Corrigendum

P.W. Hemker, Remarks on sparse-grid finite-volume multigrid, Advances in Computational Mathematics **4** (1995) 83–110.

The author is grateful to Christoph Pflaum for pointing out a mistake in the statement and proof of theorem 2.2. The correct version is:

Theorem 2.2
If we consider an expansion of a $C^3(\overline{\Omega})$-function, u, in piecewise constant functions on the grid $\Omega_{\boldsymbol{n}}$, for an arbitrary $\boldsymbol{n} \in \mathbb{Z}^3$, $\boldsymbol{n} > \boldsymbol{0}$, and if we write

$$R_{\boldsymbol{n}} u = v_0 + \sum_{\boldsymbol{0} \leq \boldsymbol{j} \leq \boldsymbol{n}} u_{\boldsymbol{j}}\,, \tag{1}$$

with $v_0 \in V_0$ and $u_{\boldsymbol{j}} \in W_{\boldsymbol{j}}$, $\boldsymbol{0} \leq \boldsymbol{j} \leq \boldsymbol{n}$, then

$$\|u_{\boldsymbol{j}}\| \leq 2^{|\boldsymbol{j}|}|u|\,, \tag{2}$$

and we get an estimate for the approximation error

$$\|u - R_{\boldsymbol{n}} u\| \leq \frac{1}{3}\sqrt{\frac{2}{3}}(h_1 + h_2 + h_3)|u|\,. \tag{3}$$

Proof
We take the normalised $\{\tilde{\psi}_{\boldsymbol{j}}^{\boldsymbol{k}}\} = \{2^{|\boldsymbol{j}-\boldsymbol{e}|/2}\psi_{\boldsymbol{k}}^{\boldsymbol{j}}\}$ as a basis in $W_{\boldsymbol{j}}$, $\boldsymbol{0} \leq \boldsymbol{j} \leq \boldsymbol{n}$, $\boldsymbol{j} \neq \boldsymbol{0}$. Clearly, all these functions are orthogonal to all functions $v_0 \in V_0$ and mutually they form an orthonormal set in $W_{\boldsymbol{j}} \subset L^2(\Omega)$. We see further $\psi_{\boldsymbol{k}}^{\boldsymbol{j}} \in W_{\boldsymbol{j}}$ and $\mathrm{support}(\psi_{\boldsymbol{k}}^{\boldsymbol{j}}) = \Omega_{\boldsymbol{j}-\boldsymbol{e},\boldsymbol{k}}$, or, in other words, $\psi_{\boldsymbol{k}}^{\boldsymbol{j}} \in V_{\boldsymbol{j}}$, but $\psi_{\boldsymbol{k}}^{\boldsymbol{j}}$ scales like a basis function in $V_{\boldsymbol{j}-\boldsymbol{e}}$. Hence

$$\int 2^{|\boldsymbol{j}-\boldsymbol{e}|/2}\psi_{\boldsymbol{k}}^{\boldsymbol{j}} 2^{|\boldsymbol{j}-\boldsymbol{e}|/2}\psi_{\boldsymbol{m}}^{\boldsymbol{j}}\, d\Omega = 0 \quad \text{for} \quad \boldsymbol{k} \neq \boldsymbol{m},$$

and

$$\int 2^{|\boldsymbol{j}-\boldsymbol{e}|/2}\psi_{\boldsymbol{k}}^{\boldsymbol{j}} 2^{|\boldsymbol{j}-\boldsymbol{e}|/2}\psi_{\boldsymbol{k}}^{\boldsymbol{j}}\, d\Omega = 2^{|\boldsymbol{j}-\boldsymbol{e}|}\int_{\Omega_{\boldsymbol{j}-\boldsymbol{e},\boldsymbol{k}}} d\Omega = 1.$$

Thus, we find (1) with

$$u_{\boldsymbol{j}} = \sum_{\boldsymbol{k}} a_{\boldsymbol{j}\boldsymbol{k}}\tilde{\psi}_{\boldsymbol{k}}^{\boldsymbol{j}} = \sum_{\boldsymbol{k}} (u, \tilde{\psi}_{\boldsymbol{k}}^{\boldsymbol{j}})\tilde{\psi}_{\boldsymbol{k}}^{\boldsymbol{j}}.$$

Now

$$a_{jk} = (u, \tilde{\psi}_k^j) = \int_\Omega u\tilde{\psi}_k^j \, d\Omega = \int_{\Omega_{j-e,k}} u\tilde{\psi}_k^j \, d\Omega.$$

By Taylor expansion around z_k^{j-e}, we have

$$\left| \int_{\Omega_{j-e,k}} u\tilde{\psi}_k^j \, d\Omega \right| \le 2^{-2|j|} 2^{|j-e|/2} |u|. \tag{4}$$

For $\boldsymbol{j} \ge \boldsymbol{e}$ the point z_k^{j-e} lies in the interior of Ω and the estimate holds with

$$|u| = \max \left| \frac{\partial^3 u(x)}{\partial x_1 \ldots \partial x_3} \right|.$$

For ψ_k^j with a $\boldsymbol{j}$-component equal to zero, the point z_k^{j-e} lies on the boundary and the function ψ_k^j is constant in one direction over the whole domain Ω, and it is of Haar-wavelet type for the non-zero indices (or index). In this situation the same estimate (4) holds with, e.g. if $j_1 = 0$,

$$|u| = \max \left| \frac{\partial^2 u(x)}{\partial x_2 \ldots \partial x_3} \right|.$$

For $\boldsymbol{j} = \boldsymbol{0}$ the relation (4) is trivially satisfied. Hence, the estimate (4) holds for $\boldsymbol{j} \ge \boldsymbol{0}$ if we use the seminorm (21), and we find

$$|a_{j,k}| \le 2^{-3/2} 2^{-3/2|j|} |u|,$$

$$\|u_j\|^2 = \sum_k |a_{jk}|^2 \le \sum_k 2^{-3|j|-3} |u|^2 = 2^{-2|j|-3} |u|^2,$$

so that

$$\|u_j\| \le 2^{|j|-3/2} |u|,$$

which leads to (2), and

$$\|u - R_n u\|^2 = \sum_{\substack{j_1 > n_1 \\ \text{or...or} \\ j_3 > n_3}} \|u_j\|^2 \le \sum_{\substack{j_1 > n_1 \\ j_2 \ge 0 \\ j_3 \ge 0}} + \cdots + \sum_{\substack{j_1 \ge 0 \\ j_2 \ge 0 \\ j_3 > n_3}} 2^{-2|j|-3} |u|^2$$

$$\le 3^{-3} 2 (2^{-2n_1} + 2^{-2n_2} + 2^{-2n_3}) |u|^2.$$

and it follows that

$$\|u - R_n u\| \le \left(\frac{2}{3^3} \right)^{1/2} (2^{-n_1} + 2^{-n_2} + 2^{-n_3}) |u| = \frac{1}{3} \sqrt{\frac{2}{3}} (h_1 + h_2 + h_3) |u|.$$

□