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ASYMPTOTIC DENSITY IN A COALESCING RANDOM WALK MODEL

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We consider a system of particles, each of which performs a continuous time random walk on $\mathbb{Z}^d$. The particles interact only at times when a particle jumps to a site at which there are a number of other particles present. If there are j particles present, then the particle which just jumped is removed from the system with probability p_j. We show that if p_j is increasing in j and if the dimension d is at least 6 and if we start with one particle at each site of $\mathbb{Z}^d$, then $p(t) := P\{\text{there is at least one particle at the origin at time } t\} \sim C(d)/t$. The constant $C(d)$ is explicitly identified. We think the result holds for every dimension $d \geq 3$ and we briefly discuss which steps in our proof need to be sharpened to weaken our assumption $d \geq 6$.

The proof is based on a justification of a certain mean field approximation for $dp(t)/dt$. The method seems applicable to many more models of coalescing and annihilating particles.

1. Introduction. Annihilating and coalescing random walks were studied as simple interacting particle systems by Bramson and Griffeath (1980), and Arratia (1981). They considered the following systems. Particles move according to a continuous time random walk on $\mathbb{Z}^d$. The particles only interact when a particle at some site x jumps to a site y which already contains a particle. At this time, the two particles annihilate each other and disappear from the system, or they coalesce to only one particle at y, which continues with its random walk until it again coincides with another particle. The former system is called *annihilating random walk* and the latter system is called *coalescing random walk*. In this paper we shall call the above models the *basic models*. These systems first arose as duals to the "antivoter model" and the "voter model" and were used as tools to analyze the voter model [see Holley and Liggett (1975), Harris (1976) and Liggett (1985), Section V.1 and Examples III.4.16, 17]. Further motivation comes from models for chemical reactions. For chemical reactions one often considers particles of two types and allows only particles of different types to annihilate each other (or to form an inert compound). Such systems have received considerable attention in the literature [see Bramson and Lebowitz (1991a, b), (1999) and Lee and Cardy

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(1995), (1997)]. Here we shall restrict ourselves to systems with particles of one type only.

Usually one starts at time 0 with one particle at each site of $\mathbb{Z}^d$, although some results are valid for more general translation invariant initial states. It is further common to let the particles move according to continuous time simple random walk. That is, the particle jumps at the times of a rate 1 Poisson process, and when it jumps from position x, then it jumps to any one of the $2d$ neighbors of x with probability $1/(2d)$. For this version of the model, Bramson and Griffeath and Arratia found the asymptotic behavior of

$$p(t) := P\{\mathbf{0} \text{ is occupied at time } t\}.$$

For the coalescing random walk in dimension $d \geq 3$ one has [Bramson and Griffeath (1980)]

$$p(t) \sim \frac{1}{\gamma_d t}, \tag{1.1}$$

where

$$\gamma_d = P\big\{\text{simple random walk on } \mathbb{Z}^d \text{ never returns to the origin after first leaving it}\big\}.$$

For annihilating random walk in $d \geq 3$, Arratia (1981) shows

$$p(t) \sim \frac{1}{2\gamma_d t}. \tag{1.2}$$

These articles also find the asymptotic behavior of $p(t)$ for $d = 1$ or 2, but we shall only be concerned with $d \geq 3$ here. In fact, the proof of our principal result requires $d \geq 6$. Bramson and Griffeath and Arratia base their proof on an ingenious derivation by Sawyer (1979) of the limit distribution of the number of particles in the voter model at time t which have taken their opinion from the same individual as the origin (the so-called patch size). Bramson and Griffeath use the so-called duality between the basic coalescing random walk and the voter model to deduce (1.1) from Sawyer's result. It is not clear how robust Sawyer's derivation is. If one wants to consider small variations in the interaction rules for the particles, then proving an analogue of (1.1) and (1.2) via Sawyer's methods seems very difficult [see also Remark (iv) after the theorem]. On the other hand, there is an intuitively appealing, heuristic derivation of (1.1) and (1.2), which will be shown below. The main purpose of this paper is to turn those heuristic arguments into a rigorous and quite robust proof. We first give this heuristic explanation.

It is not hard to see that the forward equation for $p(t)$ is

$$\frac{d}{dt}p(t) = -P\{\mathbf{0} \text{ and } e_1 \text{ are occupied at time } t\}$$

for the coalescing random walk, and

$$\frac{d}{dt}p(t) = -2P\{\mathbf{0} \text{ and } e_1 \text{ are occupied at time } t\}$$

for the annihilating random walk; here e_1 denote the site $(1, 0, \ldots, 0)$. For brevity we only discuss the coalescing random walk. Now if $\mathbf{0}$ and e_1 are occupied at time t, then the particles at these two sites must have been at some sites x and y, respectively, at the earlier time $t-\Delta$, and the paths of the particles from x to $\mathbf{0}$ and from y to e_1 must not have coincided during $[t-\Delta, t]$. One can expect that if Δ becomes large with t, then only the contributions from pairs x, y far apart will play a role. Note that, in principle, there may be several choices for x, y; we will have to choose $\Delta = o(t)$ in order to make the probability of the existence of several choices for x, y negligible. When x and y are far apart, particles which are at x and y at time $t - \Delta$ should not have "felt each other" before time $t - \Delta$. It therefore seems reasonable to believe that in this case the events

$$\{x \text{ is occupied at time } t - \Delta\} \text{ and } \{y \text{ is occupied at time } t - \Delta\}$$

are nearly independent, so that for Δ chosen properly as a function of t, the dependence between

$$\{\mathbf{0} \text{ is occupied at time } t\} \text{ and } \{e_1 \text{ is occupied at time } t\} \tag{1.3}$$

is almost entirely due to the requirement that the paths from x to $\mathbf{0}$ and from y to e_1 do not coincide during $[t - \Delta, t]$. Let $\{S_s\}_{s\geq 0}, \{S'_s\}_{s\geq 0}, \{S''_s\}_{s\geq 0}$ be independent copies of a continuous time simple random walk starting at $\mathbf{0}$. Then one is led to approximate

$$P\{\mathbf{0} \text{ and } e_1 \text{ are occupied at time } t\}$$

by

$$\begin{aligned}\sum_{x,y} & P\{x \text{ is occupied at } t - \Delta\}P\{y \text{ is occupied at } t - \Delta\} \\ & \times P\{x + S'_\Delta = \mathbf{0}, y + S''_\Delta = e_1, x + S'_s \neq y + S''_s \text{ for } 0 \leq s \leq \Delta\} \\ & = p^2(t-\Delta)\sum_{x,y} P\{x + S'_\Delta = \mathbf{0}, y + S''_\Delta = e_1, \\ & \qquad x + S'_s \neq y + S''_s \text{ for } 0 \leq s \leq \Delta\}.\end{aligned}$$

Let $\{\tilde{S}'_s\}_{s\geq 0}$ and $\{\tilde{S}''_s\}_{s\geq 0}$ be independent copies of the time-reversed random walk. For simple random walk these are again simple random walks, but in general $\tilde{S}'$ satisfies for $0 = s_0 < s_1 < \cdots < s_l = \Delta$, and Borel sets B_i,

$$P\{\tilde{S}'_{s_i} - \tilde{S}'_{s_{i-1}} \in B_i, 1 \leq i \leq l\} = P\{S_{\Delta - s_{i-1}} - S_{\Delta - s_i} \in -B_i, 1 \leq i \leq l\}. \tag{1.4}$$

The same relation holds when $\tilde{S}'$ is replaced by $\tilde{S}''$. By time reversal one then has

$$P\{x + S'_\Delta = \mathbf{0},\, y + S''_\Delta = e_1,\, x + S'_s \neq y + S''_s \text{ for } 0 \le s \le \Delta\}$$
$$= P\{\tilde{S}'_\Delta = x,\, e_1 + \tilde{S}''_\Delta = y,\ \tilde{S}'_s \neq e_1 + \tilde{S}''_s \text{ for } 0 \le s \le \Delta\}.$$

It is an exercise in random walk to show that the right-hand side here is well approximated by

$$P\{\tilde{S}'_\Delta = x\}P\{e_1 + \tilde{S}''_\Delta = y\}P\{\tilde{S}'_s \neq e_1 + \tilde{S}''_s \text{ for } 0 \le s \le \Delta\},$$

and of course, for large Δ and simple random walk,

$$P\{\tilde{S}'_s \neq e_1 + \tilde{S}''_s \text{ for } 0 \le s \le \Delta\} \sim P\{\tilde{S}'_s \neq e_1 + \tilde{S}''_s \text{ for } s \ge 0\} = \gamma_d.$$

We will explicitly estimate the errors in Lemmas 11–14, but for now we shall just ignore them. This leads to

$$\begin{aligned} &P\{\mathbf{0} \text{ and } e_1 \text{ are occupied at time } t\} \\ &\quad\approx \gamma_d \sum_x P\{\tilde{S}'_\Delta = x\}p(t-\Delta)\sum_y P\{e_1 + \tilde{S}''_\Delta = y\}p(t-\Delta) \\ &\quad= \gamma_d \sum_x P\{S'_\Delta = -x \text{ and } x \text{ is occupied at } t-\Delta\} \\ &\qquad\quad \times \sum_y P\{S''_\Delta = e_1 - y \text{ and } y \text{ is occupied at } t-\Delta\} \\ &\quad\approx \gamma_d P\{\mathbf{0} \text{ is occupied at } t\}P\{e_1 \text{ is occupied at } t\} = \gamma_d p^2(t), \end{aligned}$$

where $A \approx B$ means that $A - B$ is negligible for our purposes. From these relations we can expect $p(t)$ to behave asymptotically like the solution of the equation

$$\frac{d}{dt}y(t) = -\gamma_d y^2(t)$$

which vanishes at ∞, namely,

(1.5) $$y(t) = \frac{1}{\gamma_d t}.$$

This is the heuristic reason for (1.1).

It is precisely these approximations which our paper makes rigorous. To show the power of our method we treat the model in which the particles perform a continuous time random walk, but in which particles only coalesce with a probability which may be less than 1. As far as we know this model has not been analyzed before. Specifically, let $\{S_t\}_{t\ge 0}$ be a continuous time random walk starting at $\mathbf{0}$. We denote by $q(y)$ the probability that $S_\bullet$ has a jump of size y when it jumps; thus,

(1.6) $$q(\mathbf{0}) = 0.$$

Throughout we assume that the random walk is genuinely d-dimensional, that is,

(1.7) the support of $q(\cdot)$ contains d linearly independent vectors.

Assume that the motion of a particle starting at x is distributed like $\{x+S_t\}$, independent of the motion of all other particles. However, if a particle jumps to a site which already contains j particles, then it coalesces with one of these j particles with a certain probability p_j. For our purposes, it is simpler to say that the particle which jumps is removed from the system, and (with the exception of the proofs of Lemmas 9 and 14) we shall follow this convention. (Of course there are other problems for which one wants to keep track of the mass of particles. One then assumes that when two particles of masses m_1 and m_2 coalesce, they form a particle of mass m_1+m_2. However, we shall not do this and only consider the number of particles at a site.)

Our principal result is the following theorem.

THEOREM. *Assume that*

$$p_0 = 0, \qquad p_1 > 0 \tag{1.8}$$

and that

$$p_j \textit{ is increasing in } j. \tag{1.9}$$

Assume further that the particles perform continuous time random walks which are distributed as translates of $\{S_t\}$, *that* (1.6) *and* (1.7) *are satisfied and that*

$$ES_t = t\sum_{y\in\mathbb{Z}^d} yq(y) = \mathbf{0} \quad \textit{and} \quad \sum_{y\in\mathbb{Z}^d}\|y\|^2 q(y) < \infty. \tag{1.10}$$

Finally, assume $d \geq 6$. *Then in the above coalescing model there exists a* $\zeta = \zeta(d) > 0$ *such that*

$$p(t) - \frac{1}{C(d)t} = O\left(\frac{1}{t^{1+\zeta}}\right), \qquad t\to\infty, \tag{1.11}$$

with

$$\begin{aligned} C(d) &= p_1 \sum_{m=0}^{\infty}(1-p_1)^m \\ &\quad \times P\{S^{\sigma}_{\cdot} \textit{ returns exactly } m \textit{ times to } \mathbf{0} \textit{ after first leaving it}\} \\ &= \frac{p_1\gamma}{1-(1-p_1)(1-\gamma)}, \end{aligned} \tag{1.12}$$

where $S^{\sigma}_{\cdot}$ *is the difference of two independent copies of* $S_{\cdot}$, *and* γ *is the probability that* $S^{\sigma}_{\cdot}$ *never returns to the origin after first leaving it. Also*

$$E\{\textit{number of particles at } \mathbf{0} \textit{ at time } t\} - \frac{1}{C(d)t} = O\left(\frac{1}{t^{1+\zeta}}\right) \tag{1.13}$$

and

$$P\{\textit{there are at least } 2 \textit{ particles at } \mathbf{0} \textit{ at time } t\}$$

(1.14) $$= O\left(\frac{1}{t^2}\right), \qquad t \to \infty.$$

REMARKS. (i) It is crucial for our theorem that (1.8) holds. If $p_0 > 0$, then $p(t)$ will usually decrease exponentially in t. If $p_0 = p_1 = 0$, then $p(t)$ will usually decrease like $t^{-\rho}$ for some $\rho < 1$. Models with $p_0 = p_1 = 0$ are presently being investigated by D. M. Stephenson.

(ii) Although we think that the global structure of our proof is "what it should be," certain steps are not optimal and therefore our proof works only when $d \geq 6$. We believe that the conclusion of our theorem is valid for $d \geq 3$. This is known for the basic coalescing model with $p_0 = 0$, $p_j = 1$, $j \geq 1$ [see Bramson and Griffeath (1980)]. For the basic coalescing model our proof, too, can be improved (and even shortened) to work for all $d \geq 3$. If $p_0 = 0 < p_1 \leq p_2 \leq \cdots \leq p_M = p_{M+j} = 1$ for some finite M and all $j \geq 1$, then (with a lot of extra work) (1.11) can still be proved for $d \geq 4$. We hope to return to these improvements in a separate paper; see also Remark (vii) in Section 3.

(iii) The heuristics above form a basic outline of our proof. The principal technical tool to estimate the correlation between events such as in (1.3) is a bound on the variance of

$$\sum_x \beta(x)\xi_t(x)$$

for suitable $\beta(\cdot)$. This variance estimate is derived in Section 3 by what is sometimes called the "method of bounded differences."

(iv) We point out that we only consider the expected number of particles at the origin at time t, or the probability that there is at least one such particle. We do not keep track of how many particles have coalesced to form the particles at $\mathbf{0}$ at time t. More specifically, one can define the *mass* of a surviving particle by taking the mass of each particle at time 0 equal to 1, and by taking the mass of a particle which arises when two particles of masses m_1 and m_2 coalesce equal to $m_1 + m_2$. If $M(t)$ denotes the total mass of the particles at $\mathbf{0}$ at time t, then the result of Sawyer (1979) for the basic coalescing model is equivalent to an exponential limit law for $p(t)M(t)$, conditioned on $\{M(t) > 0\} = \{\mathbf{0}$ is occupied at time $t\}$ (when $d \geq 2$). For our more general models we do not know how to prove such a conditional limit law for $p(t)M(t)$, even though we believe that such a conditional limit theorem still holds. However, even if we could prove such a limit law, we do not see how to use the method of Bramson and Griffeath (1980) to deduce the asymptotic behavior of $E(t)$ and $p(t)$ from this. This is so because Bramson and Griffeath use the Markov property for the dual model of the coalescing random walk (see their Lemma 2). We do not know how to construct a useful dual to our more general model. We therefore have not pursued limit laws for $M(t)$, even though this is an interesting problem in its own right.

Another related interesting problem is the spatial structure of the collection of particles at time 0 which—through coalescence—end at the origin at time t. For the basic model this is investigated by Bramson, Cox and LeGall (1998).

2. Description and construction of the Markov process. Before we start any work we point out that there is no loss of generality in assuming that

(2.1) the group generated by the support of $q(\cdot)$ is all of $\mathbb{Z}^d$.

Spitzer (1976) calls a random walk with this property "aperiodic." To see that we may indeed take our random walk aperiodic, note that Proposition 7.1 of Spitzer (1976) shows that [under (1.7)] there exist linearly independent vectors $v_1, \dots, v_d$ in $\mathbb{Z}^d$ such that the group generated by $\text{supp}(q(\cdot))$ is precisely the group G generated by $v_1, \dots, v_d$, that is $G = \{k_1 v_1 + \cdots + k_d v_d: k_i \in \mathbb{Z}\}$. If a random walk with jump probabilities $q(\cdot)$ starts at a point $v_0 \in \mathbb{Z}^d$ then it will stay in $v_0 + G$ forever. Thus, $p(t)$ and $E(t) = E\{\text{number of particles at } \mathbf{0} \text{ at time } t\}$ are not influenced by any of the particles starting outside G. We may therefore start with a particle at each site of G only. If we then express the positions of all particles in the basis $v_1, \dots, v_d$, then in this new system (2.1) holds.

Since in our system of random walks there can be arbitrarily many particles at a given site, the standard existence theorems do not seem to cover the present set-up. We therefore prove in this section that there exists a Markov process which corresponds to the intuitive description given just before the Theorem in Section 1. After Lemma 1 our arguments closely follow Liggett and Spitzer (1981) or Liggett [(1985), Section IX.1]. A reader who is not worried about existence questions can safely skip the material in this section after Lemma 1.

Throughout the p_j are fixed. For the mere construction of the Markov process the monotonicity condition (1.9) is not needed. However, we do use (1.9) to establish some desirable properties of our Markov process. On the initial state and the random walks by which the particles move, we only put the weak restriction that $\xi_0 \in \Xi$ [see (2.10)] plus the dimension condition (1.7).

The state space of our Markov process will be a subset of

$$\Xi_0 := \{0, 1, \dots\}^{\mathbb{Z}^d}.$$

A generic point of Ξ_0 is denoted by $\xi = \{\xi(x): x \in \mathbb{Z}^d\}$. Here ξ_t denotes the state of our system at time t. Its x-coordinate is denoted by $\xi_t(x)$ or sometimes as $\xi(x, t)$; it represents the number of particles at position x at time t. The most useful construction of the process for our purpose is essentially one based on a graphical representation, as discussed in Griffeath (1979). Let $\tau_1(x, k) < \tau_2(x, k) < \cdots$ be the jumptimes of a Poisson process $\{\mathscr{N}_t(x, k)\}_{t\ge 0}$ (with $\mathscr{N}_0(x, k) = 0$). Set $\tau_0(x, k) = 0$. We assume that

(2.2) all processes $\mathscr{N}(x, k)$, $x \in \mathbb{Z}^d$, $k \geq 1$, are independent.

Without the interaction, each particle would perform a continuous time random walk which jumps at the times of a rate 1 Poisson process, and when it jumps from position x, then it jumps to y with probability $q(y-x) \geq 0 (q(\mathbf{0}) = 0, \sum_z q(z) = 1)$. We denote a random walk with these jump probabilities and which starts at the origin by $\{S_t\}_{t \geq 0}$.

We now attach to each jump time $\tau_n(x, k)$ of the Poisson process $\mathscr{N}(x, k)$ a position $y = y_n(x, k)$ and a collection of random variables $X(n, x, k, j)$, $j \geq 0$. The y here will specify the position to which a particle will jump from x [if any particle will jump from x at time $\tau_n(x, k)$]. $X(n, x, k, j)$ takes the value 1 or 0, and specifies whether a particle which jumps from x at time $\tau_n(x, k)$ is removed from the system or not. If there are j particles present at $y_n(x, k)$ at time τ_n (that is, $\xi(y, \tau_n -) = j$), then the particle which jumps from x to y at τ_n is removed from the system if and only if $X(n, x, k, j) = 1$. We take our sample paths right continuous, so if a particle is removed at τ, then it is not counted in ξ_τ. We assume that

$$\text{all } y_n(x, k) \text{ and } X(n, x, k, \cdot) \text{ for different } (n, x, k) \text{ are independent of each other and of all Poisson processes.} \tag{2.3}$$

Further, for fixed (n, x, k),

$$y_n(x, k) \text{ and } X(n, x, k, \cdot) \text{ are independent,} \tag{2.4}$$

but the $X(n, x, k, j)$ for different j are coupled. We let $U(n, x, k)$, $x \in \mathbb{Z}^d$, n, $k \geq 1$, be a family of uniform random variables on $[0, 1]$ which are independent of all y's and of all Poisson processes $\mathscr{N}$. We then define the joint distribution of $y_n(x, k)$ and $U(n, x, k)$ by

$$P\{y_n(x, k) = y, U(n, x, k) \leq \lambda\} = q(y - x)\lambda, \qquad 0 \leq \lambda \leq 1. \tag{2.5}$$

Further,

$$X(n, x, k, j) = 1 \quad \text{if and only if } U(n, x, k) \leq p_j. \tag{2.6}$$

In particular,

$$P\{X(n, x, k, j) = 1\} = p_j. \tag{2.7}$$

To make the description of our Markov process complete we have to tell when particles jump. The intuitive description is that if there are l particles at x at a certain time t, then the next jump from x occurs at the first jump of one of the processes $\mathscr{N}(x, k)$ with $1 \leq k \leq l$. If that jump is at time $\tau_n(x, k)$, then the particle jumps to $y = y_n(x, k)$ and is removed if and only if $X(n, x, k, j) = 1$, where $j = \xi(y, \tau_n(x, k)-)$ is the number of particles at y at time $\tau_n(x, k)-$.

If our initial state is a finite state, that is, a state with only finitely many particles present, then there is no difficulty in formalizing the above description. Indeed if we start with n_0 particles, then at all times there are at most n_0 particles present, and therefore with probability 1 the times at which any of the existing particles jumps have no finite accumulation point. On the null set on which there is an accumulation point we can give any value to ξ_t; for instance we can take $\xi_t(x) = 0$ for all x and t greater than or equal to the first accumulation point of the jump times for the existing particles. We do not give any further details but take it for granted that for any finite initial state, the Markov process $\{\xi_t\}$ is completely specified by the description in the preceding paragraph. In fact, this gives us a definition of ξ_t as a function of the initial state ξ_0, all the $\tau_n(x, k)$, $y_n(x, k)$ and the $X(n, x, k, j)$, $x \in \mathbb{Z}^d$, $n, k \geq 1$, $j \geq 0$. ξ_t is with probability 1 defined simultaneously for all finite initial states (note that there are only countably many finite states). It will be necessary on occasion to consider ξ_t for various initial states. If we have to indicate the initial state explicitly we shall write $\xi_t(\eta)$ for the process with initial state η. Of course $\xi_t(\eta)$ is also a function of the $\mathscr{N}$, y_n and the X's, but we do not indicate this in the notation. In accordance with this notation, $Ef(\xi_t(\eta))$ is the expectation of $f(\xi_t)$ over all the $\mathscr{N}$, y_n, $X(n, x, k, j)$ when the initial state $\xi_0 = \eta$. For the time being this is only meaningful for a finite state η.

Extra work is needed to define the ξ-process when we allow infinitely many particles in the system. To describe the state space when we allow infinitely many particles, we introduce the norms

$$N_t(\xi) := \sum_{x \in \mathbb{Z}^d} |\xi(x)|\alpha_t(x), \qquad t > 0, \tag{2.8}$$

where

$$\alpha_t(x) = P\{S_t = -x\} \tag{2.9}$$

(this makes sense for any $\xi \in \mathbb{Z}^{\mathbb{Z}^d}$). We take as state space for our process the space

$$\Xi := \{\xi \in \Xi_0 \colon N_t(\xi) < \infty \text{ for all } t > 0\}. \tag{2.10}$$

For any $\eta \in \Xi$ we let $\eta^{(N)}$ be the *finite* state given by

$$\eta^{(N)}(x) = \eta(x)I[|x| \leq N]. \tag{2.11}$$

For $\xi_0 \in \Xi$ we can then form the process $\xi_t(\xi_0^{(N)})$ (that is, we first truncate ξ_0 to a finite state and then construct the Markov process with the truncated state as its initial state). We are going to show that the process with the initial state ξ_0 can be defined as $\xi_t = \lim_{N\to\infty} \xi_t(\xi_0^{(N)})$. The principal estimate used to show that this makes sense is based on a comparison lemma of chains with different finite initial states. Let ξ_0', ξ_0'' and $\xi_0^{\#}$ be finite initial states which satisfy

$$\xi_0'(x) \leq \xi_0^{\#}(x) \leq \xi_0'(x) + \xi_0''(x) \quad \text{for all } x. \tag{2.12}$$

We now take $\{\xi_t'\}$ and $\{\xi_t^{\#}\}$ to be the processes $\{\xi_t(\xi_0')\}$ and $\{\xi_t(\xi_0^{\#})\}$, respectively. We also introduce a process $\{\xi_t''\}$. This will *not* be the process $\{\xi_t(\xi_0'')\}$, but an equivalent process which is coupled with the ξ'-process and the $\xi^{\#}$-process in such a way that

(2.13) the ξ'-process and the ξ''-process are independent.

The three processes are coupled in that they use the same $\mathcal{N}$, y_n and $U(n, \cdot, \cdot)$, as we now specify. In order to describe the three processes together we keep track of the system to which a particle belongs, so that we distinguish #-particles, ′-particles and ″-particles. However, we do not distinguish the particles in a single system, so we really only keep track of the number of particles of each type at each site. These numbers at x at time t are $\xi_t'(x)$, $\xi_t''(x)$ and $\xi_t^{\#}(x)$, respectively. If $\xi_t'(x) = l'$, $\xi_t''(x) = l''$ and $\xi_t^{\#}(x) = l^{\#}$, then a ′-particle jumps from x at the next jump of any of $\mathcal{N}(x, k)$, $1 \le k \le l'$, and a ″-particle jumps at the next jump of any of $\mathcal{N}(x, k)$, $l' < k \le l' + l''$. Also a #-particle jumps at the next jump of any of $\mathcal{N}(x, k)$, $1 \le k \le l^{\#}$. If a particle jumps at time $\tau_n(x, k)$, then it jumps to $y_n(x, k)$. If it is a ′-particle, then it is removed if and only if $X(n, x, k, \xi'_{\tau_n -}(y_n)) = 1$. The corresponding rules with ″ and # instead of ′ hold for ″-particles and #-particles. Note that a ′-particle and a #-particle may jump at the same time. However, with probability 1 there are no times at which both a ′-particle and a ″-particle jump. Thus the ′-process and the ″-process never use the same $y_n(x, k)$ or $U(n, x, k)$ and therefore are independent as claimed in (2.13).

LEMMA 1. *Assume* (1.9). *If* ξ_0', ξ_0'' *and* $\xi_0^{\#}$ *are finite states which satisfy* (2.12), *then, under the above coupling, it holds with probability* 1 *that for all* $t \ge 0$,

$$\xi_t'(x) \le \xi_t^{\#}(x) \le \xi_t'(x) + \xi_t''(x) \quad \text{for all } x \in \mathbb{Z}^d. \tag{2.14}$$

The left-hand inequality remains valid even without (1.9).

PROOF. We shall assume (1.9) and leave it to the reader to verify that this is only needed when proving the right-hand inequality in (2.14).

Let $s_0 = 0$ and define s_i, $i \ge 1$, recursively as follows. First let $x_1^{(i)}$, $x_2^{(i)}, \dots, x_{p(i)}^{(i)}$ be the finitely many sites with

$$\xi'_{s_i}(x) + \xi''_{s_i}(x) + \xi^{\#}_{s_i}(x) > 0.$$

Then define

$$\begin{aligned} s_{i+1} = {} & \text{first jump time} > s_i \text{ of any } \mathcal{N}(x, k) \\ & \text{with } x \in \{x_1^{(i)}, x_2^{(i)}, \dots, x_{p(i)}^{(i)}\}, \qquad k \le \xi'_{s_i}(x) + \xi''_{s_i}(x). \end{aligned}$$

Now assume that the coupling is such that (2.14) holds for all $t \le s_i$ for some i. We shall prove that (2.14) also holds for $t \le s_{i+1}$. By our construction, $\xi_t'(x)$, $\xi_t''(x)$ and $\xi_t^{\#}(x)$ are all constant for all x and $s_i \le t < s_{i+1}$. [Note that

$$\xi_t^{\#}(x_r^{(i)}) \le \xi_t'(x_r^{(i)}) + \xi_t''(x_r^{(i)})$$

for $t = s_i$, so $\xi_t^{\#}(x_r^{(i)})$ indeed does not jump for $s_i < t < s_{i+1}$.] If

$$s_{i+1} = \tau_{n(i+1)}(x_r^{(i)}, k(i+1)),$$

then some particle jumps at time s_{i+1} from $x_r^{(i)}$ to $y_{n(i+1)}(x_r^{(i)}, k(i+1))$, but for $x \neq x_r^{(i)}, y_{n(i+1)}(x_r^{(i)}, k(i+1))$, none of $\xi'_t(x)$, $\xi''_t(x)$, $\xi^\#_t(x)$ change at $t = s_{i+1}$. In order to prove (2.14) for $t \le s_{i+1}$, we therefore only have to, check that (2.14) again holds right after the jump at $t = s_{i+1}$ for $x = x_r^{(i)}$ and for $x = y_{n(i+1)}(x_r^{(i)}, k(i+1))$. We distinguish three cases:

(a) $1 \le k(i+1) \le \xi'_{s_i}(x_r^{(i)})$;

(b) $\xi'_{s_i}(x_r^{(i)}) < k(i+1) \le \xi^\#_{s_i}(x_r^{(i)})$;

(c) $\xi^\#_{s_i}(x_r^{(i)}) < k(i+1) \le \xi'_{s_i}(x_r^{(i)}) + \xi''_{s_i}(x_r^{(i)})$.

By (2.14) for $t = s_i$, these are the only possibilities.

CASE (a). In this case a ′-particle and a #-particle jump simultaneously from $x_r^{(i)}$ to $y_{n(i+1)} = y_{n(i+1)}(x_r^{(i)}, k(i+1))$ [because we also have $k(i+1) \le \xi^\#_{s_i}(x_r^{(i)})$, by virtue of (2.14)]. However, no ″-particle jumps. The particle which jumps is removed from the system in the ′-system if and only if

$$X' := X(n(i+1), x_r^{(i)}, k(i+1), \xi'_{s_i}(y_{n(i+1)})) = 1 \tag{2.15}$$

and similarly in the #-system. Therefore,

$$\xi'_{s_{i+1}}(x_r^{(i)}) = \xi'_{s_i}(x_r^{(i)}) - 1,$$
$$\xi''_{s_{i+1}}(x_r^{(i)}) = \xi''_{s_i}(x_r^{(i)}),$$
$$\xi^\#_{s_{i+1}}(x_r^{(i)}) = \xi^\#_{s_i}(x_r^{(i)}) - 1.$$

Also

$$\xi'_{s_{i+1}}(y_{n(i+1)}) = \xi'_{s_i}(y_{n(i+1)}) + 1 - X',$$
$$\xi''_{s_{i+1}}(y_{n(i+1)}) = \xi''_{s_i}(y_{n(i+1)}),$$
$$\xi^\#_{s_{i+1}}(y_{n(i+1)}) = \xi^\#_{s_i}(y_{n(i+1)}) + 1 - X^\#.$$

It is clear from the first set of these relations that (2.14) still holds at $t = s_{i+1}$, $x = x_r^{(i)}$. From the second set of relations we see immediately that the left-hand inequality in (2.14) also holds at $t = s_{i+1}$, $x = y_{n(i+1)}$ if $\xi^\#_{s_i}(y_{n(i+1)}) > \xi'_{s_i}(y_{n(i+1)})$. And if $\xi^\#_{s_i}(y_{n(i+1)}) = \xi'_{s_i}(y_{n(i+1)}) = \xi'_{s_i}$ say, for short, then

$$\begin{aligned} X' &= X(n(i+1), x_r^{(i)}, k(i+1), \xi'_{s_i}) \\ &= X(n(i+1), x_r^{(i)}, k(i+1), \xi^\#_{s_i}) = X^\#, \end{aligned} \tag{2.16}$$

so that even in this case the left-hand inequality of (2.14) holds at $t = s_{i+1}$, $x = y_{n(i+1)}$.

The right-hand inequality in (2.14) follows by noticing that under (1.9),

$$X(n, x, k, j) \text{ is increasing in } j \tag{2.17}$$

[see (2.6)]. Thus, (2.14) at $t = s_i$ and the definition of X', $X^\#$ [compare (2.15)] show that $X' \le X^\#$. Hence (2.14) holds for $t \le s_{i+1}$ in Case (a).

CASE (b). Now no $'$-particle jumps, but a #-particle and a $''$-particle jump from $x_r^{(i)}$ to $y_{n(i+1)} = y_{n(i+1)}(x_r^{(i)}, k(i+1))$. The #-particle will be removed from the system if $X^\# = 1$ and similarly for the $''$-particle. This time we therefore have

$$\begin{aligned} \xi'_{s_{i+1}}(x_r^{(i)}) &= \xi'_{s_i}(x_r^{(i)}), \\ \xi''_{s_{i+1}}(x_r^{(i)}) &= \xi''_{s_i}(x_r^{(i)}) - 1, \\ \xi^\#_{s_{i+1}}(x_r^{(i)}) &= \xi^\#_{s_i}(x_r^{(i)}) - 1. \end{aligned}$$

Also

$$\begin{aligned} \xi'_{s_{i+1}}(y_{n(i+1)}) &= \xi'_{s_i}(y_{n(i+1)}), \\ \xi''_{s_{i+1}}(y_{n(i+1)}) &= \xi''_{s_i}(y_{n(i+1)}) + 1 - X'', \\ \xi^\#_{s_{i+1}}(y_{n(i+1)}) &= \xi^\#_{s_i}(y_{n(i+1)}) + 1 - X^\#. \end{aligned}$$

The right-hand inequality in (2.14) at $t = s_{i+1}$, $x = x_r^{(i)}$ is clear from the former set of equations. The left-hand inequality can only go wrong if $\xi'_{s_i}(x_r^{(i)}) = \xi^\#_{s_i}(x_r^{(i)})$, but this is impossible in Case (b). The left-hand inequality in (2.14) at $t = s_{i+1}$, $x = y_{n(i+1)}$ is immediate from the last set of equations. Finally, the right-hand inequality in (2.14) at $t = s_{i+1}$, $x = y_{n(i+1)}$ is again obvious if $\xi'_{s_i}(y_{n(i+1)}) + \xi''_{s_i}(y_{n(i+1)}) > \xi^\#_{s_i}(y_{n(i+1)})$. If we have equality here, then $\xi''_{s_i}(y_{n(i+1)}) \le \xi^\#_{s_i}(y_{n(i+1)})$ and therefore $X'' \le X^\#$ by (2.17). Thus (2.14) at $t = s_{i+1}$, $x = y_{n(i+1)}$ again holds in this case.

CASE (c). Now only a $''$-particle jumps from $x_r^{(i)}$ to $y_{n(i+1)}$. We leave the simple verification of (2.14) at $t = s_{i+1}$ in this case to the reader.

We now have proved that (2.14) holds for $t \le s_{i+1}$ in all cases and therefore (2.14) holds by induction for all $t \le \lim_{i\to\infty} s_i$. However, let

$$\begin{aligned} \mathscr{F}_s = \sigma\text{-field generated by all } \mathscr{N}_u(x, k) \text{ for } u \le s \text{ and all } y_n(x, k) \\ \text{and } U(n, x, k) \text{ attached to some } \tau_n(x, k) \le s. \end{aligned} \tag{2.18}$$

Then the conditional distribution of $s_{i+1} - s_i$ given $\mathscr{F}_{s_i}$ is exponential with mean

$$\frac{1}{\sum_{x\in\mathbb{Z}^d}[\xi'_{s_i}(x) + \xi''_{s_i}(x)]} \ge \frac{1}{\sum_{x\in\mathbb{Z}^d}[\xi'_0(x) + \xi''_0(x)]}.$$

Consequently, with probability 1, $s_i \to \infty$ and (2.14) holds for all $t \ge 0$. □

The same argument as for the right-hand inequality of (2.14) shows that if (1.9) holds and if we have finite initial states $\xi_0(\cdot), \xi_0(\cdot;1), \dots, \xi_0(\cdot;r)$ such that

$$\xi_0(x) \le \sum_{i=1}^{r} \xi_0(x;i) \quad \text{for all } x, \tag{2.19}$$

then there exist independent processes $\xi_t(\cdot;1), \dots, \xi_t(\cdot;r)$ so that $\{\xi_t(\cdot)\}_{t\ge 0}$, $\{\xi_t(\cdot;i)\}_{t\ge 0}$ have the same distribution as $\{\xi_t(\xi_0(\cdot))\}_{t\ge 0}$ and $\{\xi_t(\xi_0(\cdot;i))\}_{t\ge 0}$, respectively, and so that

$$\xi_t(x) \le \sum_{i=1}^{r} \xi_t(x;i). \tag{2.20}$$

In particular, (2.19) implies

$$E\xi_t\Bigl(\sum_{1}^{r} \xi_0(\cdot;i)\Bigr)(x) \le \sum_{1}^{r} E\xi_t(\xi_0(\cdot;i))(x). \tag{2.21}$$

The next lemma compares processes with the same initial states, but with different collections of p_j. We shall not need the full strength of (1.9) but instead that

$$p_0 = 0. \tag{2.22}$$

The largest and smallest p_j which satisfy this side condition are

$$p_j^* := \begin{cases} 0, & \text{if } j = 0, \\ 1, & \text{if } j > 0 \end{cases} \tag{2.23}$$

and

$$\bar{p}_j := 0 \quad \text{for all } j,$$

respectively. Correspondingly, we take

$$X^*(n, x, k, j) = \begin{cases} 0, & \text{if } j = 0, \\ 1, & \text{if } j > 0 \end{cases}$$

and

$$\bar{X}(n, x, k, j) = 0, \qquad j \ge 0.$$

Based on these X^* and $\bar{X}$ we can now define processes $\{\xi_t^*(\xi_0)\}_{t\ge 0}$ and $\{\bar{\xi}_t(\xi_0)\}_{t\ge 0}$ for any finite initial state ξ_0. These will use the same $\mathcal{N}$ and y_n as the process $\{\xi_t(\xi_0)\}_{t\ge 0}$ which we have already defined [and which uses $X(n, x, k, j)$ in its construction]. The following lemma compares the coupled processes ξ^*, $\bar{\xi}$ and ξ.

LEMMA 2. *Assume that* (2.22) *holds. Then with probability* 1 *for any finite initial state* ξ_0 *and any* $x \in \mathbb{Z}^d$, $t \ge 0$,

$$\xi_t^*(x) \le \xi_t(x) \le \bar{\xi}_t(x).$$

The right-hand inequality remains valid even without (2.22).

The intuitive content of this lemma is fairly clear. In the ξ^*-process we always remove a particle which jumps to a site which is already occupied. In this process there can be at most one particle at a site and we remove particles at a maximal rate. This yields the smallest process. In the $\bar{\xi}$-process we remove as few particles as possible; that is, we never remove a particle and this process is simply a process of noninteracting random walks. It is the largest process of the type considered here. We shall not prove Lemma 2. The general outline of its proof is the same as for Lemma 1 and, in fact, various cases are easier in this lemma.

We can now show that $\lim_{N\to\infty} \xi_t(\xi_0^{(N)})$ exists with probability 1.

LEMMA 3. *Assume that* (1.9) *holds and that* $\xi_0 \in \Xi$. *With probability* 1 *it holds that for all* $x \in \mathbb{Z}^d$, $t \ge 0$,

$$\xi_t(\xi_0^{(N)})(x) \text{ increases to a finite limit, } \xi_t(x) \quad \text{say.} \tag{2.24}$$

Since $\xi_t(\xi_0^{(N)})(x)$ *is integer valued, this actually means that with probability* 1, *for fixed* x *and* t, $\xi_t(\xi_0^{(N)})(x)$ *is eventually constant in* N.

For $\eta, \lambda \in \Xi$,

$$E\{|\xi_t(\eta)(x) - \xi_t(\lambda)(x)|\} \le \sum_y |\eta(y) - \lambda(y)| P\{y + S_t = x\} \tag{2.25}$$

and

$$EN_s(\xi_t(\eta) - \xi_t(\lambda)) \le N_{s+t}(\eta - \lambda). \tag{2.26}$$

The special case $\eta = \xi_0$, $\lambda(x) \equiv 0$ *gives*

$$EN_s(\xi_t(\xi_0)) \le N_{s+t}(\xi_0) < \infty. \tag{2.27}$$

Finally,

$$P\{\xi_t(\xi_0) \in \Xi \text{ for all } t \ge 0\} = 1. \tag{2.28}$$

PROOF. By Lemma 1 we have for $N < M$ with probability 1 that

$$\xi_t(\xi_0^{(M)})(x) \ge \xi_t(\xi_0^{(N)})(x) \quad \text{for all } x, t,$$

because this inequality holds for $t = 0$. Thus $\xi_t(\xi_0^{(N)})(x)$ is increasing in N and we only have to prove that its limit $\xi_t(x)$ is with probability 1 finite for all t simultaneously, and also satisfies (2.27) and (2.28). We shall not prove that almost surely $\xi_t(x) < \infty$ and the event in (2.28) holds *simultaneously for all* $t \ge 0$. Instead, we only prove (2.27), which implies that for each fixed t and s, almost surely $\xi_t(x) < \infty$ and $N_s(\xi_t(\xi_0)) < \infty$. We then appeal to the simple inequality

$$\alpha_u(x) \ge \alpha_{s+t}(x) P\{S_{u-s-t} = \mathbf{0}\} \ge \exp(-u + s + t)\alpha_{s+t}(x), \qquad u \ge s + t. \tag{2.29}$$

In particular, this shows that

$$N_{s+t}(\xi) \le \exp(u - s - t) N_u(\xi), \qquad \xi \in \Xi, u \ge s + t. \tag{2.30}$$

Thus, if for each fixed s, $N_s(\xi_t) < \infty$ a.s., then it is even true that almost surely $N_s(\xi_t) < \infty$ for all s. Thus, (2.27) will also imply $\xi_t(\xi_0) \in \Xi$ a.s., and (2.27) will be sufficient for our purposes.

For inequality (2.25) we go back to Lemma 1. First let η and λ be finite states. We take $\xi_0'(x) = \eta(x) \wedge \lambda(x)$, $\xi_0^{\#}(x) = \eta(x)$ and $\xi_0''(x) = [\eta(x) - \lambda(x)]^+$. We then construct the processes $\{\xi_t'\}$, $\{\xi_t^{\#}\}$ and $\{\xi_t''\}$ from these initial states as in Lemma 1. We also take $\bar{\xi} = \bar{\xi}(\xi_0'')$ to be a system of noninteracting particles which starts with $\bar{\xi}_0(x) = \xi_0''(x)$ as in Lemma 2 (with ξ_0 replaced by ξ_0''). Lemma 1 then shows that

$$E|\xi_t(\eta)(x) - \xi_t'(x)| = E[\xi_t^{\#}(\eta)(x) - \xi_t'(x)] \le E\xi_t''(x) \le E\bar{\xi}_t(x)$$
$$= \sum_y [\eta(y) - \lambda(y)]^+ \alpha_t(y - x).$$

Similarly,

$$E|\xi_t(\lambda)(x) - \xi_t'(x)| \le \sum_y [\lambda(y) - \eta(y)]^+ \alpha_t(y - x).$$

Adding the last two inequalities gives (2.25), for finite initial states η, λ. In particular (2.25) holds when η and λ are replaced by $\eta^{(N)}$ and $\lambda^{(N)}$, respectively. Then (2.25) for general $\eta, \lambda \in \Xi$ follows from Fatou's lemma if we let $N \to \infty$.

We obtain (2.26) by multiplying (2.25) by $\alpha_s(x)$ and summing over x.

As mentioned in the lemma, (2.27) is a special case of (2.26), and it gives us the promised weaker version of (2.28). □

Now define

$$\mathscr{G}_n = \sigma\text{-field of subsets of } \Xi \text{ generated by the coordinate functions } \xi(x) \text{ with } |x| \le n$$

and

$$\mathscr{G} = \bigvee \mathscr{G}_n.$$

Further, for $\eta \in \Xi$, $B \in \mathscr{G}$, define

$$K_t(\eta, B) = P\{\xi_t(\eta) \in B\}.$$

We also write

$$K_t f(\eta) = K_t(\eta, f) = \int_\Xi K_t(\eta, d\xi) f(\xi),$$

when f is a $\mathscr{G}$-measurable function on Ξ which is nonnegative or for which

$$\int_\Xi K_t(\eta, d\xi)|f(\xi)| < \infty.$$

We want to show that the $K_t(\cdot, \cdot)$ are transition probability kernels which form a semigroup with the "correct" generator. For fixed η, t, $K_t(\eta, \cdot)$ is a probability measure on $\mathscr{G}$. If B is of the form $B = \{\xi \in \Xi: \{\xi(x)\}_{|x|\le n} \in C\}$ with C a subset of $\mathbb{Z}^M$ with $M =$ the number of x with $|x| \le n$, then

$$K_t(\eta, B) = P\{\{\xi_t(\eta)(x)\}_{|x|\le n} \in C\} = \lim_{N\to\infty} P\{\xi_t(\eta^{(N)}) \in B\} \quad \text{(by Lemma 3).}$$

Since $\eta^{(N)}$ can take on only countably many values, $P\{\xi_t(\eta^{(N)}) \in B\}$ is clearly a $\mathscr{G}$-measurable function of η. Therefore, for any fixed $B \in \mathscr{G}_n$, $\eta \mapsto K_t(\eta, B)$ is $\mathscr{G}$-measurable. Standard monotone class arguments show then that this remains valid for all $B \in \mathscr{G}$.

Following Liggett (1985), Section IX.1 or Liggett and Spitzer (1981) we now introduce a class $\mathscr{L}$ of Lipschitz functions. For $f: \Xi \to \mathbb{R}$ we set

$$L_t(f) := \sup_x \sup_{\eta\in\Xi} \frac{|f(\eta + e_x) - f(\eta)|}{\alpha_t(x)},$$

where e_x is the vector with $e_x(y) = 1$ if $y = x$ and 0 otherwise (here we interpret Ξ_0 as a vectorspace in the obvious way). Note that this definition implies

$$\begin{aligned}|f(\eta) - f(\lambda)| &\le L_t(f) \sum_x \alpha_t(x)|\eta(x) - \lambda(x)| \\ &= L_t(f) N_t(\eta - \lambda), \qquad \eta, \lambda \in \Xi.\end{aligned} \tag{2.31}$$

The class $\mathscr{L}$ is now defined as

$$\mathscr{L} = \{f: f \text{ is } \mathscr{G}\text{-measurable and } L_t(f) < \infty \text{ for some } t > 0\}.$$

It is not hard to check that $\mathscr{L}$ contains all bounded cylinder functions. We also note that for $f \in \mathscr{L}$ with $L_{s_0}(f) < \infty$,

$$\begin{aligned}\int_\Xi K_t(\eta, d\xi)|f(\xi)| &\le L_{s_0}(f) \int_\Xi K_t(\eta, d\xi) N_{s_0}(\xi) + f(0) \quad \text{[by (2.31)]} \\ &\le N_{s_0+t}(\eta) + f(0) \quad \text{[by (2.27)]} < \infty.\end{aligned}$$

The following simple lemma shows that K_t does preserve $\mathscr{L}$.

LEMMA 4. *Assume that* (1.9) *holds. Let* $s_0 > 0$ *and* $f \in \mathscr{L}$ *such that* $L_{s_0}(f) < \infty$ *and let* $t \ge 0$. *Then the following hold*:

(a)

$$K_t f(\eta) = \lim_{N\to\infty} Ef(\xi_t(\eta^{(N)})), \qquad \eta \in \Xi; \tag{2.32}$$

(b)

$$|K_t f(\eta) - K_t f(\lambda)| \le L_{s_0}(f) N_{s_0+t}(\eta - \lambda), \qquad \eta, \lambda \in \Xi; \tag{2.33}$$

(c) *if* $u \ge t + s_0$, *then*

$$L_u(K_t f) \le L_{s_0}(f) \exp(u - t - s_0). \tag{2.34}$$

PROOF. (a) By definition of $L_{s_0}(f)$,

$$\begin{aligned}(2.35)\qquad &|K_t f(\eta) - Ef(\xi_t(\eta^{(N)}))| \\ &\quad = |E[f(\xi_t(\eta)) - f(\xi_t(\eta^{(N)}))]| \\ &\quad \le L_{s_0}(f) EN_{s_0}(\xi_t(\eta) - \xi_t(\eta^{(N)})) \quad [\text{by } (2.31)] \\ &\quad \le L_{s_0}(f) N_{s_0+t}(\eta - \eta^{(N)}) \quad [\text{by } (2.26)]\,.\end{aligned}$$

Since $N_{s_0+t}(\eta - \eta^{(N)}) \to 0$ as $N \to \infty$, (2.32) follows.

(b) Analogously to (2.35), the left-hand side of (2.33) is equal to

$$\begin{aligned}(2.36)\qquad &|Ef(\xi_t(\eta)) - Ef(\xi_t(\lambda))| \\ &\quad \le L_{s_0}(f) EN_{s_0}(\xi_t(\eta) - \xi_t(\lambda)) \quad [\text{by } (2.31)] \\ &\quad \le L_{s_0}(f) N_{s_0+t}(\eta - \lambda) \quad [\text{by } (2.26)]\,.\end{aligned}$$

(c) The estimate (2.34) now follows from (2.33) and (2.30). □

The main fact now is that K_t defines a semigroup, as shown in the next lemma.

LEMMA 5. *Assume* (1.9). *If* $\eta \in \Xi$ *and* $f \in \mathscr{L}$, *then*

$$(2.37)\qquad K_{s+t}(\eta, f) = \int_\Xi K_s(\eta, d\xi) K_t(\xi, f).$$

PROOF. By (2.32) the left-hand side equals

$$\lim_{N\to\infty} K_{s+t}(\eta^{(N)}, f).$$

Also, by the Markov property for the process $\{\xi_t(\eta^{(N)})\}$ (which has a bounded number of particles) we have

$$K_{s+t}(\eta^{(N)}, f) = \int_\Xi K_s(\eta^{(N)}, d\xi) K_t(\xi, f).$$

However, $\xi \mapsto K_t(\xi, f)$ is a function in $\mathscr{L}$ [by (2.34)], and therefore, by (2.32) again,

$$\lim_{N\to\infty} \int_\Xi K_s(\eta^{(N)}, d\xi) K_t(\xi, f) = \int_\Xi K_s(\eta, d\xi) K_t(\xi, f). \qquad \square$$

REMARKS. (v) The preceding lemma shows that the $K_t, t \ge 0$, have the semigroup property. Therefore, there exists a Markov process which has the K_t as transition probability operators. Note that the present Lemma 5 does not quite show that the process $\{\xi_t\}$ defined in (2.24) has the Markov property.

For this we would need to show that for $0 < t_1 < t_2 < \cdots < t_k$ and bounded $f_i \in \mathscr{L}$,

$$E\left\{\prod_{i=1}^{k} f_i(\xi_{t_i}(\eta))\right\} = \int_\Xi K_{t_1}(\eta, d\lambda_1) f_1(\lambda_1) \int_\Xi K_{t_2-t_1}(\lambda_1, d\lambda_2) f_2(\lambda_2) \cdots \times \int_\Xi K_{t_k - t_{k-1}}(\lambda_{k-1}, d\lambda_k) f_k(\lambda_k).$$

This can be shown to be the case by a slight extension of the proof of Lemma 5 plus induction on k, but we shall not carry this out here.

(vi) We have not made explicit which statements can be proved without condition (1.9). However, the monotonicity and hence the existence of the limit (as $N \to \infty$) of $\xi_t(\xi_0^{(N)})$ does not rely on (1.9). Also (2.27) can be proved without (1.9) by using the right-hand inequality in Lemma 2 instead of the right-hand inequality in Lemma 1 [which is now used in the proof of (2.26)]. Therefore, the finiteness of ξ_t and (2.28) can be proved without (1.9). The same is true for Lemmas 4(a) and 5, but this requires a somewhat more elaborate argument.

In order to show that a Markov process with the K_t as transition probability operators corresponds to the description given before the theorem in Section 1 we also show that the semigroup of operators K_t has the "correct" generator, at least when applied to functions in $\mathscr{L}$. Formally, the description of our process before the theorem in Section 1 corresponds to the generator

$$\Omega f(\eta) = \sum_x \eta(x) \sum_y q(y-x) \left\{p_{\eta(y)}[f(\eta - e_x) - f(\eta)] + (1 - p_{\eta(y)})[f(\eta + e_y - e_x) - f(\eta)]\right\}. \tag{2.38}$$

We shall define $\Omega f(\eta)$ by (2.38) whenever

$$\sum_x \eta(x) \sum_y q(y-x) \left\{p_{\eta(y)}|f(\eta - e_x) - f(\eta)| + (1 - p_{\eta(y)})|f(\eta + e_y - e_x) - f(\eta)|\right\}$$

converges. We now indicate how to prove a proposition which is an analogue of Lemma 2.16 in Liggett and Spitzer (1981) and Theorem IX.1.14 in Liggett (1985).

PROPOSITION 6. *Assume that* (1.9) *holds and that* $f \in \mathscr{L}$ *satisfies* $L_{s_0}(f) < \infty$ *and that* $\eta \in \Xi$. *Then the following relations hold*:

(a) $$K_{s+t} f = K_s(K_t f);$$

(b) $\Omega(K_t f)(\eta)$ *is well defined and*

$$K_t f(\eta) = f(\eta) + \int_0^t \Omega(K_s f)(\eta)\, ds;$$

(c) $$|K_{t+s} f(\eta) - K_t f(\eta)| \le s e^{t+1} L_{s_0}(f)[N_{s_0+t+1}(\eta) + 2e N_{s_0+t+2}(\eta)],$$
for $0 \le t \le t + s \le t + 1$;

(d) $$t \mapsto \Omega(K_t f)(\eta) \text{ is continuous on } [0,\infty);$$

(e) $$\frac{d}{dt}K_t f(\eta) = \Omega(K_t f)(\eta).$$

[*Of course, at* $t = 0$ *this derivative is the right derivative only.*] *Moreover,* $t^{-1}|K_t f(\eta) - f(\eta)|$ *is bounded for* $0 < t \le 1$ *and* $N_{t+s_0+1}(\eta) \le A$, *for any fixed* $A < \infty$;

(f) $$\Omega(K_t f)(\eta) = K_t(\Omega f)(\eta) = E(\Omega f)(\xi_t(\eta)).$$

PROOF. (a) is immediate from Lemma 5. For (b) we use the Markov property for the ξ-process starting in a finite state (and which consequently has a bounded number of particles at all times). This gives [compare Dynkin (1965), I, equations I.2.1.4 and I.2.1.5]

$$Ef(\xi_t(\eta^{(N)})) = f(\eta^{(N)}) + \int_0^t \Omega(Ef(\xi_s(\eta^{(N)})))ds, \tag{2.39}$$

where $\Omega(Ef(\xi_s(\eta^{(N)})))$ stands for $\Omega(Ef(\xi_s(\cdot)))$ evaluated at $\eta^{(N)}$. We now want to take the limit $N \to \infty$ in (2.39). Note that

$$\begin{aligned}
&\Omega(Ef(\xi_s(\eta^{(N)}))) \\
&\quad = \sum_{|x|\le N} \eta(x) \sum_y q(y-x) \\
&\qquad \times \{p_{\eta^{(N)}(y)}[Ef(\xi_s(\eta^{(N)} - e_x)) - Ef(\xi_s(\eta^{(N)}))] \\
&\qquad + (1 - p_{\eta^{(N)}(y)})[Ef(\xi_s(\eta^{(N)} + e_y - e_x)) - Ef(\xi_s(\eta^{(N)}))]\}.
\end{aligned} \tag{2.40}$$

It follows from $L_{s_0}(f) < \infty$ and (2.32) that the left-hand side of (2.39) converges to $Ef(\xi_t(\eta))$, and of course $f(\eta^{(N)}) \to f(\eta)$ and $N \to \infty$. Similarly, for each fixed x, y,

$$\begin{aligned}
&p_{\eta^{(N)}(y)}[Ef(\xi_s(\eta^{(N)} - e_x)) - Ef(\xi_s(\eta^{(N)}))] \\
&\quad + (1 - p_{\eta^{(N)}(y)})[Ef(\xi_s(\eta^{(N)} + e_y - e_x)) - Ef(\xi_s(\eta^{(N)}))] \\
&\to p_{\eta(y)}[Ef(\xi_s(\eta - e_x)) - Ef(\xi_s(\eta))] \\
&\quad + (1 - p_{\eta(y)})[Ef(\xi_s(\eta + e_y - e_x)) - Ef(\xi_s(\eta))].
\end{aligned}$$

From (2.33), (2.29) and (2.30) we further have the following bound for (2.40) (when $s \le t$):

$$\begin{aligned}
&\sum_x \eta(x) \sum_y q(y-x) L_{s_0}(f)\{\alpha_{s_0+s}(x) + \alpha_{s_0+s}(y)\} \\
&\quad \le L_{s_0}(f) e^t \left\{ N_{s_0+t}(\eta) + \sum_x \eta(x) \sum_y q(y-x)\alpha_{s_0+t}(y) \right\}.
\end{aligned} \tag{2.41}$$

This is even an upper bound for (2.40) if we replace the differences between expectations in square brackets in the right-hand side of (2.40) by their absolute values. Furthermore,

$$
\begin{aligned}
&P\{x+S_u=\mathbf{0} \text{ for some } u\in[s_0+t,s_0+t+1]\}\\
&\quad\ge \sum_y P\{S_{s_0+t}=-y\}P\{S_u=y-x \text{ for some } 0\le u\le 1\\
&\quad\ge \sum_y \alpha_{s_0+t}(y)P\{\text{first jump of } S. \text{ occurs during}\\
&\qquad\qquad [0,1] \text{ and is from } \mathbf{0} \text{ to } y-x\}\\
&\quad= \sum_y \alpha_{s_0+t}(y)(1-e^{-1})q(y-x)\ge \tfrac{1}{2}\sum_y \alpha_{s_0+t}(y)q(y-x)
\end{aligned}
\tag{2.42}
$$

Consequently,

$$
\begin{aligned}
&\sum_y q(y-x)\alpha_{s_0+t}(y)\\
&\quad\le 2P\{x+S_u=\mathbf{0} \text{ for some } u\in[s_0+t,s_0+t+1]\}\\
&\quad\le 2eP\{x+S_{s_0+t+1}=\mathbf{0}\}=2e\alpha_{s_0+t+1}(x).
\end{aligned}
\tag{2.43}
$$

Substituting this estimate into (2.41) we find that (2.40) is at most

$$L_{s_0}(f)e^t\{N_{s_0+t}(\eta)+2eN_{s_0+t+1}(\eta)\}.$$

Essentially the same estimates as used to bound (2.40) show that the series for $\Omega(K_sf)$ converges and that

$$|\Omega(K_sf)(\eta)|\le L_{s_0}(f)e^t\{N_{s_0+t}(\eta)+2eN_{s_0+t+1}(\eta)\},\qquad s\le t. \tag{2.44}$$

With these bounds and the dominated convergence theorem it is easy tc that

$$\lim_{N\to\infty}\int_0^t \Omega(Ef(\xi_s(\eta^{(N)})))ds=\int_0^t \Omega(Ef(\xi_s(\eta)))ds.$$

This proves (b).

(c) follows from (b) and (2.44).

We obtain (d) by taking the limit $t\to$ some t_0 in the explicit exp for $\Omega(K_tf)(\eta)$, which is given by (2.40) with $\eta^{(N)}$ replaced by η and and with the sum over x extended over all x. The estimates used to (2.41)–(2.43) and the dominated convergence theorem justify taking th $t\to t_0$ inside the double sum over x, y.

(e) is immediate from (b), (d).

Finally, (f) is proved in essentially the same way as part (e) of Lemma in Liggett and Spitzer (1981) or part (g) in Theorem IX.1.14 in Ligget (19

3. A variance estimate. Throughout this section we take the initial state to be $\xi_0 = \mathbb{1}$, that is,

$$\xi_0(x) = 1, \qquad x \in \mathbb{Z}^d,$$

although the argument works for any ξ_0 with $\xi_0(x)$ bounded. We also use for the first time the hypothesis

$$\sum_{x \in \mathbb{Z}^d} \|x\|^2 q(x) < \infty. \tag{3.1}$$

To simplify notation somewhat, we write just ξ_t for $\xi_t(\mathbb{1})$ and $\xi_{N,t}$ for $\xi_t(\mathbb{1}^{(N)})$. The following estimate is the basic result of this section.

PROPOSITION 7. *Assume* (1.9) *and* (3.1). *Then there exists a constant* C_0, *which is independent of* β, K, t *and the* p_j, *such that for* $\beta(x) \in \mathbb{R}$ *and* $K < \infty$ *it holds that*

$$\operatorname{Var}\left\{ \sum_{|x| \le K} \beta(x)\xi_t(x) \right\} \le C_0 \log(t+2) \sum_{x \in \mathbb{Z}^d} \beta^2(x). \tag{3.2}$$

If

$$\sum_{x \in \mathbb{Z}^d} |\beta(x)| E\xi_t(x) < \infty, \tag{3.3}$$

then also

$$\operatorname{Var}\left\{ \sum_{x \in \mathbb{Z}^d} \beta(x)\xi_t(x) \right\} \le C_0 \log(t+2) \sum_{x \in \mathbb{Z}^d} \beta^2(x). \tag{3.4}$$

REMARK. (vii) The estimate (3.4) can, by quite a lot of extra work, be improved to

$$\operatorname{Var}\left\{ \sum_{x \in \mathbb{Z}^d} \beta(x)\xi_t(x) \right\} \le C_0 t^{-1/4} \log(t+2) \sum_{x \in \mathbb{Z}^d} \beta^2(x). \tag{3.5}$$

If this improved estimate is used throughout Section 4, then one obtains that (1.11) remains valid even in $d = 5$. This improvement is obtained by directly comparing the ξ' and the ξ''-processes, rather than comparing each one separately with the $\tilde{\xi}$-process [these processes are introduced a little before (3.20) below]. As we have already stated in Remark (ii), one can even prove (1.11) for $d = 4$ if one assumes that $p_M = 1$ for some M. To deal with the special case where $p_M = 1$, one needs not only (3.5), but also an improved version of Lemma 10 which shows that if $p_M = 1$ for some M, then

$$E\Lambda_t(u_1, \dots, u_p) \le C_2(p) t^{-p}. \tag{3.6}$$

[Λ_t is defined in (4.6).] In turn, (3.6) is obtained by comparing the process with $p_M = 1$ with a process which has p_j replaced by $p'_j = (j/M') \wedge 1$ for some large M' so that $p'_j \le p_j$ for all j. It can be shown that the process with parameters p'_j satisfies the analogue of Lemma 1 of Arratia (1981), to wit,

$$P\{\xi_t(x_i) \ge m_i, 1 \le i \le r\} \le \prod_{i=1}^{r} P\{\xi_t(x_i) \ge m_i\}, \qquad x_i \in \mathbb{Z}^d, m_i \ge 1, r \ge 1. \tag{3.7}$$

For such processes our proof even works for $d \ge 3$. (Note that the model with $M' = 1$ is the basic model mentioned in the beginning of this paper.)

We hope to discuss the somewhat lengthy proofs of these improvements elsewhere.

Before we can start on the proof proper of Proposition 7 we need an a priori estimate for

$$E(t) := E\xi_t(x) \tag{3.8}$$

(this is independent of x).

LEMMA 8. *Assume* (1.8) *and* (3.1). *Then, for* $d \ge 3$, *there exist constants* $0 < C_1 \le C_2 < \infty$ *such that*

$$\frac{C_1}{t} \le E(t) \le \frac{C_2}{t}, \qquad t \ge 1. \tag{3.9}$$

PROOF. These estimates basically come from Arratia (1983) and Bramson and Griffeath (1980). By Lemma 2,

$$E(t) \ge E^*(t) := E\xi_t^*(\mathbf{0}),$$

where ξ_t^* is the process with removal probabilities p_j^*, given by (2.23) (and initial state $\mathbb{1}$). This ξ^*-process is the basic coalescing random walk model, except that $S.$ does not have to be a simple random walk. We can therefore not simply use (1.1). However, by Lemma 1 of Arratia (1983) one has for $S.$ an arbitrary random walk,

$$E^*(t) \ge \frac{C_1}{t}. \tag{3.10}$$

Thus the left-hand inequality of (3.9) holds.

The right-hand inequality of (3.9) is proved in exactly the same way as the case $d \ge 3$ of Theorem 1 of Bramson and Griffeath (1980), but we nevertheless need three comments about this. The first, rather trivial comment is that for the inequality three lines below (25) in Bramson and Griffeath, we need the right-hand inequality of (2.14), or better yet, (2.20). The second comment concerns Lemma 3 of Bramson and Griffeath. Their proof is based on the fact that in the basic model, when $p_j = p_j^*$ [see (2.23)] one has for any finite initial state ξ_0 that

$$(3.11)\quad \sum_{x\in\mathbb{Z}^d} \xi_0(x) - E\left\{\sum_{x\in\mathbb{Z}^d} \xi_s(\xi_0)(x)\right\} \ge \left[\sum_{x\in\mathbb{Z}^d} \xi_0(x) - 1\right] \min_{\xi_0(u),\, \xi_0(v)>0} H_s(u-v),$$

where

$$H_s(z) = P\{S_t^\sigma = z \text{ for some } t \le s\}$$

and $\{S_t^\sigma\}$ is as in the Theorem of Section 1. The min of H_s is taken over all u, v with $\xi_0(u) > 0$, $\xi_0(v) > 0$. We need the analogue of (3.11) (with a factor p_1 in the right-hand side) for general p_j satisfying (1.8) and (1.9), not just for $p_j = p_j^*$.

In order to show that (3.11) remains valid for such p_j we have to use a construction for ξ_t other than the one used in Section 2. In this construction we distinguish the different particles and keep track of the position of the individual particles, not merely of the number of particles at each site. For the present purposes it is also better to let a particle coalesce with another particle after a jump, rather than removing it. At time 0 we label the particles at any given site x as (x, k) with $1 \le k \le \xi_0(x)$ (in some arbitrary ordering of the particles at x). We further pick for each such particle a random walk path $\{S_t^{(x,k)}\}_{t\ge 0}$. The $\{S_t^{(x,k)}\}_{t\ge 0}$ are i.i.d., each with the distribution of $\{S_t\}_{t\ge 0}$. We further attach to each particle (x, k) further random variables $\{U_n^{(x,k)}, V_{n,j}^{(x,k)}, j \ge 1, n \ge 1\}$. Random variables with different values of (x, k) or n are independent. Also, for fixed (x, k), all $U_n^{(x,k)}$ are independent of all $V_{m,j}^{(x,k)}$. All the $U_n^{(x,k)}$ are uniform on $[0, 1]$ and each $V_{n,j}^{(x,k)}$ takes values in $\{1, \dots, j\}$ with

$$P\left\{V_{n,j}^{(x,k)} = l\right\} = \frac{1}{j}, \qquad 1 \le l \le j.$$

Now the particle labeled (x, k) moves along the path $t \mapsto x + S_t^{(x,k)}$ until it first jumps to a site, y, say, which already contains another particle. At such a jump the (x, k)-particle may coalesce with one of the particles present at the site y. Whether the (x, k)-particle does coalesce, and with which particle, is a function of the $\{U_n^{(x,k)}, V_{n,j}^{(x,k)}\}$. Suppose that the (x, k)-particle did not coalesce with another particle at one of the first $n - 1$ jumps of $S^{(x,k)}$ and that at its nth jump this particle jumps to y. Suppose at that time there are j particles at y. Number these particles in some order, say in the order of their arrival times at y. Then the (x, k)-particle coalesces with one of the j particles at y if and only if $U_n^{(x,k)} \le p_j$. If this is the case, then it coalesces with the particle with the number $V_{n,j}^{(x,k)}$. After this coalescing event, the (x, k)-

particle no longer follows the path $t \mapsto S_t^{(x,k)}$, but follows the path of the particle with which it coalesced. Note that it is always the variables associated with the particle which has just jumped which determine whether coalescence takes place. It is also the particle which has just jumped which "gives up" its own trajectory and starts following the trajectory of the particle with which it coalesced.

If we start with finitely many particles, then the construction of the preceding paragraph assigns with probability 1 a unique trajectory to each particle. If the (x, k)-particle and the (y, l)-particle have coalesced, then they both move according to one of the trajectories $t \mapsto z + S_t^{(z,m)}$; (z, m) may be (x, k) or (y, l) or yet another particle with which both the (x, k)-particle and the (y, l)-particle have coalesced. This allows us to define $\xi_t(x)$ again as the number of particles present at x at time t.

We shall not prove that the preceding construction is equivalent to the one of Section 2, in the sense that the joint distribution of the $\{\xi_t(x)\}_{t\geq 0}$, $x \in \mathbb{Z}^d$, is the same under both constructions (we need this only for finite initial states). It is further left to the reader to verify that the proof of Bramson and Griffeath's lemma 3 for (3.11) (with an extra factor p_1 in the right-hand side) goes through for the newly constructed $\{\xi_t\}$. But if (3.11) holds for one of the constructions of $\{\xi_t\}$, then it holds for all constructions, since (3.11) depends only on the joint distribution of the $\{\xi_t(x)\}_{t\geq 0}$, $x \in \mathbb{Z}^d$.

Our final comment concerns the lower bound for $\inf_{\|z\|\leq r} H_{r^2}(z)$ which Bramson and Griffeath (1980) derive in their Lemma 5 when $S_{\bullet}$ is simple random walk. This lemma remains valid under condition (3.1) only, because as Bramson and Griffeath argue, in order to obtain the desired lower bound for $\inf_{\|z\|\leq r} H_{r^2}(z)$, one merely needs a lower bound (of size $C_1 r^{2-d}$) on

$$\inf_{\|z\|\leq r} \int_0^{r^2} P\{S_s^\sigma = z\}\, ds$$

$$\geq \int_{r^2/2}^{r^2} ds \sum_{k=r^2/4}^{2r^2} P\{S_{\bullet}^\sigma \text{ has } k \text{ jumps during } [0, s]\} \inf_{\|z\|\leq r} q_\sigma^{*k}(z),$$

where $q_\sigma(z) = [q(z) + q(-z)]/2 = P\{S_{\bullet}^\sigma \text{ jumps from } \mathbf{0} \text{ to } z \text{ at its first jump}\}$. The required lower bound follows directly from the local central limit theorem. [See Spitzer (1976), Proposition 7.9. Note that this may require some care, because q_σ is not necessarily strongly aperiodic (in the terminology of Spitzer). However, because $q_\sigma^{*2}(\mathbf{0}) > 0$, the proof of Proposition 5.1 in Spitzer (1976) shows that q_σ^{*2} is strongly aperiodic on some additive subgroup G_1 of $\mathbb{Z}^d$, of index 1 or 2. Moreover, when $z \notin G_1$ and $q_\sigma(w) > 0$, then $z + w \in G_1$. For each $z \in G_1$ we can therefore apply the local central limit theorem to find a lower bound for $q_\sigma^{*k}(z)$ when k is even. For $z \notin G_1$ we use $q_\sigma^{*(2l+1)}(z) \geq \sum_w q_\sigma(-w) q_\sigma^{*(2l)}(z + w)$.]

In all other respects the proof of the right-hand inequality in (3.9) follows Bramson and Griffeath (1980). □

PROOF OF PROPOSITION 7. First choose a $K < \infty$ and let

$$Z = \sum_{|x| \le K} \beta(x) \xi_t(x),$$

$$Z_N = \sum_{|x| \le K} \beta(x) \xi_{N,t}(x).$$

(Recall that $\xi_{N,t}$ is the state at time t if we start with $\xi_0(y) I[|y| \le N] = I[|y| \le N]$ particles at y.) Now

$$EZ_N = \sum_{|x| \le K} \beta(x) E\xi_{N,t}(x)$$

and, as $N \to \infty$,

$$E\xi_{N,t}(x) \uparrow E\xi_t(x) \le \sum_y P\{y + S_t = x\} = 1$$

by Lemma 3 and the monotone convergence theorem. Hence

$$EZ_N \to EZ, \qquad N \to \infty. \tag{3.12}$$

By Fatou's lemma we then get

$$\operatorname{Var}(Z) = EZ^2 - (EZ)^2 \le \liminf_{N \to \infty} \operatorname{Var}(Z_N). \tag{3.13}$$

It therefore suffices for (3.2) to prove

$$\operatorname{Var}\Bigl(\sum_{|x| \le K} \beta(x) \xi_{N,t}(x)\Bigr) \le C_0 \log(t+2) \sum_x \beta^2(x). \tag{3.14}$$

Now let $\mathscr{F}_s$ be as in (2.18) and define

$$\Delta_l = \Delta_l(p) = \Delta_l(p, N, t) = E\{Z_N \mid \mathscr{F}_{lt/p}\} - E\{Z_N \mid \mathscr{F}_{(l-1)t/p}\}.$$

Then for each integer $p \ge 1$,

$$Z_N - EZ_N = \sum_1^p \Delta_l$$

and

$$\operatorname{Var}(Z_N) = \sum_1^p E\Delta_l^2(p) = \liminf_{p \to \infty} \sum_1^p E\Delta_l^2(p)$$

$$= \liminf_{p \to \infty} \sum_1^p E\bigl\{E\{\Delta_l^2(p) \mid \mathscr{F}_{(l-1)t/p}\}\bigr\}.$$

We fix N and write $W_l = W_l(p, N)$ for the random elements which summarize all the information which becomes available between time $(l-1)t/p$ and lt/p. More precisely, W_l stands for all the increments $\mathscr{N}_u(x,k) - \mathscr{N}_{(l-1)t/p}(x,k)$ of the Poisson processes with $(l-1)t/p < u \le lt/p$, and the $y_n(x,k)$, $U(n,x,k)$ associated to jump times during $((l-1)t/p, lt/p]$ of any of these processes. We skip the tedious explicit construction of a probability space on which these random variables are defined. Whatever this probability space for the W is, we shall have

$$\mathcal{F}_{lt/p} = \sigma\{W_1, \ldots, W_l\}$$

and the W_l for different l are independent. Also, W_l has a distribution which we denote by μ_l (that is, $\mu_l(dw) = P\{W_l \in dw\}$). Now $Z_N = f(W_1, W_2, \ldots, W_p)$ for a suitably measurable function $f = f_N$ and therefore

$$E\{Z_N | \mathcal{F}_{lt/p}\} = \int \prod_{i=l+1}^{p} \mu_i(dw_i) f(W_1, \ldots, W_l, w_{l+1}, \ldots, w_p)$$
$$= \int \prod_{i=l}^{p} \mu_i(dw_i) f(W_1, \ldots, W_l, w_{l+1}, \ldots, w_p).$$

Note that the last member also includes an integration with respect to $\mu_l(dw_l)$; this integration can be added because the integrand does not depend on w_l. Therefore

$$\Delta_l = \int \prod_{i=l}^{p} \mu_i(dw_i)[f(W_1, \ldots, W_l, w_{l+1}, \ldots, w_p) - f(W_1, \ldots, W_{l-1}, w_l, w_{l+1}, \ldots, w_p)]. \tag{3.15}$$

Note that Δ_l is a function of $W_1, \ldots, W_l$, and that therefore

$$E\{\Delta_l^2 \mid \mathcal{F}_{(l-1)t/p}\} = \int \mu_l(dW_l)\Delta_l^2$$

and

$$E\Delta_l^2 = \int \prod_{j \le l} \mu_j(dW_j)\Delta_l^2.$$

By Schwarz's inequality applied to (3.15) we find

$$\Delta_l^2 \le \int \prod_{i=l}^{p} \mu_i(dw_i)[f(W_1, \ldots, W_l, w_{l+1}, \ldots, w_p) - f(W_1, \ldots, W_{l-1}, w_l, \ldots, w_p)]^2,$$

and we now turn to an estimate for

$$[f(W_1, \ldots, W_l, w_{l+1}, \ldots, w_p) - f(W_1, \ldots, W_{l-1}, w_l, \ldots, w_p)]^2. \tag{3.16}$$

The expression in square brackets here is the change in Z_N due to the change from w_l to W_l in the time interval $((l-1)t/p, lt/p]$, while keeping all other random elements in $[0, (l-1)t/p]$ fixed at $W_1, \ldots, W_{l-1}$ and the random elements in $(lt/p, t]$ fixed at $w_{l+1}, \ldots, w_p$. We shall use that at all times the number of particles present in the $\xi(\mathbb{1}^{(N)})$-process is at most

$$\sum_x \xi_{N,0}(x) = \sum_{|x| \le N} 1 = (2N+1)^d.$$

The location of these particles at time $(l-1)t/p$ is determined by $W_1, \dots, W_{l-1}$ and is therefore $\mathscr{F}_{(l-1)t/p}$-measurable. We shall write $I_l[\geq k \text{ jumps}]$ for the indicator function of the event that the particles present at time $(l-1)t/p$ have at least k jumps during $((l-1)t/p, lt/p]$. (Repeated jumps by the same particle are counted as different jumps; we anyway only keep track of the ξ's so do not know which particle jumps.) $I_l[1 \text{ jump}]$ and $I_l[\text{no jump}]$ have similar self-evident definitions. If $I_l[\geq 2 \text{ jumps}](W_1, \dots, W_{l-1}, w_l, \dots, w_p) = 1$ or if $I_l[\geq 2 \text{ jumps}](W_1, \dots, W_l, w_{l+1}, \dots, w_p) = 1$, then we simply estimate (3.16) by

$$\begin{aligned} \big[2 \sup |Z_K|\big]^2 &\leq \Big[2 \sup_{|x|\leq K} |\beta(x)| \sum_x \xi_{N,0}(x)\Big]^2 \\ &= 4(2N+1)^{2d}\Big[\sup_{|x|\leq K} |\beta(x)|\Big]^2. \end{aligned} \tag{3.17}$$

The same bound applies when there is at least one jump in *both* the configurations $W_1, \dots, W_{l-1}, w_l, \dots, w_p$ and $W_1, \dots, W_l, w_{l+1}, \dots, w_p$. We shall soon see that the contributions to $\sum E\Delta_l^2$ in all these configurations go to 0 as $p \to \infty$. When in both configurations $W_1, \dots, W_{l-1}, w_l, \dots, w_p$ and $W_1, \dots, W_l, w_{l+1}, \dots, w_p$ no particle at all jumps during $((l-1)t/p, lt/p]$, then the particle locations at time lt/p are the same in the configurations $W_1, \dots, W_l$ and $W_1, \dots, W_{l-1}, w_l$, and (3.16) equals 0. Therefore (3.16) is at most equal to the sum of the following three terms:

$$\begin{aligned} &4(2N+1)^{2d}\Big[\sup_{|x|\leq K} |\beta(x)|\Big]^2 \\ &\quad\times \big[I_l[\geq 2 \text{ jumps}](W_1, \dots, W_{l-1}, w_l, \dots, w_p) \\ &\qquad + I_l[\geq 2 \text{ jumps}](W_1, \dots, W_l, w_{l+1}, \dots, w_p) \\ &\qquad + I_l[\geq 1 \text{ jump}](W_1, \dots, W_{l-1}, w_l, \dots, w_p) \\ &\qquad\quad \times I_l[\geq 1 \text{ jump}](W_1, \dots, W_l, w_{l+1}, \dots, w_p)\big]^2; \end{aligned} \tag{3.18}$$

$$\begin{aligned} &\big[f(W_1, \dots, W_l, w_{l+1}, \dots, w_p) - f(W_1, \dots, W_{l-1}, w_l, \dots, w_p)\big]^2 \\ &\quad\times I_l[1 \text{ jump}](W_1, \dots, W_l, w_{l+1}, \dots, w_p) \\ &\quad\times I_l[\text{no jump}](W_1, \dots, W_{l-1}, w_l, \dots, w_p) \end{aligned} \tag{3.19}$$

and (3.19) with W_l and w_l interchanged.

We first show that the contribution of (3.18) to $\sum E\Delta_l^2$ becomes negligible as $p \to \infty$. The square of the sum of the indicator functions between square brackets in (3.18) is at most

$$\begin{aligned}3I_l[\geq 2 \text{ jumps}](W_1, \dots, W_{l-1}, w_l, \dots, w_p)\\ + 3I_l[\geq 2 \text{ jumps}](W_1, \dots, W_l, w_{l+1}, \dots, w_p)\\ + 3I_l[\geq 1 \text{ jump}](W_1, \dots, W_{l-1}, w_l, \dots, w_p)\\ \times I_l[\geq 1 \text{ jump}](W_1, \dots, W_l, w_{l+1}, \dots, w_p).\end{aligned}$$

We only estimate the contribution of the last term here. The other terms can be estimated in the same way (but are actually easier to treat). Note that $I_l[\geq 1 \text{ jump}](W_1, \dots W_{l-1}, w_l, \dots, w_p)$ depends on $W_1, \dots, W_{l-1}$ and w_l only, while $I_l[\geq 1 \text{ jump}](W_1, \dots W_l, w_{l+1}, \dots, w_p)$ depends on $W_1, \dots, W_l$ only. Therefore,

$$\begin{aligned}\int \mu_l(dW_l) \int \mu_l(dw_l) \int \prod_{i=l+1}^{p} \mu_i(dw_i) I_l[\geq 1 \text{ jump}](W_1, \dots W_{l-1}, w_l, \dots, w_p)\\ \times I_l\Big[\geq 1 \text{ jump}](W_1, \dots W_l, w_{l+1}, \dots, w_p)\\ = \int \mu_l(dW_l) I_l[\geq 1 \text{ jump}](W_1, \dots W_l, w_{l+1}, \dots, w_p)\\ \times \int \mu_l(dw_l) I_l[\geq 1 \text{ jump}](W_1, \dots W_{l-1}, w_l, \dots, w_p)\\ = \big[P\{ \text{ at least one jump occurs during } ((l-1)t/p, lt/p] \,|\\ W_1, \dots, W_{l-1}\}\big]^2\\ \leq \left[\sum_x \xi_{(l-1)t/p}(\mathbb{1}^{(N)})(x) \frac{t}{p}\right]^2\\ \leq \frac{t^2}{p^2}(2N+1)^{2d}.\end{aligned}$$

If we finally integrate the left-hand side also with respect to $\prod_1^{l-1} \mu(dW_j)$ and then sum over l from 1 to p we find a contribution to $\sum_1^p E\Delta_l^2$ of at most

$$\frac{t^2}{p}(2N+1)^{2d} 12\left[\sup_{|x| \leq K} |\beta(x)|\right]^2,$$

and this tends to 0 as $p \to \infty$. Thus the contribution of (3.18) can be ignored.

Because of the symmetry between W_l and w_l in our estimates, (3.19) and the term with W_l and w_l interchanged give the same contribution. We therefore only have to estimate (3.19). To this end let us write ξ'_t for the $\xi(\mathbb{1}^{(N)})$-process in the configuration $W_1, \dots, W_l, w_{l+1}, \dots, w_p$ and ξ''_t for the $\xi(\mathbb{1}^{(N)})$-process in the configuration $W_1, \dots, W_{l-1}, w_l, \dots, w_p$. Through time $(l-1)t/p$ these two processes agree, so that

$$\xi'_{(l-1)t/p} = \xi''_{(l-1)t/p}.$$

Suppose there is exactly one jump during $((l-1)t/p, lt/p]$ in the configuration $W_1, \dots, W_l, w_{l+1}, \dots, w_p$, that is, in the ξ'-process. Let this jump be from x' to y'. Assume further that there is no jump in the configuration $W_1, \ldots, W_{l-1}, w_l, \ldots, w_p$. Then

$$\begin{aligned} \xi''_{lt/p}(x) &= \xi''_{(l-1)t/p}(x) = \xi'_{(l-1)t/p}(x) \quad \text{for all } x, \\ \xi''_{lt/p}(x) &= \xi'_{lt/p}(x) \quad \text{if } x \neq x', y', \\ \xi'_{lt/p}(x') &= \xi'_{(l-1)t/p}(x') - 1, \\ \xi'_{lt/p}(y') &= \xi'_{(l-1)t/p}(y') \quad \text{or} \quad \xi'_{(l-1)t/p}(y') + 1. \end{aligned}$$

In any case, $\xi'_{lt/p}$ and $\xi''_{lt/p}$ differ at most on the two sites x', y' and there they differ by at most 1. Rather than compare ξ'_t directly with ξ''_t, we compare each of them with a third process $\tilde{\xi}_t$ which we define as the process which behaves like ξ' except that the particle which jumps from x' to y' during $((l-1)t/p, lt/p]$ is removed immediately after the jump in the $\tilde{\xi}$-process. After time lt/p, it develops by the prescribed rules in the configurations $w_{l+1}, \ldots, w_p$. Of course it may be that $\tilde{\xi} \equiv \xi'$, namely, if the particle which jumps from x' to y' is also removed in the ξ'-process. If this particle is not removed in the ξ'-process, then the ξ'-process has one particle more than the $\tilde{\xi}$-process at time lt/p, and this extra particle is located at y'. Therefore, by Lemma 1,

$$\tilde{\xi}_t(x) \leq \xi'_t(x) \leq \tilde{\xi}_t(x) + \bar{\xi}_t(y')(x), \tag{3.20}$$

where $\bar{\xi}(y')$ is a process which starts with a single particle at y' at time lt/p which moves according to the random walk but does not interact with anything. This process is not defined for times $< lt/p$. However, $\tilde{\xi}_.$ and $\bar{\xi}_.(y')$ are coupled and are defined as functions of y' and the Poisson processes $\mathcal{N}_s(x,k) - \mathcal{N}_{lt/p}(x,k)$, $x \in \mathbb{Z}^d$, $k \geq 1, s \geq lt/p$, as well as the $y_n(x,k), U(n,x,k)$ corresponding to jumps at or after time lt/p, as described for the ξ', and ξ''-processes just before Lemma 1. (Note that the present ξ', ξ'' do *not* have the same meaning as in Lemma 1.) Thus

$$\bar{\xi}_t(y')(x) = I[\text{extra particle in } \xi' \text{ which is at } y' \text{ at } lt/p \text{ moves to } x \text{ at time } t]. \tag{3.21}$$

Similarly,

$$\tilde{\xi}_t(x) \leq \xi''_t(x) \leq \tilde{\xi}_t(x) + \bar{\xi}_t(x')(x),$$

where $\bar{\xi}(x')$ is a process which starts with a single particle at x' at time lt/p and which does not interact with anything. Therefore, if there is exactly one jump in the ξ'-process and no jump in the ξ''-process, then

$$|f(W_1, \ldots, W_l, w_{l+1}, \ldots, w_p) - f(W_1, \ldots, W_{l-1}, w_l, \ldots, w_p)|$$

is at most

$$\sum_{|x|\le K} |\beta(x)||\xi_t'(x) - \xi_t''(x)|$$

(3.22)
$$\le \sum_{|x|\le K} |\beta(x)| \sum_{x',\,y'} I_l[\text{a single jump from } x' \text{ to } y' \text{ occurs during } ((l-1)t/p, lt/p]$$
$$(W_1, \dots, W_l, w_{l+1}, \dots, w_p)[\bar{\xi}_t(y')(x) + \bar{\xi}_t(x')(x)].$$

Let us estimate the contribution of the term involving $\bar{\xi}_t(y')$. Note that

$$\Bigg[\sum_{|x|\le K} |\beta(x)| \sum_{x',\,y'} I_l[\text{a single jump from } x' \text{ to } y' \text{ occurs during } ((l-1)t/p, lt/p]$$
$$(W_1, \dots, W_l, w_{l+1}, \dots, w_p)\bar{\xi}_t(y')(x)\Bigg]^2$$
$$= \sum_{x',\,y'} I_l[\text{a single jump from } x' \text{ to } y' \text{ occurs during}((l-1)t/p, lt/p]$$
$$(W_1, \dots, W_l, w_{l+1}, \dots, w_p)\Bigg[\sum_{|x|\le K} |\beta(x)|\bar{\xi}_t(y')(x)\Bigg]^2,$$

because only for one pair x', y' do we have

$$I_l[\text{a single jump from } x' \text{ to } y' \text{ occurs during } ((l-1)t/p, lt/p]$$
$$(W_1, \dots, W_l, w_{l+1}, \dots, w_p) \ne 0.$$

This yields the following contribution to $E\Delta_l^2$:

$$\int \prod_{j\le l-1} \mu_j(dW_j) \int \mu_l(dW_l) \int \mu_l(dw_l) \int \prod_{i=l+1}^{p} \mu_i(dw_i)$$

(3.23)
$$\times \sum_{x',\,y'} I_l[\text{a jump from } x' \text{ to } y' \text{ occurs during } ((l-1)t/p, lt/p]$$
$$(W_1, \dots, W_l, w_{l+1}, \dots, w_p)\Bigg[\sum_{|x|\le K} |\beta(x)|\bar{\xi}_t(y')(x)\Bigg]^2.$$

[Note that integrating over w_i, $l+1 \le i \le p$, in (3.23) includes taking the expectation over $\bar{\xi}_t(y')$, since $\bar{\xi}_t$ is a function of the processes $\mathcal{N}_s(x,k) - \mathcal{N}_{lt/p}(x,k)$, $s \ge lt/p$, as described after (3.20).] The same method will work for the term involving $\bar{\xi}_t(x')(x)$ in (3.22). We can handle (3.23) by noting that $\bar{\xi}_t(y')(x) \ne 0$ for exactly one x. Let us denote this position by z_t. Then $\bar{\xi}_t(y')(z_t) = 1$ and

$$\Bigg[\sum_{|x|\le K} |\beta(x)|\bar{\xi}_t(y')(x)\Bigg]^2 = |\beta(z_t)|^2 I[|z_t| \le K].$$

Moreover, conditionally on $\mathscr{F}_{lt/p}$, z_t is just the position of a random walk at time t which starts at y' at time lt/p. Thus

$$\int \prod_{i=l+1}^{p} \mu_i(dw_i)\left[\sum_{|x|\le K} |\beta(x)|\bar{\xi}_t(y')(x)\right]^2 \le \sum_z |\beta(z)|^2 P\{y' + S_{t-lt/p} = z\}.$$

Therefore [by (2.43)] (3.23) is at most

$$\begin{aligned}(3.24)\qquad &\int \prod_{j\le l-1} \mu_j(dW_j) \sum_{x',\,y'} \xi_{(l-1)t/p}(x')\frac{t}{p}q(y'-x') \\ &\qquad\qquad \times \textstyle\sum_z |\beta(z)|^2 P\{y' + S_{t-lt/p} = z\} \\ &\le \int \prod_{j\le l-1} \mu_j(dW_j) \sum_{x'} \xi_{(l-1)t/p}(x')\frac{t}{p} \\ &\qquad\qquad \times \textstyle\sum_z |\beta(z)|^2 2eP\{S_{t-lt/p+1} = z - x'\}.\end{aligned}$$

But, if $(l-1)t/p \ge 1$, then by Lemma 8,

$$\int \prod_{j\le l-1} \mu_j(dW_j)\xi_{(l-1)t/p}(x') = E\xi_{(l-1)t/p}(x') \le C_2\frac{p}{(l-1)t}.$$

Also, by (2.25) with $\lambda = 0$, for any $(l-1)t/p$, $E\xi_{(l-1)t/p}(x') \le \sum_y P\{S_{(l-1)t/p} = x' - y\} = 1$. Substituting these estimates into (3.24) shows that (3.23) is at most

$$C_3\frac{t}{p}\min\left\{\frac{p}{(l-1)t}, 1\right\}\sum_z |\beta(z)|^2.$$

With a similar estimate for the other term in (3.22) we finally obtain after summing over l the estimate

$$\begin{aligned}\liminf_{p\to\infty} \sum_1^P E\Delta_l^2 &\le 4C_3 \sum_z |\beta(z)|^2 t \liminf_{p\to\infty} \frac{1}{p}\left[\sum_{1\le l< p/t+1} 1 + \sum_{p/t+1\le l\le p} \frac{p}{(l-1)t}\right] \\ &\le C_0 \sum_z |\beta(z)|^2 \log(t+2)\end{aligned}$$

for some constant C_0, which is the desired inequality (3.2).

Once we have (3.2) we can obtain (3.4) under (3.3) exactly as in (3.12), (3.13). Indeed, we have

$$E\sum_{|x|\le K} \beta(x)\xi_t(x) \to E\sum_{x\in\mathbb{Z}^d} \beta(x)\xi_t(x), \qquad K\to\infty$$

and

$$\operatorname{Var}\left\{\sum_{x\in\mathbb{Z}^d} \beta(x)\xi_t(x)\right\} \le \liminf_{K\to\infty} \operatorname{Var}\left\{\sum_{|x|\le K} \beta(x)\xi_t(x)\right\}. \qquad \square$$

4. An approximate differential equation for the expected number of particles per site. Again we start with one particle at each site ($\xi_0 = \mathbb{1}$) and we write ξ_t instead of $\xi_t(\mathbb{1})$. Also $\xi_{N,t}$ stands for $\xi_t(\mathbb{1}^{(N)})$. We define

$$\gamma_t(k) = P\{\xi_t(x) = k\}.$$

Here γ_t is independent of x. Note that

$$p(t) = \sum_{k=1}^{\infty} \gamma_t(k) = P\{\xi_t(x) > 0\} \tag{4.1}$$

and

$$E(t) = \sum_{k=1}^{\infty} k\gamma_t(k). \tag{4.2}$$

We first derive a differential equation for $E(t)$.

LEMMA 9. *$E(t)$ is differentiable and*

$$\frac{d}{dt}E(t) = -\sum_{x\in\mathbb{Z}^d} E\{\xi_t(\mathbf{0})q(x)p_{\xi_t(x)}\}. \tag{4.3}$$

PROOF. A simple calculation, using (2.38) and (1.6) shows that for $f(\xi) = \xi(\mathbf{0})$ one has

$$\begin{aligned}\Omega f(\eta) &= -\eta(\mathbf{0}) + \sum_{x\neq\mathbf{0}} \eta(x)q(-x)(1 - p_{\eta(\mathbf{0})})\\ &= -\eta(\mathbf{0}) + \sum_{x\in\mathbb{Z}^d} \eta(x)q(-x) - \sum_{x\in\mathbb{Z}^d} \eta(-x)q(x)p_{\eta(\mathbf{0})}.\end{aligned}$$

Therefore,

$$E\Omega(f(\xi_t)) = -\sum_{x\in\mathbb{Z}^d} E\{\xi_t(-x)q(x)p_{\xi_t(\mathbf{0})}\} = -\sum_{x\in\mathbb{Z}^d} E\{\xi_t(\mathbf{0})q(x)p_{\xi_t(x)}\},$$

and (4.3) follows from Proposition 6(e) and (f) applied to $f(\xi) = \xi(\mathbf{0})$. □

The remainder of this section is devoted to showing that (4.3) can be replaced by

$$\frac{d}{dt}E(t) = -C(d)(1 + o(1))E^2(t), \tag{4.4}$$

where $o(1) \to 0$ and $t \to \infty$. To this end we follow the heuristic outline of the introduction to approximate $E\{\xi_t(\mathbf{0})p_{\xi_t(x)}\}$ for $x \neq \mathbf{0}$. *Throughout we assume* (1.8), (1.9) *and* $d \geq 6$. (For most lemmas $d \geq 5$ is enough.) We want the estimates to be uniform in $x \neq 0$. C_i, $i \geq 1$, will be used for various strictly positive, finite constants whose precise value is of no importance to us. The same C_i may take different values in different formulas.

Let $u_1, \dots, u_p \in \mathbb{Z}^d$ (not necessarily distinct). Define

$$\sum_{i=1}^{p}{}^{*}\xi_t(u_i) \tag{4.5}$$

to be the sum of the $\xi_t(u_i)$ only over the distinct u_i in $\{u_1, \dots, u_p\}$. Thus if a given u appears several times among the u_i, there is still only one summand $\xi_t(u)$ in (4.5). Define further

$$\begin{aligned}\Lambda_t(u_1, u_2, \dots, u_p)\\ = \Big(\sum_{i=1}^{p}{}^{*}\xi_t(u_i)\Big)\Big(\sum_{i=1}^{p}{}^{*}\xi_t(u_i) - 1\Big)\dots\Big(\sum_{i=1}^{p}{}^{*}\xi_t(u_i) - p + 1\Big).\end{aligned} \tag{4.6}$$

Here $\Lambda_t(u_1, \dots, u_p)$ represents the number of ordered p-tuples of distinct particles which we can select from the $\sum^{*}\xi_t(u_i)$ particles present at the sites $u_1, \dots, u_p$ at time t.

LEMMA 10. *Assume* (1.8), (1.9) *and* $d \geq 5$. *Then for any* $u, v \in \mathbb{Z}^d$,

$$E\Lambda_t(u, v) \leq C_1 t^{-2}. \tag{4.7}$$

Also, for any $p \geq 3, u_1, \dots, u_p \in \mathbb{Z}^d$ *and* $0 < \varepsilon < 1/2$,

$$E\Lambda_t(u_1, \dots, u_p) \leq C_2(\varepsilon, p)[t^{-p} \vee t^{-d(1-\varepsilon)/2}]. \tag{4.8}$$

PROOF. Without loss of generality we may take $u \neq v$ in (4.7) because $\Lambda_t(u, u) \leq \Lambda_t(u, v)$ for any v. Similarly, we may take the u_i distinct in (4.8). We note further that it suffices to prove (4.7) and (4.8) when $\xi_t(u_i)$ is replaced by $\xi_{N,t}(u_i)$ (with constants C_1, C_2 which are independent of N). We shall write $\Lambda_{N,t}$ instead of Λ_t when this replacement is made.

To estimate the terms $\xi_{N,t}$ which appear in Λ_t, we shall apply (2.20) to the process $\{\xi_{N,t/2+s}\}_{s\geq 0}$, conditioned on $\mathscr{F}_{t/2}$. Let $z_1, \dots, z_r$ be the positions at time $t/2$ of the particles present at time $t/2$ in $\xi_{N,t/2}$. Here each position occurs with the proper multiplicity; if $\xi_{N,t/2}(x) = k$, for some x, then k of the z_i equal x. Hence $r = \sum_x \xi_{N,t/2}(x)$. According to (2.20) there exist independent processes $\{\bar{\xi}_s(z_i)(\cdot)\}_{s\geq 0}$, $1 \leq i \leq r$, such that $\bar{\xi}_0(z_i)(x) = 1$ for $x = z_i$ and $= 0$ otherwise and such that $\{\bar{\xi}_s(z_i)(x)\}_{x\in\mathbb{Z}^d}$ has the distribution of $\{I[z_i + S_s = x]\}_{x\in\mathbb{Z}^d}$. Moreover, these processes are coupled with $\xi_{N,t/2+s}$ so that

$$\xi_{N,t/2+s}(x) \leq \sum_{i=1}^{r} \bar{\xi}_s(z_i)(x).$$

In particular,

$$\xi_{N,t}(x) \leq \sum_{i=1}^{r} \bar{\xi}_{t/2}(z_i)(x). \tag{4.9}$$

With the help of this relation we can prove (4.7). We have

$$\begin{aligned}
&\Lambda_{N,t}(u,v)\\
&\leq \left(\sum_{i=1}^{r}[\bar{\xi}_{t/2}(z_i)(u)+\bar{\xi}_{t/2}(z_i)(v)]\right)\left(\sum_{i=1}^{r}[\bar{\xi}_{t/2}(z_i)(u)+\bar{\xi}_{t/2}(z_i)(v)]-1\right)\\
&= \sum_{i=1}^{r}\sum_{\substack{j=1\\ j\neq i}}^{r}[\bar{\xi}_{t/2}(z_i)(u)+\bar{\xi}_{t/2}(z_i)(v)]\,[\bar{\xi}_{t/2}(z_j)(u)+\bar{\xi}_{t/2}(z_j)(v)].
\end{aligned}$$

The right-hand side equals the number of ordered pairs of distinct particles starting at some z_i at time $t/2$ and ending at u or v at time t. These particles are the ones counted by the $\bar{\xi}_{t/2}(z_j)$ and they just move according to random walks without interaction. At time $t/2$ we have $\xi_{N,t/2}(z)$ particles at z to choose from (for any $z \in \mathbb{Z}^d$). The number of choices for starting pairs, one from z and one from z' (with $z = z'$ allowed), is $\Lambda_{N,t/2}(z,z') \leq \xi_{N,t/2}(z)\xi_{N,t/2}(z')$. The probability that the two different particles of the pair end at u or v at time t is

$$\big(\alpha_{t/2}(z-u)+\alpha_{t/2}(z-v)\big)\big(\alpha_{t/2}(z'-u)+\alpha_{t/2}(z'-v)\big).$$

We now sum over all possible z, z' and take the conditional expectation given $\mathscr{F}_{t/2}$ to find

$$\begin{aligned}
&E\{\Lambda_{N,t}(u,v)\mid\mathscr{F}_{t/2}\}\\
&\leq \sum_{i=1}^{r}\sum_{\substack{j=1\\ j\neq i}}^{r}\big(\alpha_{t/2}(z_i-u)+\alpha_{t/2}(z_i-v)\big)\big(\alpha_{t/2}(z_j-u)+\alpha_{t/2}(z_j-v)\big)\\
&\leq \left[\sum_{x\in\mathbb{Z}^d}\xi_{N,t/2}(z)\big(\alpha_{t/2}(z-u)+\alpha_{t/2}(z-v)\big)\right]^2.
\end{aligned} \tag{4.10}$$

Finally, by virtue of (3.9) and (3.14) and the fact that $\xi_{N,t/2}(z) \leq \xi_{t/2}(z)$, we have for $d \geq 5$, and uniformly in u,

$$\begin{aligned}
&E\left\{\left[\sum_z \alpha_{t/2}(z-u)\xi_{N,t/2}(z)\right]^2\right\}\\
&\leq \left[\sum_z \alpha_{t/2}(z-u)E\xi_{N,t/2}(x)\right]^2+\operatorname{Var}\left(\sum_z \alpha_{t/2}(z-u)\xi_{N,t/2}(z)\right)\\
&\leq \left[\frac{2C_2}{t}\right]^2 + C_0\log(t/2+2)\sum_x \alpha_{t/2}^2(x)\\
&\leq \frac{4C_2^2}{t^2}+C_0\log(t+2)\sup_x \alpha_{t/2}(x)\\
&\leq C_3\big[t^{-2}+t^{-d/2}\log(t+2)\big],
\end{aligned} \tag{4.11}$$

where in the last inequality we used the estimate

$$\sup_y \alpha_s(y) = \sup_y P\{y + S_s = \mathbf{0}\} \le \frac{C_4}{(s+1)^{d/2}}, \tag{4.12}$$

which, in turn, follows from the local central limit theorem [see Spitzer (1976), Proposition 7.9 and the remark following it]. Now (4.7) is immediate from (4.10) and (4.11) (plus its analogue with u replaced by v).

The argument for (4.8) begins in the same way as for (4.7). By an application of (4.9) we can bound $\Lambda_{N,t}(u_1, \dots, u_p)$ by the number of p-tuples of distinct particles which start at some z_i at time $t/2$ and end at one the u_j at time t. Therefore, we get as in (4.10) that

$$E\Lambda_{N,t}(u_1, \dots, u_p) \le E\left\{\left[\sum_z \xi_{N,t/2}(z) \sum_{j=1}^{p} \alpha_{t/2}(z - u_j)\right]^p\right\}.$$

It therefore suffices for (4.8) to show that, uniformly in $u \in \mathbb{Z}^d$,

$$E\left\{\left[\sum_z \xi_{N,t/2}(z)\alpha_{t/2}(z-u)\right]^p\right\} \le C_2(\varepsilon, p)\left[t^{-p} \vee t^{-d(1-\varepsilon)/2}\right], \qquad p \ge 3. \tag{4.13}$$

To prove (4.13) let us use the abbreviation

$$U = \sum_z \alpha_{t/2}(z-u)\xi_{N,t/2}(z). \tag{4.14}$$

Note that $U \ge 0$. We further know from Lemma 8 that

$$EU \le C_2 t^{-1}, \tag{4.15}$$

and from Proposition 7 and (4.11) that

$$\mathrm{Var}(U) = E\{(U - EU)^2\} \le C_3 t^{-d/2} \log(t+2), \qquad E\{U^2\} \le C_4 t^{-2}. \tag{4.16}$$

Now use

$$\begin{aligned} U^p &\le C_5(p)\left[|U - EU|^p + (EU)^p\right] \\ &\le C_5(p)|U - EU|^{2-\varepsilon}|U - EU|^{p-2+\varepsilon} + C_6(p)t^{-p}. \end{aligned}$$

Combined with Hölder's inequality, this shows that

$$\begin{aligned} E\{U^p\} &\le C_5\left[E\{(U-EU)^2\}\right]^{(1-\varepsilon/2)}\left[E\{|U-EU|^{2(p-2+\varepsilon)/\varepsilon}\}\right]^{\varepsilon/2} \\ &\quad + C_6(p)t^{-p}. \end{aligned} \tag{4.17}$$

Assume for the moment that we have proven for any integer $q \ge 1$,

$$E\{U^q\} \le C_7(q), \tag{4.18}$$

(with C_7 independent of N). Then by Jensen's inequality this holds for any $q \ge 1$ and also

$$E\{|U - EU|^q\} \le C_8(q) \tag{4.19}$$

follows. Together with (4.17) this shows

$$E\{U^p\} \le C_9(\varepsilon, p)[\mathrm{Var}(U)]^{(1-\varepsilon/2)} + C_6 t^{-p}.$$

In view of (4.16) this implies (4.13) and hence (4.8).

The proof of (4.8) has therefore been reduced to (4.18). We now turn to its proof. We note that $\sum_z \alpha_{t/2}(z-u) = 1$, so that by Jensen's inequality,

$$U^q \le \sum_z \alpha_{t/2}(z-u)\xi^q_{N,t/2}(z)$$

and hence

$$E\{U^q\} \le \sup_x E\xi^q_{N,t/2}(x).$$

Next we again use (2.20) [compare also with (4.9)]. Then

$$E\{U^q\} \le \sup_x E\xi^q_{N,t/2}(x) \le \sup_x E\left[\sum_z \bar{\xi}_{t/2}(z)(x)\right]^q$$

$$= E\left[\sum_z \bar{\xi}_{t/2}(z)(\mathbf{0})\right]^q \quad \text{(by translation invariance)}$$

$$\le C_{10}(q)\sum_{k=1}^q \sum_{n_1,\dots,n_k} \sum_{\substack{z_1,\dots,z_k \\ \text{distinct}}} E\{\bar{\xi}^{n_1}_{t/2}(z_1)(\mathbf{0})\}\cdots E\{\bar{\xi}^{n_k}_{t/2}(z_k)(\mathbf{0})\},$$

where $n_1,\dots,n_k$ runs over the partitions of q into k nonzero integers. Since $\bar{\xi}_{t/2}(z)(\mathbf{0})$ can take on only the values 0 or 1 and $P\{\bar{\xi}_{t/2}(z)(\mathbf{0}) = 1\} = P\{z + S_{t/2} = \mathbf{0}\} = \alpha_{t/2}(z)$, we find that

$$E\{U^q\} \le C_{10}(q)\sum_{k=1}^q \sum_{n_1,\dots,n_k} \sum_{z_1,\dots,z_k} \prod_{i=1}^k P\{z_i + S_{t/2} = \mathbf{0}\} \le C_{11}(q),$$

as desired. □

LEMMA 11. *Assume* (1.8) *and* (1.9). *Then for* $d \ge 5$,

$$0 \le E(t) - p(t) \le E(t) - P\{\xi_t(\mathbf{0}) = 1\} \le \frac{C_1}{t^2}. \tag{4.20}$$

PROOF.

$$E(t) - p(t) = \sum_{k\ge 2}^{\infty}(k-1)P\{\xi_t(\mathbf{0}) = k\} \ge 0$$

[see (4.1) and (4.2)]. On the other hand, by (4.7),

$$E(t) - p(t) \le E(t) - P\{\xi_t(\mathbf{0}) = 1\} = E\{\xi_t(\mathbf{0}); \xi_t(\mathbf{0}) \ge 2\}$$

$$\le E\{\xi_t(\mathbf{0})[\xi_t(\mathbf{0}) - 1]\} = E\Lambda_t(\mathbf{0},\mathbf{0}) \le \frac{C_1}{t^2}.$$ □

The next lemma is an estimate for noninteracting random walks. If $s \mapsto u + S_s^{(u,k)}$ and $s \mapsto v + S_s^{(v,l)}$ are two random walk paths, then we shall say that *they meet at least m times during a time interval J* if there exist m times $s_1 < s_2 < \cdots s_m$ in J such that each s_i is a jumptime of one of these paths for which $u + S_{s_i}^{(u,k)} = v + S_{s_i}^{(v,l)}$. We say that the paths *meet exactly m times during J* if they meet at least m times during J but not at least $(m+1)$ times. (The k and l in the superscripts are introduced because we will have to allow $u = v$, and we still want to distinguish the random walk paths in this case.)

LEMMA 12. *Let $d \geq 3$ and let $\{S_t^{(u,k)}\}_{t\geq 0}$, $(u,k) \in \mathbb{Z}^d \times \{1,2,\ldots\}$, and $\{S_t^{(u)}\}_{t\geq 0}$, $u \in \mathbb{Z}^d$, be independent copies of $\{S_t\}_{t\geq 0}$. Also let $\Delta \geq 1$. Define for $u, v, y \in \mathbb{Z}^d$, $k, l \geq 1$,*

$$\begin{aligned}\mathscr{E}(u,k,v,l,m) &= \mathscr{E}(u,k,v,l,m,\Delta,y)\\ &= \big\{u + S_\Delta^{(u,k)} = \mathbf{0},\ v + S_\Delta^{(v,l)} = y \text{ and the paths } s \mapsto u + S_s^{(u,k)},\\ &\qquad s \mapsto v + S_s^{(v,l)} \text{ meet exactly } m \text{ times during } (0,\Delta]\big\}.\end{aligned}$$

Then, there exists a $\delta = \delta(d)$ with $0 < \delta(d) \leq 1$ such that uniformly in y and m,

$$\begin{aligned}\sum_{u,v\in\mathbb{Z}^d} \Big| &P\{\mathscr{E}(u,k,v,l,m,\Delta,y)\}\\ &- P\big\{s \mapsto S_s^{(\mathbf{0})} \text{ and } s \mapsto -y + S_s^{(-y)} \text{ meet}\\ &\qquad \text{exactly } m \text{ times during } (0,\infty)\big\}\,\alpha_\Delta(u)\alpha_\Delta(v-y)\Big|\\ &\leq C_{12}\Delta^{-\delta}.\end{aligned} \tag{4.21}$$

REMARK. (viii) We can take

$$\delta(d) = \frac{d-2}{3d^2 - 3d - 4}. \tag{4.22}$$

PROOF. Let $\{S_s'\}_{s\geq 0}$ and $\{S_s''\}_{s\geq 0}$ be two independent copies of $\{S_s\}_{s\geq 0}$. Also let $\{\tilde{S}_s'\}_{s\geq 0}$ and $\{\tilde{S}_s''\}_{s\geq 0}$ be two independent copies of the corresponding time reversed random walk which satisfies (1.4). We first use time reversal to write $P\{\mathscr{E}(u,k,v,l,m)\}$ as

$$\begin{aligned}P\big\{\tilde{S}_\Delta' = u,\ y + \tilde{S}_\Delta'' = v &\text{ and the paths } s \mapsto \tilde{S}_s',\ s \mapsto y + \tilde{S}_s''\\ &\text{meet exactly } m \text{ times during } (0,\Delta]\big\}.\end{aligned}$$

If we put

$$\tilde{\alpha}_s(u) = P\{\tilde{S}_s = -u\} = P\{S_s = u\} = \alpha_s(-u),$$

then

$$\alpha_\Delta(u)\alpha_\Delta(v-y) = \tilde{\alpha}_\Delta(-u)\tilde{\alpha}_\Delta(y-v).$$

Moreover,

$$P\Big\{s \mapsto S_s^{(\mathbf{0},k)} \text{ and } s \mapsto -y + S_s^{(-y,l)} \text{ meet exactly } m \text{ times during } (0,\infty)\Big\}$$
$$= P\Big\{\{S_s^{(\mathbf{0},k)} - S_s^{(-y,l)}\}_{s\geq 0} = -y \text{ for exactly } m \text{ jump times of } \{S_s^{(\mathbf{0},k)} - S_s^{(-y,l)}\}_{s\geq 0}\Big\}$$
$$= P\Big\{\{-S_s^{(\mathbf{0},k)} + S_s^{(y,l)}\}_{s\geq 0} = y \text{ for exactly } m \text{ jump times of } \{-S_s^{(\mathbf{0},k)} + S_s^{(y,l)}\}_{s\geq 0}\Big\}$$
$$= P\{s \mapsto \tilde{S}_s' \text{ and } s \mapsto y + \tilde{S}_s'' \text{ meet exactly } m \text{ times during } (0,\infty)\}.$$

Therefore, the summand in the left-hand side of (4.21) equals

$$\Big|P\{\tilde{S}_\Delta' = u,\, y + \tilde{S}_\Delta'' = v \text{ and the paths } s \mapsto \tilde{S}_s',\, s \mapsto y + \tilde{S}_s'' \text{ meet exactly } m \text{ times during } (0,\Delta]\}$$
$$- P\{s \mapsto \tilde{S}_s' \text{ and } s \mapsto y + \tilde{S}_s'' \text{ meet exactly } m \text{ times during } (0,\infty)\}\tilde{\alpha}_\Delta(-u)\tilde{\alpha}_\Delta(y-v)\Big|.$$

To simplify notation we drop the tildes and introduce

$$\nu(J) := \text{number of times } s \mapsto S_s' \text{ and } s \mapsto y + S_s'' \text{ meet during } J.$$

We shall prove, merely from assumption (1.10), that

$$\sum_{u,v}\Big|P\{S_\Delta' = u,\, y + S_\Delta'' = v,\, \nu((0,\Delta]) = m\} - P\{\nu((0,\infty)) = m\}\alpha_\Delta(-u)\alpha_\Delta(y-v)\Big| \leq C_{12}\Delta^{-\delta}. \tag{4.23}$$

If we apply this to the random walk $\{\tilde{S}_s\}$ [which also satisfies (1.10)] and reverse time we obtain (4.21).

The estimate (4.23) will be obtained by estimating various pieces. For the time being we fix an arbitrary $\delta > 0$. First we drop the sum over the terms with $\|u\| > \Delta^{(1+\delta)/2}$ or $\|v - y\| > \Delta^{(1+\delta)/2}$. Since

$$E\|S_\Delta\|^2 = \sum_{k=0}^{\infty} e^{-\Delta}\frac{\Delta^k}{k!}k\sum_x \|x\|^2 q(x) \leq C_{13}\Delta$$

for some constant $C_{13} < \infty$, we see from Chebyshev's inequality that the terms with such u, v add up to at most

$$P\{\|S_\Delta'\| > \Delta^{(1+\delta)/2}\} + P\{\|S_\Delta''\| > \Delta^{(1+\delta)/2}\} \leq 2C_{13}\Delta^{-\delta}. \tag{4.24}$$

Next we fix $1 \le \Gamma \le \Delta/2$. For the time being, Γ is otherwise arbitrary. We next replace

$$P\{S'_\Delta = u,\, y + S''_\Delta = v,\, \nu((0,\Delta]) = m\}$$

by

$$P\{S'_\Delta = u,\, y + S''_\Delta = v,\, \nu((0,\Gamma]) = m\},$$

and

$$P\{\nu((0,\infty)) = m\}$$

by

$$P\{\nu((0,\Gamma]) = m\}.$$

This changes the left-hand side of (4.23) by at most

$$\begin{aligned}
&2P\{s \mapsto S'_s \text{ and } s \mapsto y + S''_s \text{ meet at least once during } (\Gamma,\infty)\}\\
&\quad \le 4E\{\text{amount of time in } (\Gamma,\infty) \text{ that } S'_s = y + S''_s\} \qquad (4.25)\\
&\quad \le 4\int_\Gamma^\infty P\{S'_s - S''_s = y\}\,ds \le C_{14}\int_\Gamma^\infty \frac{ds}{s^{d/2}} \text{ [by (4.12)]} \le C_{15}\Gamma^{1-d/2}.
\end{aligned}$$

Combining (4.24) and (4.25) we see that the left-hand side of (4.23) is at most

$$\begin{aligned}
&2C_{13}\Delta^{-\delta} + C_{15}\Gamma^{1-d/2}\\
&\quad + \sum_{\|u\|\vee\|v-y\|\le\Delta^{(1+\delta)/2}} \Big| P\{S'_\Delta = u,\, y + S''_\Delta = v,\, \nu((0,\Gamma]) = m\} \qquad (4.26)\\
&\qquad\qquad - P\{\nu((0,\Gamma]) = m\}\alpha_\Delta(-u)\alpha_\Delta(y-v)\Big|.
\end{aligned}$$

Next we fix a $\gamma > 0$ and write

$$\begin{aligned}
&P\{S'_\Delta = u,\, y + S''_\Delta = v,\, \nu((0,\Gamma]) = m\}\\
&\quad = \sum_{a,\,b\in\mathbb{Z}^d} P\{S'_\Gamma = a,\, y + S''_\Gamma = b,\, \nu((0,\Gamma]) = m\}\alpha_{\Delta-\Gamma}(a-u)\alpha_{\Delta-\Gamma}(b-v)\\
&\quad = \sum_{\|a\|,\,\|b-y\|\le\Gamma^{(1+\gamma)/2}} P\{S'_\Gamma = a,\, y + S''_\Gamma = b,\, \nu((0,\Gamma]) = m\}\\
&\qquad\qquad \times \alpha_{\Delta-\Gamma}(a-u)\alpha_{\Delta-\Gamma}(b-v) + E_1,
\end{aligned}$$

where the error $E_1 = E_1(u,v)$ satisfies

$$\begin{aligned}
0 \le E_1 &= \sum_{\|a\|\vee\|b-y\|>\Gamma^{(1+\gamma)/2}} P\{S'_\Gamma = a,\, y + S''_\Gamma = b,\, \nu((0,\Gamma]) = m\}\\
&\qquad \times \alpha_{\Delta-\Gamma}(a-u)\alpha_{\Delta-\Gamma}(b-v) \qquad (4.27)\\
&\le P\{\|S'_\Gamma\| > \Gamma^{(1+\gamma)/2} \text{ or } \|S''_\Gamma\| > \Gamma^{(1+\gamma)/2}\}\sup_{z_1,\,z_2}\alpha_{\Delta-\Gamma}(z_1)\alpha_{\Delta-\Gamma}(z_2)\\
&\le C_{16}\Gamma^{-\gamma}\Delta^{-d} \text{ [by Chebyshev, (4.12) and } \Gamma \le \Delta/2].
\end{aligned}$$

Similarly,

$$P\{\nu((0,\Gamma]) = m\}\alpha_\Delta(-u)\alpha_\Delta(y-v)$$
$$= \sum_{\|a\|,\,\|b-y\|\le\Gamma^{(1+\gamma)/2}} P\{\nu((0,\Gamma]) = m\}$$
$$\times\, \alpha_\Gamma(-a)\alpha_\Gamma(-b+y)\alpha_{\Delta-\Gamma}(a-u)\alpha_{\Delta-\Gamma}(b-v) + E_2,$$

for an error $E_2 = E_2(u, v)$ with

$$0 \le E_2 \le C_{16}\Gamma^{-\gamma}\Delta^{-d}. \tag{4.28}$$

Finally we note that

$$\Bigg|\sum_{\|a\|,\,\|b-y\|\le\Gamma^{(1+\gamma)/2}} P\{S'_\Gamma = a,\, y + S''_\Gamma = b,\, \nu((0,\Gamma]) = m\}$$
$$- \sum_{\|a\|,\,\|b-y\|\le\Gamma^{(1+\gamma)/2}} P\{\nu((0,\Gamma]) = m\}\alpha_\Gamma(-a)\alpha_\Gamma(-b+y)\Bigg|$$
$$\le \Bigg|\sum_{\|a\|,\,\|b-y\|\le\Gamma^{(1+\gamma)/2}} P\{S'_\Gamma = a,\, y + S''_\Gamma = b,\, \nu((0,\Gamma]) = m\} - P\{\nu((0,\Gamma]) = m\}\Bigg|$$
$$+ \Bigg|\sum_{\|a\|,\,\|b-y\|\le\Gamma^{(1+\gamma)/2}} P\{\nu((0,\Gamma]) = m\}\alpha_\Gamma(-a)\alpha_\Gamma(-b+y) - P\{\nu((0,\Gamma]) = m\}\Bigg|$$
$$\le 2P\{\|S'_\Gamma\| > \Gamma^{(1+\gamma)/2} \text{ or } \|S''_\Gamma\| > \Gamma^{(1+\gamma)/2}\} \le 4C_{13}\Gamma^{-\gamma}.$$

Now for any positive measures μ_1, μ_2 of total mass less than or equal to A on some space Ω and a function $f\colon \Omega \mapsto \mathbb{R}$, one has the general and simple inequality

$$\left|\int \mu_1(d\omega)f(\omega) - \int \mu_2(d\omega)f(\omega)\right|$$
$$\le |\mu_1(\Omega) - \mu_2(\Omega)| \sup_\omega |f(\omega)| + A \sup_{\omega_1,\,\omega_2} |f(\omega_1) - f(\omega_2)|.$$

We apply this with

$$\mu_1(a,b) = P\{S'_\Gamma = a,\, y + S''_\Gamma = b,\, \nu((0,\Gamma]) = m\},$$
$$\mu_2(a,b) = P\{\nu((0,\Gamma]) = m\}\alpha_\Gamma(-a)\alpha_\Gamma(-b+y).$$

We then obtain

$$\Bigg| \sum_{a,\, b \in \mathbb{Z}^d} P\{S'_\Gamma = a,\, y + S''_\Gamma = b,\, \nu((0,\Gamma]) = m\} \alpha_{\Delta-\Gamma}(a-u)\alpha_{\Delta-\Gamma}(b-v)$$

$$- \sum_{a,\, b \in \mathbb{Z}^d} P\{\nu((0,\Gamma]) = m\}\alpha_\Gamma(-a)\alpha_\Gamma(-b+y)\alpha_{\Delta-\Gamma}(a-u)\alpha_{\Delta-\Gamma}(b-v)\Bigg|$$

$$\leq E_1 + E_2 + C_{17}\Gamma^{-\gamma}\Delta^{-d} \tag{4.29}$$

$$+ \sup_{\substack{\|a_1-a_2\| \leq 2\Gamma^{(1+\gamma)/2} \\ \|b_1-b_2\| \leq 2\Gamma^{(1+\gamma)/2}}} \Bigg| \alpha_{\Delta-\Gamma}(a_1-u)\alpha_{\Delta-\Gamma}(b_1-v) - \alpha_{\Delta-\Gamma}(a_2-u)\alpha_{\Delta-\Gamma}(b_2-v)\Bigg|.$$

We now sum (4.29) over $\|u\|$, $\|v-y\| \leq \Delta^{(1+\delta)/2}$. Since there are at most $C_{18}\Delta^{d(1+\delta)}$ points u, v satisfying these restrictions, we find by means of (4.26)–(4.29) that the left-hand side of (4.23) is at most

$$2C_{13}\Delta^{-\delta} + C_{15}\Gamma^{1-d/2} + C_{19}\Delta^{d\delta}\Gamma^{-\gamma}$$

$$+C_{18}\Delta^{d(1+\delta)} \sup_{\substack{\|a_1-a_2\| \leq 2\Gamma^{(1+\gamma)/2} \\ \|b_1-b_2\| \leq 2\Gamma^{(1+\gamma)/2} \\ u,\, v}} \Bigg| \alpha_{\Delta-\Gamma}(a_1-u)\alpha_{\Delta-\Gamma}(b_1-v) - \alpha_{\Delta-\Gamma}(a_2-u)\alpha_{\Delta-\Gamma}(b_2-v)\Bigg|. \tag{4.30}$$

Finally, denote by $\Phi_t(\theta) = E\{\exp(i\theta S_t)\}$, $\theta \in \mathbb{R}^d$, the characteristic function of S_t. Then standard arguments [compare Spitzer (1976), Propositions 7.7, 7.8] show that there exists some $C_{20}, C_{21} > 0$, $\eta > 0$ such that

$$|\Phi_t(\theta)| \leq \exp(-C_{20}t\|\theta\|^2) \quad \text{for } \|\theta\| \leq \eta$$

and

$$|\Phi_t(\theta)| \leq \exp(-C_{21}t) \quad \text{for } \eta < \|\theta\|,\, \theta \in [-\pi, \pi]^d.$$

Consequently,

$$\sup_{\substack{\|a_1-a_2\| \leq 2\Gamma^{(1+\gamma)/2} \\ \|b_1-b_2\| \leq 2\Gamma^{(1+\gamma)/2} \\ u,\, v}} \Big|\alpha_{\Delta-\Gamma}(a_1-u)\alpha_{\Delta-\Gamma}(b_1-v) - \alpha_{\Delta-\Gamma}(a_2-u)\alpha_{\Delta-\Gamma}(b_2-v)\Big|$$

$$\leq 2 \sup_{\|c_1-c_2\| \leq 2\Gamma^{(1+\gamma)/2}} |\alpha_{\Delta-\Gamma}(c_1) - \alpha_{\Delta-\Gamma}(c_2)| \sup_v \alpha_{\Delta-\Gamma}(v)$$

$$\leq C_{22}\Delta^{-d/2} \sup_{\|c_1-c_2\| \leq 2\Gamma^{(1+\gamma)/2}} \int_{\theta \in [-\pi,\pi]^d} |\exp(-i\theta c_1) - \exp(-i\theta c_2)| \times |\Phi_{\Delta-\Gamma}(\theta)|\, d\theta \tag{4.31}$$

$$\leq C_{23}\Delta^{-d/2}\Gamma^{(1+\gamma)/2}\Delta^{-(d+1)/2}.$$

Substituting this estimate into (4.30) yields the upper bound

$$C_{24}\Big[\Delta^{-\delta}+\Gamma^{1-d/2}+\Delta^{d\delta}\Gamma^{-\gamma}+\Delta^{d\delta-1/2}\Gamma^{(1+\gamma)/2}\Big]$$

for the left-hand side of (4.23). It remains to choose

$$\Gamma=\Delta^{1/(1+3\gamma)},$$
$$\gamma=\frac{(d+1)\delta}{1-3(d+1)\delta}=\frac{(d+1)(d-2)}{2},$$
$$\delta=\frac{d-2}{3d^2-3d-4},$$

to find that (4.23) holds for the given δ. □

We define

$$\rho(m,y)=P\big\{s\mapsto S_s^{(\mathbf{0})}\text{ and }s\mapsto -y+S_s^{(-y)}\text{ meet exactly } m \text{ times during }[0,\infty)\big\}\tag{4.32}$$

and

$$D(y)=p_1\sum_{m=0}^{\infty}(1-p_1)^m\rho(m,y).\tag{4.33}$$

We also define $\Lambda_t^*(u,v)$ as the number of ordered pairs of distinct particles, the first particle being present at u at time t and the second particle at v at time t. Comparison with (4.6) shows immediately that $\Lambda_t^*(u,v)\le\Lambda_t(u,v)$.

LEMMA 13. *Let $1\le\Delta\le t/2$. Then for $d\ge 5$, $0<\varepsilon<1/2$, and uniformly in $y\ne\mathbf{0}$,*

$$\begin{aligned}&\Big|E\{\xi_t(\mathbf{0})p_{\xi_t(y)}\}-D(y)\sum_{u,v\in\mathbb{Z}^d}E\{\Lambda_{t-\Delta}^*(u,v)\}\alpha_\Delta(u)\alpha_\Delta(v-y)\Big|\\&\quad\le C_{25}\Delta[t^{-3}\vee t^{-d(1-\varepsilon)/2}]+C_{25}\Delta^{-\delta(d)}t^{-2}.\end{aligned}\tag{4.34}$$

PROOF. First we observe that, by virtue of (4.6) and (4.8), it holds for $y\ne\mathbf{0}$ that

$$\begin{aligned}&\big|E\{\xi_t(\mathbf{0})p_{\xi_t(y)}\}-p_1P\{\xi_t(\mathbf{0})=\xi_t(y)=1\}\big|\\&\quad\le E\big\{\xi_t(\mathbf{0})\xi_t(y)I[\xi_t(y)\ge 2]\big\}+E\big\{\xi_t(\mathbf{0})I[\xi_t(\mathbf{0})\ge 2]\xi_t(y)\big\}\\&\quad\le E\Lambda_t(\mathbf{0},y,y)+E\Lambda_t(\mathbf{0},\mathbf{0},y)\\&\quad\le C_2(\varepsilon,3)\big[t^{-3}\vee t^{-d(1-\varepsilon)/2}\big].\end{aligned}\tag{4.35}$$

Next we approximate

$$P\big\{\xi_t(\mathbf{0})=\xi_t(y)=1\big\}.\tag{4.36}$$

It is easiest to carry out this part of the proof if we construct ξ_t as in Lemma 8, so that we can speak of the trajectory of a particle. We only know how to carry out such a construction for a system with a finite initial state. Formally, the remaining estimates in this lemma must therefore be carried out for the process ξ_N, and then the limit $N \to \infty$ must be taken in the final estimates (4.39) and (4.41) below. For simplicity we have written the proof as if it applies directly to the full process ξ. Here we want to condition on $\mathscr{F}_{t-\Delta}$, so that we think of $t-\Delta$ as the origin of the time axis. Thus, the label (x, k) refers to the kth particle at position x at time $t-\Delta$. Then $s \mapsto \{x + S_s^{(x,k)}\}_{s\ge 0}$ describes the motion of this particle until it coalesces; that is, its position at time $t-\Delta+s$ is $x + S_s^{(x,k)}$, if it did not coalesce during $(t-\Delta, t-\Delta+s]$. Of course we take the $\{S_s^{(x,k)}\}_{s\ge 0}$ to be independent copies of $\{S_s\}_{s\ge 0}$. If $\xi_t(\mathbf{0}) = \xi_t(y) = 1$, then there must be two different particles, π' and π'', say, in the system at time $t-\Delta$ which move to $\mathbf{0}$ and y, respectively, at time t, without coalescing with another particle during $(t-\Delta, t]$. Let the positions of these particles at time $t-\Delta$ be u and v, respectively. Then there must exist $1 \le k \le \xi_{t-\Delta}(u)$, $1 \le l \le \xi_{t-\Delta}(v)$ and random walk paths $s \mapsto S_s^{(u,k)}$, $s \mapsto S_s^{(v,l)}$ with $u + S_\Delta^{(u,k)} = \mathbf{0}$, $v + S_\Delta^{(v,l)} = y$.

As a first step in approximating (4.36) we bound the probability of the event $\mathscr{G}$ that there exist two different particles π', π'' with labels (u, k) and (v, l), which move along the trajectories $s \mapsto u + S_s^{(u,k)}$, $s \mapsto v + S_s^{(v,l)}$ for $0 \le s \le \Delta$, satisfying $u + S_\Delta^{(u,k)} = \mathbf{0}$, $v + S_\Delta^{(v,l)} = y$, and that there exists another particle π such that π coincides with π' or π'' at some time $s \in [0, \Delta]$. In order to estimate $P\{\mathscr{G}\}$ we write $\mathscr{G}$ as the union of several subevents. The first subevent, $\mathscr{G}_1$, is the event that there are at least two particles present at u at time $s = 0$, one of which is the particle π' and the other is distinct from π' and π''. The conditional probability of $\mathscr{G}_1$ given $\mathscr{F}_{t-\Delta}$ is at most

$$\sum_{u,v} \Lambda_{t-\Delta}(u, u, v)\alpha_\Delta(u)\alpha_\Delta(v-y).$$

Taking expectations and using (4.8) we find that

$$P\{\mathscr{G}_1\} \le \sum_{u,v} E\{\Lambda_{t-\Delta}(u, u, v)\}\alpha_\Delta(u)\alpha_\Delta(v-y) \le C_2\big[t^{-3} \vee t^{-d(1-\varepsilon)/2}\big].$$

Similarly, the subevent $\mathscr{G}_2$ of $\mathscr{G}$ on which there are two particles starting at v, one of which is the particle π'', has probability at most

$$C_2[t^{-3} \vee t^{-d(1-\varepsilon)/2}].$$

Another way in which $\mathscr{G}$ can occur is that at some time during $[0, \Delta]$ a particle π, which was at some vertex w at time 0, jumps onto the trajectory of π' or of π''. Let $\mathscr{G}_3$ be the subevent that such a jump occurs. Decomposing with respect to the time of the jump and the positions z' and z just before and after the jump we find that the conditional probability of $\mathscr{G}_3$ given $\mathscr{F}_{t-\Delta}$ is at most

$$\sum_{u,v,w} \Lambda_{t-\Delta}(u, v, w)\int_0^\Delta \Bigg[\sum_{z,z'} \alpha_s(u-z)\alpha_s(w-z')q(z-z')\alpha_{\Delta-s}(z)\alpha_\Delta(v-y) + \alpha_\Delta(u)\sum_{z,z'} \alpha_s(v-z)\alpha_s(w-z')q(z-z')\alpha_{\Delta-s}(z-y)\Bigg]ds.$$

Taking expectation we find

$$
\begin{aligned}
P\{\mathscr{G}_3\} \le \sum_{u,v,w} & E\{\Lambda_{t-\Delta}(u,v,w)\} \\
& \times \int_0^\Delta \Bigg[\sum_{z,z'} \alpha_s(u-z)\alpha_s(w-z')q(z-z')\alpha_{\Delta-s}(z)\alpha_\Delta(v-y) \\
& \qquad + \alpha_\Delta(u)\sum_{z,z'} \alpha_s(v-z)\alpha_s(w-z')q(z-z')\alpha_{\Delta-s}(z-y)\Bigg]\,ds \\
\le C_2\big[t^{-3} \vee t^{-d(1-\varepsilon)/2}\big] & \\
\times \sum_{u,v,w} \int_0^\Delta & \Bigg[\sum_{z,z'} \alpha_s(u-z)\alpha_s(w-z')q(z-z')\alpha_{\Delta-s}(z)\alpha_\Delta(v-y) \\
& \qquad + \alpha_\Delta(u)\sum_{z,z'} \alpha_s(v-z)\alpha_s(w-z')q(z-z')\alpha_{\Delta-s}(z-y)\Bigg]\,ds \\
& \qquad\qquad\qquad [\text{by } (4.8)] \\
= C_2\big[t^{-3} \vee t^{-d(1-\varepsilon)/2}\big] & \int_0^\Delta 2\,ds = 2C_2\Delta\big[t^{-3} \vee t^{-d(1-\varepsilon)/2}\big].
\end{aligned}
$$

Finally, the same estimate holds for the probability of the subevent $\mathscr{G}_4$ that at some time during $[0, \Delta]$ the particle π' or the particle π'' jumps to a position which is already occupied by a particle π which started at some position w at time $s = 0$. Thus, if $\Delta \ge 1$,

$$
P\{\mathscr{G}\} \le \sum_{i=1}^{4} P\{\mathscr{G}_i\} \le 4C_2\Delta[t^{-3} \vee t^{-d(1-\varepsilon)/2}]. \tag{4.37}
$$

Now on the complement of $\mathscr{G}$, $\{\xi_t(\mathbf{0}) = \xi_t(y) = 1\}$ occurs if and only if the following two events occur:

1. There exist $u, v \in \mathbb{Z}^d$ and a pair of particles π', π'' located at u, v, respectively, at time $t - \Delta$, which move to $\mathbf{0}$ and y, respectively, at time t.
2. At each of the jumptimes of π' or π'' at which these two particles meet during $(t - \Delta, t]$, the corresponding $U_n^{\pi'}$ or $U_n^{\pi''}$ exceeds p_1 (see proof of Lemma 8 for U_n^{π}).

In explanation of (2) we point out that we do not want π' and π'' to coalesce. However, on $\mathscr{G}^c$, neither π' nor π'' coincide with a third particle π during $[t - \Delta, t]$. Thus, when π' jumps to the position of π'', then it jumps to a site which contains exactly one particle. If this is the nth jump of π', then no coalescence takes place if and only if $U_n^{\pi'} > p_1$. A similar statement holds for π''.

Conditionally on $\mathscr{F}_{t-\Delta}$, the probability of (1) and (2) is

$$\sum_{m=0}^{\infty}(1-p_1)^m \times P\Bigg\{\bigcup_{u,\,v\in\mathbb{Z}^d}\ \bigcup_{\substack{1\le k\le\xi_{t-\Delta}(u)\\ 1\le \ell\le\xi_{t-\Delta}(v)\\ (u,k)\ne(v,l)}}\Big\{u+S_\Delta^{(u,k)}=\mathbf{0},\, v+S_\Delta^{(v,l)}=y \text{ and } s\mapsto u+S_s^{(u,k)} \text{ and } s\mapsto v+S_s^{(v,l)} \text{ meet exactly } m \text{ times during } (0,\Delta]\Big\}\Bigg\}. \tag{4.38}$$

Now (4.38) (with $\mathscr{E}$ as in Lemma 12) shows that

$$\begin{aligned}
&P\{\xi_t(\mathbf{0})=\xi_t(y)=1|\mathscr{F}_{t-\Delta}\}\\
&\quad\le P\{\mathscr{G}|\mathscr{F}_{t-\Delta}\}+\sum_{m=0}^{\infty}(1-p_1)^m\sum_{u,\,k,\,v,\,l}P\{\mathscr{E}(u,k,v,l,m\}\\
&\quad= P\{\mathscr{G}|\mathscr{F}_{t-\Delta}\}+\sum_{m=0}^{\infty}(1-p_1)^m\sum_{u,\,v}\Lambda_{t-\Delta}^*(u,v)\\
&\qquad\qquad\times\big[P\{\mathscr{E}(u,1,v,1,m,\Delta,y)\}-\rho(m,y)\alpha_\Delta(u)\alpha_\Delta(v-y)\big]\\
&\qquad+\sum_{m=0}^{\infty}(1-p_1)^m\rho(m,y)\sum_{u,\,v}\Lambda_{t-\Delta}^*(u,v)\alpha_\Delta(u)\alpha_\Delta(v-y).
\end{aligned}\tag{4.39}$$

Taking expectation once more and using (4.37) and Lemmas 10 and 12 we find

$$\begin{aligned}
&p_1P\{\xi_t(\mathbf{0})=\xi_t(y)=1\}\\
&\quad\le 4p_1C_2\Delta[t^{-3}\vee t^{-d(1-\varepsilon)/2}]+p_1\sum_{m=0}^{\infty}(1-p_1)^m\frac{C_1}{(t-\Delta)^2}\\
&\qquad\qquad\times\sum_{u,\,v}\big|P\{\mathscr{E}(u,1,v,1,m,\Delta,y)-\rho(m,y)\alpha_\Delta(u)\alpha_\Delta(v-y)\big|\\
&\qquad+p_1\sum_{m=0}^{\infty}(1-p_1)^m\rho(m,y)\sum_{u,\,v}E\{\Lambda_{t-\Delta}^*(u,v)\}\alpha_\Delta(u)\alpha_\Delta(v-y)\\
&\quad\le 4p_1C_2\Delta[t^{-3}\vee t^{-d(1-\varepsilon)/2}]+4C_1C_{12}\Delta^{-\delta(d)}t^{-2}\\
&\qquad+D(y)\sum_{u,\,v}E\{\Lambda_{t-\Delta}^*(u,v)\}\alpha_\Delta(u)\alpha_\Delta(v-y).
\end{aligned}\tag{4.40}$$

In the other direction, we have from the inclusion–exclusion principle that

$$
\begin{aligned}
&P\{\xi_t(\mathbf{0}) = \xi_t(y) = 1 \mid \mathscr{F}_{t-\Delta}\} \\
&\geq -P\{\mathscr{G} \mid \mathscr{F}_{t-\Delta}\} + \sum_{m=0}^{\infty} (1-p_1)^m \\
&\quad \times P\Bigg\{ \bigcup_{u, v \in \mathbb{Z}^d} \bigcup_{\substack{1 \leq k \leq \xi_{t-\Delta}(u) \\ 1 \leq l \leq \xi_{t-\Delta}(v) \\ (u,k) \neq (v,l)}} \Big\{ u + S_\Delta^{(u,k)} = \mathbf{0},\, v + S_\Delta^{(v,l)} = y \text{ and } s \mapsto u + S_s^{(u,k)} \text{ and } s \mapsto v + S_s^{(v,l)} \text{ meet} \\
&\qquad\qquad \text{exactly } m \text{ times during } [0, \Delta] \Big\} \Bigg\} \qquad (4.41) \\
&\geq -P\{\mathscr{G} \mid \mathscr{F}_{t-\Delta}\} + \sum_{m=0}^{\infty} (1-p_1)^m \sum_{u,k,v,l} P\{\mathscr{E}(u,k,v,l,m)\} \\
&\quad - \sum_{m=0}^{\infty} (1-p_1)^m \sum_{u_i, k_i} P\big\{\mathscr{E}(u_1,k_1,u_2,k_2,m) \cap \mathscr{E}(u_3,k_3,u_4,k_4,m)\big\},
\end{aligned}
$$

where the last sum is over all 4-tuples $(u_1, k_1), \dots, (u_4, k_4)$ with $u_i \in \mathbb{Z}^d$, $1 \leq k_i \leq \xi_{t-\Delta}(u_i)$ and $(u_1, k_1) \neq (u_2, k_2)$, $(u_3, k_3) \neq (u_4, k_4)$, $\{(u_1, k_1), (u_2, k_2)\} \neq \{(u_3, k_3), (u_4, k_4)\}$. Let us first estimate the contribution to this sum from the 4-tuples with all four (u_i, k_i) distinct. Then for given $u_1, \dots, u_4$ we get a contribution

$$
\begin{aligned}
&\sum_{\substack{k_1,\dots,k_4 \\ \text{with all } (u_i,k_i) \text{ distinct}}} P\{\mathscr{E}(u_1,k_1,u_2,k_2,m) \cap \mathscr{E}(u_3,k_3,u_4,k_4,m)\} \\
&\qquad \leq \Lambda_{t-\Delta}(u_1,u_2,u_3,u_4)\alpha_\Delta(u_1)\alpha_\Delta(u_2-y)\alpha_\Delta(u_3)\alpha_\Delta(u_4-y).
\end{aligned}
$$

After taking the expectation and multiplying by $(1-p_1)^m$ and summing over u_i, m these terms contribute at most

$$
\begin{aligned}
&\frac{1}{p_1} \sum_{u_1,\dots,u_4} E\{\Lambda_{t-\Delta}(u_1,u_2,u_3,u_4)\}\alpha_\Delta(u_1)\alpha_\Delta(u_2-y)\alpha_\Delta(u_3)\alpha_\Delta(u_4-y) \\
&\quad \leq \frac{1}{p_1} C_2(\varepsilon, 4)\big[(t/2)^{-4} \vee (t/2)^{-d(1-\varepsilon)/2}\big] \\
&\qquad \times \sum_{u_1,\dots,u_4} \alpha_\Delta(u_1)\alpha_\Delta(u_2-y)\alpha_\Delta(u_3)\alpha_\Delta(u_4-y) \\
&\quad \leq C_{26}\big[(t/2)^{-4} \vee (t/2)^{-d(1-\varepsilon)/2}\big].
\end{aligned}
$$

Similarly the sum of the $P\{\mathscr{E}(u_1,k_1,u_2,k_2,m) \cap \mathscr{E}(u_3,k_3,u_4,k_4,m)\}$ over the (u_i, k_i) with only three distinct pairs contributes at most $C_{26}\big[(t/2)^{-3} \vee (t/2)^{-d(1-\varepsilon)/2}\big]$. Combining these estimates and taking expectation again, we obtain

$$P\{\xi_t(\mathbf{0}) = \xi_t(y) = 1\}$$

$$\geq -P\{\mathscr{G}\} + \sum_{m=0}^{\infty} (1-p_1)^m \sum_{u,k,v,l} P\{\mathscr{E}(u,k,v,l,m)\} - C_{27}[t^{-3} \vee t^{-d(1-\varepsilon)/2}].$$

Continuing as in (4.39) and (4.40) this yields

$$(4.42)\quad \begin{aligned} p_1 P\{\xi_t(\mathbf{0}) = \xi_t(y) = 1\} \geq D(y) \sum_{u,v} E\{\Lambda^*_{t-\Delta}(u,v)\}\alpha_\Delta(u)\alpha_\Delta(v-y) \\ - C_{28}\Delta[t^{-3} \vee t^{-d(1-\varepsilon)/2}] - C_{29}\Delta^{-\delta(d)}t^{-2}. \end{aligned}$$

Together with (4.35) and (4.40) this gives (4.34). □

PROOF OF THEOREM. Let $d \geq 6$. Then choose $\Delta = t^{1-\eta}$ with $0 < \eta < 1$ so small that, for all large t,

$$(4.43)\quad \log(t+2)\Delta^{-d/2} \leq t^{-5/2}.$$

After that choose $\varepsilon \in (0, 1/2)$ so small that

$$(4.44)\quad \Delta t^{-d(1-\varepsilon)/2} \leq t^{-2-\eta/2}.$$

Lemmas 9 and 13 then show that there exists some $\zeta = \zeta(d) \in (0, \eta \wedge \frac{1}{2})$ and some constant $C_{30} < \infty$ such that

$$(4.45)\quad \begin{aligned} \left|\frac{d}{dt}E(t) + \sum_y q(y)D(y) \sum_{u,v} E\{\Lambda^*_{t-\Delta}(u,v)\}\alpha_\Delta(u)\alpha_\Delta(v-y)\right| \\ \leq C_{30}t^{-2-\zeta}. \end{aligned}$$

In addition, by the definition of $\Lambda^*_{t-\Delta}(u,v)$,

$$\sum_{u,v} \Lambda^*_{t-\Delta}(u,v)\alpha_\Delta(u)\alpha_\Delta(v-y)$$

$$= \sum_u \alpha_\Delta(u)\xi_{t-\Delta}(u) \sum_v \alpha_\Delta(v-y)\xi_{t-\Delta}(v) - \sum_u \alpha_\Delta(u)\alpha_\Delta(u-y)\xi_{t-\Delta}(u).$$

Therefore, by (3.9), (3.4) and (4.12), there exists a constant C_{31}, independent of y such that

$$\begin{aligned}
&\Bigg|\sum_{u,v} E\{\Lambda^*_{t-\Delta}(u,v)\}\alpha_\Delta(u)\alpha_\Delta(v-y) \\
&\quad - E\Big\{\sum_u \alpha_\Delta(u)\xi_{t-\Delta}(u)\Big\} E\Big\{\sum_v \alpha_\Delta(v-y)\xi_{t-\Delta}(v)\Big\}\Bigg| \\
&\quad \le \sigma\Big(\sum_u \alpha_\Delta(u)\xi_{t-\Delta}(u)\Big)\sigma\Big(\sum_v \alpha_\Delta(v-y)\xi_{t-\Delta}(v)\Big) \\
&\qquad + \frac{C_2}{t}\sum_u \alpha_\Delta(u)\alpha_\Delta(u-y) \\
&\quad \le C_0 \log(t+2)\sum_u \alpha_\Delta^2(u) + \frac{C_2}{t}\sup_u \alpha_\Delta(u) \\
&\quad \le C_{31}\frac{\log(t+2)}{\Delta^{d/2}}.
\end{aligned} \tag{4.46}$$

Substitution of this estimate into (4.45) and use of (4.43) yields

$$\begin{aligned}
&\left|\frac{d}{dt}E(t) + \sum_y q(y)D(y)E\Big\{\sum_u \alpha_\Delta(u)\xi_{t-\Delta}(u)\Big\}E\Big\{\sum_v \alpha_\Delta(v-y)\xi_{t-\Delta}(v)\Big\}\right| \\
&\quad \le C_{30}t^{-2-\zeta} + C_{31}\frac{\log(t+2)}{\Delta^{d/2}} \le 2C_{30}t^{-2-\zeta}.
\end{aligned} \tag{4.47}$$

Moreover,

$$\sum_y q(y)D(y) = C(d). \tag{4.48}$$

Now for $\xi_t(y) \neq 0$ to occur, there must be at least one particle in the system at time $t-\Delta$ which moves to y during $[t-\Delta, t]$ without coalescing. The same arguments as in Lemma 13 (but easier) now show that

$$\begin{aligned}
&\left|E\xi_t(y) - E\Big\{\sum_v \alpha_\Delta(v-y)\xi_{t-\Delta}(v)\Big\}\right| \\
&\quad \le \sum_v E\{\text{number of particles } \pi' \text{ which are at } v \text{ at time } t-\Delta \\
&\qquad\quad \text{and reach } y \text{ at time } t, \text{ but which do coincide with} \\
&\qquad\quad \text{some other particle } \pi \text{ during } [t-\Delta, t]\}
\end{aligned}$$

$$\le \sum_v E\{\Lambda_{t-\Delta}(v,v)\}\alpha_\Delta(v-y)$$

$$(4.49)\qquad + 2\sum_v \int_0^\Delta \sum_{z,z',w} E\{\Lambda_{t-\Delta}(v,w)\}\alpha_s(v-z)\alpha_s(w-z') \times q(z-z')\alpha_{\Delta-s}(z-y)ds$$

$$\le C_{32}\Delta t^{-2} = C_{32}t^{-1-\eta} \le C_{32}t^{-1-\zeta}.$$

This estimate is uniform in $y \in \mathbb{Z}^d$, by translation invariance. Combined with (4.47), (4.48) and (3.9) this yields

$$\left|\frac{d}{dt}E(t) + C(d)E^2(t)\right| \le C_{33}t^{-2-\zeta} \le C_{34}t^{-\zeta}E^2(t), \qquad t \ge 1.$$

Integration now gives

$$\frac{1}{E(t)} - \frac{1}{E(0)} = -\int_0^t E^{-2}(s)\frac{dE(s)}{ds}ds = C(d)t + O(t^{1-\zeta}),$$

from which (1.13) follows. Then (1.11) and (1.14) follow from Lemma 11. □

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