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The Excluded Minors for *GF*(4)-Representable Matroids[1]

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There are exactly seven excluded minors for the class of $GF(4)$-representable matroids.

1. INTRODUCTION

We prove the following theorem.

THEOREM 1.1. *A matroid M is $GF(4)$-representable if and only if M has no minor isomorphic to any of $U_{2,6}$, $U_{4,6}$, P_6, F_7^-, $(F_7^-)^*$, P_8, and P_8''.*

The definitions of these matroids, with a summary of their interesting properties, can be found in the Appendix. Except for P_8'', they were all known to be excluded minors for $GF(4)$-representability (see Oxley [15, 17]). The matroid P_8'' is obtained by relaxing the unique pair of disjoint circuit-hyperplanes of P_8.

Ever since Whitney's introductory paper [28] on matroid theory, researchers have sought ways to distinguish the representable matroids. For any field **F**, the class of **F**-representable matroids is closed under taking minors. Thus, it is natural to characterize the minor-minimal matroids that are not **F**-representable; we refer to such matroids as *excluded minors.* Tutte [27] showed that $U_{2,4}$ is the only excluded minor for the class of binary matroids. Tutte also showed that the excluded minors for the class

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of regular matroids (the matroids representable over all fields) are $U_{2,4}$, F_7, and F_7^*. Reid [20] announced that the excluded minors for the class of ternary matroids are $U_{2,5}$, $U_{3,5}$, F_7 and F_7^*; this result was later published by Bixby [2] and Seymour [23]. (See also Kahn and Seymour [11], Kahn [9], and Truemper [25].) Following these results, Rota [21] conjectured that, for every finite field $GF(q)$, there are just finitely many excluded minors for the class of $GF(q)$-representable matroids. This conjecture is in stark contrast with the result of Lazarson [14] that, for fields of characteristic zero, there are infinitely many excluded minors.

Rota's conjecture is one of the more important open problems in matroid theory. So far, it has only been proven for the fields $GF(2)$, $GF(3)$, and, in the present paper, for $GF(4)$. The current approach for each of these cases relies heavily on unique representability (see Section 2 for the exact meaning of "unique"). Representations over $GF(2)$ and $GF(3)$ are unique. Although this is no longer the case for $GF(4)$, Kahn [10] proved that $GF(4)$-representations are unique under certain connectivity assumptions (3-connectivity, essentially). This result allows us to extend the existing approach for $GF(3)$ to $GF(4)$. The fact that the proof in the present paper is so much longer than the current proofs for $GF(3)$ lies entirely in the fact that 3-connectivity becomes an issue here.

The next case, $GF(5)$, is still open and is of great interest because there much of the uniqueness is lost: Oxley, Vertigan, and Whittle [19] showed that 3-connected matroids may have up to six inequivalent representations over $GF(5)$. Oxley, Vertigan, and Whittle [19] also showed that for larger fields no such bound exists. This seems to indicate that current approaches are doomed for all fields with more than five elements. We fear that Rota's conjecture may fail for those fields.

The matroids $U_{2,6}$, $U_{4,6}$, P_6, and P_8'' and $\mathbf{F}$-representable if and only if $|\mathbf{F}| \geqslant 5$, while the matroids F_7^-, $(F_7^-)^*$, and P_8 are $\mathbf{F}$-representable if and only if $\mathbf{F}$ has characteristic different from 2. Hence the following result of Whittle [29] is an immediate consequence of Theorem 1.1.

COROLLARY 1.2. *If M is a ternary matroid that is representable over some field of characteristic* 2, *then M is $GF(4)$-representable.*

Whittle [29] has characterized the ternary matroids that are representable over some field of characteristic different from 3. The class of matroids that are representable over both $GF(3)$ and $GF(4)$ play a significant role in Whittle's characterization; he calls such matroids $\sqrt[6]{1}$-*matroids* or *sixth-root-of-unity matroids.*

THEOREM 1.3 (Whittle [29]). *The following are equivalent for a matroid M.*

• *M is representable over both $GF(3)$ and $GF(4)$.*

• *M is representable over all finite fields $GF(q)$ where q is not congruent to $2 \bmod 3$.*

• *M can be represented over the complex numbers by a matrix whose nonzero subdeterminants are all sixth-roots of unity.*

By combining Theorem 1.1 with Reid's characterization of ternary matroids, we get the excluded minors for the class of $\sqrt[6]{1}$-matroids. The excluded minors are exactly those conjectured by Oxley, Vertigan, and Whittle [18]. (In the same paper Oxley, Vertigan, and Whittle conjecture a list of excluded minors for the class of dyadic matroids. That list is incomplete, as the matroid T_8, see Oxley [17, p. 511], is also an excluded minor.)

COROLLARY 1.4. *M is a $\sqrt[6]{1}$-matroid if and only if M has no minor isomorphic to any of $U_{2,5}$, $U_{3,5}$, F_7, F_7^*, F_7^-, $(F_7^-)^*$, and P_8.*

We assume that the reader is familiar with elementary notions in matroid theory, including representability, minors, duality, connectivity, direct sums, and 2-sums. For an excellent introduction to the subject see Oxley [17].

2. UNIQUE REPRESENTABILITY

As is the case with many excluded-minor characterizations, we rely heavily on unique representability. Two **F**-representations of a matroid are *equivalent* if they can be obtained, one from the other, by elementary row operations, column scaling, and applying automorphisms of **F**. We say that a matroid M is *uniquely representable* over a field **F** if any two representations of M over **F** are equivalent. The 2-sum of two copies of $U_{2,4}$ has inequivalent representations over $GF(4)$. However, this is, in some sense, the only way to obtain matroids with inequivalent representations over $GF(4)$. We call a matroid *stable* if it cannot be expressed as the direct sum or the 2-sum of two nonbinary matroids.

THEOREM 2.1 (Kahn [10]). *A $GF(4)$-representable matroid is uniquely $GF(4)$-representable if and only if it is stable.*

Whittle [30] has recently developed techniques that enable results like Theorem 2.1 to be proven by elementary case checking.

The following proposition demonstrates the importance of unique representability in obtaining an excluded-minor characterization. Similar ideas led to an elementary proof of Tutte's excluded-minor characterization of regular matroids [8].

LEMMA 2.2. *Let M be a matroid, and let u, v be a coindependent pair of elements of M such that $M\backslash u$, $M\backslash v$, and $M\backslash u, v$ are all stable, and $M\backslash u, v$ is connected and nonbinary. If $M\backslash u$ and $M\backslash v$ are both $GF(4)$-representable, then there exists a unique $GF(4)$-representable matroid N such that $N\backslash u = M\backslash u$ and $N\backslash v = M\backslash v$.*

Proof. Let B be a basis of M containing neither u nor v. Consider $GF(4)$-representations A_1 and A_2 of $M\backslash u$ and $M\backslash v$. By row operations we can put these representations into the following forms:

$$A_1 = \begin{array}{c} \begin{array}{ccc} B & & u \end{array} \\ \begin{pmatrix} I & C_1 & x \end{pmatrix} \end{array} \quad \text{and} \quad A_2 = \begin{array}{c} \begin{array}{ccc} B & & v \end{array} \\ \begin{pmatrix} I & C_2 & y \end{pmatrix} \end{array}.$$

Then (I, C_1) and (I, C_2) are both $GF(4)$-representations of $M\backslash u, v$. By Theorem 2.1, we may assume, by possibly scaling and applying an automorphism of $GF(4)$ to A_2, that $C_2 = C_1$. Now let N be the matroid represented over $GF(4)$ by the following matrix:

$$\begin{array}{c} \begin{array}{cccc} B & & u & v \end{array} \\ \begin{pmatrix} I & C_1 & x & y \end{pmatrix} \end{array}.$$

Certainly $N\backslash u = M\backslash u$ and $N\backslash v = M\backslash v$. We are required to prove that N is the only $GF(4)$-representable matroid having these properties. Let N' be another $GF(4)$-representable matroid such that $N'\backslash u = M\backslash u$ and $N'\backslash v = M\backslash v$. Consider a $GF(4)$-representation of N' of the following form:

$$A' = \begin{array}{c} \begin{array}{cccc} B & & u & v \end{array} \\ \begin{pmatrix} I & C' & x' & y' \end{pmatrix} \end{array}.$$

Then (I, C', x') and (I, C_1, x) both represent $M\backslash v$. By Theorem 2.1, we may assume, by possibly scaling and applying an automorphism of $GF(4)$ to A', that $C' = C_1$ and $x' = x$. So we may assume that $A' = (I, C_1, x, y')$. Now we have two representations (I, C_1, y) and (I, C_1, y') of $M\backslash u$. By Theorem 2.1 these representations are equivalent. Consider the operations required to transform (I, C_1, y') into (I, C_1, y). We have at our disposal elementary row operations, column scaling, and applying an automorphism of $GF(4)$. The common identity matrix in the representations limits the row operations to row scaling. Since $M\backslash u, v$ is nonbinary, we cannot apply a nontrivial automorphism of $GF(4)$, because otherwise we would be unable to recover the matrix (I, C_1) using scaling. However, since $M\backslash u, v$ is connected, the only scalings that we can apply to (I, C_1) without changing it are trivial (that is, we may multiply all rows by a constant α and divide all columns by α). Therefore, y' is just a scaling of y. Consequently, $N' = N$, as required. ∎

Remark. If in Lemma 2.2 we replace the condition that $M \backslash u, v$ is nonbinary by the condition that $M \backslash u$ is binary, the conclusion of Lemma 2.2 remains true. The proof, left to the reader, is a slight modification of the one above.

An intermediate consequence of Lemma 2.2 is the following result.

LEMMA 2.3. *Let M and N be matroids on a common ground set S, where N is $GF(4)$-representable, and let u and v be distinct elements of S such that $M \backslash u = N \backslash u$ and $M \backslash v = N \backslash v$. Suppose that there exists disjoint sets X, $Y \subseteq S - \{u, v\}$ such that:*

(1) *$(M \backslash X/Y) \backslash u$ and $(M \backslash X/Y) \backslash v$ are stable,*

(2) *$(M \backslash X/Y) \backslash u, v$ is connected, stable, and nonbinary, and*

(3) *$M \backslash X/Y \neq N \backslash X/Y$.*

Then $M \backslash X/Y$ is not $GF(4)$-representable.

Proof. It follows from (1), (2) and Lemma 2.2 that $N \backslash X/Y$ is the only $GF(4)$-representable matroid $\tilde{N}$ with $\tilde{N} \backslash u = (M \backslash X/Y) \backslash u$ and $\tilde{N} \backslash v = (M \backslash X/Y) \backslash v$. Hence, as $M \backslash X/Y \neq N \backslash X/Y$, the matroid $M \backslash X/Y$ is not $GF(4)$-representable. ∎

Lemmas 2.2 and 2.3 summarize the strategy employed in the proof of Theorem 1.1. We begin with a "large" minor-minimal non-$GF(4)$-representable matroid M. (Smaller matroids are deferred to the case analysis in Section 6.) In Section 3 we show that, by possibly dualizing, we can find elements u and v satisfying the conditions of Theorem 2.2. Then, by Theorem 2.2, there is a $GF(4)$-representable matroid N such that $M \backslash u = N \backslash u$ and $M \backslash v = N \backslash v$. Next we "build" a proper minor $M' := M \backslash X/Y$ of M that satisfies conditions (1), (2), and (3) of Lemma 2.3. By Lemma 2.3, M' is not $GF(4)$-representable. As M' is a proper minor of M this yields a contradiction. So no minor-minimal non-$GF(4)$-representable matroid is "large." Actually, it is relatively easy to find a minor M' that satisfies (2) and (3); most of the work is in introducing property (1) without losing (2) or (3).

3. DELETING A PAIR

We now seek the elements required to invoke Lemma 2.2. A pair $\{a, b\}$ of elements of a matroid M is a *deletion pair* of M if $M \backslash a, b$ is connected, and each of $M \backslash a$, $M \backslash b$, and $M \backslash a, b$ is a 0-, 1-, or 2-element coextension of a 3-connected nonbinary matroid. (Matroid N_1 is a *k-element coextension* of matroid N_2, if the ground set of N_1 has a k-element subset Y such that $N_2 = N_1/Y$; if $N_2 = N_1 \backslash Y$ for some k-element subset Y of the ground set of N_1, then we say that N_1 is a *k-element extension* of N_2.) A *contraction pair* is a deletion pair for the dual matroid.

In this section we prove the following result.

THEOREM 3.1. *A 3-connected matroid has a deletion pair or a contraction pair if and only if it is nonbinary and has rank or corank at least* 4.

This theorem has been derived independently by Whittle [29]. We include our proof for the sake of completeness. Whittle's result is more general than Theorem 3.1. However, our proof techniques provide a shorter proof of his result. One of our main tools is the following theorem of Seymour [24].

THEOREM 3.2 (Splitter Theorem). *Let N be a 3-connected proper minor of a 3-connected matroid M. If M is not a wheel or a whirl, then it contains an element x such that either $M \backslash x$ or M/x is 3-connected and has a minor isomorphic to N.*

Let $\mathscr{L}$ denote the collection of matroids $\{U_{2,5}, F_7^-, P_7, O_7\}$ (see Fig. 1) and $\mathscr{L}^* := \{M^*: M \in \mathscr{L}\}$. The next lemma is helpful in proving Theorem 3.1.

LEMMA 3.3. *Each 3-connected nonbinary matroid that is not a whirl has a minor in $\mathscr{L} \cup \mathscr{L}^*$.*

Proof. It has been proven by Coullard [5] (cf. Coullard and Oxley [6], Oxley [17, p. 370]) that each 3-connected nonbinary matroid that is not a whirl has a 3-connected 1-element extension or coextension of $U_{2,4}$ or $\mathscr{W}^3$ as a minor. The only 3-connected 1-element extension of $U_{2,4}$ is $U_{2,5}$. So, by duality, we need only prove that any 3-connected 1-element

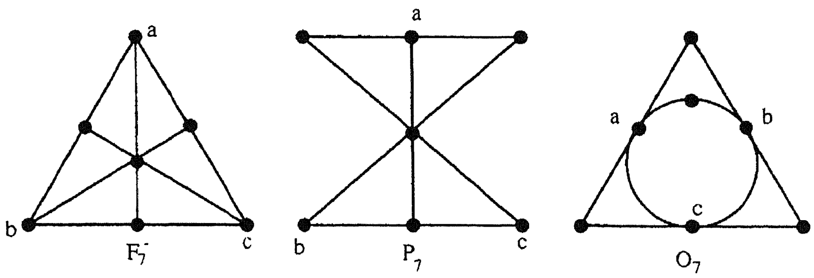

FIG. 1. The three 7-element members of $\mathscr{L}$.

extension M of $\mathcal{W}^3$ without $U_{2,5}$- or $U_{3,5}$-minors is in $\mathcal{L}$. As M is non-binary, it is neither F_7 nor F_7^*. Hence, as M has seven elements, it is ternary. From this it is easy to check that M is isomorphic to F_7^-, P_7, or O_7. ∎

Before we turn to the proof of Theorem 3.1, we first give some preliminary results. The first one is well known and easy.

PROPOSITION 3.4. *Let M be a connected matroid not isomorphic to $U_{1,4}$ and with ground set S. If $x \in S$ such that $M\backslash x$ is 3-connected, then either M is 3-connected or there exists a unique element p_x in S such that $\{x, p_x\}$ is a circuit. Moreover, if M is not 3-connected, then $(\{x, p_x\}, S\backslash\{x, p_x\})$ is the unique 2-separation in M.*

The next one is only a little bit more involved.

PROPOSITION 3.5. *Let N be a matroid with at least six elements and let x and y be two elements of the ground set S of N such that $N\backslash x/y$ is 3-connected and such that $N\backslash x$ and N/y are connected, and N, $N\backslash x$, and N/y are not 3-connected. Let p_x be the unique element such that $\{x, p_x\}$ is a circuit in N/y and p_y be the unique element such that $\{y, p_y\}$ is a cocircuit in $N\backslash x$.*

If $p_x \neq p_y$, then x is parallel to p_x in N and y is in series with p_y in N; moreover, there are no 2-separations in N other than $(\{x, p_x\}, S\backslash\{x, p_x\})$ and $(\{y, p_y\}, S\backslash\{y, p_y\})$.

If $p_x = p_y$, then $(\{x, y, p_x\}, S\backslash\{x, y, p_x\})$ is a 2-separation of N and there exists at most one other 2-separation, which, if it exists, is either $(\{x, p_x\}, S\backslash\{x, p_x\})$ or $(\{y, p_y\}, S\backslash\{y, p_y\})$.

Proof. Let (X, Y) be a 2-separation of N, such that X and Y both have at least three elements. Assuming $y \in X$, the partition $(X\backslash\{y\}, Y)$ is a 2-separation of N/y. As N has at least six elements, N/y is not isomorphic to $U_{1,4}$. Hence, by Proposition 3.4, $X\backslash\{y\} = \{x, p_x\}$. So, $x \in X$, and by symmetry between x and y (under duality), $X\backslash\{x\} = \{y, p_y\}$. Hence, $p_x = p_y$ and $X = \{x, y, p_x\}$.

On the other hand, if $p_x = p_y$, then both the rank and corank of $\{x, y, p_x\}$ are at most 2, so $(\{x, y, p_x\}, S\backslash\{x, y, p_x\})$ is a 2-separation.

It remains to check the 2-separations (X, Y) with $|X| \leqslant 2$. As $N\backslash x$ and N/y are connected, so is N. Hence, X is a pair of series or parallel elements in N. By duality, we may assume that X is a parallel pair. Then $X\backslash\{y\}$ is dependent in N/y. Hence $X = \{x, p_x\}$. As $\{y, p_y\}$ is a cocircuit in $N\backslash x$, exactly one of $\{y, p_y\}$ and $\{x, y, p_y\}$ is a cocircuit in N. The intersection of a circuit and a cocircuit cannot consist of exactly one element. So, if

$p_x \neq p_y$, $\{y, p_y\}$ is a cocircuit in N and if $p_x = p_y$, then $\{y, p_y\}$ is not a cocircuit in N. ∎

Now we get to the proof of Theorem 3.1.

Proof of Theorem 3.1. Clearly, 3-connected matroid with a contraction pair are nonbinary and have rank at least 4. So assume that there exists a 3-connected nonbinary matroid M with rank or corank at least 4 that has no deletion or contraction pair. It is easy to check that M is not a whirl. Hence, by Lemma 3.3, M has a minor in $\mathscr{L} \cup \mathscr{L}^*$.

For a matroid N, we define $\Lambda(N)$ as the set of elements q in N such that $N \backslash q$ is 3-connected and nonbinary; $\Lambda^*(N) := \Lambda(N^*)$. In this proof we will repeatedly use the following three facts:

(1) *If N is 3-connected and $q \in \Lambda(N)$, then $\Lambda(N \backslash q) \subseteq \Lambda(N) \backslash q$.*

(2) *Each $L \in \mathscr{L}$ satisfies $|\Lambda(L)| \geqslant 3$ and $\Lambda^*(L) = \emptyset$.*

(3) *If N is 3-connected and $\Lambda(N) = \{q\}$, then $\Lambda^*(N \backslash q) \neq \emptyset$.*

Assertion (1) is an obvious consequence of Proposition 3.4 and (2) is easily checked. We prove (3) by contradiction. Assume that N is 3-connected, that $\Lambda(N) = \{q\}$, and that $\Lambda^*(N \backslash q) = \emptyset$. Then, by (1), $\Lambda(N \backslash q) = \emptyset$. So it follows from the Splitter Theorem that $N \backslash q$ is a whirl. Hence N is not a whirl and thus, by Lemma 3.3, it has a minor in $\mathscr{L} \cup \mathscr{L}^*$. As $|\Lambda(N)| = 1$, it follows from (2) that this minor is proper. Because $\Lambda(N) = \{q\}$ and the whirl $N \backslash q$ has no minor in $\mathscr{L} \cup \mathscr{L}^*$, it now follows from the Splitter Theorem that $\Lambda^*(N) \neq \emptyset$. In fact, $\Lambda^*(N) = \{q\}$, as for each element x of the whirl $N \backslash q$, the matroid $N \backslash q/x$, and therefore also N/x, has parallel elements. As the rank and corank of N/q differ by exactly 2, it follows from Lemma 3.3 that N/q has a proper minor in $\mathscr{L} \cup \mathscr{L}^*$. So, as $\Lambda^*(N/q) = \emptyset$, there exists an element $y \neq q$ in N such that $N/q \backslash y$ is 3-connected. As $y \notin \{q\} = \Lambda(N)$, there exists an element z that is in series with q in $N \backslash y$. So $\{y, z, q\}$ is a cocircuit in N. As this contradicts the 3-connectivity of $N \backslash q$, (3) follows.

(4) $M \notin \mathscr{L} \cup \mathscr{L}^*$.

As the rank or corank of M is at least 4, $M \notin \{U_{2,5}, U_{2,5}^*\}$. So, to prove (4), it suffices to prove that each of F_7^-, P_7, and O_7 has a deletion pair. Therefore, consider the geometric representations of these three matroids depicted in Fig. 1. It is easy to check that, in each of these pictures, the indicated elements a and b form a deletion pair.

By duality, the Splitter Theorem, and (4), we may assume that, for some element e_1 of M, $M_1 := M/e_1$ is 3-connected and has a minor in $\mathscr{L} \cup \mathscr{L}^*$.

As M has no contraction pair, it follows from (1) that $\Lambda^*(M_1) = \emptyset$. So $M_1 \notin \mathscr{L}^*$. Hence, $M_1 \in \mathscr{L}$ or M_1 has a proper minor in $\mathscr{L} \cup \mathscr{L}^*$. In either case, $\Lambda(M_1) \neq \emptyset$. If $M_1 \notin \mathscr{L}$, choose $e_2 \in \Lambda(M_1)$ such that $M_1 \backslash e_2$ contains a minor in $\mathscr{L} \cup \mathscr{L}^*$. If $M_1 \in \mathscr{L}$, choose e_2 arbitrarily in $A(M_1)$. In either case, we define $M_2 := M_1 \backslash e_2$. (Soon we will see that, in fact, M_2 will have a minor in $\mathscr{L} \cup \mathscr{L}^*$.)

A subset X of the ground set S of M is *deletable* if $M \backslash X$ is a 0-, 1-, or 2-element coextension of a 3-connected nonbinary matroid. A subset X of S is *contractible* if M/X is a 0-, 1-, or 2-element extension of a 3-connected nonbinary matroid.

(5) *If $M_2 \backslash f$ is a 0- or 1-element coextension of a 3-connected nonbinary matroid and $M_1 \backslash f$ is 3-connected, then $\{e_1, e_2, f\}$ is a cocircuit in M.*

Indeed, as $\{f\}$, $\{e_2\}$, and $\{e_2, f\}$ are deletable and M has no deletion pair, $M \backslash e_2, f$ is not connected. Hence, e_1 is a coloop in $M \backslash e_2, f$ (it cannot be a loop in that matroid as M is connected). So $\{e_1, e_2, f\}$ is a cocircuit in M.

(6) *M_2 has a minor in $\mathscr{L} \cup \mathscr{L}^*$.*

If not, $M_1 \in \mathscr{L}$. As M has rank or corank at least 4, $M_1 \in \{F_7^-, P_7, O_7\}$. As $e_2 \in A(M_1)$, we may assume, by symmetry, that e_2 is the element denoted by a in the geometric representation of M_1 in Fig. 1. It is easy to check that $M_1 \backslash b \cong M_1 \backslash c \cong \mathscr{W}^3$ and $M_1 \backslash a, b$ and $M_1 \backslash a, c$ are connected 1-element coextensions of $U_{2,4}$. From (5) it follows that $\{e_1, e_2, b\}$ and $\{e_1, e_2, c\}$ are cocircuits in M. Then, by the circuit exchange axiom, $\{a, b, c\} = \{e_2, b, c\}$ is a cocircuit of M, and, hence, also of M_1. By Fig. 1, this is nonsense. So (6) follows.

(7) $e_2 \notin A(M)$.

Suppose $e_2 \in A(M)$. Then $e_2 \in A^*(M^*)$, $e_1 \in A^*(M \backslash e_2) = A(M^*/e_2)$, and, by (6), $M^*/e_2 \backslash e_1$ has a minor in $\mathscr{L} \cup \mathscr{L}^*$. So, if we turn from M to M^*, e_1 and e_2 switch roles. Hence, by duality and (6), we may assume that, for some f, $M_2 \backslash f$ is 3-connected and nonbinary. Then, $M_1 \backslash f$ is 3-connected (because M_1 and $M_1 \backslash e_2, f = M_2 \backslash f$ are 3-connected). Hence, by (5), $\{e_1, e_2, f\}$ is a cocircuit. But then e_1 and f are in series in the 3-connected matroid $M \backslash e_2$. As this is absurd, (7) follows.

So there exists an element $e_{12} \in S \backslash \{e_1, e_2\}$ such that e_{12} is in series with e_1 in $M \backslash e_2$; in other words, such that $\{e_1, e_2, e_{12}\}$ is a cocircuit in M. As M_2 is 3-connected, the element e_{12} is unique and it follows from (5) that $A(M_2) \subseteq \{e_{12}\}$. The following fact will be used repeatedly throughout the rest of this proof.

(8) *If $q, p \in S \backslash \{e_1, e_2, e_{12}\}$, then $M_2/q \backslash p$ is binary or has a 2-separation.*

Suppose this is false, so that $M_2/q \backslash p$ is 3-connected and nonbinary. We first argue that

(8.1) *The matroids* $M\backslash p, e_2/e_1$, $M\backslash p/q, e_1$, *and* $M\backslash p, e_2/q$ *are connected.*

As M_2 is 3-connected, $M\backslash p, e_2/e_1 = M_2\backslash p$ has no loops or coloops. As M_1 is 3-connected, $M\backslash p/q, e_1 = M_1\backslash p/q$ has no loops or coloops. So, as $M\backslash p, e_2/e_1, q = M_2\backslash p/q$ is 3-connected, both $M\backslash p, e_2/e_1$ and $M\backslash p/q, e_1$ are connected.

As M is 3-connected, e_1 is not a loop in $M\backslash p, e_2/q$. Moreover, e_1 is not a coloop in $M\backslash p, e_2/q$, because $\{e_1, e_2, e_{12}\}$ is the unique 3-element cocircuit in M containing $\{e_1, e_2\}$. So, as $M_2/q\backslash p$ is 3-connected, $M\backslash p, e_2/q$ is connected. Thus (8.1) holds.

(8.2) *Each of* $M\backslash p, e_2/e_1$, $M\backslash p/q, e_1$, $M\backslash p, e_2/q$, $M\backslash p/q$, *and* $M\backslash p/e_1$ *has a 2-separation.*

The sets $\{e_1\}$ and $\{e_1, q\}$ are both contractible. As M_1 is 3-connected, $M/e_1, q$ has no loops or coloops. Hence, as $M\backslash p/q, e_1$ is connected, so is $M/e_1, q$. So we see that $\{q\}$ is not contractible, since otherwise $\{e_1, q\}$ would be a contraction pair. Hence, as $M\backslash p, e_2/q$ and $M\backslash p/q$ are nonbinary, neither of the two is 3-connected.

The sets $\{e_2\}$ and $\{e_2, p\}$ are both deletable. As $M\backslash p, e_2/q$ and $M\backslash p, e_2/e_1$ are connected, so is $M\backslash p, e_2$. Hence, as $\{e_2, p\}$ is not a deletion pair, the set $\{p\}$ is not deletable. So, as $M\backslash p/q, e_1$ and $M\backslash p/e_1$ are nonbinary, neither of the two is 3-connected.

Finally, as $\Lambda(M_2) \subseteq \{e_{12}\}$, $M\backslash p, e_2/e_1 = M_2\backslash p$ is not 3-connected. Thus (8.2) holds.

(8.3) *The elements* q *and* e_{12} *are in series in* $M\backslash e_2, p$.

To see this, apply Proposition 3.5 to the two triples

$$\begin{cases} N_1 := M/e_1\backslash p \\ x_1 := e_2 \\ y_1 := q \end{cases} \quad \text{and} \quad \begin{cases} N_2 := M/q\backslash p \\ x_2 := e_2 \\ y_2 := e_1 \end{cases}.$$

As M_1 is 3-connected, N_1 has no parallel elements. So by Proposition 3.5, $p_{x_1} = p_{y_1}$. As $\{e_1, e_2, e_{12}\}$ is a cocircuit in M, it is a cocircuit in N_2 as well. Hence, $p_{y_2} = e_{12}$ and y_2 and p_{y_2} are not in series in N_2. So by Proposition 3.5, $p_{x_2} = p_{y_2} = e_{12}$. Finally, as $N_1/y_1 = N_2/y_2$ and $x_1 = x_2$, we have that $p_{x_1} = p_{x_2}$. So we conclude that $p_{y_1} = e_{12}$. In other words, $q(= y_1)$ and e_{12} are in series in $N_1\backslash x_1 = M_1\backslash e_2, p$, hence also in $M\backslash e_2, p$. So (8.3) follows.

By (8.3), $(M/e_1, e_{12})\backslash p, e_2$ is isomorphic to the 3-connected matroid $(M/e_1, q)\backslash p, e_2$. Hence $\{e_1, e_{12}\}$ is contractible. Moreover, as M/e_1 is 3-connected, $M/e_1, e_{12}$ is connected. As $M/e_{12}\backslash e_2 \cong M/e_1\backslash e_2$ is 3-connected,

$\{e_{12}\}$ and $\{e_1\}$ are contractible as well. So $\{e_1, e_{12}\}$ is a contraction pair. As M has no such pair, (8) follows.

(9) *If* $e_{12} \in \Lambda(M_2)$, *then* $M_2 \backslash e_{12}$ *is a whirl.*

If not, then by Lemma 3.3, $M_2 \backslash e_{12}$ is 3-connected and has a minor in $\mathscr{L} \cup \mathscr{L}^*$. Hence, e_{12} satisfies the properties required from e_2 when it was defined. So there is a symmetry between e_2 and e_{12}. As $\Lambda(M_2) = \{e_{12}\}$, it follows from (3) that $\Lambda^*(M_2 \backslash e_{12}) \neq \varnothing$. Let $q \in \Lambda^*(M_2 \backslash e_{12})$.

(9.1) $\{e_2, e_{12}\}$ *is the unique parallel pair in* $M/e_1, q$.

Indeed, as $\{e_1, q\}$ is not a contraction pair, $M/e_1, q$ is not 3-connected. As $M/e_1, q$ is connected and $M/e_1, q \backslash e_2, e_{12}$ is 3-connected, there exist parallel pairs in $M/e_1, q$; moreover, each of those involve at least one of e_2 and e_{12}. By the symmetry between e_2 and e_{12} noted above, we may assume that e_{12} is parallel in $M/e_1, q$ with an another element p. If p were different from e_2, then $M_2 \backslash p/q$ would be isomorphic to the 3-connected nonbinary matroid $M_2 \backslash e_{12}/q$, contradicting (8). So $p = e_2$, which proves (9.1).

By (9.1), M_2/q is 3-connected. Now $\Lambda(M_2/q) = \{e_{12}\}$, as if $p \in \Lambda(M_2/q) \backslash \{e_{12}\}$, then p and q would falsify (8). So, by (3), $\Lambda^*(M_2/q \backslash e_{12}) \neq \varnothing$. Then, by (1), $\Lambda^*(M_2 \backslash e_{12}) \neq \{q\}$; let $q' \in \Lambda^*(M_2 \backslash e_{12}) \backslash \{q\}$. By (9.1), $\{e_2, e_{12}, q\}$ and $\{e_2, e_{12}, q'\}$, hence also $\{e_{12}, q, q'\}$, are circuits in M_1. However, that means that e_{12} and q' are parallel in M_1/q, contradicting (9.1). This proves (9).

Recall that $\Lambda(M_2) \subseteq \{e_{12}\}$. Hence, by (2), the matroid M_2 is not in $\mathscr{L}$; and, by (9), M_2 has no 3-connected proper deletion minor with a minor in $\mathscr{L} \cup \mathscr{L}^*$. Hence, as M_2 has a minor in $\mathscr{L} \cup \mathscr{L}^*$, it follows from the Splitter Theorem that M_2 is a member of $\mathscr{L}^*$ or has a 3-connected proper contraction minor with a minor in $\mathscr{L} \cup \mathscr{L}^*$. In either case, $\Lambda^*(M_2)$ is not empty; let q be one of its members.

As $\{e_1, q\}$, $\{e_1\}$ are contractible and $M/e_1, q$ is connected, $\{q\}$ is not contractible. Hence M/q is not 3-connected. Moreover, as $M/e_{12} \backslash e_2 \cong M/e_1 \backslash e_2$, the set $\{e_{12}\}$ is contractible, so $q \neq e_{12}$.

Apply Proposition 3.5 to the triple

$$\begin{cases} N := M/q \\ x := e_2 \\ y := e_1 \end{cases}.$$

As M is 3-connected, N has no series elements. So by Proposition 3.5, $p_x = p_y$. As $\{y, x, e_{12}\} = \{e_1, e_2, e_{12}\}$ is a cocircuit in M, $p_y = e_{12}$. Hence, e_2 and e_{12} are parallel in M_1/q. Suppose, there existed a second element q' in $\Lambda^*(M_2)$. Then e_2 and e_{12} would be parallel in M_1/q' as well. So $\{e_2, e_{12}, q\}$ and $\{e_2, e_{12}, q'\}$, hence also $\{e_{12}, q', q\}$, would be circuits in M_1. This implies that e_{12} and q' would be parallel in M_2/q, which is absurd. So we see that $\Lambda^*(M_2) = \{q\}$.

Hence, by (3) and (8), $\Lambda(M_2/q) = \{e_{12}\}$. As $M_2\backslash e_{12}/q$ is 3-connected, it follows from (9), that $e_{12} \notin \Lambda(M_2)$. Hence q is in series in $M_2\backslash e_{12}$ with some other element q''. The matroid $M_2/q''\backslash e_{12} \cong M_2/q\backslash e_{12}$ is 3-connected. As $q'' \notin \Lambda^*(M_2)$, e_{12} is parallel in M_2/q'' to an element e''_{12}. As $M_2/q''\backslash e''_{12} \cong M_2/q''\backslash e_{12} = M_2\backslash e_{12}/q'' \cong M_2\backslash e_{12}/q$, the two elements q'' and e''_{12} contradict (8). So Theorem 3.1 follows. ∎

4. TWISTED MATROIDS AND BLOCKING SEQUENCES

This section provides notions and preliminaries needed in our proof of Theorem 1.1. The most important notions are "twisted matroids" and "blocking sequences."

Twisted Matroids and Fundamental Graphs

Let $\mathscr{B}$ be the set of bases of a matroid M with ground set S. For $B \in \mathscr{B}$, define $M_B := (S, \mathscr{F}_B)$, where $\mathscr{F}_B := \{B \Delta B' \colon B' \in \mathscr{B}\}$. Members of $\mathscr{F}_B$ are called *feasible sets* of the *twisted matroid* M_B. We endow M_B with a rank function r_B: if $X \subseteq S$, then $r_B(X)$ is half the size of the largest feasible set in X. Equivalently, $r_B(X) := r(X \Delta B) - |B\backslash X|$. Note that duality is absorbed in the definition of a twisted matroid, since $M_B = (M^*)_{S\backslash B}$.

The notion of a twisted matroid is not new. Twisted matroids are essentially the same as "linking systems" (Schrijver [22]), "bimatroids" (Kung [13]), or "abstract matrices" (Truemper [26]). The notion of a twisted matroid as a matroid viewed with respect to a fixed basis resembles that of a fundamental graph. In fact, fundamental graphs are easily defined in terms of twisted matroids. The *fundamental graph* of M_B is the bipartite graph $G_B = (S, E_B)$, where $E_B := \{ij \colon \{i, j\} \in \mathscr{F}_B\}$. We denote by $\text{nigh}_B(x)$ the neighbour set of vertex x in G_B. Equivalently, $\text{nigh}_B(x) := \{y \in S : B\Delta\{x, y\} \in \mathscr{B}\}$. For $X \subseteq S$, $G_B[X]$ denotes the subgraph of G_B induced by X. Our proof techniques in the subsequent sections are mainly graphic, acting on fundamental graphs. One reason to use fundamental graphs is that they reveal a lot about the connectivity of the matroid. However, on the other hand, they also suppress much information about the matroid, also regarding connectivity. The reason to work with twisted matroids is to allow graph-theoretical reasoning without losing contact with the actual matroid.

Representability is quite natural for twisted matroids. An *X by Y matrix over* a field $\mathbf{F}$ is a matrix in $\mathbf{F}^{X \times Y}$. If A is an X by Y matrix, $X' \subseteq X$, and $Y' \subseteq Y$, then we denote the X' by Y' submatrix of A by $A[X', Y']$. A B by $S\backslash B$ matrix A over $\mathbf{F}$ is an *$\mathbf{F}$-representation* of M_B if the rank of the matrix

$A[X, Y]$ is equal to $r_B(X \cup Y)$ for each $X \subseteq B$ and $Y \subseteq S \backslash B$. Equivalently, A is an $\mathbf{F}$-representation of M_B if and only if (I, A) is an $\mathbf{F}$-representation of M. So we see that twisted matroids match very well with the common practice in matroid theory of considering standard representations with respect to a fixed basis; these are representations of the form (I, A), where I represents the fixed basis. A subset X of S is a feasible set X of M_B if and only if the submatrix $A[X \cap B, X \backslash B]$ is nonsingular. One way to visualize an $\mathbf{F}$-representation of a twisted matroid is as a labeling of the edges of the fundamental graph with nonzero elements from $\mathbf{F}$.

The following propositions are well-known and straightforward to prove; in fact, they are trivial for representable matroids.

PROPOSITION 4.1 (Brualdi [4]). *If X is a feasible set of M_B, then $G_B[X]$ has a perfect matching.*

PROPOSITION 4.2 (Krogdahl [12]). *If $G_B[X]$ has a unique perfect matching, then X is feasible in M_B.*

Restrictions and Minors

Given $X \subseteq S$, we define the *restriction of M_B to X* as $M_B[X] := (X, \mathscr{F}')$, where $\mathscr{F}' := \{F \subseteq X : F \in \mathscr{F}_B\}$. It is easy to prove that $M_B[X]$ is a twisted matroid again, namely, the twisted matroid $M'_{B'}$ with $B' := B \cap X$ and with M' the minor of M obtained by deleting $(S \backslash B) \backslash X$ and contracting $B \backslash X$. We stress that we never have to specify the actual restriction when we write that some set is feasible or not; a set is feasible in a restriction if and only if it is feasible in the original matroid. Also note that "restriction of a twisted matroid" is not the same as "restriction of a matroid"; the latter is just a deletion minor.

Clearly, the rank function of the restriction of M_B to X is the restriction of the rank function of M_B to subsets of X. Moreover, if A is an $\mathbf{F}$-representation of M_B, then the submatrix $A[X \cup B, X \backslash B]$ is an $\mathbf{F}$-representation of $M_B[X]$. We denote by $M_B - X$ the twisted matroid $M_B[S \backslash X]$.

Finally, note that, although restrictions of a twisted matroid are twisted minors, it is not true that each minor of M corresponds to a restriction of M_B. To make a minor "visible" as a restriction we might have to change the basis.

Pivoting

Usually we work with a fixed basis B, but sometimes it will be necessary to change bases, for instance to make a minor "visible" as a restriction. It is straightforward to see that, for any feasible set X, $\mathscr{F}_{B \Delta X} = \{F \Delta X : F \in \mathscr{F}_B\}$. Typically we will change to a basis $B \Delta \{x, y\}$ for an edge xy of G_B. We call such a shift from M_B to $M_{B \Delta \{x, y\}}$ a *pivot on xy*. Let B' denote $B \Delta \{x, y\}$.

A pivot is also a matrix operation. Indeed, if M_B is represented by a matrix A over $\mathbf{F}$, then *pivoting on xy* in A yields an $\mathbf{F}$-representation A' of $M_{B'}$, where

$$A=\begin{array}{cc} & y \\ x & \\ & \end{array}\!\!\begin{pmatrix} \alpha & v^T \\ w & D \end{pmatrix} \quad \text{and} \quad A'=\begin{array}{cc} & x \\ y & \\ & \end{array}\!\!\begin{pmatrix} -\alpha & v^T \\ w & D-\alpha^{1}wv^T \end{pmatrix}.$$

Much of the structure of $G_{B'}$ is determined by G_B. The following observations are trivial for represented twisted matroids. For general twisted matroids, representable or not, they are easy consequences of Propositions 4.1 and 4.2.

(i) $\operatorname{nigh}_{B'}(x)=\operatorname{nigh}_B(y)\,\Delta\{x,y\}$ and $\operatorname{nigh}_{B'}(y)=\operatorname{nigh}_B(x)\,\Delta\{x,y\}$,

(ii) if $v\notin \operatorname{nigh}_B(x)\cup\operatorname{nigh}_B(y)$, then $\operatorname{nigh}_{B'}(v)=\operatorname{nigh}_B(v)$, and

(iii) if $v\in\operatorname{nigh}_B(x)$, $w\in\operatorname{nigh}_B(y)\backslash\operatorname{nigh}_B(v)$, then vw is an edge of $G_{B'}$.

Thus we can account for most edges of $G_{B'}$. The only pairs $\{v,w\}$ for which G_B does not reveal whether or not vw is an edge of $G_{B'}$ are the ones for which $\{x,y,v,w\}$ induces a circuit in G_B. In that case, vw is an edge in $G_{B'}$ whenever $\{x,y,v,w\}$ is feasible in M_B.

Twirls

A twisted matroid M_B is a *twirl* if G_B is an induced circuit and S is feasible. Note that a twirl is a twisted whirl, for an appropriate choice of the distinguished basis. (Consider a whirl constructed from a wheel in the usual way and take the set of spokes as the distinguished basis.)

As mentioned before we will often work with fundamental graphs. One major disadvantage of these graphs is that they do not reveal whether the matroid is binary or not. The following lemma says that, for $GF(4)$-representable matroids, the fundamental graph plus a list of the twirls provide all the information we need to determine which restrictions of a twisted matroid are nonbinary. The lemma is crucial to our proof of Theorem 1.1.

LEMMA 4.3. *Let B be a basis in a $GF(4)$-representable matroid M. Then M is nonbinary if and only if some restriction of M_B is a twirl.*

Proof. As the "if"-direction is trivial, we only prove the "only if"-direction. Let A be a representation of M_B over $GF(4)$, and let T be a spanning forest of G_B. We interpret the entries of A as edge-weights for G_B. By scaling rows and columns of A, we may assume each edge ij of T has weight one. Since M is not binary, A is not a $(0,1)$-matrix. Therefore there exists a circuit in G_B having exactly one edge of weight different from one. Let C be such a circuit having minimum length, and let X be the set of vertices of C. Then C is an induced circuit, and $M_B[X]$ is a twirl. ∎

The following lemma is proven in much the same way; the details are left to the reader.

LEMMA 4.4. *Let B be a basis of a $GF(4)$-representable matroid M. Suppose, for $X \subseteq S$, that $M_B[X]$ is a twirl, and $x \in S \backslash X$ such that $|\text{nigh}_B(x) \cap X| \geqslant 2$. Then, there exists a twirl $M_B[X']$ with $x \in X' \subset X \cup \{x\}$.*

The previous two lemmas are interesting in that they hold for $GF(4)$-representable matroids, but they fail in general. Indeed, Lemma 4.4 fails for the non-Fano (F_7^-), and Lemma 4.3 fails for some 8-element matroids. It can in fact be shown that both results hold for all matroids that contain neither the non-Fano nor its dual as a minor (Geelen [7]).

The following proposition describes the effect of pivoting on twirls.

PROPOSITION 4.5. *Let M_B be a twirl, and let xy be an edge of G_B.*

(i) *If $|S| = 4$, then $M_{B \Delta \{x, y\}}$ is a twirl.*
(ii) *If $|S| > 4$, then $M_{B \Delta \{x, y\}}[S \backslash \{x, y\}]$ is a twirl.*

A consequence of Proposition 4.5 is that the fundamental graph resulting from a pivot on $xy \in E_B$ is completely determined by G_B and all the 4-element twirls through xy.

Connectivity

Next we extend the connectivity function of M. Given subsets X and Y of S, we define

$$\lambda_B(X, Y) := r_B((X \cap B) \cup (Y \backslash B)) + r_B((Y \cap B) \cup (X \backslash B)).$$

We call λ_B the *connectivity function* of M_B. (The function $\lambda(X) := \lambda_B(X, S \backslash X)$ is the usual connectivity function of a matroid.) Note that the restriction of the connectivity function of M_B to the subsets of $S' \subseteq S$ is the connectivity function of $M_B[S']$.

For representable matroids there is an easy description of λ_B. Suppose that A is a representation of M_B. Let T denote the skew-symmetric matrix

$$\begin{array}{cc} & \begin{array}{cc} B & S\backslash B \end{array} \\ \begin{array}{c} B \\ S\backslash B \end{array} & \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix}. \end{array}$$

Then $\lambda_B(X, Y) = \text{rank } T[X, Y]$. For represented matroids, many of the results in this section can easily be verified using T.

The connectivity function has the following properties:

Symmetry. For subsets X, Y of S, $\lambda_B(X, Y) = \lambda_B(Y, X)$.

Monotonicity. For subsets X, X', Y of S, where $X \subseteq X'$, $\lambda_B(X, Y) \leqslant \lambda_B(X', Y)$.

Unit-Increase. For subsets X, X', Y of S, where $X \subseteq X'$, $\lambda_B(X, Y) \geqslant \lambda_B(X', Y) - |X' \backslash X|$.

Linking-Submodularity. For subsets X_1, X_2, Y_1, Y_2 of S,

$$\lambda_B(X_1, Y_1) + \lambda_B(X_2, Y_2) \geqslant \lambda_B(X_1 \cap X_2, Y_1 \cup Y_2) + \lambda_B(X_1 \cup X_2, Y_1 \cap Y_2).$$

The edges of G_B are easily characterized in terms of λ_B.

PROPOSITION 4.6. *If $x, y \in S$, then $\lambda_B(\{x\}, \{y\}) \leqslant 1$. Moreover, $\lambda_B(\{x\}, \{y\}) = 1$ if and only if xy is an edge of G_B.*

The following proposition explicitly describes the effect that pivoting has on the connectivity function. Again, the calculation is left to the reader.

PROPOSITION 4.7. *Let F be a feasible set of M_B, and let X, Y be subsets of S. Now let $X' := (X \backslash F) \cup (F \backslash Y)$, and $Y' := (Y \backslash F) \cup (F \backslash X)$. Then*

$$\lambda_{B \Delta F}(X, Y) = \lambda_B(X', Y') - |X'| + |X|.$$

Let (X, Y) be a partition of S such that $|X|, |Y| \geqslant k$. If $\lambda_B(X, Y) \leqslant k-1$, then we call (X, Y) a *k-separation* of M_B; if $\lambda_B(X, Y) = k-1$ we call the k-separation *exact*. Note that (X, Y) is a k-separation of M_B if and only if (X, Y) is a k-separation of M, in the usual sense. We call a twisted matroid *k-connected* if it has no $(k-1)$-separation, in other words, if the underlying matroid is k-connected.

PROPOSITION 4.8. *Let X, Y be subsets of S, and let $X' \subseteq X$ and $Y' \subseteq Y$ be such that $\lambda_B(X', Y') = k-1$. Then, $\lambda_B(X, Y) \geqslant k$ if and only if there exist $x \in X$ and $y \in Y$ such that $\lambda_B(X' \cup \{x\}, Y' \cup \{y\}) = k$.*

Proof. Firstly, it is clear that if $\lambda_B(X, Y) = k-1$, then, for each $x \in X$ and $y \in Y$, we have $\lambda_B(X' \cup \{x\}, Y' \cup \{y\}) = k-1$. Conversely, suppose that $\lambda_B(X, Y) \geqslant k$. Choose minimal sets X'', Y'' such that $X' \subseteq X'' \subseteq X$, $Y' \subseteq Y'' \subseteq Y$, and $\lambda_B(X'', Y'') \geqslant k$. We are required to prove that $|X''| \leqslant |X'| + 1$ and $|Y''| \leqslant |Y'| + 1$. Suppose not. By the symmetry between X and Y, we may assume that $|X''| \geqslant |X'| + 2$. Let x_1, x_2 be distinct elements in $X'' \backslash X'$. By our choice of X'', we have

$$\lambda_B(X'' - x_1, Y'') = \lambda_B(X'' - x_2, Y'') = \lambda_B(X'' - x_1 - x_2, Y'') = k - 1.$$

However, by the submodularity of λ_B, we have

$$\lambda_B(X'' - x_1, Y'') + \lambda_B(X'' - x_2, Y'') \geqslant \lambda_B(X'' - x_1 - x_2, Y'') + \lambda_B(X'', Y''),$$

which is a contradiction. ▮

PROPOSITION 4.9. *Let X and Y be subsets of S and let $x \in S \backslash X$ such that $\lambda(X \cup \{x\}, Y) > \lambda(X, Y)$. Then, in G_B, x is adjacent to a node in Y.*

Proof. By submodularity: $\lambda(X, Y) + \lambda(\{x\}, Y) \geqslant \lambda(X \cup \{x\}, Y) + \lambda(\varnothing, Y)$. Hence, $\lambda(\{x\}, Y) > 0 = \lambda(\varnothing, \varnothing)$. So, by Propositions 4.6 and 4.8, x is adjacent to a node in Y in G_B. ▮

We are primarily interested in 1- and 2-separations. We now consider how such separations can be identified in the fundamental graph. The following propositions are straightforward corollaries of Propositions 4.6 and 4.8.

PROPOSITION 4.10. *Let (X, Y) be a partition of S with $|X|$, $|Y| \geqslant 1$. Then (X, Y) is a 1-separation of M_B if and only if there are no edges from X to Y in G_B.*

For the next proposition we need some more definitions. A partition (X, Y) of S is called a *split* of G_B if $|X|$, $|Y| \geqslant 2$ and the edges from X to Y induce a complete bipartite graph. (A vertex in X need not be adjacent to each vertex in Y; in fact, if there are no edges from X to Y, then (X, Y) is a split.)

PROPOSITION 4.11. *If (X, Y) is a 2-separation of M_B, then (X, Y) is a split in G_B.*

The converse is not true. As stated below, the only splits that actually yield 2-separations are the ones without twirls.

PROPOSITION 4.12. *Let (X, Y) be a split in G_B and let $x_1 y_1$ be an edge of G_B with $x_1 \in X$ and $y_1 \in Y$. Then, (X, Y) is not a 2-separation of M_B if and only if there exist $x_2 \in X$ and $y_2 \in Y$ such that $M_B[\{x_1, x_2, y_1, y_2\}]$ is a twirl.*

Consider a 2-separation (X, Y) of M_B with $|X| = 2$. Let x_1, x_2 be the elements of X. By Proposition 4.11, in the graph G_B either x_1 and x_2 have the same neighbours, or one of x_1, x_2 has no neighbours in Y. Elements

$a, b \in S$ are called *twins* of M_B if they have the same neighbours in G_B and $(\{a, b\}, S \backslash \{a, b\})$ is a 2-separation. An element $a \in S$ is said to be *pendant* to an element $b \in S$ if b is the only neighbour of a in G_B. If a is pendant to b, then, by Proposition 4.8, $(\{a, b\}, S \backslash \{a, b\})$ is a 2-separation of M_B. The following proposition is straightforward; its proof is left to the reader.

PROPOSITION 4.13. *Suppose that a is pendant to b in $M_B[X]$. Then X is feasible if and only if $X \backslash \{a, b\}$ is feasible.*

Suppose (X, Y) is an exact 2-separation of M_B. It is well known that in that case the matroid M is a 2-sum of two proper minors of M. In fact, the parts of this 2-sum are easily recognized from M_B and (X, Y). Indeed, let $x \in X$ and $y \in Y$ be adjacent members of G_B. Then M is a 2-sum of the matroids underlying the twisted matroids $M_B[X \cup \{y\}]$ and $M_B[Y \cup \{x\}]$. (Note that the particular choice of x and y is irrelevant modulo isomorphism.)

Note that because of the previous observations, the fundamental graph and the twirls of a twisted matroid exhibit all the 1- and 2-separations and all the nonbinary restrictions. Moreover, they show whether or not the underlying matroid is stable (in which case we call the twisted matroid *stable* as well).

Blocking Sequence

In proving Theorem 1.1 we will frequently encounter 2-separations of minors of 3-connected matroids and nonstable minors of stable matroids. Intuitively, one might expect that in such a situation the parts of the 2-separation of the minors are connected one way or another by a certain structure that establishes that the 2-separation does not extend to the whole matroid. Such structures indeed exist, namely "blocking sequences." Blocking sequences were initially used in the study of delta-matroids [3].

Let $X, Y \subseteq S$ be disjoint sets. We call (X, Y) a *k-subseparation* of M_B if (X, Y) is a k-separation of $M_B[X \cup Y]$, in other words: if $|X|, |Y| \geqslant k$ and $\lambda_B(X, Y) < k$. A k-subseparation (X, Y) is *exact* if $\lambda_B(X, Y) = k-1$, and (X, Y) is *induced* if there exists a k-separation (X', Y') with $X \subseteq X'$ and $Y \subseteq Y'$. A "blocking sequence" is a certificate proving that an exact k-subseparation is not induced. Specifically, let (X, Y) be an exact k-subseparation of M_B; a sequence $v_1, \ldots, v_p$ of elements in $S \backslash (X \cup Y)$ is a *blocking sequence for* (X, Y) if

(i) (a) $\lambda_B(X, Y \cup \{v_1\}) = k$,

(b) $\lambda_B(X \cup \{v_i\}, Y \cup \{v_{i+1}\}) = k$, for $i = 1, \ldots, p-1$,

(c) $\lambda_B(X \cup \{v_p\}, Y) = k$, and

(ii) no proper subsequence of $v_1, \ldots, v_p$ satisfies (i).

There is a natural directed graph $D(X, Y)$ associated with the problem of finding a blocking sequence for (X, Y). Fix some $x \in X$ and some $y \in Y$; the particular choices are irrelevant. Then $D(X, Y)$ has vertex set $\{x, y\} \cup (S \backslash (X \cup Y))$ and arc set

$$\{uv: \lambda_B(X \cup \{u\}, Y \cup \{v\}) = k\}.$$

Clearly, $v_1, ..., v_p$ is a blocking sequence for (X, Y) if and only if $x, v_1, ..., v_p, y$ is a minimal directed (x, y)-path in $D(X, Y)$.

THEOREM 4.14. *Let (X, Y) be an exact k-subseparation of M_B. Then there exists a blocking sequence for (X, Y) if and only if (X, Y) is not induced.*

Proof. Suppose that (X, Y) is induced, and let (X', Y') be a k-separation in M_B with $X \subseteq X'$ and $Y \subseteq Y'$. Then, for all $x' \in X'$ and $y' \in Y'$, $\lambda_B(X \cup \{x'\}, Y \cup \{y'\}) = k - 1$. Consequently there exists no blocking sequence.

Conversely, suppose there exists no blocking sequence. Then there is no directed xy-path in $D(X, Y)$. Hence, there exists a partition (X', Y') of S such that, for all $x' \in X'$ and $y' \in Y'$, $\lambda_B(X \cup \{x'\}, Y \cup \{y'\}) = k - 1$. By Proposition 4.8, (X', Y') is a k-separation. ∎

The following proposition summarizes some nice properties of blocking sequences.

PROPOSITION 4.15. *Let $v_1, ..., v_p$ be a blocking sequence for an exact k-subseparation (X, Y) of M_B. Then the following properties hold.*

(i) *For $1 \leqslant i \leqslant j \leqslant p$, $v_i, ..., v_j$ is a blocking sequence for the exact k-subseparation $(X \cup \{v_1, ..., v_{i-1}\}, Y \cup \{v_{j+1}, ..., v_p\})$.*

(ii) *If $x_1 x_2$ is an edge of G_B, and $x_1, x_2 \in X \cup Y$, then $v_1, ..., v_p$ is a blocking sequence for the exact k-subseparation (X, Y) of $M_{B \Delta \{x_1, x_2\}}$.*

(iii) *If Y' is a subset of Y such that $|Y'| \geqslant k$ and $\lambda_B(X, Y') = k - 1$ and $\lambda_B(X \cup \{v_p\}, Y') > k - 1$, then $v_1, ..., v_p$ is a blocking sequence for the k-subseparation (X, Y') in M_B.*

(iv) *The sequence $v_1, v_2, ..., v_p$ alternates between elements of B and $S \backslash B$.*

Proof. For all assertions we may assume that S (the ground set of M) is equal to $X \cup Y \cup \{v_1, ..., v_p\}$.

Part (i). This follows immediately from definitions and Proposition 4.8.

Part (ii). Let X', Y' be disjoint subsets of S such that $X \subseteq X'$ and $Y \subseteq Y'$. By Proposition 4.7, we have $\lambda_{B \Delta \{x_1, x_2\}}(X', Y') = \lambda_B(X', Y')$. Then the result follows immediately from definitions.

Part (iii). Choose $v_0 \in X$. Then, for $i = 0, ..., p-1$, we have $\lambda_B(X \cup \{v_i\}, Y \cup \{v_{i+1}\}) = k$. Hence, as $\lambda_B(X, Y') = \lambda_B(X \cup \{v_i\}, Y) = \lambda_B(X, Y \cup \{v_{i+1}\}) = k-1$, it follows from Proposition 4.8 that $\lambda_B(X \cup \{v_i\}, Y' \cup \{v_{i+1}\}) \geqslant k$. Therefore, some subsequence of $v_1, ..., v_p$ is a blocking sequence for (X, Y'). By monotonicity, $v_1, ..., v_p$ is the blocking sequence, as required.

Part (iv). Suppose that the claim is false. Then, by part (i) and duality, we may assume that $p = 2$ and that v_1 and v_2 are both in B. We have $\lambda_B(X \cup \{v_1\}, Y \cup \{v_2\}) > \lambda_B(X, Y)$. By definition,

$$\lambda_B(X, Y) = r_B((X \cap B) \cup (Y \backslash B)) + r_B((Y \cap B) \cup (X \backslash B)),$$

and

$$\begin{aligned}
&\lambda_B(X \cup \{v_1\}, Y \cup \{v_2\}) \\
&\quad = r_B(((X \cup \{v_1\}) \cap B) \cup (Y \backslash B)) + r_B(((Y \cup \{v_2\}) \cap B) \cup (X \backslash B)).
\end{aligned}$$

Therefore, either

$$r_B(((X \cup \{v_1\}) \cap B) \cup (Y \backslash B)) > r_B((X \cap B) \cup (Y \backslash B)),$$

or

$$r_B(((Y \cup \{v_2\}) \cap B) \cup (X \backslash B)) > r_B((Y \cap B) \cup (X \backslash B)).$$

By symmetry, we assume that

$$r_B(((X \cup \{v_1\}) \cap B) \cup (Y \backslash B)) > r_B((X \cap B) \cup (Y \backslash B)).$$

Therefore,

$$\begin{aligned}
&\lambda_B(X \cup \{v_1\}, Y) \\
&\quad = r_B(((X \cup \{v_1\}) \cap B) \cup (Y \backslash B)) + r_B((Y \cap B) \cup (X \backslash B)) \\
&\quad > r_B((X \cap B) \cup (Y \backslash B)) + r_B((Y \cap B) \cup (X \backslash B)) \\
&\quad = \lambda_B(X, Y).
\end{aligned}$$

However, this contradicts the minimality of the blocking sequence. ∎

It is obviously desirable to find short blocking sequences. The following proposition describes ways to reduce the length of blocking sequences. Using these reductions it is often possible to reduce blocking sequences to length 1 or 2.

PROPOSITION 4.16. *Let $v_1, ..., v_p$ be a blocking sequence for an exact k-subseparation (X, Y) of M_B. Then the following properties hold.*

(i) *Let Y' be a subset of Y such that $|Y'| \geqslant k$ and $\lambda_B(X, Y') = k-1$. If $p > 1$, then $v_1, \ldots, v_{p-1}$ is a blocking sequence for the exact k-subseparation $(X, Y' \cup \{v_p\})$.*

(ii) *Let $y \in Y$ be a neighbour in G_B of v_p such that $\lambda_B(X \cup \{y\}, Y) = k$. If $p > 1$, then $v_1, \ldots, v_{p-1}$ is a blocking sequence for the exact k-subseparation $(X, Y \Delta \{v_p, y\})$ in $M_{B\Delta\{y, v_p\}}$.*

(iii) *If v_i has no neighbours in $X \cup Y$ in G_B, then $1 < i < p$, $v_i v_{i-1}$ is an edge, and $v_1, \ldots, v_{i-2}, v_{i+1}, \ldots, v_p$ is a blocking sequence for the exact k-subseparation (X, Y) in $M_{B\Delta\{v_{i-1}, v_i\}}$.*

Proof. For all assertions we may assume that S (the ground set of M) is equal to $X \cup Y \cup \{v_1, \ldots, v_p\}$.

Part (i). By Proposition 4.15 (part (i)), $v_1, \ldots v_{p-1}$ is a blocking sequence for $(X, Y \cup \{v_p\})$. Furthermore, $k-1 = \lambda_B(X, Y \cup \{v_p\}) \geqslant \lambda_B(X, Y' \cup \{v_p\}) \geqslant \lambda_B(X, Y') = k-1$, so $\lambda_B(X, Y' \cup \{v_p\}) = k-1$. Moreover, as $\lambda_B(X \cup \{v_{p-1}\}, Y \cup \{v_p\}) = k$ and $\lambda_B(X, Y') = \lambda_B(X \cup \{v_{p-1}\}, Y) = \lambda_B(X, Y \cup \{v_p\}) = k-1$, it follows from Proposition 4.8 that $\lambda_B(X \cup \{v_{p-1}\}, Y' \cup \{v_p\}) \geqslant k$. So, by Proposition 4.15 (part (iii)), we see that $v_1, \ldots, v_{p-1}$ is blocking sequence for $(X, Y' \cup \{v_p\})$.

Part (ii). By Proposition 4.15 (parts (i) and (ii)), $v_1, \ldots, v_{p-1}$ is a blocking sequence for the k-subseparation $(X, Y \cup \{v_p\})$ in $M_{B\Delta\{y, v_p\}}$. By Proposition 4.7, $\lambda_{B\Delta\{y, v_p\}}(X, Y\Delta\{y, v_p\}) = \lambda_B(X \cup \{y\}, Y \cup \{v_p\}) - 1$. Hence, as $k = \lambda_B(X \cup \{y\}, Y) \leqslant \lambda_B(X \cup \{y\}, Y \cup \{v_p\}) \leqslant \lambda_B(X, Y \cup \{v_p\}) + 1 = k$, we have that $\lambda_{B\Delta\{y, v_p\}}(X, Y\Delta\{y, v_p\}) = k-1$. So, by Proposition 4.15 (part (iii)), it suffices to prove that $\lambda_{B\Delta\{y, v_p\}}(X \cup \{v_{p-1}\}, Y\Delta\{v_p, y\}) > k-1$. By Proposition 4.7, $\lambda_{B\Delta\{y, v_p\}}(X \cup \{v_{p-1}\}, Y\Delta\{v_p, y\}) = \lambda_B(X \cup \{y, v_{p-1}\}, Y \cup \{v_p\}) - 1$. By submodularity, we have

$$
\begin{aligned}
&\lambda_B(X \cup \{y, v_{p-1}\}, Y \cup \{v_p\}) + \lambda_B(X \cup \{v_{p-1}\}, Y) \\
&\qquad \geqslant \lambda_B(X \cup \{v_{p-1}\}, Y \cup \{v_p\}) + \lambda_B(X \cup \{v_{p-1}, y\}, Y).
\end{aligned}
$$

However, $\lambda_B(X \cup \{v_{p-1}\}, Y) = k-1$, $\lambda_B(X \cup \{v_{p-1}\}, Y \cup \{v_p\}) = k$, and $\lambda_B(X \cup \{v_{p-1}, y\}, Y) \geqslant \lambda_B(X \cup \{y\}, Y) = k$. Therefore, $\lambda_B(X \cup \{y, v_{p-1}\}, Y \cup \{v_p\}) \geqslant k+1$, as required.

Part (iii). Clearly, by Proposition 4.9, in G_B there exists an edge from v_1 to X and from v_p to Y. So, $1 < i < p$. As v_i has no neighbours in $X \cup Y$ and $\lambda_B(X \cup \{v_i\}, Y \cup \{v_{i+1}\}) = k > k-1 = \lambda_B(X, Y \cup \{v_{i+1}\})$, it follows from Proposition 4.9, that $v_i v_{i+1}$ is and edge of G_B. By symmetry, $v_i v_{i-1}$ is also an edge. We denote by B' the basis $B\Delta\{v_i, v_{i-1}\}$. Let $X_0 := X$, $Y_{p+1} := Y$, and, for $j = 1, \ldots, p$, we let $X_j := X \cup \{v_1, \ldots, v_j\}$ and $Y_j := Y \cup \{v_j, \ldots, v_p\}$.

We first prove that $\lambda_{B'}(X_{i-3}, Y_{i+1}) = k-1$ (in case $i > 2$). Indeed, by the minimality of the blocking sequence, $\lambda_B(X_{i-3}, Y_{i-1}) = k-1$. Hence, by Proposition 4.7,

$$\begin{aligned}\lambda_{B'}(X_{i-3}, Y_{i+1}) &= \lambda_B(X_{i-3} \cup \{v_{i-1}, v_i\}, Y_{i-1}) - 2 \\ &\leqslant \lambda_B(X_{i-3}, Y_{i-1}) = k-1.\end{aligned}$$

So our claim follows.

Next we prove that v_i has no neighbours in G_B other than v_{i-1} and v_{i+1}. By submodularity, we have

$$\lambda_B(X \cup \{v_i\}, Y_{i+2}) + \lambda_B(\{v_i\}, Y) \geqslant \lambda_B(X \cup \{v_i\}, Y) + \lambda_B(\{v_i\}, Y_{i+2}).$$

However, $\lambda_B(\{v_i\}, Y) = 0$, $\lambda_B(X \cup \{v_i\}, Y) = k-1$, and, by Proposition 4.15, part (i), $\lambda_B(X \cup \{v_i\}, Y_{i+2}) = k-1$, so $\lambda_B(\{v_i\}, Y_{i+2}) = 0$. Therefore, v_i has no neighbours in Y_{i+2} in G_B. By symmetry, v_i has no neighbours in X_{i-2}. Hence, as $S = X \cup Y \cup \{v_1, ..., v_p\}$, node v_i is adjacent to only v_{i-1} and v_{i+1}, as claimed.

As v_i is pendant to v_{i-1} in $M_B - v_{i+1}$, it follows from Proposition 4.7 and 4.13, that $M_B - \{v_{i-1}, v_i, v_{i+1}\}$ is identical to $M_{B'} - \{v_{i-1}, v_i, v_{i+1}\}$. Hence (X, Y) and (X_{i-2}, Y) are exact k-subseparations of $M_{B'}$. Another consequence of $M_{B'} - \{v_{i-1}, v_i, v_{i+1}\}$ being identical with $M_B - \{v_{i-1}, v_i, v_{i+1}\}$ and of $\lambda - B'(X_{i-3}, Y_{i-1}) = k-1$ (in case $i > 2$), is that (X, Y) is an induced k-subseparation in $M_{B'}$ if and only if (X_{i-2}, Y) is an induced k-subseparation of $M_{B'}$, and that the blocking sequences of (X, Y) in M'_B are exactly the sequences starting with $v_1, ..., v_{i-2}$, followed by a blocking sequence of (X_{i-2}, Y) in $M_{B'}$. Hence, to prove that $v_1, ..., v_{i-2}$, $v_{i+1}, ..., v_p$ is a blocking sequence for (X, Y) in $M_{B'}$, it remains to prove that $v_{i+1}, ..., v_p$ is a blocking sequence for (X_{i-2}, Y) in $M_{B'}$.

By Proposition 4.15 (part (i)), $v_i, ..., v_p$ is a blocking sequence for the k-subseparation (X_{i-1}, Y) in M_B. As $k \geqslant \lambda_B(X_{i-1}, Y \cup \{v_{i-1}\}) \geqslant \lambda_B(X_{i-2}, Y \cup \{v_{i-1}\}) \geqslant k$, we have that $\lambda_B(X_{i-1}, Y \cup \{v_{i-1}\}) = k$. Hence, by part (ii), $v_{i+1}, ..., v_p$ is a blocking sequence for the k-subseparation $(X_{i-2} \cup \{v_i\}, Y)$ in $M_{B'}$. Recall that $\lambda_{B'}(X_{i-2}, Y) = k-1$. Hence, by Proposition 4.15 (part (iii)), it suffices to prove that $\lambda_{B'}(X_{i-2}, Y \cup \{v_{i+1}\}) = k$.

By submodularity, we have

$$\lambda_B(X_{i-1}, Y_{i+1}) + \lambda_B(X_i, Y_i) \geqslant \lambda_B(X_i, Y_{i+1}) + \lambda_B(X_{i-1}, Y_i).$$

Hence, as $\lambda_B(X_{i-1}, Y_{i+1}) = k-1$, $\lambda_B(X_i, Y_{i+1}) = k$, and $\lambda_B(X_{i-1}, Y_i) = k$, we have that $\lambda_B(X_i, Y_i) \geqslant k+1$. Similarly, $\lambda_B(X_{i-1}, Y_{i-1}) \geqslant k+1$.

Again, by submodularity, we have

$$\lambda_B(X_i, Y_{i-1}) + \lambda_B(X_{i-1}, Y_i) \geqslant \lambda_B(X_i, Y_i) + \lambda_B(X_{i-1}, Y_{i-1}) \geqslant 2k+2.$$

Then, since $\lambda_B(X_{i-1}, Y_i)=k$, we have $\lambda_B(X_i, Y_{i-1})\geqslant k+2$. Therefore, by Proposition 4.7, $\lambda_{B'}(X_{i-2}, Y_{i+1})\geqslant k$. So, as $\lambda_{B'}(X_{i-2}, Y)=\lambda_{B'}(X_{i-2}, Y_{i+2})=k-1$, it follows from Proposition 4.8 that $\lambda_{B'}(X_{i-2}, Y\cup\{v_{i+1}\})=k$, as required. Hence, part (iii) follows. ∎

While the previous results are stated for arbitrary values of k, we are interested only in 2-subseparations. We now introduce a result that is particular to this special case.

Two partitions (X_1, X_2) and (Y_1, Y_2) of a common set are said to *cross* if $X_i\cap Y_j$ is nonempty for each $i, j\in\{1, 2\}$. A 2-subseparation (X_1, X_2) of M_B is *crossed* (otherwise *uncrossed*) if there exists a 2-separation (Y_1, Y_2) of $M_B[X_1\cup X_2]$ such that the partitions (Y_1, Y_2) and (X_1, X_2) cross.

PROPOSITION 4.17. *Let $v_1, ..., v_p$ be a blocking sequence for an uncrossed 2-subseparation (X_1, X_2) of M_B, and let (Y_1, Y_2) be a 2-separation of $M_B[X_1\cup X_2\cup\{v_1, ..., v_p\}]$. Then, for some $i, j\in\{1, 2\}$, $X_i\cup\{v_1, ..., v_p\}\subseteq Y_j$.*

The proof requires the following proposition.

PROPOSITION 4.18. *Let (X_1, X_2) be an uncrossed 2-subseparation of $M_B[X_1\cup X_2]$, let $v\in S\backslash(X_1\cup X_2)$ be such that $\lambda_B(X_1\cup\{v\}, X_2)=2$, and let (Y_1, Y_2) be a 2-separation of $M_B[X_1\cup X_2\cup\{v\}]$ such that $X_2\subseteq Y_2$. Then $v\in Y_2$.*

Proof. Suppose that $v\in Y_1$. By submodularity we have

$$\lambda_B(X_1, X_2)+\lambda_B(Y_1, Y_2)\geqslant\lambda_B(X_1\cap Y_1, Y_2)+\lambda_B(X_1\cup\{v\}, X_2).$$

Hence, $\lambda_B(X_1\cap Y_1, Y_2)=0$. Note that $X_1\cap Y_1=Y_1\backslash\{v\}$, and that Y_2 strictly contains X_2. Fix any $a\in X_2$. Then $((Y_1\backslash\{v\})\cup\{a\}, Y_2\backslash\{a\})$ crosses (X_1, X_2). However, $((Y_1\backslash\{v\})\cup\{a\}, Y_2\backslash\{a\})$ is a 2-separation of $M_B[X_1\cup X_2]$, as $\lambda_B((Y_1\backslash\{v\})\cup\{a\}, Y_2\backslash\{a\})\leqslant\lambda_B(Y_1\backslash\{v\}, Y_2)+1=1$. This contradiction completes the proof. ∎

Proof of Proposition 4.17. Note that $\lambda_B(Y_1\cap(X_1\cup X_2), Y_2\cap(X_1\cup X_2))\leqslant 1$. Therefore, $(Y_1\cap(X_1\cup X_2), Y_2\cap(X_1\cup X_2))$, and (X_1, X_2) do not cross. Hence, there exists $i, j\in\{1, 2\}$ such that $X_i\subseteq Y_j$. By swapping X_1 and X_2 and swapping Y_1 and Y_2, if necessary, we assume that $X_2\subseteq Y_2$.

We prove the result by induction on p. The case that $p=1$ is proven in Proposition 4.18. We assume that $p>1$ and that the result holds for all smaller cases. By Proposition 4.18, it follows that $(X_1, X_2\cup\{v_p\})$ is uncrossed. By Proposition 4.15 (part (i)), $v_1, ..., v_{p-1}$ is a blocking sequence for $(X_1, X_2\cup\{v_p\})$.

First suppose $v_p \in Y_2$. Then $X_2 \cup \{v_p\} \subseteq Y_2$, so, by induction and as $(Y_1, Y_2) \neq (X_1 \cup \{v_1, ..., v_{p-1}\}, \{v_p\} \cup X_2)$, it follows that $v_1, ..., v_{p-1} \in Y_2$. Hence, the conclusion of Proposition 4.17 follows when $v_p \in Y_2$.

Hence we suppose $v_p \in Y_1$. Then, since $(X_1, X_2 \cup \{v_p\})$ is uncrossed, either $X_1 \subseteq Y_1$ or $X_1 \subseteq Y_2$. Clearly $X_1 \not\subseteq Y_1$, since $\lambda_B(X_1 \cup \{v_p\}, X_2) = 2$. Hence, $X_1 \subseteq Y_2$. Now $v_{p-1}, ..., v_1$ is a blocking sequence for the 2-sub-separation $(X_2 \cup \{v_p\}, X_1)$, and $X_1 \subseteq Y_2$. So, by induction, $v_1, ..., v_{p-1} \in Y_2$. However, this implies that $Y_1 = \{v_p\}$, contradicting that (Y_1, Y_2) is a 2-separation. ∎

The following corollary is an easy consequence of Proposition 4.17.

COROLLARY 4.19. *If (X_1, X_2) is the unique 2-separation in $M_B[X_1 \cup X_2]$, and $v_1, ..., v_p$ is a blocking sequence for (X_1, X_2), then $M_B[X_1 \cup X_2 \cup \{v_1, ..., v_p\}]$ is 3-connected.*

5. REDUCTION TO A FINITE LIST OF EXCLUDED MINORS

Theorem 5.1 below constitutes the main part of the proof of Theorem 1.1. In particular, it says that all excluded minors have at most eight elements. The final case analysis, establishing the excluded minors explicitly, is deferred to Section 6.

THEOREM 5.1. *Minor-minimal non-GF(4)-representable matroids have rank and corank at most 4.*

Proof. Suppose the theorem fails. Let M be a minor-minimal non-$GF(4)$-representable matroid with rank or corank at least 5. Clearly, M is 3-connected and nonbinary. Hence, by Theorem 3.1, there exists $M' \in \{M, M^*\}$ and elements u, v such that $M' \backslash u$, $M' \backslash v$ are stable, and $M' \backslash u, v$ is connected, stable, and contains a 3-connected nonbinary minor M'' of size at least $|S| - 4$. Our first assumption is that

(1) *M', M'', u, v are chosen os that M'' is as large as possible.*

By duality, we may also assume that $M = M'$. As all proper minors of M are $GF(4)$-representable, it follows from Lemma 2.2, that there exists a unique $GF(4)$-representable matroid N on S such that $N \backslash u = M \backslash u$ and $N \backslash v = M \backslash v$. As M is not $GF(4)$-representable, M and N are not isomorphic. Let B be a basis of M disjoint from $\{u, v\}$. Since, $N_B \neq M_B$, there exists a set that is feasible in exactly one of M_B and N_B; such a set is said to *distinguish* M_B from N_B. As $M \backslash u, v$ is nonbinary, $M_B - u - v$ has a twirl. Our first goal will be to establish that we may choose B such that both this twirl and distinguishing set can be chosen small (of size equal to four) and close to each other in the fundamental graph G_B. (Note that M_B and N_B have the same fundamental graph.)

(2) *M has a basis B and elements a and b such that B avoids $\{u, v\}$ and $\{u, v, a, b\}$ distinguishes M_B from N_B.*

Since M is 3-connected, there exists a basis disjoint from $\{u, v\}$. Let B' be a basis of exactly one of N and M, and choose a basis B of $M\backslash u, v$ minimizing $|B\Delta B'|$. Note that $u, v \in B'$ and that B is a basis of N. If $|B\Delta B'| = 4$, then (2) follows (with a and b the two elements in $B\backslash B'$). If $|B\Delta B'| > 4$, take $x \in (B'\backslash B)\backslash\{u, v\}$. By the basis exchange axiom, there exists a $y \in B\backslash B'$ such that $B\Delta\{x, y\}$ is a basis of at least one of N and M. However $u, v \notin B\Delta\{x, y\}$, so $B\Delta\{x, y\}$ is a basis in both N and M. In particular, $B\Delta\{x, y\}$ is a basis of $M\backslash u, v$, and $|(B\Delta\{x, y\})\,\Delta B'| < |B\Delta B'|$, contradicting our choice of B. This proves (2).

Henceforth we assume that B, a, b are as in (2). To switch between the various choices for B, a, b, we pivot extensively, though we are cautious and make sure that u and v stay out of the basis B and that there is still a distinguishing set of size 4. To be precise, for an edge xy of $G_B - u - v$, the pivot on xy is *allowable* if either

(i) $x \in \{a, b\}$,

(ii) $y \in \{a, b\}$, or

(iii) $\{u, v, a, b, x, y\}$ distinguishes M_B from N_B.

Note that $M_{B\Delta\{x, y\}}$ and $N_{B\Delta\{x, y\}}$ are indeed distinguished by a set of size 4; namely, by $\{a, b, u, v\}$ if the pivot is allowable of type (iii) and by $\{a, b, u, v\}\ \Delta\{x, y\}$ if the pivot is allowable of type (i) or (ii). While allowable pivots of types (i) and (ii) are clear from the fundamental graph, this is not the case for allowable pivots of type (iii). However, from Proposition 4.13, we have the following sufficient conditions:

(i) *If xy is an edge of $G_B[S\backslash\{u, v, a, b\}]$, and neither x nor y is adjacent to either a or b, then the pivot on xy is allowable.*

(ii) *If xy is an edge of $G_B[S\backslash\{u, v, a, b\}]$ and neither x nor y is adjacent to either u or v, then the pivot on xy is allowable.*

Given elements x and y, we denote by $d_B(x, y)$, or just $d(x, y)$, the distance between x and y in $G_B - u - v$. If U is a set of vertices in $G_B - u - v$, then $d(x, U)$ denotes the length of a shortest path from x to a vertex in U.

We now refine our choice of B, a, and b. We choose B, a, b, and C in $S\backslash\{u, v\}$ such that

(3) *$(|C|, d(a, C), d(b, C))$ is lexicographically minimal subject to the following conditions: B is a basis of $M\backslash\{u, v\}$, the set $\{u, v, a, b\}$ distinguishes M_B from N_B, and $M_B[C]$ is a twirl.*

This choice has the following three consequences.

(4) *Let* $x \in (S\backslash\{u, v\})\backslash C$. *Then* $|\text{nigh}(x) \cap C| \leqslant 2$. *Furthermore,*

(i) *If* $a \notin C$, *then* $|\text{nigh}(a) \cap C| \leqslant 1$,

(ii) *If* $b \notin C$ *and* $|\text{nigh}(b) \cap C| = 2$, *then* $a \in C$.

Suppose $x \in (S\backslash\{u, v\})\backslash C$ and $|\text{nigh}(x) \cap C| \geqslant 2$. We are required to prove that $|\text{nigh}(x) \cap C| = 2$, $x \neq a$, and that $a \in C$ if $x = b$. By Lemma 4.4, there exists a twirl $M_B[C']$ such that $x \in C' \subset C \cup \{x\}$. By (3), we must have $|C'| = |C|$, which is only possible if $|\text{nigh}_B(x) \cap C| = 2$. Again by (3), $d(a, C) \leqslant d(a, C')$. Hence, as $d(x, C) = 1 > 0 = d(x, C')$, it follows that $x \neq a$. Finally, if $x = b$, then, by (3), $d(a, C) \leqslant d(b, C') = d(x, C') = 0$, so $a \in C$. So (4) follows.

(5) $|C| = 4$.

Suppose that $|C| > 4$. No edge of $G_B[C]$ is an allowable pivot, since otherwise, by Proposition 4.5, pivoting on such an edge would yield a shorter twirl, contradicting (3). Therefore, neither a nor b is contained in C. Hence, by (4), both a and b have at most one neighbour in C. However, since $|C| \geqslant 6$, there exists an edge xy of $G_B[C]$ such that neither x nor y is adjacent to either a or b. So xy is allowable. This contradiction proves (5).

(6) $d(a, C) \leqslant 1$.

Suppose that $d(a, C) > 1$, and let $x_1, ..., x_k$ be the internal vertices of a shortest path from a to C in $G_B - u - v$. If x_k has at least two neighbours in C, then, by Lemma 4.4, there is a twirl $M_B[C']$ of size 4 that contains x_k. As $d(a, C') < d(a, C)$, this contradicts (3). So x_k has exactly one neighbour, say x, in C. Let y a neighbour of x in $G_B[C]$ and let z be the neighbour of y in $G_B[C]$ different from x. Then, xy is an allowable pivot, since neither x nor y is adjacent to either a or b. If we pivot on xy, then C remains a twirl, $x_1, ..., x_k$ remain the internal vertices of a path from a to C, but x_k becomes adjacent to y and z. So then we are back in the earlier excluded case that x_k has at least two neighbours in C. This proves (6).

(2), (5), and (6) establish our first goal: the existence of a basis B with a small distinguishing set $\{u, v, a, b\}$ and a 4-element twirl $M_B[C]$ in $M_B - u - v$, that is close to $\{u, v, a, b\}$ in G_B. Figure 2 lists the possible subgraphs of G_B induced by $\{u, v, a, b\}$ and C. (That $G_B[\{u, v, a, b\}]$ is a circuit, follows from Propositions 4.1 and 4.2.)

One of the main tools from now on is Lemma 2.3. In terms of twisted matroids it reads:

(7) *Let* $X \subseteq S$ *such that* $M_B[X] - u$ *and* $M_B[X] - v$ *are stable,* $M_B[X] - u - v$ *is connected, stable, and nonbinary, and there exists* $Y \subseteq X$

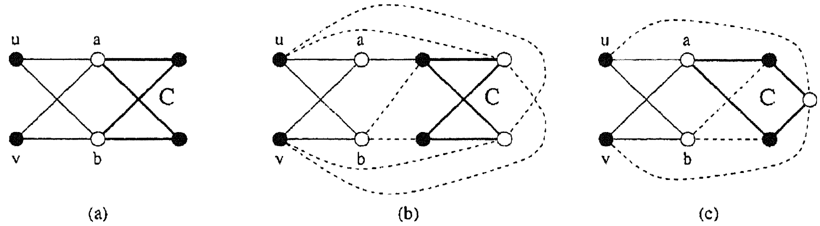

FIG. 2. The subgraph of G_B induced by $\{u, v, a, b\}$ and by C (indicated by bold edges). Dashed edges might or might not exist.

distinguishing M_B and N_B. Then $M_B[X]$ is not $GF(4)$-representable. Consequently $M_B = M_B[X]$.

By applying (7), we make short work of the first case in Fig. 2a:

(8) $b \notin C$.

Suppose that $b \in C$. Then by (3) also $a \in C$. We define $X := \{u, v\} \cup C$. Then $M_B[X] - u - v$ is 3-connected and nonbinary. Hence, $M_B[X] - u$, $M_B[X] - v$, and $M_B - u - v$ are all stable. So, by (7), $M_B = M_B[X]$. This contradicts the fact that M has rank or corank at least five. So (8) follows.

Before we proceed with the other cases we derive a simple fact that we will use several times.

(9) *If $x \in S \backslash \{u, v, a, b\}$ such that $a, b \in \text{nigh}_B(x)$, then $M_B[\{u, a, b, x\}]$ and $M_B[\{v, a, b, x\}]$ are both twirls.*

By Lemma 4.4, if any of $M_B[\{u, a, b, x\}]$, $M_B[\{v, a, b, x\}]$, and $M_B[\{a, b, u, v\}]$ is a twirl, then at least two are. Similarly, if any of $M_B[\{u, a, b, x\}] = N_B[\{u, a, b, x\}]$, $M_B[\{v, a, b, x\}] = N_B[\{v, a, b, x\}]$, and $N_B[\{a, b, u, v\}]$ is a twirl, then at least two are. However $\{a, b, u, v\}$ distinguishes M_B and N_B, so exactly one of $M_B[\{a, b, u, v\}]$ and $N_B[\{a, b, u, v\}]$ is a twirl. Thus $M_B[\{u, a, b, x\}]$ and $M_B[\{v, a, b, x\}]$ are both twirls, which proves (9).

Next we rule out the possibility in Fig. 2b.

(10) $a \in C$.

Suppose that $a \notin C$. Let the elements of C be sequentially labeled 1, 2, 3, 4, where 1 is a neighbour of a.

(10.1) *3 is adjacent to neither a nor b in G_B.*

By (4), a is not adjacent to 3, and b has at most one neighbour in C. Suppose that b is adjacent to 3, and hence 3 is the only neighbour of b in C. Let $X := \{u, v, a, b, 1, 2, 3, 4\}$. Then, $M_B[X] - u - v$ is connected, stable, and nonbinary. Furthermore, by Proposition 4.11, $M_B[X] - u - 2$ is 3-connected. Therefore, $M_B[X] - u$ is stable. By symmetry, $M_B[X] - v$ is stable as well. So, by (7), $M_B = M_B[X]$, which contradicts the fact that M has rank or corank at least 5. This proves (10.1).

(10.2) *$(\{a, b, 1\}, \{2, 3, 4\})$ is an induced 2-subseparation of $M_B - u - v$.*

By (10.1), $(\{a, b, 1\}, \{2, 3, 4\})$ is a 2-subseparation of M_B. Suppose that $(\{a, b, 1\}, \{2, 3, 4\})$ is not induced and let $v_1, ..., v_p$ be a blocking sequence. We prove (10.2) by induction on p. We consider separately the cases given by the colour class of v_p in G_B; these are depicted in Fig. 3.

We first consider the case where v_p is in the same colour class of G_B as 3. As v_p is the last vertex in the blocking sequence, $(\{a, b, 1, v_p\}, \{2, 3, 4\})$ is not a 2-separation. Consequently, by Proposition 4.12, v_p is adjacent to either 2 or 4 in G_B. By pivoting on 23 or 34, if necessary, we may assume that v_p is adjacent to both 2 and 4 (cf. Proposition 4.15 (part (ii))). Since $(\{a, b, 1, v_p\}, \{2, 3, 4\})$ is a split of G_B but not a 2-subseparation, Proposition 4.12 implies that $M_B[\{1, 2, v_p, 4\}]$ is a twirl. Consider replacing 3 by v_p (so C by $\{1, 2, v_p, 4\}$). If $p = 1$, then v_p is adjacent to a or b, which contradicts (10.1). If $p > 1$, then, by Proposition 4.16 (part (i)), $v_1, ..., v_{p-1}$ is a blocking sequence for the 2-subseparation $(\{a, b, 1\}, \{2, v_p, 4\})$, and (10.2) follows inductively.

Now we suppose that v_p is in the same colour class of G_B as 2. As v_p is the last vertex in the blocking sequence, $(\{a, b, 1, v_p\}, \{2, 3, 4\})$ is not a 2-separation. Consequently v_p is adjacent to 3 in G_B. By pivoting on 23, if necessary, we may assume that v_p is also adjacent to 1. By Lemma 4.4, either $M_B[\{v_p, 1, 2, 3\}]$ or $M_B[\{v_p, 1, 3, 4\}]$ is a twirl. We suppose that $M_B[\{v_p, 1, 3, 4\}]$ is a twirl. Consider replacing 2 by v_p. Since $(\{a, b, 1\}, \{v_p, 2, 3, 4\})$ is a 2-subseparation, we must have $p > 1$. Then, by Proposition 4.16 (part (i)), $v_1, ..., v_{p-1}$ is a blocking sequence for the 2-subseparation $(\{a, b, 1\}, \{v_p, 3, 4\})$, and (10.2) follows inductively.

(10.3) *$M_B - \{a, b, u, v\}$ is 3-connected and a and b are pendant with 1 in $G_B - u - v$.*

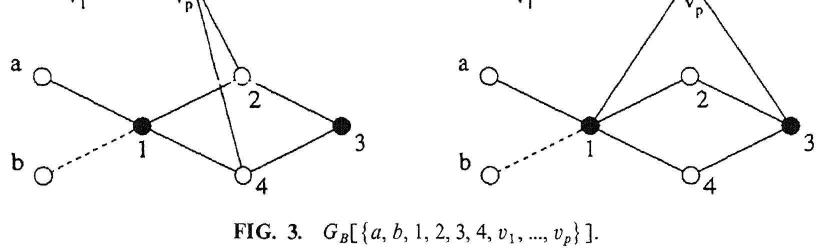

FIG. 3. $G_B[\{a, b, 1, 2, 3, 4, v_1, ..., v_p\}]$.

By (10.2), there exists a 2-separation (X, Y) of $M_B - u - v$ such that a, b, $1 \in X$ and $2, 3, 4 \in Y$. However, $M_B - u - v$ is stable and has a 3-connected nonbinary minor of size at least $|S| - 4$, so $X = \{a, b, 1\}$. Then, since (X, Y) is a split in $G_B - u - v$, neither a nor b has neighbours in Y. However, $M_B - u - v$ is connected, so a and b are pendant with 1 in $G_B - u - v$. Moreover, it follows that $M_B - \{a, b, u, v\}$ is 3-connected. So (10.3) follows.

Note that both 2 and 4 are adjacent to either u or v. Indeed, if 2 were not adjacent to either u or v, we could pivot on 12, making 3 adjacent to a and b and thus contradicting (10.1).

(10.4) *If $b' \in \{a, b\}$ and $v' \in \{u, v\}$, then $M_B - b' - v'$ is 3-connected.*

By symmetry we may assume that $b' = b$ and $v' = v$. As $M_B - \{a, b, u, v\}$ is 3-connected, $(\{a, 1\}, S \backslash \{a, b, u, v, 1\})$ is the unique 2-separation in $M_B - \{b, u, v\}$.

Now suppose that $(\{u, a, 1\}, S \backslash \{b, v, a, u, 1\})$ is a 2-separation in $M_B - b - v$. Since the only neighbours of b in G_B are u, v, 1, it follows that $(\{u, a, b, 1\}, S \backslash \{b, v, a, u, 1\})$ is a 2-separation in $M_B - v$. However, $M_B[\{1, 2, 3, 4\}]$ is a twirl and, by (9), $M_B[\{u, a, b, 1\}]$ is also a twirl. This contradicts the fact that $M_B - v$ is stable. Consequently $(\{u, a, 1\}, S \backslash \{b, v, a, u, 1\})$ is not a 2-separation in $M_B - b - v$. Moreover, as au is an edge of G_B, $(\{a, 1\}, S \backslash \{b, v, a, 1\})$ is not a 2-separation in $M_B - b - v$. So we may conclude that u is a blocking sequence for the 2-subseparation $(\{a, 1\}, S \backslash \{a, b, u, v, 1\})$ in M_B. Then, by Corollary 4.19, $M_B - b - v$ is 3-connected. This proves (10.4).

As $M_B - \{u, v, a, b\}$ is 3-connected and nonbinary, $M_B - a - b$ is stable and nonbinary. Moreover, by (10.4), $M_B - a$ and $M_B - b$ are both stable. Also by (10.4), $M_B - a - b - u$ and $M_B - a - b - v$ are connected, so $M_B - a - b$ is connected. Hence, a, b is a contraction pair in M.

Since a is pendant to 1 in $M_B - u - v$, $M_B - a - u - v$ is connected and stable. Moreover $M_B - a - u - v$ is clearly nonbinary. Furthermore, by (10.4), $M_B - a - v$ and $M_B - a - u$ are both stable. Therefore, by (7), every set that distinguishes M_B and N_B must contain a. Hence, $M_B - a = N_B - a$, and thus, by symmetry, $M_B - b = N_B - b$.

Recall that 2 and 4 are both adjacent to either u or v. So after replacing M by M^*, $\{u, v\}$ by $\{a, b\}$; $\{a, b\}$ by $\{u, v\}$ and 2, 3, 4, 1 by 1, 2, 3, 4, we contradict (10.1). This completes the proof of (10).

It remains to consider the possibility in Fig. 2c. We label the elements of C so that, $C = \{a, 1, 2, 3\}$, where 1, 2 are the vertices adjacent to a. Let $x_0, \ldots, x_{k+1}$ be the vertices of a shortest path connecting b to C in $G_B - u - v$ with $x_0 = b$ and $x_{k+1} \in C$. Moreover, we let $A = (\alpha_{ij})$ be a $GF(4)$-representation of N_B, and we assign to each edge ij of G_B the weight α_{ij}.

(11) *$d(b, C) = k + 1$ is odd.*

Suppose not. Then x_k is in the same colour class as 1 and 2. First assume that x_k is adjacent to a. By pivoting on $a1$, if necessary, we may assume that x_k is also adjacent to 3. By Lemma 4.4, one of $M_B[\{a, 1, 3, x_k\}]$ and $M_B[\{a, 2, 3, x_k\}]$ is a twirl, contradicting our choice of C. Thus x_k is not adjacent to a, and hence is adjacent to 3. Let $X := \{u, v, a, 1, 2, 3, x_0, ..., x_k\}$. By Proposition 4.11, $M_B[X] - u - 1$ and $M_B[X] - v - 1$ are both 3-connected. Hence $M_B[X] - u$ and $M_B[X] - v$ are both stable. Furthermore, $M_B[X] - u - v$ is clearly nonbinary, connected, and stable, so, by (7), $M_B = M_B[X]$. By scaling lines of A, we may assume that

$$\alpha_{av} = \alpha_{bv} = \alpha_{a1} = \alpha_{a2} = \alpha_{x_0 x_1} = \cdots = \alpha_{x_{k-1} x_k} = \alpha_{3 x_k} = 1.$$

Since $M_B[C]$ is a twirl, $\alpha_{31} \neq \alpha_{32}$. Then, by interchanging the labels 1 and 2, if necessary, we may assume that $\alpha_{32} \neq 1$. Hence, $M_B - u - 1$ is nonbinary. As argued above, $M_B - u - 1$ is also 3-connected. However, since $M_B - u - v$ is not 3-connected, this contradicts (1). Hence (11) follows.

Since $d(b, C)$ is odd, x_k is in the same colour class as b, and hence x_k is adjacent to either 1 or 2. By pivoting on $a1$ or a_2, if necessary, we assume that x_k is adjacent to both 1 and 2. Note that $k \in \{0, 2\}$, since otherwise we could reduce $d(b, C)$ by pivoting on $x_2 x_3$. Also note that $M_B[\{a, 1, 2, x_k\}]$ is not a twirl, since otherwise we could replace 3 by x_k, contradicting (3). $G_B[\{u, v, 1, 2, 3, a, x_0, ..., x_k\}]$ is depicted in Fig. 4.

(12) *For $w \in \{u, v\}$, $M_B[\{w, a, 1, 2, 3, x_0, ..., x_k\}]$ is 3-connected if and only if w is adjacent to 3. Furthermore, if w is not adjacent to 3, then $(\{w, a, x_0, ..., x_k\}, \{1, 2, 3\})$ is the only 2-separation of $M_B[\{w, a, 1, 2, 3, x_0, ..., x_k\}]$.*

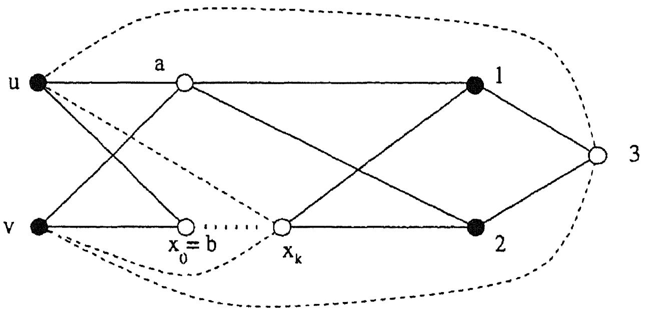

FIG. 4. $G_B[\{u, v, 1, 2, 3, a, x_0, ..., x_k\}]$. Dashed edges might or might not exist; the dotted $x_0 x_k$-path denotes $x_0, x_1, ..., x_k$.

Let $X := \{w, a, 1, 2, 3, x_0, ..., x_k\}$. By (9) and Proposition 4.11, $M_B[X] - 2 - 3$ is 3-connected. Therefore, $(\{w, a, x_0, ..., x_k\}, \{1, 2\})$ is the unique 2-separation in $M_B[X] - 3$. Since $M_B[\{a, 1, 2, 3\}]$ is a twirl, $(\{3, w, a, x_0, ..., x_k\}, \{1, 2\})$ is not a 2-subseparation. Furthermore, $(\{w, a, x_0, ..., x_k\}, \{1, 2, 3\})$ is a 2-subseparation if an only if 3 is not adjacent to w. Therefore if 3 is adjacent to w, then 3 is a blocking sequence for $(\{w, a, x_0, ..., x_k\}, \{1, 2\})$, and thus, by Corollary 4.19, $M_B[X]$ is 3-connected. Otherwise, when 3 is not adjacent to w, $(\{w, a, x_0, ..., x_k\}, \{1, 2, 3\})$ is a 2-separation in $M_B[X]$. Furthermore, it is straightforward to deduce, from Proposition 4.18, that this is the only 2-separation of $M_B[X]$. This proves (12).

(13) *We may assume that v is adjacent to* 3.

Suppose v is not adjacent to 3. We may also suppose that u is not adjacent to 3, since otherwise we would swap u and v. Since 3 is adjacent to neither u nor v, for any neighbour x of 3 in G_B, $3x$ is an allowable pivot. By (12), $(\{v, a, x_0, ..., x_k\}, \{1, 2, 3\})$ is a 2-subseparation. If $k = 0$, then $(\{v, a, x_0, ..., x_k\}, \{1, 2, 3\})$ is not induced in $M_B - u$, since $M_B - u$ is stable and $M_B[\{v, a, b, 1\}]$ and $M_B[\{a, 1, 2, 3\}]$ are both twirls. If $k = 2$, then $(\{v, a, x_0, ..., x_k\}, \{1, 2, 3\})$ is not induced in $M_B - u$, since $M_B - u - v$ contains a 3-connected nonbinary minor of size at least $|S| - 4$. In either case, there exists a blocking sequence $v_1, ..., v_p$ for $(\{v, a, x_0, ..., x_k\}, \{1, 2, 3\})$ in $M_B - u$. We prove (13) by induction on p.

We first consider the case that v_p is in the same colour class as 3. As $(\{v, a, x_0, ..., x_k, v_p\}, \{1, 2, 3\})$ is not a 2-subseparation, v_p is adjacent to 1 or 2. By pivoting on 13 or 23, if necessary, we may assume that v_p is adjacent to both 1 and 2. By Proposition 4.13, since $(\{v, a, x_0, ..., x_k, v_p\}, \{1, 2, 3\})$ is not a 2-subseparation, $M_B[\{v_p, a, 1, 2\}]$ is a twirl. Consider replacing 3 by v_p. If $p = 1$, then v_p is adjacent to either v or x_1. If v_p is adjacent to v, then we are done. If v_p is adjacent to x_1, then $d(b, \{a, 1, 2, v_p\}) = 2$, contradicting (3). Thus $p > 1$. Then, by Proposition 4.16 (part (i)), $v_1, ..., v_{p-1}$ is a blocking sequence for the 2-subseparation $(\{v, a, x_0, ..., x_k\}, \{1, 2, v_p\})$. So (13) follows inductively.

We now consider the case that v_p is in the same colour class as 1, and hence $v_p 3$ is an edge of G_B. By pivoting on 23, if necessary, we may assume that v_p is adjacent to a as well. Therefore at least one of $M_B[\{a, 1, 3, v_p\}]$ and $M_B[\{a, 2, 3, v_p\}]$ is a twirl. By swapping 1 and 2, if necessary, we may assume that $M_B[\{a, 1, 3, v_p\}]$ is a twirl. By pivoting on 13, if necessary, we may assume that v_p is adjacent to x_k. Consider replacing 2 by v_p. If $p > 1$, then, by Proposition 4.16 (part (i)), $v_1, ..., v_{p-1}$ is a blocking sequence for the 2-subseparation $(\{v, a, x_0, ..., x_k\}, \{1, 3, v_p\})$, so (13) follows inductively. Thus we may assume that $p = 1$. Recall that $v_1 = v_p$ is adjacent to a and x_k. Since $(\{v, a, x_0, ..., x_k\}, \{1, 2, 3, v_1\})$ is not a 2-subseparation,

$M_B[\{a, 1, x_k, v_1\}]$ is a twirl. However $d(b, \{a, x_k, 1, v_1\}) < d(b, C)$ which contradicts (3). This proves (13).

(14) *u is not adjacent to* 3.

Suppose u and 3 are adjacent. Let $X := \{u, v, a, x_0, ..., x_k, 1, 2, 3\}$. By (12), $M_B[X] - u$ and $M_B[X] - v$ are both 3-connected, and, hence, stable. Also $M_B[X] - u - v$ is connected, nonbinary, and stable. So, by (7), $M_B = M_B[X]$. Hence, $k \neq 0$, since M has rank or corank at least 5. Thus $k = 2$, contradicting that $M - u - v$ has a 3-connected nonbinary minor of size at least $|S| - 4$. This proves (14).

So, by (12), $(\{u, a, x_0, ..., x_k\}, \{1, 2, 3\})$ is a 2-subseparation. If $k = 0$, then $(\{u, a, x_0, ..., x_k\}, \{1, 2, 3\})$ is not induced in $M_B - v$, since $M_B - v$ is stable and $M_B[\{u, a, b, 1\}]$ and $M_B[\{a, 1, 2, 3\}]$ are both twirls. If $k = 2$, then $(\{u, a, x_0, ..., x_k\}, \{1, 2, 3\})$ is not induced in $M_B - v$, since $M_B - u - v$ contains a 3-connected nonbinary minor of size at least $|S| - 4$. In either case, there exists a blocking sequence $v_1, ..., v_p$ for $(\{u, a, x_0, ..., x_k\}, \{1, 2, 3\})$ in $M_B - v$. Assume that, subject to everything deduced so far, B, a, b, C, $x_1, ..., x_k$ and $v_1, ..., v_p$ have been chosen so that p is as small as possible.

(15) $p \neq 1$.

Suppose that $p = 1$. Let $X := \{u, v, a, x_0, ..., x_k, 1, 2, 3, v_1\}$. By (13), $M_B[X] - u - v_1$ is 3-connected, so $M_B[X] - u$ is stable. Since $(\{u, a, x_0, ..., x_k\}, \{1, 2, 3\})$ is the only 2-separation in $M_B[X] - v - v_1$, and v_1 is a blocking sequence, it follows from Corollary 4.19 that $M_B[X] - v$ is 3-connected. Furthermore, it is easy to check that $M_B[X] - u - v$ is stable, connected, and nonbinary. Hence, by (7), $M_B = M_B[X]$. (Those readers whose primary interest is seeing that the list of excluded minors is finite may choose to skip the rest of the proof of (15).)

We begin by considering the case that $k = 0$. Since M_B has rank or corank at least 5, v_1 is in the same colour class of G_B as 1. Since v_1 is a blocking sequence for the 2-subseparation $(\{u, a, b\}, \{1, 2, 3\})$, v_1 is adjacent to 3, and v_1 is adjacent to either a or b. Furthermore, if v_1 is adjacent to both a and b, then, by Proposition 4.12, $M_B[\{a, b, v_1, 1\}]$ is a twirl, which contradicts (3). Therefore v_1 is adjacent to exactly one of a and b. By relabeling, if necessary, we assume that v_1 is adjacent to a.

Note that $M_B - u - v$ is 3-connected. For this case, we assume that u and v have been chosen such that $M_B - u - v$ is 3-connected, nonbinary, and, if possible, $M \backslash u, v$ contains a $U_{2,5}$- or $U_{3,5}$-minor. (Note that such a choice of u, v implies that $M_B - u$, $M_B - v$, and $M_B - u - v$ are all stable.) We scale lines of A so that all edges of G_B that are incident with either a or 1 have weight one. Note that $M_B[\{a, b, 1, 2\}]$ is not a twirl, so the edge

$2b$ also has weight one. Let $x := \alpha_{32}$. Since $M_B[\{a, 1, 2, 3\}]$ is a twirl, $x \notin \{0, 1\}$.

Choose $w \in \{1, 2\}$ and let w' be the remaining element in $\{1, 2\} \backslash \{w\}$. Suppose that $M_B[\{a, w, 3, v_1\}]$ and $M_B[\{a, w, 3, v\}]$ are both twirls. In this case it is easily checked that $M_B - w' - u - v$ is stable, connected, and nonbinary, and that $M_B - w' - u$ and $M_B - w' - v$ are both stable. Then, by (7), $M_B - w'$ is not $GF(4)$-representable, which is a contradiction. Therefore, either $M_B[\{a, w, 3, v_1\}]$ or $M_B[\{a, w, 3, v\}]$ is not a twirl. Since $M_B[\{a, 1, 2, 3\}]$ is a twirl, then, by Proposition 4.4, either $M_B[\{v_1, a, 1, 3\}]$ or $M_B[\{v_1, a, 2, 3\}]$ is a twirl. By relabeling, if necessary, we assume that $M_B[\{v_1, a, 1, 3\}]$ is a twirl. Then, $M_B[\{v, a, 1, 3\}]$ is not a twirl, and hence $\alpha_{3v} = 1$. This implies that $M_B[\{v, a, 2, 3\}]$ is a twirl, and, hence, $M_B[\{v_1, a, 2, 3\}]$ is not a twirl. Therefore $\alpha_{3v_1} = x$.

Now that we have an explicit $GF(4)$-representation of $M_B - u - v$, it is easily checked that $M \backslash \{u, v\}$ has no $U_{2,5}$- or $U_{3,5}$-minor. (Indeed, $M \backslash \{u, v\}$ is ternary.) If $M_B[\{v, b, 2, 3\}]$ is a twirl, then $M_B[\{v, a, b, 2, 3\}]$ is a twisted $U_{3,5}$ and $M_B - 1 - v_1$ is 3-connected, which contradicts our choice of u and v. Therefore, $M_B[\{v, b, 2, 3\}]$ is not a twirl, and, hence, $\alpha_{bv} = x + 1$. By (9), both $M_B[\{1, a, b, u\}]$ and $M_B[\{1, a, b, v\}]$ are twirls. If, in addition, $M_B[\{u, v, a, b\}]$ is a twirl, then $M_B[\{u, v, a, b, 1\}]$ is a twisted $U_{2,5}$ and $M_B - 2 - v_1$ is 3-connected, which contradicts our choice of u, v. Therefore, $M_B[\{u, v, a, b\}]$ is not a twirl, and, hence, $N_B[\{u, v, a, b\}]$ is a twirl. Thus $\alpha_{bu} \neq \alpha_{bv}$. Furthermore, since $M_B[\{1, a, b, u\}]$ is a twirl, $\alpha_{bu} \notin \{0, 1\}$. Hence $\alpha_{bu} = x$.

We now have an explicit $GF(4)$-representation for N_B. The graphs G_B and $G_{B \Delta \{a, 1\}}$ are depicted in Fig. 5. Then, $M_{B \Delta \{1, a\}}[\{u, b, 1, v_1, 3\}]$ is a twisted $U_{3,5}$, and $M_{B \Delta \{1, a\}} - 2 - a$ is 3-connected, which contradicts our choice of u and v. This completes the case where $k = 0$.

Now consider the case where $k = 2$. We divide this case into two further cases. We first consider the case in which v_1 is in the same colour class as 1 in G_B. Since v_1 is a blocking sequence for $(\{u, a, b, x_1, x_2\}, \{1, 2, 3\})$, v_1 is adjacent to 3 and to at least one of a, b, and x_2. However, since $d(b, C) = 3$, v_1 is not adjacent

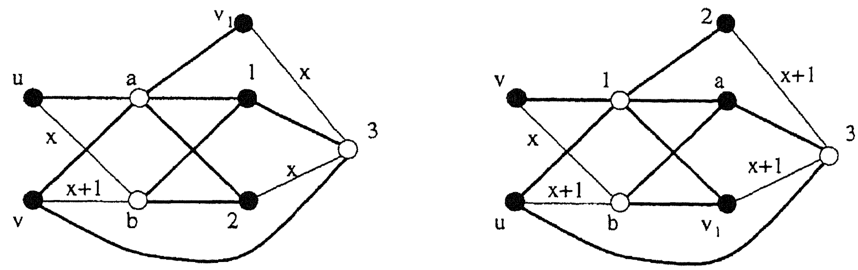

FIG. 5. G_B and $G_{B \Delta \{a, 1\}}$ (bold edges are labeled 1).

to b. Since $M_B[\{a, 1, 2, 3\}]$ and $M_B[\{x_2, 1, 2, 3\}]$ are both twirls, by Proposition 4.4, either $M_B[\{a, x_2, 1, 3, v_1\}]$ or $M_B[\{a, x_2, 2, 3, v_1\}]$ is nonbinary. By swapping 1 and 2, if necessary, we assume that $M_B[\{a, x_2, 2, 3, v_1\}]$ is nonbinary. Now $(\{u, a, b, x_1, x_2\}, \{2, 3\})$ is the only 2-separation in $M_B[\{u, a, b, x_1, x_2, 2, 3\}]$, and v_1 is a blocking sequence for this 2-subseparation. So, by Corollary 4.19, $M_B - v - 1$ is 3-connected. Thus we have that $M_B - v$, $M_B - 1$, and $M_B - 1 - v$ are all stable, and $M_B - 1 - v$ is connected and nonbinary. As $M_B - u - v$ is not 3-connected, this contradicts (1).

Now we consider the more difficult case that v_1 is in the same colour class as 3 in G_B. Since $M_B - u - v$ contains a 3-connected nonbinary minor of size at least $|S| - 4$, v_1 must be a blocking sequence for the 2-subseparation $(\{b, x_1, x_2, a\}, \{1, 2, 3\})$. Hence v_1 is adjacent to x_1. The 2-subseparation $(\{a, b, x_1, x_2\}, \{1, 2\})$ is uncrossed in $M_B[\{a, b, x_1, x_2, 1, 2\}]$ and v_1 is a blocking sequence for this 2-subseparation. Hence, by Proposition 4.17, $(\{b, x_1\}, \{x_2, a, 1, 2, v_1\})$ is the only 2-separation of $(M_B - 3) - u - v$. So $(M_B - 3) - u - v$ is stable. Furthermore, both u and v are blocking sequences for this 2-separation of $M_B - 3 - u - v$. Hence, by Proposition 4.17, $(M_B - 3) - u$ and $(M_B - 3) - v$ are both stable. Therefore, by (7), $M_B - 3 - u - v$ is binary. Since $(\{u, a, b, x_1, x_2, v_1\}, \{1, 2, 3\})$ is not a 2-subseparation, v_1 is adjacent to either 1 or 2. If v_1 is adjacent to both 1 and 2, then, by Proposition 4.12, $M_B[\{v_1, 1, 2, a\}]$ is a twirl, contradicting that $M_B - 3 - u - v$ is binary. Therefore, v_1 is adjacent to exactly one of 1 and 2. By relabeling, if necessary, we assume that v_1 is adjacent to 2.

Next we show that $M_B[\{v, a, 1, 3\}]$ is a twirl. Note that $M_B - x_2 - u - v$ is stable, nonbinary, and connected. By Proposition 4.11, $M_B - x_2 - v$ is 3-connected. If $M_B[\{v, a, 2, 3\}]$ is a twirl, then, by Propositions 4.11 and 4.12, $M_B - x_2 - u$ is 3-connected. Therefore, by (7), $M_B[\{v, a, 2, 3\}]$ cannot be a twirl. Since $M_B[\{a, 1, 2, 3\}]$ is a twirl, then, by Proposition 4.4, $M_B[\{v, a, 1, 3\}]$ is a twirl, as claimed.

We scale lines of A so that all edges in $G_B - u - v - 3$ have weight one (which is possible since $M_B - u - v - 3$ is binary). By further scaling we assume that edges ua, va, and 13 also have weight one. Now consider

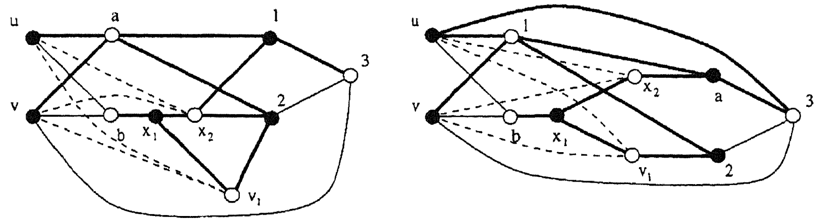

FIG. 6. G_B and $G_{B'}$.

pivoting on $1a$. Let $B' := B\Delta\{a, 1\}$. The graphs G_B and $G_{B'}$ are depicted in Fig. 6, the bold edges are those whose weight is known to be one. Let $A' = (\alpha'_{ij})$ be the representation of $M_{B'}$. Note that $M_{B'}[\{a, 1, 2, 3\}]$ is a twirl, and hence $\alpha'_{23} \neq 1$.

If $\alpha'_{ux_2} \notin \{0, 1\}$, then $M_{B'}[\{u, x_2, a, 3\}]$ is a twirl. If $\alpha'_{uv_1} \notin \{0, \alpha'_{23}+1\}$, then $M_{B'}[\{u, v_1, 2, 3\}]$ is a twirl. As $\alpha'_{23} \notin \{0, 1\}$, this implies that by pivoting on bx_1 and swapping b and x_1, if necessary, we may assume that either $M_{B'}[\{u, x_2, a, 3\}]$ or $M_{B'}[\{u, v_1, 2, 3\}]$ is a twirl. Define C' to be either $\{u, x_2, a, 3\}$ or $\{u, v_1, 2, 3\}$ such that $M_{B'}[C']$ is a twirl. (We will show that we can choose 1, b, u, v, B', C, in place of u, v, a, b, B, C and that this choice is in fact better.)

Note that $M_{B'} - b - u - v$ is 3-connected and nonbinary. Therefore, $M_{B'} - b - u$ and $M_{B'} - b - v$ are both stable. Hence, by (7), $M_{B'} - b = N_{B'} - b$. Now consider $M_{B'} - 1$. $(\{b, x_1\}, \{v_1, 2, 3, a, x_2\})$ is the only 2-separation in $M_{B'} - 1 - u - v$. So $M_{B'} - 1 - u - v$ is stable, and, clearly, nonbinary. Furthermore, both u and v are blocking sequences for this 2-separation. So, by Corollary 4.19, both $M_{B'} - 1 - u$ and $M_{B'} - 1 - v$ are 3-connected. Hence, by (7), $M_{B'} - 1 = N_{B'} - 1$.

Since $M_{B'}[C']$ is a twirl, we have that $M_{B'} - 1 - b - v$ is 3-connected. Therefore, $M_{B'} - 1 - b$ is stable, connected, and contains a 3-connected nonbinary matroid of size at least $|S| - 3$. Since $M_{B'} - 1 - u$ is 3-connected, $M_{B'} - 1$ is stable. By Proposition 4.11, $M_{B'} - v - b$ is 3-connected, and hence $M_{B'} - b$ is stable. Hence, by Lemma 2.2, N is the unique $GF(4)$-representable matroid such that $N_{B'} - 1 = M_{B'} - 1$ and $N_{B'} - b = N_{B'} - b$. Moreover, $\{1, b, u, v\}$ distinguishes $M_{B'}$ from $N_{B'}$. Hence we may choose 1, b, u, v, B', C' in place of u, v, a, b, B, C. As $d_B(b, C) = 3 > 1 = d_{B'}(v, C')$, this contradicts (3). So (15) follows.

(16) *v_p is in the same colour class of G_B as* 3.

Suppose not. Then, since $(\{u, a, x_0, ..., x_k, v_p\}, \{1, 2, 3\})$ is not a 2-subseparation, v_p is adjacent to 3. Since $p > 1$, $(\{u, a, x_0, ..., x_k\}, \{1, 2, 3, v_p\})$ is a 2-subseparation. Hence $(\{u, a, x_0, ..., x_k\}, \{1, 2, 3, v_p\})$ is a split in $G_B[\{u, a, x_0, ..., x_k, 1, 2, 3, v_p\}]$. Consequently, v_p is either adjacent to both a and x_k or nonadjacent to both a and x_k. First, suppose that v_p is adjacent to both a and x_k. Either $M_B[\{v_p, a, 1, 3\}]$ or $M_B[\{v_p, a, 2, 3\}]$ is a twirl. By interchanging 1 and 2, if necessary, we assume that $M_B[\{v_p, a, 1, 3\}]$ is a twirl. Consider replacing 2 by v_p. By Proposition 4.16 (part (i)), $v_1, ..., v_{p-1}$ is a blocking sequence for $(\{u, a, x_0, ..., x_k\}, \{1, v_p, 3\})$, which contradicts the minimality of p. Hence, v_p is adjacent to neither a nor x_k. Then v_p is pendant to 3 in $M_B[\{u, v, a, x_0, ..., x_k, 1, 2, 3, v_p\}]$. Consider pivoting on v_p3. We have that $M_{B\Delta\{3, v_p\}}[\{u, v, a, x_0, ..., x_k, 1, 2, v_p\}]$ is isomorphic to $M_B[\{u, v, a, x_0, ..., x_k, 1, 2, 3\}]$. Furthermore $\lambda_B(\{u, a, x_0, ..., x_k, 3\}, \{1, 2, 3\}) \geqslant \lambda_B(\{u, a, x_0, ..., x_k, 3\}, \{1, 2\}) = 2$, so, by Proposition 4.16

(part (ii)), $v_1, ..., v_{p-1}$ is a blocking sequence for the 2-subseparation $(\{u, a, x_0, ..., x_k\}, \{1, 2, v_p\})$ in $M_{B\Delta\{3, v_p\}}$. As this contradicts the minimality of p, (16) follows.

(17) $p \neq 2$.

Suppose that $p=2$. Then, by (16), v_2 is in the same colour class as 3. Hence, by Proposition 4.15 (part (iv)), v_1 is in the same colour class as 1. Then, the only possible neighbours of v_1 among $\{u, v, a, x_0, ..., x_k, 1, 2, 3\}$ are x_0, x_2, and a. First we suppose that v_1 is adjacent to just one of x_0, x_2, and a and let $z \in \{x_0, x_2, a\}$ be the neighbour of v_1. Consider pivoting on zv_1. Note that zv_1 is an allowable pivot. Then $M_{B\Delta\{z, v_1\}}[\{u, v, a, x_0, ..., x_k, 1, 2, 3\} \Delta\{z, v_1\}]$ is isomorphic to $M_B[\{u, v, a, x_0, ..., x_k, 1, 2, 3\}]$. Furthermore $\lambda_B(\{u, a, x_0, ..., x_k\}, \{1, 2, 3, z\}) \geqslant \lambda_B(\{u, a, x_0, ..., x_k\} \backslash \{z\}, \{1, 2, 3, z\}) = 2$, so, by Proposition 4.16 (part (ii)), v_2 is a blocking sequence for the 2-subseparation $(\{u, a, x_0, ..., x_k\} \Delta\{z, v_1\}, \{1, 2, 3\})$ in $M_{B\Delta\{z, v_1\}}$. As this contradicts the minimality of p, v_1 has at least two neighbours among x_0, x_2, and a. We consider the case where $k=2$ and v_1 is adjacent to x_0. Since $d(b, C)=3$, v_1 is not adjacent to a. Hence v_1 is adjacent to x_2. Then, by Proposition 4.16 (part (i)), v_2 is a blocking sequence for $(\{u, a, x_0, v_1, x_2\}, \{1, 2, 3\})$. Hence, replacing x_1 by v_1 yields a contradiction against the minimality of p. Thus, if $k=2$, then v_1 is not adjacent to x_0. Hence, with $k=0$ or $k=2$, v_1 is adjacent to both a and x_k. Since $(\{u, a, x_0, ..., x_k\}, \{1, 2, 3, v_1\})$ is not a 2-separation, $M_B[\{v_1, 1, a, x_k\}]$ is a twirl. However, $d_B(b, \{v_1, 1, a, x_k\}) < d_B(b, C)$, which contradicts (3). So (17) follows.

Let $X := \{u, v, a, x_0, ..., x_k, 1, 2, 3, v_{p-1}, v_p\}$. By (16), v_p is in the same colour class as 3. Hence, by Proposition 4.15 (part (iv)), v_{p-1} is in the same colour class as 1. As v_{p-1} is the next to last element of the blocking sequence, it is adjacent to v_p but not to 3. Since $(\{u, a, x_0, ..., x_k\}, \{1, 2, 3, v_{p-1}\})$ and $(\{u, a, x_0, ..., x_k, v_{p-1}\}, \{1, 2, 3\})$ are both 2-subseparations, the only possible neighbours of v_{p-1} in X are v_p, a, and x_k; furthermore v_{p-1} is adjacent either to neither or to both of a and x_k. Suppose that v_p is adjacent to neither a nor x_k. Hence v_{p-1} has no neighbours in $X \backslash \{v_p\}$. Consider pivoting on $v_{p-1}v_p$. $M_B[X] - v_{p-1} - v_p$ is isomorphic to $M_{B\Delta\{v_{p-1}, v_p\}}[X] - v_{p-1} - v_p$, and, by Proposition 4.16 (part (iii)), $v_1, ..., v_{p-2}$ is a blocking sequence for the 2-subseparation $(\{u, a, x_0, ..., x_k\}, \{1, 2, 3\})$ in $M_{B\Delta\{v_{p-1}, v_p\}}$, which contradicts the minimality of p. Therefore, v_{p-1} is adjacent to both a and x_k. Since v_p is the end of the blocking sequence, it must be adjacent to either 1 or 2. By interchanging 1 and 2, if necessary, we assume that v_p is adjacent to 1. $G_B[X]$ is depicted in Fig. 7.

(18) $M_B[\{a, 1, v_{p-1}, v_p\}]$ *is not a twirl.*

Suppose $M_B[\{a, 1, v_{p-1}, v_p\}]$ is a twirl.

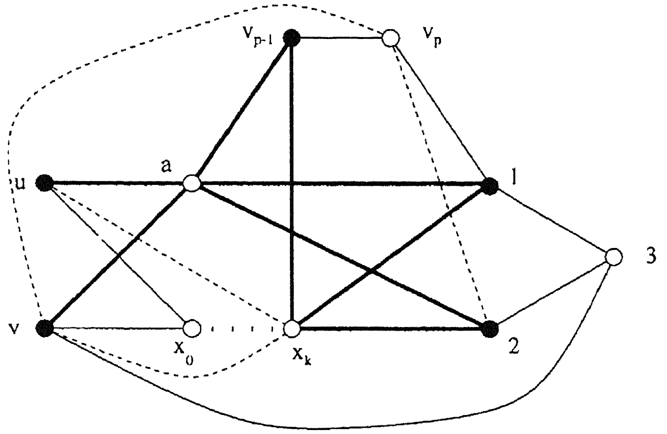

FIG. 7. $G_B[X]$.

By Proposition 4.16 (part (i)), $v_1, ..., v_{p-2}$ is a blocking sequence for the 2-subseparation $(\{u, a, x_0, ..., x_k\}, \{1, v_{p-1}, v_p\})$. Hence, as $M_B[\{a, 1, v_{p-1}, v_p\}]$ is a twirl, it follows from the minimality of p that v and v_p are not adjacent.

Now, $1v_p$ is an allowable pivot. $G_{B\Delta\{1, v_p\}}[X\backslash\{2\}]$ is depicted in Fig. 8. Since 3 is pendant to 1 in $M_B[\{a, 1, 3, v_{p-1}, v_p\}]$, 1 and 3 are twins in $M_{B\Delta\{1, v_p\}}[\{a, 1, 3, v_{p-1}, v_p\}]$. Furthermore, as $M_B[\{a, 1, v_{p-1}, v_p\}]$ is a twirl, so is $M_{B\Delta\{1, v_p\}}[\{a, 1, v_{p-1}, v_p\}]$. Hence, $M_{B\Delta\{1, v_p\}}[\{a, 3, v_{p-1}, v_p\}]$ is a twirl as well. Since v is adjacent to neither 1 nor v_p in G_B, v remains adjacent to 3 in $G_{B\Delta\{1, v_p\}}$. By Proposition 4.15 (parts (i) and (ii)), $v_1, ..., v_{p-1}$ is a blocking sequence for the 2-subseparation $(\{u, a, x_0, ..., x_k\}, \{v_p, 1, 2, 3\})$ in $M_{B\Delta\{1, v_p\}}$. Then, by Proposition 4.16 (part (i)), $v_1, ..., v_{p-2}$ is

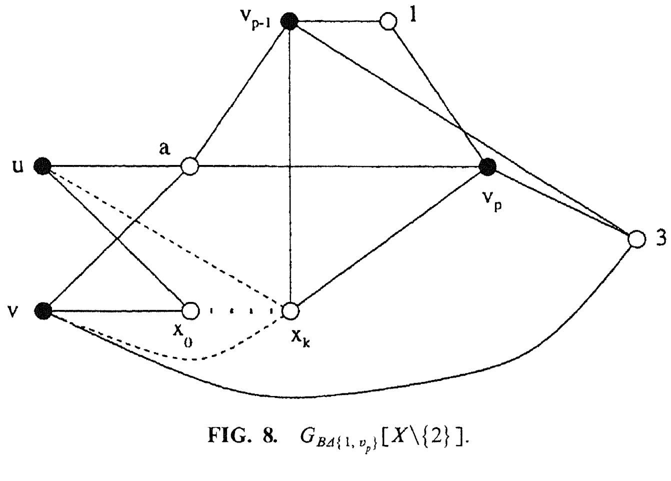

FIG. 8. $G_{B\Delta\{1, v_p\}}[X\backslash\{2\}]$.

a blocking sequence for the 2-subseparation $(\{u, a, x_0, ..., x_k\}, \{v_{p-1}, v_p, 3\})$ in $M_{B\Delta\{1, v_p\}}$. Hence, replacing B by $B\Delta\{1, v_p\}$ and C by $\{a, v_p, v_{p-1}, 3\}$ yields a contradiction against the minimality of p. So (18) follows.

(19) *v_p is not adjacent to* 2.

Suppose that v_p is adjacent to 2. Since $(\{u, a, x_0, ..., x_k, v_p\}, \{1, 2, 3\})$ is not a 2-subseparation, $M_B[\{a, 1, 2, v_p\}]$ is a twirl. Hence, either $M_B[\{v_{p-1}, v_p, a, 1\}]$ or $M_B[\{v_{p-1}, v_p, a, 2\}]$ is a twirl. By interchanging 1 and 2, if necessary, we obtain a contradiction to (18). This proves (19).

(20) *v is adjacent to v_p.*

Suppose not. Then v_p is pendant to 1 in $M_B[X] - v_{p-1}$. Hence $M_{B\Delta\{v_p, 1\}}[X] - v_{p-1} - 1$ is isomorphic to $M_B[X] - v_{p-1} - v_p$. Furthermore $\lambda_B(\{u, a, x_0, ..., x_k, 1\}, \{1, 2, 3\}) \geqslant \lambda_B(\{u, a, x_0, ..., x_k, 1\}, \{2, 3\}) = 2$, so, by Proposition 4.16 (part (ii)), $v_1, ..., v_{p-1}$ is a blocking sequence for the 2-subseparation $(\{u, a, x_0, ..., x_k\}, \{v_p, 2, 3\})$ of $M_{B\Delta\{1, v_p\}}$. As this contradicts the minimality of p, (20) follows.

We scale the columns of A so that $\alpha_{ai} = 1$ for each $i \in \text{nigh}_B(a)$. Also by scaling we may assume that $\alpha_{x_k, 1} = \alpha_{v_p 1} = \alpha_{32} = 1$, and, if $k = 2$, $\alpha_{x_0 x_1} = \alpha_{x_2 x_1} = 1$. Since $(\{u, a, x_0, ..., x_k\}, \{1, 2, 3, v_{p-1}\})$ is a 2-subseparation, $\alpha_{x_k v_{p-1}} = \alpha_{x_k 2} = 1$, and, by (18), we also have $\alpha_{v_p v_{p-1}} = 1$. Now $G_B[X]$ is depicted in Fig. 9; the bold edges indicate entries in A that are known to be one. Let A' be the matrix obtained from A by applying the automorphism of $GF(4)$ to the elements in column u.

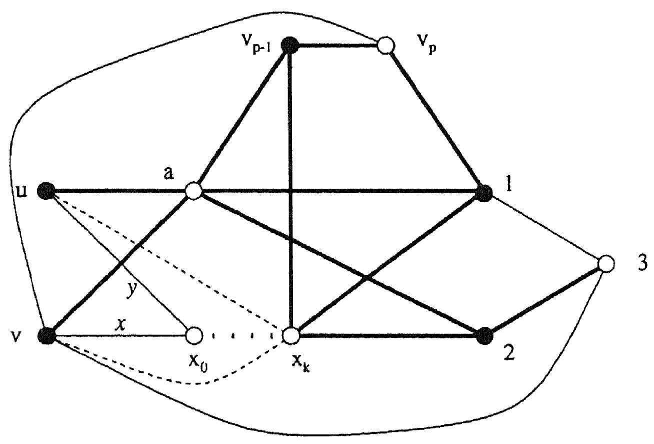

FIG. 9. $G_B[X]$.

(21) *$A'[B \cap X, X \backslash B]$ is a GF(4)-representation of $M_B[X]$.*

Since $p > 2$, $S \neq X$, and, hence, $M_B[X]$ is $GF(4)$-representable. Let $A'' = (\alpha''_{ij})$ be a $GF(4)$-representation of $M_B[X]$. By (12) and (13), $M_B[X] - u - v_{p-1} - v_p$ is 3-connected. Since v_{p-1} has no twin in $M_B[X] - u - v_p$, $M_B[X] - u - v_p$ is also 3-connected. Similarly, since v_p has no twin in $M_B[X] - u$, $M_B[X] - u$ is 3-connected. Therefore, $M_B[X] - u$ is stable. Since $A[X \cap B, (X \backslash B) \backslash \{u\}]$ is a $GF(4)$-representation of $M_B[X] - u$, we may assume that $A''[X \cap B, (X \backslash B) \backslash \{u\}] = A[X \cap B, (X \backslash B) \backslash \{u\}]$. By (12), $(\{u, a, x_0, ..., x_k\}, \{1, 2, 3\})$ is the only 2-separation of $M_B[X] - v - v_{p-1} - v_p$. So, by Proposition 4.18, $(\{u, a, x_0, ..., x_k\}, \{1, 2, 3, v_{p-1}, v_p\})$ is the only 2-subseparation in $M_B[X] - v$. Therefore, there are at most two distinct $GF(4)$-representations of $M_B[X] - v$. Hence any $GF(4)$-representation of $M_B[X] - v$ is equivalent to $A[X \cap B, (X \backslash B) \backslash \{v\}]$ or $A'[X \cap B, (X \backslash B) \backslash \{v\}]$. Therefore, we may assume that $A''[X \cap B, (X \backslash B) \backslash \{v\}]$ is one of these two matrices. However, since $M_B[X] \neq N_B[X]$, it must be the case that $A''[X \cap B, (X \backslash B) \backslash \{v\}] = A'[X \cap B, (X \backslash V) \backslash \{v\}]$. So $A'' = A'[X \cap B, (X \backslash B)]$, which proves (21).

Let $x := \alpha_{bv}$, $y := \alpha_{bu}$, and let y' be the image of y under the automorphism of $GF(4)$. We use (21) to determine subsets of X that distinguish M_B and N_B. For instance $\{u, v, a, b\}$ distinguishes M_B and N_B, so $\det(A_B[\{u, v, a, b, \}]) = 0$ if and only if $\det(A'_B[\{u, v, a, b\}]) \neq 0$. Now $\det(A_B[\{u, v, a, b\}]) = x + y$, and $\det(A'_B[\{u, v, a, b\}]) = x + y'$. Hence $y \neq y'$, so neither y nor y' is either zero or one. Furthermore, either $x = y$ or $x = y'$, so x is neither zero nor one. Hence $\{y, y'\} = \{x, x+1\}$. Let $\varepsilon := \alpha_{x_0 1}$. Thus $\varepsilon \in \{0, 1\}$, and $\varepsilon = 0$ if and only if $k = 2$. Note that $\alpha_{bv_{p-1}} = \alpha_{b2} = \alpha_{b1} = \varepsilon$.

(22) *Let $z \in \{v_{p-1}, 1, 2\}$, and let $w \in \{v_p, 3\}$ be adjacent to z. If $\alpha_{wv}/\alpha_{wz} = x + \varepsilon$, then wz is an allowable pivot.*

We have

$$A[\{a, b, w\}, \{u, v, z\}] = \begin{array}{c} \\ a \\ b \\ w \end{array} \begin{array}{c} \begin{array}{ccc} u & v & z \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ y & x & \varepsilon \\ 0 & \alpha_{vw} & \alpha_{wz} \end{pmatrix} \end{array}.$$

Therefore,

$$\det(A[\{a, b, w\}, \{u, v, z\}]) = \alpha_{wv}(y + \varepsilon) + \alpha_{wz}(x + y)$$
$$= \alpha_{wz}((x + \varepsilon + 1)(y + \varepsilon + 1) + 1).$$

Similarly $\det(A'[\{a, b, w\}, \{u, v, z\}]) = \alpha_{wz}((x+\varepsilon+1)(y'+\varepsilon+1)+1)$. Recall that $\{y, y'\} = \{x, x+1\}$. Now $(x+\varepsilon+1)(x+\varepsilon+1) = x+\varepsilon$, while $(x+\varepsilon+1)((x+1)+\varepsilon+1) = 1$. Thus, exactly one of $A[\{a, b, w\}, \{u, v, z\}]$ and $A'[\{a, b, w\}, \{u, v, z\}]$ is nonsingular. So $\{a, b, u, v, w, z\}$ distinguishes M_B and N_B. Hence, wz is an allowable pivot, which proves (22).

(23) $\alpha_{v_p v} \in \{1, x+\varepsilon+1\}$.

By (20), $\alpha_{v_p v} \neq 0$. Suppose that $\alpha_{v_p v} \notin \{1, x+\varepsilon+1\}$, and, hence, $\alpha_{v_p v} = x+\varepsilon$. By (22), $1v_p$ is an allowable pivot. Now suppose that $M_B[\{1, 3, v, v_p\}]$ is not a twirl. Then $0 = \det(A[\{3, v_p\}, \{1, v\}]) = \alpha_{3v} + (x+\varepsilon)\alpha_{31}$. Hence, by (22), 31 is an allowable pivot. By pivoting on 31, v_p becomes adjacent to 2, which contradicts (19). Hence, $M_B[\{1, 3, v, v_p\}]$ is a twirl.

Consider pivoting on $1v_p$ and replacing 1 by v_p. Since $M_B[\{1, 3, v, v_p\}]$ is a twirl, v remains adjacent to 3 in $G_{B\Delta\{1, v_p\}}$. v_p is pendant to 1 in $M_B[X] - v_{p-1} - v$. Hence $M_{B\Delta\{v_p, 1\}}[X] - v_{p-1} - v - 1$ is isomorphic to $M_B[X] - v_{p-1} - v - v_p$. Furthermore $\lambda_B(\{u, a, x_0, ..., x_k, 1\}, \{1, 2, 3\}) \geqslant \lambda_B(\{u, a, x_0, ..., x_k, 1\}, \{2, 3\}) = 2$, so, by Proposition 4.16, $v_1, ..., v_{p-1}$ is a blocking sequence for the 2-subseparation $(\{u, a, x_0, ..., x_k\}, \{v_p, 2, 3\})$ of $M_{B\Delta\{1, v_p\}}$. As this contradicts the minimality of p, (23) follows.

(24) $\alpha_{3v} \in \{1, x+\varepsilon+1\}$.

By (13), $\alpha_{3v} \neq 0$. Suppose that $\alpha_{3v} \notin \{1, x+\varepsilon+1\}$, and, hence, $\alpha_{3v} = x+\varepsilon$. By (22), 23 is an allowable pivot. Consider pivoting on 23 and interchanging 2 and 3. The pivot changes α_{a1} from 1 to $1+\alpha_{13}$. Hence $M_{B\Delta\{2,3\}}[\{1, a, v_{p-1}, v_p\}]$ is a twirl, contradicting (18). This proves (24).

(25) $\{u, v, a, b, 2, 3, v_{p-1}, v_p\}$ *distinguishes* M_B *and* N_B.

Let $Y_1 := \{a, b, 3, v_p\}$ and $Y_2 := \{u, v, 2, v_{p-1}\}$. We have

$$A[Y_1, Y_2] = \begin{array}{c} \\ a \\ b \\ 3 \\ v_p \end{array} \!\!\begin{array}{c} \begin{array}{cccc} u & v & 2 & v_{p-1} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ y & x & \varepsilon & \varepsilon \\ 0 & \alpha_{3v} & 1 & 0 \\ 0 & \alpha_{v_p v} & 0 & 1 \end{pmatrix} \end{array}.$$

Therefore $\det(A[Y_1, Y_2]) = (y+\varepsilon)(\alpha_{3v} + \alpha_{v_p v}) + x + y$, and $\det(A'[Y_1, Y_2]) = (y'+\varepsilon)(\alpha_{3v} + \alpha_{v_p v}) + x + y'$. By (23) and (24), $\alpha_{3v} + \alpha_{v_p v}$ is either zero or $x+\varepsilon$. First suppose that $\alpha_{3v} + \alpha_{v_p v} = x+\varepsilon$. Thus $\det(A[Y_1, Y_2]) = (x+\varepsilon+1)(y+\varepsilon+1)+1$, and $\det(A'[Y_1, Y_2]) = (x+\varepsilon+1)(y'+\varepsilon+1)+1$. Recall that $\{y, y'\} = \{x, x+1\}$. Now $(x+\varepsilon+1)(x+\varepsilon+1) = x+\varepsilon$, while $(x+\varepsilon+1)((x+1)+\varepsilon+1) = 1$. So exactly one of $A[Y_1, Y_2]$ an

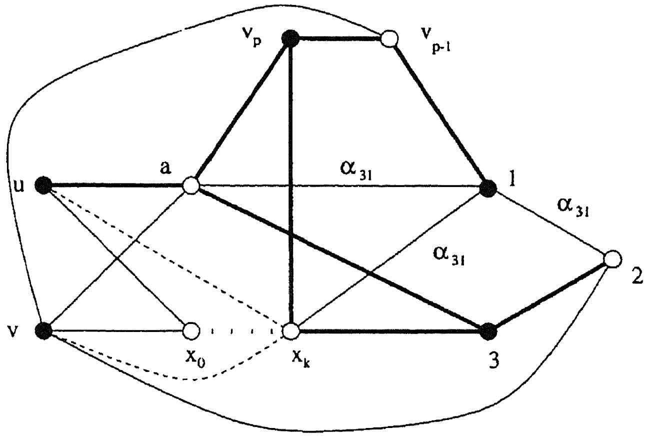

FIG. 10. $G_{B\Delta X'}[X]$.

$A'[Y_1, Y_2]$ is singular. Hence $Y_1 \cup Y_2$ distinguishes M_B and N_B, as required. Now suppose that $\alpha_{3v} + \alpha_{v_p v} = 0$. Thus $\det(A[Y_1, Y_2]) = x + y$, and $\det(A'[Y_1, Y_2]) = x + y'$. So exactly one of $A[Y_1, Y_2]$ and $A'[Y_1, Y_2]$ is singular. Hence $Y_1 \cup Y_2$ distinguishes M_B and N_B, which proves (25).

Let $X' := \{2, 3, v_{p-1}, v_p\}$. Consider pivoting on 23, and pivoting on $v_{p-1}v_p$. Figure 10 depicts $G_{B\Delta X'}[X]$. The key observations are that v is adjacent to v_{p-1} in $G_{B\Delta X'}$ and that $M_{B\Delta X'}[\{a, v_p, v_{p-1}, 1\}]$ is a twirl. Now, by Proposition 4.15 (parts (i) and (ii)), $v_1, ..., v_{p-1}$ is a blocking sequence for the 2-subseparation $(\{u, a, x_0, ..., x_k\}, \{1, 2, 3, v_p\})$ in $M_{B\Delta\{2,3\}}$. Then, by Proposition 4.16 (part (i)) and Proposition 4.15 (part (ii)), $v_1, ..., v_{p-2}$ is a blocking sequence for the 2-subseparation $(\{u, a, x_0, ..., x_k\}, \{1, v_p, v_{p-1}\})$ in $M_{B\Delta X'}$. Hence, replacing B and C by $B\Delta X'$ and $\{a, 1, v_p, v_{p-1}\}$, yields a, final, contradiction to the minimality of p. This completes the proof of Theorem 5.1. ∎

6. CASE ANALYSIS

We now complete the proof of Theorem 1.1 by analyzing the matroids with rank and corank at most 4. This requires a lot of case checking, much of which we leave to the reader. Figure 11 contains geometric representations of matroids used frequently in the case analysis. Throughout this section, M is a minor-minimal non-$GF(4)$-representable matroid. We have already seen that M has rank and corank at most 4.

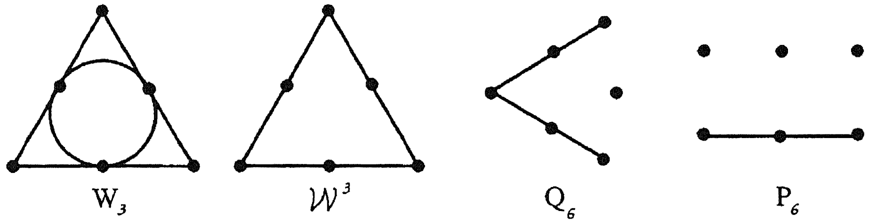

FIG. 11. All nonuniform 3-connected matroids with 6 elements.

M is certainly nonbinary and 3-connected. Below we list all small 3-connected nonbinary matroids:

4 elements: $U_{2,4}$,

5 elements: $U_{2,5}$ and $U_{3,5}$,

6 elements: $\mathcal{W}^3$, $U_{3,6}$, Q_6, P_6, $U_{2,6}$, and $U_{4,6}$.

Among these matroids, the only non-$GF(4)$-representable matroids are $U_{2,6}$, $U_{4,6}$, and P_6. In what follows, we assume that M has at least 7 elements. If M has rank or corank 2, then M is uniform and so has a $U_{2,6}$- or $U_{4,6}$-minor. Hence, M has rank and corank at least 3.

From the above list of matroids, we see that $U_{2,5}$ is a splitter for the family of matroids without $U_{2,6}$- or $U_{3,5}$-minors. By duality, $U_{3,5}$ is a splitter for the family of matroids without $U_{4,6}$- or $U_{2,5}$-minors. Therefore M contains a $U_{2,5}$-minor if and only if M contains a $U_{3,5}$-minor.

In what follows, we occasionally use assertions from the proof of Theorem 5.1; in particular, we use (7), (2), and (9). (In this section, each time we mention one of (7), (2), and (9), we mean (7), (2), and (9) in Section 5.) Strictly speaking, such assertions are subject to the conditions of Theorem 5.1 and to preceding assumptions in its proof. However, the reader can easily verify the validity of the assertion when it is applied.

Case 1. *M contains a $U_{3,5}$-minor.* We break this into two further cases.

Case 1.1. *M has 7 elements.*

The three matroids depicted in Fig. 12 are the only 3-connected 7-element rank-3 matroids having a $U_{3,5}$-minor but no $U_{2,6}$- for P_6-minors. (This is easily checked by trying to add a point to the representations of either $U_{3,6}$ or Q_6.)

To save the reader checking that the three matroids in Fig. 12 are $GF(4)$-representable, we give an alternative proof. By duality, we may assume that M has rank 3 and corank 4. Hence there exist elements u, v such that $M\backslash u, v$ is isomorphic to $U_{3,5}$. Then $M\backslash u$, $M\backslash v$, and $M\backslash u, v$ are all stable, nonbinary, and connected. Hence, by Lemma 2.2, there exists a unique $GF(4)$-representable matroid N such that $M\backslash u = N\backslash u$ and $M\backslash v = N\backslash v$.

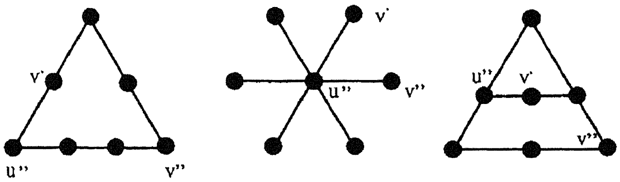

FIG. 12. The only 3-connected 7-element rank-3 matroids with $U_{3,5}$-minors, but without $U_{2,6}$- or P_6-minors.

Furthermore, as we showed in (2), there exists a basis B of $M\backslash u, v$, and elements a, b such that $\{u, v, a, b\}$ distinguishes M_B and N_B. Clearly a, $b \in B$; let c be the third element of B. Note that $M\backslash u, v/c$ is isomorphic to $U_{2,4}$. Hence $M_B - c - u$, $M_B - c - v$, and $M_B - c - u - v$ are all stable. Then, by (7), $M_B - c$ is not $GF(4)$-representable, which is a contradiction.

Case 1.2. *M has* 8 *elements.*

Note that M must have rank and corank both equal to 4. We begin by proving that there exist $M' \in \{M, M^*\}$ and distinct elements u, v such that $M'\backslash u$, $M'\backslash v$, and $M'\backslash u, v$ are all stable, connected, and have a $U_{3,5}$-minor. By the Splitter Theorem and duality, we may assume that there exists an element u' such that M/u' is 3-connected and has a $U_{2,5}$- or $U_{3,5}$-minor. In fact, M/u' has both $U_{2,5}$- and $U_{3,5}$-minors. Figure 12 depicts all the candidates for M/u'. Also depicted in Fig. 12 are elements v', u'', v'' satisfying the following conditions.

(i) $M/u', v'\backslash u''$ is isomorphic to $U_{2,5}$.

(ii) $M\backslash u'', v''/u'$ is isomorphic to $U_{3,5}$, and

(iii) u'', v'' are not parallel in $M/u', v'$.

$M\backslash u''/u'$, $M\backslash v''/u'$, and $M\backslash u'', v''/u'$ are 3-connected, so $M\backslash u''$, $M\backslash v''$, and $M\backslash u'', v''$ are all stable and have $U_{3,5}$-minors. If, in addition, $M\backslash u'', v''$ is connected, then u'' and v'' satisfy the requirements for the two desired elements u and v. Therefore, we may assume that $M\backslash u'', v''$ is not connected. Thus, u' is a coloop of $M\backslash u'', v''$. Now, M/u' and $M/u', v'$ are both stable, connected, and have a $U_{2,5}$-minor. If M/v' is stable then $u := u'$ and $v := v'$ satisfy the required properties with respect to $M' := M^*$. So we may assume that M/v' is not stable. Then M/v', $M/v'\backslash u''$, and $M/v', u'$ are not 3-connected. Furthermore, it is straightforward to see that $M/v'\backslash u''$ and $M/v', u'$ are connected, and that $M/v', u'\backslash u''$ is 3-connected. We now apply Proposition 3.5 to the matroid $N := M/v'$ and the elements $x := u''$ and $y := u'$. Since M is 3-connected, N has no series pairs. Hence, in the notation of Proposition 3.5, $p_x = p_y$. Since $y = u'$ is a coloop in $M\backslash u'', v''$, the elements u' and v'' are in series in $M\backslash u''$, and hence also in $N\backslash u''$.

So $p_y = v''$. Hence, $u'' = x$ and $v'' = p_y = p_x$ are in parallel in $N/y = N/u'$, contradicting (iii).

So we conclude that the desired pair u and v does exist. Replacing M by M^*, if necessary, we assume that $M\backslash u$, $M\backslash v$, and $M\backslash u, v$ are all stable, connected, and have a $U_{3,5}$-minor.

There exists a unique $GF(4)$-representable matroid N such that $M\backslash u = N\backslash u$ and $M\backslash v = N\backslash v$. As we showed in (2), there exists a basis B of $M\backslash u, v$ and elements $a, b \in B$ such that $\{a, b, u, v\}$ distinguishes M_B and N_B. Let $S\backslash B = \{u, v, 1, 2\}$ and $B = \{a, b, 3, 4\}$. Let G_B be the fundamental graph of M_B, and let $A = (\alpha_{ij})$ be a representation of N_B. By Propositions 4.1 and 4.2, $\{a, b, u, v\}$ induces a 4-circuit in G_B. If $M\backslash u, v/3, 4$ is isomorphic to $U_{2,4}$, then $M\backslash u/3, 4$, $M\backslash v/3, 4$, and $M\backslash u, v/3, 4$ are all stable, nonbinary, and connected. Hence, by (7), $M_B - 3 - 4$ is not $GF(4)$-representable. Thus, $M\backslash u, v/3, 4$ is not isomorphic to $U_{2,4}$. $M\backslash u, v$ has a $U_{3,5}$-minor, but it contains no $U_{3,5}$-minor using both a and b. By possibly interchanging a, b, we may assume that $M\backslash u, v/a$ is isomorphic to $U_{3,5}$. Since $M\backslash u, v/3, 4$ is not isomorphic to $U_{2,4}$, a is in series with either 1, 2, or b in $M\backslash u, v$. By possibly pivoting on one of $(b, 1)$ or $(b, 2)$ in M_B and relabeling, we may assume that a, b are series elements of $M\backslash u, v$. Then, by scaling, we may assume that A has the following form:

$$\begin{array}{c} \\ a \\ b \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} u & v & 1 & 2 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ \alpha_{bu} & \alpha_{bv} & 1 & 1 \\ \alpha_{3u} & \alpha_{3v} & 1 & x \\ \alpha_{4u} & \alpha_{4v} & 1 & x+1 \end{pmatrix} \end{array}.$$

By (9), we have $\alpha_{bu} \in \{x, x+1\}$ and $\alpha_{bv} \in \{x, x+1\}$. Since $M\backslash v$ is stable, α_{3u} and α_{4u} cannot both be zero. Similarly, α_{3v} and α_{4v} cannot both be zero. Suppose that α_{3u} and α_{3v} are both nonzero. Then $(M_B - 4) - u$, $(M_B - 4) - v$, and $(M_B - 4) - u - v$ are all stable, nonbinary, and connected. By (7), this is a contradiction. Hence one of α_{3u} and α_{3v} is zero. Similarly, one of α_{4u} and α_{4v} is zero. By possibly interchanging 3, 4, we may assume that $\alpha_{3v} = \alpha_{4u} = 0$.

We proceed by showing that $\{u, v, a, b\}$ is the only set distinguishing M_B and N_B. Certainly every distinguishing set contains both u and v. Note that $(M_B - a) - u$, $(M_B - a) - v$, and $(M_B - a) - u - v$ are all 3-connected and nonbinary. Hence, by (7), every distinguishing set contains a. Similarly, every distinguishing set contains b. For some $i \in \{3, 4\}$ and $j \in \{1, 2\}$, suppose that $\{u, v, a, b, i, j\}$ is a distinguishing set. Then the pivot on ij is allowable, and by performing the pivot and interchanging i and j, we get $\alpha_{i'u} \neq 0$ and $\alpha_{i'v} \neq 0$ for some $i' \in \{3, 4\}$. This contradicts an earlier finding, thus $\{u, v, a, b, i, j\}$ is not a distinguishing set. In similar fashion, by pivoting on both 13 and 24, we can show that S is not a distinguishing set. Hence, as claimed $\{a, b, u, v\}$ is the only set distinguishing M_B from N_B.

Recall that $M_B[\{a, b, u, 1\}]$ is a twirl. Hence, by Lemma 4.4, either $M_B[\{a, 3, u, 1\}]$ or $M_B[\{b, 3, u, 1\}]$ is a twirl. We claim that exactly one of these is a twirl. Suppose, to the contrary, that $M_B[\{a, 3, u, 1\}]$ and $M_B[\{b, 3, u, 1\}]$ are both twirls. Then the following matrices are the only two plausible $GF(4)$-representations for either $M_B - 2 - 4$ or $N_B - 2 - 4$:

$$\begin{array}{c@{}c} & \begin{array}{ccc} u & v & 1 \end{array} \\ \begin{array}{c} a \\ b \\ 3 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ x & x & 1 \\ x+1 & 0 & 1 \end{pmatrix} \end{array}, \qquad \begin{array}{c@{}c} & \begin{array}{ccc} u & v & 1 \end{array} \\ \begin{array}{c} a \\ b \\ 3 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ x & x+1 & 1 \\ x+1 & 0 & 1 \end{pmatrix} \end{array}.$$

Since $\{a, b, u, v\}$ distinguishes $M_B - 2 - 4$ and $N_B - 2 - 4$, one of these matrices represents $M_B - 2 - 4$ and the other one represents $N_B - 2 - 4$. However, the above matrices have determinants x and 0 respectively. Thus $\{a, b, u, v, 1, 3\}$ distinguishes M_B and N_B. This contradiction verifies that exactly one of $M_B[\{a, 3, u, 1\}]$ and $M_B[\{b, 3, u, 1\}]$ is a twirl. So, by symmetry, for each $i \in \{3, 4\}$, $j \in \{1, 2\}$, and $w \in \{u, v\}$ such that $\alpha_{iw} \neq 0$, exactly one of $M_B[\{a, i, w, j\}]$ and $M_B[\{b, i, w, j\}]$ is a twirl.

By possible interchanging a and b, we can assume that $M_B[\{a, 3, u, 1\}]$ is not a twirl. Hence $\alpha_{3u} = 1$. Then $M_B[\{a, 3, u, 2\}]$ is a twirl, and, consequently, $M_B[\{b, 3, u, 2\}]$ is not a twirl. Thus $\alpha_{bu} = x + 1$. Now exactly one of $M_B[\{a, 4, v, 1\}]$ and $M_B[\{b, 4, v, 1\}]$ is a twirl. Considering these two cases separately, and using the fact that exactly one of $M_B[\{a, 4, v, 2\}]$ and $M_B[\{b, 4, v, 2\}]$ is a twirl, we get the following two candidates for A.

$$A_1 := \begin{array}{c@{}c} & \begin{array}{cccc} u & v & 1 & 2 \end{array} \\ \begin{array}{c} a \\ b \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ x+1 & x & 1 & 1 \\ 1 & 0 & 1 & x \\ 0 & 1 & 1 & x+1 \end{pmatrix} \end{array}, \qquad A_2 := \begin{array}{c@{}c} & \begin{array}{cccc} u & v & 1 & 2 \end{array} \\ \begin{array}{c} a \\ b \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ x+1 & x+1 & 1 & 1 \\ 1 & 0 & 1 & x \\ 0 & x+1 & 1 & x+1 \end{pmatrix} \end{array}.$$

We now consider the cases that $A = A_1$ and $A = A_2$. Note that, in either case, we know N explicitly and, since $\{a, b, u, v\}$ is the only set distinguishing N_B and M_B, we know M explicitly. If $A = A_1$, then M/u is isomorphic to F_7^-; if $A = A_2$, then M is isomorphic to P_8''.

Case 2. *M contains no $U_{2,5}$- or $U_{3,5}$-minor.*

The matroids depicted in Fig. 13 are the only 3-connected, rank-3 matroids on seven elements without a $U_{3,5}$-minor. (This is easily shown by trying to add a point to the geometric representations of $\mathcal{W}^3$ and W_3.) Among these, F_7^- is the only matroid that is not $GF(4)$-representable. Hence the only excluded minors on seven elements are F_7^- and its dual. In what remains, we assume that M has eight elements. Thus, M has rank and corank both 4.

We begin by showing that M is ternary. Suppose otherwise. Recall that M has no $U_{2,5}$- nor $U_{3,5}$-minors. Thus, by Reid's characterization of $GF(3)$-representable matroids, the only nonternary minors of M are F_7 and its dual, which are binary. By duality, we may suppose that $M\backslash u = F_7^*$. Since M is nonbinary there exists an element v such that $M\backslash v$ is not binary. Observe that deleting a single element from a nonstable matroid cannot yield a connected binary matroid. Hence, as $M\backslash u, v = F_7^*\backslash v$ is binary and connected, each of $M\backslash u, v$, $M\backslash v$, and $M\backslash u$ is stable. Therefore, as $M\backslash u$ is binary, it follows from the remark just below the proof of Lemma 2.2 that there exists a unique $GF(4)$-representable matroid N such that $M\backslash u = N\backslash u$ and $M\backslash v = N\backslash v$. As we showed in (2), there exists a basis B of $M\backslash u, v$ and elements $a, b \in B$ such that $\{a, b, u, v\}$ distinguishes M_B and N_B. Choose an element c in $B\backslash\{a, b\}$. F_7^* cannot be disconnected by performing one deletion and one contraction; hence, $M/c\backslash u, v$ is connected. As $M/c\backslash u, v$ is also binary, each of $M/c\backslash u$, $M/c\backslash v$, and $M/c\backslash u, v$ is stable. Hence, as $M/c\backslash u$ is binary, it follows from the remark just below the proof of Lemma 2.2 there exists a unique $GF(4)$-representable matroid N' such that $M/c\backslash u = N'\backslash u$ and $M/c\backslash v = N'\backslash v$. Clearly $N' = N/c$. However $\{u, v, a, c\}$ distinguishes $N_B - c$ and $M_B - c$, so M/c is not $GF(4)$-representable. This contradiction implies that M is ternary.

Case 2.1. *M contains a W_3-minor.*

By the Splitter Theorem, and duality, we may assume that there exists an element x such that M/x is 3-connected, and contains a W_3-minor. Then M/x is one of the matroids in Fig. 13. P_7 has no W_3-minor, F_7 is not

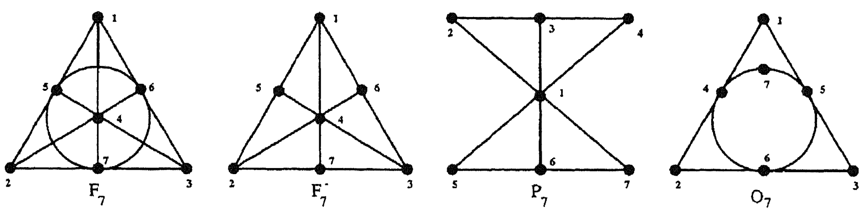

FIG. 13. Seven-element rank-3 matroids without $U_{3,5}$- or $U_{2,5}$-minors.

$GF(3)$-representable, and F_7^- is not $GF(4)$-representable. Hence $M/x = O_7$. The following matrix is a representation of O_7 over $GF(3)$:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \end{array}$$

$O_7 \backslash 7 = W_3$, so either $M\backslash 7$ is isomorphic to O_7^* or $M\backslash 7$ is not 3-connected. Note that there are automorphisms of O_7 that realize any permutation of $\{4, 5, 6\}$. Taking these permutations into account, there are just a few ways to extend our $GF(3)$-representation of O_7 to a possible representation of M, namely

$$\begin{array}{c} \\ x \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & \alpha \\ -1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \end{array}, \quad \begin{array}{c} \\ x \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & \alpha \\ -1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \end{array},$$

$$\begin{array}{c} \\ x \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & \alpha \\ -1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \end{array}, \quad \begin{array}{c} \\ x \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & \alpha \\ -1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \end{array}.$$

Label the above matrices $A_1(\alpha), ..., A_4(\alpha)$, and let $M_i(\alpha)$ be the ternary matroid represented by $A_i(\alpha)$.

$M_3(-1)\backslash 6$ is isomorphic to $(F_7^-)^*$. Moreover, for $i = 1, 2$, or 4, $M_i(-1)$ is isomorphic to $M_i(1)$. Indeed, if $i = 1$, 2 or 4, then $A_i(1)$ can be obtained from $A_i(-1)$ by negating lines 4, 5, 6, x and then interchanging 4 with 6 and 1 with 3. $M_3(1)$ and $M_4(0)$ are not 3-connected. So we are left with the cases: $M_1(0)$, $M_1(1)$, $M_2(0)$, $M_2(1)$, $M_3(0)$, and $M_4(1)$. They are all $GF(4)$-representable, with the following $GF(4)$-representations:

$$M_1(0): \begin{array}{c} \\ x \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & z & z+1 & 0 \\ 1 & 1 & 0 & z+1 \\ 1 & 0 & 1 & z \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}, \quad M_1(1): \begin{array}{c} \\ x \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} z+1 & z & 1 & 1 \\ 1 & 1 & 0 & z+1 \\ 1 & 0 & 1 & z \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array},$$

$$M_2(0): \begin{array}{c} \\ x \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} z & 0 & 1 & 0 \\ 1 & 1 & 0 & z+1 \\ 1 & 0 & 1 & z \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}, \quad M_2(1): \begin{array}{c} \\ x \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} z+1 & 0 & 1 & 1 \\ 1 & 1 & 0 & z+1 \\ 1 & 0 & 1 & z \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array},$$

$$M_3(0): \begin{array}{c} \\ x \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & z+1 \\ 1 & 0 & 1 & z \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}, \quad M_4(1): \begin{array}{c} \\ x \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & z+1 \\ 1 & 1 & 0 & z+1 \\ 1 & 0 & 1 & z \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

Case 2.2. *M contains no W_3-minors.*

(This case follows easily from results of Oxley [16], who gives a complete characterization of the ternary matroids without a W_3-minor. However, for completeness, we provide a direct proof.)

We may assume that M is not isomorphic to $\mathcal{W}^4$ and has no F_7^- or O_7-minor. (O_7 has a W_3-minor.) So, by Lemma 3.3 and duality, we may assume that M has an element u such that $M/u \cong P_7$. Let v be the unique element in M/u such that $M/u\backslash v \cong U_{2,4} \oplus_2 U_{2,4}$. Consider the ternary matroids $M_i(\alpha)$ with the following ternary representations:

$$M_1(\alpha): \begin{array}{c} \\ u \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & v \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \end{array}, \quad M_2(\alpha): \begin{array}{c} \\ u \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & v \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & \alpha \\ 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \end{array},$$

$$M_3(\alpha): \begin{array}{c} \\ u \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & v \end{array} \\ \begin{pmatrix} 1 & 1 & -1 & \alpha \\ 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \end{array}, \quad M_4(\alpha): \begin{array}{c} \\ u \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & v \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & \alpha \\ 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

Then $M \cong M_i(\alpha)$ for some $i = 1, ..., 4$ and $\alpha \in GF(3)$. Indeed, $M\backslash v$ is either a series-extension of $U_{2,4} \oplus_2 U_{2,4}$ or isomorphic to P_7^*. As the automorphism group of M/u is transitive on $\{1, ..., 6\}$, we may assume that in the first case u is in series with 4 in $M\backslash v$, so that $M \cong M_1(\alpha)$ for some α. As the automorphism group of M/u is transitive on pairs of lines through

v, there are, up to symmetry, three possibilities for $M\backslash v$ to be isomorphic to P_7^*. These lead to $M \cong M_2(\alpha)$, $M \cong M_3(\alpha)$, or $M \cong M_4(\alpha)$.

Now, $M_1(0)$ is not 3-connected, $M_3(0) \cong P_8$, and $M_3(-1) \cong M_3(1)^*$. Moreover, $M_1(1)/1\backslash 2$, $M_1(-1)/v\backslash 5$, $M_2(0)/3\backslash 5$, $M_2(-1)/v\backslash 6$, $M_3(1)/1\backslash u$, $M_4(0)/3\backslash 1$, $M_4(1)/6\backslash 5$, and $M_4(-1)/2\backslash 4$ are all isomorphic to W_3. Finally, $M_2(1)$ is $GF(4)$-representable with representation

$$\begin{array}{c} \\ u \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & v \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & z & 1 & 1 \\ z+1 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

APPENDIX

Below we describe the excluded minors, as well as some of their interesting properties. The class of excluded minors for **F**-representability is closed not only under duality, as observed by Akkari and Oxley [1], also under delta–wye (and wye–delta) transformations. The only non-$GF(4)$-representable matroids that are minimal with respect to taking minors and performing wye-delta transformations are $U_{2,6}$, F_7^-, P_8, and P_8''.

$\mathbf{U_{2,6}}$, $\mathbf{U_{4,6}}$. The 6-point line and its dual. $U_{2,6}$ has the following (standard) **F**-representation, where a, b, c are distinct elements of $\mathbf{F}\backslash\{0, 1\}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & a & b & c \end{pmatrix}.$$

- **F**-representable if and only if $|\mathbf{F}| \geqslant 5$.
- $U_{4,6}$ can be obtained from $U_{2,6}$ by two delta–wye transformations.

$\mathbf{P_6}$ has the following **F**-representation, where a, b, and c are elements of $\mathbf{F}\backslash\{0, 1\}$ and c is not equal to a, b, or ab:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & a \\ 1 & b & c \end{pmatrix}.$$

- The 6-element simple rank-3 matroid with a single 3-point line (see Fig. 14).
- **F**-representable if and only if $|\mathbf{F}| \geqslant 5$.

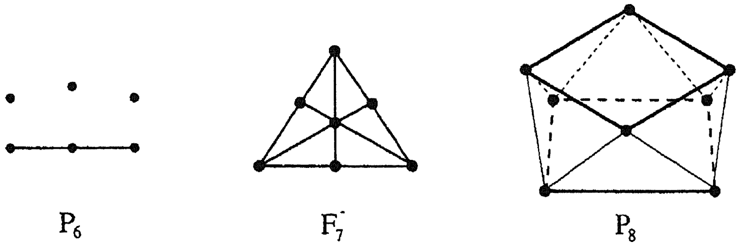

FIG. 14. Some excluded minors.

- P_6 can be obtained from $U_{2,6}$ by a delta–wye transformation.
- Self-dual.

$\mathbf{F}_7^-, (\mathbf{F}_7^-)^*$. The non-Fano and its dual. F_7^- has the following $\mathbf{F}$-representation, where $\mathbf{F}$ is a field of characteristic different from 2:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

- See Fig. 14 for a geometric representation of F_7^-.
- $\mathbf{F}$-representable if and only if $\mathbf{F}$ has characteristic different from two.
- F_7^- is the unique relaxation of the Fano matroid (F_7).
- $(F_7^-)^*$ can be obtained from F_7^- by a delta-wye transformation.

$\mathbf{P}_8$ has the following $\mathbf{F}$-representation,[2] where $\mathbf{F}$ is a field of characteristic different from 2:

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}.$$

- To obtain a geometric representation of P_8 over the reals, take a 3-dimensional cube and rotate a face of the cube 45° (in its plane); then the vertices become points of P_8 (see Fig. 14).
- $\mathbf{F}$-representable if and only if $\mathbf{F}$ has characteristic different from 2 (see Oxley [15]).

[2] The $GF(3)$-representation of P_8 on page 512 of [17] has a misprint.

- Self-dual.
- Transitive automorphism group.

$\mathbf{P}_8''$ has the following (standard) $\mathbf{F}$-representation, where a and b are distinct elements of $\mathbf{F}\backslash\{0, 1\}$ and $a \neq b^{-1}$:

$$A := \begin{array}{c c} & \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & b^{-1} & a \\ 1 & a & 0 & a \\ 1 & b & 1 & 0 \end{pmatrix} \end{array}.$$

- P_8'' can be obtained by relaxing the unique pair of disjoint circuit-hyperplanes of P_8.
- $\mathbf{F}$-representable if and only if $|\mathbf{F}| \geqslant 5$.
- Self-dual.
- Transitive automorphism group.

We conclude by showing that P_8 and P_8'' are in fact excluded minors. Let $M_{a,b}$ denote the matroid represented by the matrix A (above), where a, b are elements of $\mathbf{F}\backslash\{0, 1\}$, but where we possibly allow $a = b$ and/or $a = b^{-1}$. By considering the 1×1 and 2×2 singular submatrices of A, it is clear that, by elementary row operations and column scaling, we can put any representation of $M_{a,b}$ into the same form as A. There are just two square submatrices of A that are singular for some, but not all, choices of a and b from $\mathbf{F}\backslash\{0, 1\}$; these are

$$A_1 := \begin{array}{c c} & \begin{array}{cc} 5 & 6 \end{array} \\ \begin{array}{c} 3 \\ 4 \end{array} & \begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix} \end{array} \quad \text{and} \quad A_2 := \begin{array}{c c} & \begin{array}{cc} 7 & 8 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 1 \\ b^{-1} & a \end{pmatrix} \end{array}.$$

A_1 is singular if and only if $a = b$, and A_2 is singular if and only if $a = b^{-1}$. Exactly one of the two equations $a = b$ and $a = b^{-1}$ is satisfied by a given pair $a, b \in GF(4)\backslash\{0, 1\}$. P_8 is the matroid obtained by insisting that both equations are satisfied, and P_8'' is the matroid obtained when neither is satisfied. Therefore neither P_8 nor P_8'' is $GF(4)$-representable.

It remains to check that proper minors of P_8 and P_8'' are $GF(4)$-representable. Note that any such minor is a minor of one of the two matroids, $M_{a,a}$ and $M_{a,a^{-1}}$ $(a \neq 0, 1)$, obtained by insisting that exactly one of A_1 and A_2 is singular. By the discussion above, these matroids are both $GF(4)$-representable (in fact they are isomorphic). Hence, all proper minors of P_8 and P_8'' are $GF(4)$-representable as well.

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