Designs, Codes and Cryptography, 21, 149–164, 2000.

Equilateral Dimension of the Rectilinear Space

JACK KOOLEN jkoolen@mathematik.uni-bielefeld.de
CWI, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

MONIQUE LAURENT monique@cwi.nl
CWI, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

ALEXANDER SCHRIJVER
CWI, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

Abstract. It is conjectured that there exist at most $2k$ equidistant points in the k-dimensional rectilinear space. This conjecture has been verified for $k \leq 3$; we show here its validity in dimension $k = 4$. We also discuss a number of related questions. For instance, what is the maximum number of equidistant points lying in the hyperplane: $\sum_{i=1}^{k} x_i = 0$? If this number would be equal to k, then the above conjecture would follow. We show, however, that this number is $\geq k+1$ for $k \geq 4$.

Keywords: Touching number, rectilinear space, equidistant set, cut metric, design, touching simplices

1. Introduction

1.1. The Equilateral Problem

Following Blumenthal [2], a subset X of a metric space M is said to be *equilateral* (or *equidistant*) if any two distinct points of X are at the same distance; then, the *equilateral dimension* $e(M)$ of M is defined as the maximum cardinality of an equilateral set in M.

Equilateral sets have been extensively investigated in the literature for a number of metric spaces, including spherical, hyperbolic, elliptic spaces and real normed spaces. Their structure is well understood in the Euclidean, spherical and hyperbolic spaces (cf. [2]) and results about equiangular sets of lines are given by van Lint and Seidel [25] and Lemmens and Seidel [16]. As we will see below some bounds are known for the equilateral dimension of a normed space but its exact value is not known (except for the Euclidean and ℓ_∞-norms). In this paper we focus on the rectilinear space $\ell_1(k)$; that is, the real space $\mathbb{R}^k$ equipped with the ℓ_1-norm. (For $x \in \mathbb{R}^k$, its ℓ_1-*norm* is $\|x\|_1 = \sum_{i=1}^{k} |x_i|$.). Clearly,

$$e(\ell_1(k)) \geq 2k$$

as the unit vectors and their opposites form an equilateral set. It is generally believed (see, in particular, Kusner [12]) that $2k$ is the right value for the equilateral dimension.

CONJECTURE 1 *For each* $k \geq 1, e(\ell)_1(k)) = 2k$.

This conjecture has been shown to hold for $k \geq 3$ [1]. Our main result in this paper is to show its validity in the next case $k = 4$. (Cf. Theorem 9.)

What plays an essential role in our proof is the fact that the equilateral problem in the rectilinear space $\ell_1(k)$ can be reformulated as a discrete 0-1 problem, which permits a direct search attack to the problem; namely, proving Conjecture 1 for given k reduces to checking the nonexistence of a certain set system on $2k + 1$ elements. Moreover, we formulate a stronger version of Conjecture 1 (cf. Conjecture 6) which allows a further simplification in the proof since it suffices to consider certain set systems on $2k - 1$ elements (instead of $2k + 1$). This reformulation is presented in Section 2 and the proof of Conjecture 6 in the case $k = 4$ is given in Section 4.

In Section 3, we discuss several further questions related to the equilateral problem in the rectilinear space. In particular, what is the maximum cardinality of an equilateral set lying in a hyperplane $\sum_{i=1}^{k} x_i = 0$ of the rectilinear space $\mathbb{R}^k$? (Is it k?) What is the maximum number of pairwise touching translates of a k-dimensional simplex? (Is it $k + 1$?) (Call two convex bodies *touching* is they meet but have disjoint interiors.) Does every design on n points contain an antichain of size n? These questions are in some sense equivalent and a positive answer to any of them would imply a proof of our basic Conjecture 1 (cf. Proposition 11). However, except for small k or n, the answers proposed above are not correct. Indeed, for any $n \geq 5$, there exists a design on n points having no antichain of size n; for any $k \geq 3$, there exist $k + 2$ pairwise touching translates of a k-dimensional simplex (cf. Proposition 13).

1.2. Related Geometric Questions

The problem of determining the equilateral dimension of a normed space V arises in particular when studying singularities of minimal surfaces and networks (cf. [17], [9], [15]). This problem has the following interesting geometric interpretation. Let K denote the unit ball of the normed space V and let $t(K)$ denote the maximum number of translates of K that pairwise touch, called the *touching number* of K. Given $x_1, \ldots, x_n \in V$, the set $\{x_1, \ldots, x_n\}$ is equilateral with common distance 2 if and only if the translated bodies $K + x_1, \ldots, K + x_n$ are pairwise touching. Hence, the equilateral dimension $e(V)$ of the normed space V is equal to the touching number $t(K)$ of its unit ball K.

Upper Bound. A simple volume argument shows that $e(V) \leq 3^k$ if V is k-dimensional; indeed, $A := \cup_{i=1}^{n}(K + x_i)$ is contained in the ball of center x_1 and radius 3. As noted in [9], this upper bound can be refined to 2^k by observing that A has diameter 2 and using the isodiametric inequality which states that the volume of a body with diameter ≤ 2 is less than or equal to the volume of the unit ball. The 2^k upper bound had been obtained earlier by Petty [18] who showed the following structural characterization for equilateral sets: A set $X \subseteq \mathbb{R}^k$ is equilateral with respect to some norm if and only if X is an antipodal set (that is, for any distinct points $x, x' \in X$ there exist two parallel supporting hyperplanes H, H' for X such that $x \in H, x' \in H'$); the 2^k bound now follows from the fact established in [3] that an antipodal set in $\mathbb{R}^k$ has at most 2^k points. Clearly, the 2^k upper bound in attained for the ℓ_∞-norm (as $\{0, 1\}^k$ is equilateral); moreover, an equilateral set of size 2^k exists only when the unit ball K is affinely equivalent to the k-cube [18].

Lower Bound. Petty [18] shows that one can find four equidistant points in any normed space of dimension ≥ 3. It is still an open question to decide whether one can find an equilateral set of cardinality $k+1$ in a normed space of dimension $k \geq 4$ (cf. [17], [15], [20] or [24] (problem 4.1.1 page 308)). Note, however, that the answer is obviously positive for the ℓ_p-norm (as $e_1, \dots, e_k, (a, \dots, a)$ form an equilateral set, where $e_1, \dots, e_k$ are the unit vectors and a satisfies $|a-1|^p + (k-1)|a|^p = 2$). In the Euclidean case ($p = 2$), $k+1$ is the right value for the equilateral dimension [2].

Hadwiger's Problem. The equilateral problem has interesting connections to several other problems in combinatorial geometry. In particular, it is related to a classic problem posed by Hadwiger [14] which asks for the maximum number $m(K)$ of translates of a convex body K that all meet K and have pairwise disjoint interiors. (See p. 149 in [4] for history, results and precise references on Hadwiger's problem.) It can be shown that $m(K) = H(K) + 1$, where $H(K)$ is the maximum number of translates of K that all touch K and have pairwise disjoint interiors; $H(K)$ is known as the *Hadwiger number* (or *translative kissing number*) of K. In other words, when K is centrally symmetric with associated norm $\|.\|$, $H(K)$ is the maximum number n of vectors $x_1, \dots, x_n$ satisfying: $\|x_i\| = 2$ and $\|x_i - x_j\| \geq 2$ for all $i \neq j = 1, \dots, n$. The touching and Hadwiger numbers are related by the inequality: $t(K) \leq H(K) + 1$.

Let K be a k-dimensional convex body; the following is known: $H(K) \leq 3^k - 1$ (Hadwiger [14]; simple volume computation); $H(K) = 3^k - 1$ if and only if K is a parallelotope (Grünbaum [11] for $k = 2$ and Groemer [10] for general k); $H(K) = 6$ when K is a 2-dimensional convex body different from a parallelogram [10]; $H(K) \geq k^2 + k$ (Sinnerton-Dyer [21]). The previous lower bound was recently improved by Talata [22] who showed the existence of a constant $c > 0$ such that $H(K) \geq 2^{ck}$ for any k-dimensional convex body K. Determining the Hadwiger number for any k-dimensional Euclidean ball B_k is a longstanding famous open problem which has surged intensive research; in particular, it is known that $H(B_k) = k^2 + k$ for $k \leq 3$. The Hadwiger number of the tetrahedron was recently shown to be equal to 18 (Talata [23]).

Other related combinatorial problems are investigated in [9], [19], [20]. For instance, if $x_1, \dots, x_n \in \mathbb{R}^k$ are unit vectors (with respect to some norm) satisfying $\|x_i + x_j\| \leq 1$ for all $i \neq j$, then $n < 2^{k+1}$; moreover, $n \leq 2k$ if 0 belongs to the relative interior of the convex hull of the x_i's, or if $\| \sum_{i \in I} x_i \| \leq 1$ for all $I \subseteq [1, n]$. Further geometric questions (like the problem of finding large antichains in designs or the problem of determining the maximum number of pairwise touching translates of a simplex) will be discussed in Section 3.

2. Reformulating the Equilateral Problem in the Rectilinear Space

We present here some reformulations of the equilateral problem in the rectilinear space $\ell_1(k)$ in terms of set systems.

We introduce some definitions. Given $X = \{x_1, \dots, x_n\} \subseteq \mathbb{R}_+$, let $a_1 < \cdots < a_p$ denote the distinct values taken by $x_1, \dots, x_n$ and set

$$S_q := \{i \in [1, n] \mid x_i \geq a_q\} \text{ for } q = 1, \dots, p.$$

Then $\mathcal{B}(X)$ denotes the weighted set system on $V := [1, n]$ consisting of the sets S_q with weight $\alpha_{S_q} := a_q - a_{q-1}$ for $q = 1, \ldots, p$ (setting $a_0 := 0$). Then, $S_p \subseteq \cdots \subseteq S_1$ and the following holds for $i \neq j \in V$:

$$\text{(i) } x_i = \sum_{S \in \mathcal{B}(X) | i \in S} \alpha_S, \quad \text{(ii) } |x_i - x_j| = \sum_{S \in \mathcal{B}(X): |S \cap \{i,j\}| = 1} \alpha_S. \tag{1}$$

Generally, given $X = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}^k_+$, we let $\mathcal{B}(X)$ denote the weighted set system defined as the union of the k weighted set systems $\mathcal{B}(\{x_1(h), \ldots, x_n(h)\})$ for $h = 1, \ldots, k$. Then, $\mathcal{B}(X)$ can be covered by k chains and the following holds for $i \neq j \in V$:

$$\text{(i) } e^T x_i = \sum_{S \in \mathcal{B}(X) | i \in S} \alpha_S, \quad \text{(ii) } \|x_i - x_j\|_1 = \sum_{S \in \mathcal{B}(X): |S \cap \{i,j\}| = 1} \alpha_S. \tag{2}$$

When all vectors in X are nonnegative *integral*, $\mathcal{B}(X)$ can be viewed as a multiset if we replace a weighted set S with weight a (a positive integer) by a occurrences of S. Note that the correspondence $X \mapsto \mathcal{B}(X)$ is many-to-one (as there may be several ways of partitioning a set system into chains). For instance, consider

$$M_1 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

and let X_1, X_2 denote the sets in $\mathbb{R}^3$ whose points are the rows of M_1 and M_2, respectively. Then, $\mathcal{B}(X_1) = \mathcal{B}(X_2)$ is the multiset given by the columns of A; X_1 and X_2 correspond to two distinct partitions of the columns of A into chains, namely with parts $\{1, 2\}, \{3, 4\}, \{5, 6\}$, and with parts $\{1, 2\}, \{3\}, \{4, 5, 6\}$.

Given a subset $S \subseteq V$, the *cut* $\delta(S)$ is the vector of $\{0, 1\}^{\binom{n}{2}}$ defined by $\delta(S)_{ij} = 1$ if and only if $|S \cap \{i, j\}| = 1$ for $1 \leq i < j \leq n$. Let $\mathbb{1}_n$ denote the all-ones vector in $\mathbb{R}^{\binom{n}{2}}$. A cut family $\mathcal{S}$ is said to be *nested* if its members can be ordered as $\delta(S_1), \delta(S_2), \ldots, \delta(S_m)$ in such a way that $S'_1 \subseteq S'_2 \subseteq \cdots \subseteq S'_m$ where $S'_j \in \{S_j, V/S_j\}$ for each $j = 1, \ldots, m$; $\mathcal{S}$ is said to be a *k-nested*[1] if it can be decomposed as a union of k nested subfamilies. A cut family $\mathcal{S}$ is said to be *equilateral* if there exist positive scalars $\alpha_S (\delta(S) \in \mathcal{S})$ for which the following relation holds:

$$\mathbb{1}_n = \sum_{\delta(S) \in \mathcal{S}} \alpha_S \delta(S). \tag{3}$$

Clearly, (3) holds if and only if the rows of the matrix whose columns are the vectors $\alpha_S \chi^S (\delta(S) \in \mathcal{S})$ form an equilateral set. (Given a set $S \subseteq V$, $\chi^S \in \{0, 1\}^V$ denotes its characteristic vector defined by $\chi^S_i = 1$ if and only if $i \in S$, for $i \in V$.) For instance,

$$\sum_{i=1}^{n} \delta(i) = 2\mathbb{1}_n,$$

which shows that the cut family $\{\delta(i) \mid i = 1, \ldots, n\}$ is equilateral; this cut family is called the *trivial* cut family. Finally, note that in (3) we can assume that the scalars α_S are rational numbers; similarly, when looking for equilateral sets we can restrict our attention to nonnegative integral ones. To summarize, we have shown:

PROPOSITION 2 *The following assertions are equivalent.*

(i) *There exists an equilateral set in* $\ell_1(k)$ *of cardinality* n.

(ii) *There exists a multiset* $\mathcal{B}$ *on* $[1, n]$ *which is covered by* k *chains and satisfies* $|\{S \in \mathcal{B} : |S \cap \{i, j\}| = 1\}| = r$ *for all* $i \neq j \in V$, *for some* $r > 0$.

(iii) *There exists a* k*-nested equilateral cut family on* n *elements.*

For small n one can make an exhaustive search of all the equilateral cut families on n points. For instance, the trivial cut family is the only equilateral cut family on 3 points and for $n = 4$ the following result can be easily verified.

LEMMA 3 *For* $n = 4$, *any decomposition (3) has the form:*

$$\mathbb{1}_4 = \alpha \sum_{i=1}^{4} \delta(i) + \left(\frac{1}{2} - \alpha\right) \sum_{i=2}^{4} \delta(1i) \text{ where } 0 \leq \alpha \leq \frac{1}{2}.$$

We now state some results that will enable us to formulate some strengthenings of Conjecture 1.

LEMMA 4 *Consider the assertions:*

(i) *Any* k*-nested equilateral cut family on* $2k - 1$ *elements is trivial.*

(ii) *Any* k*-nested equilateral cut family on* $2k$ *elements is trivial.*

(iii) *There does not exist a* k*-nested equilateral cut family on* $2k + 1$ *elements, i.e.,* $e(\ell_1(k)) \leq 2k$.

Then, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

Proof. (i) $\Longrightarrow$ (ii) Let $\mathcal{S}$ be a k-nested equilateral cut family on V, $|V| = 2k$. Assume that $\mathcal{S}$ is not trivial and let $\delta(S) \in \mathcal{S}$ with $2 \leq |S| \leq 2k - 2$. For each $i \in V$, the induced cut family on $V \backslash \{i\}$ is trivial which implies that $|S \cap (V \backslash \{i\})| = 1$ or $2k - 2$. Choosing $i \in V \backslash S$, we obtain that $|S| = 2k - 2$ and choosing $i \in S$ that $|S| = 2$. Therefore, $k = 2$. In view of Lemma 3, $\mathcal{S}$ contains the three cuts $\delta(12)$, $\delta(13)$, $\delta(14)$, contradicting the assumption that $\mathcal{S}$ is 2-nested. The proof for implication (ii) $\Longrightarrow$ (iii) is similar and thus omitted. ∎

Given $x_0 \in \mathbb{R}^k$ and $\lambda > 0$, the set $X := \{x_0 \pm \lambda e_i \mid i = 1, \ldots, k\}$ is obviously equilateral; any set of this form is called a *trivial* equilateral set in $\mathbb{R}^k$. Given $x, y, z \in \mathbb{R}^k$, their *median* is the point $m \in \mathbb{R}^k$ whose hth coordinate is the median value of x_h, y_h, z_h for $h = 1, \ldots, k$. As is well known, the median m is the unique point lying on the three geodesics between any two of the points x, y, z; the geodesics being taken with respect to the ℓ_1-distance, and the geodesic between x and y consisting of all points $u \in \mathbb{R}^k$ satisfying $\|x - y\|_1 = \|x - u\|_1 + \|u - y\|_1$. We now reformulate Lemma 4 (ii) in more geometric terms.

LEMMA 5 *Consider the assertions:*

(i) *Any k-nested equilateral cut family on $2k$ elements is trivial.*

(ii) *Any equilateral set in $\mathbb{R}^k$ of cardinality $2k$ is trivial.*

(iii) *If X is an equilateral set in $\mathbb{R}^k$ of cardinality $2k$, and with common distance 2, then there exists $x_0 \in \mathbb{R}^k$ such that $\|x_0 - x\|_1 = 1$ for all $x \in X$.*

Then, (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii).

Proof. (i) $\Longrightarrow$ (ii) Let $X \subseteq \mathbb{R}^k$ be equilateral of cardinality $2k$ and with common distance 2; up to translation we can suppose that $\min(x_i(h) \mid i = 1, \ldots, 2k) = 0$ for $h = 1, \ldots, k$. Let $\mathcal{B}(X)$ denote the associated weighted set system as explained earlier in this section. By (i), we know that every set in $\mathcal{B}(X)$ is a singleton or the complement of a singleton. Therefore, we find (up to permutation on $1, \ldots, 2k$) that $\mathcal{B}(X)$ consists of the sets $\{2i - 1\}$ and $V\backslash\{2i\}$ for $i = 1, \ldots, k$, each with multiplicity 1. Using relation (1)(i), this implies that X consists of the points $e \pm e_i (i = 1, \ldots, k)$, where e is the all-ones vector; that is, X is trivial.

(ii) $\Longrightarrow$ (iii) holds trivially.

(iii) $\Longrightarrow$ (i) Let $\mathcal{S}$ be a k-nested equilateral cut family on $2k$ points. Then, a suitable choice of S or $V\backslash S$ for each cut $\delta(S) \in \mathcal{S}$ yields a weighted set system $\mathcal{B}$ on $V = [1, 2k]$ which is covered by k chains an such that $\mathbb{1}_n = \sum_{S \in \mathcal{B}} \alpha_S \delta(S)$. Let $X = \{x_1, \ldots, x_{2k}\} \subseteq \mathbb{R}^k$ denote an equilateral set corresponding to $\mathcal{B}$ (defined using (1)(i) and given partition of $\mathcal{B}$ into k chains). By (iii), we obtain that any three distinct points of X have the same median. Therefore, for every $h = 1, \ldots, k$, the vector $(x_1(h), \ldots, x_{2k}(h)$ is of the form $a\chi^i + b\chi^{V\backslash j}$ where $i \neq j \in V$ and $a, b \geq 0$. From this we see that $\mathcal{S}$ is the trivial cut family. ∎

To summarize, we can formulate the following conjectures:

CONJECTURE 6 *Any k-nested equilateral cut family on $2k - 1$ elements is trivial.*

CONJECTURE 7 *Any k-nested equilateral cut family on $2k$ elements is trivial. Equivalently, any equilateral set in $\mathbb{R}^k$ of cardinality $2k$ is trivial.*

PROPOSITION 8 *Conjecture 6 $\Longrightarrow$ Conjecture 7 $\Longrightarrow$ Conjecture 1.*

Conjecture 6 holds for $k = 2$ (trivial) and for $k = 3$) ([1]). We show that it also holds for $k = 4$; the proof is delayed till Section 4.

THEOREM 9 *Conjecture 6 holds for $k = 4$.*

3. Connections to Other Geometric Problems

3.1. Touching Cross-Polytopes

Let $\beta_k = \{x \in \mathbb{R}^k : \|x\|_1 \leq 1\}$ denote the unit ball of the k-dimensional rectilinear space; β_k is also known as the k-dimensional cross-polytope. As mentioned in the introduction, the equilateral dimension of $\ell_1(k)$ is equal to the touching number of β_k. A more restrictive question is to determine the maximum number of pairwise touching translates of β_k that share a common point. This question can be answered easily.

LEMMA 10 *The maximum number of pairwise touching translates of the cross-polytope β_k sharing a common point is equal to $2k$.*

Proof. Clearly, $2k$ is a lower bound (since the $\beta_k \pm e_i$'s ($i = 1, \ldots, k$) all meet at the origin). The fact that $2k$ is an upper bound follows from results in [13], [7] on the ℓ_1-embedding dimension of trees. (It can also be checked directly using the same reasoning as for the implication (iii) $\Longrightarrow$ (i) of Lemma 5.) ■

Hence, we find again that Conjecture 1 holds if one can show that there are at most $n < 2k$ pairwise touching translates of β_k having no common point (that is, if Conjecture 7 holds) (this is, in fact, the proof technique used in [1] in the case $k = 3$).

Let us observe that touching translates of the cross-polytope enjoy a strong Helly type property. Namely, if $B_i := \beta_k + x_i (i = 1, \ldots, n)$ are n pairwise touching translates of β_k, then $B_i \cap B_j \cap B_h$ is reduced to a single point (the median of x_i, x_j, x_h) for any distinct $i, j, h \in [1, n]$; therefore, $\bigcap_{i=1}^n B_i \neq \emptyset$ if and only if $B_1 \cap B_2 \cap B_3 \cap B_i \neq \emptyset$ for all $i = 4, \ldots, n$.

3.2. Antichains in Designs and Touching Simplices

We present here some variations on the equilateral problem in the rectilinear space, dealing with equilateral sets on a hyperplane, antichains in designs and touching simplices.

A first variation asks for the maximum cardinality $h(k)$ of an equilateral set $X \subseteq \mathbb{R}^k$ lying in a hyperplane $H_r := \{x \in \mathbb{R}^k \mid e^t x = r\}$ (for some $r \in \mathbb{R}$). (Recall that e denotes the all-ones vector.) Clearly, $h(k) \geq k$ (considering the k unit vectors).

The weighted set systems $\mathcal{B}(X)$ corresponding to integral equilateral sets X lying in a hyperplane H_r lead naturally to the notion of designs. Recall that, given positive integers $r > \lambda$, a multiset $\mathcal{B}$ on $V = [1, n]$ is called an (r, λ)-*design* if every point $i \in V$ belongs to r members (blocks) of $\mathcal{B}$ and any two distinct points $i, j \in V$ belong to λ common members of $\mathcal{B}$. An *antichain* in $\mathcal{B}$ is a subset of $\mathcal{B}$ whose members are pairwise incomparable. Let $a(n)$ denote the maximum integer such that every design on n points has an antichain of cardinality $a(n)$ (equivalently, by Dilworth's theorem, $a(n)$ is the minimum taken over all designs $\mathcal{B}$ on n points of the minimum number of chains needed to cover $\mathcal{B}$). Clearly, $a(n) \leq n$ (considering the design consisting of all singletons). Equality $a(n) = n$ would mean that every design on n points has an antichain of size n. It is well-known that every design on n points contains at least n distinct blocks (cf. [5]; this fact is also known as Fisher's inequality). Therefore, any *pairwise balanced incomplete* design (that is, a design $\mathcal{B}$ whose blocks all have the same cardinality) contains obviously an antichain of size n.

Call a design $\mathcal{B}$ on a set V *self-complementary* if, for every $B \subseteq V$, the set B and its complement $V \setminus B$ appear with the same multiplicity in $\mathcal{B}$. Denote by $a'(n)$ the maximum cardinality of an antichain in a self-complementary design on n points. Hence, $a(n) \leq a'(n) \leq n$.

Finally, we consider the touching number $t(\alpha_k)$ of the k-dimensional regular simplex α_k (that is, the maximum number of pairwise touching translates of α_k). We have: $t(\alpha_k) \geq k+1$. Indeed, induction on k shows easily the existence of $k+1$ translates of α_k that are pairwise touching and share a common point. (Cf. Remark 14.)

PROPOSITION 11 *The following holds for integers* $k, n \geq 1$.

(i) $h(k) \geq n \iff a(n) \leq k$.

(ii) $h(k) = t(\alpha_{k-1})$.

(iii) $a(n) \geq 2k \Longrightarrow e(\ell_1(k)) \leq n$.

(iv) $a'(n+1) \geq 2k+1 \Longrightarrow e(\ell_1(k)) \leq n$.

Proof. (i) Let $X = \{x_1, \ldots, x_n\} \subseteq \mathbb{Z}_+^k$ and let $\mathcal{B}(X)$ be its associated multiset on $V = [1, n]$. Using relation (2), we deduce that $\mathcal{B}(X)$ is a (r, λ)-design if and only if X is contained in the hyperplane H_r and X is equilateral with common distance $\mu = 2(r-\lambda)$. Moreover, $\mathcal{B}(X)$ is covered by k chains by construction. This shows (i).

(ii) We need the following notation. Given $x, y \in \mathbb{R}^k$, let $x \vee y$ denote the vector of $\mathbb{R}^k$ whose h-th component is equal to $\max(x_h, y_h) = \frac{1}{2}(x_h + y_h + |x_h - y_h|)$ for $h = 1, \ldots, k$. We have:

$$e^T(x \vee y) = \frac{1}{2}(e^T x + e^T y + \|x - y\|_1).$$

Let $S_1, \ldots, S_n$ be pairwise touching translates of the regular $(k-1)$-dimensional simplex. We can suppose that the S_i's are all translates of the simplex $S_0 := \{x \in \mathbb{R}^k \mid x \geq 0, e^T x = 1\}$ and that they lie in the hyperplane H_1. Then, $S_i = S_0 + x_i = \{x \in \mathbb{R}^k \mid x \geq x_i, e^T x = 1\}$ where the x_i's lie in H_0. As $S_i \cap S_j = \{x \mid x \geq x_i \vee x_j, e^T x = 1\}$ and S_i, S_j are touching, we deduce that $e^T(x_i \vee x_j) = 1$, which implies that $\|x_i - x_j\|_1 = 2$. Therefore, the set $\{x_1, \ldots, x_n\}$ is equilateral in H_0. Conversely, if $X = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}^k$ is an equilateral set with common distance 2 and lying in H_0, then the n simplices $S_i := \{x \in \mathbb{R}^k \mid x \geq x_i, e^T x = 1\}$ $(i = 1, \ldots, n)$ are pairwise touching. This shows that $h(k) = t(\alpha_{k-1})$.

We prove (iii) and (iv) together. For this, let $\mathcal{B}$ be a multiset of $[1, N]$ which is covered by k chains and satisfies:

$$|\{S \in \mathcal{B} : |S \cap \{i, j\}| = 1\}| = r$$

for all $i \neq j \in [1, N]$. We show that, if $a(n) \geq 2k$ or $a'(n+1) \geq 2k+1$, then $N \leq n$ (recall Proposition 2(ii)). Say, $\mathcal{B} = \cup_{h=1}^k \mathcal{B}_h$ where each $\mathcal{B}_h$ is a chain. Without loss of generality we can suppose that the element N belongs to all sets $S \in \mathcal{B}_1$. We define two new multisets $\mathcal{B}'$ on $[1, N-1]$ and $\mathcal{B}''$ on $[1, N]$ in the following manner:

$$\mathcal{B}' := \{S \in \mathcal{B} \mid N \notin S\} \cup \{[1, N] \setminus S \mid N \in S\},$$

$$\mathcal{B}'' := \mathcal{B} \cup \{[1, N] \setminus B \mid B \in \mathcal{B}\}.$$

Obviously, $\mathcal{B}'$ is covered by $1 + 2(k-1) = 2k-1$ chains and $\mathcal{B}''$ by $2k$ chains. Moreover, one can verify that $\mathcal{B}'$ is a $(r, \frac{r}{2})$-design on $N-1$ points and that $\mathcal{B}''$ is a $(|\mathcal{B}|, |\mathcal{B}|-r)$-design on N points. Therefore, we find $N-1 \leq n-1$, i.e., $N \leq n$ when $a(n) \geq 2k$, and $N \leq n$ when $A'(n+1) \geq 2k+1$. ■

Therefore, Conjecture 1 would hold if one could show that every design on n points has an antichain of size n. One can show that the latter holds for $n \leq 4$; however, for each $n \geq 5$, one can construct a design $\mathcal{B}_n$ on n points having no antichain of size n (cf. Proposition 13 below). For $n = 5$, one can show that $\mathcal{B}_5$ is the only design having no antichain of size 5 (unique up to addition of the full set $[1, 5]$). This permits to show that any design on 6 points has an antichain of size 5. To summarize, we have:

$$\begin{gathered} h(k) = t(\alpha_{k-1}) = k \text{ for } k \leq 3;\ a(n) = n \text{ for } n \leq 4; \\ h(k) = t(\alpha_{k-1}) \geq k+1 \text{ for } k \geq 4;\ a(n) \leq n-1 \text{ for } n \geq 5; \\ a(n) = k \text{ for } h(k-1) < n \leq h(k). \end{gathered}$$

In particular,

$$a(5) = 4, \quad a(6) = 5, \quad h(4) = t(\alpha_3) = 5.$$

Moreover, we have checked that

$$a'(n) = n \text{ for } n \leq 7.$$

Example 12. We describe here two designs $\mathcal{B}_n$ on $n = 5, 6$ points which are covered by $n-1$ chains, as well as the associated equilateral sets in $\mathbb{R}^{n-1}$ (vectors are the rows of the arrays) of cardinality n.

$$\mathcal{B}_5 : \quad \begin{array}{cccc} 1 & 2 & 34 & 35 \\ 145 & 245 & 1234 & 1235 \end{array} \qquad \begin{array}{cccc} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{array}$$

$$\mathcal{B}_6 : \quad \begin{array}{lllll} 1(\times 2) & 2(\times 2) & 3(\times 2) & 5(\times 2) & 4(\times 2) \\ 16 & 26 & 346 & 356 & 12345(\times 2) \\ 1456 & 2456 & 12346 & 12356 & \end{array} \qquad \begin{array}{ccccc} 4 & 0 & 1 & 1 & 2 \\ 0 & 4 & 1 & 1 & 2 \\ 0 & 0 & 4 & 2 & 2 \\ 1 & 1 & 2 & 0 & 4 \\ 1 & 1 & 0 & 4 & 2 \\ 2 & 2 & 2 & 2 & 0 \end{array}$$

Another design on 6 points covered by 5 chains:

1(×2)	2(×2)	4(×2)	5(×2)	36(×2)	
16	26	34	35		
1456	2456	1234	1235		

$$\begin{matrix} 4 & 0 & 1 & 1 & 0 \\ 0 & 4 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 1 & 1 & 4 & 0 & 0 \\ 1 & 1 & 0 & 4 & 0 \\ 2 & 2 & 0 & 0 & 2 \end{matrix}$$

PROPOSITION 13 *For each $n \geq 5$, there exists a design on n points which is covered by $n-1$ chains.*

Proof. Using induction on $n \geq 6$ we construct a design $\mathcal{B}_n$ on n points which is covered by $n-1$ chains and with parameters r_n, λ_n satisfying:

$$|\mathcal{B}_n| > 2r_n - \lambda_n \text{ and } \{i\} \in \mathcal{B}_n \text{ for all } i = 1, \ldots, n-1. \tag{4}$$

Design $\mathcal{B}_6$ is as described in Example 12; it satisfies (4). Given $\mathcal{B}_n$ satisfying (4), we let $\mathcal{B}_{n+1}$ consist of the following sets: $B \cup \{n+1\}$ for $B \in \mathcal{B}_n$, $\{1, \ldots, n\}$ repeated $r_n - \lambda_n$ times and, for $i = 1, \ldots, n$, $\{i\}$ repeated $|\mathcal{B}_n| - 2r_n + \lambda_n$ times. Then, $\mathcal{B}_{n+1}$ is a design with parameters $r_{n+1} = |\mathcal{B}_n|$, $\lambda_{n+1} = r_n$. Moreover,

$$|\mathcal{B}_{n+1}| = |\mathcal{B}_n| + (r_n - \lambda_n) + n(|\mathcal{B}_n| - 2r_n + \lambda_n)$$

which implies that

$$|\mathcal{B}_{n+1}| - 2r_{n+1} + \lambda_{n+1} = (n-1)(|\mathcal{B}_n| - 2r_n + \lambda_n) > 0.$$

Hence, (4) holds for $\mathcal{B}_{n+1}$. Finally, $\mathcal{B}_{n+1}$ can be covered by n chains since one can assign the singletons $\{i\}(i = 1, \ldots, n-1)$ to the $n-1$ chains covering $\{B \cup \{n+1\} \mid B \in \mathcal{B}_n\}$ and put $\{1, \ldots, n\}$ and $\{n\}$ together in a new chain. ∎

Remark 14. The maximum number of pairwise touching translates of the $(k-1)$-dimensional simplex that share a common point is equal to k. (Indeed, similarly to the proof of Proposition 11, one can show that there exist n touching translates of α_{k-1} sharing a common point if and only if there exists an equilateral set $X = \{x_1, \ldots, x_n\}$ in a hyperplane H_r of $\mathbb{R}^k$ such that $x_i \vee x_j$ is a constant vector for all $i \neq j$; this in turn means that the associated multiset $\mathcal{B}(X)$ consists of copies of $V = [1, n]$ and of $V \setminus i$ for $i \in V$, which implies that $n \leq k$ since $\mathcal{B}(X)$ is covered by k chains.)

4. Proofs of Theorem 9

Let $\mathcal{S}$ be a cut family on V. Call two cuts $\delta(S)$, $\delta(T)$ *crossing* if the four sets S, T, $V \setminus S$, $V \setminus T$ are pairwise incomparable and *cross-free* otherwise; in other words, two cuts are cross-free if and only if they form a nested pair. Given $t \leq \frac{|V|}{2}$, a cut $\delta(S)$ is called a *t-split* if S has cardinality t or $|V| - t$. Given a subset $X \subseteq V$, let $\mathcal{S}_X$ denote the induced cut family on X, consisting of the cuts $(S \cap S, X \backslash S)$.

In what follows, $V = \{1, \ldots, 7\}$ and $\mathcal{S}$ is assumed to be a nontrivial equilateral cut family on V which is 4-nested; moreover, we choose such $\mathcal{S}$ minimal with respect to inclusion.

If $X \subseteq V$ with $|X| = 4$ then, by Lemma 3, $\mathcal{S}_X$ either contains all 1-splits or contains no 1-split. The first step of the proof consists of showing that the former always holds.

PROPOSITION 15 *For every $X \subseteq V$ with $|X| = 4$, $\mathcal{S}_X$ contains all 1-splits.*

Proof. Assume that the result from Proposition 15 does not hold for some subset $X \subseteq V$; say, $X := \{1, 2, 3, 4\}$. By Lemma 3, $\mathcal{S}_X$ contains no 1-split and, thus, $\mathcal{S}_X$ contains all the three 2-splits on X. Hence, $\mathcal{S}$ can be partitioned into

$$\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$$

where all cuts in $\mathcal{S}_0$ (resp. $\mathcal{S}_i, i = 2, 3, 4$) are of the form $\delta(S)$ (resp. $\delta(1iS)$) for some $S \subseteq W := V \backslash X = \{5, 6, 7\}$ and $\mathcal{S}_i \neq \emptyset$ for $i = 2, 3, 4$. Note that any two cuts belonging to distinct families $\mathcal{S}_i, \mathcal{S}_j (i \neq j = 2, 3, 4)$ are crossing. Therefore, as $\mathcal{S}$ is 4-nested, we deduce that

$$\text{at least two of the families } \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4 \text{ are nested.} \tag{5}$$

As $\mathcal{S}$ is equilateral we have:

$$\mathbb{1}_7 = \sum_{S \subseteq W} \alpha_S^0 \delta(S) + \sum_{i=2,3,4} \sum_{S \subseteq W} \alpha_S^i \delta(1iS)$$

for some nonnegative scalars α_S^0, α_S^i; $\mathcal{S}$ consisting of those cuts having a positive coefficient. For $x \neq y \in W$ and $i = 0, 2, 3, 4$, set

$$\alpha_i(x) := \sum_{S \subseteq W | x \in S} \alpha_S^i, \alpha_i(\overline{x}) := \sum_{S \subseteq W | x \notin S} \alpha_S^i, \alpha_i(xy) := \sum_{S \subseteq W | x, y \in S} \alpha_S^i.$$

By evaluating coordinatewise the right hand side of the above decomposition of $\mathbb{1}_7$ we find the relations:

$$\alpha_0(x) = \alpha_i(x) = \alpha_i(\overline{x}) = \frac{1}{4} \text{ for } i = 2, 3, 4 \text{ and } x \in W, \tag{6}$$

$$\alpha_0(xy) + \sum_{i=2,3,4} \alpha_i(xy) = \frac{1}{2} \text{ for } x \neq y \in W. \tag{7}$$

We claim that if $\mathcal{S}_i$ is nested for some $i = 2, 3, 4$, then

$$\mathcal{S}_i = \{\delta(1i), \delta(1iW)\}.$$

Indeed, assume that $\mathcal{S}_i$ consists of the cuts $\delta(1iA_1), \ldots, \delta(1iA_p)$ where $A_1 \subseteq \cdots \subseteq A_p \subseteq W$. Using relation (6), we find $A_1 = \emptyset$ (as $\alpha_i(\overline{x}) = 0$ for $x \in A_1$), $A_p = W$

(as $\alpha_i(x) = 0$ for $x \in W\backslash A_p$) and $p = 2$ (if $p \geq 3$ we would have $\alpha_i(x) < \alpha_i(y)$ for $x \in A_p\backslash A_2$ and $y \in A_2$).

By relation (5), we can suppose that $\mathcal{S}_2$ and $\mathcal{S}_3$ are both nested. Therefore, $\mathcal{S}_i$ consists of the cuts $\delta(1i)$ and $\delta(1iW)$ for $i = 2, 3$. If follows that $\alpha_2(xy) = \alpha_3(xy) = \frac{1}{4}$ for $x \neq y \in W$. Using relation (7), we obtain: $\alpha_4(xy) = 0$ for $x \neq y \in W$. Therefore, all cuts $\delta(14u)$ belong to $\mathcal{S}$ for $u \in W$. Together with $\delta(12)$ and $\delta(13)$ they form a set of five pairwise crossing cuts, which contradicts the assumption that $\mathcal{S}$ is 4-nested. ■

As $\mathcal{S}$ is not trivial, the minimality assumption on $\mathcal{S}$ implies that one of the 1-splits is not present in $\mathcal{S}$; say, $\delta(1) \notin \mathcal{S}$. Let $A_1, \ldots, A_p$ denote the (inclusionwise) minimal subsets of $V\backslash\{1\}$ for which $\delta(A_1 \cup \{1\}), \ldots, \delta(A_p \cup \{1\})$ belong to $\mathcal{S}$ and set

$$\mathcal{S}_{\min} := \{\delta(1A_1), \ldots, \delta(1A_p)\}.$$

A set $T \subseteq V\backslash\{1\}$ is said to be *transversal* if T meets each of the sets $A_1, \ldots, A_p$.

PROPOSITION 16 *$p = 4$ and the sets $A_1, \ldots, A_p$ are pairwise disjoint.*

Proof. We first claim that

$$\text{every transversal } T \text{ has cardinality } |T| \geq 4 \tag{8}$$

Indeed, if $|T| \leq 3$ then, in view of Proposition 15, there exists $\delta(S) \in \mathcal{S}$ for which $S \cap (T \cup \{1\}) = \{1\}$. Then, T is disjoint from the set A_i for which $1A_i \subseteq S$, contradicting the assumption that T is transversal.

If $\delta(1A_i), \delta(1A_j) \in \mathcal{S}_{\min}$ are two cross-free cuts, then the following holds:

$$A_i \cap A_j = \emptyset \text{ and } |A_i| = |A_j| = 3. \tag{9}$$

Indeed, $A_i \cup A_j = V\backslash\{1\} = [2, 7]$, since $\delta(1A_i)$ and $\delta(1A_j)$ are cross-free. Moreover, $|A_i\backslash A_j| \geq 3$ (else, the set $(A_i\backslash A_j) \cup \{x\}$ where $x \in A_j\backslash A_i$ would be a transversal of cardinality ≤ 3, contradicting (8)) and, similarly, $|A_j\backslash A_i| \geq 3$. Relation (9) now follows from the above observations and the identity: $6 = |A_i \cup A_j| = |A_i\backslash A_j| + |A_j\backslash A_i| + |A_i \cap A_j|$. We now show that

$$\text{every two cuts among } \delta(1A_1), \ldots, \delta(1A_p) \text{ are crossing.} \tag{10}$$

For, suppose not. Then, by (9), the cuts are of the form: $\delta(1A_i), \delta(1A'_i)$ for $i = 1, \ldots, q$ and $\delta(1A_j)$ for $j = q + 1, \ldots, m$, where $A'_i := V\backslash(A_i \cup \{1\})$ and $p = m + q$. Clearly, $m \leq 4$ since the cuts $\delta(1A_1), \ldots, \delta(1A_m)$ are pairwise crossing. We claim that we can find a transversal of cardinality 3, thus contradicting (8) and proving (10). For this, we use the fact that $A_i \cap A_j, A_i \cap A'_j, A'_j \cap A'_h \neq \emptyset$ for $1 \leq i \leq m$, $1 \leq j, h \leq q$. Indeed let us suppose that $q = 4$ (the case when $q \leq 3$ is analogue). Then, by the above observation, one of the two sets $A_1 \cap A_2 \cap A_3$ and $A'_1 \cap A_2 \cap A_3$ is not empty; similarly, one of the two sets $A'_2 \cap A'_3 \cap A_4$ and $A'_2 \cap A'_3 \cap A'_4$ is not empty. We can assume without loss of generality that $A_1 \cap A_2 \cap A_3, A'_2 \cap A'_3 \cap A'_4 \neq \emptyset$. Then choosing $x \in A_1 \cap A_2 \cap A_3$, $y \in A'_2 \cap A'_3 \cap A'_4$ and $z \in A'_1 \cap A_4$, the set $\{x, y, z\}$ is transversal.

We can conclude the proof of Proposition 16. Indeed, $p \geq 4$ by (8) and $p \leq 4$ by (10); hence, $p = 4$. Moreover, the sets $A_1, \ldots, A_4$ are pairwise disjoint for, otherwise, we would find a transversal of cardinality less than 4. ■

We now conclude the proof of Theorem 9 by analyzing various possibilities for the family $\mathcal{S}_{\min}$. The following notation will be useful: Given two disjoint sets S and A, S_A denotes a set of the form $S \cup B$ where $B \subseteq A$.

We first assume that the family $\mathcal{S}_{\min}$ contains a cut $\delta(1A_i)$ with $|A_i| \geq 2$. Then, we can assume that the cuts in $\mathcal{S}_{\min}$ are of the form

$$\delta(12B_2), \delta(13B_3), \delta(14B_4), \delta(156B_5)$$

where $B_2, \ldots, B_5$ are pairwise disjoint subsets of $\{7\}$. Let

$$\mathcal{S} = \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$$

be a decomposition of $\mathcal{S}$ into four nested families where $\delta(1iA_i) \in \mathcal{C}_i$ for $i = 2, 3, 4$ and $\delta(156A_5) \in \mathcal{C}_5$.

Consider the $X := 1256$. All induced 2-splits on X must be present in $\mathcal{S}_X$; therefore,

$$\delta(15_{347}), \delta(16_{347}) \in \mathcal{S}.$$

The above two cuts are crossing; moreover, they are crossing with $\delta(12A_2)$ (obvious) and with $\delta(156A_5)$ (use here the minimality assumption on $56A_5$) and, thus, they must be assigned to $\mathcal{C}_3 \cup \mathcal{C}_4$. By considering the sets $X := 1356$ and 1456, we obtain in the same manner that $\delta(15_{247}), \delta(16_{247})$ belong to $\mathcal{C}_2 \cup \mathcal{C}_4$ and that $\delta(15_{237}), \delta(16_{237})$ belong to $\mathcal{C}_2 \cup \mathcal{C}_3$. Without loss of generality, let us assign $\delta(15_{347})$ to $\mathcal{C}_3$ and $\delta(16_{347})$ to $\mathcal{C}_4$; then, necessarily, $\delta(15_{247}) \in \mathcal{C}_2, \delta(16_{247}) \in \mathcal{C}_4$ and we reach a contradiction when trying to assign $\delta(16_{237})$ to $\mathcal{C}_2 \cup \mathcal{C}_3$.

We can now assume that $|A_i| = 1$ for every cut $\delta(1A_i) \in \mathcal{S}_{\min}$. Therefore, $\mathcal{S}_{\min}$ consists of the cuts

$$\delta(12), \delta(13), \delta(14), \delta(15)$$

and, thus, $\delta(16), \delta(17) \notin \mathcal{S}$. Let

$$\mathcal{S} = \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$$

be a decomposition of $\mathcal{S}$ into four nested families where $\delta(1i) \in \mathcal{C}_i$ for $i = 2, \ldots, 5$.

For every element $k \in V$ for which $\delta(k) \notin \mathcal{S}$, we find similarly that $\mathcal{S}$ contains four cuts of the form $\delta(ki)$ $(i \in V \backslash \{k\})$. It follows that at least one of $\delta(6), \delta(7)$ belongs to $\mathcal{S}$. Say, $\delta(7) \in \mathcal{S}$ and we can suppose that

$$\delta(7) \in \mathcal{C}_2.$$

The following observation will be repeatedly used: Any cut belonging to $\mathcal{C}_2$ and distinct from $\delta(2)$ is of the form $\delta(S)$ where $12 \subseteq S$ and $7 \notin S$.

For each of the sets $X := 1347$, 1357, and 1457, all 2-splits are present in $\mathcal{S}_X$; therefore, $\delta(17_{256}), \delta(17_{246}), \delta(17_{236}) \in \mathcal{S}$. It is easy to verify that these cuts must be assigned in the following manner to the classes $\mathcal{C}_i$ composing $\mathcal{S}$:

$$\delta(17_{256}) \in \mathcal{C}_5, \quad \delta(17_{246}) \in \mathcal{C}_4, \quad \delta(17_{236}) \in \mathcal{C}_3.$$

Considering the set $X := 1267$, we see that $\delta(16_{345}) \in \mathcal{S}$. We can assume that $\delta(16_{345}) \in \mathcal{C}_5$. Then, $\delta(15), \delta(17_{256}), \delta(16_{345})$ are nested which implies that

$$\delta(156) \in \mathcal{C}_5.$$

This yields $\delta(6) \in \mathcal{S}$. (Indeed, if $\delta(6) \notin \mathcal{S}$, then $\mathcal{S}$ contains four cuts of the form $\delta(i6)$; we reach a contradiction since any cut $\delta(i6)$ is crossing with $\delta(156)$ and thus cannot be assigned to $\mathcal{C}_5$.) Without loss of generality,

$$\delta(6) \in \mathcal{C}_3.$$

Considering the set $X := 1256$, we derive analogously that

$$\delta(16_{347}) \in \mathcal{C}_4.$$

We will use the following fact:

$$\text{For } X := 2367\text{, the induced cut family } \mathcal{S}_X \text{ contains no 2-split.} \tag{11}$$

For, if not, then $\delta(27_{145}) \in \mathcal{S}$, yielding a contradiction as this cut cannot be assigned to any class $\mathcal{C}_i$.

In particular, we obtain that the cut $\delta(17_{236})$ (which belongs to $\mathcal{C}_3$) is equal to $\delta(1237)$. Considering the set $X := 1267$, we obtain that $\delta(17_{345})$ belongs to $\mathcal{S}$. Moreover,

$$\delta(17_{345}) \in \mathcal{C}_4.$$

(Indeed, $\delta(17_{345}) \notin \mathcal{C}_2 \cup \mathcal{C}_5$ since it crosses $\delta(12)$ and $\delta(156)$. If $\delta(17_{345}) \in \mathcal{C}_3$, then it is nested with $\delta(17_{236})$ which implies that $\delta(137) \in \mathcal{S}$ contradicting (11).)

Considering the set $X := 1247$, we obtain that $\delta(17_{356}) \in \mathcal{S}$. We now reach a contradiction since we cannot assign this cut to any class $\mathcal{C}_i$. Indeed, $\delta(17_{356}) \notin \mathcal{C}_2 \cup \mathcal{C}_4$ (obviously) and $\delta(17_{356}) \notin \mathcal{C}_3 \cup \mathcal{C}_5$ (for, otherwise, $\delta(17_{356})$ is nested, either with $\delta(1237)$, or with $\delta(156)$ and $\delta(17_{256})$, which implies that one of the cuts $\delta(137)$, $\delta(1567)$ belongs to $\mathcal{S}$, contradicting (11)). This concludes the proof of Theorem 9.

5. Conclusions

We have presented some relations between Conjecture 1 (dealing with the maximum cardinality of an equilateral set in the k-dimensional rectilinear space) and some other geometric questions, like the maximum size $a(n)$ of an antichain in a design on n points, or the touching numbers of the cross-polytope and the simplex. We mention here some further related problems.

Consider the sequence $(n - a(n))_{n \geq 1}$. Is it monotone nondecreasing? Does it converge to ∞? (If the sequence would be bounded by a constant C, it would imply the upper bound $2k + C$ for $e(\ell_1(k))$.)

It would be interesting to evaluate the touching number $t(P)$ of a k-dimensional polytope. Conjecture 1 asserts that, for $P = \beta_k$ (the k-dimensional cross-polytope), this number is equal to $2k$ (the number of vertices of β_k). If P is the k-dimensional cube, then $t(P) = 2^k$ (the number of vertices). On the other hand, for $P = \alpha_k$ (the k-dimensional simplex), this number is $\geq k+2$ if $k \geq 3$ (thus, greater than the number of vertices). One may wonder for which polytopes P, the number of vertices of P is an upper bound for $t(P)$. Is it true when P is centrally symmetric? The answer is obviously positive when the number of vertices of P exceeds 2^k which is the case, for instance, if P is a k-dimensional zonotope. Given a polytope P and its symmetrization $P^* := P - P$, observe that $t(P)$ is equal to $t(P^*)$. Hence, if the answer to the above question is positive, we find that $t(\alpha_k) \leq k(k+1)$.

Notes

1. As is well known, the minimum number of chains needed for covering a set system (more generally, a partially ordered set) is equal to the maximum cardinality of an antichain (by Dilworth's theorem) and can be determined in polynomial time (using a maximum flow algorithm). Fleiner [8] has given a minimax formula for the minimum number k of nested subfamilies needed to cover a cut family (more generally, for a symmetric poset) and shown that it can be determined in polynomial time (be a reduction to the matching problem).

References

1. H.-J. Bandelt, V. Chepoi and M. Laurent, Embedding into rectilinear spaces, *Discrete and Computational Geometry*, Vol. 19 (1998) pp. 595–604.
2. L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Clarendon Press, Oxford (1953).
3. L. Danzer and B. Grünbaum, Über zwei Probleme bezüglich knovexer Körper von P. Erdös und von V. L. Klee, *Mathematische Zeitschrift*, Vol. 79 (1962) pp. 95–99.
4. L. Danzer, B. Grünbaum and V. Klee, Helly's theorem and its relatives, *Proceedings of Symposia in Pure Mathematics*, American Mathematical Society, Providence, Rhodes Island, VII (1963) pp. 101–181.
5. N. G. de Bruijn and P. Erdös, On a combinatorial problem, *Indagationes Mathematicae*, Vol. 10 (1948) pp. 421–423.
6. M. Deza and M. Laurent, Geometry of cuts and metrics, *Algorithms and Combinatorics*, Springer Verlag, Berlin, 15 (1997).
7. B. Fichet, Dimensionality problems in L_1-norm representations, *Classification and Dissimilarity Analysis*, Lecture Notes in Statistics, Springer-Verlag, Berlin, 93 (1994) pp. 201–224.
8. T. Fleiner, Covering a symmetric poset by symmetric chains, *Combinatorica*, Vol. 17 (1997) pp. 339–344.
9. Z. Füredi, J. C. Lagarias, and F. Morgan, Singularities of minimal surfaces and networks and related extremal problems in Minkowski space, *Discrete and Computational Geometry* (J. E. Goodman et al.,eds.) Vol. 6 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science* (1991) pp. 95–109.
10. H. Groemer, Abschätzungen für die Anzahl der knovexen Körper die einen konvexen Körper berühren, *Monatshefte für Mathematik*, Vol. 65 (1961) pp. 74–81.
11. B. Grünbaum, On a conjecture of H. Hadwiger, *Pacific Journal of Mathematics*, Vol. 11 (1961) pp. 215–219.
12. R. K. Guy and R. B. Kusner, An olla podrida of open problems, often oddly posed, *American Mathematical Monthly*, Vol. 90 (1983) pp. 196–199.
13. F. Hadlock and F. Hoffman, Manhatten trees, *Utilitas Mathematica*, Vol. 13 (1978) pp. 55–67.
14. H. Hadwiger, Ueber Treffanzahlen bei translationsgleichen Eikörpern, *Archiv der Mathematik*, Vol. 8 (1957) pp. 212–213.

15. G. R. Lawlor and F. Morgan, Paired calibrations applied to soap films, immiscible fluids, and surfaces and networks minimizing other norms, *Pacific Journal of Mathematics*, Vol. 166 (1994) pp. 55–82.
16. P. W. H. Lemmens and J. J. Seidel, Equiangular sets of lines, *Journal of Algebra*, Vol. 24 (1973) pp. 494–512.
17. F. Morgan, Minimal surfaces, crystals, networks, and ungraduate research, *Mathematical Intelligencer*, Vol. 14 (1992) pp. 37–44.
18. C. M. Petty, Equilateral sets in Minkowski spaces, *Proceedings of the American Mathematical Society*, Vol. 29 (1971), pp. 369–374.
19. K. J. Swanepoel, Extremal problems in Minkowski space related to minimal networks, *Proceedings of the American Mathematical Society*, Vol. 124 (1996) pp. 2513–2518.
20. K. J. Swanepoel, Cardinalities of k-distance sets in Minkowki spaces, Technical report UPWT 97/4, University of Pretoria (1997).
21. H. P. F. Swinnerton-Dyer, Extremal lattices of convex bodies, *Proceedings of the Cambridge Philosophical Society*, Vol. 49 (1953) pp. 161–162.
22. I. Talata, Exponential lower bound for the translative kissing numbers of d-dimensional convex bodies, *Discrete and Computational Geometry*, Vol. 19 (1998) pp. 447–455.
23. I. Talata, The translative kissing number of tetrahedra is 18, *Discrete and Computational Geometry*, Vol. 22 (1999) pp. 231–248.
24. A. C. Thompson, Minkowski Geometry, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 63 (1996).
25. J. H. van Lint and J. J. Seidel, Equilateral point sets in elliptic geometry, *Indagationes Mathematicae*, Vol. 28 (1966) pp. 335–348.