Edge-reinforced random walk on finite graphs

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Abstract

We consider reinforced random walk, in which the transition probabilities at each step are influenced by the number of crossings of the edges. For finite graphs we prove a limit theorem for the joint distribution of the normalized occupation time of the edges and the normalized cycle numbers (number of times the random walk winds around the individual cycles of the graph). The limiting distribution is calculated explicitely.

1 Introduction

Edge-reinforced random walk was introduced in 1987 by Coppersmith and Diaconis as a simple model of exploring a new city. Imagine a person walking in a new city. At first all streets are equally unfamiliar and she chooses at random between them. As time goes on, streets that have been traversed more often in the past are more familiar and more likely to be traversed.

Let G be a finite connected graph. We consider reinforced random walk on G defined as follows. Each edge is given a strictly positive real number as initial weight. In each step the random walker jumps from the current vertex to an adjacent vertex by traversing an edge with probability proportional to the weight of that edge. Each time an edge is traversed, its weight is increased by 1. This process has an infinite memory. It remembers where it has been, and it prefers to traverse edges that have been traversed often in the past.

In [Dia88], Diaconis states without proof that the normalized occupation measure on the edges converges almost surely to a random vector which has a density with respect to Lebesgue measure on the simplex. The density is given up to a normalizing constant. For a proof Diaconis refers to an upcoming paper together with Coppersmith, but this paper seems never to have been published. In [Dia88], there is a hint concerning the proof of Diaconis' statement, namely that it involves "a difficult combinatorics calculation". A proof for the special case where the underlying graph is a triangle has been given by one of the authors [Kea90].

In this paper we extend Diaconis' statement: We study the joint asymptotic behaviour of (α_n, β_n) where α_n denotes the proportion of time spent on the individual edges and β_n denotes the normalized cycle numbers (the number of times the individual cycles of the graph are traversed where traversals of the same cycle in different directions are counted with different signs). We show that the distributions under consideration converge weakly to an absolutely continuous distribution. The density of the limit is calculated explicitely. As a corollary we obtain Diaconis' statement and we give the normalizing constant of his density.

The exposition is organized as follows: In Section 2 we introduce some basic notation and state the result. Section 3 gives an important property of a reinforced random walk on a finite graph: it is a mixture of Markov chains. In Section 4, a counting problem is solved that allows us to calculate the limiting distributions. Section 5 contains the asymptotics, and in Section 6 the main result is proved.

2 Result

Let $G = (V, E)$ be a finite connected graph with vertex set V and edge set E. We denote the cardinalities of V and E with l and m, respectively. We assume that G has no loops, i.e. each edge has two distinct endpoints. Parallel edges are allowed, so two edges may have the same pair of endpoints. For an edge e we denote the set of its endpoints by $\bar{e}$. The edges are given positive weights. At time 0 the weights are non-random; edge e has weight $a_e > 0$. We denote by $w_n(e)$ the weight of edge e at time n (just after the n^{th} step) and by $w_n(v)$ the sum of the weights of the edges incident to vertex v. Let $v_0 \in V$.

We define reinforced random walk with starting point v_0 to be a sequence $X_0, Y_1, X_1, Y_2, X_2, \dots$ with X_i taking values in V, Y_i taking values in E and $\bar{Y_i} = \{X_{i-1}, X_i\}$ for all $i \in \mathbb{N} = \{1, 2, \dots\}$. Furthermore $P(X_0 = v_0) = 1$ and

$$P(Y_{n+1} = e, X_{n+1} = v | X_0, Y_1, X_1, \dots, Y_n, X_n) = \begin{cases} \dfrac{w_n(e)}{w_n(X_n)} & \text{if } \bar{e} = \{X_n, v\} \\ 0 & \text{otherwise,} \end{cases}$$

and the weights satisfy $w_0(e) = a_e$,

$$w_{n+1}(e) = \begin{cases} w_n(e) + 1 & \text{if } Y_{n+1} = e \\ w_n(e) & \text{otherwise.} \end{cases}$$

We denote by $k_n(e)$ the number of times the reinforced random walk traverses the edge e up to time n:

$$k_n(e) := |\{i \in \{1, \dots, n\} : Y_i = e\}|.$$

Here and in the following, $|S|$ denotes the cardinaliy of a set S. We denote by $\alpha_n(e) := \frac{k_n(e)}{n}$ the proportion of crossings of edge e up to time n. We write k_n and α_n for the m-dimensional vectors $(k_n(e); e \in E)$ and $(\alpha_n(e); e \in E)$. Clearly, all coordinates of α_n are nonnegative, and they sum to 1. So α_n takes values in the $(m-1)$-dimensional simplex

$$\Delta := \left\{ x = (x_e; e \in E) \in [0,1]^E : \sum_{e \in E} x_e = 1 \right\}.$$

We denote by $a_v := \sum_{\{e: v \in \bar{e}\}} a_e$ the sum of the initial weights of the edges incident to vertex v, and we set $x_v := \sum_{\{e: v \in \bar{e}\}} x_e$ for any $x \in \Delta$.

Let $c_1, \dots, c_{m-l+1}$ be a fundamental system of cycles of G, and let the cycles be oriented in an arbitrary way. (For definition see Section 4, Definition 4.) We define for $x \in \Delta$ an $(m-l+1) \times (m-l+1)$-dimensional matrix $A(x) = (a_{i,j}(x))$ by

$$a_{i,i}(x) = \sum_{e \in c_i} \frac{1}{x_e}, \quad a_{i,j}(x) = \sum_{e \in c_i \cap c_j} \pm \frac{1}{x_e} \text{ for } i \neq j,$$

where the signs in the last sum are chosen to be $+1$ or -1 depending on whether the edge e has in c_i and c_j the same orientation or not.

Cycle numbers are $(m-l+1)$-dimensional vectors counting how often the walker traverses the fundamental cycles of the graph, where traversals of the same cycle in different directions are counted with different signs. We denote by j_n the cycle numbers at time n, and we set $\beta_n = \frac{j_n}{\sqrt{n}}$. Furthermore, we denote by $\Gamma(a)$ the value of the Gamma function at a, and we set

$$C = \frac{\Gamma(\frac{a_{v_0}}{2}) \prod_{v \in V \setminus \{v_0\}} \Gamma(\frac{a_v + 1}{2})}{\prod_{e \in E} \Gamma(a_e)}. \tag{1}$$

Theorem 1 *The sequence $\{\alpha_n; n \in \mathbb{N}\}$ converges almost surely. The distribution of the limit is absolutely continuous with respect to surface measure on Δ with density given by*

$$\phi(x) = C\frac{2^{1-l+\sum_{e\in E} a_e}}{(m-1)!\pi^{\frac{l-1}{2}}}\frac{\prod_{e\in E} x_e^{a_e-\frac{1}{2}}}{x_{v_0}^{\frac{a_{v_0}}{2}}\prod_{v\in V\setminus\{v_0\}} x_v^{\frac{a_v+1}{2}}}\sqrt{det(A(x))}.$$

Furthermore, the sequence $\{(\alpha_n, \beta_n); n \in \mathbb{N}\}$ converges weakly. The limit is absolutely continuous with respect to the product of surface measure on Δ and Lebesgue measure on $\mathbb{R}^{m-l+1}$ with density given by

$$\varphi(x,y) = C\frac{2^{\frac{1-m-l}{2}+\sum_{e\in E} a_e}}{(m-1)!\pi^{\frac{m}{2}}}\frac{\prod_{e\in E} x_e^{a_e-\frac{3}{2}}}{x_{v_0}^{\frac{a_{v_0}}{2}}\prod_{v\in V\setminus\{v_0\}} x_v^{\frac{a_v+1}{2}}}\sum_{T\in\mathcal{T}}\prod_{e\in E(T)} x_e \exp\left(-\frac{1}{2}y^t A(x) y\right).$$

3 Reinforced random walk as a mixture of Markov chains

For technical reasons, we choose in an arbitrary manner for each edge an orientation that we call positive. If e is an edge with $\bar{e} = \{u, v\}$, we call one endpoint (say v) *positive* and the other endpoint *negative*. We say the arc induced by e with starting point u and endpoint v is *positively oriented* and we denote this arc by e^+ or $e(u,v)$. Similarly we call the other arc induced by e *negatively oriented* and denote it by e^- or $e(v,u)$. We denote by $k_n(e^+)$ respectively $k_n(e^-)$ the number of times the reinforced random walk traverses the arc e^+ respectively e^- up to time n, and we set $k_n^+ := (k_n(e^+); e \in E)$, $k_n^- := (k_n(e^-); e \in E)$.

Proposition 1 *The reinforced random walk traverses almost surely every edge infinitely often in both directions.*

Proof. Let e be an edge with $\bar{e} = \{u,v\}$ and $e^- = e(v,u)$. We denote the time of the i^{th} visit to vertex v by τ_i with the convention $\tau_i = \infty$ if v is visited at most $i-1$ times. Clearly,

$$\begin{aligned} & P(v \text{ is visited infinitely often}, e^- \text{ is traversed at most finitely often}) \\ = & \lim_{i_0\to\infty}\lim_{I\to\infty} P\left(\bigcap_{i_0\le i\le I}\{\tau_i < \infty, X_{\tau_i+1} \neq u\}\right). \end{aligned}$$

Since between two successive visits to v the sum of the weights of the edges incident to v increases by 2, $w_{\tau_i}(v) \leq a_v + 2i$. Therefore given the past up to time τ_i, the probability of not traversing e^- at time $\tau_i + 1$ equals $1 - \frac{w_{\tau_i}(e)}{w_{\tau_i}(v)} \leq 1 - \frac{a_e}{a_v+2i} \leq \exp(-\frac{a_e}{a_v+2i})$, so by induction

$$P\left(\bigcap_{i_0 \leq i \leq I} \{\tau_i < \infty, X_{\tau_i+1} \neq u\}\right) \leq \prod_{i_0 \leq i \leq I} \exp\left(-\frac{a_e}{a_v + 2i}\right) = \exp\left(-\sum_{i_0 \leq i \leq I} \frac{a_e}{a_v + 2i}\right).$$

Taking limits $I \to \infty$ and then $i_0 \to \infty$ in the last inequality shows that $P(v \text{ is visited infinitely often}, e^- \text{ is traversed at most finitely often}) = 0$. Using induction we conclude $P(v$ is visited infinitely often, an arc is traversed only finitely often$) = 0$. Since the graph is finite, at least one vertex is visited infinitely often. Thus $P($an arc is traversed at most finitely often$) = 0$. □

We call a sequence $p := (u_0, e_1, u_1, e_2, \ldots, e_n, u_n)$ with $n \geq 1$, $u_i \in V$, $e_i \in E$ and $\bar{e}_i = \{u_{i-1}, u_i\}$ for all $i \in \{1, \ldots, n\}$ a *path* of length n with *starting point* u_0 and *endpoint* u_n. Observe that an edge can occur more than once in a path. Let $k_e^+(p)$, $k_e^-(p)$ denote the number of transitions of the arcs e^+, e^- in the path p, i.e. $k_e^+ = |\{i \in \{1, \ldots n\} : (u_{i-1}, u_i) = e^+\}|$. If it is clear which path we consider, we write briefly k_e^+ and k_e^-. We set $k^+ := (k_e^+; e \in E)$, $k^- := (k_e^-; e \in E)$. Clearly, $k_e := k_e^+ + k_e^-$ equals the number of transitions of the edge e. We call p a (k^+, k^-)*-path* or a k*-path.*

Definition 1 *A stochastic process with finite state space I is called* partially exchangeable *if all (k^+, k^-)-paths with the same starting point have the same probability. Here a path is understood with respect to the graph with vertex set I and edge set $I \times I$.*

Clearly, every Markov chain is partially exchangeable.

Let I be a finite set. Consider the set $\mathcal{P}$ of stochastic matrices on $I \times I$ with the topology of coordinate convergence. The set $I \times \mathcal{P}$ is compact. We denote the coordinates of $p \in \mathcal{P}$ by $p(i,j)$. A stochastic process $\{Z_n\}$ with state space I is called a *mixture of Markov chains* if for each $i \in I$ there exists a probability measure $\mu(i, \cdot)$ on the Borel sets of $\mathcal{P}$ such that

$$P(Z_j = i_j \text{ for } j = 0, \ldots, n) = \int_{\mathcal{P}} \prod_{j=0}^{n-1} p(i_j, i_{j+1}) \mu(i_0, dp).$$

Theorem 2 (Diaconis and Freedman [DF80], p.117, theorem (7)) *Let $\{Z_n; n \in \mathbb{N}_0\}$ be a stochastic process with finite state space. Suppose $P(Z_n = Z_0$ for infinitely many $n) = 1$. If Z is partially exchangeable, then Z is a mixture of Markov chains.*

Let $p = (u_0, e_1, u_1, e_2, \dots, e_n, u_n)$ be a path, and let $v \in V$. We denote by $n_v(p)$ the *number of departures from* v and by $\bar{n}_v(p)$ the *number of arrivals to* v in the path p. Formally, we set

$$n_v = |\{i \in \{0, 1 \dots, n-1\} : u_i = v\}|, \quad \bar{n}_v = |\{i \in \{1 \dots, n\} : u_i = v\}|.$$

Clearly, the number of departures and arrivals is determined by k^+ and k^-, and

$$n_v + \bar{n}_v = \sum_{\{e: v \in \bar{e}\}} k_e. \tag{2}$$

If we denote by v_0 the starting point of the path and by v_1 its endpoint, then

$$n_v + \delta_{v_1}(v) = \bar{n}_v + \delta_{v_0}(v), \tag{3}$$

where $\delta_u(v)$ takes the values 0 or 1 depending on whether $u \neq v$ or $u = v$. From (2) and (3) we conclude

$$n_v = \frac{1}{2}\left(\delta_{v_0}(v) - \delta_{v_1}(v) + \sum_{\{e: v \in \bar{e}\}} k_e\right) \tag{4}$$

and

$$\bar{n}_v = \frac{1}{2}\left(\delta_{v_1}(v) - \delta_{v_0}(v) + \sum_{\{e: v \in \bar{e}\}} k_e\right). \tag{5}$$

We see from the last equation that $(k_e \bmod 2; e \in E)$ together with the starting point v_0 determines uniquely the endpoint of a k-path: If $\sum_{\{e: v \in \bar{e}\}} k_e$ is even for all vertices, then the endpoint equals the starting point. If there exists a vertex $v_1 \neq v_0$ such that $\sum_{\{e: v \in \bar{e}\}} k_e$ is even for $v \in V \setminus \{v_0, v_1\}$ and odd for $v \in \{v_0, v_1\}$, then the endpoint equals v_1. In all other cases, no k-path is possible.

Lemma 1 *A k-path with starting point v_0 and endpoint v_1 exists iff*

1. $v_0 = v_1$ *and* $\sum_{\{e:v\in\bar{e}\}} k_e$ *is even for all* $v \in V$, *or*

2. $v_0 \neq v_1$, $\sum_{\{e:v\in\bar{e}\}} k_e$ *is even for all* $v \in V \setminus \{v_0, v_1\}$ *and odd for* $v \in \{v_0, v_1\}$.

In particular $(k_e \bmod 2; e \in E)$ *together with the starting point* v_0 *determines the endpoint of the path uniquely. Furthermore the number of departures and arrivals are given by (4) and (5).*

Proof. It remains to show that the given conditions are sufficient for the existence of a k-path with starting point v_0 and endpoint v_1. This will be Lemma 5. □

Lemma 2 *Two* k*-paths of reinforced random walk with the same starting point have the same probability. In particular, reinforced random walk is partially exchangeable (with respect to the state space* $E \times V$*).*

Proof. We compute the probability of a fixed k-path of length n with starting point v_0. This probability is given by a product with n factors, one for each transition. The first time the reinforced random walk visits a vertex v different from the starting point, the sum of the weights of the edges incident to v equals $a_v + 1$. From this observation it follows easily that the probability under consideration is given by

$$\frac{\prod_{e\in E} \prod_{i=0}^{k_e-1} (a_e + i)}{\prod_{i=0}^{n_{v_0}-1} (a_{v_0} + 2i) \prod_{v\in V\setminus\{v_0\}} \prod_{i=0}^{n_v-1} (a_v + 1 + 2i)}. \tag{6}$$

Since the number of crossings k and the starting point v_0 determine the endpoint of a k-path uniquely, we see from (4) that this probability depends only on k and v_0. □

We denote by $\partial\Delta$ the boundary of Δ:

$$\partial\Delta = \{(x_e; e \in E) : x_e = 0 \text{ for some } e \in E\}$$

Set $\alpha_n^+ = \frac{k_n^+}{n}$, $\alpha_n^- = \frac{k_n^-}{n}$.

Lemma 3 *The sequences* $\{\alpha_n^+; n \in \mathbb{N}\}$, $\{\alpha_n^-; n \in \mathbb{N}\}$ *converge almost surely. The limits* α_∞^+, α_∞^- *satisfy*

$$P(\alpha_\infty^+(e) = 0 \text{ for some } e \in E) = P(\alpha_\infty^-(e) = 0 \text{ for some } e \in E) = 0.$$

In particular, $P(\alpha_\infty \in \partial\Delta) = 0$.

Proof. Reinforced random walk with starting point v_0 is a stochastic process $Z_0 = X_0$, $Z_n = (X_n, Y_n)$, $n \in \mathbb{N}$ with state space $I := \{v_0\} \cup V \times E$. We have to add $\{v_0\}$ because the starting value plays a special role. We denote by Q_p the distribution of a Markov chain with state space I and transition matrix p. Reinforced random walk is partially exchangeable, so by Theorem 2 there exists a probability measure μ on the set $\mathcal{P}$ of stochastic matrices on I such that the distribution P of reinforced random walk with starting point v_0 satisfies

$$P(\cdot) = \int_{\mathcal{P}} Q_p(\cdot)\mu(dp). \tag{7}$$

Since for every Markov chain $\{\alpha_n^+; n \in \mathbb{N}\}$, $\{\alpha_n^-; n \in \mathbb{N}\}$ converge almost surely, we conclude from (7) that the same is true for the reinforced random walk.

We denote by $\mathcal{P}' \subset \mathcal{P}$ the set of all irreducible stochastic matrices. For $p \notin \mathcal{P}'$, there exist $i, j \in I$ such that $p_{i,j}^n = 0$ for all $n \in \mathbb{N}$, i.e. if the corresponding Markov chain reaches state i at some time n_0, then it will not reach j after time n_0. Observe that each element of $V \times E$ corresponds uniquely to an arc in the graph G. Suppose i and j correspond to the arcs e^+ and f^+ respectively. Then $Q_p(k_n^+(e) > 0, \lim_{n\to\infty} k_n^+(f) = +\infty) = 0$, so by (7)

$$\begin{aligned} & P(\lim_{n\to\infty} k_n^+(e) = \lim_{n\to\infty} k_n^-(e) = +\infty \text{ for all } e \in E) \\ = & \int_{\mathcal{P}'} Q_p(\lim_{n\to\infty} k_n^+(e) = \lim_{n\to\infty} k_n^-(e) = +\infty \text{ for all } e \in E)\mu(dp) \le \mu(\mathcal{P}'). \end{aligned}$$

Since reinforced random walk traverses almost surely each arc infinitely often, $P(\lim_{n\to\infty} k_n^+(e) = \lim_{n\to\infty} k_n^-(e) = +\infty \text{ for all } e \in E) = 1$, and we conclude $\mu(\mathcal{P}') = 1$. Hence μ is concentrated on the set of irreducible stochastic matrices. For an irreducible Markov chain with finite state space, all states are positive recurrent, so for all $p \in \mathcal{P}'$ $Q_p(\alpha_\infty^+(e) = 0 \text{ for some } e \in E) = 0$, and the lemma follows from another application of (7). □

4 The number of k-paths

For the proof of Theorem 1, we compute explicitely $P(\alpha_n = \frac{k}{n})$, the probability of all k-paths, and $P(\alpha_n = \frac{k}{n}, \beta_n = \frac{j}{\sqrt{n}})$, the probability of all k-paths with cycle number j. Using Lemma 2, we know that the first probability equals the probability of a fixed k-path multiplied by the number of

k-paths. It will turn out that there is a one-to-one correspondence between (k, j) and (k^+, k^-), so the second probability equals the probability of a fixed (k^+, k^-)-path multiplied by the number of (k^+, k^-)-paths. We have already computed the probability of a fixed k-path, so it remains to find the number of k-paths and (k^+, k^-)-paths. Throughout we denote the starting point by v_0 and the endpoint by v_1.

We call a path with the same starting and endpoint that contains no edge more than once a *closed path.* The collection of vertices and edges constituting a closed path considered as a circular sequence, i.e. as a sequence defined relative to circular order is called a *circuit* or an *oriented cycle.* A circuit has no distinguished starting point, and this is the only difference between a closed path and a circuit. If we reverse the order of the vertices and edges constituting a circuit, we obtain a different circuit. If we disregard the orientation of a circuit, we call the collection of vertices and edges constituting the circuit a *cycle.*

A *cycle graph* is a graph with the property that its vertex set together with its edge set constitute a cycle. A *tree* is a connected graph that contains no cycle graph. A *spanning tree* of the graph G is a maximal tree contained in G, i.e. a subtree of G that is not contained in any larger subtree. It is easy to see that a subtree of G is a spanning tree iff it has $|V| - 1$ edges.

For a vertex v in a directed graph, we call the number of arcs having v as an endpoint the *out-valency* of v. A *tree converging to* v is a directed tree, in which v has out-valency 0 and all other vertices have out-valency 1.

Given a path $p = (u_0, e_1, u_1, e_2, \dots, e_n, u_n)$, we can for each vertex $v \in V$ determine the sequence of edges chosen to leave the vertex, ordered in time. We call this sequence of edges the *exit order* of v. The last edge in the exit order of v is called the *exit edge* of v. Analogously, we can define the *exit arc* of v.

Lemma 4 *Given a k-path that traverses each edge at least once, the set of exit arcs from all vertices except the endpoint form a spanning tree converging to the endpoint of the path.*

Proof. The set of exit edges contains $|V|-1$ elements: If not, the exit edges of two different vertices agree. But then the last departure from one vertex yields to a vertex from which a last departure has already been performed, so this vertex must be the designated endpoint, contradicting the fact that we didn't consider the exit edge of the endpoint.

Suppose the set of exit edges contains a cycle. Consider the edges in the order they have been traversed. The edge that closes the cycle originates from a departure towards a vertex from which the last departure has already been performed. This is a contradiction by the same argument as before. So we have shown that the set of exit edges under consideration forms a spanning tree. Clearly, the exit arcs are directed towards the endpoint.□

We are now able to give a scheme to construct all (k^+, k^-)-paths. The method we use basically amounts to counting Euler circuits in directed graphs with parallel edges. This idea goes back to van Aardenne-Ehrenfest and de Bruijn ([vd51] section 6). See also [Kas67], section IV, C. For a description in terms of urns see [Zam84].

Lemma 5 *Let k_e^+, k_e^-, $e \in E$, be non-negative integers such that $k_e := k_e^- + k_e^+$ are strictly positive and condition 1 or 2 of Lemma 1 is satisfied. Then n_v and $\bar{n}_v$ defined by (4) and (5) are strictly positive integers and all (k^+, k^-)-paths starting at v_0 can be constructed in the following way: Construct an exit order of length n_v for every vertex v by first choosing, for each vertex different from v_1, the exit edge in such a way that all exit arcs form a spanning tree converging to v_1. Then choose all the remaining edges in an arbitrary order, and construct a path as follows: The path starts at v_0, traverses the first edge in the exit order of v_0, goes to the indicated vertex, traverses the first edge in the exit order of this vertex etc.*

Proof. Given a (k^+, k^-)-path, we can determine for each vertex the exit order, and by Lemma 4 the exit arcs chosen from the vertices except the endpoint give a spanning tree converging to the endpoint. Conversely, if we are given an exit order for each vertex, we can start to construct a path as indicated above. Eventually, one arrives at a vertex where there are no more departures left. From (4) and (5) we conclude that (3) holds, and from (3) it follows that the vertex where there are no departures left must be the endpoint v_1, and there cannot be any arrival to v_1 left. Let v be adjacent to v_1 and suppose that v and v_1 are connected by an edge in the spanning tree induced by the designated last departures. Since there are no arrivals to v_1 left and the last departure from v must be performed on the spanning tree towards v_1, there cannot be any departure from v left. By (3), there cannot be any arrivals to v left. Since any vertex can be reached from v_1 via a finite path on the spanning tree, we conclude by induction that there cannot be any crossings left. So the construction yields indeed a (k^+, k^-)-path. □

It is now easy to write down a formula for the number of (k^+, k^-)-paths.We denote by $\mathcal{T}$ the set of all spanning trees of G.

Lemma 6 *The number of (k^+, k^-)-paths with starting point v_0 is given by*

$$\frac{\prod_{v\in V} n_v!}{\prod_{e\in E} k_e^-!k_e^+!}\frac{\sum_{T\in\mathcal{T}}\prod_{e\in E(T)} k_e^\pm}{\prod_{v\in V\setminus\{v_1\}} n_v},$$

where $\prod_{e\in E(T)} k_e^\pm$ is a product over all edges in the tree T, and k_e^- or k_e^+ is chosen depending on whether the edge e^- or e^+ is directed toward the endpoint v_1, and n_v is defined by (4).

Proof. This follows easily from Lemma 5. □

Now we want to find out how to determine the number of k-paths from the number of (k^+, k^-)-paths. First we recall definitions and results from graph theory. For a more detailed exposition we refer to [Big93] and [Bol98]. We enumerate the vertices and edges in an arbitrary way, $V = \{v_1, \dots, v_l\}$, $E = \{e_1, \dots, e_m\}$. Recall that we have chosen an (arbitrary) orientation for each edge, and for $e^+ = e(u, v)$ we call u the negative and v the positive end of the edge e.

Definition 2 *The* incidence matrix $D = (d_{ij})$ *of the graph G is the $(l \times m)$-matrix defined by*

$$d_{ij} = \begin{cases} +1, & \text{if } v_i \text{ is the positive end of } e_j \\ -1, & \text{if } v_i \text{ is the negative end of } e_j \\ 0, & \text{otherwise.} \end{cases}$$

For a (k^+, k^-)-path we call $k^+ - k^-$ the *transition vector* of the path. The transition vector of a cycle is an m-dimensional column vector c satisfying $c(i) = 1$ if e_i^+ belongs to the cycle, $c(i) = -1$ if e_i^- belongs to the cycle and $c(i) = 0$ otherwise. It is easy to see that $Dc = 0$. This motivates the following definition.

Definition 3 *The* cycle space *of the graph is the set of all column vectors $c \in \mathbb{Z}^m$ with $Dc = 0$.*

Fact 1 *The cycle space is an $(m - l + 1)$-dimensional vector space.*

Fact 2 *Let T be a spanning tree of the graph G. For each edge e of the graph not belonging to T there exists a unique cycle c_e that contains e and edges from T only. The transition vectors c_e of these $m - l + 1$ cycles form a basis of the cycle space.*

For a proof of these Facts, we refer to [Big93], sections 4 and 5.

Definition 4 *The collection of cycles induced by a spanning tree as stated in Fact 2 is called a* fundamental system of cycles *of the graph G.*

We call a path $(u_0, e_1, u_2, \dots, e_n, u_n)$ without cycles that contains no edge more than once, a *simple path* from u_0 to u_n. A *loop* is a path $(u_0, e_1, u_1, e_2, u_2)$ with $u_0 = u_2$ and $e_1 = e_2$. Clearly, the transition vector of a loop equals 0.

A (k^+, k^-)-path p from v_0 to v_1 can be decomposed in cycles, loops and a simple path from v_0 to v_1. Clearly, the transition vector $k^+ - k^-$ of the path equals the sum of the transition vectors of the cycles, loops and the simple path. Denote the transition vector of the simple path by s. Let $\{c_1, \dots, c_{m-l+1}\}$ be a basis of the cycle space. Then the sum of the transition vectors of the cycles can be written in exactely one way as a linear combination of this basis, so we get

$$k^+ - k^- = \sum_{i=1}^{m-l+1} j_i c_i + s \tag{8}$$

with integers j_i. Suppose we decompose the path p in a different way in cycles, loops and a simple path from v_0 to v_1, so that we get a representation $k^+ - k^- = \sum_{i=1}^{m-l+1} j'_i c_i + s'$. Then $s' = s + (s' - s)$ and $s' - s$ is the transition vector of paths with the same starting and endpoint, which can be decomposed in cycles and loops, and we get $k^+ - k^- = \sum_{i=1}^{m-l+1} j''_i c_i + s$. Since $\{c_i\}$ is a basis of the cycle space, $j_i = j''_i$ for all i, and we have shown the following lemma.

Lemma 7 *Let $\{c_1, \dots, c_{m-n+1}\}$ be a basis of the cycle space, and let s be the transition vector of a simple path from v_0 to v_1. Given k^+ und k^- such that a (k^+, k^-)-path from v_0 and v_1 is possible there exist uniquely determined integers $j_1, \dots, j_{m-l+1}$ such that (8) holds.*

Definition 5 *The uniquely determined integers $j := (j_1, \dots, j_{m-l+1})$ from Lemma 7 are called the* cycle numbers *of the (k^+, k^-)-path (relative to the basis $\{c_1, \dots, c_{m-n+1}\}$).*

From (8) and $k = k^+ + k^-$, we conclude

$$k^+ = \frac{1}{2}\left(k + s + \sum_{i=1}^{m-l+1} j_i c_i\right) \text{ and } k^- = \frac{1}{2}\left(k - s - \sum_{i=1}^{m-l+1} j_i c_i\right). \tag{9}$$

Given k, we want to determine all possible decompositions $k = k^+ + k^-$ such that a (k^+, k^-)-path exists. Let s be the transition vector of a simple path from v_0 to v_1, and let $j_1, \dots, j_{m-l+1}$ be integers such that all components of k^+ and k^- defined by (9) are non-negative integers. We claim that there exists a (k^+, k^-)-path. To show this we first verify (3) for the number of departures and arrivals induced by k^+, k^-. Each cycle contributes the same number of arrivals and departures to a vertex, so the same is true for any path with transition vector $\sum_{i=1}^{m-l+1} j_i c_i$. The simple path contributes an additional departure to v_0, and an additional arrival to v_1. So (3) is satisfied. Together with (4) and (5) it follows that the assumptions of Lemma 5 are satisfied, and we have shown that a (k^+, k^-)-path exists.

It remains to determine conditions on the cycle numbers j that are necessary and suffcient for k^+ and k^- defined by (9) to have non-negative integer-valued components. Let $\{c_1, \dots, c_{m-l+1}\}$ be a basis of the cycle space induced by a spanning tree as described in Fact 2. There exists for each c_i exactely one edge e_{l_i} in the corresponding cycle that does not belong to the spanning tree.

Lemma 8 *Given k and a simple path s from v_0 to v_1, let k^+ and k^- be defined by (9). There exist (k^+, k^-)-paths from v_0 to v_1 iff the cycle numbers j satisfy the following conditions:*

1. *$j_i \equiv k_{l_i} + s_{l_i} \bmod 2$ for $i = 1, \dots m - l + 1$.*
2. *$|s + \sum_{i=1}^{m-l+1} j_i c_i| \leq k$ where the absolute value is taken component-wise, and the inequality is also component-wise.*

Proof. Since $n \equiv -n \bmod 2$ for any integer n, a necessary condition for (k^+, k^-) to have integer-valued components is $k \equiv s + \sum_{i=1}^{m-l+1} j_i c_i \bmod 2$, where the equivalence is componentwise. Since c_i is the only cycle with a non-zero l_i^{th} component, the l_i^{th} equation is $k_{l_i} \equiv s_{l_i} + j_i \bmod 2$, so the first condition is necessary. Since there exists a k-path, we know that for some integers j_i', $j_i' \equiv k_{l_i} + s_{l_i} \bmod 2$. From this equality it follows $j_i' \equiv j_i \bmod 2$, and we conclude $k \equiv s + \sum_{i=1}^{m-l+1} j_i' c_i \equiv s + \sum_{i=1}^{m-l+1} j_i c_i \bmod 2$. Hence the first condition is sufficient.

Clearly, the second condition is necessary and sufficient for k^+ and k^- to have non-negative components. □

Definition 6 *Integers j satisfying the conditions from Lemma 8 are called* possible cycle numbers *(relative to the basis $\{c_1, \dots, c_{m-n+1}\}$).*

5 Asymptotics

We recall Stirling's formula and the corresponding asymptotic formula for the Gamma function.

Lemma 9 *1. For $n \in \mathbb{N}$, $n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+\frac{\theta(n)}{12n}}$ with $0 < \theta(n) < 1$.*

2. For real-valued $x > 0$, $\Gamma(x) = \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x+\frac{\theta(x)}{12x}}$ with $0 < \theta(x) < 1$.

For a proof we refer to [Art64], p.24 formulas (3.9).

Let $\{c_1, \dots, c_{m-l+1}\}$ be a basis of the cycle space induced by a spanning tree as described in Fact 2. Let $j_n = (j_n(1), \dots, j_n(m-l+1))$ denote the cycle numbers at time n. We set $\beta_n = (\frac{j_n(1)}{\sqrt{n}}, \dots, \frac{j_n(m-l+1)}{\sqrt{n}})$,and write briefly $\beta_n = \frac{j_n}{\sqrt{n}}$. For $\epsilon > 0$ and $n \in \mathbb{N}$, we set

$$\begin{aligned} \Delta_\epsilon &:= \{x \in \Delta : x_e \geq \epsilon \text{ for all } e \in E\}, \\ \Delta_\epsilon^n &:= \left\{x \in \Delta_\epsilon : x_e = \frac{k_e}{n} \text{ with } k_e \in \mathbb{N}_0 \text{ for all } e \in E\right\}. \end{aligned}$$

Furthermore, we set

$$g(j) := s + \sum_{i=1}^{m-l+1} j_i c_i.$$

Fix $\gamma \in]\frac{1}{2}, \frac{2}{3}[$, and set

$$\begin{aligned} J_n &= \left\{\frac{j}{\sqrt{n}} : j \text{ is a possible cycle number}\right\} \\ Q_n &= \left\{\frac{j}{\sqrt{n}} \in J_n : |g_e(j)| \leq n^\gamma \text{ for all } e \in E\right\} \\ Q_n' &= J_n \setminus Q_n. \end{aligned}$$

By Lemma 1, $(k_e \bmod 2; e \in E)$ together with the starting point v_0 determines the endpoint v_1 uniquely. Lemma 8 gives the possible cycle numbers for k-paths. We denote by $\Delta_\epsilon^n \times Q_n(v_1)$ the subset of $\Delta_\epsilon^n \times Q_n$ consisting of all pairs $(\frac{k}{n}, \frac{j}{\sqrt{n}})$ such that a k-path with endpoint v_1 and cycle number j is possible. We write $d := \frac{m-l+1}{2}$ for the dimension of the cycle space multiplied by $\frac{1}{2}$.

Lemma 10 *1. Let $\psi(v,x,y) = 2^{m-1}x_v\varphi(x,y)$. For all $v_1 \in V$ and for all $\epsilon > 0$,*

$$\lim_{n\to\infty} \sup_{(x,y)\in\Delta_\epsilon^n\times Q_n(v_1)} \left| \frac{n^{m+d-1}}{(m-1)!} P(\alpha_n = x, \beta_n = y) - \psi(v_1,x,y) \right| = 0,$$

2. For all $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$\sup_{(x,y)\in\Delta_\epsilon^n\times Q'_n} n^{m+d-1} P(\alpha_n = x, \beta_n = y) \le C_\epsilon \exp\left(-\frac{1}{2}(\epsilon n)^{2\gamma-2}\right).$$

Proof. First we compute the asymptotic behaviour of the probability p of a fixed k-path given by (6). We use

$$\prod_{i=0}^{k-1}(a+i) = \frac{\Gamma(a+k)}{\Gamma(a)} \sim \frac{\sqrt{2\pi}}{\Gamma(a)} k^{k+a-\frac{1}{2}} e^{-k} \text{ as } k \to \infty$$

for the enumerator, and

$$\prod_{i=0}^{n-1}(a+2i) = 2^n \frac{\Gamma(\frac{a}{2}+n)}{\Gamma(\frac{a}{2})} \sim \frac{\sqrt{2\pi}}{\Gamma(\frac{a}{2})} 2^n n^{n+\frac{a-1}{2}} e^{-n} \text{ as } n \to \infty$$

for the denominator, to obtain

$$p \sim (2\pi)^{\frac{m-l}{2}} C \frac{\prod_{e\in E} k_e^{a_e-\frac{1}{2}}}{n_{v_0}^{\frac{a_{v_0}-1}{2}} \prod_{v\in V\setminus\{v_0\}} n_v^{\frac{a_v}{2}}} \frac{\prod_{e\in E} k_e^{k_e}}{\prod_{v\in V}(2n_v)^{n_v}} \tag{10}$$

with C given by (1) as $k_e \to \infty$ for all $e \in E$. The convergence is uniform in k for $\frac{k}{n} \in \Delta_\epsilon$ because in this case $k_e \ge n\epsilon$.

Next we compute for $x = \frac{k}{n}$ the asymptotic behaviour of $P(\alpha_n = x, \beta_n = 0)$, the probability of all k-paths with cycle numbers equal to zero. If all cycle numbers equal zero, then by (9), $k^\pm = \frac{k\pm s}{2}$. In particular, $k_e^\pm$ differs from $\frac{k_e}{2}$ in absolute value by at most $\frac{1}{2}$. Lemma 6 gives the corresponding number of (k^+, k^-)-paths, so we get

$$P(\alpha_n = x, \beta_n = 0) = p \frac{\prod_{v\in V} n_v!}{\prod_{e\in E} \frac{k_e+s}{2}! \frac{k_e-s}{2}!} \frac{\sum_{T\in\mathcal{T}} \prod_{e\in E(T)} \frac{k_e\pm s}{2}}{\prod_{v\in V\setminus\{v_1\}} n_v}.$$

Using Stirling's formula and (10) we obtain

$$\begin{aligned} P(\alpha_n = x, \beta_n = 0) \quad &\sim \quad p\frac{(2\pi)^{\frac{l}{2}}}{\pi^m} 2^n \frac{\prod\limits_{v\in V} n_v^{n_v+\frac{1}{2}}}{\prod\limits_{e\in E} k_e^{k_e+1}} \frac{\sum\limits_{T\in\mathcal{T}} \prod\limits_{e\in E(T)} k_e}{\prod\limits_{v\in V\setminus\{v_1\}} 2n_v} \\ &= \quad C\left(\frac{2}{\pi}\right)^{\frac{m}{2}} \frac{\prod\limits_{e\in E} k_e^{a_e-\frac{3}{2}}}{n_{v_0}^{\frac{a_{v_0}-2}{2}} \prod\limits_{v\in V\setminus\{v_0\}} n_v^{\frac{a_v-1}{2}}} \frac{\sum\limits_{T\in\mathcal{T}} \prod\limits_{e\in E(T)} k_e}{\prod\limits_{v\in V\setminus\{v_1\}} 2n_v}, \end{aligned}$$

where again the convergence is uniform in k for $\frac{k}{n} \in \Delta_\epsilon$. With $\frac{k_e}{n} = x_e$ and $\frac{2n_v}{n} = x_v$, we obtain

$$P(\alpha_n = x, \beta_n = 0) \sim n^{\frac{l-3m+1}{2}} C \frac{2^{\frac{1}{2}(m-l-1+\sum_{v\in V} a_v)}}{\pi^{\frac{m}{2}}} \frac{x_{v_1} \prod\limits_{e\in E} x_e^{a_e-\frac{3}{2}}}{x_{v_0}^{\frac{a_{v_0}}{2}} \prod\limits_{v\in V\setminus\{v_0\}} x_v^{\frac{a_v+1}{2}}} \sum_{T\in\mathcal{T}} \prod_{e\in E(T)} x_e.$$

It remains to compute for $y = \frac{j}{\sqrt{n}}$ the asymptotic behaviour of

$$\frac{P(\alpha_n = x, \beta_n = y)}{P(\alpha_n = x, \beta_n = 0)} = \prod_{e\in E} \frac{\frac{k_e}{2}!\frac{k_e}{2}!}{\frac{k_e-g_e}{2}!\frac{k_e+g_e}{2}!} \frac{\sum\limits_{T\in\mathcal{T}} \prod\limits_{e\in E(T)} \frac{k_e\pm g_e}{2}}{\sum\limits_{T\in\mathcal{T}} \prod\limits_{e\in E(T)} \frac{k_e\pm s}{2}}. \tag{11}$$

By Stirling's formula,

$$\frac{\frac{k_e}{2}!\frac{k_e}{2}!}{\frac{k_e-g_e}{2}!\frac{k_e+g_e}{2}!} \sim \frac{k_e}{\sqrt{k_e^2-g_e^2}} \left(1-\frac{g_e}{k_e}\right)^{-\frac{k_e-g_e}{2}} \left(1+\frac{g_e}{k_e}\right)^{-\frac{k_e+g_e}{2}} \tag{12}$$

and the convergence is uniform in k and j for $(\frac{k}{n}, \frac{j}{\sqrt{n}}) \in \Delta_\epsilon \times Q_n$ because in this case $k_e \pm g_e \geq k_e - n^\gamma \geq \epsilon n - n^\gamma$ and $\gamma < 1$. To compute the asymptotic behaviour of the last term, we take logarithms and use Taylor's formula

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3\theta(x)^3} \text{ with } |\theta(x) - 1| < x \text{ for } |x| < 1,$$

to obtain

$$\begin{aligned}
\left(1-\frac{g_e}{k_e}\right)^{-\frac{k_e-g_e}{2}} &= \left(1+\frac{g_e}{k_e}\right)^{-\frac{k_e+g_e}{2}} \\
&= \exp\left[-\frac{k_e-g_e}{2}\log\left(1-\frac{g_e}{k_e}\right)-\frac{k_e+g_e}{2}\log\left(1+\frac{g_e}{k_e}\right)\right] \\
&= \exp\left[-\frac{g_e^2}{2k_e}+r(g_e,k_e)\right] \\
&\qquad \text{with } r(g,k)=\frac{(k-g)g^3}{6k^3\theta(-\frac{g}{k})}-\frac{(k+g)g^3}{6k^3\theta(\frac{g}{k})}.
\end{aligned}$$

For $(\frac{k}{n},\frac{j}{\sqrt{n}})\in\Delta_\epsilon\times Q_n$ we have $|\frac{g_e}{k_e}|\le\frac{n^\gamma}{\epsilon n}\le\frac{1}{2}$ for n sufficiently large, so $\frac{1}{2}\le\theta(\pm\frac{g_e}{k_e})\le\frac{3}{2}$ and consequently,

$$\exp\left(-\frac{2n^{3\gamma-2}}{3\epsilon^2}\right)\le\exp(r(j,k))\le\exp\left(\frac{2n^{3\gamma-2}}{3\epsilon^2}\right).$$

Thus $\exp(r(j,k))\to 1$ uniformly on $\Delta_\epsilon\times Q_n$, and we conclude

$$\frac{P(\alpha_n=x,\beta_n=y)}{P(\alpha_n=x,\beta_n=0)}\sim\prod_{e\in E}\exp\left(-\frac{g_e^2}{2k_e}\right)=\exp\left(-\sum_{e\in E}\frac{g_e^2}{2nx_e}\right).$$

Let $\text{diag}(\frac{1}{x_e};e\in E)$ denote the $|E|$-dimensional diagonal matrix with entries $\frac{1}{x_e}$, and set $N=(c_1,\dots,c_{m-l+1})^t$, so N is an $(m-l+1)\times m$-matrix. Then

$$\sum_{e\in E}\frac{g_e^2}{nx_e}=\frac{g^t}{\sqrt{n}}\text{diag}\left(\frac{1}{x_e};e\in E\right)\frac{g}{\sqrt{n}}.$$

Observe that $\frac{g}{\sqrt{n}}=\frac{s}{\sqrt{n}}+\sum_{i=1}^d\frac{j_i}{\sqrt{n}}c_i=\frac{s}{\sqrt{n}}+N^ty$ and

$$N\text{diag}\left(\frac{1}{x_e};e\in E\right)N^t=A(x),$$

where $a_{i,i}(x)=\sum_{e\in c_i}\frac{1}{x_e}$ and $a_{i,j}(x)=\sum_{e\in c_i\cap c_j}\pm\frac{1}{x_e}$ for $i\neq j$ and the signs in the last sum are chosen to be $+1$ or -1 depending on whether the edge e has in c_i and c_j the same orientation or not. It follows $\sum_{e\in E}\frac{g_e^2}{nx_e}=y^tA(x)y+O(\frac{1}{\sqrt{n}})$, and we conclude

$$\frac{P(\alpha_n=x,\beta_n=y)}{P(\alpha_n=x,\beta_n=0)}\sim\exp\left(-\frac{1}{2}y^tA(x)y\right).$$

Putting everything together, the first part of Lemma 10 follows.

For the proof of the second part of Lemma 10, we observe that the error term $e^{\frac{\theta(n)}{12n}}$ in Stirling's formula is uniformly bounded: $1 \leq e^{\frac{\theta(n)}{12n}} \leq e^{\frac{1}{12}}$. Thus using the same argument as before, we get

$$n^{m+d-1} P(\alpha_n = x, \beta_n = 0) \leq \text{const}\varphi(x, 0).$$

It remains to analyze (11). Clearly,

$$\frac{P(\alpha_n = x, \beta_n = y)}{P(\alpha_n = x, \beta_n = 0)} \leq \prod_{e \in E} \frac{\frac{k_e}{2}! \frac{k_e}{2}!}{\frac{k_e - g_e}{2}! \frac{k_e + g_e}{2}!}.$$

Clearly, the factors are decreasing in g_e and ≤ 1. For $(\frac{k}{n}, \frac{j}{\sqrt{n}}) \in \Delta_\epsilon \times Q'_n$, there exists an $e \in E$ with $|g_e(j)| > n^\gamma$, so

$$\begin{aligned} \frac{P(\alpha_n = x, \beta_n = y)}{P(\alpha_n = x, \beta_n = 0)} &\leq \frac{\frac{k_e}{2}! \frac{k_e}{2}!}{\frac{k_e - n^\gamma}{2}! \frac{k_e + n^\gamma}{2}!} \\ &\leq \text{const} \frac{1}{\sqrt{1 - \frac{n^{2\gamma-2}}{\epsilon}}} \exp\left(-\frac{n^{2\gamma-1}}{2\epsilon}\right) \exp\left(\frac{2n^{3\gamma-2}}{3\epsilon^2}\right), \end{aligned}$$

and the second part of Lemma 10 follows. □

6 Proof of Theorem 1

First we prove the weak convergence of $\{(\alpha_n, \beta_n)\}$. Let $\epsilon > 0$, and let $f : \Delta_\epsilon \times \mathbb{R}^{2d} \to \mathbb{R}$ be bounded and continuous. We set for $M > 0$,

$$Q_{n,M} = \left\{ \frac{j}{\sqrt{n}} \in J_n : |j_i| \leq M\sqrt{n} \text{ for all } i \in \{1, \dots, m - l + 1\} \right\}.$$

For all $M > 0$ we get

$$\begin{aligned} E[f(\alpha_n, \beta_n)] &= E[f(\alpha_n, \beta_n); \beta_n \in Q_{n,M}] + E[f(\alpha_n, \beta_n); \beta_n \in Q_n \setminus Q_{n,M}] \\ &\quad + E[f(\alpha_n, \beta_n); \beta_n \in Q'_n]. \end{aligned} \tag{13}$$

The first term in (13) equals

$$\sum_{v_1 \in V} E[f(\alpha_n, \beta_n); (\alpha_n, \beta_n) \in \Delta_\epsilon^n \times Q_{n,M}(v_1)]$$

Using the first part of Lemma 10 and the fact that the number of points in $\Delta_\epsilon^n \times Q_{n,M}(v_1)$ is of the order n^{m-1+d}, we see that the last expressions has the same asymptotic behaviour as

$$\sum_{v_1 \in V} \frac{(m-1)!}{n^{m-1+d}} \sum_{(x,y)\in\Delta_\epsilon^n \times Q_{n,M}(v_1)} f(x,y)\psi(v_1,x,y).$$

This is a finite sum of Riemannian sums. Taking into account that the individual cycle numbers are either odd or even (condition 1 from Lemma 8) gives a factor 2^{-2d}. There are roughly $\frac{((1-\epsilon)n)^{m-1}}{(m-1)!}$ points in Δ_ϵ^n and points of the order n^d in $Q_{n,M}(v_1)$. The restrictions mod 2 on the number of crossings of the edges gives a factor $2^{-(m-(m-l+1))} = 2^{1-l}$. Using this we see that taking the limit $n \to \infty$ and then $M \to \infty$ yields

$$\begin{aligned} \sum_{v_1 \in V} \frac{(1-\epsilon)^{m-1}}{2^m} & \int_{\Delta_\epsilon}\int_{\mathbb{R}^{2d}} f(x,y)\psi(v_1,x,y)dy\sigma(dx) \\ &= \frac{(1-\epsilon)^{m-1}}{2^m}\int_{\Delta_\epsilon}\int_{\mathbb{R}^{2d}} f(x,y)\sum_{v_1\in V}\psi(v_1,x,y)dy\sigma(dx). \end{aligned}$$

Here we denote by σ normalized Lebesgue measure on Δ.

Observe that part 1 of Lemma 10 implies

$$n^{m+d-1}P(\alpha_n = x, \beta_n = y) \leq 2\psi(v_1,x,y)$$

for all $(x,y) \in \Delta_\epsilon^n \times Q_n(v_1)$ and all n sufficiently large. Using this, the integrability of ψ on $\Delta_\epsilon \times \mathbb{R}^{2d}$ and the fact that $\Delta_\epsilon \times \{y \in \mathbb{R}^{2d} : |y_i| > M \text{ for some } i \in \{1,\dots,m-l+1\}\} \downarrow \emptyset$ as $M \to \infty$, we can argue in a similar way as above that the second term in (13) tends to zero if we take the limit $n \to \infty$ and then $M \to \infty$.

Using the last part of Lemma 10 and the fact that the number of points in $\Delta_\epsilon^n \times Q_n'$ is at most of the order n^{m-1+2d} it follows easily that the last term in (13) tends to zero as $n \to \infty$. So we have shown

$$\begin{aligned} \lim_{n\to\infty} E[f(\alpha_n,\beta_n)] &= \frac{(1-\epsilon)^{m-1}}{2^m}\int_{\Delta_\epsilon}\int_{\mathbb{R}^{2d}} f(x,y)\sum_{v_1\in V}\psi(v_1,x,y)dy\sigma(dx) \\ &= (1-\epsilon)^{m-1}\int_{\Delta_\epsilon}\int_{\mathbb{R}^{2d}} f(x,y)\varphi(x,y)dy\sigma(dx). \end{aligned}$$

Here we have used $\sum_{v\in V}\psi(v,x,y) = 2^m\varphi(x,y)\sum_{v\in V}x_v$ and $\sum_{v\in V}x_v = 2$.

To prove weak convergence of $\{(\alpha_n, \beta_n)\}$, let $h : \Delta \times \mathbb{R}^{2d} \to \mathbb{R}$ be bounded and continuous. For all $\epsilon > 0$,

$$E[h(\alpha_n, \beta_n)] = E[h(\alpha_n, \beta_n); \alpha_n \in \Delta_\epsilon] + E[h(\alpha_n, \beta_n); \alpha_n \in \Delta \setminus \Delta_\epsilon]. \quad (14)$$

For $h \geq 0$, we have by the above argument and by the monotone convergence theorem,

$$\begin{aligned} \lim_{\epsilon \to 0} \lim_{n \to \infty} E[h(\alpha_n, \beta_n); \alpha_n \in \Delta_\epsilon] &= \lim_{\epsilon \to 0} (1-\epsilon)^{m-1} \int_{\Delta_\epsilon} \int_{\mathbb{R}^{2d}} h(x,y)\varphi(x,y) dy \sigma(dx) \\ &= \int_{\Delta} \int_{\mathbb{R}^{2d}} h(x,y)\varphi(x,y) dy \sigma(dx). \end{aligned}$$

Since $0 \leq E[h(\alpha_n, \beta_n)] \leq ||h||_\infty$, the last integral is finite. So for general h we get the same decomposing h in its positive and negative part.

The second term in (14) is dominated by a constant times the probability $P(\alpha_n \in \Delta \setminus \Delta_\epsilon)$, and

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} P(\alpha_n \in \Delta \setminus \Delta_\epsilon) = \lim_{\epsilon \to 0} P(\alpha_\infty \in \Delta \setminus \Delta_\epsilon) = P(\alpha_\infty \in \partial\Delta) = 0,$$

so this term converges to zero as $n \to \infty$ and $\epsilon \to 0$, and we have shown the weak convergence.

Almost sure convergence of $\{\alpha_n; n \in \mathbb{N}\}$ follows form Lemma 3 and the identity $\alpha_n = \alpha_n^+ + \alpha_n^-$. To compute $\phi(x)$, we have to integrate $\varphi(x, y)$ with respect to y. The only factor that depends on y is the exponential term, and

$$\int_{\mathbb{R}^{2d}} \exp(-\frac{1}{2} y^t A(x) y) dy = (2\pi)^d [\det A(x)]^{-\frac{1}{2}}.$$

Now the crucial step is to apply the following matrix-tree-theorem.

Theorem 3 ([Mau76], p. 145, theorem 3')

$$detA(x) = \sum_{T \in \mathcal{T}} \prod_{e \notin E(T)} \frac{1}{x_e}$$

This equation can be rewritten as

$$\det A(x) = \frac{\sum\limits_{T \in \mathcal{T}} \prod\limits_{e \in E(T)} x_e}{\prod\limits_{e \in E} x_e},$$

and putting everything together, it follows that ϕ has the form indicated in Theorem 1. □

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