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A THEOREM OF TRUEMPER*

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An important theorem due to Truemper characterizes the graphs whose edges can be labeled so that all chordless cycles have prescribed parities. This theorem has proven to be an essential tool in the study of various objects like balanced matrices, graphs with no even length chordless cycle and graphs with no odd length chordless cycle with at least five edges. In this paper we prove this theorem in a novel and elementary way and derive some of its consequences. In particular, we show an easy way to obtain Tutte's characterization of regular matrices.

1. Truemper's theorem

Let β be a 0,1 vector indexed by the chordless cycles of an undirected graph $G=(V,E)$. In this paper, we consider the following system of linear equations over $GF(2)$:

(1) $l(C)=\beta_C \bmod 2$ *for every chordless cycle* C *of* G,

where $l(C):=\sum_{e\in E(C)} l(e)$. A 0,1 labeling l of the edges of G satisfying (1) is called a *β-balancing* of G. If G admits a β-balancing it is called *β-balanceable*.

We denote by β^H the restriction of the vector β to the chordless cycles of an induced subgraph H of G. In [14], Truemper showed the following theorem:

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Theorem 1.1. *A graph G is β-balanceable if and only if every induced subgraph H that is a 3-path configuration or a wheel (Figure 1.) is β^H-balanceable.*

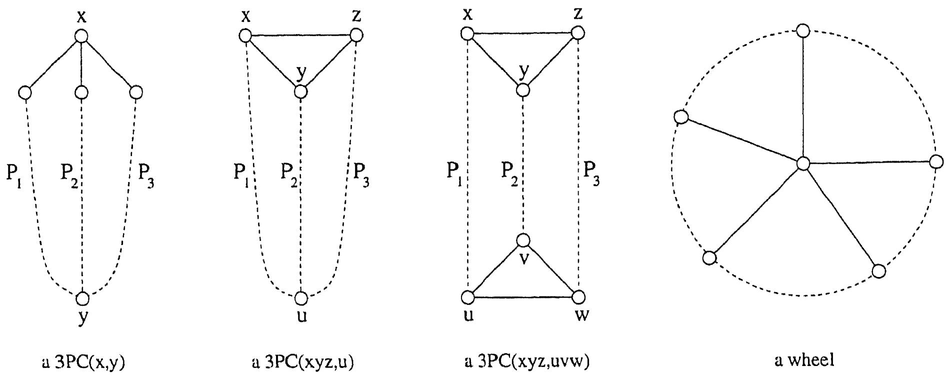

Fig. 1. 3-path configurations and a wheel

There are three types of *3-path configurations* ($3PC$'s): a $3PC(x,y)$, where node x and node y are connected by three internally disjoint paths P_1, P_2 and P_3; a $3PC(xyz,u)$, where xyz is a triangle and P_1, P_2 and P_3 are three internally disjoint paths with endnodes x, y and z respectively and a common endnode u; and a $3PC(xyz,uvw)$, which consists of two node disjoint triangles xyz and uvw and three disjoint paths P_1, P_2 and P_3 with endnodes x and u, y and v, and z and w respectively. In all three cases the nodes of $P_i \cup P_j$, $i \neq j$, must induce a chordless cycle. This implies that all paths P_1, P_2, P_3 of a $3PC(x,y)$ have length greater than one. A *wheel* is a graph (C,x) consisting of a chordless cycle C and a node $x \notin V(C)$ that has at least three neighbors on C. We call C the *rim* and x the *center* of the wheel (C,x). Note that a $3PC(xyz,u)$ may also be a wheel.

From standard linear algebra it follows that the system of linear equations (1) is infeasible if and only if

(2) *G contains chordless cycles $C_1, \ldots, C_k$ such that $C_1 \Delta \cdots \Delta C_k = \emptyset$ and* $\beta_{C_1} + \cdots + \beta_{C_k} = 1 \bmod 2$.

So, (2) provides a co-NP characterization of β-balanceability of G. Theorem 1.1 states that the chordless cycles of G satisfying (2) can be chosen so that their support graph is a 3-path configuration or a wheel, which substantially sharpens (2).

In this paper, we give an alternative simple proof of Theorem 1.1 and we highlight its importance by deriving some well known theorems, such as Tutte's characterization of regular matrices, the characterization of balanceable matrices, and of even, odd and universally signable graphs.

A derivation of Tutte's characterization of regular matroids from Theorem 1.1 has already been given by Truemper in [13]. In fact, in [14], he derived from Theorem 1.1 Reid's characterization of ternary matroids [1], [11], which generalizes Tutte's result. Our derivation of Tutte's result is more direct. Truemper's theorem also played a role in the proof of another extension of Tutte's result, namely Geelen's characterization of the symmetric $0, \pm 1$ matrices in which all principal submatrices have $0, \pm 1$ determinants [9].

Our proof of Theorem 1.1 is divided into two parts. First we derive two graph-theoretic lemmas on the occurrence of 3-path configurations and wheels. Next these results are used in the second part of the proof, which is more explicitly concerned with the linear algebra involved in solving the linear system (1). Throughout the paper, $N(v)$ will denote the set of neighbors of node v.

Lemma 1.1. *Let C be a chordless cycle of G with $G \neq C$ such that $V(C)$ contains no K_2 cutset of G. Then C is contained in a 3-path configuration or a wheel in G.*

Proof. Let G and C form a counterexample. First assume that C is not a triangle. Choose two nonadjacent nodes u^* and w^* in C and a u^*w^*-path $P = u^*, u, \ldots, w, w^*$ whose intermediate nodes and edges are $G \setminus V(C)$ such that P is as short as possible. The existence of such a pair of nodes u^* and w^* follows because $G \neq C$ and $V(C)$ contains no K_2 cutset. As C is not contained in a 3-path configuration or a wheel, u and v are distinct. For the same reason, both $U := N(u) \cap V(C)$ and $W := N(w) \cap V(C)$ consist of a single node or two adjacent nodes.

Let Y be the set of nodes in C that have a neighbor in $V(P) \setminus \{u^*, u, w, w^*\}$. Y is nonempty as otherwise $P \cup C$ induces a 3-path configuration (if $U \cap W = \emptyset$) or a wheel (if $U \cap W \neq \emptyset$). By the minimality of P, the nodes of $Y \cup U$ are pairwise adjacent. Hence, $|Y \cup U| \leq 2$. So, as $u^* \notin Y$, we have that $|Y| = |U \setminus Y| = 1$ and, by symmetry, also $|W \setminus Y| = 1$. But then $C \cup P$ induces a wheel with the single node in Y as its center, a contradiction.

So $C = c_1, c_2, c_3$ is a triangle. As $G \neq C$ and as $\{c_1, c_2\}$ and $\{c_1, c_3\}$ are no cutsets of G, the edge c_2c_3 is not an edge cutset of $G \setminus \{c_1\}$. Hence, there exists a chordless cycle C' in $G \setminus \{c_1\}$ containing c_2c_3. As $\{c_2, c_3\}$ is not a K_2 cutset, there exists for each such C' a c_1x-path Q in $G \setminus V(C'')$ such that x

is adjacent to a node in $V(C_1)\setminus\{c_2,c_3\}$. Now select C' and Q such that Q is as short as possible. As $C\cup C'$ is not a wheel, $N(c_1)\cap V(C')=\{c_2,c_3\}$; in particular, $x\neq c_1$. By the minimality of Q, x has at most two neighbors in C' and if it has two, they are adjacent. There exists a $y\in V(Q)\setminus\{c_1\}$ adjacent to c_2 or c_3, because otherwise $C\cup C'\cup Q$ would be a 3-path configuration or, in case x is adjacent to c_2 or c_3, a wheel. Choose such y closest to c_1 in Q and assume that y is adjacent to c_2. Any c_1c_3-path with nodes in $(V(Q)\cup V(C'))\setminus\{c_2\}$ and not using edge c_1c_3 contains y, so a shortest such path induces with C a wheel with center c_2, a contradiction. ∎

For $e\in E(G)$, G^e denotes the graph whose node set represents the chordless cycles of G containing e and whose edges are the pairs C_1,C_2 in $V(G^e)$ for which there exists a 3-path configuration or a wheel containing both C_1 and C_2.

Lemma 1.2. *If $e=uv$ is not a K_2 cutset of G, G^e is connected.*

Proof. Assume not. Choose two chordless cycles C_1 and C_2 of G in different components of G^e with the distance $d(C_1,C_2)$ of $V(C_1)\setminus\{u,v\}$ and $V(C_2)\setminus\{u,v\}$ in $G\setminus\{u,v\}$ minimal and, subject to this, $|V(C_1)\cup V(C_2)|$ minimal. Choose an st-path P in $C_1\setminus\{e\}$ with $V(P)\cap V(C_2)=\{s,t\}$. Let Q be the st-path in C_2 through e.

We first prove $P\cup Q=C_1$. If not, both $V(C_1)\cup V(P)\cup V(Q)$ and $V(C_2)\cup V(P)\cup V(Q)$ are properly contained in $V(C_1)\cup V(C_2)$. Let C be a chordless cycle through e with nodes in $V(P)\cup V(Q)$. Then $C\neq C_1$, $C\neq C_2$, $d(C_1,C)=d(C,C_2)=0$, and $|V(C)\cup V(C_2)|$ and $|V(C_1)\cup V(C)|$ are both smaller than $|V(C_1)\cup V(C_2)|$. Now C and C_2 or C_1 and C contradict the choice of C_1 and C_2. So $P\cup Q=C_1$.

Let T be a shortest path from $V(C_1)\setminus\{u,v\}$ to $V(C_2)\setminus\{u,v\}$ in $G\setminus\{u,v\}$. (Note that, T may be a single node in $V(C_1)\cap V(C_2)\setminus\{u,v\}$.) C_1 contains no K_2 cutset of the graph G' induced by C_1, C_2 and T. Hence by Lemma 1.1, G' contains a chordless cycle $\tilde{C}_1$ adjacent to C_1 in G^e. $V(\tilde{C}_1)\setminus V(C_1)$ is obviously nonempty, so by the choice of C_1 and C_2, $d(C_1,C_2)=d(\tilde{C}_1,C_2)=0$. As $P\cup Q=C_1$, all intermediate nodes of Q have degree 2 in G', so $\tilde{C}_1$ contains Q. As $\tilde{C}_1\neq C_1$, $V(\tilde{C}_1)\cup V(C_2)$ is properly contained in $V(C_1)\cup V(C_2)$, contradicting the choice of C_1 and C_2. ∎

The rest of the proof of Theorem 1.1 is mainly algebraic, concerning the solvability of the linear system (1). For this we need two easy facts from the linear algebra over $GF(2)$ of circuits and cuts in a graph. By $\chi_{\delta(U)}$ we will denote the characteristic vector of the subset $\delta(U)$ of $E(G)$ consisting of the edges leaving node set $U\subseteq V(G)$.

Lemma 1.3. *If l is a β-balancing and l' a $0,1$ labeling of the edges of G, then l' is a β-balancing of G if and only if $l' = l + \chi_{\delta(U)} \bmod 2$ for some $U \subseteq V(G)$.*

Proof. l' is a β-balancing of G if and only if $\nu := l + l'$ satisfies $\nu(C) = 0 \bmod 2$ for each chordless cycle C in G. As each cycle of G is the symmetric difference of chordless cycles in G, the latter is equivalent to $\nu(C) = 0 \bmod 2$ for each cycle C of G. Now it is easy to see that this is equivalent to $\nu = \chi_{\delta(U)}$ for some $U \subseteq V(G)$. ∎

Corollary 1.1. *If G' is an induced subgraph of a β-balanceable graph G, then each $\beta^{G'}$-balancing of G' extends to a β-balancing of G.*

Proof. Let l be a β-balancing of G and l' be a $\beta^{G'}$-balancing of G'. Then the restriction $l^{G'}$ of l to G' is a $\beta^{G'}$-balancing. By Lemma 1.3, $l' = l^{G'} + \chi_{\delta_{G'}(U)}$ for some $U \subseteq V(G') \subseteq V(G)$. Hence, again by Lemma 1.3, the extension $l + \chi_{\delta_G(U)}$ of l' is a β-balancing of G. (Sums taken modulo 2). ∎

Assume G is connected and contains a clique cutset K_t with t nodes and let $G'_1, G'_2, \ldots, G'_n$ be the components of the subgraph induced by $V(G) \setminus K_t$. The *blocks* of G are the subgraphs G_i induced by $V(G'_i) \cup K_t$, $i = 1, \ldots, n$.

Corollary 1.2. *If G contains a K_t cutset, then G is β-balanceable if and only if each block G_i is β^{G_i}-balanceable.*

Proof. The "only if" part is obvious. We prove the „if" statement. Fix a β^{K_t}-balancing l of the clique K_t. By Corollary 1.1, in each block G_i we may extend l to a β^{G_i}-balancing of G_i. As each chordless cycle lies entirely in one of the blocks, we thus get a β-balancing of G. ∎

Proof of Theorem 1.1. The necessity of the condition is obvious. We prove the sufficiency by induction on $V(G)$. Let uv be an edge of G. By Corollary 1.2, we may assume that G is connected and has no K_1 or K_2 cutset.

Fix a $\beta^{G \setminus \{u\}}$-balancing of $G \setminus \{u\}$. By Corollary 1.1, we may extend its restriction to $G \setminus \{u, v\}$ to a $\beta^{G \setminus \{v\}}$-balancing of $G \setminus \{v\}$. Thus we obtained a labeling of all the edges except uv. Assigning label 0 to uv, we obtain a labeling, ℓ say, of $E(G)$. We call a chordless circuit C *correct* if $\ell(C) = \beta_C \bmod 2$; otherwise we call C *incorrect*. All chordless cycles C not containing uv are correct. Furthermore at least one chordless cycle C_1 containing uv is incorrect (else ℓ is a β-balancing of G) and at least one chordless cycle C_2 is correct (else by resetting $\ell(uv)$ to 1, we have a β-balancing of G).

As $\{u,v\}$ is not a K_2 cutset of G, by Lemma 1.2, we may choose C_1 and C_2 to be adjacent in G^{uv}. Hence there is a 3-path configuration or a wheel G' containing both C_1 and C_2. Since every edge of G' (and in particular uv) is in exactly two chordless cycles of G', C_2 is the only incorrect chordless cycle of G'. So, denoting the set of chordless cycles in G' by $\mathcal{C}'$, we get $\sum_{C\in\mathcal{C}'}\beta_C = 1+\sum_{C\in\mathcal{C}'}\sum_{e\in E(C)}\ell(e) = 1+\sum_{e\in E(G')}\sum_{C\in\mathcal{C}',C\ni e}\ell(e) = 1+\sum_{e\in E(G')}2\ell(e) = 1 \bmod 2$. Hence, by (2), G' is not $\beta^{G'}$-balanceable. ∎

2. Even and odd-signable graphs

A *hole* is a chordless cycle of length greater than three. Graphs with no odd holes are related to perfect graphs since the famous strong perfect graph conjecture states that a graph G is perfect if and only if G and its complement contain no odd hole.

A graph G is *even-signable* if G is β-balanceable for the vector $\beta_C=1$ if C is a triangle of G and $\beta_C=0$ if C is a hole of G. Even-signable graphs were introduced in [7] and they generalize graphs with no odd holes, for G contains no odd hole if and only if G is even-signable with all labels equal to one. By checking which 3-path configurations and wheel are not even-signable, we get from Theorem 1.1 the following characterization of even-signable graphs.

Theorem 2.2. *A graph is even-signable if and only if it contains no genuine $3PC(xyz,u)$ and no odd wheel.*

Here, a $3PC(xyz,u)$ is *genuine* if in all paths P_1, P_2, P_3 has length greater than one, and a wheel is *odd* if it contains an odd number of triangles.

Theorem 1.1 might turn out useful in understanding graphs with no odd holes. That this is not inconceivable could be argued from the fact that in [4] a polynomial time recognition algorithm is given to test if a graph contains no even hole and that heavily relies on Theorem 2.3 below. We call a graph *odd-signable* if it is β-balanceable for the vector β of all ones. Note that a graph has no even holes if and only if it is odd-signable with all labels equal to one.

Theorem 2.3. *A graph is odd-signable if and only if it contains no $3PC(x,y)$, no $3PC(xyz,uvw)$, and no even wheel.*

Here, a wheel (C,x) is *even* if x has an even number of neighbors on C. (Note that a wheel may be both even and odd and that K_4 is a wheel that is neither even nor odd). Theorem 2.3 follows immediately from Theorem 1.1 by checking which 3-path configurations and wheels are not odd-signable.

The recognition problem for both even-signable and odd-signable graphs is still open. In [5] both problems are solved for graphs that do not contain a cap as induced subgraph. (A cap is a hole H plus a node that has two neighbors in H and these neighbors are adjacent).

3. Universally signable graphs

Let G be a graph that is β-balanced for all 0,1 vectors β that have an entry of 1 corresponding to the triangles of G. Such a graph we call *universally signable.* Clearly triangulated graphs, i.e. graphs that do not contain a hole, are universally signable. In [6] these graphs are shown to generalize many of the structural properties of triangulated graphs. From Theorem 1.1 it follows that G is universally signable if and only if no hole of G belongs to a 3-path configuration or a wheel. Hence we get the following result.

Theorem 3.4. *A graph G is universally signable if and only if G contains no 3-path configuration and no wheel that is distinct from K_4.*

As a consequence of Theorem 3.4 and Lemma 1.1 we have the following decomposition theorem.

Theorem 3.5. *A connected universally signable graph that is not a hole and is not a triangulated graph contains a K_1 or K_2 cutset.*

It was the above decomposition theorem that prompted us to look for a new proof for Theorem 1.1.

4. α-balanced graphs, regular and balanceable matrices

Let α be a vector with entries in $\{0,1,2,3\}$ indexed by the chordless cycles of a graph G. A graph G is *α-balanceable* if its edges can be labeled with labels -1 and $+1$ so that for every chordless cycle C of G, $l(C) = \alpha_C \bmod 4$. Such a labeling is an *α-balancing* of G. As we shall see there is a strong relationship between α- and β-balanceability. In fact, Truemper proved Theorem 1.1 (on β-balanceability) by first proving Theorem 4.6 below (on α-balanceablility) and then showing that the two statements are equivalent.

Theorem 4.6. *A graph is α-balanceable if and only if α_C is even for all even length chordless cycles C and odd otherwise and every induced subgraph H of G that is a 3-path configuration or a wheel is α^H-balanceable.*

To see that the two theorems are equivalent indeed, note that an α-balancing of G with labels of 1 and -1, is implied by a β-balancing with $\beta_C := \frac{\alpha_C - |E(C)|}{2} \bmod 2$, by replacing the 0's by -1's. Similarly the β-balancing of G with labels of 0 and 1 is implied by an α-balancing with $\alpha_C := 2\beta_C + |E(C)| \bmod 4$, by replacing the -1's by 0's.

Balanceable and balanced matrices

The *bipartite graph* $G(A)$ of a matrix A has the row and column sets of A as color classes and an edge ij with label $l_A(ij) := a_{ij}$ for each nonzero entry a_{ij} of A. A $0, \pm 1$ matrix A is *balanced* if $G(A)$ is α-balanced for the vector α of all zeroes. A $0,1$ matrix A is *balanceable* if $G(A)$ is α-balanceable for the vector α of all zeroes. From now on, *signing* means replacing some of the $1's$ with $-1's$. By straightforward checking, we can now derive from Theorem 4.6 the following characterization of balanceable matrices.

Theorem 4.7. *A $0,1$ matrix A is balanceable if and only if $G(A)$ contains no wheel with an odd number of spokes and no $3PC(x,y)$ such that x and y belong to opposite sides of the bipartition.*

In [3] a polynomial algorithm is given to recognize if a matrix is balanceable or balanced. Balanced $0, \pm 1$ matrices have interesting polyhedral properties and have recently been the subject of several investigations, see [8] for a survey.

By the same argument used to obtain Theorem 4.6 from Theorem 1.1, we get from Lemma 1.3 the following result.

Lemma 4.4. *(Camion [2]) The balanced signings of a balanceable graph are unique up to multiplication of some rows and columns by -1.*

Totally unimodular and regular matrices: A theorem of Tutte

A matrix is *totally unimodular* if all of its square submatrices have determinant $0, \pm 1$. Consequently a totally unimodular matrix is a $0, \pm 1$ matrix. If $\tilde{A}$ is a $0, \pm 1$ matrix such that $G(\tilde{A})$ is a chordless cycle C, then $\det(\tilde{A}) = 0$ if $l_A(C) = 0 \bmod 4$ and $\det(\tilde{A}) = \pm 2$ if $l_A(C) = 2 \bmod 4$. So, totally unimodular matrices are balanced.

A $0,1$ matrix is *regular* if it can be signed to be totally unimodular. Clearly, regular matrices are balanceable. Moreover, as total unimodularity is invariant under multiplication of rows and columns by -1, the following lemma follows from Lemma 4.4.

Lemma 4.5. *Every balanced signing of a regular matrix is totally unimodular.*

To state the theorem of Tutte characterizing regular matrices, we need to introduce the notion of pivoting a matrix: *pivoting* a matrix A on a nonzero entry a_{ij} yields the matrix B with entries defined as follows:

$$b_{kl} := \begin{cases} -a_{kl} & \text{if } k=i, j=l \\ a_{kl} & \text{if } k=i, j\neq l \text{ or } k\neq i, j=l \\ a_{kl} - a_{ij}^{-1} a_{il} a_{kj} & \text{if } k\neq i, j\neq l. \end{cases}$$

Lemma 4.6. *Let B be the result of pivoting $A = \begin{bmatrix} \epsilon & y^T \\ x & D \end{bmatrix}$ on the nonzero entry ϵ. Then the following hold:*

i) $B = \begin{bmatrix} -\epsilon & y^T \\ x & D - \epsilon^{-1} x y^T \end{bmatrix}$.

ii) *Pivoting B on $-\epsilon$ yields A.*

iii) *If A is square, $\det(A) = \epsilon \det(D - \epsilon^{-1} x y^T)$ and $\det(B) = -\epsilon \det(D)$.*

iv) *If $\epsilon = \pm 1$, then the set of absolute values of the subdeterminants of A is equal to the set of absolute values of the subdeterminants of B.*

Proof. i) and ii) are obvious from the definition of pivoting. As the matrix $\begin{bmatrix} \epsilon & y^T \\ 0 & D - \epsilon^{-1} x y^T \end{bmatrix}$ follows from A by row operations, we get that $\det(A) = \epsilon \det(D - \epsilon^{-1} x y^T)$. Combining this with ii), yields $\det(B) = -\epsilon \det(D)$. So iii) follows as well. Remains to prove iv); assume $\epsilon = \pm 1$. By ii) it follows that it suffices to prove that if M is a square submatrix of A, then there exists a subdeterminant of B with value $\pm \det(M)$. By taking transposes, if necessary, we may assume that M contains ϵ or is disjoint from the top row. Moreover, we may delete from A all rows and columns that do not contain ϵ and do not intersect M. In other words, we may assume that $M = A$, $M = D$ or $M = [x|D]$. If $M = A$ or $M = D$ then, by i) and iii), the determinant of M occurs, up to a sign, in B. If $M = [x|D]$, then it can be turned into the the submatrix $[x, D - \epsilon^{-1} x y^T]$ of B by column operations. Hence, also in this case the determinant of M occurs up to a sign in B. ∎

We will pivot matrices both over the reals ($\mathbb{R}$-pivoting) and over $GF(2)$ ($GF(2)$-pivoting).

Lemma 4.7. *Let $\tilde{A}$ be a balanced signing of a 0,1 matrix A. Let B be the result of $GF(2)$-pivoting A on an entry a_{ij}. Then $\mathbb{R}$-pivoting $\tilde{A}$ on the corresponding entry $\tilde{a}_{ij}$ yields a (not necessarily balanced!) $0, \pm 1$ signing $\tilde{B}$ of B.*

Proof. As, obviously, $\tilde{B}$ and B are congruent modulo 2, it suffices to show that $\tilde{B}$ is a $0,\pm 1$ matrix. If not, then for some $k \neq i$ and $l \neq j$, $\tilde{a}_{kl} - \tilde{a}_{ij}^{-1}\tilde{a}_{il}\tilde{a}_{kj} \neq 0,\pm 1$. But, then the four entries $\tilde{a}_{ij}, \tilde{a}_{il}, \tilde{a}_{kj}$, and $\tilde{a}_{kl}$ make up an unbalanced submatrix of $\tilde{A}$, a contradiction. ∎

Lemma 4.8. *Every nonregular* $0,1$ *matrix can be* $GF(2)$*-pivoted into a nonbalanceable matrix.*

Proof. Let A be a counterexample. We may assume that A is minimally nonregular (minimal under taking submatrices and pivoting). We first prove the following:

(∗) *If* u *and* w *are in different color classes of* $G(A)$*, then* w *has degree 2 in* $G(A) \setminus \{u\}$.

To prove this, let v be adjacent to u and different from w (as A is minimally nonregular, v exists). Let B be the result of $GF(2)$-pivoting A on a_{uv}. B is also minimal nonregular and balanceable. Let $\hat{B}$ be a balanced signing of B. Then as all proper submatrices of B are regular and all submatrices of $\hat{B}$ are balanced, it follows from Lemma 4.5 that $\det(\hat{B})$ is the only subdeterminant of $\hat{B}$ that is not $0,\pm 1$. Let $\hat{A}$ be the result of $\mathbb{R}$-pivoting $\hat{B}$ on $\hat{b}_{uv}$; as $\hat{B}$ is balanced, by Lemma 4.7, $\hat{A}$ is a signing of A. By Lemma 4.6, iii) and iv), the only subdeterminant of $\hat{A}$ that is not $0,\pm 1$ is the determinant of the submatrix $\hat{A} - \{u,v\}$ corresponding to $G(A) \setminus \{u,v\}$. As $A - \{u,v\}$ is regular, and $\hat{A} - \{u,v\}$ is not totally unimodular, it follows from Lemma 4.5 that $\hat{A} - \{u,v\}$ is not balanced. As all proper subdeterminants are $0,\pm 1$, $G(A) \setminus \{u,v\}$ is a chordless cycle. So, as v is not adjacent to w, (∗) follows.

By (∗), $G(A)$ is 3-regular (each node w has a neighbor u). But now, again by (∗), $G(A)$ is the complete bipartite graph $K_{3,3}$. As A is nonregular, this is impossible. ∎

The next remark follows from the definition of pivoting.

Remark 4.1. Let B be the result of $GF(2)$-pivoting a $0,1$ matrix A on $a_{ij} = 1$. Then $G(B)$ is obtained from $G(A)$ by picking each pair $k \in N(i) \setminus \{j\}$, $l \in N(j) \setminus \{i\}$, adding edge kl if k and l are nonadjacent in $G(A)$ and removing edge kl if k and l are adjacent in $G(A)$.

Tutte [16], [17] proves the following:

Theorem 4.8. *A* $0,1$ *matrix* A *is regular if and only if for no matrix* B*, obtained from* A *by* $GF(2)$*-pivoting,* $G(B)$ *contains a wheel whose rim has length 6.*

Proof. Assume A is a regular matrix and let $\tilde{A}$ be a totally unimodular signing of A. Let $\tilde{B}$ be the result of $\mathbb{R}$-pivoting $\tilde{A}$ on a nonzero entry $\tilde{a}_{ij}$ and let B be the result of $GF(2)$-pivoting A on entry a_{ij}. By Lemma 4.7, $\tilde{B}$ is a signing of B and by Lemma 4.6 iv), $\tilde{B}$ is totally unimodular. So B is regular and the necessity follows.

For the sufficiency part, let A be a nonregular $0,1$ matrix. Then, by Lemma 4.8, we can $GF(2)$-pivot A into a nonbalanceable $0,1$ matrix B. By Theorem 4.7, $G(B)$ contains a $3PC(x,y)$ where x and y belong to distinct color classes, or a wheel (C,x) where x has an odd number, greater than one, of neighbors in C.

If $G(B)$ contains a $3PC(x,y)$, then, by Remark 4.1, we can perform a series of $GF(2)$-pivots on B so that in the end all three xy-paths in the $3PC(x,y)$ have length three. When that is achieved, $GF(2)$-pivoting on an entry corresponding to an edge incident with x, will yield a wheel whose rim has length 6.

If $G(B)$ contains a wheel (C,x) and x has an odd number of neighbors in the rim C, then, by Remark 4.1, we can perform a series of $GF(2)$-pivots on B so that all the *sectors* of (C,x), i.e. the subpaths of C between two consecutive neighbors of x, have length two. When all sectors do have length 2 and x has more than three neighbors in C, a $GF(2)$-pivot on an entry corresponding to an edge of C, yields a wheel (C',x) such that x has two less neighbors in C' than in C. The new wheel (C',x) has one sector of length 4 now, but that can be reduced to length 2 by a single pivot, as before. So ultimately, we will obtain a wheel whose rim has length 6. ∎

Tutte's original proof of the above theorem is quite difficult. A short, self-contained proof can be found in [10]. In [12], a polynomial algorithm is given to recognize if a matrix is regular or totally unimodular. For a faster algorithm, see [15].

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