Analytical methods for an elliptic singular perturbation problem In a circle

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April 2, 2006

Abstract

We consider an elliptic perturbation problem in a circle by using the analytical solution that is given by a Fourier series with coefficients in terms of modified Bessel functions. By using saddle point methods we construct asymptotic approximations with respect to a small parameter. In particular we consider approximations that hold uniformly in the boundary layer, which is located along a certain part of the boundary of the domain.

2000 Mathematics Subject Classification: 35B25, 35C05, 35C20, 35J25, 41A60.
Keywords & Phrases: singular perturbations, elliptic equations, boundary value problem, series solution, uniform asymptotic expansion.

1 Introduction

Consider the elliptic partial differential equation

$$\varepsilon\Delta\Phi(x,y) - \frac{\partial\Phi}{\partial y}(x,y) = f(x,y), \quad x^2+y^2<1, \tag{1.1}$$

where $\varepsilon>0$. The boundary condition reads

$$\Phi(\cos\theta,\sin\theta) = g(\theta) \tag{1.2}$$

on the boundary of the circle $r=1$, where we introduced the polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad 0\le r\le 1, \quad -\pi<\theta\le\pi. \tag{1.3}$$

The problem is to find the asymptotic behavior of Φ as $\varepsilon\to 0$. The solution to equation (1.1) has a boundary layer at the boundary where y is positive. In

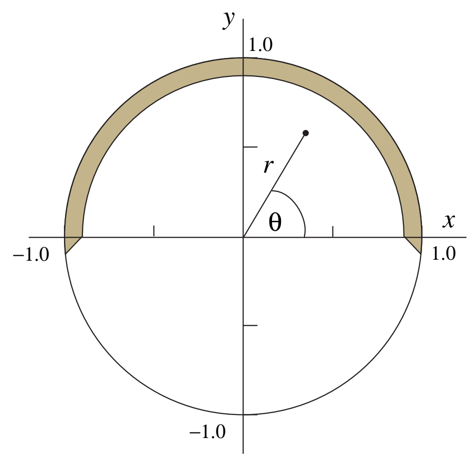

Figure 1: Boundary layer inside the circle along the upper boundary $r = 1, y > 0$ and near the points $(\pm 1, 0)$.

particular it is of interest to find the behavior of Φ in small neighborhoods of the points $(x, y) = (\pm 1, 0)$, the places of birth of the boundary layer.

When $\varepsilon \to 0$ the second order elliptic operator in (1.1) reduces (in the limit $\varepsilon = 0$) to a first order operator. The solution of the reduced equation cannot satisfy the boundary condition on the whole circle. The capricious behavior of the solution occurs in the boundary layer, whereas along the part with $y < 0$ the solution behaves very regularly; see [6, Theorem IV].

This type of singular perturbation problems is well studied in the literature. Rather early publications are [26] and [11], who have pointed out that the boundary layer occurs near $r = 1, y > 0$ and the points $(\pm 1, 0)$; see Figure 1. A classical paper for the construction of the asymptotic expansion is [6], where general operators in a general convex domain are considered. Further research is done in numerous papers and books, for example in [9]. The most common method to obtain the approximations is based on stretching the variables in the boundary layers and substituting local expansions in the transformed equations. Matching the solutions from one domain into other domains yield values for integration constants, from which uniform approximations can be obtained.

When we choose simple operators (as in (1.1)) and domains it is possible to solve the problems analytically and to use other methods for obtaining asymptotic approximations. In [23] we have used analytical methods based on integral representations to study a problem as in (1.1)–(1.2), with simple functions f and g, in a sector

$$\left\{x = r\cos\theta,\ y = r\sin\theta \;\middle|\; r \ge 0,\ 0 \le \theta \le \alpha < 2\pi\right\}, \tag{1.4}$$

and we were able to describe in detail the behavior of the solution in the boundary layer $\theta \sim \alpha$ and in the internal layer $\theta \sim \frac{1}{2}\pi$. It was also possible to describe precisely the behavior of the solution in the case of the transition of the sector into the quarter plane $\alpha \to \frac{1}{2}\pi$, in which case a boundary layer of different type arises (a so-called parabolic layer is present when $\alpha = \frac{1}{2}\pi$, whereas the boundary layer is of a linear character when $\alpha \neq \frac{1}{2}\pi$). For further recent studies on this type of problems in which the asymptotic approximations are derived from explicit representations of the solutions, we refer to [13], [14], [12], [15], [16], and [17]. See also [21] and [22].

In [19] the circle problem is considered with the same simple differential equation and boundary condition as in our case. A detailed analysis is given for the boundary layer near the points $(\pm 1, 0)$ by using boundary layers coordinates. Integrals of ratios of Airy functions are used to obtain the approximations. In [25] the problem is considered in a domain exterior to the unit circle. In that very instructive paper the exact solution is used with saddle point methods for integrals and residue series in terms of zeros of modified Bessel functions. In [10] these results are summarized and numerical aspects of this problem are discussed.

In [8], [19] and [20] problems from mathematical physics are given that lead to the elliptic singular perturbation problem considered here. The equation (1.1) arises in magnetohydrodynamics, where ε measures the importance of viscous force relative to the electromagnetic force, and in the theory of plate-membranes under tension in the $y-$direction, where ε measures the bending stiffness.

The purpose of the paper is to construct approximations of $\Phi(x, y)$ by using the analytic representation of the solution, which in the present case can be given in terms of a Fourier series of which the coefficients can be written in terms of Bessel functions.

2 Singular perturbation methods

We give a few steps on the construction of the asymptotic solution of the singular perturbation problem by substituting an asymptotic expansion. We consider the problem defined in (1.1)–(1.2) with simple choices of f and g. In order to be able to construct an explicit series solution later in this paper we consider

$$\begin{aligned} &\varepsilon \Delta \Phi(x, y) - \frac{\partial \Phi}{\partial y}(x, y) = 1, \quad x^2 + y^2 < 1, \\ &\Phi(\cos\theta, \sin\theta) = 0. \end{aligned} \tag{2.1}$$

To give a first impression of what is happening in the boundary layer we consider an example of a singular perturbation problem for an ordinary differential equation. It is known that the influence of the term w_{xx} in (1.1) is not very great in the interior of the circle and when x is bounded away from ± 1. This is due to the influence of the characteristic lines $x =$ constant of the linear operator in the equation. The solution of the equation

$$\varepsilon \frac{d^2 w}{dy^2} - \frac{dw}{dy} = 1, \quad w = 0 \ \text{ if } \ y = \pm 1, \tag{2.2}$$

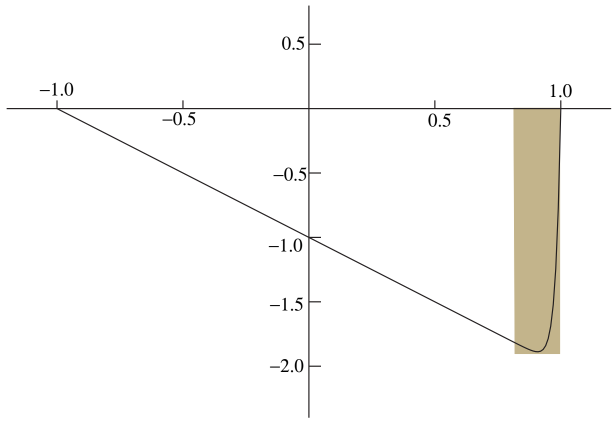

Figure 2: The solution $w(y)$ of (2.3) on the interval $[-1, 1]$, with boundary layer near $y = 1$.

is given by

$$w(y) = -1 - y + \frac{e^{y/\varepsilon} - e^{-1/\varepsilon}}{\sinh 1/\varepsilon}. \tag{2.3}$$

We observe that on the interval $[-1, 1-\delta]$, where δ is a fixed small positive number, for small values of ε the function w is equal to $w_0(y) = -1 - y$ plus a function that is exponentially small. Near the boundary $y = 1$ the solution w drops from the value -2 to its proper boundary value 0; see Figure 2. In this example we see that the boundary layer occurs at $y = 1$. Similarly, for the circle problem (1.1) the boundary layer occurs at the upper semi-circle. Inside the circle the solution of the circle problem can be approximated by

$$w_0(x, y) = -y - \sqrt{1 - x^2}, \tag{2.4}$$

which satisfies the condition on the lower semi-circle, but not on the upper semi-circle. Using the singular perturbation method (cf. for instance [6]) we can construct more terms in an expansion. When we substitute the formal series

$$\Phi(x, y) \sim \sum_{n=0}^{\infty} \varepsilon^n w_n(x, y) \tag{2.5}$$

into (2.1) and equate equal powers of ε, we find

$$\begin{aligned} \frac{\partial w_0(x,y)}{\partial y} &= -1, \\ \frac{\partial w_n(x,y)}{\partial y} &= \Delta w_{n-1}(x,y), \quad n = 1, 2, \ldots, \end{aligned} \tag{2.6}$$

and all w_n should vanish at the lower part of the unit circle. This gives w_0 given in (2.4) and

$$w_n(x,y) = \int_{-\sqrt{1-x^2}}^{y} \Delta w_{n-1}(x,\eta)\, d\eta, \quad n = 1, 2, \ldots, . \tag{2.7}$$

It is easily verified that

$$\begin{aligned} w_1(x,y) &= \frac{y+R}{R^3}, \qquad w_2(x,y) = \frac{y+R}{2R^7}\left(3y + 12yx^2 + R\right), \\ w_3(x,y) &= \frac{y+R}{2R^{11}} \times \\ &\quad \left(15(8x^4+12x^2+1)y^2 + 3(3+20x^2)yR - 2(12x^4+6x^2-1)R^2\right), \end{aligned} \tag{2.8}$$

where $R = \sqrt{1-x^2}$. We observe that these w_n become singular at the points $(\pm 1, 0)$ and that they do not satisfy the boundary condition $w_n = 0$ on the upper part of the unit circle.

To satisfy the boundary conditions along the upper part of the unit circle so-called boundary layer terms are introduced. These functions have the property of being of order $\mathcal{O}(\varepsilon^n)$ for all n everywhere inside the unit circle, except for a small neighborhood of the upper part of the circle. Following the construction of the boundary layer term given in [6], we can write in first approximation

$$\Phi(x,y) = -y - \sqrt{1-x^2} + 2\sin\theta\, e^{-\frac{1}{\varepsilon}(1-r)\sin\theta}\, \psi(x,y) + z_0(x,y,\varepsilon), \tag{2.9}$$

where $z_0(x,y,\varepsilon) = \mathcal{O}(\varepsilon)$, uniformly inside the unit disk, with the exception of small neighborhoods of the points $(\pm 1, 0)$. The function ψ is a $C^\infty-$function, a smoothing factor, on the disc, which equals unity on a neighborhood of the upper part of the circle, say the domain given by $\frac{2}{3} < r \le 1, y > y_0$, where y_0 is a fixed positive small number, and ψ vanishes in the lower part of the disc.

In [6] an iteration process is given for obtaining any number of terms w_m and boundary layer functions v_m in the expansion

$$\Phi(x,y) = \sum_{m=0}^{n} \varepsilon^m w_m(x,y) + \psi(x,y) \sum_{m=0}^{n+1} \varepsilon^m v_m(x,y,\varepsilon) + z_n(x,y,\varepsilon), \tag{2.10}$$

where $w_m(x,y)$ and $v_m(x,y,\varepsilon)$ are uniformly bounded for $\varepsilon > 0$ for $r \le 1$, with the exception of small neighborhoods of the points $(\pm 1, 0)$, and where $\psi(x,y)$ is a smoothing factor. For a description of the properties of the remainder $z_n(x,y,\varepsilon)$ we refer to [6, Theorem VI].

3 The solution of the boundary value problem

We construct the solution of equation (1.1) and boundary condition (1.2) with the simple functions $f(x,y) = 1, g(\theta) = 0$. A first substitution

$$\Phi(x,y) = -y - e^{\omega y} F(x,y), \tag{3.1}$$

gives the problem

$$\Delta F(x,y) - \omega^2 F(x,y) = 0, \quad \omega = \frac{1}{2\varepsilon}, \tag{3.2}$$

with boundary condition

$$F(\cos\theta, \sin\theta) = -\sin\theta\, e^{-\omega\sin\theta}. \tag{3.3}$$

The Helmholtz equation in (3.2) can be solved in terms of modified Bessel functions by using the polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{3.4}$$

In terms of r and θ, (3.2) reads

$$r^2 F_{rr} + rF_r - \omega^2 r^2 F + F_{\theta\theta} = 0. \tag{3.5}$$

The modified Bessel function $I_n(\omega r)$ satisfies the differential equation

$$r^2 G'' + rG' - \left(\omega^2 r^2 + n^2\right) G = 0, \tag{3.6}$$

where differentiation is with respect to r. It is straightforward to verify that the Fourier series

$$F(x,y) = \sum_{n=-\infty}^{\infty} a_n I_n(\omega r) e^{in\theta} \tag{3.7}$$

satisfies (3.2), where the coefficients a_n follow from the well-known Bessel function series (cf. [1, 9.6.33])

$$e^{z\cos t} = \sum_{n=-\infty}^{\infty} I_n(z)\cos nt. \tag{3.8}$$

This gives

$$a_n = \frac{I_n'(\omega)}{I_n(\omega)}\, e^{in\frac{1}{2}\pi}. \tag{3.9}$$

By using the symmetry $I_n(z) = I_{-n}(z)$, we obtain

$$F(x,y) = 2{\sum_{n=0}^{\infty}}' \frac{I_n'(\omega)}{I_n(\omega)}\, I_n(\omega r)\, \cos n(\theta + \tfrac{1}{2}\pi), \tag{3.10}$$

where the prime in the summation symbol means that the first term of the sum is to be halved.

4 The Poisson summation formula

In the previous section we have derived the solution of the boundary value problem (3.2)–(3.3) in terms of the Fourier series given in (3.10). In this section we investigate this Fourier series by transforming it by using the Poisson summation formula. In this way we obtain a series of integrals, and the first term of the new series can be used for obtaining the asymptotic expansion of $\Phi(x, y)$ outside the boundary layer.

We apply the Poisson summation formula to the series in (3.10). We have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \widehat{f}(2\pi m), \tag{4.1}$$

where $\widehat{f}$ is the Fourier transform of f:

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)\, e^{ixy}\, dx. \tag{4.2}$$

This result holds if f is of bounded variation and absolutely integrable on $\mathbb{R}$ (cf. [29, p. 68]). For cosine transforms we have (by assuming that f is even)

$$\sum_{n=0}^{\infty}{}' f(n) = \sum_{n=0}^{\infty}{}' \widehat{f}(2\pi m), \quad \widehat{f}(y) = 2\int_0^{\infty} f(x)\, \cos(xy)\, dx. \tag{4.3}$$

Applying this to (3.10) we obtain

$$F(x, y) = 2\sum_{n=0}^{\infty}{}' F_m(x, y), \tag{4.4}$$

where

$$F_m(x, y) = 2\int_0^{\infty} \frac{I'_\nu(\omega)}{I_\nu(\omega)}\, I_\nu(\omega r)\, \cos\nu(\theta + \tfrac{1}{2}\pi)\, \cos(2\pi m\nu)\, d\nu. \tag{4.5}$$

The function $I_\nu(z)$ is an analytic function of ν, with the the asymptotic behavior as given in (A.2). For small values of ν and large z the estimates given in (A.3) are applicable, and $I_\nu(z)$ is positive if z and ν are positive. It follows that all functions $F_m(x, y)$ in (4.5) are well-defined, and that the Poisson summation formula can be applied. The integrals that define $F_m(x, y)$ converge fast for fixed values of ω, as follows from (A.2).

4.1 The asymptotic behavior of $F_0(x, y)$

To investigate the asymptotic behavior of $F_0(x, y)$ defined in (4.5) we use the Debye uniform approximation of $I_\nu(\omega)$ given in Appendix A. We have

$$\frac{I'_\nu(\omega)}{I_\nu(\omega)}\, I_\nu(\omega r) = \frac{\sqrt{\nu^2 + \omega^2}}{\sqrt{2\pi}\omega}\, \frac{e^{\nu\eta}}{(\nu^2 + \ \omega^2 r^2)^{\frac{1}{4}}}\, \left[1 + \mathcal{O}(1/\omega)\right], \tag{4.6}$$

as $\omega \to \infty$, uniformly with respect to $\nu \in [0, \infty)$. The quantity η is given by

$$\eta = \sqrt{1+z^2} + \ln \frac{z}{1+\sqrt{1+z^2}}, \quad z = \frac{\omega r}{\nu}. \tag{4.7}$$

Using this in F_0 of (4.5), putting $\nu = \omega r \sinh t$, replacing the cosine by an exponential function, we obtain

$$F_0(x,y) \sim \sqrt{\frac{\omega r}{2\pi}} \int_{-\infty}^{\infty} \sqrt{1 + r^2 \sinh^2 t}\, \sqrt{\cosh t}\, e^{-\omega r f(t)}\, dt, \tag{4.8}$$

where

$$f(t) = (t - i\theta - i\tfrac{1}{2}\pi) \sinh t - \cosh t. \tag{4.9}$$

The saddle points follow from the equation $f'(t) = (t - i\theta - \frac{1}{2}\pi i)\cosh t = 0$. The zeros of $\cosh t$ are $i(\frac{1}{2}\pi + k\pi)$. The other one is $i\theta + \frac{1}{2}\pi i$. For details on the saddle point method for integrals we refer to [28]

Several interesting aspects can be observed:

- when $\theta \to 0$, the two saddle points and a singularity of the integrand coalesce.
- when $\theta \to 0$ and $r \to 1$, another singularity coalesces with the two coalescing saddle points.

There is no standard method in uniform asymptotic methods for integrals available to handle the second case. In the first case Airy functions can be used, although the term $\sqrt{\cosh t}$ causes a difficulty.

This first orientation in the asymptotic phenomena demonstrates the complicated situation that arises in the points $(\pm 1, 0)$. On the other hand, the asymptotic estimate (4.6) is not valid in the neighborhood of the points $\nu = \pm i\omega$ and $\nu = \pm i\omega r$. The first points correspond with saddle points due to zeros of $\cosh t$.

By using standard methods we obtain the first terms in the asymptotic expansion of F_0 for the case that $y < 0$. Because our singular perturbation problem is symmetric with respect to x we can always assume that $x \geq 0$.

Let $\theta \in [-\frac{1}{2}\pi, 0)$. The saddle point of interest is $t_0 = i(\theta + \frac{1}{2}\pi)$, and we can deform the interval of integration $(-\infty, \infty)$ into the path of steepest descent through the saddle point t_0; see Figure 3.

We have $f(t_0) = \sin\theta$ and transform

$$f(t) = f(t_0) + \tfrac{1}{2}u^2, \quad \operatorname{sign}(t) = \operatorname{sign}(u), \tag{4.10}$$

which gives

$$F_0(x,y) \sim \sqrt{\frac{\omega r}{2\pi}}\, e^{-\omega r \sin\theta} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\omega r u^2}\, g(u)\, du, \tag{4.11}$$

where

$$g(u) = \sqrt{1 + r^2 \sinh^2 t}\, \sqrt{\cosh t}\, \frac{dt}{du}. \tag{4.12}$$

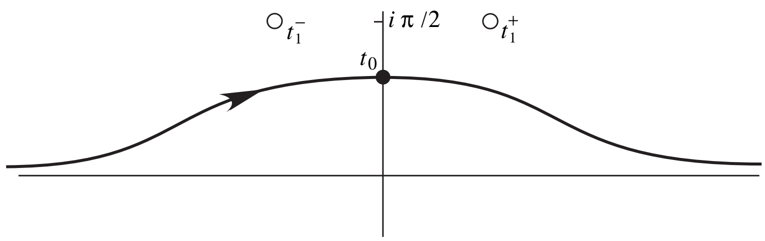

Figure 3: Saddle point contour through the saddle point $t_0 = i(\theta+\frac{1}{2}\pi)$ if $\theta \in [-\frac{1}{2}\pi, 0)$. The points $t_1^{\pm} = \pm\mathrm{arccosh}\,\frac{1}{r} + \frac{1}{2}\pi i$ are singularities of $\sqrt{1+r^2\sinh^2 t}$. The point $\frac{1}{2}\pi i$ is also a singularity, because of the factor $\sqrt{\cosh t}$. When $\theta \to 0$ and $r \to 1$ the three singularities at $t_1^{\pm}$ and $\frac{1}{2}\pi i$ coincide with the saddle point t_0, and the standard saddle point method cannot be used.

The expansion of g in powers of u can be obtained by inverting the relation

$$\begin{aligned} \tfrac{1}{2}u^2 = &-\sin\theta\left[\tfrac{1}{2}(t-t_0)^2 + \tfrac{1}{144}(t-t_0)^4 + \tfrac{1}{8}(t-t_0)^6 + \ldots\right] \\ &+i\cos\theta\left[\tfrac{1}{3}(t-t_0)^3 + \tfrac{1}{30}(t-t_0)^5 + \tfrac{1}{840}(t-t_0)^7 + \ldots\right], \end{aligned} \tag{4.13}$$

that is

$$t = t_0 + \frac{u}{\sqrt{-\sin\theta}} - \frac{i\cos\theta}{3\sin^2\theta}\,u^2 - \frac{9\sin^2\theta + 20\cos^2\theta}{72(-\sin\theta)^{\frac{7}{2}}}\,u^3 + \ldots\,. \tag{4.14}$$

This gives

$$g(u) = g_0 + g_1 u + g_2 u^2 + \ldots, \tag{4.15}$$

with

$$\begin{aligned} g_0 &= \sqrt{1-x^2}, \\ g_2 &= \frac{r}{24\,y^3\theta\,(1-x^2)^{\frac{3}{2}}}\left[12r^2\sin^4\theta - (5\cos^2\theta + 3\sin^2\theta)(1-x^2)^2\right]. \end{aligned} \tag{4.16}$$

When we take into account the term g_0 only, we obtain

$$F_0(x,y) \sim e^{-\omega r\sin\theta}\sqrt{1-x^2}, \tag{4.17}$$

which gives

$$\Phi(x,y) \sim -y - \sqrt{1-x^2} = w_0(x,y), \tag{4.18}$$

cf. (2.4). To obtain a second term, which should be compared with w_1 of (2.8), we need g_2 and a coefficient that comes from a second term in the Debye-type expansions of the modified Bessel function $I_\nu(\omega)$; cf. Appendix A. That is, we use

$$\begin{aligned} \frac{I_\nu'(\omega)}{I_\nu(\omega)}\,I_\nu(\omega r) = &\frac{\sqrt{\nu^2+\omega^2}}{\sqrt{2\pi}\omega}\,\frac{e^{\nu\eta}}{(\nu^2+\ \omega^2 r^2)^{\frac{1}{4}}}\,\times \\ &\left[1 + \frac{3\cosh^2 t - 5\sinh^2 t}{24\omega r\cosh^3 t} - \frac{1}{2\omega(1+r^2\sinh^2 t)^2} + \mathcal{O}(1/\omega^2)\right], \end{aligned} \tag{4.19}$$

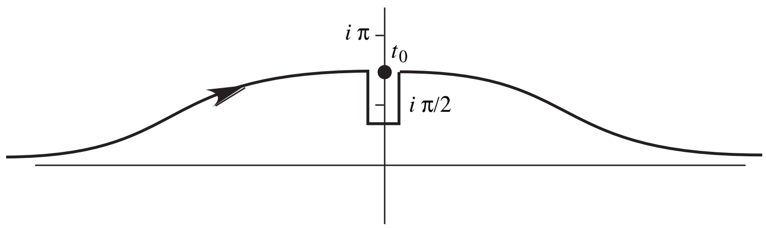

Figure 4: Saddle point contour through the saddle point $t_0 = i(\theta + \frac{1}{2}\pi)$ if $\theta \in (0, \frac{1}{2}\pi)$.

where t and ν are related by $\nu = \omega r \sinh t$. We have

$$F_0(x,y) \sim \sqrt{\frac{\omega r}{2\pi}}\, e^{-\omega r \sin\theta} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\omega r u^2}\, g(u)\, h(u)\, du, \tag{4.20}$$

where $g(u)$ is given by (4.12) and $h(u)$ corresponds with the function between square brackets in (4.19). We expand

$$h(u) = 1 + \left[h_0 + h_1 u + \mathcal{O}(u^2)\right] \frac{1}{\omega} + \mathcal{O}(\omega^{-2}), \tag{4.21}$$

where

$$h_0 = \frac{3\cosh^2 t_0 - 5\sinh^2 t_0}{24 r \cosh t_0} - \frac{1}{2(1 + r^2 \sinh^2 t_0)^2}. \tag{4.22}$$

It follows that

$$F_0(x,y) = e^{-\omega r \sin\theta} \left[g_0 + \left(g_0 h_0 + \frac{1}{r} g_2\right) \frac{1}{\omega} + \mathcal{O}(\omega^{-2})\right]. \tag{4.23}$$

By using the values of the coefficients g_0, g_2 and h_0 given above, we obtain

$$F_0(x,y) = e^{-\omega r \sin\theta} \left[\sqrt{1-x^2} + \frac{-y - \sqrt{1-x^2}}{2\omega(1-x^2)^{\frac{3}{2}}} + \mathcal{O}(\omega^{-2})\right]. \tag{4.24}$$

For Φ this gives

$$\Phi(x,y) = -y - \sqrt{1-x^2} + \frac{y + \sqrt{1-x^2}}{2\omega(1-x^2)^{\frac{3}{2}}} + \mathcal{O}(\omega^{-2}), \quad \omega = \frac{1}{2\varepsilon}, \tag{4.25}$$

which is the same as the two-term expansion that is obtained by using singular perturbation methods (cf. (2.4), (2.5) and (2.8)).

We expect that all higher terms w_n of (2.5) follow from the saddle point method, by using more terms in the Debye-type expansions of the Bessel functions and in the expansion of $g(u)$.

We have derived the above results for negative values of y (that is, $\theta \in [-\frac{1}{2}\pi, 0)$), although the results appear to hold inside the unit disk, with the exception of the boundary layers. When $y > 0$, the saddle point t_0 is located above the other saddle point at $\frac{1}{2}\pi i$, which is also a singularity of the integrand (because of the factor $\sqrt{\cosh t}$). The point t_0 is still the relevant saddle point in this case, and we have to integrate around the branch cut of $\sqrt{\cosh t}$; see Figure 4. This yields the same terms as in (4.25). Again, this follows from standard saddle methods, but the details will not be given.

It is of interest to see how the results for $y < 0$ and for $y > 0$ can both be obtained by using uniform asymptotics, that is, by using a cubic transformation in (4.8). We will not work out this in the present paper.

The approximations derived so far are not valid in the boundary layers. In §4.2 we discuss the behavior of F_0 on the periphery of the circle, and we show that, if $r = 1$, with exception of the point$(0, 1)$,

$$F_0(x, y) = -\sin\theta\, e^{-\omega\sin\theta} + \mathcal{O}\left(e^{-\omega}\right), \quad \omega \to \infty. \tag{4.26}$$

So, apart from the point $(0, 1)$, F_0 satisfies the boundary relation for F (cf. (3.3)) up to an exponentially small term.

A further analysis is needed to show how this behavior can be obtained from the integral in (4.5) with $m = 0$.

In connection with this we mention the following points.

- Equations (4.8) and (4.20) are obtained by replacing the modified Bessel functions in (4.5) with their Debye-type approximations. These approximations are excellent if ν is positive (and ω is large). In the saddle point analysis we encounter saddle points near $\nu = i\omega$, which is a turning point of the modified Bessel function. Near the turning point the Debye-type approximations are not valid, and the singularities in the approximations of the Bessel functions are in fact not related with singular points of the Bessel functions themselves.

- By replacing in (4.5) the Bessel functions with Debye-type approximations we cannot take into account the $\nu-$zeros of $I_\nu(\omega)$, which cause poles near the saddle points. From Appendix B it follows that the zeros occur if

  $$\nu = \pm i\omega\left[1 + \mathcal{O}(\omega^{-\frac{2}{3}})\right] \tag{4.27}$$

  (cf. (B.10)). The transformation of variables $\nu = \omega r \sinh t$, which leads to the integral in (4.8), maps the $\nu-$zeros that are near $i\omega$ to points in the $t-$plane near $\frac{1}{2}\pi i$. At this point interesting phenomena occur when we consider (x, y) in (4.8) near the points $(\pm 1, 0)$.

- We might replace the Bessel functions in (4.5) with the Airy-type approximations as given in Appendix A, which are based on results in [5]. The Airy-type expansions, which are valid near the turning point, do not show singular terms. However, it is quite difficult to manipulate these approximations in a saddle point analysis. In [19] integrals with ratios of Airy functions are used to approximate the solution of $\Phi(x, y)$ near the points $(\pm 1, 0)$; the analysis is based on singular perturbation methods.

4.2 The behavior of F_m at the boundary

We investigate the behavior of the quantities $F_m(x,y)$ introduced in (4.5) at $r = 1$. This gives insight in the role of each $F_m(x,y)$ in constituting the boundary value of $F(x,y)$ given in (3.3). We have to evaluate integrals of the type

$$G'_\sigma(\omega) = \int_0^\infty I'_\nu(\omega)\, \cos\sigma\nu\, d\nu, \tag{4.28}$$

where σ is a real number. In order to do this we first consider the integral

$$G_\sigma(\omega) = \int_0^\infty I_\nu(\omega)\, \cos\sigma\nu\, d\nu. \tag{4.29}$$

We use the Sommerfeld-type integral representation (cf. [24], page 235 or [27], page 181)

$$I_\nu(z) = \frac{1}{2\pi i}\int_{-i\pi+\infty}^{i\pi+\infty} e^{z\cosh t - \nu t}\, dt. \tag{4.30}$$

For the contour we take the two half lines $t = \pm i\pi + v, v \geq \delta > 0$ and the segment that runs from $t = \delta - i\pi$ to $t = \delta + i\pi$. With this choice we can substitute the integral in (4.30) into (4.29), interchange the order of integration, and obtain

$$G_\sigma(\omega) = \frac{1}{2\pi i}\int_{-i\pi+\infty}^{i\pi+\infty} e^{\omega\cosh t}\, \frac{t}{t^2+\sigma^2}\, dt. \tag{4.31}$$

We let $\delta \to 0$ and obtain

$$G_\sigma(\omega) = A_\sigma(\omega) - \frac{1}{2\pi}\int_0^\infty e^{-\omega\cosh t}\left[\frac{\pi-\sigma}{t^2+(\pi-\sigma)^2} + \frac{\pi+\sigma}{t^2+(\pi+\sigma)^2}\right] dt, \tag{4.32}$$

where

$$A_\sigma(\omega) = \frac{1}{2\pi i}\int_{-i\pi}^{i\pi} e^{\omega\cosh t}\, \frac{t}{t^2+\sigma^2}\, dt. \tag{4.33}$$

This integral vanishes if $|\sigma| > \pi$, whereas if $-\pi < \sigma < \pi$ the poles at $t = \pm i\sigma$ are avoided by integrating along small semi-circles (with $\Re t \geq 0$) around the poles. In the latter case the integral can be evaluated by using the residues. It follows that

$$A_\sigma(\omega) = \begin{cases} 0, & \text{if } |\sigma| > \pi, \\ \frac{1}{2}\, e^{\omega\cos\sigma}, & \text{if } -\pi < \sigma < \pi. \end{cases} \tag{4.34}$$

We observe that when $|\sigma| \leq \pi - \delta < \pi$,

$$G_\sigma(\omega) = \frac{1}{2}\, e^{\omega\cos\sigma} + \mathcal{O}\left(e^{-\omega}\right), \quad \omega \to \infty. \tag{4.35}$$

When $\sigma = \pm\pi$ we have

$$G_{\pm\pi}(\omega) = \tfrac{1}{4}\, e^{-\omega} - \int_0^\infty e^{-\omega\cosh t}\frac{1}{t^2+4\pi^2}\, dt. \tag{4.36}$$

The function $G_\sigma(\omega)$ is continuous (even analytic) at $\sigma = \pm\pi$, which cannot be seen from the above representations.

We proceed with the integrals $F_m(x,y)$ of (4.5) at $r=1$. Writing

$$\begin{aligned} &2\cos\nu(\theta+\tfrac{1}{2}\pi)\,\cos(2\pi m\nu) = \\ &\qquad \cos\nu(\theta+\tfrac{1}{2}\pi-2\pi m)+\cos\nu(\theta+\tfrac{1}{2}\pi+2\pi m) \end{aligned} \tag{4.37}$$

we obtain

$$\begin{aligned} F_0(x,y)\big|_{r=1} =& -\sin\theta\, e^{-\omega\sin\theta} \\ &+\frac{1}{\pi}\int_0^\infty e^{-\omega\cosh t}\left[\frac{\pi-\sigma}{t^2+(\pi-\sigma)^2}+\frac{\pi+\sigma}{t^2+(\pi+\sigma)^2}\right]\cosh t\,dt, \end{aligned} \tag{4.38}$$

where $\sigma = \theta + \frac{1}{2}\pi$. This result is valid for $\theta < \frac{1}{2}\pi$. In particular, it is not valid at the summit of the circle, the point (0,1). Similar results hold for the other quantities $F_m(x,y)$. When $\theta < \frac{1}{2}\pi$ we have

$$\begin{aligned} &F_m(x,y)\big|_{r=1} = \\ &\quad \sum_{(+,-)}\frac{1}{2\pi}\int_0^\infty e^{-\omega\cosh t}\left[\frac{\pi-\sigma_\pm}{t^2+(\pi-\sigma_\pm)^2}+\frac{\pi+\sigma_\pm}{t^2+(\pi+\sigma_\pm)^2}\right]\cosh t\,dt, \end{aligned} \tag{4.39}$$

where the sum contains two terms with $\sigma_\pm = \theta + \frac{1}{2}\pi \pm 2\pi m$. Because the functions $\Phi(x,y), F(x,y)$ and $F_m(x,y)$ are even functions of x, we can use (4.38) and (4.39) also for negative values of x.

We see that from (3.3), (3.10) and (4.38) that, when we stay away from the point (0,1), $F_0(x,y)$ satisfies the boundary condition along the circle up to an exponentially small term. Also, every $F_m(x,y)$ satisfies equation (3.2). Therefore, we expect that the function $F_0(x,y)$ can be used as an excellent approximation for $F(x,y)$ in the whole unit disc, with exception of a small neighborhood of the point $(0,1)$. Near that point part of the function $F_1(x,y)$ is needed to satisfy the boundary condition (up to exponentially small order).

5 The Watson transformation

Another approach is based on replacing the Fourier series in (3.10) with an integral in the complex plane, where we integrate with respect to complex orders of the Bessel functions.

For example, we can write:

$$F(x,y) = -i\int_{\mathcal{C}} \frac{I'_\nu(\omega)}{I_\nu(\omega)}\, I_\nu(\omega r)\, \frac{\cos\nu(\theta-\frac{1}{2}\pi)}{\sin(\pi\nu)}\, d\nu, \tag{5.1}$$

where $\mathcal{C}$ is a contour around the positive poles of $1/\sin(\pi\nu)$ and through the pole at $\nu = 0$ (which gives half of the residue of this pole); see Figure 5. The

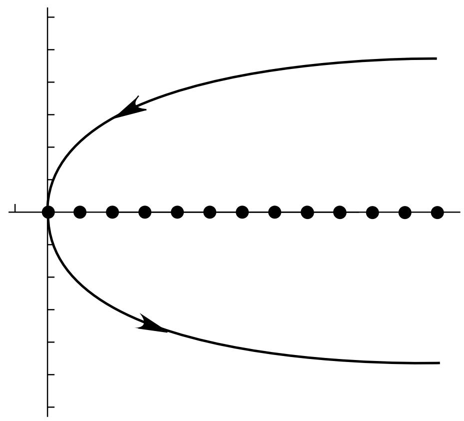

Figure 5: Contour of integration around the poles of $1/\sin(\pi\nu)$.

residues of the poles of $1/\sin(\pi\nu)$ at $\nu = n$ are $(-1)^n$, and $(-1)^n \cos n(\theta - \frac{1}{2}\pi) = \cos n(\theta + \frac{1}{2}\pi)$. This gives the series in (3.10).

For this approach it is needed to know the location of the zeros of the modified Bessel function $I_\nu(\omega)$, and the possibility of using these zeros for obtaining an expansion in the form of a residue series. When we use a contour as shown in Figure 5 we can avoid the complex $\nu-$zeros of $I_\nu(\omega)$, which are located in the half-plane $\Re\nu \le -\frac{3}{2}$ (cf. Lemma 1, Appendix B). Hence, as long as the contour is in the half-plane $\Re\nu \ge 0$, no poles other than those of $1/\sin(\pi\nu)$ can be taken into account.

The convergence of (5.1) for large $\Re\nu$ follows from (A.2). We conclude that (5.1) holds for any value of θ, as long as $\Re\nu \to \infty$ on the upper and lower parts of the contour. When we deform the contour along the imaginary $\nu-$axis, we need to restrict the values of θ to $(0, \pi)$ because (cf. [1, Eq. 6.1.31])

$$\frac{1}{|\Gamma(1 + i\mu)|} = \sqrt{\frac{\sinh \pi\mu}{\pi\mu}}. \tag{5.2}$$

Before discussing asymptotic methods it is of interest to see how the integral gives the boundary value if $r = 1$. From (3.3) it follows that we have to verify the relation

$$-i \int_{\mathcal{C}} I'_\nu(\omega)\, \frac{\cos \nu(\theta - \frac{1}{2}\pi)}{\sin(\pi\nu)}\, d\nu = -\sin\theta\, e^{-\omega \sin\theta}. \tag{5.3}$$

The following steps are valid if $\theta \in (0, \pi)$.

$$
\begin{aligned}
&-i \int_{\mathcal{C}} I_\nu(\omega) \, \frac{\cos\nu(\theta - \frac{1}{2}\pi)}{\sin(\pi\nu)} \, d\nu = \\
&-i \int_{+i\infty}^{-i\infty} I_\nu(\omega) \, \frac{\cos\nu(\theta - \frac{1}{2}\pi)}{\sin(\nu\pi)} \, d\nu = \\
&-i \int_0^\infty [I_{-i\mu}(\omega) - I_{i\mu}(\omega)] \, \frac{\cosh\mu(\theta - \frac{1}{2}\pi)}{\sinh(\pi\mu)} \, d\mu = \\
&\frac{2}{\pi} \int_0^\infty K_{i\mu}(\omega) \cosh\mu(\theta - \tfrac{1}{2}\pi) \, d\mu = e^{-\omega\sin\theta}.
\end{aligned}
\tag{5.4}
$$

The result in (5.3) follows from differentiating with respect to ω. In the final step in (5.4) we have interpreted the integral as a Kontorovich-Lebedev transform, and used a result in [7, Vol. 2, p. 175]. We have also used the well-known relation for the modified Bessel functions:

$$
I_{-\nu}(\omega) = I_\nu(\omega) + \frac{2}{\pi} \sin\pi\nu \, K_\nu(\omega). \tag{5.5}
$$

A different proof can be based by substituting in (5.3) the Sommerfeld contour integral for the Bessel function, see (4.30). This gives

$$
\begin{aligned}
&-i \int_{\mathcal{C}} I_\nu(\omega) \, \frac{\cos\nu(\theta - \frac{1}{2}\pi)}{\sin(\pi\nu)} \, d\nu = \\
&\quad \frac{-i}{2\pi i} \int_{-i\pi+\infty}^{i\pi+\infty} e^{\omega\cosh t} \int_{\mathcal{C}} e^{-\nu t} \, \frac{\cos\nu(\theta - \frac{1}{2}\pi)}{\sin(\pi\nu)} \, d\nu \, dt = \\
&\quad \frac{1}{2\pi i} \int_{-i\pi+\infty}^{i\pi+\infty} e^{\omega\cosh t} \, \frac{\sinh t}{\cosh t - \cos(\theta + \frac{1}{2}\pi)} \, dt,
\end{aligned}
\tag{5.6}
$$

where the $\nu-$integral in the second line is evaluated by using residues and using the series

$$
\sum_{n=0}^{\infty} z^n \, \cos nt = \frac{1 - z\cos t}{1 - 2z\cos t + z^2}, \quad |z| < 1. \tag{5.7}
$$

By using the method for evaluating (4.31) it can easily be shown that the final integral in (5.6) equals $e^{-\omega\sin\theta}$.

A remarkable point is that the derivation in (5.4) is valid only for $\theta \in (0, \pi)$. For the singular perturbation problem this is the domain of interest, in particular if $r \to 1$. This brings us round to investigate (5.1) also if $r < 1$.

As in (5.3) we take $\theta \in (0, \pi)$, and write

$$
F(x, y) = -i \int_{+i\infty}^{-i\infty} \frac{I'_\nu(\omega)}{I_\nu(\omega)} \, I_\nu(\omega r) \, \frac{\cos\nu(\theta - \frac{1}{2}\pi)}{\sin(\pi\nu)} \, d\nu. \tag{5.8}
$$

In (5.4) a crucial step was the introduction of the $K-$function. As a consequence, the dominant term at $\nu = 0$ in the first and second line, that is, the function $I_\nu(\omega)$, has been replaced with an exponentially small term $K_{i\mu}(\omega)$ in the fourth line. As an extra advantage, the troublesome term $\sin\pi\nu$ has been removed.

In the present case we replace in (5.8) the function $\frac{I_\nu'(\omega)}{I_\nu(\omega)}\, I_\nu(\omega r)$ (considered as a function of ν) by its odd part, because the even part does not contribute in the integral. The odd part equals

$$\begin{aligned}\frac{1}{2I_\nu(\omega)I_{-\nu}(\omega)}\left[I_{-\nu}(\omega)I_\nu'(\omega)I_\nu(\omega r)-I_{-\nu}'(\omega)I_\nu(\omega)I_\nu(\omega r)\right]=\\ \frac{\sin\pi\nu}{\pi I_\nu(\omega)I_{-\nu}(\omega)}\left[\omega^{-1}\, I_\nu(\omega r)-I_{-\nu}'(\omega)I_\nu(\omega)K_\nu(\omega r)\right],\end{aligned} \tag{5.9}$$

where we have used the Wronskian (cf. [24, p. 248] or [1, p. 375])

$$I_\nu(\omega)I_{-\nu}'(\omega)-I_\nu'(\omega)I_{-\nu}(\omega)=-\frac{2\sin\pi\nu}{\pi\omega}. \tag{5.10}$$

It follows that (5.8) can be written as

$$F(x,y)=\frac{i}{\pi}\int_{-i\infty}^{+i\infty}\frac{\omega^{-1}\, I_\nu(\omega r)-I_{-\nu}'(\omega)I_\nu(\omega)K_\nu(\omega r)}{I_\nu(\omega)I_{-\nu}(\omega)}\,\cos\nu(\theta-\tfrac12\pi)\,d\nu. \tag{5.11}$$

When $r=1$ the Bessel functions fraction reduces to $-K_\nu'(\omega)$, as follows from another Wronskian:

$$I_{-\nu}(\omega)K_\nu'(\omega)-I_{-\nu}'(\omega)K_\nu(\omega)=-\frac{1}{\omega}. \tag{5.12}$$

Hence, for $r=1$, (5.11) reduces to the boundary value $-\sin\theta e^{-\omega\sin\theta}$; cf. the fourth line in (5.3) and (3.3).

5.1 Asymptotic analysis in and near the boundary layer

We analyze the two parts forming the integral in (5.11). We write

$$F(x,y)=F_B(x,y)+F_I(x,y), \tag{5.13}$$

where

$$\begin{aligned}F_B(x,y)&=\frac{i}{\omega\pi}\int_{-i\infty}^{+i\infty}\frac{I_\nu(\omega r)}{I_\nu(\omega)I_{-\nu}(\omega)}\,\cos\nu(\theta-\tfrac12\pi)\,d\nu\\ &=\frac{-1}{\omega\pi}\int_{-\infty}^{\infty}\frac{I_{i\mu}(\omega r)}{I_{i\mu}(\omega)I_{-i\mu}(\omega)}\,\cosh\mu(\theta-\tfrac12\pi)\,d\mu,\\ F_I(x,y)&=\frac{1}{\pi i}\int_{-i\infty}^{+i\infty}\frac{I_{-\nu}'(\omega)}{I_{-\nu}(\omega)}\,K_\nu(\omega r)\,\cos\nu(\theta-\tfrac12\pi)\,d\nu\\ &=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{I_{-i\mu}'(\omega)}{I_{-i\mu}(\omega)}\,K_{i\mu}(\omega r)\,\cosh\mu(\theta-\tfrac12\pi)\,d\mu.\end{aligned} \tag{5.14}$$

The function $F_I(x,y)$ yields the asymptotic expansion outside the boundary layer, which may be compared with the contributions given by F_0 in (4.24).

The function $F_B(x,y)$ is the boundary layer function: in the interior of the disk it is exponentially small compared with $F_I(x,y)$, except near the upper boundary where it gives the correction to $F_I(x,y)$, in order to produce the correct boundary value. In other words, $F_I(x,y)$ can be compared with the linear term $-1-y$ in the solution (2.3) of example (2.2), and $F_B(x,y)$ plays the part of the other term in (2.3). See also (2.9), where the boundary layer function contains the smoothing factor $\psi(x,y)$.

5.1.1 The contributions from $F_I(x,y)$

In the fourth integral of (5.14) the $K-$Bessel function and the hyperbolic function are even functions of μ. The ratio of $I-$functions is not even; we make it even and write the integral on the interval $[0,\infty)$:

$$F_I(x,y) = \frac{1}{\pi}\int_0^\infty \left[\frac{I'_{i\mu}(\omega)}{I_{i\mu}(\omega)} + \frac{I'_{-i\mu}(\omega)}{I_{-i\mu}(\omega)}\right] K_{i\mu}(\omega r)\ \cosh\mu(\theta-\tfrac12\pi)\,d\mu. \qquad (5.15)$$

On the interval $[\omega r,\infty)$ the $K-$function is strongly oscillating, and the main contributions for large ω come from the interval $[0,\omega r)$. We replace the hyperbolic function by the dominant exponential term. Because of the symmetry $F_I(x,y) = F_I(-x,y)$, we consider $\theta \in (0,\frac12\pi]$, and replace $\cosh\mu(\theta-\frac12\pi)$ by $\frac12 e^{\mu(\frac12\pi-\theta)}$. Next we replace the Bessel functions by asymptotic forms that are valid on $[0,\omega r)$. The best approximations are those based on the Airy functions; see (A.14) and (A.16). However, when r is not close to unity, and θ not close to zero, we can use the Debye type approximations given in (A.6) with ν replaced by $i\mu$ and z by $-i\omega/\mu$ (for the $I-$functions), and z by $-i\omega r/\mu$ (for the $K-$function).

Summarizing, we use,

$$\begin{aligned} &\frac{I'_{i\mu}(\omega)}{I_{i\mu}(\omega)} \sim \frac{\sqrt{\omega^2-\mu^2}}{\omega} - \frac{1}{2\omega}\,\frac{\omega^2}{\omega^2-\mu^2},\\ &K_{i\mu}(\omega r) \sim \sqrt{\frac{\pi}{2\mu}}\,\frac{e^{-\mu\frac12\pi-\mu[\sqrt{z^2-1}-\arctan\sqrt{z^2-1}]}}{(z^2-1)^{\frac14}}\left(1-\frac{u_1(t)}{i\mu}\right), \end{aligned} \qquad (5.16)$$

where $z=\omega r/\mu$, $t = i\mu/\sqrt{\omega^2r^2-\mu^2}$, and $u_1(t)$ is given in (A.7).

For $I'_{-i\mu}(\omega)/I_{-i\mu}(\omega)$ we can use the same estimate. This follows from (5.10), after dividing by $I_{-i\mu}(\omega)/I_{-i\mu}(\omega)$, and observing that then the right-hand side becomes exponentially small on $[0,\omega r)$. We can use these estimates in the following asymptotic analysis because the main contributions to the integral in (5.15) come from a saddle point well inside $[0,\omega r)$.

Using the asymptotic estimates of the Bessel functions given in (5.16) in (5.15) and substituting $\mu=\omega r\cos\beta$, with $0<\beta\le\frac12\pi$, we obtain

$$F_I(x,y) \sim \sqrt{\frac{\omega r}{2\pi}}\int_0^{\frac12\pi} \sqrt{1-r^2\cos^2\beta}\,\sqrt{\sin\beta}\,g(\beta)e^{-\omega r f(\beta)}\,d\beta, \qquad (5.17)$$

where

$$f(\beta) = \sin\beta + (\theta - \beta)\cos\beta, \quad f'(\beta) = -(\theta - \beta)\sin\beta, \tag{5.18}$$

$$g(\beta) = \left(1 - \frac{1}{2\omega(1 - r^2\cos^2\beta)^{3/2}}\right)\left(1 - \frac{u_1(t)}{i\omega r\cos\beta}\right), \quad t = i\cot\beta. \tag{5.19}$$

The saddle point at θ gives the dominant contributions, and $f(\theta) = f''(\theta) = \sin\theta$. As long as β is bounded away from 0, the above estimates for the Bessel functions are valid. We find

$$F_I(x,y) \sim e^{-\omega r\sin\theta}\left[\sqrt{1-x^2} + \frac{-y - \sqrt{1-x^2}}{2\omega(1-x^2)^{\frac{3}{2}}} + \mathcal{O}(\omega^{-2})\right]. \tag{5.20}$$

By using more terms in the Debye expansions we can obtain a complete asymptotic expansion of F_I that holds for large ω, uniformly for θ in a compact set of $(0,\pi)$.

Comparing this result with the estimate of F_0 given in (4.24), we conclude that, although the integrals in (5.17) and (4.8) are not the same, they have the same asymptotic expansion.

5.1.2 The contributions from $F_B(x,y)$

We write F_B of (5.14) in the form

$$F_B(x,y) = \frac{-1}{\omega\pi}\int_{-\infty}^{\infty} \frac{L_{i\mu}(\omega r)}{I_{i\mu}(\omega)I_{-i\mu}(\omega)}\cosh\mu(\theta - \tfrac{1}{2}\pi)\,d\mu, \tag{5.21}$$

where $L_{i\mu}(z)$ is the even part of $I_{i\mu}(z)$ (see (A.12)). We proceed as in the treatment of $F_I(x,y)$. Observe that

$$I_{i\mu}(z)I_{-i\mu}(z) = L_{i\mu}^2(z) + \frac{\sinh^2(\pi\mu)}{\pi^2}K_{i\mu}^2(z), \tag{5.22}$$

and that on $[0,\omega r)$ the $L-$part is dominant compared with the $K-$part, because of the different Airy functions in (A.14) for those functions. We use the first term approximation of the Airy function Bi when $\zeta < 0$ (see (A.20)), and obtain if $0 \le \mu < r\omega$

$$L_{i\mu}(\omega) \sim \frac{e^{\frac{1}{2}\pi\mu + \mu\eta_1}}{\sqrt{2\pi}(\omega^2 - \mu^2)^{\frac{1}{4}}}, \quad L_{i\mu}(\omega r) \sim \frac{e^{\frac{1}{2}\pi\mu + \mu\eta_r}}{\sqrt{2\pi}(\omega^2 r^2 - \mu^2)^{\frac{1}{4}}}, \tag{5.23}$$

where

$$\begin{aligned} \eta_1 &= \sqrt{z_1^2 - 1} - \arctan\sqrt{z_1^2 - 1}, \quad z_1 = \frac{\omega}{\mu}, \\ \eta_r &= \sqrt{z_r^2 - 1} - \arctan\sqrt{z_r^2 - 1}, \quad z_r = \frac{\omega r}{\mu}. \end{aligned} \tag{5.24}$$

This gives, on substituting $\mu = \omega r\cos\beta$,

$$F_B(x,y) \sim -\sqrt{\frac{2\omega r}{\pi}}\int_0^{\frac{1}{2}\pi} \sqrt{1 - r^2\cos^2\beta}\,\sqrt{\sin\beta}\,e^{-\omega g(\beta)}\,d\beta, \tag{5.25}$$

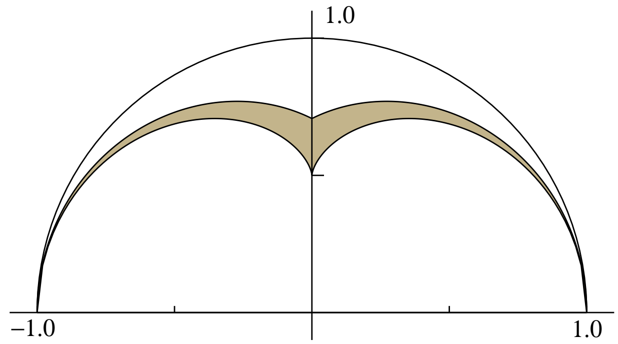

Figure 6: In the domain near the upper part of the boundary, equation in (5.28), that defines the saddle points of the integral in (5.25), has one real saddle point; in the shaded domain two real saddle points occur; in the domain below the shaded domain two complex saddle points occur.

where

$$g(\beta) = 2\sin\alpha + r(\theta+\beta-2\alpha)\cos\beta - r\sin\beta, \quad \cos\alpha = r\cos\beta. \tag{5.26}$$

When $r = 1$ we have $g(\beta) = f(\beta)$, the function of (5.18). The saddle points follow from

$$g'(\beta) = -r\sin\beta\,(\theta+\beta-2\alpha) = 0. \tag{5.27}$$

The relevant saddle point β_0 is the one that follows from the equation

$$2\arccos(r\cos\beta) - \beta = \theta, \quad \beta \in (0, \tfrac{1}{2}\pi). \tag{5.28}$$

This equation has one real solution $\beta_0 \in (0, \frac{1}{2}\pi)$ for values of (x, y) in the domain along the upper part of the boundary of the unit disc; see Figure 6. In the shaded part of the disc there are two real saddle points, in the lower part there are no real saddle points. The curves along the upper part of the shaded domain are defined by $r = \cos\frac{1}{2}\theta, 0 \le \theta \le \frac{1}{2}\pi$, with a symmetric part for $x < 0$. This easily follows from drawing the curves of $r\cos\beta$ and $\cos\frac{1}{2}(\theta+\beta)$. The curves along the lower part follow from the equation

$$\tan\theta = \frac{\sqrt{1-r^2}(1+8r^2)}{(4r^2-1)^{\frac{3}{2}}}, \quad \tfrac{1}{2} \le r \le 1, \quad 0 \le \theta \le \tfrac{1}{2}\pi, \tag{5.29}$$

with a symmetric part for $x < 0$. This equation follows from putting the derivative of $2\arccos(r\cos\beta)$ equal to unity, which gives $r\cos\beta = \sqrt{\frac{1}{3}(4r^2-1)}$, and using this relation in (5.28).

We concentrate on the domain along the boundary. In that case a unique real saddle point $\beta_0 \in (0, \frac{1}{2}\pi)$ satisfies (5.28). We have

$$
\begin{aligned}
g(\beta_0) &= 2\sin\alpha - r\sin\beta_0 = 2\sqrt{1-r^2\cos^2\beta_0} - r\sin\beta_0, \\
g''(\beta_0) &= \frac{2r^2\sin^2\beta_0}{\sin\alpha} - r\sin\beta_0 = \frac{2r^2\sin^2\beta_0}{\sqrt{1-r^2\cos^2\beta_0}} - r\sin\beta_0.
\end{aligned} \tag{5.30}
$$

It follows that a first order saddle point approximation reads

$$
F_B(x,y) \sim -2\sqrt{\frac{r\sin\beta_0(1-r^2\cos^2\beta_0)}{g''(\beta_0)}}\, e^{-\omega g(\beta_0)}. \tag{5.31}
$$

This is the requested boundary layer term; it compensates the wrong behavior of $F_I(x,y)$ at the upper boundary. Namely, it easily follows that, if $r = 1, y > 0$:

$$
F_B(x,y) \sim -2ye^{-\omega y}. \tag{5.32}
$$

Hence, if $r = 1, y > 0$, by (3.1) and (5.13),

$$
\begin{aligned}
\Phi(x,y) &= -y - e^{\omega y}F(x,y) = -y - e^{\omega y}\left[F_B(x,y) + F_I(x,y)\right] \\
&\sim -y - e^{\omega y}\left[-2ye^{-\omega y} + ye^{-\omega y}\right] = 0.
\end{aligned} \tag{5.33}
$$

We conclude that the splitting of the function F into F_B and F_I yields the asymptotic behavior of $\Phi(x,y)$ inside the upper half of the unit disc (including the upper boundary), with exception of small neighborhoods of the points $(\pm 1, 0)$. The uniform approximations can be given in terms of elementary functions. We expect that near the points $(\pm 1, 0)$ integrals containing ratios of Airy functions are needed for the local approximations.

A Asymptotic expansions of modified Bessel functions

We summarize a few properties and asymptotic expansions of the modified Bessel functions. These can be found in [1, Ch. 9] or in other mentioned references.

From the expansion

$$
I_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!\,\Gamma(\nu+k+1)} \tag{A.1}
$$

it follows that

$$
I_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\left[1 + \mathcal{O}(\nu^{-1})\right], \quad \nu \to \infty, \tag{A.2}
$$

with z fixed. Asymptotic expansions for large arguments are

$$
\begin{aligned}
I_\nu(z) &\sim \frac{e^z}{\sqrt{2\pi z}}\left[1-\frac{\alpha_1}{z}+\frac{\alpha_2}{z^2}-\frac{\alpha_3}{z^3}+\ldots\right],\\
K_\nu(z) &\sim \sqrt{\frac{\pi}{2z}}e^{-z}\left[1+\frac{\alpha_1}{z}+\frac{\alpha_2}{z^2}+\frac{\alpha_3}{z^3}+\ldots\right],\\
I_\nu'(z) &\sim \frac{e^z}{\sqrt{2\pi z}}\left[1-\frac{\beta_1}{z}+\frac{\beta_2}{z^2}-\frac{\beta_3}{z^3}+\ldots\right],\\
K_\nu'(z) &\sim -\sqrt{\frac{\pi}{2z}}e^{-z}\left[1+\frac{\beta_1}{z}+\frac{\beta_2}{z^2}+\frac{\beta_3}{z^3}+\ldots\right],
\end{aligned}
\tag{A.3}
$$

where

$$
\begin{aligned}
\alpha_1 &= \frac{\mu-1}{8}, \quad \alpha_2 = \frac{(\mu-1)(\mu-9)}{2!\,8^2}, \quad \alpha_3 = \frac{(\mu-1)(\mu-9)(\mu-25)}{3!\,8^3},\\
\beta_1 &= \frac{\mu+3}{8}, \quad \beta_2 = \frac{(\mu-1)(\mu+15)}{2!\,8^2}, \quad \beta_3 = \frac{(\mu-1)(\mu-9)(\mu+35)}{3!\,8^3},
\end{aligned}
\tag{A.4}
$$

and $\mu = 4\nu^2$. The expansions for the $K-$functions hold for $|\mathrm{ph}\, z| < \frac{3}{2}\pi$, those for the $I-$functions for $|\mathrm{ph}\, z| < \frac{1}{2}\pi$.

A.1 Debye-type expansions

Let

$$
t = 1/\sqrt{1+z^2}, \quad \eta = \sqrt{1+z^2} + \ln\frac{z}{1+\sqrt{1+z^2}}. \tag{A.5}
$$

Then

$$
\begin{aligned}
I_\nu(\nu z) &\sim \frac{1}{\sqrt{2\pi\nu}}\frac{e^{\nu\eta}}{(1+z^2)^{\frac14}}\left[1+\frac{u_1(t)}{\nu}+\frac{u_2(t)}{\nu^2}+\ldots\right],\\
K_\nu(\nu z) &\sim \sqrt{\frac{\pi}{2\nu}}\frac{e^{-\nu\eta}}{(1+z^2)^{\frac14}}\left[1-\frac{u_1(t)}{\nu}+\frac{u_2(t)}{\nu^2}+\ldots\right],\\
I_\nu'(\nu z) &\sim \frac{1}{\sqrt{2\pi\nu}}\frac{(1+z^2)^{\frac14}}{z}\,e^{\nu\eta}\left[1+\frac{v_1(t)}{\nu}+\frac{v_2(t)}{\nu^2}+\ldots\right],\\
K_\nu'(\nu z) &\sim -\sqrt{\frac{\pi}{2\nu}}\frac{(1+z^2)^{\frac14}}{z}\,e^{-\nu\eta}\left[1-\frac{v_1(t)}{\nu}+\frac{v_2(t)}{\nu^2}+\ldots\right],
\end{aligned}
\tag{A.6}
$$

where

$$
u_1 = \tfrac{1}{24}t(3-5t^2), \quad v_1 = \tfrac{1}{24}t(-9+7t^2). \tag{A.7}
$$

The higher coefficients follow, for $k = 2, 3, \ldots$, from

$$
\begin{aligned}
u_k(t) &= \tfrac12\, t^2\,(1-t^2)\,u_{k-1}'(t) + \tfrac18\int_0^t (1-5\tau^2)\,u_{k-1}(\tau)\,d\tau,\\
v_k(t) &= u_k(t) + t(t^2-1)\left[\tfrac12 u_{k-1}(t) + t\,u_{k-1}'(t)\right].
\end{aligned}
\tag{A.8}
$$

The expansions in (A.6) hold as $\nu \to +\infty$, uniformly with respect to z in the sector $|\mathrm{ph} z| \le \frac12\pi - \delta$, where δ is a small positive number. They have a double asymptotic property: they hold when one of the parameters z, ν tends to infinity, uniformly with respect to the other parameter.

A.2 Airy-type expansions

We give Airy-type asymptotic expansions of the modified Bessel functions and concentrate on functions with purely imaginary order. We summarize the results of [5].

Let ζ be defined by

$$\begin{aligned} \tfrac{2}{3}\zeta^{\frac{3}{2}} &= \operatorname{arctanh}\sqrt{1-z^2} - \sqrt{1-z^2}, \quad 0 < z \leq 1, \\ \tfrac{2}{3}(-\zeta)^{\frac{3}{2}} &= \sqrt{z^2-1} - \arctan\sqrt{z^2-1}, \quad z \geq 1. \end{aligned} \tag{A.9}$$

By expanding the arctanh-function:

$$\tfrac{2}{3}\zeta^{\frac{3}{2}} = \tfrac{1}{3}(1-z^2)^{\frac{3}{2}} + \tfrac{1}{5}(1-z^2)^{\frac{5}{2}} + \ldots, \tag{A.10}$$

which gives

$$\zeta = 2^{\frac{1}{3}}(1-z)\left[1 + \mathcal{O}(1-z)\right], \quad z \to 1. \tag{A.11}$$

This defines the relation near $z = 1$. For complex values of z this relation should be used with analytic continuation to define which branch of the multi-valued function $\zeta^{\frac{3}{2}}$ is used.

The function $K_{i\nu}(z)$ is real for real values of ν and $z, z > 0$. The function $I_{i\nu}(z)$ is complex in that case. Therefore it is convenient to introduce the function

$$L_{i\nu}(z) = \tfrac{1}{2}\left[I_{-i\nu}(z) + I_{i\nu}(z)\right]. \tag{A.12}$$

This function is real for real values of ν and $z, z > 0$ and the definition may be compared with the relation

$$K_{i\nu}(z) = \frac{\pi}{2i\sinh(\pi\nu)}\left[I_{-i\nu}(z) - I_{i\nu}(z)\right], \tag{A.13}$$

which is defined at $\nu = 0$, with limit $K_0(z)$.

We have the following representations

$$\begin{aligned} K_{i\nu}(\nu z) &= \pi\,\nu^{-\frac{1}{3}}\,e^{-\nu\frac{1}{2}\pi}\left(\frac{4\zeta}{1-z^2}\right)^{\frac{1}{4}} \times \\ &\quad \left[\mathrm{Ai}(-\nu^{\frac{2}{3}}\zeta)\,F_\nu(\zeta) + \nu^{-\frac{4}{3}}\mathrm{Ai}'(-\nu^{\frac{2}{3}}\zeta)\,G_\nu(\zeta)\right], \\ L_{i\nu}(\nu z) &= \tfrac{1}{2}\nu^{-\frac{1}{3}}\,e^{\nu\frac{1}{2}\pi}\left(\frac{4\zeta}{1-z^2}\right)^{\frac{1}{4}} \times \\ &\quad \left[\mathrm{Bi}(-\nu^{\frac{2}{3}}\zeta)\,F_\nu(\zeta) + \nu^{-\frac{4}{3}}\mathrm{Bi}'(-\nu^{\frac{2}{3}}\zeta)\,G_\nu(\zeta)\right], \end{aligned} \tag{A.14}$$

where the functions $F_\nu(\zeta), G_\nu(\zeta)$ can be expanded:

$$F_\nu(\zeta) \sim \sum_{s=0}^{\infty}\frac{A_s(-\zeta)}{\nu^{2s}}, \quad G_\nu(\zeta) \sim \sum_{s=0}^{\infty}\frac{B_s(-\zeta)}{\nu^{2s}}, \tag{A.15}$$

as $\nu \to \infty$, $A_0(\zeta) = 1$. If $\nu > 0$ the expansions hold uniformly with respect to z in the sector $|\mathrm{ph} z| \le \pi - \delta$, where δ is a small positive number. $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are the well-known Airy functions; see [1, Ch. 10]. We also have

$$I_{i\nu}(\nu z) = \frac{e^{\nu \frac{1}{2}\pi}}{2\,\nu^{\frac{1}{3}}} \left(\frac{4\zeta}{1-z^2}\right)^{\frac{1}{4}} \times \\ \left[\widehat{\mathrm{Bi}}(-\nu^{\frac{2}{3}}\zeta)\, F_\nu(\zeta) + \nu^{-\frac{4}{3}} \widehat{\mathrm{Bi}}'(-\nu^{\frac{2}{3}}\zeta)\, G_\nu(\zeta)\right], \tag{A.16}$$

where $\widehat{\mathrm{Bi}}(z)$ is a linear combination of two Airy functions:

$$\widehat{\mathrm{Bi}}(z) = \mathrm{Bi}(z) - i \tanh(\pi\nu)\, \mathrm{Ai}(z). \tag{A.17}$$

The coefficients $A_s(\zeta)$ and $B_s(\zeta)$ in (A.15) are analytic functions in a large domain of the $\zeta-$ plane and are the same as those used in the Airy-type expansions of ordinary Bessel functions (cf. [18, p. 421]). In [5] the expansions like (A.15) are supplied with remainders, and bounds on the remainders can be obtained from [18, p. 418]. [1]

When in (A.14) and (A.16) the arguments of the Airy functions are large, these functions can be replaced by their asymptotic expansion, and the expansions are in terms of elementary functions. For example, when $z > 1$, ζ is negative (see (A.9)), and the arguments of the Airy functions in (A.14) are positive. By using

$$\mathrm{Ai}(z) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{\frac{3}{2}}}, \quad \mathrm{Bi}(z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{\frac{2}{3} z^{\frac{3}{2}}}, \tag{A.18}$$

as $z \to \infty$, we obtain for large μ and $z > 1$

$$K_{i\mu}(\mu z) \sim \sqrt{\frac{\pi}{2\mu}}\, \frac{e^{-\frac{1}{2}\pi\mu - \mu[\sqrt{z^2-1} - \arctan\sqrt{z^2-1}]}}{(z^2-1)^{\frac{1}{4}}}, \tag{A.19}$$

and

$$L_{i\mu}(\mu z) \sim \frac{1}{\sqrt{2\pi\mu}}\, \frac{e^{\frac{1}{2}\pi\mu + \mu[\sqrt{z^2-1} - \arctan\sqrt{z^2-1}]}}{(z^2-1)^{\frac{1}{4}}}. \tag{A.20}$$

These results follow also formally form the Debye expansion of $K_\nu(\nu z)$ and $I_\nu(\nu z)$ given in (A.6) by replacing $\nu \to i\mu$ and $z \to -iz$.

B On the zeros of $K_{i\nu}(x)$ and $I_{i\nu}(x)$ with respect to ν

The function $K_{i\nu}(x)$ is an even function of ν. If $x > 0$ it has an infinite number of simple real $\nu-$zeros and no complex zeros (cf., for instance, [4]). There are

[1] In Dunster's formula (4.6) there seems to be an error, and in (4.7), (4.14), and (4.14) he should have used the coefficients $A_s(-\zeta), B_s(-\zeta)$ instead of $(-1)^s A_s(\zeta), (-1)^s B_s(\zeta)$. Also, in (4.15) the arguments of the Airy functions should read $-\nu^{\frac{2}{3}}\zeta$.

two infinite strings of zeros inside the intervals $(-\infty,-x]$ and $[x,\infty)$. For large values of x the zeros can be obtained by using the asymptotic representation in (A.14). The Airy function $\mathrm{Ai}(x)$ has real negative zeros, and consequently $K_{i\nu}(\nu z)$ has zeros if $\zeta>0$. From (A.9) we see that z should satisfy $0<z<1$, that is, to have zeros the order of $K_{i\nu}(x)$ should be larger (in absolute value) than the argument x.

The location of the zeros of the function $I_\nu(x)$ is more complicated. First we mention the following result from [3].

Lemma 1 *The function $I_\nu(x)$, with $x>0$, cannot vanish if $\Re\nu>-\frac{3}{2}$.*

Proof. The proof is based on the following representation (cf. [27, p. 150])

$$I_\mu(x)\,I_\nu(x)=\tfrac{2}{\pi}\int_0^{\frac{1}{2}\pi} I_{\mu+\nu}(2x\cos\theta)\,\cos[(\mu-\nu)\theta]\,d\theta, \tag{B.1}$$

which holds if $\Re(\mu+\nu)>-1$. Let $\nu=\nu_1+i\nu_2$ be given with $\nu_2\neq 0$; we try to find $\mu=\mu_1+i\mu_2$ such that the following conditions hold:

- $\mu+\nu$ is real and larger than -1, which gives $\mu_2=-\nu_2, \mu_1+\nu_1>-1$; from the series expansion of the Bessel function it follows that the function $I_{\mu+\nu}(2x\cos\theta)$ is positive.
- The imaginary part of $\cos(\mu-\nu)\theta$, which is $\sin(\nu_1-\mu_1)\theta\sinh 2\mu_2\theta$, should have a fixed sign on the interval of integration (if $\mu_1=\nu_1$ this part is zero, but then the real part is positive). When $|\mu_1-\nu_1|<2$ this condition is satisfied.

Combining the two conditions we conclude that $I_\nu(z)$ cannot vanish if $\nu_1>-\frac{3}{2}$. ■

When x is large the location of the zeros of $I_\nu(\nu x)$ can be derived from (A.16): we need the zeros of the Airy function $\widehat{\mathrm{Bi}}(z)$ defined in (A.17). We can eliminate the functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ by using

$$\begin{aligned}\mathrm{Ai}(z)&=-e^{\frac{2}{3}\pi i}\mathrm{Ai}\left(ze^{\frac{2}{3}\pi i}\right)-e^{-\frac{2}{3}\pi i}\mathrm{Ai}\left(ze^{-\frac{2}{3}\pi i}\right),\\ \mathrm{Bi}(z)&=e^{\frac{1}{6}\pi i}\mathrm{Ai}\left(ze^{\frac{2}{3}\pi i}\right)+e^{-\frac{1}{6}\pi i}\mathrm{Ai}\left(ze^{-\frac{2}{3}\pi i}\right),\end{aligned} \tag{B.2}$$

(cf. [1, p. 446]). The result is

$$\begin{aligned}\widehat{\mathrm{Bi}}(z)=e^{-\frac{1}{6}\pi i}&[1+\tanh(\pi\nu)]\ \times\\ &\mathrm{Ai}\left(ze^{-\frac{2}{3}\pi i}\right)+e^{\frac{1}{6}\pi i}[1-\tanh(\pi\nu)]\,\mathrm{Ai}\left(ze^{\frac{2}{3}\pi i}\right),\end{aligned} \tag{B.3}$$

from which follows that, if ν is large, the zeros of $\widehat{\mathrm{Bi}}(z)$ are approximately given by those of the Airy function $\mathrm{Ai}\left(ze^{-\frac{2}{3}\pi i}\right)$, which lie on the half-line with

$\mathrm{ph}\, z = -\frac{1}{3}\pi$. From (A.16) and (B.3) we infer that the zeros of $I_{i\nu}(\nu z)$ can be derived from those of $\mathrm{Ai}(-\nu^{\frac{2}{3}}\zeta e^{-\frac{2}{3}\pi i})$.

Before giving more details on the $\nu-$zeros of $I_\nu(z)$ we introduce a different notation in (A.16), because the order ν occurs also in the argument. Let

$$z = \frac{\omega}{\nu}, \quad \nu = \mu\omega. \tag{B.4}$$

Then, the first line of (A.9) becomes

$$\tfrac{2}{3}\zeta^{\frac{3}{2}} = \mathrm{arctanh}\sqrt{1-1/\mu^2} - \sqrt{1-1/\mu^2}. \tag{B.5}$$

A first approximation of the $\mu-$zeros of $I_{i\mu\omega}(\omega)$ if ω is large can be obtained as follows. When we denote the negative zeros of $\mathrm{Ai}(z)$ by a_s, $s = 1, 2, \ldots$, then the zeros μ_s of the dominant Airy function in (A.16), that is, of $\widehat{\mathrm{Bi}}(-\nu^{\frac{2}{3}}\zeta)$ with $\nu = \omega\mu$, are approximately given by

$$\mu_s^{\frac{2}{3}} e^{-\frac{2}{3}\pi i}\zeta_s \sim -a_s\omega^{-\frac{2}{3}}, \tag{B.6}$$

where ζ_s is given in (B.5) with $\mu = \mu_s$. Using the relation in (A.11), that is,

$$\zeta = 2^{\frac{1}{3}}(\mu-1)\left[1+\mathcal{O}(\mu-1)\right], \quad \mu\to 1, \tag{B.7}$$

we obtain for the early zeros (small values of s) of $I_{i\mu\omega}(\omega)$ the estimate

$$\mu_s = 1 - a_s\, 2^{-\frac{1}{3}}\,\omega^{-\frac{2}{3}}\, e^{\frac{2}{3}\pi i} + \mathcal{O}\left(\omega^{-\frac{4}{3}}\right), \quad \omega\to\infty. \tag{B.8}$$

When we consider the function $I_{i\nu}(\omega)$, we see that for large values of ω the $\nu-$zeros near ω have the expansion

$$\nu_s = \omega - a_s\left(\tfrac{1}{2}\omega\right)^{\frac{1}{3}}\, e^{\frac{2}{3}\pi i} + \mathcal{O}\left(\omega^{-\frac{1}{3}}\right), \quad \omega\to\infty, \tag{B.9}$$

and that $I_\nu(\omega)$ has $\nu-$zeros near $i\omega$ with the expansion

$$\nu_s = i\omega - a_s\left(\tfrac{1}{2}\omega\right)^{\frac{1}{3}}\, e^{\frac{7}{6}\pi i} + \mathcal{O}\left(\omega^{-\frac{1}{3}}\right), \quad \omega\to\infty, \tag{B.10}$$

with conjugate values in the lower half plane.

In Figure 7 the first 25 zeros of the function $\mathrm{Ai}(-\nu^{\frac{2}{3}}\zeta e^{-\frac{2}{3}\pi i})$ are given. This Airy function is used via (B.3) in (A.16), with ζ defined in (A.9). The first zeros correspond with the $\nu-$zeros of $I_{i\nu}(\omega)$, and an approximation of these zeros is given in(B.9).

The expansions in (B.9) and (B.10) agree with the analysis in [2], where the $\nu-$zeros of the Hankel function $H_\nu^{(1)}(w)$ are investigated. However, with the information in (B.9), (B.10) and Figure 7, the description of the zeros of the modified Bessel function $I_{i\nu}(\omega)$ is not complete. To see this, we observe that the hyperbolic functions in (B.3) may have poles when ν becomes an imaginary

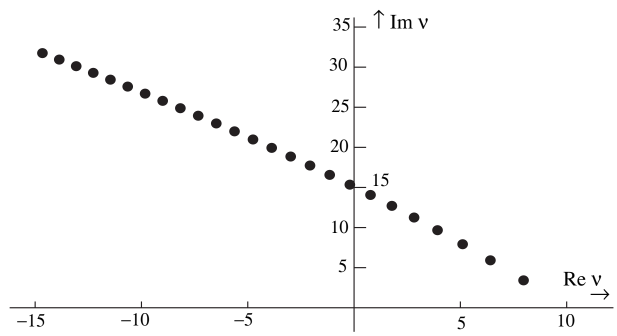

Figure 7: First 25 zeros of the Airy function $\mathrm{Ai}(-\nu^{\frac{2}{3}}\zeta e^{-\frac{2}{3}\pi i})$ corresponding with some $\nu-$zeros of $I_{i\nu}(\omega)$, of which an approximation is given in (B.9).

number, and the first term in (B.3) is no longer the dominant one. For describing the zero distribution also the role of the hyperbolic functions should be taken into account. The zeros of $I_{i\nu}(\omega)$ near the positive imaginary $\nu-$axis correspond with zeros of $I_\nu(\omega)$ near the negative $\nu-$axis.

It is better to describe these zeros near and on the negative $\nu-$axis by using the relation

$$I_{-\nu}(\omega) = I_\nu(\omega) + \frac{2}{\pi}\sin\pi\nu\, K_\nu(\omega). \tag{B.11}$$

By using (cf. [1], page 375)

$$I_\nu(\omega) \sim \left(\tfrac{1}{2}\omega\right)^\nu/\Gamma(\nu+1), \quad K_\nu(\omega) \sim \tfrac{1}{2}\Gamma(\nu)\left(\tfrac{1}{2}\omega\right)^{-\nu}, \quad \nu\to\infty, \tag{B.12}$$

with ω fixed, we see that large $\nu-$zeros occur near the large positive zeros of $\sin\pi\nu$.

A better description follows by using the Debye-type expansions of (A.6). It follows that approximations of the $\mu-$zeros of $I_{\mu\omega}(\omega)$ can be obtained from the equation

$$e^{2\mu\omega\eta} = -2\sin\mu\omega\pi, \tag{B.13}$$

where

$$\eta = \sqrt{1+1/\mu^2} - \ln\left(\mu + \sqrt{\mu^2+1}\right). \tag{B.14}$$

When we write $\mu = \sinh\kappa$, we see that $\mu\eta = \cosh\kappa - \kappa\sinh\kappa$, and it is not difficult to compute the zeros from the above analysis. It is easily verified that real negative zeros of $I_\nu(\omega)$ occur if $\nu < -\omega\sinh\kappa_0$, where $\kappa_0 = 1.19968\ldots$ is the positive solution of the equation $\eta = 0$, that is of $\cosh\kappa - \kappa\sinh\kappa = 0$. In Figure 8 we show the $\nu-$zeros of $I_\nu(\omega)$ for $\omega = 10$. We see that left from -15.0 real negative zeros occur (observe that $\omega\sinh\kappa_0 \doteq 15.0888$ if $\omega = 10$).

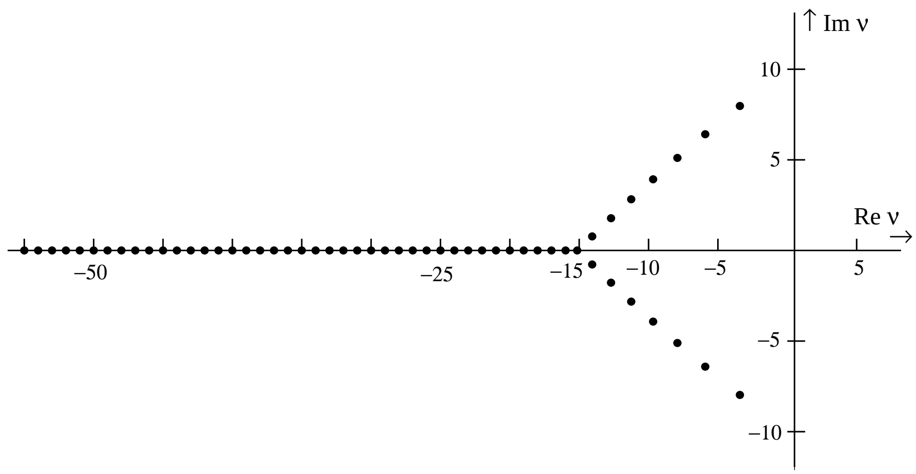

Figure 8: The $\nu-$zeros of $I_{i\nu}(\omega)$ for $\omega = 10$. Left from $\nu \doteq -15.0888\omega$ real negative zeros occur.

Epilogue

My thesis supervisor Hans Lauwerier suggested me to study this type of problems when the topic of matched asymptotic expansions for singular perturbation methods became very popular in my country, with as main actors Wiktor Eckhaus, Eduard de Jager, and Johan Grasman. Lauwerier liked simple non-trivial model problems which could be studied by using the explicit solutions in the form of integrals and series, and by exploiting the role of special functions and the well-known techniques of asymptotic analysis, such as saddle point methods.

Our aim was to obtain new insight in certain phenomena of singular perturbation problems by studying these model problems. My paper [23] provides a few results that, perhaps, cannot be obtained by using matched asymptotic analysis.

In my younger days the results were not suitable as part of my thesis. Later I studied this circle problem time and again, and came back to it quite often, and I was not earlier satisfied with the results. Quite recently a few steps gave the desired breakthrough, although the treatment of the asymptotics near the points $(\pm 1, 0)$ is still missing in the present paper. This can be done by replacing the modified Bessel functions in (5.14) by their approximations in terms of Airy functions, which I don't like because it gives a rather messy result.

This paper summarizes, somehow, a lifetime activity in asymptotic analysis and special functions. I appreciate that it can be included in the proceedings of the Santander conference.

Acknowledgments

The author wishes to thank the referees for many comments and suggestions which have resulted in an improved version of the paper. The author acknowledges financial support from the Spanish *Ministry of Education and Science* (Project MTM2004–01367).

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