THE USE OF DEFECT CORRECTION FOR THE SOLUTION OF A SINGULARLY PERTURBED O.D.E.

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ABSTRACT

The effect of a defect correction process with a central - and an upwind - difference operator is shown for a singularly perturbed two-point boundary-value problem.
A 'mixed defect correction process' is introduced which is both stable and accurate for the smooth components in the solution. Application in an adaptive procedure is mentioned.

1. INTRODUCTION

In this paper we describe an iterative method for the accurate solution of a singular perturbation problem (SPP). As a model problem for more complex situations we take the linear two-point boundary-value problem

$$(1.1)\qquad \varepsilon y'' + a_1 y' + a_2 y = f \quad \text{on } \Omega = (a,b),$$
$$0 < \varepsilon \ll 1, \qquad a_1 \neq 0,$$

with the Dirichlet boundary conditions $y(a) = y_a$, $y(b) = y_b$.
This problem is written in symbolic form as

$$L_\varepsilon y = f.$$

It is well-known, that for such problems with a strongly asymmetric differential operator, the usual discretizations are either unstable (central differences, finite element discretizations) or inaccurate (one-sided differences, artificial viscosity). Many methods are proposed to overcome these difficulties (see e.g. [1,8,9]). However, if we look for a discretization that is both accurate and direction independent (i.e. independent of the sign of a_1/ε), none of the available methods is appropriate.

In the following sections we first briefly show the disadvantages of the simple central and one-sided or artificial viscosity discretization. Then we study the combination of these discretizations in a straightforward defect correction (DCP) algorithm. Further, we show a combination of a "defect correction step" and a "smoothing step" to obtain a "mixed defect correction iteration" (MDCP), which solves the problem accurately $O(h^2)$ for the smooth components of the solution and with a numerical boundary layer of width $O(h)$.

It is the advantage of this method that it does *not* make use of particular a priori knowledge about the solution, such as the shape or the location of interior or boundary layers. It only uses the fact that a small parameter multiplies the highest derivative. The method has no directional bias and it is able to locate the special regions. Thus, it can be used in an adaptive procedure to refine the mesh in the non-smooth parts and, in this way, to resolve the special regions in the solution.

The same iteration method (MDCP) can also be applied to the solution of singularly perturbed partial differential equations, such as the convection diffusion equation [7]. In this paper, however, we restrict ourselves to a more detailed discussion of the model equation

(1.2) $$\varepsilon y'' + 2y' = 0,$$
$$y(0) = 0, \quad y(1) = 1.$$

2. CENTRAL AND UPWIND DIFFERENCES FOR THE SINGULARLY PERTURBED PROBLEM

The possible instability of the central difference discretization for the problem (1.1) is easily shown for the example (1.2). We take a uniform partition $\{0 = x_0 < x_1 <\ldots< x_N = 1\}$ of the interval (0,1). The solution of the central difference equation

(2.1) $$L^c_{h,\varepsilon} \equiv \varepsilon(y_{i+1}-2y_i+y_{i-1})/h^2 + (y_{i+1}-y_{i-1})/h = 0,$$
$$y_0 = 0, \quad y_N = 1,$$

reads

(2.2) $$y_i = (1-r^i)/(1-r^N),$$

with $r := (\varepsilon-h)/(\varepsilon+h)$. The exact solution of the differential equation is

(2.2) with $r := \exp(-2h/\varepsilon)$. From this we derive that

(2.3) $$|y(x_i)-y_i| \le c(h/\varepsilon)^2 \quad \text{for } (h/\varepsilon) \to 0.$$

For a fixed ε the method is $O(h^2)$ accurate, but the error may explode for $\varepsilon \to 0$ as is seen from (2.2), because (with even N)

$$\lim_{\varepsilon\to 0} y_i = i/N \quad \text{(for even i)},$$
$$\approx h/\varepsilon N \quad \text{(for odd i)},$$

and $\lim_{\varepsilon\to 0} y(x_i) = 1$ for $x_i \neq 0$. This large error is clearly due to instability since the eigenvalues of the discrete operator are

(2.4) $$\lambda_i = -\frac{\varepsilon}{h} + \frac{1}{h^2}\sqrt{\varepsilon^2-h^2}\cos\left(\frac{i\pi}{N}\right), \qquad i = 1,2,\ldots,N-1.$$

The simplest cure against this instability is the use of "upwind differencing"

(2.5) $$L^u_{h,\varepsilon} \equiv \varepsilon(y_{i+1}-2y_i+y_{i-1})/h^2 + 2(y_{i+1}-y_i)/h = 0,$$

i.e. one-sided differences are used to approximate the term $2y'$ in (1.2). The solution of this difference equation reads (2.2) with $r := \varepsilon/(\varepsilon+2h)$. Now we find

$$|y(x_i)-y_i| \le c(h/\varepsilon) \quad \text{for} \quad (h/\varepsilon) \to 0$$

and, moreover, we find $\lim_{\varepsilon\to 0} y_i = 1$ for all $i > 0$, i.e. the discrete solution has the same asymptotic $\varepsilon \to 0$ behaviour as the differential equation.

The success of the upwind scheme depends crucially on the upstream approximation in (2.5), an approximation $2(y_i-y_{i-1})/h$ would have yielded a completely wrong solution.

We note that the "upwind differencing" is equivalent to the central difference approximation with an enlarged ε:

(2.6) $$L^u_{h,\varepsilon} = L^c_{h,\varepsilon+h}.$$

In this form the difference scheme (2.5) is called the "artifical viscosity" discretization and in this form the discretization method is independent of the sign of a_1. The stability of this discretization, uniform in ε, easily follows from (2.4) and (2.6).

Another way to study the (in-)stability of the above difference schemes is by Fourier or Local Mode Analysis [2]. We consider the equation on $\Omega = \mathbb{R}$; we discretize the equation on a uniform partition $\{x_i = ih \mid i \in \mathbb{Z}\}$. As a forcing function we take the "mode"

$$f_h(jh) = e^{ijh\omega}, \qquad \mathrm{Re}(h\omega) \in [-\pi,\pi].$$

With

$$FT(y_h) = \hat{y}_h = \frac{h}{\sqrt{2\pi}} \sum_{j\in\mathbb{Z}} e^{-ijh\omega} \; y_h(jh),$$

the Fourier Transform is a norm-preserving bijection between the function spaces $\ell_2(\mathbb{Z})$ and $L_2(-\pi,+\pi)$, i.e. $\|y_h\| = \|\hat{y}_h\|$.

Considering the equation $L_{h,\alpha} y_h = L^c_{h,\alpha}\, y_h = f_h$ (e.g. $\alpha=\varepsilon$ or $\alpha=\varepsilon+h$), we see

$$(2.7) \qquad \hat{L}_{h,\alpha}(\omega) = \hat{f}_h(\omega)/\hat{y}_h(\omega) = \frac{-4}{h^2} S[\alpha S - ihC],$$

where $S = \sin(\omega h/2)$ and $C = \cos(\omega h/2)$. From this we derive the 2nd order consistency of $L^c_{h,\varepsilon}$:

$$|\hat{L}_{h,\varepsilon} - \hat{L}_\varepsilon| = |\hat{L}^c_{h,\varepsilon} - (\varepsilon\omega^2 + 2i\omega)| \le ch^2 |\varepsilon\omega^4 + i\omega^3|,$$

where $\hat{L}_\varepsilon(\omega) = -\varepsilon\omega^2 + 2i\omega$ is the characteristic polynomial of (1.2). The (in-)stability of $L^c_{h,\alpha}$ is shown by

$$|\hat{L}_{h,\alpha}| = \left| \frac{\sin(\omega h/2)}{h/2} \right| \sqrt{\left(\frac{2\alpha}{h}\right)^2 \sin^2(\omega h/2) + 4\cos^2(\omega h/2)}$$

$$\ge |\omega| \, \frac{2}{\pi} \, 2 \min\left(\left(\frac{\alpha}{h}\right), 1\right).$$

We see that $\hat{L}_{h,\alpha}(\omega)$ has one real zero ($\omega=0$) in common with $\hat{L}_\alpha(\omega)$. A spurious zero appears for $\alpha \to 0$, viz. (cf.(2.7))

$$\lim_{\varepsilon\to 0} \hat{L}_{h,\varepsilon}(\omega) = 0 \quad \text{for} \quad \omega h/2 = \pm\pi/2.$$

This shows that, with central differences, an instable mode appears in the solution as $\varepsilon/h \to 0$, which is of the form

$$(2.8) \qquad y_j = y_h(jh) = e^{ij\pi} = (-1)^j.$$

We also see that the upwind discretization, $\alpha = \varepsilon+h$, is 1st order accurate and has no instability

$$|\hat{L}^u_{h,\varepsilon} - \hat{L}_\varepsilon| \le |\hat{L}_{h,\alpha} - \hat{L}_{h,\varepsilon}| + |\hat{L}_{h,\varepsilon} - \hat{L}_\varepsilon|$$

$$\le \frac{4(\alpha-\varepsilon)}{h^2} \sin^2(\omega h/2) + O(h^2) \le ch\omega^2$$

and

$$|\hat{L}^u_{h,\varepsilon}| = |\hat{L}_{h,\varepsilon+h}| \ge |\omega|\pi/4$$

i.e. $\hat{L}^u_{h,\varepsilon}$ has no spurious zero.

Illustrative in this respect are the solutions of (2.1) and (2.5) on the interval $(0,\infty)$ with $y(0) = 1$, $y(\infty) = 0$, where we find the solution $y_i = r^i$ with $r = (\varepsilon-h)/(\varepsilon+h)$ and $r = \varepsilon/(\varepsilon+2h)$, respectively. Now, for $(\varepsilon/h) \to 0$, $L^c_{h,\varepsilon}$ yields the oscillating solution (2.8) whereas $L^u_{h,\varepsilon}$ yields $y_i = 0$ for $i > 0$.

3. STRAIGHTFORWARD DEFECT CORRECTION

Defect correction is a general technique to solve an equation in an iterative process by means of the repeated direct solution of a nearby simpler problem. Is the "target" problem to solve

$$Lu = f,$$

and can an "approximate" problem

$$\tilde{L}\tilde{u} = \tilde{f}$$

be solved directly, then the iterative process reads

(3.1) $\tilde{L}u^{(i+1)} = \tilde{L}u^{(i)} - Lu^{(i)} + f,$

(in case of a nonlinear operator $\tilde{L}$ other variants are possible). Many of these processes are well-known in numerical mathematics, e.g. iterative refinement of linear systems, Newton-like methods etc.

It can be derived under rather general smoothness conditions that, if L and $\tilde{L}$ are discretizations of the same differential problem and if $\tilde{L}$ is stable and L and $\tilde{L}$ are consistent of the order p and $q < p$, respectively, then $u^{(k)}$ is an approximation with accuracy $\mathcal{O}(h^{\min(p,kq)})$ (cf.[3,4,10]), without a stability requirement for L. Hence, we apply (3.1) for the solution of (1.1), using $L = L^c_{h,\varepsilon}$ and $\tilde{L} = L^u_{h,\varepsilon} = L_{h,\alpha}$, with $\alpha = \varepsilon + \mathcal{O}(h)$. Starting with $u^{(0)} = 0$, we find the artificial viscosity solution as $u^{(1)}$. Since $\tilde{L}$ is 1st and L is 2nd order consistent, a single iteration step is already sufficient to obtain 2nd order accuracy. If the iteration (3.1) is continued and converges to a fixed point $u^{(\infty)}$, then, clearly, $u^{(\infty)}$ is the unstable solution of $Lu = f$.

We find that the solution after one iteration step, $u^{(2)}$, satisfies

$$Q_{h,\varepsilon}u^{(2)} \equiv L_{h,\alpha}(2L_{h,\alpha}-L_{h,\varepsilon})^{-1}L_{h,\alpha}u^{(2)} = f.$$

By Fourier analysis we find, analogous to (2.7),

$$\hat{Q}_{h,\varepsilon}(\omega) = \frac{-4S(\alpha S-ihC)^2}{h^2[(2\alpha-\varepsilon)S-ihC]} ,$$

from which we derive that Q_h is stable, uniformly for small ε:

$$|\hat{Q}_{h,\varepsilon}| \geq \frac{2}{\pi}\,|\omega|\,\frac{\min^2(1,\alpha/h)}{\max(1,2\alpha/h)} .$$

For the smooth components of the solution the solution is accurate of order two:

$$|\hat{Q}_{h,\varepsilon}-\hat{L}_{h,\varepsilon}| = \frac{4S^2}{h^2}\,\frac{(\alpha-\varepsilon)^2}{\{(2\alpha-\varepsilon)^2S^2+h^2C^2\}}\,|S|\cdot|(2\alpha-\varepsilon)S+ihC| = \mathcal{O}(h^2).$$

We find $u^{(2)}$ to be a 2nd order accurate solution , uniformly in $\varepsilon > 0$, for the smooth components in the solution. This is also found experimentally in the second part of Table 1.

	$\max\|y_i-y(x_i)\|$			$\|y_i-y(x_i)\|$, $j = N/2$		
N	10	20	40	10	20	40
$u^{(1)}$	0.3303	0.1665	0.0831	0.0698	0.02931	0.01326
$u^{(2)}$	0.6213	0.5714	0.5384	0.1037	0.02707	0.00687
$u^{(3)}$	0.7770	0.7791	0.7677	0.0544	0.01188	0.00284

Table 1. Errors in the numerical solution of
$\varepsilon y'' + y' = f$ on $(0,1)$,
by straightforward DCP. $(\varepsilon=10^{-6})$.
Boundary conditions and f are such that $y(x) = \sin(4x)$.

Near the boundary at $x = 0$, where the solution is not smooth at all, the solution is not well represented. Here the accuracy is only $O(1)$. However, on a mesh with meshwidth h such a sharp boundary layer cannot be represented, anyway. For boundary layer resolution a finer mesh is necessary.

In order to see the effect of the boundary layer in the numerical solution with $h \gg \varepsilon$, we consider again problem (1.2) on the interval $(0,\infty)$ with $y(0) = 1$, $y(\infty) = 0$. On a uniform mesh we find for the iterands:

$$u_h^{(1)}(jh) = \left(\frac{\varepsilon}{\varepsilon+2h}\right)^j ,$$

$$u_h^{(2)}(jh) = \left(\frac{\varepsilon}{\varepsilon+2h}\right)^j \left[1-j\,\frac{2h^2}{\varepsilon(\varepsilon+2h)}\right] ,$$

$$u_h^{(3)}(jh) = \left(\frac{\varepsilon}{\varepsilon+2h}\right)^j \quad 1-j\,\frac{2h^2}{\varepsilon(\varepsilon+2h)}\left\{1-\frac{jh^2-h(\varepsilon+h)}{\varepsilon(\varepsilon+2h)}\right\}\Big] .$$

The general solution is

$$u_h^{(k+1)}(jh) = \left(\frac{\varepsilon}{\varepsilon+2h}\right)^j P_k(j;\varepsilon/h),$$

where P_k is a k-th degree polynomial in j, with a parameter ε/h. For small ε/h, $y^{(k+1)}$ changes the sign k times. The influence of the numerical boundary layer decreases exponentially in the interior of the domain, but the instability creeps into the solution further and further as the iteration proceeds. A single DCP step gives already 2nd accuracy. More DCP steps introduce more instability.

4. MIXED DEFECT CORRECTION

Although a single step with the straightforward DCP gives already good results for particular linear 1-dimensional problems, we don't get the same favourable results for the convection-diffusion equation (cf. [6,7]). Moreover, for nonlinear problems and for the 2-D problems we prefer the numerical solution to be a fixed point of an iteration process. Therefore we construct an iterative process of "mixed defect correction" (MDCP-) type

$$\begin{cases} \tilde{L}_1\, y^{(i+\frac{1}{2})} = \tilde{L}_1\, y^{(i)} - L_1\, y^{(i)} + f_1, & (4.1a) \\ \tilde{L}_2\, y^{(i+1)} = \tilde{L}_2\, y^{(i+\frac{1}{2})} - L_2\, y^{(i+\frac{1}{2})} + f_2. & (4.1b) \end{cases}$$

If this iteration converges, we obtain two different solutions viz. $y^A = \lim_{i\to\infty} y^{(i)}$ and $y^B = \lim_{i\to\infty} y^{(i+\frac{1}{2})}$. For our purpose we choose $L_1 = L^c_{h,\varepsilon}$, $L_2 = \tilde{L}_1 = L^c_{h,\alpha}$ and $\tilde{L}_2 = 2\,\mathrm{diag}(L^c_{h,\alpha}) =: D_{h,\alpha}$, $f_1 = f_2 = f_h$. Thus, with $\alpha = \varepsilon$ + artificial viscosity, (4.1a) is a defect correction step and (4.1b) is a damped Jacobi relaxation step. This relaxation sweep is introduced to reduce the high frequencies that are introduced by the DCP-step near the boundary layer.

The fixed point y^A of the iteration (4.1) can be characterized as the solution of the equation

$$[L_{h,\varepsilon} + L_{h,\alpha} D^{-1}_{h,\alpha}(L_{h,\alpha} - L_{h,\varepsilon})]\,y^A = f_h\,, \qquad (4.2)$$

which ~~equation~~ we briefly denote by $M_{h,\varepsilon} y^A = f_h$.

For the model equation (1.2), by Fourier analysis we find with $\alpha = \varepsilon+h$, analogous to (2.7),

$$\hat{M}_{h,\varepsilon}(\omega) = \frac{-4\varepsilon}{h^2} S^2[1+\frac{h}{\varepsilon} S^2] + \frac{4i}{h} SC[1+\frac{h}{\varepsilon+h} S^2].$$

For $\varepsilon \to 0$ there is no spurious zero:

$$\lim_{\varepsilon\to 0} |\hat{M}_{h,\varepsilon}| = |\frac{4}{h} S^4 + \frac{4i}{h} SC[1+S^2]| = \frac{4}{h} |S| \sqrt{1+S^2C^2},$$

i.e. the discretization is stable, uniformly for $\varepsilon \to 0$. Further it is consistent of 2nd order:

$$|\hat{M}_{h,\varepsilon}(\omega)-\hat{L}_{h,\varepsilon}| = \frac{4}{h} S^2 |S^2 + \frac{ih}{\varepsilon+h} SC| \leq O(h^2).$$

Hence the solution y^A is accurate $O(h^2)$ in the smooth parts. Results are shown in Table 2.

	$\max\|y_i-y(x_i)\|$			$\|y_i-y(x_i)\|$, $j = N/2$		
N	10	20	40	10	20	40
y^A	0.208	0.227	0.233	0.02507	0.00653	0.00165
y^B	0.565	0.604	0.614	0.05953	0.01556	0.00392

Table 2. Errors in the numerical solution of mixed DCP; the same problem as for table 1.

For sufficiently smooth solutions, we can also derive error estimates in a global norm using the stability of the operators $L_{h,\alpha}$ and $D_{h,\alpha}$ and the relative consistency between $L_{h,\varepsilon}$ and $L_{h,\alpha}$, and $L_{h,\alpha}$ and $D_{h,\alpha}$. (cf. [10]).

In order to study the boundary layer behaviour, we set $\lambda = e^{i\omega h}$ and compute the four roots of $\hat{M}_{h,\varepsilon}(\omega) = 0$. Asymptotically, for $\frac{\varepsilon}{h} \to 0$, these are

$$\lambda_1 = -\varepsilon/2h, \ \lambda_{2,3} = 2\pm\sqrt{5} + (2\pm 2/\sqrt{5})\frac{\varepsilon}{h}, \ \lambda_4 = 1.$$

Hence, for a problem on the right half line the boundary layer is of the form

$$y_i = A\lambda_1^i + B\lambda_2^i, \qquad |\lambda_1| < 1, \ |\lambda_2| < 1.$$

A closer analysis shows

$$y_i^A = (2-\sqrt{5})^i + O(\tfrac{\varepsilon}{h}),$$
$$y_i^B = \tfrac{1}{2}(3+\sqrt{5})(2-\sqrt{5})^i + O(\tfrac{\varepsilon}{h}).$$

This shows that, for small ε/h, the error in the boundary layer is $O(1)$, but the influence of the boundary data decreases at a fixed rate per mesh-point. Hence, also in this case, the width of the numerical boundary layer is $O(h)$.

Convergence of the MDCP iteration is proved by showing that

$$D_{h,\alpha}^{-1}(D_{h,\alpha}-L_{h,\alpha})L_{h,\alpha}^{-1}(L_{h,\alpha}-L_{h,\varepsilon})$$

has eigenvalues less than 1. A good impression of the convergence behaviour is obtained again by local mode analysis, viz.

$$|\hat{D}_{h,\alpha}^{-1}(\hat{D}_{h,\alpha}-\hat{L}_{h,\alpha})\hat{L}_{h,\alpha}^{-1}(\hat{L}_{h,\alpha}-\hat{L}_{h,\varepsilon})|$$
$$= \frac{\alpha-\varepsilon}{\alpha}\, SC\sqrt{\frac{\alpha^2C^2+h^2S^2}{\alpha^2S^2+h^2C^2}} \le \frac{\alpha-\varepsilon}{2\alpha}\, A(\omega h)\ \sin(\omega h),$$

where $C_1\frac{h}{\alpha} \le A(\omega h) \le C_2\frac{\alpha}{h}$. With $\alpha = \varepsilon+h$, we find for $\varepsilon \ll 1$ a convergence factor $\le \frac{1}{2}$ per iteration sweep. (Note: for the two-dimensional problem, the convergence of the iteration is essentially more complex to analyze, cf. [7]).

As a result of our iteration process we obtain two solutions: y^A and y^B. The difference between these solutions is

$$y^A - y^B = \frac{\alpha-\varepsilon}{2} D_{h,\alpha}^{-1} \Delta_h y^A,$$

which is proportional to the amount of artificial viscosity $\alpha-\varepsilon$ and to the 2nd differences in y^A. These differences are particularly large in those regions where the special layers exist. Hence we can use $y^A - y^B$ to detect these regions and - if necessary - to refine the mesh locally. On this basis an adaptive procedure has been constructed, which halves the mesh size in those regions where $y^A - y^B$ exceeds a given tolerance. By this procedure boundary layers are resolved automatically. Results are shown in Table 3.

In the adaptive procedure used, only discretizations with a fixed mesh size were constructed. Where refinement is necessary a new problem on a sub-interval is generated with half the mesh size; after the solution of this new problem (h/2) the coarse grid problem (h) is corrected for the relative truncation error between both meshes. This procedure is made recursively to create finer and finer meshes, if necessary. In this way an hierarchy of submeshes is generated (cf. [5]).

	Number of intervals in the mesh				
h \ ε	0.1	0.05	0.025	0.0125	0.00625
1/2	2	2	2	2	2
1/4	4	4	4	4	4
1/8			8	8	8
1/16				8	8
1/32					8
NP	5	5	9	13	17
ME	0.040	0.088	0.088	0.088	0.088

Table 3. Automatic mesh refinement for
$\varepsilon y'' + 2y' = 0, \quad y(0) = 1, \ y(1) = 0.$

Shown is the number of intervals on each mesh, the total number of meshpoints (NP) and the maximal error over all meshpoints (ME). The criterium for mesh-refinement is

$$\{|y_i^A - y_i^B| < 0.025, \quad i = 0,2,4,\ldots,N\}.$$

All local mesh-refinements appear at the left end of the interval (i.e. in the boundary layer).

REMARK. For the one-dimensional model problem there is no clear advantage of the "mixed defect correction" over the straightforward defect correction. For the two-dimensional convection diffusion equation, however, there is a difference. Here, the MDCP again shows $\mathcal{O}(h^2)$ accuracy in the smooth parts, whereas straightforward DCP does not. For 2-D problems this higher order accuracy is very important because the computational work is at least proportional to the number of meshpoints, which is $\mathcal{O}(h^{-2})$.

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