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High-Order Accurate Decomposition of Richardson's Method for a Singularly Perturbed Elliptic Reaction–Diffusion Equation

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Abstract—The Dirichlet boundary value problem for a singularly perturbed elliptic reaction–diffusion equation is considered in a strip. For this problem, special difference schemes are available that converge ε-uniformly with up to the second order of accuracy. Special schemes on piecewise uniform meshes and Richardson's technique are used to construct a scheme whose solutions converge ε-uniformly with the third order of accuracy (up to a logarithmic factor) and with the fourth order of accuracy with respect to the orthogonal and tangential (to the boundary) variables, respectively. For Richardson's scheme, a decomposition scheme (with domain decomposition into overlapping subdomains) is proposed, which preservers the ε-uniform accuracy of the former.

1. INTRODUCTION

At present, special numerical methods have been constructed and examined for sufficiently large classes of singularly perturbed boundary value problems. These methods, in contrast to those developed for regular boundary value problems (e.g., see [1, 2]), give solutions that converge uniformly in the perturbation parameter ε (in other words, ε-uniformly convergent solutions). The order of ε-uniform convergence of well-known regular methods on special condensing meshes does not exceed two in the case of boundary value problems for elliptic reaction–diffusion equations and does not exceed unity for convection–diffusion equations even if the data in the problem are smooth (e.g., see [3–6] and the bibliography therein; see [7, 8] for fitted operator methods on uniform meshes). This circumstance motivates interest in designing special schemes for reaction–diffusion and convection–diffusion problems that converge ε-uniformly with orders higher than the second and first, respectively.

In the case of regular boundary value problems, the defect-correction and Richardson methods are used to increase the order of accuracy of approximate solutions (e.g., see [1, 9, 10] and the bibliography therein). These methods are also used to improve the time-accuracy of solutions in singularly perturbed problems (see, e.g., [11–13]). Note that a uniform time mesh was used in these studies, which greatly simplifies the construction and analysis of schemes of high-order accuracy in time. For large classes of singularly perturbed problems, the use of meshes condensing in a boundary layer (with respect to a spatial variable in the direction across the boundary layer) is a necessary requirement for constructing ε-uniformly convergent schemes (e.g., see [4, 14]). Thus, it is of interest to develop schemes of high-order ε-uniform accuracy with respect to variables over which the mesh size varies sharply.

In this paper, we consider a boundary value problem for a singularly perturbed elliptic reaction–diffusion equation in a vertical strip. By using Richardson's method, we construct a special scheme that converges ε-uniformly with an order of accuracy $\mathcal{O}(N_1^{-3}\ln^3 N_1 + N_2^{-4})$, where N_1 determines the number of mesh points across the strip (across the boundary layer) and N_2 determines the number of mesh points on a unit segment along the strip. The scheme is constructed by using piecewise uniform meshes condensing in the neighborhood of the boundary layer. For Richardson's scheme, we construct a decomposition scheme (with domain decomposition into overlapping subdomains) that preservers the ε-uniform accuracy of Richardson's scheme. The technique developed can be used to construct parallel Richardson schemes, i.e., high-order accurate schemes intended for parallel computations.

Note that Richardson's method was used in [15] to develop a technique for improving the accuracy of solutions to elliptic convection–diffusion equations on a strip in the construction of schemes that converge ε-uniformly at a rate of $\mathbb{O}(N_1^{-2}\ln^3 N_1 + N_2^{-2})$.

2. STATEMENT OF THE PROBLEM

In a vertical strip $\overline{D}$, where

$$D = \{x : x_1 \in (0, d), x_2 \in \mathbb{R}\}, \tag{2.1}$$

we consider the following boundary value problem for a singularly perturbed elliptic equation:

$$Lu(x) \equiv \left\{\varepsilon^2 \sum_{s=1,2} a_s(x)\frac{\partial^2}{\partial x_s^2} - c(x)\right\}u(x) = f(x), \quad x \in D, \qquad u(x) = \varphi(x), \quad x \in \Gamma. \tag{2.2}$$

Here, $\Gamma = \overline{D}\setminus D$; the functions $a_s(x)$, $c(x)$, $f(x)$, and $\varphi(x)$ are assumed to be sufficiently smooth on $\overline{D}$ and Γ, respectively, and to satisfy

$$a_0 \le a_s(x) \le a^0, \quad s = 1, 2, \quad c_0 \le c(x) \le c^0, \quad a_0, c_0 > 0, \qquad |f(x)| \le M, \quad x \in \overline{D}, \quad |\varphi(x)| \le M, \quad x \in \Gamma, \tag{2.3}$$

and ε is a parameter taking arbitrary values in the half-open interval (0, 1].

Here and below, M and M_i (m) are used to denote different sufficiently large (small) positive constants independent of ε or the parameters of difference schemes. The notation $L_{(j.k)}(M_{(j.k)}, G_{h(j.k)})$ means that these operators (constants, meshes) were introduced in formula ($j.k$).

As ε tends to zero, a regular boundary layer arises in the neighborhood of Γ.

For boundary value problem (2.2), (2.1), it is necessary to construct a difference scheme that converges ε-uniformly with an order of accuracy higher than two in each variable.

3. BASIC SCHEME FOR PROBLEM (2.2), (2.1)

We describe a classical difference scheme and a special (basic) scheme that converges ε-uniformly at a rate of $O(N_1^{-2}\ln^2 N_1 + N_2^{-2})$, i.e., a scheme that is second-order accurate in x_2.

On $\overline{D}$, we introduce a rectangular grid

$$\overline{D}_h = \overline{\omega}_1 \times \omega_2, \tag{3.1}$$

where $\overline{\omega}_1$ and ω_2 are arbitrary, generally nonuniform meshes on [0, d] and the x_2-axis, respectively. We set $h_s^i = x_s^{i+1} - x_s^i$, where $x_s^i, x_s^{i+1} \in \overline{\omega}_1$ for $s = 1$ and $x_s^i, x_s^{i+1} \in \omega_2$ for $s = 2$. Let $h_s = \max_i h_s^i$ and $h = \max_s h$. It is assumed that $h \le MN^{-1}$, where $N = \min[N_1, N_2]$, $N_1 + 1$ is the number of mesh points in $\overline{\omega}_1$, and $N_2 + 1$ is the minimum number of mesh points on a unit segment in ω_2.

Problem (2.2), (2.1) is approximated by the difference scheme

$$\Lambda z(x) \equiv \left\{\varepsilon^2 \sum_{s=1,2} a_s(x)\delta_{\overline{xs}\widehat{xs}} - c(x)\right\}z(x) = f(x), \quad x \in D_h, \qquad z(x) = \varphi(x), \quad x \in \Gamma_h. \tag{3.2}$$

Here, $D_k = D \cap \overline{D}_h$, $\Gamma_h = \Gamma \cap \overline{D}_h$, and $\delta_{\overline{xs}\widehat{xs}}z(x) = z_{\overline{xs}\widehat{xs}}(x)$ is the second (central) difference derivative on a nonuniform mesh [2]; for example,

$$\delta_{\overline{x1}\widehat{x1}}z(x) = 2(h_1^i + h_1^{i-1})^{-1}[\delta_{x1}z(x) - \delta_{\overline{x1}}z(x)], \quad x = (x_1^i, x_2) \in D_h.$$

Scheme (3.2), (3.1) is ε-uniformly monotone (see [2]).

The solutions of the difference scheme satisfy the estimate

$$|u(x) - z(x)| \le M[(\varepsilon + N_1^{-1})^{-1} N_1^{-1} + N_2^{-1}], \quad x \in \overline{D}_h. \tag{3.3}$$

For meshes

$$\overline{D}_h^u = \overline{\omega}_1 \times \omega_2, \tag{3.4}$$

that are uniform in both variables (with mesh sizes $h_1 = dN_1^{-1}$ and $h_2 = N_2^{-1}$), we have

$$|u(x) - z(x)| \le M[(\varepsilon + N_1^{-1})^{-2} N_1^{-2} + N_2^{-2}], \quad x \in \overline{D}_h^u. \tag{3.5}$$

Let us construct a piecewise uniform mesh on which scheme (3.2) converges ε-uniformly [4, 5]. On $\overline{D}$, we introduce a special mesh condensing in the neighborhood of the boundary layer:

$$\overline{D}_h = \overline{\omega}_1^* \times \omega_2, \tag{3.6a}$$

where $\omega_2 = \omega_{2(3.4)}$ and $\overline{\omega}_1^*$ is a piecewise uniform mesh constructed as follows. Let σ be a mesh parameter depending on ε and N_1 such that $\sigma \le 4^{-1}d$. The interval $[0, d]$ is partitioned into three subintervals: $[0, \sigma]$, $[\sigma, d - \sigma]$, and $[d - \sigma, d]$. On each subinterval the mesh size is constant and is equal to $h_1^{(1)} = 4\sigma N_1^{-1}$ on $[0, \sigma]$ and $[d - \sigma, d]$ and to $h_1^{(2)} = 2(d - 2\sigma)N_1^{-1}$ on $[\sigma, d - \sigma]$. The parameter σ is defined by the relation

$$\sigma = \sigma(\varepsilon, N_1, l) = \min[4^{-1}d, lm^{-1}\varepsilon \ln N_1], \tag{3.6б}$$

where $m = m_{(7.2)}$. Here, $l = 2$; in other meshes, this parameter will be chosen. The construction of $\overline{\omega}_1^*$ and $\overline{D}_h = \overline{D}_h(l = 2)$ is completed.

The solutions of scheme (3.2), (3.6) satisfy the estimate

$$|u(x) - z(x)| \le M\{N_1^{-2}\min[\varepsilon^{-2}, \ln^2 N_1] + N_2^{-2}\}, \quad x \in \overline{D}_h, \tag{3.7}$$

and the ε-uniform estimate

$$|u(x) - z(x)| \le M\{N_1^{-2}\ln^2 N_1 + N_2^{-2}\}, \quad x \in \overline{D}_h. \tag{3.8}$$

Theorem 1. *Suppose that the solution $u(x)$ of problem* (2.2), (2.1) *satisfies the estimates in Theorem* 5 *in Section* 7, *where $K = 4$. Then the difference scheme* (3.2), (3.6) (*schemes* (3.2), (3.1) *and* (3.2), (3.4)) *converges ε-uniformly* (*for fixed values of* ε). *The grid solutions satisfy estimates* (3.3), (3.5), (3.7), *and* (3.8).

4. RICHARDSON'S SCHEME FOR PROBLEM (2.2), (2.1)

Richardson's method is used to improve the accuracy of solutions to the special difference scheme, and it can be described as follows.

4.1. On $\overline{D}$ we introduce grids

$$\overline{D}_h^i = \overline{\omega}_1^{*i} \times \omega_2^i, \quad i = 1, 2, \tag{4.1a}$$

where $\overline{D}_h^1 = \overline{D}_{h(3.6)}(l)$ for $l = 4$; $\overline{\omega}_1^{*2}$ is a piecewise uniform mesh whose mesh size on $[0, \sigma]$, $[\sigma, d - \sigma]$, and $[d - \sigma, d]$, where $\sigma = \sigma_{(3.6б)}(\varepsilon, N_1 = N_{1(3.6)}, l = 4)$, is k times smaller than the mesh size of $\overline{\omega}_1^{*1}$; the mesh size of ω_2^2 is k times smaller than that of ω_2^1; $kN_1 + 1$ and $kN_2 + 1$ are the number of mesh points in $\overline{\omega}_1^{*2}$ and on a unit segment of ω_2^2, respectively. Let

$$\overline{D}_h^0 = \overline{D}_h^1 \cap \overline{D}_h^2. \tag{4.1b}$$

We have $\overline{D}_h^0 = \overline{D}_h^1$ for an integer k ($k \ge 2$) and $\overline{D}_h^0 \ne \overline{D}_h^1$ for a noninteger k.

Let $z^i(x)$ ($x \in \overline{D}_h^i$, $i = 1, 2$) be solutions of the difference schemes

$$\Lambda_{(3.2)} z^i(x) = f(x), \quad x \in D_h^i, \quad z^i(x) = \varphi(x), \quad x \in \Gamma_h^i, \quad i = 1, 2. \tag{4.2a}$$

We set

$$z^0(x) = \gamma z^1(x) + (1-\gamma)z^2(x), \quad x \in \bar{D}_h^0, \tag{4.2b}$$

where $\gamma = \gamma(k) = -(k^2-1)^{-1}$. The function $z^0(x)$ ($x \in \bar{D}_h^0$) is called a solution of scheme (4.2), (4.1), a scheme based on Richardson's technique with two nested grids.

4.2. To prove the convergence of scheme (4.2), (4.1), it is convenient to consider the expansions of $z^i(x)$ ($x \in \bar{D}_h^i$, $i = 1, 2$) in powers of N_1^{-1} and N_2^{-1}:

$$z^i(x) = u(x) + k^{-2(i-1)}[N_1^{-2}u_1(x) + N_2^{-2}u_2(x)] + v^i(x), \quad x \in \bar{D}_h^i, \quad i = 1, 2, \tag{4.3}$$

where $v^i(x)$ is the residual term. The function $u_2(x)$ solves the problem

$$L_{(2.2)}u_2(x) = -12^{-1}\varepsilon^2 a_2(x)\frac{\partial^4}{\partial x_2^4}u(x), \quad x \in D, \quad u_2(x) = 0, \quad x \in \Gamma.$$

The function $u_1(x)$ is represented as a sum of functions:

$$u_1(x) = u_{11}(x) + u_{12}(x), \quad x \in \bar{D}, \tag{4.4}$$

where $u_{1j}(x)$ ($x \in \bar{D}$, $i = 1, 2$) solve the problems

$$L_{(2.2)}u_{11}(x) = -12^{-1}\varepsilon^2 N_1^2 a_1(x)\frac{\partial^4}{\partial x_1^4}u(x)\begin{Bmatrix} (h_1^{(1)})^2, & x_1 \in (0, d)\backslash[\sigma, d-\sigma] \\ (h_1^{(2)})^2, & x_1 \in (\sigma, d-\sigma) \end{Bmatrix}, \quad x \in D,$$

$$x_1 \neq \sigma, d-\sigma,$$

$$[u_{11}(x)] = \left[\frac{\partial}{\partial x_1}u_{11}(x)\right] = 0, \quad x \in D, \quad x_1 = \sigma, d-\sigma,$$

$$u_{11}(x) = 0, \quad x \in \Gamma;$$

$$L_{(2.2)}u_{12}(x) = 0, \quad x \in D, \quad x_1 \neq \sigma, d-\sigma,$$

$$u_{12}(x) = \psi(x), \quad x \in D, \quad x_1 = \sigma, d-\sigma,$$

$$u_{12}(x) = 0, \quad x \in \Gamma.$$

Here, $\psi(x) = w(x)$, $x \in D$, $x_1 = \sigma$, $d - \delta$, where $w(x)$ solves the problem of the method of lines in x_2:

$$L^h w(x) \equiv \left\{\varepsilon^2\left(\alpha_1(x)\delta_{\overline{x1}\widehat{x1}} + a_2(x)\frac{\partial^2}{\partial x_2^2}\right) - c(x)\right\}w(x)$$

$$= \begin{Bmatrix} 0, \quad x_1 \neq \sigma, d-\sigma, \\ -\frac{\varepsilon^2}{3}N_1^2 a_1(x)\frac{\partial^3}{\partial x_1^3}U(x)\begin{cases} h_1^{(2)} - h_1^{(1)}, & x_1 = \sigma, \\ -h_1^{(2)} + h_1^{(1)}, & x_1 = d-\sigma \end{cases} \end{Bmatrix}, \quad x \in D, \quad x_1 \in \omega_1^{*1},$$

$$w(x) = 0, \quad x \in \Gamma, \quad h_1^{(1)} = h_{1(3.6)}^{(1)}, \quad h_1^{(2)} = h_{1(3.6)}^{(2)}, \quad U(x) = U_{(7.1)}(x).$$

The functions $u_{11}(x)$, $u_{12}(x)$, and $u_2(x)$ ($x \in \bar{D}$) satisfy the estimates

$$|u_2(x)| \le M\varepsilon^2, \quad |u_{11}(x)| \le M\min[\varepsilon^{-2}, \ln^2 N_1], \quad |u_{12}(x)| \le M\varepsilon, \quad x \in \bar{D}.$$

The component $u_1(x)$ is sufficiently smooth on $\bar{D}$; and $u_2(x)$, on $\bar{D}$ in the strips $x_1 \le \sigma$, $\sigma \le x_1 \le d - \sigma$, and $x_1 \geq d - \sigma$. Taking into account the estimates for $u_1(x)$ and $u_2(x)$, we estimate $v^i(x)$:

$$|v^i(x)| \leq M\{N_1^{-3}\min[\varepsilon^{-3}, \ln^3 N_1] + N_2^{-4}\}, \quad x \in \bar{D}_h^i, \quad i = 1, 2. \tag{4.5}$$

In view of (4.5), we find

$$|u(x) - z^0(x)| \leq M\{N_1^{-3}\min[\varepsilon^{-3}, \ln^3 N_1] + N_2^{-4}\}, \quad x \in \bar{D}_h^0, \tag{4.6}$$

and the ε-uniform estimate

$$|u(x) - z^0(x)| \leq M[N_1^{-3}\ln^3 N_1 + N_2^{-4}], \quad x \in \bar{D}_h^0. \tag{4.7}$$

Theorem 2. *Suppose that the data in the boundary value problem* (2.1), (2.2) *satisfy conditions* (2.3) *and, additionally,* a_s, $c, f \in C^{6+\alpha}(\bar{D})$, $\varphi \in C^{6+\alpha}(\Gamma)$; *and* $\alpha > 0$. *Suppose that the components of solutions to the problem in* (7.1) *satisfy a priori estimates* (7.2), *where* $K = 6$. *Then the solution to the difference scheme* (4.2), (4.1) *converges* ε*-uniformly to the solution of the boundary value problem at a rate of* $\mathcal{O}(N_1^{-3}\ln^3 N_1 + N_2^{-4})$. *The discrete solutions satisfy estimate* (4.6), (4.7).

5. DECOMPOSITION OF SCHEME (3.2), (3.1)

The decomposition scheme based on difference scheme (3.2), (3.1) can be described as follows.

5.1. Suppose that D is covered with subdomains $D^{(k)}$:

$$D = D^{(1)} \cup D^{(2)}, \quad D^{(1)} = (0, d_1) \times R, \quad D^{(2)} = (d_2, d) \times R, \quad 0 < d_2 < d_1 < d, \tag{5.1a}$$

which have the overlap $\delta = d_1 - d_2 > 0$. On $\bar{D}^{(k)}$ $(k = 1, 2)$ we introduce the grids

$$\bar{D}_h^{(k)} = \bar{D}^{(k)} \cap \bar{D}_h, \quad k = 1, 2, \tag{5.1b}$$

where $\bar{D}_h = \bar{D}_{h(3.1)}$. Assume that the boundaries of $D^{(1)}$ and $D^{(2)}$ pass through nodes of $\bar{\omega}_1$.

Let $z_0(x)$ be an arbitrary bounded function defined on $\bar{D}_h$, and let $z_1(x), \ldots, z_{n-1}(x)$ $(x \in \bar{D}_h)$ have been determined and $z_i(x) = \varphi(x)$, $x \in \Gamma_h$. We want to find $z_n(x)$. To this end, we solve the problems

$$\begin{gathered} \Lambda z_n^{(1)}(x) = f(x), \quad x \in D_h^{(1)}, \quad z_n^{(1)}(x) = z_{n-1}(x), \quad x \in \Gamma_h^{(1)}, \\ \Lambda z_n^{(2)}(x) = f(x), \quad x \in D_h^{(2)}, \quad z_n^{(2)}(x) = \begin{cases} z_n^{(1)}(x), & x \in \Gamma_h^{(2)} \setminus \Gamma, \\ z_{n-1}(x), & x \in \Gamma_h^{(2)} \cap \Gamma. \end{cases} \end{gathered} \tag{5.2a}$$

Let

$$z_n(x) = \left\{ \begin{array}{ll} z_n^{(1)}(x), & x \in \bar{D}_h^{(1)} \setminus \bar{D}^{(2)} \\ z_n^{(2)}(x), & x \in \bar{D}_h^{(2)} \end{array} \right\}, \quad x \in \bar{D}_h. \tag{5.2b}$$

The function $z_n(x)$ $(x \in \bar{D}_h, n = 1, 2, \ldots)$ is called a solution of the difference scheme (5.2), (5.1), i.e., the scheme for domain decomposition into overlapping subdomains.

The difference scheme (5.2), (5.1) is an approximation to the following continual scheme for the Schwarz method.

Let $u_0(x)$ $(x \in \bar{D})$ be an arbitrary bounded function, and let $u_1(x), \ldots, u_{n-1}(x)$ $(x \in \bar{D})$ have been constructed and $u_i(x) = \varphi(x)$, $x \in \Gamma$. We construct $u_n(x)$. Preliminarily, we solve the problems

$$\begin{gathered} Lu_n^{(1)}(x) = f(x), \quad x \in D^{(1)}, \quad u_n^{(1)}(x) = u_{n-1}(x), \quad x \in \Gamma^{(1)}, \\ Lu_n^{(2)}(x) = f(x), \quad x \in D^{(2)}, \quad u_n^{(2)}(x) = \begin{cases} u_n^{(1)}(x), & x \in \Gamma^{(2)} \setminus \Gamma, \\ u_{n-1}(x), & x \in \Gamma^{(2)} \cap \Gamma. \end{cases} \end{gathered} \tag{5.3a}$$

Next, we set

$$u_n(x) = \begin{Bmatrix} u_n^{(1)}(x), & x \in \overline{D}^{(1)} \backslash \overline{D}^{(2)} \\ u_n^{(2)}(x), & x \in \overline{D}^{(2)} \end{Bmatrix}, \quad x \in \overline{D}. \tag{5.3b}$$

The function $u_n(x)$ ($x \in \overline{D}$, $n = 1, 2, \ldots$) is called a solution of the continual scheme for the Schwarz method (on overlapping subdomains).

When $z_0(x) = u_0(x)$ for $x \in \overline{D}_h$, scheme (5.2), (5.1) is an approximation to problem (5.3), (5.1a).

5.2. Note that the overlap δ of subdomains can depend on ε: $\delta = \delta(\varepsilon)$.

When

$$\inf_{\varepsilon}(\varepsilon^{-1}\delta) > 0 \tag{5.4}$$

the solutions of the difference schemes (3.2), (3.1) and (5.2), (5.1), (3.1) satisfy the estimate

$$|z(x) - z_n(x)| \le Mq^n, \quad x \in \overline{D}_{h(3.1)}, \tag{5.5}$$

(see [4]), where $q \le 1 - m$. On the special mesh (3.6), we obtain the estimate

$$|u(x) - z_n(x)| \le M\{N_1^{-2}\min[\varepsilon^{-2}, \ln^2 N_1] + N_2^{-2} + q^n\}, \quad x \in \overline{D}_{h(3.6)}, \tag{5.6}$$

(see [4]) and the ε-uniform estimate

$$|u(x) - z_n(x)| \le M[N_1^{-2}, \ln^2 N_1 + N_2^{-2} + q^n], \quad x \in \overline{D}_{h(3.6)}. \tag{5.7}$$

Scheme (5.2), (5.1) with $n = n_*$ is said to be consistent with respect to both the accuracy of the limit solution $z_\infty(x)$ at $n = \infty$ (of the basic scheme) and the number n of iterations (or, briefly, consistent) if

$$\max_{\overline{D}_h}|z_\infty(x) - z_n(x)| \le M\max_{\overline{D}_h}|u(x) - z(x)|, \quad n \ge n^*,$$

where $z_\infty(x) = z(x)$ solves scheme (3.2) on $\overline{D}_h$.

The consistent scheme (5.2), (5.1), (3.6) satisfies the estimate

$$\begin{gathered} |u(x) - z_n(x)| \le M\{N_1^{-2}\min[\varepsilon^{-2}, \ln^2 N_1] + N_2^{-2}\} \le M[N_1^{-2}\ln^2 N_1 + N_2^{-2}], \\ x \in \overline{D}_{h(3.6)}, \quad n \ge n^*, \end{gathered} \tag{5.8a}$$

and $n^* = n^*_{(5.8)}$ satisfies

$$n^* \le M\ln(\min[N_1, N_2]) \le M\ln N. \tag{5.8b}$$

Theorem 3. *Under the hypotheses of Theorem* 1, *the solution of the difference scheme* (5.2), (5.1), (3.6) *converges ε-uniformly to the solution of problem* (2.2), (2.1) *as N_1, N_2, $n \longrightarrow \infty$. The discrete solutions satisfy estimates* (5.5)–(5.8).

6. DECOMPOSITION OF SCHEME (4.2), (4.1)

Let us describe the decomposition scheme based on difference scheme (4.2), (4.1), i.e., Richardson's scheme of fourth-order accuracy in x_2.

6.1. On $\overline{D}^{(1)}$ and $\overline{D}^{(2)}$, we introduce the grids

$$\overline{D}_h^{(k)i} = \overline{D}^{(k)} \cap \overline{D}_h^i, \quad k, i = 1, 2, \tag{6.1}$$

where $\overline{D}_h^i = \overline{D}_{h(4.1)}^i$ and $\overline{D}^{(k)} = \overline{D}_{(5.1)}^{(k)}$. Assume that the boundaries of $\overline{D}^{(k)}$ pass through nodes of $\overline{D}_{h(4.1)}^0$.

Let $z_0^i(x)$ ($x \in \overline{D}_h^i$, $i = 1, 2$) be a given function such that $z_0^i(x) = \varphi(x)$ for $x \in \Gamma_h^i$, and $z_0^1(x) = z_0^2(x)$ for $x \in \overline{D}_h^0$. Suppose that $z_1^i(x), \ldots, z_{n-1}^i(x)$ ($x \in \overline{D}_h^i$) have been determined. We solve the problems

$$\Lambda z_n^{(1)i}(x) = f(x), \quad x \in D_h^{(1)i}, \quad z_n^{(1)i}(x) = z_{n-1}^i(x), \quad x \in \Gamma_h^{(1)i},$$

$$\Lambda z_n^{(2)i}(x) = f(x), \quad x \in D_h^{(2)i}, \quad z_n^{(2)i}(x) = \begin{cases} z_n^{(1)i}(x), & x \in \Gamma_h^{(2)i} \setminus \Gamma, \\ z_{n-1}^i(x), & x \in \Gamma_h^{(2)i} \cap \Gamma; \quad i = 1, 2. \end{cases} \tag{6.2a}$$

Next, we set

$$z_n^i(x) = \left\{ \begin{matrix} z_n^{(1)i}(x), & x \in \overline{D}_h^{(1)i} \setminus \overline{D}^{(2)} \\ z_n^{(2)i}(x), & x \in \overline{D}_h^{(2)i} \end{matrix} \right\}, \quad x \in \overline{D}_h^i, \quad i = 1, 2. \tag{6.2b}$$

The function $z_n^i(x)$ ($x \in \overline{D}_h^i$, $i = 1, 2$) is a solution of problem (6.2a), (6.1). This iterative process converges as $n \longrightarrow \infty$. The functions $z_n^i(x)$ ($x \in \overline{D}_h^i$, $i = 1, 2$) are used to find

$$z_n^0(x) = \gamma z_n^1(x) + (1 - \gamma) z_n^2(x), \quad x \in \overline{D}_h^0, \quad n = 1, 2, \ldots. \tag{6.2c}$$

The function $z_n^0(x)$ ($x \in \overline{D}_h^0$, $n = 1, 2, \ldots$) is called a solution of the difference scheme (6.2), (6.1), i.e., the decomposition of Richardson's scheme (4.2), (4.1).

6.2. By virtue of (5.5), we have

$$\left| z^i(x) - z_n^i(x) \right| \le M q^n, \quad x \in \overline{D}_h^i, \quad i = 1, 2. \tag{6.3}$$

Taking into account representation (4.3) and estimates (4.6), (4.7), and (6.3), we obtain the following estimate for solutions of scheme (6.2), (6.1):

$$\left| u(x) - z_n^0(x) \right| \le M \{ N_1^{-3} \min[\varepsilon^{-3}, \ln^3 N_1] + N_2^{-4} + q^n \}, \quad x \in \overline{D}_h^0, \tag{6.4}$$

and the ε-uniform estimate

$$\left| u(x) - z_n^0(x) \right| \le M [N_1^{-3} \ln^3 N_1 + N_2^{-4} + q^n], \quad x \in \overline{D}_h^0. \tag{6.5}$$

In (6.4) and (6.5), $q \le 1 - m$.

The consistent scheme (6.2), (6.1) satisfies the estimate

$$\left| u(x) - z_n^0(x) \right| \le M \{ N_1^{-3} \min[\varepsilon^{-3}, \ln^3 N_1] + N_2^{-4} \} \le M [N_1^{-3} \ln^3 N_1 + N_2^{-4}],$$
$$x \in \overline{D}_h^0, \quad n \ge n^*, \tag{6.6a}$$

and $n^* = n^*_{(6.6)}$ satisfies an estimate similar to (5.8b):

$$n^* \le M \ln(\min[N_1, N_2]) \le M \ln N. \tag{6.6b}$$

Theorem 4. *Let the hypotheses of Theorem* 2 *and condition* (5.4) *be fulfilled. Then the solution of the difference scheme* (6.2), (6.1) *converges* ε*-uniformly to the solution of problem* (2.2), (2.1) *as* $N_1, N_2, n \longrightarrow \infty$. *The discrete solutions satisfy estimates* (6.4)–(6.6).

Remark. The technique for constructing scheme (6.2), (6.1) and the technique described in [16–18] can be used to construct parallel Richardson schemes, i.e., high-order accurate decomposition schemes intended for parallel computations on $P > 1$ processors.

7. SUPPLEMENT: A PRIORI ESTIMATES

Below, we present the a priori estimates for the solution of problem (2.2), (2.1) that were used in our constructions (see, e.g., [4]). These estimates are established by using internal a priori estimates and estimates up to the boundary (see [19]).

The solution to the problem under consideration is represented as the sum of functions:

$$u(x) = U(x) + V(x), \quad x \in \overline{D}, \tag{7.1}$$

where $U(x)$ and $V(x)$ are the regular and singular components of the solution to the problem. The components in (7.1) satisfy the estimates

$$\left|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}}U(x)\right| \le M,$$
$$\left|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}}V(x)\right| \le M\varepsilon^{-k_1}\exp[-m\varepsilon^{-1}r(x,\Gamma)], \quad x \in \overline{D}, \quad k = k_1 + k_2 \le K, \tag{7.2}$$

where $r(x, \Gamma)$ is the distance from a point x to the set Γ and m is an arbitrary number in the range $(0, m_0)$, with $m_0 = \min\limits_{\overline{D}}[a_1^{-1}(x)c(x)]^{1/2}$. The parameter K depends on the smoothness of the data in the problem.

The function $u(x)$ also satisfies the estimate

$$\left|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}}u(x)\right| \le M\varepsilon^{-k}, \quad x \in \overline{D}, \quad k \le K.$$

Theorem 5. *Suppose that the data in the boundary value problem* (2.2), (2.1) *satisfy conditions* (2.3), *and let* $a_s, c, f \in C^{K+\alpha}(\overline{D})$, $\varphi \in C^{K+\alpha}(\Gamma)$, $s = 1, 2$, $K \ge 0$; *and* $\alpha > 0$. *Then the components in* (7.1) *satisfy estimates* (7.2).

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