High-Order Space-Time Accurate Defect Correction Schemes for Semilinear Singularly Perturbed Parabolic Problems with Convection*

Programme

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1. Introduction

A boundary value problem for semilinear singularly perturbed parabolic PDEs with convection is considered on a vertical strip; the highest space derivatives in the equation are multiplied by the perturbation parameter ε. When ε tends to zero, the solution of such a problem typically exhibit a boundary layer. It is well known that traditional numerical methods give errors in the solutions which grow and become comparable with the exact solution when ε become small. That is why special numerical methods, i.e. methods for which errors are independent of the parameter ε (or in short, ε-uniform, or robust, methods), are very important. At present robust numerical methods were developed for various problems with boundary layers (see, e.g., [1, 2]). However, for convection-diffusion problems the order of ε-uniform accuracy for these numerical methods is too low and does not exceed one. This is a main restriction to use such methods.

The defect correction method was applied in order to increase accuracy of solutions with respect to the time variable for nonstationary singularly perturbed boundary value problems of reaction-diffusion in [3, 4] and convection-diffusion in [5, 6].

The nonlinearity of the discrete problem requires a special approach in order to solve this problem. Here we use such finite difference schemes, where the unknown function, which is involved in the nonlinear term, is taken from the previous time level. The defect correction technique, based on the results from [7] where accuracy of solutions with respect to the time and space variables was increased for parabolic singularly perturbed convection-diffusion problems, allows us to construct the improved schemes, whose solutions converge ε-uniformly at the rate $\mathcal{O}(N_1^{-k}\ln^k N_1 + N_2^{-k} + N_0^{-k_0})$, $k = 1, 2$, $k_0 = 1, 2, 3$, where N_1+1 and N_0+1 denote the number of mesh points in x_1 and t respectively, and N_2+1 denotes the number of mesh points in x_2-axis on the segment of unit length. The efficiency of the constructed high-order accurate schemes is illustrated by adequate numerical experiments.

2. The studied class of initial boundary value problems

On the domain $G = D \times (0, T]$, $D = (0,1) \times I\!R$ with the boundary $S = \overline{G} \setminus G$ we consider the following singularly perturbed parabolic equation[1]:

$$L_{(2.1)}\,(u(x,t)) = 0, \qquad (x,t) \in G, \tag{2.1a}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{2.1b}$$

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[1] The notation is such that the operator $L_{(a.b)}$ is first introduced in equation $(a.b)$.

Here $\qquad L_{(2.1)}\,(u(x,t)) = L^{(2)}_{(2.1)} u(x,t) + L^{(0)}_{(2.1)} (u(x,t)),$

$$L^{(2)}_{(2.1)} u(x,t) \;\equiv\; \left\{ \varepsilon \sum_{s=1,\,2} a_s(x,t) \frac{\partial^2}{\partial {x_s}^2} + \sum_{s=1,\,2} b_s(x,t) \frac{\partial}{\partial x_s} - c(x,t) - p(x,t) \frac{\partial}{\partial t} \right\} u(x,t),$$

$$L^{(0)}_{(2.1)}\,(u(x,t)) \;\equiv\; -g\,(x,t,\,u(x,t))\,.$$

For $S = S_0 \cup S^L$, we distinguish the lateral boundary $S^L = \{(x,t)\colon\; x_1 = 0 \text{ or } x_1 = 1,\, 0 < t \le T\}$ and the initial boundary $S_0 = \{(x,t)\colon\; x \in [0,1] \times I\!R,\; t = 0\}$. In (2.1) $a_s(x,t)$, $b_s(x,t)$, $c(x,t)$, $p(x,t)$, $(x,t) \in \overline{G}$, $g(x,t,\,u)$, $(x,t,\,u) \in \overline{G} \times I\!R$ and $\varphi(x,t)$, $(x,t) \in S$ are sufficiently smooth and bounded functions satisfying $\; a_s(x,t) \ge a_0 > 0, \;\; b_1(x,t),\; |b_2(x,t)| \ge b_0 > 0, \;\; p(x,t) \ge p_0 > 0,$ $c(x,t) \ge 0, \;\; (x,t) \in \overline{G}, \;\; \left| \dfrac{\partial}{\partial u}\, g(x,t,\,u) \right| \le g^0, \;\; (x,t,\,u) \in \overline{G} \times I\!R; \;\; s = 1,\,2.$ The perturbation parameter ε may take any values from $(0,1]$. When ε tends to zero, the solution exhibits a regular boundary layer in a neighbourhood of the outflow boundary $S^L_1 = \{(x,t)\colon\; x_1 = 0,\; 0 \le t \le T\}$.

3. The ε-uniformly convergent scheme

Here we discuss an ε-uniformly convergent method for problem (2.1) by taking a special mesh condensed in the neighbourhood of the boundary layer. The way to construct the mesh is the same as in [1]–[7]. More specifically, we take

$$\overline{G}_h = \overline{\omega}_1(\sigma) \times \omega_2 \times \overline{\omega}_0\;, \tag{3.1a}$$

where $\overline{\omega}_0$ and ω_2 are uniform meshes on $[0,T]$ and in the x_2-axis with step-sizes $\tau = T/N_0$ and $h_2 = N_2^{-1}$ respectively, and $\overline{\omega}_1 = \overline{\omega}_1(\sigma)$ is a special *piecewise* uniform mesh of nodal points, x_1^i, in $[0,1]$, depending on the parameter $\sigma \in I\!R$ which depends on ε and N_1; N_1 and N_0 are the numbers of intervals in the meshes $\overline{\omega}$ and $\overline{\omega}_0$ respectively, and N_2 is the number of intervals on a unit segment in the x_2-axis. We choose

$$\sigma = \sigma_{(3.1)}(\varepsilon,\, N_1) = \min\left\{\, 1/2,\; m^{-1}\, \varepsilon\, \ln N_1 \,\right\}, \tag{3.1b}$$

where m is an arbitrary number from the interval $(0, m_0)$, $m_0 = \min_{\overline{G}} \left[a_1^{-1}(x,t)\, b_1(x,t) \right]$. The mesh $\overline{\omega}_1(\sigma)$ is constructed as follows. The interval $[0,1]$ is divided in two parts $[0,\sigma]$, $[\sigma,1]$, $\sigma \le 1/2$. In each part we use a uniform mesh, with $N_1/2$ subintervals in $[\,0,\sigma\,]$ and $[\,\sigma,1\,]$. We define $h_1^i = x_1^{i+1} - x_1^i$, $h_1 = \max_i h_1^i$, $h_1 \le M/N_1$, $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$.

For problem (2.1) we use the difference scheme [8]

$$\Lambda_{(3.2)}\,(z(x,t)) = 0, \;\;\; (x,t) \in G_h, \;\;\;\; z(x,t) = \varphi(x,t), \;\;\; (x,t) \in S_h. \tag{3.2}$$

Here $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{S}_h$, $\quad \Lambda_{(3.2)}\,(z(x,t)) = \Lambda^{(2)}_{(3.2)} z(x,t) + \Lambda^{(0)}_{(3.2)}\,(z(x,t))$,

$$\Lambda^{(2)}_{(3.2)} z(x,t) \;\equiv\; \Big\{ \varepsilon\, [\, a_1(x,t)\, \delta_{\overline{x_1}\widehat{x_1}} + a_2(x,t)\, \delta_{x_2\,\overline{x_2}}\,] + b_1(x,t)\, \delta_{x_1} + {} + \left[\, b_2^+(x,t)\, \delta_{x_2} + b_2^-(x,t)\, \delta_{\overline{x_2}} \,\right] - c(x,t) - p(x,t) \delta_{\overline{t}} \Big\} z(x,t),$$

$$\Lambda^{(0)}_{(3.2)}\,(z(x,t)) \;\equiv\; -g(x,t,\,z(x,t)), \;\;\; (x,t) \in G_h,$$

$\delta_{\overline{x_1}\widehat{x_1}}\, z(x,t) \;=\; 2\left(h_1^{i-1} + h_1^i\right)^{-1} (\delta_{x_1} z(x,t) \;-\; \delta_{\overline{x_1}}\, z(x,t))), \quad \delta_{\overline{x_1}}\, z(x,t)) \;=\; \left(h_1^{i-1}\right)^{-1} (z(x,t) \;-\; z(x_1^{i-1}, x_2, t))$, $\delta_{x_1} z(x,t) = \left(h_1^i\right)^{-1} (z(x_1^{i+1}, x_2, t) - z(x,t))$, $\delta_{\overline{t}}\, z(x,t) = \tau^{-1}(z(x,t) - z(x,t-\tau))$, $x \;=\; (x_1^i, x_2)$; operators $\delta_{x_2} z(x,t)$ and $\delta_{\overline{x_2}}\, z(x,t)$ are defined by a similar way; $\delta_{x_s} z(x,t)$

and $\delta_{\overline{x_s}}\, z(x,t)$, $\delta_{\overline{t}}\, z(x,t)$ are the forward and backward differences, and the difference operators $\delta_{\overline{x_1}\widehat{x_1}}\, z(x,t)$ and $\delta_{x2\,\overline{x_2}}\, z(x,t)$ are approximations of the operators $(\partial^2/\partial x_1^2)u(x,t)$ and $(\partial^2/\partial x_2^2)u(x,t)$, $\delta_{x2\,\overline{x_2}}\, z(x,t)$ is the second difference derivative with respect to x_2 on an uniform mesh; $b_s^+(x,t) = 2^{-1}\ (b_s(x,t) + |b_s(x,t)|)$, $b_s^-(x,t) = 2^{-1}\ (b_s(x,t) - |b_s(x,t)|)$, $s = 1, 2$.

For the one-dimensional problem in [5] the theorem is given which states that the discrete solution of the linear convection-diffusion problem converges ε-uniformly. Using the same technique as in [5], we justify that the solution of (3.2), (3.1) converges ε-uniformly to the solution of (2.1) and the following error estimate holds:

$$|\, u(x,t) - z(x,t)\, | \le\ M\left[N_1^{-1}\ln N + N_2^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h, \tag{3.3}$$

i.e. the convergence order of such a scheme does not exceed 1.

4. High-order schemes based on defect correction

Defect correction techniques proved to be efficient for constructing ε-uniformly convergent schemes of high-order accuracy for singularly perturbed parabolic linear problems (see, e.g., [3–7]). Therefore, this technique seems attractive to be used also for the construction of high-order accurate schemes in x and t for semilinear problems under consideration.

The idea of the defect correction method is the following. For the difference scheme (3.2), (3.1) the error in the approximation of the partial derivative $(\partial/\partial t)\, u(x,t)$ is caused by the divided difference $\delta_{\overline{t}}\, z(x,t)$ and is associated with the truncation error given by

$$\frac{\partial u}{\partial t}(x,t) - \delta_{\overline{t}}\, u(x,t) = 2^{-1}\,\tau\,\frac{\partial^2 u}{\partial t^2}(x,t) - 6^{-1}\,\tau^2\,\frac{\partial^3 u}{\partial t^3}(x,t-\theta), \quad \theta \in [0,\tau].$$

The truncation error for the $\delta_{x_1} z(x,t)$ is defined by the formula

$$\frac{\partial u}{\partial x_1}(x,t) - \delta_{x_1}\, u(x,t) = -2^{-1} h_1^i\, \frac{\partial^2}{\partial x_1^2}\, u(x_1+\theta_1, x_2, t), \quad \theta_1 \in [0, h_1^i], \quad x_1 = x_1^i,$$

and similar formulae can be written out for the differences $\delta_{x_2} z(x,t)$ and $\delta_{\overline{x_2}} z(x,t)$. Therefore, for the approximation of $(\partial/\partial t)u(x,t)$ we now use the expression $\delta_{\overline{t}}u(x,t) + \tau\delta_{\overline{tt}}u(x,t)/2$, where $\delta_{\overline{tt}}u(x,t) \equiv \delta_{t\overline{t}}u(x,t-\tau)$, $\delta_{t\overline{t}}u(x,t)$ is the second central divided difference, and for the approximation of $(\partial/\partial x_s)u(x,t)$ we use the relation $\delta_{x_1}u(x,t) - h_1^i\delta_{\overline{x_1}\widehat{x_1}}u(x,t)/2$ for $s = 1$ and the relations $\delta_{x_2}u(x,t) - h_2\delta_{x_2\,\overline{x_2}}u(x,t)/2$, $\delta_{\overline{x_2}}u(x,t) + h_2\delta_{x_2\,\overline{x_2}}u(x,t)/2$ for $s = 2$. We can evaluate a better approximation than (3.2) by defect correction

$$\Lambda_{(3.2)}(z^c(x,t)) = 2^{-1}\,\tau\, p(x,t)\,\frac{\partial^2 u}{\partial t^2}(x,t) + 2^{-1}\, h_1^i\, b_1(x,t)\,\frac{\partial^2 u}{\partial x_1^2}(x,t), \quad x_1 = x_1^i,$$

where $x \in \omega_1 \times \omega_2$ and $t \in \overline{\omega}_0$, $z^c(x,t)$ is the "corrected" solution. Instead of $(\partial^2/\partial t^2)\, u(x,t)$ and $(\partial^2/\partial x_1^2)\, u(x,t)$, $(\partial^2/\partial x_2^2)\, u(x,t)$, we will use $\delta_{\overline{tt}}\, z(x,t)$ and $\delta_{\overline{x_1}\widehat{x_1}}\, z(x,t)$, $\delta_{x_2\overline{x_2}}\, z(x,t)$, respectively, where $z(x,t)$, $(x,t) \in \overline{G}_h$ is the solution of the difference scheme (3.2), (3.1).

For *fixed* values of ε, the new solution $z^c(x,t)$ has a consistency error with respect to x and t of order $\mathcal{O}(N_2^{-2} + N_1^{-2}\ln^2 N_1 + N_0^{-2})$. As we shall see further in the case of special piecewise uniform meshes (3.1) the order of *ε-uniform* convergence with respect to the variables x and t is $\mathcal{O}(N_1^{-2}\ln^2 N_1 + N_2^{-2} + N_0^{-2})$. In a similar way we can construct a difference approximation with a convergence order higher than two with respect to the variables x_2 and t ε-uniformly (see similar results for one-dimensional linear problems in [3]–[7]).

5. Schemes with improved convergence in time and space

5.1. On the mesh $\overline{G}_h$ we write the finite difference scheme (3.2) in the form

$$\Lambda_{(3.2)}\left(z^{(1)}(x,t)\right) = 0, \quad (x,t) \in G_h, \quad z^{(1)}(x,t) = \varphi(x,t), \quad (x,t) \in S_h \tag{5.1}$$

where $z^{(1)}(x,t)$ is the uncorrected solution. We define this scheme as the base scheme. Further we denote by $\delta_{k\bar{t}} z(x,t)$ the backward difference of order k:

$$\delta_{k\bar{t}}\, z(x,t) = \left(\delta_{k-1\,\bar{t}}\, z(x,t) - \delta_{k-1\,\bar{t}}\, z(x,t-\tau)\right)/\tau, \quad t \ge k\tau, \quad k \ge 1;$$
$$\delta_{0\bar{t}}\, z(x,t) = z(x,t), \quad (x,t) \in \overline{G}_h.$$

Using the idea from the previous section, we approximate boundary value problem (2.1) by the discrete problem

$$\begin{aligned} \Lambda_{(3.2)}\left(z^{[2,2]}(x,t)\right) &= \psi^{[1]}(x,t) + \psi^{(1)}(x,t), \quad (x,t) \in G_h, \\ z^{[2,2]}(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{5.2}$$

Here

$$\psi^{(1)}(x,t) \equiv \left\{ \begin{array}{ll} 2^{-1}\, p(x,t)\, \tau\, \dfrac{\partial^2}{\partial t^2}\, u(x,0), & t = \tau \\ 2^{-1}\, p(x,t)\, \tau\, \delta_{2\,\bar{t}}\, z^{(1)}(x,t), & t \ge 2\tau \end{array} \right\}, \quad (x,t) \in G_h;$$

the derivative $(\partial^2/\partial t^2)\, u(x,0)$ can be obtained from (2.1a), $\delta_{2\bar{t}}\, z^{(1)}(x,t)$ is the second divided difference of the solution $z^{(1)}(x,t)$ for the base problem (5.1), (3.1);

$$\psi^{[1]}(x,t) \equiv 2^{-1}\, \left[h_1^{i-1}\, b_1(x,t)\, \delta_{\overline{x_1}\widehat{x_1}} + h_2\, \left[b_2^{+}(x,t) - b_2^{-}(x,t)\right]\, \delta_{x_2\overline{x_2}}\right]\, z^{(1)}(x,t), \quad (x,t) \in G_h.$$

We call $z^{[2,2]}(x,t)$, $(x,t) \in G_h$ the solution of scheme (5.2), (5.1), (3.1). As opposed to mesh (3.1), for scheme (5.2), (5.1) we will use the special mesh

$$\overline{G}_h = \overline{G}_{h(3.1a)}, \tag{5.3a}$$

where $\overline{\omega}_1 = \overline{\omega}_{1(3.1a)}(\sigma)$ but under the different condition (as compared with (3.1b))

$$\sigma = \sigma_{(5.3)}(\varepsilon, N) = \min\{1/2,\ l\, m^{-1}\, \varepsilon\, \ln N_1\}, \tag{5.3b}$$

$m = m_{(3.1)}$, $l \ge 3$ is an arbitrary number.

For simplicity, we suppose

$$a_s(x,t) = a_s(x),\ b_s(x,t) = b_s(x), \quad (x,t) \in \overline{G}, \quad \varphi(x,t) = 0, \quad (x,t) \in S_0, \quad s = 1,\, 2. \tag{5.4}$$

Under condition (5.4) we have the ε-uniform estimate for the solution of problem (5.2), (5.3)

$$|u(x,t) - z^{[2,2]}(x,t)| \le M\, \left[N_1^{-2}\, \ln^2 N_1 + N_2^{-2} + N_0^{-2}\right], \quad (x,t) \in \overline{G}_h. \tag{5.5}$$

5.2. In order to find a solution of the finite difference scheme (3.2), (3.1) (scheme (5.2), (5.3)), it is required to solve a nonlinear discrete equation on each time level. It would be attractive to use such finite difference schemes, where the unknown function, which is involved in the nonlinear term, is taken from the previous time level [8]. Let us give such a scheme.

On mesh (3.1), for problem (2.1) we use the finite difference scheme

$$\begin{aligned} \Lambda\ (z(x,t)) &\equiv \Lambda^{(2)}_{(3.2)} z(x,t) + \Lambda^{(0)}_{(3.2)}(\breve{z}(x,t)) = 0, \quad (x,t) \in G_h, \\ z(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{5.6}$$

Here $\Lambda^{(0)}_{(3.2)}\, (\breve{z}(x,t)) = -g\,(x,t,\, z(x,t-\tau))$, $(x,t) \in G_h$, For solutions of the finite difference scheme (5.6), (3.1) we have the estimate

$$|\, u(x,t) - z(x,t)\,| \le M\, \left[N_1^{-1}\, \ln N_1 + N_2^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h.$$

Let $z^{(1)}(x,t)$, $(x,t)\in\overline{G}_h$ be a solution of the finite difference scheme (5.6), (3.1). Then we find the function $z^{[2,2]}(x,t)$, $(x,t)\in\overline{G}_{h(5.3)}$, by solving the discrete problem

$$\begin{aligned}\Lambda_{(5.6)}\left(z^{[2,2]}(x,t)\right) &= \psi^{[1]}(x,t)+\psi^{(1)}(x,t), \quad (x,t)\in G_h, \\ z^{[2,2]}(x,t) &= \varphi(x,t), \quad (x,t)\in S_h.\end{aligned} \tag{5.7a}$$

Here

$$\psi^{[1]}(x,t)=\psi^{[1]}_{(5.2)}(x,t;\,z^{(1)}(x,t)), \tag{5.7b}$$

$$\psi^{(1)}(x,t)\equiv\left\{\begin{array}{ll}\tau\left[2^{-1}p(x,t)\dfrac{\partial^2}{\partial t^2}u(x,0)-\dfrac{\partial}{\partial t}g(x,t,u(x,0))\dfrac{\partial}{\partial t}u(x,0)\right], & t=\tau\\ \tau\left[2^{-1}p(x,t)\delta_{2\bar t}z^{(1)}(x,t)-\dfrac{\partial}{\partial t}g(x,t,z^{(1)}(x,t))\delta_{\bar t}z^{(1)}(x,t)\right], & t\ge 2\tau\end{array}\right\},\ (x,t)\in G_h;$$

$z^{(1)}(x,t)$ in (5.7) is the solution of problem (5.6), (3.1). We call the function $z^{[2,2]}(x,t)$, $(x,t)\in G_h$ the solution of the difference scheme (5.7), (5.6), (3.1).

For the solution of the difference scheme (5.7), (5.6), (5.3) we have the following estimate, which is similar to estimate (5.5):

$$\left|u(x,t)-z^{[2,2]}(x,t)\right| \le M\ \left[N_1^{-2}\ln^2 N_1+N_2^{-2}+N_0^{-2}\right], \quad (x,t)\in\overline{G}_h.$$

The approach, based on the defect correction, allow us to construct a discrete solution convergent ε-uniformly at the rate $\mathcal{O}(N_1^{-2}\ln^2 N_1+N_2^{-k}+N_0^{-k})$, where k, $k_0\ge 2$.

6. Numerical results

We find the solution of the following boundary value problem

$$\begin{aligned}L_{(6.1)}u(x,t) &\equiv \left\{\varepsilon\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right\}u(x,t)-g(x,t,u(x,t))=0, \quad (x,t)\in G,\\ u(x,t) &= \varphi(x,t), \qquad (x,t)\in S.\end{aligned} \tag{6.1}$$

Here $\overline{G}=G\cup S$, $D\times(0,T]$, $D=(0,1)$, $g(x,t,u)=g_\alpha(x,t,u)=-f_0(x,t)+\alpha f_1(t,u)$, $(x,t,u)\in\overline{G}\times R$, α takes the value **0** or **1**; $f_0(x,t)=12\varepsilon t x^2+4tx^3-x^4-5t^4$, $f_1(t,u)=1+t^5u^4$,

$$u(0,t)=\alpha+t^4+t^5,\quad u(1,t)=\alpha+t+t^5,\quad 0<t\le T=1;\qquad u(x,0)=\alpha,\quad 0<x\le 1.$$

The solution of this problem is singular. When $\alpha=0$, the solution of problem (6.1) is linear; such a problem was considered in [7]. The analytical solution of problem (6.1) used for the computation of the errors in the approximate solution is unknown. We find the maximum pointwise errors $E(N,N_0,\varepsilon)$ for the functions $z^{(1)}(x,t)$ and $z^{[2,2]}(x,t)$ by the formulae

$$E(N,N_0,\varepsilon)=\max_{(x,t)\in\overline{G}_h}\left|z^{(1)}(x,t)-u^*(x,t)\right|, \tag{6.2}$$

$$E(N,N_0,\varepsilon)=\max_{(x,t)\in\overline{G}_h}\left|z^{[2,2]}(x,t)-u^{**}(x,t)\right|. \tag{6.3}$$

Here $u^*(x,t)$ and $u^{**}(x,t)$ are linear interpolants obtained from the reference solution $u_\varepsilon^{2048}(x,t)$ corresponding to the numerical solution $z^{(1)}(x,t)$ and $z^{[2,2]}(x,t)$ for $N=N_0=2048$, $N=2^i$, $i=2,3,\ldots,10$, $N_0=2^j$, $j=2,3,\ldots,10$. We calculate the ratios of the maximum pointwise errors for the functions $z^{(1)}(x,t)$ and $z^{[2,2]}(x,t)$ as follows

$$R_{z^j}(N)=\frac{\overline{E}_{z^j}(2^{-1}N)}{\overline{E}_{z^j}(N)},\quad j=(1),\ [2,2],\quad \text{where}\ \ \overline{E}_{z^j}(N)=\max_\varepsilon E(N,N_0,\varepsilon),$$

and give them in Table 1. We can see from the table that the order of convergence is almost one for the function $z^{(1)}(x,t)$ and almost two for the function $z^{[2,2]}(x,t)$, which corresponds to the theoretical results (see estimates (3.3) and (5.5)).

Table 1: Ratios of the maximum pointwise errors for the functions $z^{(1)}$ and $z^{[2,2]}$

N	8	16	32	64	128	256	512	1024
$R_{z^{(1)}}(N)$	1.73	1.70	1.71	1.71	1.73	1.78	1.80	1.82
$R_{z^{[2,2]}}(N)$	2.58	2.97	3.14	2.85	2.93	3.06	3.16	3.13

Conclusion

1. In this paper we have shown theoretically that the use of a defect correction technique for solving a boundary value problem in the case of a semilinear singularly perturbed parabolic convection-diffusion equation allows us to construct effectively ε-uniformly convergent schemes with the second (up to a logarithmic factor) order of accuracy with respect to x_1 and with the second order of accuracy with respect to x_2 and t.

2. The numerical example is given where it is shown that for the improved scheme the rate of convergence with respect to the space and time variables is corresponding to the theoretical results.

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