^{\infty}|\mathscr{F}^{-1}\{(1-\psi(2\xi))|\xi|^{-\beta}\mathscr{F}\Lambda_{2r}(\xi-l)\}(w)|\,dw$$
$$\le C_1\sum_{l\in\mathbb{Z}}\|(1-\psi(2\xi))|\xi|^{-\beta}\mathscr{F}\Lambda_{2r}(\xi-l)|B^{1/2}_{2,1}(\mathbb{R})\|$$
$$\le C_2\|(\mathscr{M}^{-}_{2r}(\xi))^{-1}|C^1(\mathbb{R})\|\sum_{l\in\mathbb{Z}}\|(1-\psi(2\xi))|\xi|^{-\beta}\mathscr{F}\mathscr{M}_{2r}(2\pi(\xi-l))|B^{1/2}_{2,1}(\mathbb{R})\|$$
$$\le C_3\sum_{l\in\mathbb{Z}}\|(1-\psi(2\xi))|\xi|^{-\beta}\mathscr{F}\mathscr{M}_{2r}(2\pi(\xi-l))|W^1_2(\mathbb{R})\|,$$

where we have used the embedding $W^1_2(\mathbb{R})\hookrightarrow B^{1/2}_{2,1}(\mathbb{R})$. First, notice the norm in case $l=0$ is finite. Let $l>0$. Observe

$$\int_{1/2}^{\infty}|\xi|^{-2\beta}|\mathscr{F}\mathscr{M}_{2r}(2\pi(\xi-l))|^2\,d\xi$$
$$=\int_{1/2}^{l/2}|\xi|^{-2\beta}|\mathscr{F}\mathscr{M}_{2r}(2\pi(\xi-l))|^2\,d\xi$$
$$+\int_{l/2}^{3l/2}|\xi|^{-2\beta}|\mathscr{F}\mathscr{M}_{2r}(2\pi(\xi-l))|^2\,d\xi$$
$$+\int_{3l/2}^{\infty}|\xi|^{-2\beta}|\mathscr{F}\mathscr{M}_{2r}(2\pi(\xi-l))|^2\,d\xi$$
$$\le C_1\left(l^{-4r}\int_{1/2}^{\infty}|\xi|^{-2\beta}\,d\xi+l^{-2\beta}\int_0^{l/2}|\mathscr{F}\mathscr{M}_{2r}(2\pi\xi)|^2\,d\xi\right)\tag{30}$$
$$\le C_2(l^{-2\beta}+l^{-4r}).$$

By obvious modifications, we obtain the same estimate for the integral $\int_{-\infty}^{-1/2}\cdots d\xi$ and, of course, also if $l<0$. Next, we consider

$$\frac{d}{d\xi}(1-\psi(2\xi))|\xi|^{-\beta}\mathscr{F}\mathscr{M}_{2r}(2\pi(\xi-l))$$
$$=-2\psi'(2\xi)|\xi|^{-\beta}\mathscr{F}\mathscr{M}_{2r}(2\pi(\xi-l))$$
$$+(-\beta)\operatorname{sign}\xi(1-\psi(2\xi))|\xi|^{-\beta-1}\mathscr{F}\mathscr{M}_{2r}(2\pi(\xi-l))$$
$$+2r(1-\psi(2\xi))|\xi|^{-\beta}\left(\frac{\sin(\pi(\xi-l))}{\pi(\xi-l)}\right)^{2r-1}$$
$$\times\frac{\pi(\xi-l)\cos(\pi(\xi-l))-\sin(\pi(\xi-l))}{\pi(\xi-l)^2}$$
$$=T_1+T_2+T_3.$$

In order to estimate the L_2-norm of this expression, one observes that in case of the term T_1 one may proceed exactly as in (30) because ψ' is vanishing in a neighborhood of the origin. The estimate of T_2 can also be done as there (only β has to be replaced by $\beta+1$). What concerns T_3, then the result follows from the boundedness of

$$\left(\frac{\sin(\pi(\xi-l))}{\pi(\xi-l)}\right)^{2r-1}\frac{\pi(\xi-l)\cos(\pi(\xi-l))-\sin(\pi(\xi-l))}{\pi(\xi-l)^2}$$

in the neighborhood of l and the decay properties which are the same as of $\mathcal{F}\mathcal{M}_{2r}(2\pi(\xi-l))$. If $\min(\beta,2r)>1$, then this yields the finiteness of (16). □

Corollary 1. Let $r\in\mathbb{N}$, $1<p<\infty$, and suppose

$$\frac{1}{p}<s<2r. \tag{31}$$

Let I_N^{2r} be the interpolation operator induced by Λ_{2r}. Then periodic spline interpolation has the following properties: for all $f\in B^s_{p,\infty}(\mathbb{T})$, it holds

$$\|f-I_N^{2r}f|L_p(\mathbb{T})\|\leq CN^{-s}\|f|B^s_{p,\infty}(\mathbb{T})\|, \tag{32}$$

where C denotes a constant independent of N and f; vice versa, if $f\in C(\mathbb{T})$ satisfies

$$\sup_{N\in\mathbb{N}} N^s\|f-I_N^{2r}f|L_p(\mathbb{T})\|<\infty,$$

then f belongs to $B^s_{p,\infty}(\mathbb{T})$.

Proof.

Step 1. We shall prove (32). By the above lemma, Proposition 2 yields the result as long as $1<s<2r$, because of $B^s_{p,\infty}(\mathbb{T})\hookrightarrow\mathcal{A}(\mathbb{T})$ under this condition. It remains to investigate $1/p<s\leq 1$. Let $1<p_0<2$. Then (31) implies $B^{s_0}_{p_0,\infty}(\mathbb{T})\hookrightarrow B^{1/2}_{2,1}(\mathbb{T})\hookrightarrow\mathcal{A}(\mathbb{T})$. Hence, if p_0 is close to 1, then Proposition 2 and Lemma 4 prove (32) as long as $1/p_0<1<s_0<2r$. If $p_1=\infty$, Proposition 1(ii) can be applied as long as $0<s_1<2r-1$, $r>1$. If $r=1$, then Proposition 1(i) can be applied for $0<s_1<1$. Now we proceed by complex interpolation. To this end, let $1<p<\infty$ and let s be as in (31). Then there exists some Θ, $0<\Theta<1$, $p_0>1$, $1<s_0<2r$, and $0<s_1<2r$ such that

$$\frac{1}{p}=\frac{1-\Theta}{p_0}\qquad\text{and}\qquad s=(1-\Theta)s_0+s_1.$$

Complex interpolation yields

$$[b^{s_0}_{p_0,\infty}(\mathbb{T}),\, b^{s_1}_{\infty,\infty}(\mathbb{T})]_\Theta = b^s_{p,\infty}(\mathbb{T}) \qquad \text{and} \qquad [L_{p_0}(\mathbb{T}),\, L_\infty(\mathbb{T})]_\Theta = L_p(\mathbb{T}),$$

(cf. e.g. Triebel [29, Chap. 1.18.1, 1.18.4, 2.4.1]). Here $b^s_{p,\infty}(\mathbb{T})$ denotes the closure of the set of trigonometric polynomials in $B^s_{p,\infty}(\mathbb{T})$. This leads to

$$\begin{aligned}&\|(E - I^{2r}_N)|\mathscr{L}(b^s_{p,\infty}(\mathbb{T}), L_p(\mathbb{T}))\| \\ &\qquad \le \|(E - I^{2r}_N)|\mathscr{L}(b^{s_0}_{p_0,\infty}(\mathbb{T}), L_{p_0}(\mathbb{T}))\|^{1-\Theta} \|(E - I^{2r}_N)|\mathscr{L}(b^{s_1}_{\infty,\infty}(\mathbb{T}), L_\infty(\mathbb{T}))\|^{\Theta}.\end{aligned}$$

Consequently,

$$\|f - I^{2r}_N f\,|L_p(\mathbb{T})\| \le CN^{-s}\|f\,|B^s_{p,\infty}(\mathbb{T})\| \tag{33}$$

for all $f \in b^s_{p,\infty}(\mathbb{T})$. Hence, the above inequality is true at least for all polynomials f. Replacing f in (33) by its Fourier partial sum $S_M f$, we find

$$\|S_M f - I^{2r}_N S_M f\,|L_p(\mathbb{T})\| \le C_1 N^{-s}\|S_M f\,|B^s_{p,\infty}(\mathbb{T})\| \le C_2 N^{-s}\|f\,|B^s_{p,\infty}(\mathbb{T})\|.$$

On the left-hand side, we have a convergent sequence (with respect to M). This may be seen as follows. Because of $f \in B^{s-\delta}_{p,1}(\mathbb{T})$ we may apply (33) with $s - \delta > 1/p$ (if δ is sufficiently small) and end up with

$$\begin{aligned}&\|S_{M_1} f - S_{M_2} f - I^{2r}_N(S_{M_1} f - S_{M_2} f)|L_p(\mathbb{T})\| \\ &\qquad \le CN^{-s+\delta}\|S_{M_1} f - S_{M_2} f\,|B^{s-\delta}_{p,1}(\mathbb{T})\| < \varepsilon\end{aligned}$$

if M_1, M_2 are large enough. This proves (33) for all $f \in B^s_{p,\infty}(\mathbb{T})$.

Step 2. The second statement of the corollary is a consequence of (27). □

Remark 9. Results as stated in the Corollary 1 are not new (cf. e.g. Oswald [16]).

3.4.2. de la Vallée Poussin Means

Let $0 < \lambda < 1/2$ be fixed. We put

$$\varphi_\lambda(x) = 2\frac{\sin\frac{1}{2}x \sin\lambda x}{\lambda x^2}.$$

This implies

$$\mathscr{F}\varphi_\lambda(\xi) = \sqrt{2\pi}\begin{cases}1 & \text{if } |\xi| \le \frac{1}{2} - \lambda, \\ (1/2\lambda)(\frac{1}{2} + \lambda - \xi) & \text{if } \frac{1}{2} - \lambda < |\xi| < \frac{1}{2} + \lambda, \\ 0 & \text{if } \frac{1}{2} + \lambda \le |\xi|,\end{cases}$$

Some calculations yield

$$\Lambda^{\pi}_{\lambda,N}(x) = \frac{2}{N}\frac{1}{K_{\lambda,N}} \sum_{(1/2-\lambda)N \le l < (1/2+\lambda)N} D_l(x),$$

where D_l denotes the Dirichlet kernel of order l and $K_{\lambda,N}$ the number of summands in $\sum_{(1/2-\lambda)N \le l < (1/2+\lambda)N}$. Proposition 1 is applicable for all s. Moreover, because of the fact that $\mathscr{F}\varphi_r$ is identical $\sqrt{2\pi}$ in a neighborhood of the origin the singularity of $|\xi|^{-\alpha}$ does not enter the picture, so (14) is finite for all $\alpha > 0$. Finally, the compactness of the support of $\mathscr{F}\varphi_r$ and its weak singularities in $-1/2-\lambda$, $-1/2+\lambda$, $1/2-\lambda$, $1/2+\lambda$ yield the finiteness of (16) for all $\beta > 1$. Complementing Propositions 1 and 2 by complex interpolation (as in proof of Corollary 1), we end up with the following.

Corollary 2. Let $0 < \lambda < 1/2$. Let $1 < p < \infty$ and suppose $s > 1/p$. Let I^{λ}_N be the periodic interpolation operator induced by $\Lambda^{\pi}_{\lambda,N}$. Then: for all $f \in B^s_{p,\infty}(\mathbb{T})$, it holds that

$$\|f - I^{\lambda}_N f | L_p(\mathbb{T})\| \le C N^{-s} \|f | B^s_{p,\infty}(\mathbb{T})\|,$$

where C denotes a constant independent of N and f; vice versa, if $f \in C(\mathbb{T})$ satisfies

$$\sup_{N \in \mathbb{N}} N^s \|f - I^{\lambda}_N f | L_p(\mathbb{T})\| < \infty,$$

then f belongs to $B^s_{p,\infty}(\mathbb{T})$.

4. Interpolation on Sparse Grids

4.1. Preliminaries

For this, we need some preparations. By E, we denote always the identity (even on different spaces without indicating this). First, we introduce the parameter-dependent extension of our interpolation operators. We put

$$I^x_K f(x, y) = \sum_{l \in J_K} f\left(\frac{2\pi l}{K}, y\right) \Lambda_K\left(x - \frac{2\pi l}{K}\right),$$

$$I^y_N f(x, y) = \sum_{l \in J_N} f\left(x, \frac{2\pi l}{N}\right) \Lambda_N\left(y - \frac{2\pi l}{N}\right).$$

What is important for us are the following identities

$$I^x_K f = (I_K \otimes E) f, \qquad \text{and} \qquad I^y_N f = (E \otimes I_N) f, \qquad f \in C(\mathbb{T}) \otimes C(\mathbb{T}).$$

4.2. Interpolation on Sparse Grids

We follow Delvos and Schempp [6]. A sequence of projections $\{P_j\}_{j=1}^{\infty}$ is called a chain if P_j commutates with P_{j+1} and $P_{j+1}P_j = P_jP_{j+1} = P_j$ holds for all j. The interpolation operators I_N, considered on $C(\mathbb{T})$, form a chain if the range spaces satisfy the inclusion relation

$$\mathscr{R}(I_N) \subset \mathscr{R}(I_M), \qquad N \leq M, \tag{34}$$

and if $\mathscr{T}_N \subset \mathscr{T}_M$, $N \leq M$. The second condition will be always satisfied if we concentrate on a sequence $N_j = d2^j$ for some natural number d. The first condition has to be checked separately in each example.

Now, having a chain I_{N_j}, $j \in \mathbb{N}$, of interpolation operators, then the parameter dependent extensions $I^x_{N_j}$ and $I^y_{N_j}$ also form chains. The Boolean sum of order j is defined as

$$B_j = \bigoplus_{r=0}^{j} I^x_{N_r} I^y_{N_{j-r}} = \sum_{r=0}^{j} I^x_{N_r} I^y_{N_{j-r}} - \sum_{r=0}^{j-1} I^x_{N_r} I^y_{N_{j-r-1}}.$$

Then B_j form a chain (cf. Baszenski and Delvos [3] or Delvos and Schempp [6, Prop. 1.3.1]). They have range space

$$\mathscr{R}(B_j) = \sum_{r=0}^{j} \mathscr{R}(I_{N_r}) \otimes \mathscr{R}(I_{N_{j-r}})$$

and satisfy the interpolation conditions with respect to the sparse grid

$$\mathscr{T}(B_j) = \bigcup_{r=0}^{j} \mathscr{T}_{N_r} \times \mathscr{T}_{N_{j-r}}.$$

The total number of knots in this grid is equal to $d^2(j2^{j-1} + 2^j)$ and is much less than the number of knots $d^2 2^{2j}$ in the full grid $\mathscr{T}_{N_j} \times \mathscr{T}_{N_j}$. Finally, the remainder operator $E - B_j$ has the representation

$$\begin{aligned} E - B_j = (E - I^x_{N_j}) + (E - I^y_{N_j}) &- \sum_{r=0}^{j} (E - I^x_{N_r})(E - I^y_{N_{j-r}}) \\ &+ \sum_{r=0}^{j-1} (E - I^x_{N_r})(E - I^y_{N_{j-r-1}}), \end{aligned} \tag{35}$$

(cf. [6, Prop. 1.4.2]).

Now we are ready to formulate the main result of this paper.

Theorem 1. Suppose that I_{N_j}, $j \in \mathbb{N}$, form a chain and satisfy

$$\sup_{j \in \mathbb{N}} N_j^{s_k} \| f - I_{N_j} f | L_p(\mathbb{T}) \| \leq C_k \| f | B^{s_k}_{p,\infty}(\mathbb{T}) \| \tag{36}$$

with some constants C_k independent of f and for some fixed s_1, s_2, p such that $1 < p < \infty$ and $\min(s_1, s_2) > 1/p$.

Then, in case $s_1 = s_2 = s$, we find

$$\|f - B_j f | L_p(\mathbb{T}^2)\| \le C(j+1) N_j^{-s} \|f | B^s_{p,\infty}(\mathbb{T}) \otimes_{\alpha_p} B^s_{p,\infty}(\mathbb{T})\|, \tag{37}$$

whereas, in case $s_1 \neq s_2$, it holds that

$$\|f - B_j f | L_p(\mathbb{T}^2)\| \le C N_j^{-\min(s_1,s_2)} \|f | B^{s_1}_{p,\infty}(\mathbb{T}) \otimes_{\alpha_p} B^{s_2}_{p,\infty}(\mathbb{T})\|. \tag{38}$$

In both situations C denotes a constant independent on j and f.

Proof. We employ (35) and split the error into

$$\begin{aligned} \|f - B_j f | L_p(\mathbb{T}^2)\| \le{}& \|f - I^x_{N_j} f | L_p(\mathbb{T}^2)\| + \|f - I^y_{N_j} f | L_p(\mathbb{T}^2)\| \\ &+ \sum_{r=0}^{j} \|(E - I^x_{N_r})(E - I^y_{N_{j-r}}) f | L_p(\mathbb{T}^2)\| \\ &+ \sum_{r=0}^{j-1} \|(E - I^x_{N_r})(E - I^y_{N_{j-r-1}}) f | L_p(\mathbb{T}^2)\|. \end{aligned} \tag{39}$$

By means of $E - I^x_K = (E - I_K) \otimes E$ and (4), we have

$$\begin{aligned} \|f - I^x_{N_j} f | L_p(\mathbb{T}^2)\| \le{}& C \|E - I_{N_j} | \mathscr{L}(B^{s_1}_{p,\infty}(\mathbb{T}), L_p(\mathbb{T}))\| \, \|E | \mathscr{L}(B^{s_2}_{p,\infty}(\mathbb{T}), L_p(\mathbb{T}))\| \\ &\times \|f | B^{s_1}_{p,\infty}(\mathbb{T}) \otimes_{\alpha_p} B^{s_2}_{p,\infty}(\mathbb{T})\|. \end{aligned} \tag{40}$$

Next we use

$$\begin{aligned} (E - I^x_{N_r})(E - I^y_{N_{j-r}}) &= ((E - I_{N_r}) \otimes E)(E \otimes (E - I_{N_{j-r}})) \\ &= (E - I_{N_r}) \otimes (E - I_{N_{j-r}}). \end{aligned}$$

Analogously to (40), we derive

$$\begin{aligned} &\|(E - I^x_{N_r})(E - I^y_{N_{j-r}}) f | L_p(\mathbb{T}^2)\| \\ &\quad \le C \|E - I_{N_j} | \mathscr{L}(B^{s_1}_{p,\infty}(\mathbb{T}), L_p(\mathbb{T}))\| \, \|E - I_{N_{j-r}} | \mathscr{L}(B^{s_2}_{p,\infty}(\mathbb{T}), L_p(\mathbb{T}))\| \\ &\quad \times \|f | B^{s_1}_{p,\infty}(\mathbb{T}) \otimes_{\alpha_p} B^{s_2}_{p,\infty}(\mathbb{T})\|. \end{aligned} \tag{41}$$

Inserting (36) into (40) and into (41) and making use of the obtained estimates in (39), the desired result then follows. □

Remark 10. We do not know more explicit characterizations of the space $B^{s_1}_{p,\infty}(\mathbb{T}) \otimes_{\alpha_p} B^{s_2}_{p,\infty}(\mathbb{T})$ as it would be desirable from Theorem 1. However, (3) gives at least some information about.

Remark 11. Most results of the type of Theorem 1 are either given for function spaces of continuous functions, e.g. [2], or in case that the considered function spaces are Hilbert spaces, e.g. [5,6]. In both cases, the choice of a uniform tensor norm is more or less natural. For more general function spaces, this is not clear at all. Theorem 1 extends some earlier results of the second named author [18,26]. There the fractional order Sobolev spaces $S_p^{s_1,s_2}H(\mathbb{T}^2)$ appear instead of $B^{s_1}_{p,\infty}(\mathbb{T})\otimes_{\alpha_p}B^{s_2}_{p,\infty}(\mathbb{T})$ [cf. (3)] for a comparison. However, we do not know about optimality of Theorem 1. In view of our method, it would be desirable to enlarge the tensor product space $B^{s_1}_{p,\infty}(\mathbb{T})\otimes_{\alpha_p}B^{s_2}_{p,\infty}(\mathbb{T})$ by switching from the p-nuclear norm to a more appropriate but still uniform one.

4.3. Examples

4.3.1. Periodized B-Splines

First, we have to check whether the corresponding sequences I_{N_j} form a chain. But this is clear from the nestedness of the underlying sets of the spline nodes (which equal the interpolation knots $\mathcal{T}_{N_j}$ here). Now, Theorem 1 together with Corollary 1 yields the following.

Theorem 2. Let $r\in\mathbb{N}$, $1<p<\infty$, and suppose

$$\frac{1}{p}<s_1,\, s_2<2r.$$

Let I_N^{2r} be the interpolation operator induced by Λ_{2r}. The corresponding Boolean sum of order j we denote by B_j^{2r}. Then periodic spline interpolation on the sparse grids $\mathcal{T}(B_j^{2r})$ satisfies

$$\|f-B_j^{2r}f|L_p(\mathbb{T}^2)\|\le C\|f|B^{s_1}_{p,\infty}(\mathbb{T})\otimes_{\alpha_p}B^{s_2}_{p,\infty}(\mathbb{T})\|\begin{cases}N_j^{-\min(s_1,s_2)} & \text{if } s_1\neq s_2,\\ (j+1)N_j^{-s} & \text{if } s_1=s_2=s,\end{cases}$$

where C denotes a constant independent of j and f.

4.3.2. de la Vallée Poussin Means

In case of the de la Vallée Poussin means, the chain condition (34) is, in general, violated. Lemma 1(iii) offers a simple possibility to obtain a sufficient condition to guarantee (34). To this end, we investigate the refinement equation (12) in the Fourier image:

$$\mathcal{F}\Lambda(2\xi)=\left(\sum_{k\in\mathbb{Z}}h_k\,e^{-i2\pi k\xi}\right)\mathcal{F}\Lambda(\xi),\qquad \xi\in\mathbb{R}.$$

In our particular situation, it is obvious that if $\lambda \le 1/6$, then

$$\mathscr{F}_{\varphi\lambda}(2\xi) = \mathscr{F}_{\varphi\lambda}(2\xi)\mathscr{F}_{\varphi\lambda}(\xi) = \left(\sum_{k\in\mathbb{Z}} h_k\, e^{-i2\pi k\xi}\right)\mathscr{F}_{\varphi\lambda}(\xi)$$

holds, where

$$h_k = \int_0^1 \mathscr{F}_{\varphi\lambda}(2\xi)\, e^{i2\pi k\xi}\, d\xi.$$

The 1-periodic extension of $\mathscr{F}\varphi_\lambda(2\xi)$ is a Lipschitz function. Hence, its Fourier coefficients are absolutely summable. Thus we may apply Lemma 1. Thus, Corollary 2 and Theorem 1 guarantee the following.

Theorem 3. Let $1 < p < \infty$ and suppose $1/p < s_1, s_2$. Let $0 < \lambda \le 1/6$. Let I_N^λ be interpolation operator associated to φ_λ and denote by B_j^λ its Boolean sum of order j. Then we have

$$\|f - B_j^\lambda f | L_p(\mathbb{T}^2)\| \le C \|f | B^{s_1}_{p,\infty}(\mathbb{T}) \otimes_{\alpha_p} B^{s_2}_{p,\infty}(\mathbb{T})\| \begin{cases} N_j^{-\min(s_1,s_2)} & \text{if } s_1 \ne s_2, \\ (j+1)N_j^{-s} & \text{if } s_1 = s_2 = s, \end{cases}$$

where C denotes a constant independent of j and f.

Remark 12. Investigations of periodic fundamental interpolants of de la Vallée Poussin type have been made in a multiresolution setting by Prestin and Selig [19].

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