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Triangle Sets in PHIGS PLUS: a Valuable Link with Finite Element Modeling

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ABSTRACT

At the ISO/IEC editing meeting on PHIGS PLUS[2] in Villars sur Ollon, Switzerland from 9 to 14 September 1990, an additional output primitive was added to PHIGS PLUS [2]: TRIANGLE SET 3 WITH DATA together with its 2D shorthand. The principal reason for that addition was that only triangular primitives admit uniquely defined interpolation of data (colours, normals, etc.) across the facet.
However, an additional advantage of this new primitive is its value for the visualization of scientific computing results. It corresponds in a very pleasant way with the Finite Element Method using piecewise linear blending functions on a triangular partition of the function domain.
In this paper, an outline will be given of the possible use of this function by Finite Element engineers.

1983 CR Categories: G.1.1, G.1.2, I.3.2, I.3.5, I.3.7, J.6

Keywords and Phrases: computer graphics standardization, PHIGS, PHIGS PLUS, computer aided geometric design, finite element modeling, surfaces in computer graphics.

Note: the present text will be published in *Computer Graphics Forum.*

1. Introduction

At the editing meeting on the PHIGS PLUS [2] Committee Draft (Committee Draft or CD is the new name for Draft Proposal), a new surface primitive was added: TRIANGLE SET 3 WITH DATA and its 2D shorthand TRIANGLE SET WITH DATA. The functional specification of this primitive is

TRIANGLE SET 3 WITH DATA **(PHOP,*,STOP,*)**

Parameters:

In	data per facet flag		E
In	data per edge flag		E
In	data per vertex flag		E
[In	colour type		I]
[In	trios of edge flags	(OFF, ON)	L(E, E, E)]
[In	facet data		L(colours, facet normals, reals)]
In	vertex data		L(P3[, colour][, vertex normal][, reals])
In	trios of vertex indices	(1..n)	L(I, I, I)

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For the 2D shorthand TRIANGLE SET WITH DATA, a similar definition holds with P3 replaced by P2.

Effect: The graphical element TRIANGLE SET (3) WITH DATA is stored in the currently open structure. At traversal time, the triangles specified by the list of trios of vertex indices and by the list of vertices are rendered on the open workstations, using the current rendering attributes, the current interior and edge attributes and the rendering data within the primitive's parameter list.

> **Comparison with Triangle Strip with Data**
>
> PHIGS PLUS supports another triangular primitive:
> Triangle Strip with Data. The difference between these two primitives is the way the triangles are defined.
>
> In the Triangle Strip primitive, each i-th triangle is specified by the i-th, (i+1)-th and (i+2)-th member from the list of vertices. In the Triangle Set, each triangle is specified by a trio of indices, each of which refers to a vertex. Both primitives have their pros and cons. The Triangle strip has a more compact parameter list and guarantees that a closed and connected region is specified, but is less suitable for parallel rendering than the Triangle Set. The Triangle Set, on the other hand, does not guarantee that the result is a closed and connected figure.

2. Triangle Sets and Triangular Splines

2.1. Spline primitives in PHIGS PLUS

PHIGS PLUS [2] has added two classes of new output primitives to PHIGS[1]:

1) Extension of linear and facial primitives with and without rendering data; examples are POLYLINE 3 SET WITH DATA, FILL AREA SET 3 WITH DATA, QUADRILATERAL MESH 3 WITH DATA and the triangular primitives TRIANGULAR STRIP 3 WITH DATA and TRIANGLE SET 3 WITH DATA;
2) Non-Uniform Rational B-Spline (NURB) curves and surfaces with and without rendering data; these primitives play a central role in Finite Element Modeling (also called Finite Element Method, henceforth called FEM) and are being used in a host of areas like Simulation of Systems, Computer Aided Geometric Design (CAGD) and Optimal Control.

FEM, however, knows another important class of parametric surfaces: the Triangular Splines. These splines have not been adopted by PHIGS PLUS, except the simplest one: the linear triangular spline, which is a special case of Triangle Set 3 with Data. We return to that primitive in section 2.3.

Reasons for not adopting triangular splines were:

1) They are widely used but not as generally as NURBs;
2) They can be simulated, albeit clumsily, by degenerated NURBs;
3) None of the US-based workstation manufacturers represented in the PHIGS PLUS committee was ready to implement this type of splines.

2.2. Definition of Triangular Splines

Let $\{ T_j \}_{j=1}^{N}$ be a set of *two-dimensional* triangles in the (u,v) parameter space with the following properties:

1) The interiors of two different triangles are disjoint:

$$\mathrm{int}\,(T_i) \cap \mathrm{int}\,(T_j) = \emptyset \text{ , if } i \neq j; \qquad (1)$$

2) No vertex points of adjacent triangles on the *interior* of any triangle edge;
3) The intersection of two different triangles is

a) empty,

or

b) a common vertex point;

or

c) a common edge;

4) The union of all these triangles is a closed and connected region: the boundary of

$$D = \bigcup_{j=1}^{N} T_j \qquad (2)$$

is a *closed non-self-intersecting* Polyline.

These properties are essential to make the Triangle Set function significant for FEM.

By definition a set of triangles that satisfies these four properties is called a *Triangular Partition* or *Triangulation* of D (see figure 1) and is denoted by Δ (D) or by Δ shortly.

Mesh generators

Many FEM packages support mesh generators, i.e. modules that partition a given domain into a list of triangles or quadrilaterals, given some partition criterion. That criterion can be the maximum diameter of the triangle, the maximum number of triangles, etc.

These mesh generators also *approximate* domains with *curved* boundaries by a list of triangles :

$$\bigcup_{j=1}^{N} T_j = D_\Delta \approx D$$

with some partition criterion, e.g.

$$| \text{area}((D \cup D_\Delta) - (D \cap D_\Delta)) | \leq \varepsilon * \text{area}(D)$$

A possible specification of a tri-mesh generator could be (using PHIGS terminology)

Generate Triangle Set

In	Domain Boundary (Parametric Curve)
In	Domain Partition Criterion
Out	List of Vertices $\{P_i\}_{i=1}^{M}$
Out	List of Index Trios $\{ I_j \}_{j=1}^{N}$

See figures 1 and 3.

Given such a partition $\Delta = \{ T_j \}_{j=1}^{N}$, we can define the set

$$S_{k,p}(\Delta) \; ; p \geq 0 \; ; k \geq 1; \qquad (3)$$

of *Triangular Splines* as the set of parametric surfaces F(u,v) defined on D, which are

1) p times differentiable with respect to u and v:

$$\frac{\partial^l F(u,v)}{\partial^m u \, \partial^{l-m} v} \text{ continuous;}$$

$$m = 0, \ldots, l; \qquad (4)$$

$$l = 0, \ldots, p;$$

for p=0, this means that F(u,v) is continuous on D; for p=1, this means that F, $\partial F/\partial u$ and $\partial F/\partial v$ are continuous;

2) parametric polynomial of degree k on each triangle T_j:

$$F(u,v) = \sum_{m,\,n=0}^{m+n \le k} \alpha^{j}_{m,n}\, u^m\, v^n;$$

$$(u,v) \in T_j; \qquad (5)$$

$$j = 1, \ldots, N;$$

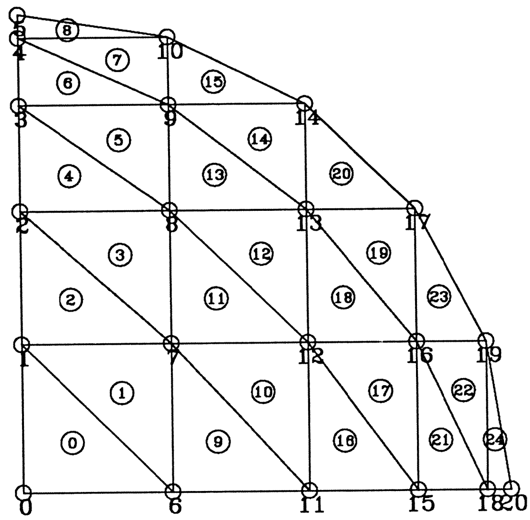

Figure 1. Triangulation of the interior of the quarter circle $u^2 + v^2 \le 1$; u, v $\ge$ 0; the encircled numbers refer to the triangles, the other numbers refer to the vertices

Use of Triangular Splines

This class of splines is used in FEM and CAGD, although not as frequently as the NURBs. They are mainly used for the design or approximation of surfaces which are defined on topologically nonrectangular domains, for example on polygonal approximations of domains with curved boundaries. The most popular examples of triangular splines in FEM are the Legendre and Hermite splines [4], the most popular examples in CAGD are the triangular Bezier splines [3].

2.3. The linear triangular spline and the Triangle Set

It is not hard to show that the linear triangular spline (i.e. an element of $S_{1,0}(\Delta)$) is a special case of a Triangle Set 3 with Data. In fact, it is a continuous piecewise linear mapping from a 2D Triangle Set with the properties (1)-(4) from section 2.2 to a 3D Triangle Set with the properties (1)-(3) from section 2.2.

Let D be a closed and connected region in the (u,v) parameter space with a polygonal non-self-intersecting boundary, let

$$\Delta = \{ T_j \}_{j=1}^{N} \qquad (6)$$

be a triangular partition of D, i.e.

$$D = \bigcup_{j=1}^{N} T_j \; ; \tag{7}$$

let

$$\{P_j\}_{j=1}^{M} \; ; \; P_j = (u_j, v_j); \tag{8}$$

be the list of vertices of Δ and let $\{ I_j \}_{j=1}^{N}$ be the list of index trios, specified by

$$I_j = \begin{bmatrix} i_{1,j} \\ i_{2,j} \\ i_{3,j} \end{bmatrix} ; j = 1, \ldots, N; \tag{9}$$

For $S_{1,0}(\Delta)$, we define the piecewise linear blending functions (also called pyramid functions or chapeau functions, see figure 2) $\{\phi_i(u,v)\}_{i=1}^{M}$ by

$$\phi_i(P_j) = \begin{cases} 0, & i \neq j; \\ 1, & i = j. \end{cases} \tag{10}$$

It is standard (see e.g.[3, 4]) that each function $F \in S_{1,0}(\Delta)$ can be represented by

$$F(u,v) = \sum_{j=1}^{M} F(u_j, v_j)\phi_j(u,v) = \sum_{j=1}^{M} F_j \phi_j(u,v) \tag{11}$$

It is now obvious from the preceding that F continuously maps the two-dimensional Triangle Set Δ onto the three-dimensional Triangle Set Δ^{3D} defined by

$$\Delta^{3D} = \{T_j^{3D}\}_{j=1}^{N} \tag{12}$$

where

$$\begin{aligned} T_j^{3D} &= \mathrm{TRIANGLE}(P_{i_{1,j}}^{3D}, P_{i_{2,j}}^{3D}, P_{i_{3,j}}^{3D}) \\ & j = 1, \ldots, N; \\ P_i^{3D} &= \begin{bmatrix} u_i \\ v_i \\ F(u_i, v_i) \end{bmatrix}, i = 1, \ldots, M. \end{aligned} \tag{13}$$

This concludes the proof of

THEOREM 1. Let $\Delta = \{ T_j \}_{j=1}^{N}$ be a 2D Triangle Set within the (u,v) parameter space specified by a list $\{I_j\}_{j=1}^{N}$ of trios of vertex indices and by a list $\{P_i\}_{i=1}^{M}$ of vertices, with the properties (1)-(4) from section 2.2 and let $S_{1,0}(\Delta)$ be the set of triangular splines defined on Δ (see section 2.2). Then any $F \in S_{1,0}(\Delta)$ continuously maps Δ onto a 3D Triangle Set Δ^{3D} defined by (6-13) □

Consequently, the functions Triangle Set with Data and Triangle Set 3 with Data can be used to visualize scientific computing results from FEM using piecewise linear triangular splines (see table 1).

Using the same reasoning, Theorem 1 can easily be extended to

THEOREM 2. Let $\Delta = \{ T_j \}_{j=1}^{N}$ be a 2D Triangle Set within the (u,v) parameter space specified by a list $\{I_j\}_{j=1}^{N}$ of trios of vertex indices and by a list $\{P_i\}_{i=1}^{M}$ of vertices, with the properties from section 2.2.

Let F_1, F_2, F_3 be three elements of $S_{1,0}(\Delta)$ and let $\{P_i^{3D}\}_{i=1}^{M}$ be the list of 3D control points defined by

$$P_i^{3D} = \begin{bmatrix} F_1(u_i,v_i) \\ F_2(u_i,v_i) \\ F_3(u_i,v_i) \end{bmatrix}, \; i = 1, \ldots, M.$$

Then the parametric polynomial surface

$$S(u,v) = \begin{bmatrix} F_1(u,v) \\ F_2(u,v) \\ F_3(u,v) \end{bmatrix} = \sum_{i=1}^{M} P_i^{3D} \phi_i(u,v)$$

maps continuously the 2D Triangle Set $\{T_j\}_{j=1}^{N}$ onto the 3D Triangle Set $\Delta^{3D} = \{T_j^{3D}\}_{j=1}^{N}$ with

$$T_j^{3D} = \text{TRIANGLE}(P_{i_{1,j}}^{3D}, P_{i_{2,j}}^{3D}, P_{i_{3,j}}^{3D})$$

□

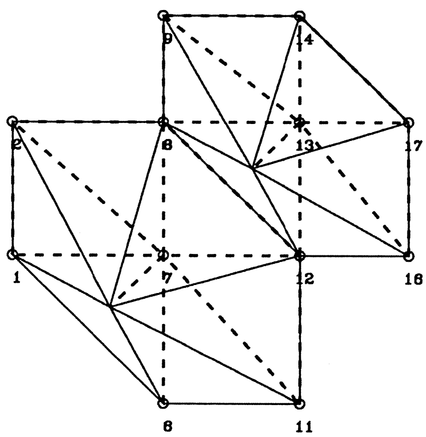

Figure 2. Graph of the pyramid functions $\phi_7(u,v)$ and $\phi_{13}(u,v)$ corresponding with the triangulation of figure 1.

3. Conclusions

The functions TRIANGLE SET 3 WITH DATA and its 2D subset form a powerful and robust visualization tool for FEM applications using other splines than NURBs.

The Triangle Set primitives can be used directly for the visualization of triangular partitions of arbitrary domains and for the display of piecewise linear splines defined on these partitions.

These primitives can be used indirectly for the visualization of other splines than NURBs, when these splines are approximated by linear triangular splines on application level.

The addition of these primitives is a reasonable compensation for the missing of other splines in PHIGS PLUS and by their adoption, PHIGS PLUS has become considerably more valuable for FEM users. Also,

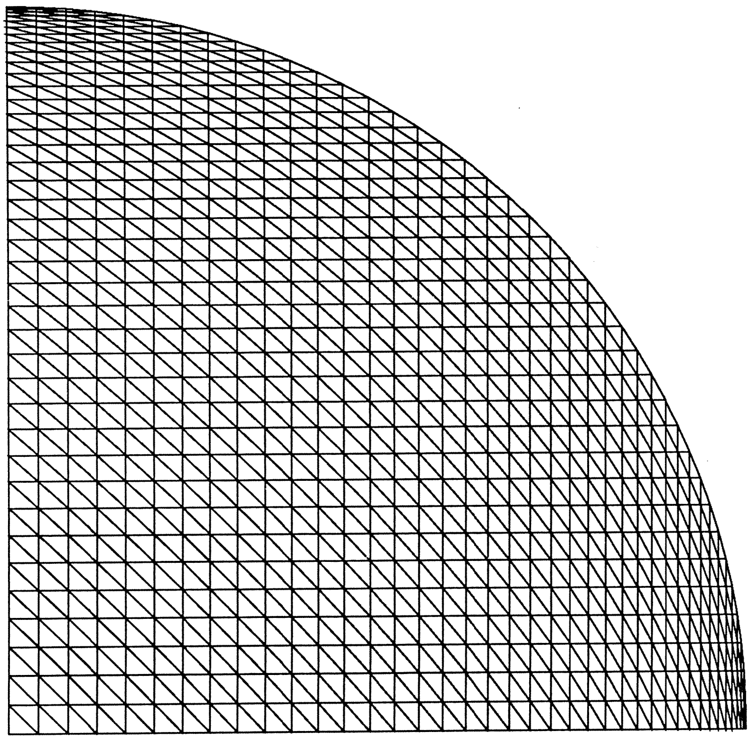

Figure 3. Finer triangulation of the interior of the quarter circle $u^2 + v^2 \leq 1$; u, $v \geq 0$;

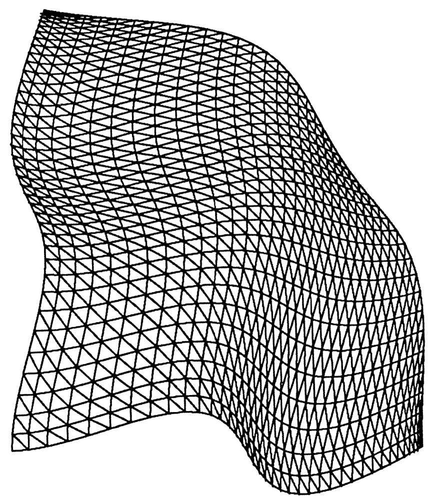

Figure 4. Graph of a surface defined on the interior of the quarter circle $u^2 + v^2 \leq 1$; u, $v \geq 0$;
The surface is approximated by a Triangle Set 3

the addition of these new primitives may be an encouragement to add more classes of widely used spline surfaces to PHIGS PLUS.

Generate Triangle Set		
	In	Domain Boundary (Polygon or Parametric Curve)
	In	Domain Partition Criterion
	Out	List of Vertices $\{P_i\}_{i=1}^{M}$
	Out	List of Index Trios $\{I_j\}_{j=1}^{N}$
Solve FEM Problem		
	In	List of Vertices $\{P_i\}_{i=1}^{M}$
	In	List of Index Trios $\{I_j\}_{j=1}^{N}$
	In	FEM Problem Data Record
	Out	List of Vertices $\{P_i^{3D}\}_{i=1}^{M}$
	[Out	List of vertex normals $\{Vnorm_i\}_{i=1}^{M}$]
	[Out	List of facet normals $\{Fnorm_i\}_{i=1}^{N}$]
<display domain; see figure 3 >		
Triangle Set with Data		
	In	List of Vertices $\{P_i\}_{i=1}^{M}$
	In	List of Index Trios $\{I_j\}_{j=1}^{N}$
<display solution of FEM problem; see figure 4 >		
Triangle Set 3 with Data		
	In	List of Vertices $\{P_i^{3D}\}_{i=1}^{M}$
	In	List of Index Trios $\{I_j\}_{j=1}^{N}$
	[In	List of vertex normals $\{Vnorm_i\}_{i=1}^{M}$]
	[In	List of facet normals $\{Fnorm_i\}_{i=1}^{N}$]

Table 1. Computing scheme of FEM application using Triangle Set and Triangle Set 3

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