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Distributing the numerical solution of parabolic singularly perturbed problems with defect correction over independent processes*

P.W. Hemker, G.I. Shishkin and L.P. Shishkina

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На отрезке изучается первая краевая задача типа реакции-диффузии для сингулярно возмущенного параболического уравнения. Для аппроксимации краевой задачи используются разностные схемы высокого (ε-равномерно) порядка точности по времени, разработанные ранее на основе коррекции невязки. Новым в этой статье является введение разделения области для таких ε-равномерных схем. Указаны условия, при которых разностные схемы, используемые независимо на подобластях, могут ускорить ε-равномерно решение краевой задачи без потери точности исходных схем. Следовательно, одновременное решение задачи на разных подобластях может быть в принципе использовано для распараллеливания вычислительного метода.

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For a singularly perturbed parabolic equation on an interval, the first boundary value problem of reaction-diffusion type is studied. For the approximation of the boundary value problem we use previously developed finite difference schemes, of high ε-uniform order of accuracy in time, based on defect correction. The new approach developed in this paper is the introduction of a partitioning of the domain for these ε-uniform schemes. We determine conditions under which the difference schemes applied independently on the subdomains can accelerate (ε-uniformly) the solution of the boundary value problem without losing the accuracy of the original schemes. Hence, the simultaneous solution on the subdomains can in principle be used for parallelization of the computational method.

1. Introduction

Special ε-uniformly convergent difference schemes for singularly perturbed boundary value problems for elliptic and parabolic equations have been well developed, see e.g., [1–4, 8, 9, 13]. If the problem data are sufficiently smooth for parabolic equations without convection terms, then the order of ε-uniform convergence for the scheme studied in [1, 2] will be $O(N^{-2}\ln^2 N + N_0^{-1})$, where N and N_0 denote, respectively, the number of intervals in the space and time discretization. For this scheme the amount of computational work is primarily determined by the time discretization, which is of first order accuracy only. In [3, 4] we

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developed an algorithm based on the defect correction principle which achieves a high order of accuracy with respect to time and preserves second-order accuracy in space.

To improve the efficiency of the algorithm, we also need efficient high order methods for solving discretized problems. In paper [14] parallel computational methods were proposed that make it possible to accelerate the numerical solution of singularly perturbed boundary value problems for parabolic reaction-diffusion equations. In the present paper we develop a new, related, computational method to solve the system of discrete equations that arises when the defect correction technique is used to improve the accuracy of the discrete problem. In this way, we achieve a high order of accuracy for the time variable, maintaining ε-uniform convergence and second-order accuracy in space, as well as high efficiency of the algorithms due to possible parallel computations. It should be noted that this parallel method is not iterative within its time steps.

The schemes developed for parallel computation can be considered as domain decomposition variants of schemes based on the defect correction technique. The domain decomposition introduces additional errors (perturbations) in the solutions obtained by the qschemes. In this paper we determine conditions (both for the derivatives of the solutions and for the parameters of the difference schemes) under which parallel computation does not affect the accuracy of the solution. Thus, by means of parallel computation we can achieve an acceleration of the solution process and maintain the high order of accuracy of the schemes based on defect correction technique.

In this paper we use the convention that symbol $L_{(k.l)}$ denotes the symbol L introduced in formula $(k.l)$. Wherever no confusion is possible, additional subscripts may be omitted.

2. Problem formulation

In the domain $G = (0,1) \times (0,T]$, with boundary $S = \overline{G} \setminus G$ we consider the following singularly perturbed parabolic equation with Dirichlet boundary conditions:

$$L_{(2.1)}u(x,t) \equiv \left\{\varepsilon^2 a(x,t)\frac{\partial^2}{\partial x^2} - c(x,t) - p(x,t)\frac{\partial}{\partial t}\right\}u(x,t) = f(x,t), \quad (x,t) \in G, \tag{2.1a}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{2.1b}$$

For $S = S_0 \cup S_1$, we distinguish the initial boundary $S_0 = \{(x,t) : x \in [0,1], \ t = 0\}$ and the lateral boundary $S_1 = \{(x,t) : x = 0 \text{ or } x = 1, \ 0 < t \le T\}$. In (2.1) $a(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$, $(x,t) \in \overline{G}$, and $\varphi(x,t)$, $(x,t) \in S$, are sufficiently smooth and bounded functions which satisfy

$$0 < a_0 \le a(x,t), \qquad 0 < p_0 \le p(x,t), \qquad c(x,t) \ge 0, \qquad (x,t) \in \overline{G}. \tag{2.1c}$$

The real parameter ε may take any positive value in the interval

$$\varepsilon \in (0,1]. \tag{2.1d}$$

When the parameter ε tends to zero in (2.1a) in the neighborhood of the lateral boundary S_1, boundary layers appear in the solution. These layers are described by a parabolic equation (parabolic boundary layers).

For problem (2.1), we construct a numerical method that has a high order of accuracy with respect to time and, in addition, allows for parallel solution of the difference equations.

3. The difference scheme

To solve problem (2.1) we first consider a classical finite difference method. On the set $\overline{G}$ we introduce a rectangular grid

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{3.1}$$

where $\overline{\omega}$ is (possibly) a non-uniform grid of nodal points x^i on $[0,1]$, $\overline{\omega}_0$ is a uniform grid on the interval $[0,T]$; N and N_0 are the numbers of intervals in the grids $\overline{\omega}$ and $\overline{\omega}_0$, respectively. We denote $\tau = T/N_0$, $h^i = x^{i+1} - x^i$, $h = \max_i h^i$, $h \le M/N$, $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$.

Here and below we denote by M (or m) sufficiently large (or small) positive constants which do not depend on parameters ε and N.

For problem (2.1), we use the difference scheme [10]

$$\Lambda_{(3.2)} z(x,t) = f(x,t), \quad (x,t) \in G_h, \tag{3.2a}$$

$$z(x,t) = \varphi(x,t), \quad (x,t) \in S_h, \tag{3.2b}$$

where

$$\begin{aligned}
\Lambda_{(3.2)} z(x,t) &\equiv \left\{ \varepsilon^2 \, a(x,t)\delta_{\overline{x}\widehat{x}} - c(x,t) - p(x,t)\delta_{\overline{t}} \right\} z(x,t), \\
\delta_{\overline{x}\widehat{x}} z(x^i,t) &= 2(h^{i-1} + h^i)^{-1} \left[\delta_x z(x^i,t) - \delta_{\overline{x}} z(x^i,t) \right], \\
\delta_x z(x^i,t) &= (h^i)^{-1} \left(z(x^{i+1},t) - z(x^i,t) \right), \\
\delta_{\overline{x}} z(x^i,t) &= (h^{i-1})^{-1} \left(z(x^i,t) - z(x^{i-1},t) \right), \\
\delta_{\overline{t}} z(x^i,t) &= \tau^{-1} \left(z(x^i,t) - z(x^i,t-\tau) \right),
\end{aligned}$$

$\delta_x z(x,t)$ and $\delta_{\overline{x}} z(x,t)$, $\delta_{\overline{t}} z(x,t)$ are forward and backward differences, and $\delta_{\overline{x}\widehat{x}} z(x,t)$ is an approximation of the derivative $(\partial^2/\partial x^2)u(x,t)$ on a non-uniform grid.

From [10], we know that the difference scheme (3.2), (3.1) is monotone. By means of the maximum principle, and taking into account a-priori estimates of the derivatives (see Theorem 11.1 in the Appendix), we find that the solution of the difference scheme (3.2), (3.1) converges for a fixed value of the parameter ε as

$$|u(x,t) - z(x,t)| \le M(\varepsilon^{-1}N^{-1} + \tau), \quad (x,t) \in \overline{G}_h. \tag{3.3}$$

Our proof of (3.3) is similar to the classical convergence proof for monotone difference schemes [10, 13]. Taking into account an a-priori estimate for the solution (see the Appendix), this results in the following theorem:

Theorem 3.1. *Let us assume that estimate* (11.2), *where* $n = 0$, *holds for the solution of* (2.1). *Then, for a fixed value of the parameter* ε, *the solution of* (3.2), (3.1) *converges to the solution of* (2.1) *with the error bound given by* (3.3).

4. The ε-uniformly convergent method

In this section we discuss an ε-uniformly convergent method for (2.1) by taking a special grid condensed in the neighborhood of the boundary layers. The distribution of nodes is derived from *a priori* estimates of the solution and its derivatives. We follow the approach described in [3, 8, 11, 13], i.e., we take

$$\overline{G}_h = \bar{\omega}^*(\sigma) \times \bar{\omega}_0, \tag{4.1}$$

where $\bar{\omega}_0$ is a uniform grid with step-size $\tau = TN_0^{-1}$, and $\bar{\omega}^* = \bar{\omega}^*(\sigma)$ is a special *piecewise* uniform grid depending on a parameter $\sigma \in IR$, which depends on ε and N. We take $\sigma = \sigma(\varepsilon, N) = \min[1/4, m\varepsilon \ln N]$, where m is an arbitrary positive number. The grid $\bar{\omega}^*(\sigma)$ is constructed as follows. The interval $[0,1]$ is divided in three parts $[0,\sigma]$, $[\sigma, 1-\sigma]$, $[1-\sigma, 1]$, $0 < \sigma \le 1/4$. In each part we use a uniform grid, with $N/2$ subintervals in $[\sigma, 1-\sigma]$ and with $N/4$ subintervals in each interval $[0,\sigma]$ and $[1-\sigma, 1]$.

Theorem 4.1. *If the solution of problem* (2.1) *satisfies the hypotheses of Theorem 11.1* (*see Appendix*), *where* $n = 0$, *then the solution of* (3.2), (4.1) *converges* ε*-uniformly to the solution of* (2.1), *and the following estimate holds*:

$$|u(x,t) - z(x,t)| \le M(N^{-2}\ln^2 N + \tau), \qquad (x,t) \in \overline{G}_h^*. \tag{4.2}$$

The proof of this theorem can be found in [12, 13].

Remark. Under the conditions of Theorem 4.1, where $n = K \ge 0$, for the derivatives $(\partial^{k_0}/\partial t^{k_0})u(x,t)$ and the divided differences $\delta_{l\bar{t}}\, z(x,t)$, the following estimates hold:

$$\left|\frac{\partial^{k_0}}{\partial t^{k_0}} u(x,t)\right| \le M_{(4.3)}^{(k_0)}, \quad (x,t) \in \overline{G}, \;\; k_0 \le K+2; \tag{4.3}$$

$$|\delta_{l\bar{t}}\, z(x,t)| \le M_{(4.4)}^{(l)}, \quad (x,t) \in \overline{G}_{h(4.1)}, \;\; t \ge l\tau, \;\; l \le K+1. \tag{4.4}$$

Here we denote by $\delta_{l\bar{t}} z(x,t)$ the backward difference of order l:

$$\delta_{l\bar{t}}\, z(x,t) = \left(\delta_{l-1\,\bar{t}}\, z(x,t) - \delta_{l-1\,\bar{t}}\, z(x,t-\tau)\right)/\tau,$$

$$\delta_{0\bar{t}}\, z(x,t) = z(x,t), \quad (x,t) \in \overline{G}_h, \;\; t \ge l\tau, \;\; l \ge 1.$$

5. Schwarz method for parabolic equations

In this section we modify Schwarz' domain decomposition method for the boundary value problem (2.1), and for the solutions obtained we give the necessary and sufficient conditions for ε-uniform convergence.

5.1. We first describe Schwarz' classical method for problem (2.1). Let a set of open subdomains

$$D^k, \quad k = 1, \ldots, K \tag{5.1a}$$

with piecewise smooth boundaries Γ^k, $\Gamma^k = \Gamma(D^k) = \overline{D}^k \setminus D^k$, cover the domain D: $D = \bigcup_{k=1}^{K} D^k$, and let

$$G^k = D^k \times (0,T], \quad k = 1, \ldots, K. \tag{5.1b}$$

We denote by $D^{[k]}$ the union of the subdomains $D^1, \ldots, D^K$ which does not include D^k:

$$D^{[k]} = \bigcup_{i=1,\, i\ne k}^{K} D^i. \tag{5.1c}$$

We denote the minimal overlap of the sets D^k and $D^{[k]}$ by δ^k, and by δ the smallest value of δ^k, i.e.,

$$\min_{k,x^1,x^2} \rho(x^1,x^2) = \delta, \tag{5.2}$$

$$x^1 \in \overline{D}^k, \qquad x^2 \in \overline{D}^{[k]}, \qquad x^1, x^2 \notin \left\{D^k \cap D^{[k]}\right\}, \qquad k = 1,\dots,K,$$

where $\rho(x^1,x^2)$ is the distance between points $x^1, x^2 \in \overline{D}$. In general, the value δ may depend on the parameter ε.

Let

$$u^0(x,t), \quad (x,t) \in \overline{G}, \tag{5.3a}$$

be an arbitrary function satisfying the condition (2.1b). We seek a sequence of functions $u^r(x,t)$, $(x,t) \in \overline{G}$, $r = 1,2,\dots$. Let a function $u^r(x,t)$ be known. The function $u^{r+1}(x,t)$ is determined in the following way. First we find functions $u^{r+\frac{k}{K}}(x,t)$. These are solutions of the following problems:

$$L_{(5.4)}(u^{r+\frac{k}{K}}(x,t)) = 0, \qquad (x,t) \in G^k, \tag{5.3b}$$

$$u^{r+\frac{k}{K}}(x,t) = u^{r+\frac{k-1}{K}}(x,t), \qquad (x,t) \in \overline{G} \setminus G^k, \quad k = 1,\dots,K.$$

The required function is defined by the relation

$$u^{r+1}(x,t) = u^{r+\frac{K}{K}}(x,t), \quad r = 0,1,2,\dots\ . \tag{5.3c}$$

In the case of boundary value problem (2.1) the operator $L_{(5.4)}$ in (5.3b) is defined as

$$L_{(5.4)}(u(x,t)) \equiv L_{(2.1)}u(x,t) - f(x,t), \quad (x,t) \in G. \tag{5.4}$$

Each function $u^{r+\frac{k}{K}}(x,t)$, $(x,t) \in \overline{G}$, is the solution of a Dirichlet problem on the set $\overline{G}^k$ and coincides with the function $u^{r+\frac{k-1}{K}}(x,t)$ on the set $\overline{G} \setminus G^k$. This process is a natural generalization of the classical Schwarz "alternating" method.

In principle, we could give conditions under which process (5.3), (5.4), and (5.1) converges to the solution of boundary value problem (2.1) as $r \to \infty$, where r is the number of iterations. However, in this paper we are interested in a non-iterative variant solver based on a modified Schwarz method.

5.2. Now we describe the modified Schwarz method. Let

$$\overline{\omega}_0 \tag{5.5a}$$

be a uniform grid, like $\overline{\omega}_{0(3.1)}$, on [0,T] with a stepsize τ. By $G(t_1)$ we denote the strip

$$G(t_1) = \{\, (x,t) : \ (x,t) \in G, \ t_1 < t \le t_1 + \tau \}, \quad t_1, t_1 + \tau \in \overline{\omega}_0.$$

Let $S(t_1) = \overline{G}(t_1) \setminus G(t_1)$ be the boundary of $G(t_1)$ and let $v(x,t) = v(x,t;t_1)$ be defined on $S(t_1)$. We denote the extension of the function $v(x,t)$ onto the whole set $\overline{G}(t_1)$ by $\overline{v}(x,t;t_1)$. The function $\overline{v}(x,t;t_1)$ is assumed to satisfy a Lipschiz condition with respect to t. We subdivide the strip $G(t_1)$ into sections $G^k(t_1) = G^k \cap G(t_1)$, $S^k(t_1) = \overline{G}^k(t_1) \setminus G^k(t_1)$.

Suppose the function $u(x,t)$, $(x,t) \in ovlG$, for $t^n \in \overline{\omega}_0$, $t \le t^n < T$, $n = 0,1,\dots,N_0 - 1$, has already been constructed. Now we construct the function $u(x,t)$ for $t \le t^{n+1}$, i.e., we find the function $u(x,t)$ on the strip $G(t^n)$. This is done in the following way. First we find functions $u^{k/K}(x,t)$ on the sections $\overline{G}^k(t^n)$ solving the boundary value problems

$$\left.\begin{array}{ll} L_{(5.4)}(u^{\frac{k}{K}}(x,t)) = 0, & (x,t) \in G^k(t^n), \\ u^{\frac{k}{K}}(x,t) = \left\{\begin{array}{ll} \overline{u}(x,t;t^n), & k=1, \\ u^{\frac{k-1}{K}}(x,t), & k \geq 2 \end{array}\right\}, & (x,t) \in S^k(t^n) \end{array}\right\} \text{ for } (x,t) \in \overline{G}^k(t^n), \tag{5.5b}$$

$$k = 1, \dots, K, \quad t^n \in \bar{\omega}_0, \quad n \leq N_0 - 1.$$

Having $u^{k/K}(x,t)$ on $\overline{G}^k(t^n)$, we extend these functions for each value of k onto the whole strip $\overline{G}(t^n)$ in the following way:

$$u^{\frac{k}{K}}(x,t) = \left\{\begin{array}{ll} u^{\frac{k}{K}}(x,t), & (x,t) \in \overline{G}^k(t^n), \\ \left.\begin{array}{ll} \overline{u}(x,t;t^n), & k=1, \\ u^{\frac{k-1}{K}}(x,t), & k \geq 2 \end{array}\right\}, & (x,t) \in \overline{G}(t^n) \setminus \overline{G}^k(t^n) \end{array}\right\} \text{ for } (x,t) \in \overline{G}(t^n), \tag{5.5c}$$

$$k = 1, \dots, K, \quad t^n \in \bar{\omega}_0.$$

Having $u^{k/K}(x,t)$, for $k = K$ we define the function $u(x,t)$ on the whole strip $\overline{G}(t^n)$ by

$$u(x,t) = u^{\frac{K}{K}}(x,t), \quad (x,t) \in \overline{G}(t^n), \quad t^n \in \bar{\omega}_0. \tag{5.5d}$$

Thereby we have the function $u(x,t)$ defined on the domain $\overline{G}$ for $t \in [0, t^{n+1}]$.

In the relations (5.5b), (5.5c) the function $\overline{u}(x,t;t^n)$ is constructed on the basis of a function $\overline{v}(x,t;t^n)$,

$$\overline{u}(x,t;t^n) = \overline{v}(x,t;t^n), \quad (x,t) \in \overline{G}(t^n). \tag{5.5e}$$

Using $v(x,t;t^n)$, which is defined on the boundary $S(t^n)$ in (5.5g), we find the function

$$\overline{v}(x,t;t^n), \qquad (x,t) \in \overline{G}(t^n), \tag{5.5f}$$

supposing $\overline{v}(x,t;t^n) = v(x,t;\, t^n)$ for $(x,t) \in S(t^n)$ and $\overline{v}(x,t;t^n) = v(x,t^n;\, t^n)$ for $(x,t) \in G(t^n)$. Here

$$v(x,t;t^n) = \left\{\begin{array}{lll} \varphi(x,t), & (x,t) \in S(t^n), & t^n = t^0 = 0, \\ \left.\begin{array}{lll} \varphi(x,t), & (x,t) \in S(t^n) \cap S, & t \geq t^n, \\ u(x,t), & (x,t) \in S(t^n) \setminus S, & t = t^n \end{array}\right\}, & & t^n > 0, \; (x,t) \in S(t^n), \end{array}\right. \tag{5.5g}$$

$$n = 0, 1, \dots, N_0 - 1.$$

Thus, the function $\overline{u}(x,t;t^n)$ on $\overline{G}(t^n)$ have been constructed.

The function $u^{\frac{k}{K}}(x,t)$ on each strip $\overline{G}(t^n)$ is the solution of a Dirichlet problem on the section $\overline{G}^k(t^n)$, whereas on the set $\overline{G}(t^n) \setminus G^k(t^n)$ it coincides with the function $\overline{u}(x,t;t^n)$, $(x,t) \in \overline{G}(t^n)$ for $k=1$, and with the function $u^{\frac{k-1}{K}}(x,t)$, $(x,t) \in \overline{G}(t^n)$ for $k \geq 2$. Thus we have found a function $u(x,t)$, $(x,t) \in \overline{G}$, the solution of the process (5.5), (5.4), (5.1) which we call the modified Schwarz method.

Note that the process (5.5), (5.4), (5.1), "the modified Schwarz method" is not an iterative process in a strict sense. The boundary value problems in (5.5), (5.4), (5.1) are solved only once at those points of $\overline{G}$ which do not belong to the intersection of the subdomains. The boundary value problem is solved twice only on the intersection of the subdomains.

In the continuous domain decomposition method (5.5), (5.4), (5.1) the intermediate problems on the subsets $\overline{D}^k_{(5.1)}$, $k = 1, \dots, K$ are solved sequentially.

Using comparison theorems [5, 6], we obtain the estimate

$$|u(x,t) - u_{(5.5)}(x,t)| \le Q(\varepsilon,\delta)N_0^{-1}, \qquad (x,t) \in \overline{G},$$

where $u_{(5.5)}(x,t)$ is the solution of the process (5.5), (5.4), (5.1), $\delta = \delta_{(5.2)}(\varepsilon)$, i.e., the function $u_{(5.5)}(x,t)$ converges, as $N_0 \to \infty$, to the solution of boundary value problem (2.1) for each fixed value of the parameter ε. Note that the function $u_{(5.5)}(x,t)$ for $\delta = 0$ does not converge to the solution of boundary value problem (2.1) as $N_0 \to \infty$. Under the condition

$$\delta = \delta_{(5.2)}(\varepsilon) > 0, \qquad \varepsilon \in (0,1], \quad \inf_{\varepsilon\in(0,1]} [\varepsilon^{-1}\delta_{(5.2)}(\varepsilon)] > 0 \tag{5.6}$$

which is equivalent to the condition $\delta = \delta_{(5.2)}(\varepsilon) \ge m_{(5.6)}\varepsilon$, $\varepsilon \in (0,1]$, the function $u_{(5.5)}(x,t)$ converges ε-uniformly as $N_0 \to \infty$:

$$|u(x,t) - u_{(5.5)}(x,t)| \le MN_0^{-1}, \quad (x,t) \in \overline{G}.$$

If condition (5.6) is violated and the value δ satisfies the condition

$$\delta = \delta_{(5.2)}(\varepsilon) > 0, \qquad \varepsilon \in (0,1], \quad \inf_{\varepsilon\in(0,1]} [\varepsilon^{-1}\ \delta_{(5.2)}(\varepsilon)] = 0, \tag{5.7}$$

the function $u_{(5.5)}(x,t)$ does not converge ε-uniformly.

5.3. Here we describe a continuous variant of the modified Schwarz method that allows parallel computations on $P \ge 1$ processors.

Let D^k, $k = 1,\dots,K$, be the subdomains from (5.1a) and let each D^k be partitioned in P disjoint (possibly empty) parts

$$D^k = \bigcup_{p=1}^{P} D_p^k, \qquad k = 1,\dots,K, \quad \overline{D}_i^k \cap \overline{D}_j^k = \emptyset,\ i \ne j. \tag{5.8a}$$

Here we assume that the non-empty D_p^k do overlap, but generally D^k do not. We set

$$G_p^k = D_p^k \times (0,T], \qquad p = 1,\dots,P, \quad k = 1,\dots,K. \tag{5.8b}$$

We find the function $u(x,t)$ by the solution of problems (5.9) similar to (5.5) but now on the set $\overline{G}_p^k(t^n)$ instead of $\overline{G}^k(t^n)$

$$L_{(5.4)}(u_p^{\frac{k}{K}}(x,t) = 0, \quad (x,t) \in G_p^k(t^n), \tag{5.9a}$$

$$u_p^{\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} \overline{u}(x,t;t^n), & k=1, \\ u^{\frac{k-1}{K}}(x,t), & k \ge 2 \end{array} \right\}, \quad (x,t) \in S_p^k(t^n), \quad p = 1,\dots,P$$

$$\text{for} \quad (x,t) \in \overline{G}_p^k(t^n), \quad k = 1,\dots,K, \quad t^n \in \overline{\omega}_0, \quad n \le N_0 - 1;$$

$$u^{\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} u_p^{\frac{k}{K}}(x,t), & (x,t) \in \overline{G}_p^k(t^n),\ p = 1,\dots,P, \\ \left. \begin{array}{ll} \overline{u}(x,t;t^n), & k=1, \\ u^{\frac{k-1}{K}}(x,t), & k \ge 2 \end{array} \right\}, & (x,t) \in \overline{G}(t^n) \setminus \bigcup_{p=1}^{P} \overline{G}_p^k(t^n) \end{array} \right\} \tag{5.9b}$$

$$\text{for} \quad (x,t) \in \overline{G}(t^n), \quad k = 1,\dots,K, \quad t^n \in \overline{\omega}_0.$$

$$u(x,t) = u^{\frac{K}{K}}(x,t), \quad (x,t) \in \overline{G}(t^n), \quad t^n \in \overline{\omega}_0. \tag{5.9c}$$

The function $\overline{u}(x,t;t^n) = \overline{v}(x,t;t^n)$, $(x,t) \in \overline{G}(t^n)$, $t^n \in \overline{\omega}_0$. The function $\overline{v}(x,t;t^n)$, $(x,t) \in \overline{G}(t^n)$ is determined as in (5.5f).

Stepwise, for $n = 1, 2, \dots$, we find the function $u_{(5.9)}(x,t)$, $(x,t) \in \overline{G}$, i.e., the solution of process (5.9), (5.8). This we call the modified continuous Schwarz method for P "processors".

The scheme (5.9) with the decomposition (5.8) can be written in "operator" form

$$Q(u(x,t);\ \omega_0,\ f(\cdot), \varphi(\cdot), \psi(\cdot)) = 0, \qquad (x,t) \in G. \tag{5.9d}$$

Here the function $\psi(x,t;t^n)$, $(x,t) \in G(t^n)$, defines the prolonged function $\overline{u}(x,t;t^n)$:

$$\overline{u}(x,t;t^n) = \left\{ \begin{array}{ll} v(x,t;t^n), & (x,t) \in S(t^n), \\ v(x,t^n;t^n) + \psi(x,t;t^n), & (x,t) \in G(t^n) \end{array} \right\}, \quad (x,t) \in \overline{G}(t^n), \tag{5.9e}$$

so that in the case of conditions (5.5e), (5.5f), simply, $\psi(x,t;t^n) \equiv 0$. The problem (5.9), (5.8) for $P = 1$ is identical with problem (5.5), (5.1).

In the continuous domain decomposition method (5.9), (5.8) the intermediate problems on the subsets $\overline{D}^k_{p(5.8)}$, $p = 1, \dots, P$, $k = 1, \dots, K$ can be solved independently of each other, for all $p = 1, \dots, P$. For this construction the following theorem [14] is useful.

Theorem 5.1. *The condition* (5.6) *is necessary and sufficient for ε-uniform convergence* (*as $N_0 \to \infty$*) *of $u_{(5.9)}(x,t)$, i.e., the solution of process* (5.9), (5.8) *with $P \geq 1$, to $u(x,t)$, i.e., the solution of boundary value problem* (2.1).

6. Difference schemes based on the Schwarz method

6.1. Here we construct a difference scheme based on the process (5.5), (5.1) and give necessary and sufficient conditions for ε-uniform convergence of this scheme. We introduce rectangular grids on each set $\overline{G}^k$ and $\overline{G}^k_p$:

$$\overline{G}^k_h = \overline{G}^k \cap \overline{G}_{h(3.1)}, \qquad \overline{G}^k_{ph} = \overline{G}^k_p \cap \overline{G}_{h(3.1)}, \tag{6.1}$$

or

$$\overline{G}^{k*}_h = \overline{G}^k \cap \overline{G}^*_{h(4.1)}, \qquad \overline{G}^{k*}_{ph} = \overline{G}^k_p \cap \overline{G}^*_{h(4.1)}, \tag{6.2}$$

where $\overline{G}^k_{ph} = \overline{G}^k_{p,h}$. We assume that the boundaries of $\overline{G}^k$ and $\overline{G}^k_p$ pass through nodes of grids $\overline{G}_h$ and $\overline{G}^*_h$, respectively.

Now we introduce a discrete function $v(x,t) = v(x,t;t_1)$ defined on the boundary of a discrete strip $S_h(t_1) = S(t_1) \cap \overline{G}_h$, $t_1 \in \overline{\omega}_0$. By $\overline{v}(x,t;t_1)$ we denote the extension of this function $v(x,t)$ to the discrete set $\overline{G}_h(t_1) = \overline{G}(t_1) \cap \overline{G}_h$. The function $\overline{v}(x,t;t_1)$ is considered to satisfy a Lipschitz condition with respect to t. The "strip" $\overline{G}_h(t_1)$ consists of only two time levels $\overline{G}_h(t_1) = \{\overline{\omega} \times [t = t_1]\} \cup \{\overline{\omega} \times [t = t_1 + \tau]\}$, where $\overline{\omega}$ was introduced in (3.1).

Now we find discrete solutions $z^{\frac{k}{K}}(x,t)$ by a procedure similar to (5.5). That is, assuming that $z(x,t)$, $t \leq t^n$ is computed, we solve on the strip $\overline{G}_h(t^n)$ the following problems:

$$\left.\begin{array}{ll} \Lambda_{(6.3)}(z^{\frac{k}{K}}(x,t)) = 0, & (x,t) \in G^k_h(t^n), \\ z^{\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} \overline{z}(x,t;t^n), & k = 1, \\ z^{\frac{k-1}{K}}(x,t), & k \geq 2 \end{array} \right\}, & (x,t) \in S^k_h(t^n) \end{array}\right\} \text{ for } (x,t) \in \overline{G}^k_h(t^n), \tag{6.3a}$$

$$k = 1, \dots, K, \quad t^n \in \overline{\omega}_0, \quad n \leq N_0 - 1;$$

$$z^{\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} z^{\frac{k}{K}}(x,t), & (x,t) \in \overline{G}_h^k(t^n), \\ \left. \begin{array}{ll} \overline{z}(x,t;t^n), & k=1, \\ z^{\frac{k-1}{K}}(x,t), & k \geq 2 \end{array} \right\}, & (x,t) \in \overline{G}_h(t^n) \setminus \overline{G}_h^k(t^n) \end{array} \right\} \text{ for } (x,t) \in \overline{G}_h(t^n), \quad (6.3b)$$

$$k = 1, \dots, K, \quad t^n \in \overline{\omega}_0.$$

The required function $z(x,t)$ on the strip $\overline{G}_h(t^n)$ is determined by the relation

$$z(x,t) = z^{\frac{K}{K}}(x,t), \qquad (x,t) \in \overline{G}_h(t^n), \quad t^n \in \overline{\omega}_0. \tag{6.3c}$$

In relations (6.3a), (6.3b)

$$\overline{z}(x,t;t^n) = \overline{v}(x,t;t^n), \qquad (x,t) \in \overline{G}_h(t^n), \quad t^n \in \overline{\omega}_0. \tag{6.3d}$$

The function $\overline{v}(x,t;t^n)$, $(x,t) \in \overline{G}_h(t^n)$ is found, using $v(x,t;t^n)$, $(x,t) \in S_h(t^n)$,

$$\overline{v}(x,t;t^n) = \left\{ \begin{array}{ll} v(x,t\,;t^n), & (x,t) \in S_h(t^n), \\ v(x,t^n;t^n), & (x,t) \in G_h(t^n) \end{array} \right\}, \quad (x,t) \in \overline{G}_h(t^n), \tag{6.3e}$$

where

$$v(x,t;t^n) = \left\{ \begin{array}{ll} \varphi(x,t), \quad (x,t) \in S_h(t^n), & t^n = t^0 = 0, \\ \left. \begin{array}{lll} \varphi(x,t), & (x,t) \in S_h(t^n) \cap S_h, & t \geq t^n, \\ z(x,t), & (x,t) \in S_h(t^n) \setminus S_h, & t = t^n \end{array} \right\}, & t^n > 0 \end{array} \right\}, \tag{6.3f}$$

$$(x,t) \in S_h(t^n), \quad n = 0, 1, \dots, N_0 - 1.$$

On each strip $\overline{G}_h(t^n)$ the function $z^{\frac{k}{K}}(x,t)$ is the solution of a discrete Dirichlet problem on the set $\overline{G}_h^k(t^n)$. On the remaining part $\overline{G}_h(t^n) \setminus G_h^k(t^n)$, for $k=1$ it coincides with the function $\overline{z}(x,t;t^n)$, $(x,t) \in \overline{G}_h(t^n)$, and for $k \geq 2$ with the function $z^{\frac{k-1}{K}}(x,t)$, $(x,t) \in \overline{G}_h(t^n)$. We define an operator $\Lambda_{(6.3)}$ by the relation

$$\Lambda_{(6.3)}(z(x,t)) \equiv \Lambda_{(3.2)} z(x,t) - f(x,t), \quad (x,t) \in G_h. \tag{6.4}$$

We seek a function $z_{(6.3)}(x,t)$, $(x,t) \in \overline{G}_h$, i.e., the solution of difference scheme (6.3) either on the grid (4.1) or on the grid (3.1) . The difference scheme (6.3) can symbolically be written in operator form as

$$Q_{(6.3)}(z_{(6.3)}(x,t); f(\cdot), \varphi(\cdot), \psi(\cdot)) = 0, \quad (x,t) \in \overline{G}_h. \tag{6.3g}$$

Similarly to (5.9e), here the function $\psi(x,t;t^n)$, $(x,t) \in G_h(t^n)$ determines the function $\overline{z}(x,t;t^n)$:

$$\overline{z}(x,t;t^n) = \left\{ \begin{array}{ll} v(x,t;t^n), & (x,t) \in S_h(t^n), \\ v(x,t^n;t^n) + \psi(x,t;t^n), & (x,t) \in G_h(t^n) \end{array} \right\}, \quad (x,t) \in \overline{G}_h(t^n). \tag{6.3h}$$

In the above case of conditions (6.3d), (6.3e) we have $\psi(x,t;t^n) \equiv 0$.

In the discrete domain decomposition method (6.3), the intermediate problems on the subsets $\overline{D}_h^k = \overline{D}_{(5.1)}^k \cap \overline{D}_h$ are solved sequentially. Thus, to solve boundary value problem (2.1), here we used difference scheme (6.3), (3.1), which is the discrete equivalent of (5.5), (5.1). In the following section we extend this to the "parallel" case (5.9).

6.2. To describe the difference scheme that approximates process (5.9), (5.8) with P parallel processors, assume that $z(x,t)$ is known for $t \le t^n$; then we solve the following problems:

$$\Lambda_{(6.3)}(z_p^{\frac{k}{K}}(x,t)) = 0, \quad (x,t) \in G^k_{ph}(t^n), \tag{6.5a}$$

$$z_p^{\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} \overline{z}(x,t;t^n), & k=1, \\ z^{\frac{k-1}{K}}(x,t), & k \ge 2 \end{array} \right\}, \quad (x,t) \in S^k_{ph}(t^n), \quad p = 1,\dots,P$$

$$\text{for} \quad (x,t) \in \overline{G}^k_{ph}(t^n), \quad k = 1,\dots,K, \quad t^n \in \overline{\omega}_0, \quad n \le N_0 - 1;$$

$$z^{\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} z_p^{\frac{k}{K}}(x,t), & (x,t) \in \overline{G}^k_{ph}(t^n), \quad p = 1,\dots,P, \\ \left. \begin{array}{ll} \overline{z}(x,t;t^n), & k=1, \\ z^{\frac{k-1}{K}}(x,t), & k \ge 2 \end{array} \right\}, & (x,t) \in \overline{G}(t^n) \setminus \bigcup_{p=1}^{P} \overline{G}^k_p(t^n) \end{array} \right\}$$

$$\text{for} \quad (x,t) \in \overline{G}_h(t^n), \quad k = 1,\dots,K, \quad t^n \in \overline{\omega}_0.$$

We define a function $z_{(6.5)}(x,t)$ on the strip $\overline{G}_h(t^n)$ by the relation

$$z_{(6.5)}(x,t) = z^{\frac{K}{K}}(x,t), \quad (x,t) \in \overline{G}_h(t^n), \quad t^n \in \overline{\omega}_0. \tag{6.5b}$$

In (6.5a) the function $\overline{z}(x,t;t^n) = \overline{v}(x,t;t^n)$, $(x,t) \in \overline{G}_h(t^n)$. The function $\overline{v}(x,t;t^n)$, $(x,t) \in \overline{G}_h(t^n)$ is found, using $v(x,t;t^n)$, $(x,t) \in S_h(t^n)$, which is determined by relation (6.3e). Thus the function $z_{(6.5)}(x,t)$, $(x,t) \in \overline{G}_h$, i.e., the solution of difference scheme (6.5), (3.1) is found.

The difference scheme (6.5) can be written in operator form

$$Q_{(6.5)}(z_{(6.5)}(x,t); f(\cdot), \varphi(\cdot), \psi(\cdot)) = 0, \quad (x,t) \in \overline{G}_h, \tag{6.5c}$$

with $\psi(x,t;t^n) \equiv 0$.

In the discrete domain decomposition method (6.5), (3.1) the intermediate problems on the subsets $\overline{D}^k_{ph} = \overline{D}^k_{p(5.8)} \cap \overline{D}_h$ are solved independently of each other ("in parallel") for all $p = 1,\dots,P$. For $P = 1$ the difference scheme (6.5), (3.1) reduces to (6.3), (3.1).

Under condition (5.6), using the standard technique of comparison theorems, we get the following estimate:

$$|z_{(3.2)}(x,t) - z_{(6.5)}(x,t)| \le M N_0^{-1}, \quad (x,t) \in \overline{G}_h, \tag{6.6}$$

where $z_{(3.2)}(x,t)$ and $z_{(6.5)}(x,t)$ are solutions of difference schemes (3.2), (3.1) and (6.5), (3.1), respectively.

6.3. A technique similar to the one explained in [3, 4] gives us error bounds for the discrete solutions that are obtained by the difference schemes described above. Under condition (5.6), using the difference schemes (6.5), (3.1) and (6.5), (4.1), we obtain the following error estimates for the solution of boundary value problem (2.1):

$$|u(x,t) - z_{(6.5)}(x,t)| \le M(\varepsilon^{-1}N^{-1} + \tau), \qquad (x,t) \in \overline{G}_{h(3.1)}, \tag{6.7a}$$

$$|u(x,t) - z_{(6.5)}(x,t)| \le M(N^{-2}\ln^2 N + \tau), \quad (x,t) \in \overline{G}^*_{h(4.1)}. \tag{6.7b}$$

The above formulation allows us to summarize a result obtained in [14] as follows:

Theorem 6.1. *Let the hypotheses of Theorem 4.1 hold for the data of boundary value problem* (2.1) *and its solution. Then, under condition* (5.6) *and for* N, $N_0 \to \infty$, *the solution of the difference scheme* (6.5), (4.1) (*or scheme* (6.5), (3.1)) *converges to the solution of* (2.1) ε*-uniformly* (*for a fixed value of* ε). *The estimates* (6.6), (6.7) *hold for the solutions of these difference schemes.*

Remark. If the condition $n = 0$ of Theorem 4.1 is replaced by $n = K$, $K \geq -1$, the following estimate holds:

$$\left|\delta_{l\bar{t}}\left(z_{(3.2)}(x,t) - z_{(6.5)}(x,t)\right)\right| \leq M\, M_{(4.3)}^{(l+1)}\, N_0^{-1}, \quad (x,t) \in \overline{G}_h, \quad t \geq l\tau, \quad l \leq K+1.$$

7. Improved time accuracy

7.1. A scheme based on defect correction. The technique used in this paper to improve time-accuracy is based on the one in [3]. For the difference scheme (3.2), (4.1) the error in the approximation of the partial derivative $(\partial/\partial t)\,u(x,t)$ is caused by the divided difference $\delta_{\bar{t}}\, z(x,t)$ and is associated with the truncation error given by the relation

$$\frac{\partial}{\partial t}u(x,t) - \delta_{\bar{t}}\,u(x,t) = 2^{-1}\,\tau\frac{\partial^2}{\partial t^2}u(x,t) - 6^{-1}\,\tau^2\frac{\partial^3}{\partial t^3}u(x,t-\vartheta), \tag{7.1}$$

where $\vartheta \in [0,\tau]$. Therefore, we now use for the approximation of $(\partial/\partial t)\,u(x,t)$ the expression $\delta_{\bar{t}}\,u(x,t) + \tau\delta_{\bar{t}\bar{t}}\,u(x,t)/2$, where $\delta_{\bar{t}\bar{t}}\,u(x,t) \equiv \delta_{t\bar{t}}\,u(x,t-\tau)$, $\delta_{t\bar{t}}\,u(x,t)$ is the second central divided difference. We can obtain a better approximation than (3.2a) by the defect correction

$$\Lambda_{(3.2)}z^c(x,t) = f(x,t) + 2^{-1}p(x,t)\tau\,\frac{\partial^2}{\partial t^2}u(x,t), \tag{7.2}$$

with $x \in \overline{\omega}$ and $t \in \overline{\omega}_0$, where $\overline{\omega}$ and $\overline{\omega}_0$ are as in (3.1); τ is the step-size of the grid $\overline{\omega}_0$; $z^c(x,t)$ is the "corrected" solution. Instead of $(\partial^2/\partial t^2)\,u(x,t)$ we shall use $\delta_{\bar{t}\bar{t}}\,z(x,t)$, where $z(x,t)$, $(x,t) \in G_{h(4.1)}$, is the solution of the difference scheme (3.2), (4.1). The new solution $z^c(x,t)$ has a consistency error of the order of $O(\tau^2)$.

7.2. The defect correction scheme of second-order accuracy in time. Constructing the difference scheme in (7.2), instead of $(\partial^2/\partial t^2)u(x,t)$ we use $\delta_{2\bar{t}}\,z(x,t)$, the second divided difference of the solution to the discrete problem (3.2), (4.1). On $\overline{G}_h$ we write the finite difference scheme (3.2) as

$$\Lambda_{(3.2)}z^{(1)}(x,t) = f(x,t), \quad (x,t) \in G_h, \qquad z^{(1)}(x,t) = \varphi(x,t), \quad (x,t) \in S_h, \tag{7.3}$$

where $z^{(1)}(x,t)$ is the uncorrected solution. For the corrected solution $z^{(2)}(x,t)$ we solve the problem for $(x,t) \in G_h$

$$\Lambda_{(3.2)}z^{(2)}(x,t) = f(x,t) + \left\{\begin{array}{ll} p(x,t)2^{-1}\,\tau\dfrac{\partial^2}{\partial t^2}u(x,0), & t=\tau, \\ p(x,t)2^{-1}\,\tau\delta_{2\bar{t}}\,z^{(1)}(x,t), & t \geq 2\tau \end{array}\right\}, \quad (x,t) \in G_h,$$

$$z^{(2)}(x,t) = \varphi(x,t), \quad (x,t) \in S_h. \tag{7.4}$$

Here the derivative $\dfrac{\partial^2}{\partial t^2}u(x,0)$ is obtained from equation (2.1a).

To clarify the construction, in the remainder of this section we consider a homogeneous initial condition:

$$\varphi(x,0) = 0, \quad x \in \overline{D}. \tag{7.5}$$

Under this condition, the following estimate [4] holds for the solution of problem (7.4), (4.1):

$$\left| u(x,t) - z^{(2)}(x,t) \right| \le M \left[N^{-2} \ln^2 N + \tau^2 \right], \quad (x,t) \in \overline{G}_h. \tag{7.6}$$

This is more properly formulated in the following theorem [4]:

Theorem 7.1. *Let condition* (7.5) *hold and assume in equation* (2.1) *that* $a,\ c,\ p,\ f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n = K$, $K \ge 1$ *and let condition* (11.3) *and the estimates* (11.6), (11.7) *be satisfied for* $n = K$. *Then for the solution of difference scheme* (7.4), (4.1) *estimate* (7.6) *holds.*

7.3. The defect correction scheme of third-order accuracy in time. The above procedure can be used to obtain an arbitrary large order of accuracy in time. Here we only show how to construct a difference scheme of third order accuracy. On the grid $\overline{G}_h$ we consider the difference scheme

$$\Lambda_{(3.2)}\, z^{(3)}(x,t) = f(x,t) + \left\{ \begin{array}{ll} p(x,t)\left(C_{11}\tau \frac{\partial^2}{\partial t^2} u(x,0) + C_{12}\tau^2 \frac{\partial^3}{\partial t^3} u(x,0) \right), & t = \tau, \\ p(x,t)\left(C_{21}\tau \frac{\partial^2}{\partial t^2} u(x,0) + C_{22}\tau^2 \frac{\partial^3}{\partial t^3} u(x,0) \right), & t = 2\tau, \\ p(x,t)\left(C_{31}\tau \delta_{2\bar{t}} z^{(2)}(x,t) + C_{32}\tau^2 \delta_{3\bar{t}} z^{(1)}(x,t) \right), & t \ge 3\tau \end{array} \right\}, \quad (x,t) \in G_h, \tag{7.7a}$$

$$z^{(3)}(x,t) = \varphi(x,t), \quad (x,t) \in S_h.$$

Here $z^{(1)}(x,t)$ and $z^{(2)}(x,t)$ are the solutions of problems (7.3) and (7.4), respectively, the derivatives $(\partial^2/\partial t^2)u(x,0)$, $(\partial^3/\partial t^3)u(x,0)$ are again obtained from (2.1a). The coefficients C_{ij} are determined below. They are chosen such that they satisfy the following conditions:

$$\frac{\partial}{\partial t} u(x,t) = \delta_{\bar{t}} u(x,t) + C_{11}\tau \frac{\partial^2}{\partial t^2} u(x,t-\ \tau) + C_{12}\tau^2 \frac{\partial^3}{\partial t^3} u(x,t-\ \tau) + O(\tau^3),$$

$$\frac{\partial}{\partial t} u(x,t) = \delta_{\bar{t}} u(x,t) + C_{21}\tau \frac{\partial^2}{\partial t^2} u(x,t-2\tau) + C_{22}\tau^2 \frac{\partial^3}{\partial t^3} u(x,t-2\tau) + O(\tau^3),$$

$$\frac{\partial}{\partial t} u(x,t) = \delta_{\bar{t}} u(x,t) + C_{31}\tau \delta_{2\bar{t}} u(x,t) + C_{32}\tau^2 \delta_{3\bar{t}} u(x,t) + O(\tau^3).$$

It follows that

$$C_{11} = C_{21} = C_{31} = 1/2, \quad C_{12} = C_{32} = 1/3, \quad C_{22} = 5/6. \tag{7.7b}$$

Again, for simplicity, we assume the homogeneous initial condition

$$\varphi(x,0) = 0, \quad f(x,0) = 0, \quad x \in \overline{D}. \tag{7.8}$$

It is proved in [4] that under condition (7.8) the following error estimate holds for the solution $z^{(3)}(x,t)$ of scheme (7.7):

$$\left| u(x,t) - z^{(3)}(x,t) \right| \le M \left[N^{-2} \ln^2 N + \tau^3 \right], \quad (x,t) \in \overline{G}_h. \tag{7.9}$$

This is more properly formulated in the following theorem:

Theorem 7.2. *Let conditions* (7.8) *hold and assume in equation* (2.1) *that* $a,\ c,\ p,\ f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n = K$, $K \ge 2$ *and let condition* (11.3) *and the estimates* (11.6), (11.7) *be satisfied for* $n = K$. *Then for the solution of scheme* (7.7), (4.1) *the estimate* (7.9) *is valid.*

Illustrative numerical results without domain decomposition are discussed in [3, 4]. These results demonstrate the efficiency of the defect correction technique in improving the accuracy with respect to the time variable. However, in this paper we are interested in distributing the above algorithm over a number of independent ("parallel") processes.

8. Parallel method based on defect correction

8.1. Difference schemes of second-order accuracy in τ. Now we describe a finite difference scheme (6.3) constructed for the modified Schwarz method (5.5) with $P = 1$ in the case of defect correction. To approximate the alternating process (5.5), we apply the defect correction scheme (7.4), (4.1) to the discrete equations (6.3), (6.2).

First we find a function $z^{(1)}(x,t)$, $(x,t) \in \overline{G}_h(t^n)$, solving problem (6.3), (6.2)

$$z^{(1)}(x,t) \equiv z_{(6.3;6.2)}(x,t) \tag{8.1}$$

where $z^{\frac{k}{K}}(x,t)$ and z(x,t) are now denoted by $z^{(1)\frac{k}{K}}(x,t)$ and $z^{(1)}(x,t)$ respectively. To make a precise reference later, we write the procedure (6.3), (6.2) now as

$$\Lambda_{(6.3;6.2)}\left(z^{(1)\frac{k}{K}}(x,t)\right) = 0, \quad (x,t) \in G_h^k(t^n), \tag{8.2a}$$

$$z^{(1)\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} \overline{z}^{(1)}(x,t;t^n), & k = 1, \\ z^{(1)\frac{k-1}{K}}(x,t), & k \ge 2 \end{array} \right\}, \quad (x,t) \in S_h^k(t^n)$$

$$\text{for} \quad (x,t) \in \overline{G}_h^k(t^n), \quad k = 1,\dots,K, \quad t^n \in \overline{\omega}_0;$$

$$z^{(1)\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} z^{(1)\frac{k}{K}}(x,t), & (x,t) \in \overline{G}_h^k(t^n), \\ \left. \begin{array}{ll} \overline{z}^{(1)}(x,t;t^n), & k = 1, \\ z^{(1)\frac{k-1}{K}}(x,t), & k \ge 2 \end{array} \right\}, & (x,t) \in \overline{G}_h(t^n) \setminus \overline{G}_h^k(t^n) \end{array} \right\}$$

$$\text{for} \quad (x,t) \in \overline{G}_h(t^n), \quad k = 1,\dots,K, \quad t^n \in \overline{\omega}_0.$$

$$z^{(1)}(x,t) = z^{(1)\frac{K}{K}}(x,t), \quad (x,t) \in \overline{G}_h(t^n), \quad t^n \in \overline{\omega}_0. \tag{8.2b}$$

Here

$$\overline{z}^{(1)}(x,t;t^n) = \overline{v}^{(1)}(x,t;t^n) = \overline{v}_{(6.3e)}(x,t;t^n), \quad (x,t) \in \overline{G}_h(t^n), \quad t^n \in \overline{\omega}_0.$$

Now we find $z^{(2)\frac{k}{K}}(x,t)$ for $(x,t) \in \overline{G}_h(t^n)$, solving the corrected problem

$$\Lambda_{(8.3)}\left(z^{(2)\frac{k}{K}}(x,t)\right) = 0, \quad (x,t) \in G_h^k(t^n), \tag{8.3a}$$

$$z^{(2)\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} \overline{z}^{(2)}(x,t;t^n), & k=1, \\ z^{(2)\frac{k-1}{K}}(x,t), & k \geq 2 \end{array} \right\}, \quad (x,t) \in S_h^k(t^n)$$

$$\text{for} \quad (x,t) \in \overline{G}_h^k(t^n), \quad k = 1,\dots,K, \quad t^n \in \overline{\omega}_0;$$

$$z^{(2)\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} z^{(2)\frac{k}{K}}(x,t), & (x,t) \in \overline{G}_h^k(t^n), \\ \left. \begin{array}{ll} \overline{z}^{(2)}(x,t;t^n), & k=1, \\ z^{(2)\frac{k-1}{K}}(x,t), & k \geq 2 \end{array} \right\}, & (x,t) \in \overline{G}_h(t^n) \setminus \overline{G}_h^k(t^n) \end{array} \right\}$$

$$\text{for} \quad (x,t) \in \overline{G}_h(t^n), \quad k = 1,\dots,K, \quad t^n \in \overline{\omega}_0.$$

The function $z^{(2)}(x,t)$ on the strip $\overline{G}_h(t^n)$ is defined by the relation

$$z^{(2)}(x,t) = z^{(2)\frac{K}{K}}(x,t), \quad (x,t) \in \overline{G}_h(t^n), \quad t^n \in \overline{\omega}_0. \tag{8.3b}$$

Here

$$\begin{aligned} \overline{z}^{(2)}(x,t;t^n) &= \overline{v}^{(2)}(x,t;t^n), \quad (x,t) \in \overline{G}_h(t^n), \\ \overline{v}^{(2)}(x,t;t^n) &= \overline{v}^{(2)}(x,t;t^n,z^{(1)}(\cdot)) \\ &= v(x,t^n;t^n) + z^{(1)}(x,t^{n+1}) - z^{(1)}(x,t^n), \quad (x,t) \in G_h(t^n), \\ v(x,t;t^n) &= v_{(6.3f)}(x,t;t^n), \quad (x,t) \in S_h(t^n); \end{aligned}$$

$$\begin{aligned} \Lambda_{(8.3)}\left(z^{(2)}(x,t)\right) &\equiv \Lambda_{(3.2)} z^{(2)}(x,t) - f^{(2)}(x,t), \quad (x,t) \in G_h^k, \\ f^{(2)}(x,t) &= f^{(2)}(x,t;\, z^{(1)}(\cdot)) \\ &= f(x,t) + \left\{ \begin{array}{ll} p(x,t)2^{-1}\tau \dfrac{\partial^2}{\partial t^2} u(x,0), & t = \tau, \\ p(x,t)2^{-1}\tau \delta_{2\bar{t}} z^{(1)}(x,t), & t \geq 2\tau \end{array} \right\}, \quad (x,t) \in G_h^k. \end{aligned} \tag{8.3c}$$

We call the function $z_{(8.3)}^{(2)}$ the solution of the domain decomposition – defect correction scheme. The difference scheme (8.3), (4.1) symbolically can be written in operator form

$$\begin{aligned} Q_{(6.3)}(z^{(1)}(x,t); f^{(1)}(\cdot), \varphi(\cdot), \psi^{(1)}(\cdot)) = 0, \quad (x,t) \in \overline{G}_h, \\ Q_{(6.3)}(z^{(2)}(x,t); f^{(2)}(\cdot), \varphi(\cdot), \psi^{(2)}(\cdot)) = 0, \quad (x,t) \in \overline{G}_h, \end{aligned} \tag{8.3d}$$

where

$$f^{(1)}(x,t) = f(x,t), \quad f^{(2)}(x,t) = f_{(8.3c)}^{(2)}(x,t;z^{(1)}(\cdot)), \quad \psi^{(1)}(x,t;t^n) \equiv 0,$$

$$\psi^{(2)}(x,t;t^n) = \psi^{(2)}(x,t;t^n,z^{(1)}(\cdot)) = z^{(1)}(x,t^{n+1}) - z^{(1)}(x,t^n), \quad (x,t) \in G_h(t^n), \quad t = t^{n+1}.$$

For the solutions of difference scheme (8.3), (4.1) the estimate (7.6) holds also (assuming that condition (5.6) and the hypotheses of Theorem 7.1 are satisfied).

In the case of $P > 1$ we discretize the process (5.9), (5.8). In grid constructions (8.2a) and (8.3a), solving the finite difference boundary problems on the G_h^k, the functions $z^{(1)\frac{k}{K}}(x,t)$

and $z^{(2)\frac{k}{K}}(x,t)$ are replaced by the functions $z_p^{(1)\frac{k}{K}}(x,t)$ and $z_p^{(2)\frac{k}{K}}(x,t)$, and the set G_h^k is replaced by the set G_{ph}^k

$$\Lambda_{(8.4)}^{(i)}\left(z_p^{(i)\frac{k}{K}}(x,t)\right) = 0, \quad (x,t) \in G_{ph}^k(t^n), \tag{8.4a}$$

$$z_p^{(i)\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} \overline{z}^{(i)}(x,t;t^n), & k=1, \\ z^{(i)\frac{k-1}{K}}(x,t), & k \geq 2 \end{array} \right\}, \quad (x,t) \in S_{ph}^k(t^n), \quad p = 1,\dots,P$$

$$\text{for } \ (x,t) \in \overline{G}_{ph}^k(t^n), \quad k = 1,\dots,K, \quad t^n \in \overline{\omega}_0, \quad i = 1,2;$$

$$z^{(i)\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} z_p^{(i)\frac{k}{K}}(x,t), & (x,t) \in \overline{G}_{ph}^k(t^n), \quad p = 1,\dots,P, \\ \left. \begin{array}{ll} \overline{z}^{(i)}(x,t;t^n), & k=1, \\ z^{(i)\frac{k-1}{K}}(x,t), & k\geq 2 \end{array} \right\}, & (x,t) \in \overline{G}_h(t^n) \setminus \bigcup_{p=1}^P \overline{G}_{ph}^k(t^n) \end{array} \right\}$$

$$\text{for } \ (x,t) \in \overline{G}_h(t^n), \quad k = 1,\dots,K, \quad t^n \in \overline{\omega}_0, \quad i = 1,2;$$

$$z^{(i)}(x,t) = z^{(i)\frac{k}{K}}(x,t), \quad (x,t) \in \overline{G}_h(t^n), \quad t^n \in \overline{\omega}_0, \quad i = 1,2.$$

Here

$$\overline{z}^{(i)}(x,t;t^n) = \overline{v}^{(i)}(x,t;t^n), \quad (x,t) \in \overline{G}_h(t^n),$$

$$\overline{v}^{(i)}(x,t;t^n) = \overline{v}^{(i)}(x,t;t^n,\psi^{(i)}(\cdot))$$

$$= \left\{ \begin{array}{ll} v(x,t^n;t^n) + \psi^{(i)}(x,t;t^n), & (x,t) \in G(t^n), \\ v(x,t;t^n), & (x,t) \in S(t^n) \end{array} \right\}, \quad (x,t) \in \overline{G}_h(t^n);$$

$$v(x,t;t^n) = v_{(6.3f)}(x,t;t^n), \quad (x,t) \in S_h(t^n),$$

$$\psi^{(1)}(x,t;t^n) \equiv 0, \quad \psi^{(2)}(x,t;t^n) = \psi_{(8.3)}^{(2)}(x,t;t^n,z_{(8.4)}^{(1)}(\cdot)), \quad (x,t) \in G_h(t^n), \quad t = t^{n+1};$$

$$\Lambda_{(8.4)}^{(i)}(z_p^{(i)\frac{k}{K}}(x,t)) \equiv \Lambda_{(3.2)} z_p^{(i)\frac{k}{K}}(x,t) - f^{(i)}(x,t), \quad (x,t) \in G_{ph}^k,$$

$$f^{(1)}(x,t) = f(x,t), \quad f^{(2)}(x,t) = f_{(8.3)}^{(2)}(x,t;z_{(8.4)}^{(1)}(\cdot)).$$

In operator form the difference scheme (8.4), (6.2) is written

$$\begin{array}{l} Q_{(6.5)}(z^{(1)}(x,t); \quad f^{(1)}(\cdot),\, \varphi(\cdot),\, \psi^{(1)}(\cdot)) = 0, \quad (x,t) \in \overline{G}_h, \\ Q_{(6.5)}(z^{(2)}(x,t); \quad f^{(2)}(\cdot),\, \varphi(\cdot),\, \psi^{(2)}(\cdot)) = 0, \quad (x,t) \in \overline{G}_h, \end{array} \tag{8.4b}$$

where

$$f^{(1)}(x,t) = f(x,t), \quad f^{(2)}(x,t) = f_{(8.3)}^{(2)}(x,t;z_{(8.4)}^{(1)}(\cdot)),$$

$$\psi^{(1)}(x,t;t^n) \equiv 0, \quad \psi^{(2)}(x,t;t^n) = \psi_{(8.3)}^{(2)}(x,t;t^n,z_{(8.4)}^{(1)}(\cdot)).$$

Following the reasoning given in [3, 4, 14] the following theorem can be derived:

Theorem 8.1. *Let the boundary value problem* (2.1) *and its solution satisfy the assumptions of Theorem 7.1. Then, under condition* (5.6), *the solutions of the difference schemes* (8.3), (4.1) *and* (8.4), (4.1) *converge, as* N, $N_0 \to \infty$, *to the solution of the boundary value problem* ε*-uniformly (the solutions of schemes* (8.3), (3.1) *and* (8.4), (3.1) *converge for a fixed value of* ε*). For the solutions of the difference schemes on grid* (4.1) *the estimate* (7.6) *holds.*

Remark. If the conditions of Theorem 7.1, where $n = K+1$, $K \geq -1$, are satisfied, the following estimate holds:

$$\left|\delta_{l\bar{t}}\left(z_{(7.4)}(x,t) - z_{(8.4)}(x,t)\right)\right| \leq M M^{(l)}_{(4.3)} N_0^{-2}, \quad (x,t) \in \overline{G}_{h(4.1)}, \quad l \leq K+1.$$

8.2. Difference schemes of third-order accuracy in τ. We approximate the boundary value problem by the alternating scheme with one processor

$$\Lambda^{(i)}_{(8.5)}\left(z^{(i)\frac{k}{K}}(x,t)\right) = 0, \quad (x,t) \in G^k_h(t^n), \tag{8.5}$$

$$z^{(i)\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} \overline{z}^{(i)}(x,t;t^n), & k=1, \\ z^{(i)\frac{k-1}{K}}(x,t), & k \geq 2 \end{array} \right\}, \quad (x,t) \in S^k_h(t^n)$$

for $(x,t) \in \overline{G}^k_h(t^n)$, $k = 1,\dots,K$, $t^n \in \overline{\omega}_0$, $i = 1,2,3$;

$$z^{(i)\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} z^{(i)\frac{k}{K}}(x,t), & (x,t) \in \overline{G}^k_{ph}(t^n), \\ \left\{ \begin{array}{ll} \overline{z}^{(i)}(x,t;t^n), & k=1, \\ z^{(i)\frac{k-1}{K}}(x,t), & k \geq 2 \end{array} \right\}, & (x,t) \in \overline{G}_h(t^n) \setminus \overline{G}^k_h(t^n) \end{array} \right\}$$

for $(x,t) \in \overline{G}_h(t^n)$, $k = 1,\dots,K$, $t^n \in \overline{\omega}_0$, $i = 1,2,3$;

$$z^{(i)}(x,t) = z^{(i)\frac{K}{K}}(x,t), \quad (x,t) \in \overline{G}_h(t^n), \quad t^n \in \overline{\omega}_0, \quad i = 1,2,3.$$

Here

$$\overline{z}^{(i)}(x,t;t^n) = \overline{v}^{(i)}_{(8.4)}(x,t;t^n,\psi^i(\cdot)), \quad (x,t) \in \overline{G}_h(t^n), \quad i = 1,2,3;$$

$$\overline{\psi}^{(s)}(x,t;t^n) = \overline{\psi}^{(s)}_{(8.4)}(x,t;t^n), \quad s = 1,2,$$

$$\overline{\psi}^{(3)}(x,t;t^n) = z^{(2)}(x,t^{n+1}) - z^{(2)}(x,t^n);$$

$$\Lambda^{(s)}_{(8.5)}(z^{(s)\frac{k}{K}}(x,t)) = \Lambda^{(s)}_{(8.4)}(z^{(s)\frac{k}{K}}(x,t)), \quad s = 1,2,$$

$$\Lambda^{(3)}_{(8.5)}(z^{(3)\frac{k}{K}}(x,t)) = \Lambda_{(3.2)} z^{(3)\frac{k}{K}}(x,t) - f^{(3)}(x,t),$$

$$f^{(3)}(x,t) = f^{(3)}(x,t;z^{(1)}(\cdot),z^{(2)}(\cdot))$$

$$= f(x,t) + \left\{ \begin{array}{ll} p(x,t)\left(C_{11}\tau \frac{\partial^2}{\partial t^2}u(x,0) + C_{12}\tau^2 \frac{\partial^3}{\partial t^3}u(x,0)\right), & t = \tau, \\ p(x,t)\left(C_{21}\tau \frac{\partial^2}{\partial t^2}u(x,0) + C_{22}\tau^2 \frac{\partial^3}{\partial t^3}u(x,0)\right), & t = 2\tau, \\ p(x,t)\left(C_{31}\tau \delta_{2\bar{t}} z^{(2)}(x,t) + C_{32}\tau^2 \delta_{3\bar{t}} z^{(1)}(x,t)\right), & t \geq 3\tau \end{array} \right\},$$

$$(x,t) \in G^k_h, \quad C_{ij} = C_{ij\,(7.7)}.$$

In the case of $P > 1$ processors we use the scheme

$$\Lambda^{(i)}_{(8.5)}\left(z_p^{(i)\frac{k}{K}}(x,t)\right) = 0, \quad (x,t) \in G^k_{ph}(t^n), \tag{8.6}$$

$$z_p^{(i)\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} \overline{z}^{(i)}(x,t;t^n), & k=1, \\ z^{(i)\frac{k-1}{K}}(x,t), & k \geq 2 \end{array} \right\}, \quad (x,t) \in S^k_{ph}(t^n), \quad p = 1,\dots,P$$

for $(x,t) \in \overline{G}^k_{ph}(t^n)$, $k = 1,\dots,K$, $t^n \in \overline{\omega}_0$, $i = 1,2,3$;

$$z^{(i)\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} z_p^{(i)\frac{k}{K}}(x,t), & (x,t) \in \overline{G}_{ph}^{k}(t^n), \quad p = 1,\dots,P, \\ \left. \begin{array}{ll} \overline{z}^{(i)}(x,t;t^n), & k = 1, \\ z^{(i)\frac{k-1}{K}}(x,t), & k \geq 2 \end{array} \right\}, & (x,t) \in \overline{G}_h(t^n) \setminus \bigcup_{p=1}^{P} \overline{G}_{ph}^{k}(t^n) \end{array} \right\}$$

$$\text{for } (x,t) \in \overline{G}_h(t^n), \quad k = 1,\dots,K, \quad t^n \in \bar{\omega}_0, \quad i = 1,2,3;$$

$$z^{(i)}(x,t) = z^{(i)\frac{K}{K}}(x,t), \quad (x,t) \in \overline{G}_h(t^n), \quad t^n \in \bar{\omega}_0, \quad i = 1,2,3.$$

Here

$$\overline{z}^{(i)}(x,t;\, t^n) = \overline{v}^{(i)}(x,t;\, t^n), \quad (x,t) \in \overline{G}_h(t^n), \qquad \overline{v}^{(i)}(x,t;\, t^n) = \overline{v}^{(i)}_{(8.4)}\left(x,t;\, t^n,\, \psi^i(\cdot)\right),$$

$$\psi^{(i)}(x,t;\, t^n) = \psi^{(i)}_{(8.5)}(x,t;\, t^n) = \psi^{(i)}_{(8.5)}\left(x,t;\, t^n,\, z^{(j)}_{(8.6)}(\cdot),\, 0 < j < i\right),$$

$$f^{(i)}(x,t) = f^{(i)}_{(8.5)}\left(x,t;\, z^{(j)}_{(8.6)}(\cdot),\, 0 < j < i\right).$$

Now the following theorem can be derived:

Theorem 8.2. *Let the boundary value problem* (2.1) *and its solution satisfy the assumptions of Theorem 7.2. Then, under condition* (5.6), *the solutions of the difference schemes* (8.5), (4.1) *and* (8.6), (4.1) *converge, as* N, $N_0 \to \infty$, *to the solution of the boundary value problem* ε*-uniformly* (*the solutions of schemes* (8.5), (3.1) *and* (8.6), (3.1) *converge for a fixed value of* ε). *For the solutions of the difference schemes on grid* (4.1) *the estimate* (7.9) *holds.*

Proof of Theorems 8.1 and 8.2 can be obtained by using the technique developed in [3, 4, 14].

9. Distribution of the scheme (3.2), (4.1) over independent processes

In this section we compare basic scheme (3.2), (4.1) with scheme (6.5), (4.1), that is, a decomposition scheme for P parallel solvers. These schemes have the same order of accuracy. For problem (6.5), (4.1) the computation time (of solving the grid problem) for sufficiently large number of solvers P can be essentially less than that for problem (3.2), (4.1).

For simplicity, we consider that the time of work for one solver (the time of solving a problem and/or the intermediate subproblem) is determined only by the number of nodes in the grid set where the boundary value problem (subproblem) is solved; all the rest operations are realized instantly. In the case of scheme (6.3), (4.1) for sequential solvers, the computation time is greater than that for scheme (3.2), (4.1) because of the covering of the subdomains. For relatively small overlapping of the subdomains the computation times for problems (3.2), (4.1) and (6.3), (4.1) are close. For scheme (6.5), (4.1) with parallel solvers the computation time, depending on the number P of the solvers, can be less in many cases than that for schemes (3.2), (4.1) and (6.3), (4.1).

If one and the same generative grid (4.1) is used for all the schemes, it can turn out that the errors introduced by the domain decomposition (with the same order of accuracy) essentially exceed the errors of the basic scheme (3.2), (4.1). This fact reduces the effect of acceleration of the solution process with the use of parallel solvers. Thus, in the case of basic difference scheme (3.2), (4.1), the formal application of the schemes for parallel solvers (on the same grid (4.1)) leads to reducing the computation time and also to automatic growth of the error in the approximate solution, i.e., the basic scheme and the scheme for parallel solvers on the same grid (4.1) turn out to be incomparable.

It is convenient to compare the computation time of the basic difference scheme (3.2), (4.1) with a scheme for parallel solvers, but already on another (denser) grid, in case when the errors of the discrete solutions for both the schemes are equal.

It is appropriate to speak that the use of parallel computations leads to acceleration of the solution process (compared to the basic scheme (3.2), (4.1)), if such a scheme with $P > 1$ parallel solvers can be found for which the computation time turns out to be smaller, and the accuracy of the approximate solution is not less than that for the basic scheme.

9.1. The error of the basic scheme (3.2), (4.1) can be represented as the sum of two components generated by the errors of approximation of the space and time derivatives, respectively:

$$\delta(x,t) = \delta_1(x,t) + \delta_2(x,t), \quad (x,t) \in \overline{G}_h,$$

where $\delta(x,t) = u(x,t) - z_{(3.2)}(x,t)$, $(x,t) \in \overline{G}_h$, and, by assumption, the component $\delta_2(x,t)$ is generated by the error of approximation of the derivative w.r.t. time. The function $\delta_2(x,t)$ is the solution of the problem

$$\Lambda_{(3.2)}\delta_2(x,t) = \psi_2(x,t), \quad (x,t) \in G_h, \qquad \delta_2(x,t) = 0, \quad (x,t) \in S_h,$$

where

$$\psi_2(x,t) = p(x,t)\left[\frac{\partial}{\partial t}u(x,t) - \delta_{\bar{t}}u(x,t)\right], \qquad (x,t) \in G_h.$$

We consider difference schemes for P parallel solvers on the grids

$$\overline{G}_{ph}^{kP} = \overline{G}_p^k \cap \overline{G}_h^P, \quad \overline{G}_h = \overline{G}_h^P = \overline{D}_h \times \overline{\omega}_0^P, \tag{9.1}$$

where $\overline{D}_{h(9.1)} = \overline{D}_{h(4.1)}$, $\overline{\omega}_0^P$ is a uniform grid on $[0,T]$ with number of nodes $N_0^P + 1$ and grid step τ^P; generally speaking, $\overline{\omega}_{0(9.1)}^P \neq \overline{\omega}_{0(4.1)}^P$. For these difference schemes the error of grid solution $\delta^P(x,t) = u(x,t) - z_{(6.5)}^P(x,t)$, $(x,t) \in \overline{G}_{h(9.1)}$, where $z_{(6.5)}^P(x,t)$ is the solution of scheme (6.5) on grid (9.1), can be represented as the sum of functions,

$$\delta^P(x,t) = \delta_1^P(x,t) + \delta_2^P(x,t) + \delta_3^P(x,t), \quad (x,t) \in \overline{G}_h,$$

where the components $\delta_1^P(x,t)$, $\delta_2^P(x,t)$, and $\delta_3^P(x,t)$ are generated, respectively, by the errors of approximation of the space and time derivatives and by the discrepancy of the functions $z_{(3.2;9.1)}(x,t)$ and $z_{(6.5;9.1)}^P(x,t)$ on the sets $\Gamma_{ph}^k \times \overline{\omega}_{0(9.1)}$, where $z_{(3.2;9.1)}(x,t)$ and $z_{(6.5;9.1)}^P(x,t)$ are the solutions of problem (3.2) and (6.5) on the grids $\overline{G}_{h(9.1)}^P$ and $\overline{G}_{ph(9.1)}^{kP}$, respectively, $\Gamma_p^k = \overline{D}_p^k \setminus D_p^k$. The function $\delta_2^P(x,t)$ is the solution of the problem

$$\Lambda_{(3.2)}\delta_2^P(x,t) = \psi_2^P(x,t), \quad (x,t) \in G_h, \qquad \delta_2^P(x,t) = 0, \quad (x,t) \in S_h,$$

where $\overline{G}_h = \overline{G}_{h(9.1)}$,

$$\psi_2^P(x,t) = p(x,t)\left[\frac{\partial}{\partial t}u(x,t) - \delta_{\bar{t}}u(x,t)\right], \quad (x,t) \in G_h^P.$$

The function $\delta_3^P(x,t)$ is defined by the relation

$$\delta_3^P(x,t) = z_{(3.2;9.1)}(x,t) - z_{(6.5;9.1)}^P(x,t), \quad (x,t) \in G_h^P.$$

We say that difference scheme (6.5), (9.1) for P parallel solvers accelerates (for a fixed value of the parameter ε or ε-uniformly) the solution of the boundary value problem, if the duration of computations (for the fixed value of the parameter ε or ε-uniformly) by scheme (6.5), (9.1) turns out to be less than that by the basic scheme, and, besides, the following condition holds:

$$\max_{\overline{G}_h^P} |\delta_2^P(x,t) + \delta_3^P(x,t)| \le \max_{\overline{G}_h} |\delta_2^P(x,t)|, \tag{9.2}$$

where $\overline{G}_h = \overline{G}_{h(4.1)}$, $\overline{G}_h^P = \overline{G}_{h(9.1)}$.

9.2. We compare the duration of computations for schemes (3.2), (4.1) and (6.5), (9.1) in case when the difference derivatives $\delta_{\bar{t}} z_{(3.2)}(x,t)$ are sufficiently large as compared with the derivatives $(\partial^2/\partial t^2)u(x,t)$.

Further we need estimates for the quantities $\delta_2(x,t)$, $\delta_2^P(x,t)$ and $\delta_3^P(x,t)$.

For the quanitities $\delta_2^P(x,t)$ and $\delta_2(x,t)$ the following estimates hold:

$$|\delta_2^P(x,t)| \le 2^{-1}T\tau^P \max_{\overline{G}} \left| \frac{\partial^2}{\partial t^2} u(x,t) \right| \le M_1 \tau^P M_{(4.3)}^{(2)}, \quad (x,t) \in \overline{G}_{h(9.1)}^P, \tag{9.3a}$$

$$|\delta_2(x,t)| \le M_1 \tau M_{(4.3)}^{(2)}, \quad (x,t) \in \overline{G}_{h(4.1)}, \tag{9.4}$$

where $M = 2^{-1}T$. For the quantity $\delta_3^P(x,t)$, we have the estimate

$$|\delta_3^P(x,t)| \le M_2 M_{(4.4)}^{(1)} \tau^P (1 + \varepsilon^2 \delta^{-2}), \quad (x,t) \in \overline{G}_{h(9.1)}^P. \tag{9.3b}$$

We give also some bounds from below for the values of $|\delta_2(x,t)|$ and $|\delta_3^P(x,t)|$.

Let on a certain set

$$\overline{G}^0 = \left\{ (x,t) : \; x_0^1 \le x \le x_0^2, \; t_0^1 \le t \le t_0^2 \right\}, \quad \overline{G}^0 \subseteq \overline{G}, \tag{9.5a}$$

the following condition hold:

$$\left| \frac{\partial^2}{\partial t^2} u(x,t) \right| \ge m^{(2)}, \quad (x,t) \in \overline{G}^0. \tag{9.5b}$$

Then, for the quantity $\delta_2(x,t)$, the following estimate is valid:

$$\max_{\overline{G}_h} |\, \delta_2(x,t) \,| \ge m_1 m^{(2)} \tau = m_1 m^{(2)} T N_0^{-1}, \tag{9.6}$$

where $\overline{G}_h = \overline{G}_{h(4.1)}$, $\tau = \tau_{(4.1)}$, $m^{(2)} = m_{(9.5)}^{(2)}$,

$$m_1 = 8^{-1} \min \left\{ 2^{-1}(x_0^2 - x_0^1)^2 \min_{\overline{G}} a^{-1}(x,t), \; (t_0^2 - t_0^1) \min_{\overline{G}} p^{-1}(x,t) \right\}.$$

We now investigate the behavior of the quantity $\delta_3(x,t)$. For this we consider problem (3.2), (4.1) with $\varepsilon = 1$. In this case the grid (4.1) is uniform. Assume that for the solution $z(x,t)$ of scheme (3.2), (4.1) on the set

$$\overline{G}_h^1 = \overline{G}^1 \cap \overline{G}_h, \qquad \overline{G}^1 = \left\{ (x,t) : \; x_1^1 \le x \le x_1^2, \; t_1^1 \le t \le t_1^2 \right\}, \tag{9.7a}$$

the following estimate holds:

$$\min_{\overline{G}_h, t>0} \left| \delta_{\bar{t}}\, z_{(3.2)}(x,t) \right| \geq m^{(1)}. \tag{9.7b}$$

We wish to estimate max $_{\overline{G}_h}$ $|\delta_3(x,t)|$ from below under these conditions.

Let a decomposition of the domain G onto subdomains G_p^k be such that the number of nodes in the grid sets $\overline{G}_{ph}^k$ is equal, the overlapping width of the neighboring subdomains is equal, and, besides, condition (5.6) holds. In this case $\delta_3(x,t)$ satisfies the estimate

$$\max_{\overline{G}_h} |\, \delta_3(x,t)\, | \geq m_1 m^{(1)} \tau^P, \tag{9.8}$$

where m_1 does not depend on K, P, and $m_1 = m_1(x_1^2 - x_1^1, t_1^2 - t_1^1)$.

Note that the same estimate for $\delta_3(x,t)$ is satisfied if $\varepsilon \in (0,1]$, when condition (5.6) is true for the minimal overlapping width of the subdomains, and also the number of nodes in the grids $\overline{G}_{ph}^k$ for all the subdomains $\overline{G}_p^k$ is comparable (the solvers are loaded effectively).

It follows from estimates (9.3a), (9.4) and (9.8) that in the class of schemes (6.5), (9.1) for parallel solvers, if the condition

$$P < 2^{-1} m_{1(9.8)}\, M_{1(9.3)}^{-1}\, m_{(9.7)}^1 \left(M_{(4.3)}^{(2)}\right)^{-1} \equiv P_* = P_*\left(m_{(9.7)}^1, M_{(4.3)}^{(2)}\right) \tag{9.9}$$

is valid, there do not exist ε-uniform convergent schemes with effective w.r.t. loading parallel solvers (i.e., the number of solvers P, the grids $\overline{\omega}_0^P$ and the decompositions of the domain G onto the subdomains $\overline{G}_p^k$ under condition (5.6)) for which the estimate (9.2) holds. Note that the quantity P_* from (9.9), under condition (9.7), infinitely grows for $M_{(4.3)}^{(2)} \to 0$.

Thus, in case when the difference derivatives $\delta_{\bar{t}} z_{(3.2)}(x,t)$, $(x,t) \in \overline{G}_{h(4.1)}$, on the set $\overline{G}^1 \subseteq \overline{G}$ are sufficiently large as compared to the derivative $(\partial^2/\partial t^2)u(x,t)$ on $\overline{G}$, and the number P of solvers is not too large (for example, in case of condition (9.9)), the use of parallel computations does not bring to acceleration of the solution process in comparison with the basic scheme (3.2), (4.1).

Hence, conditions (4.3), (4.4), i.e., restrictions on the derivatives $(\partial^2/\partial t^2)u(x,t)$ and $\delta_{\bar{t}} z_{(3.2)}(x,t)$, are not sufficient in order to accelerate the solution of the boundary value problem by means of parallel computations.

Theorem 9.1. *Assume the derivatives* $(\partial^2/\partial t^2)u(x,t)$ *and* $\delta_{\bar{t}} z_{(3.2)}(x,t)$ *satisfy only conditions* (4.3), (4.4). *Then the use of difference schemes* (6.5), (6.4), (9.1) *for parallel solvers, with their effective loading, does not allow us to accelerate the solution of boundary value problem* (2.1) *for fixed values of the parameter* ε *and* N. *For difference scheme* (6.5), (9.1) *acceleration of the solution process is not achieved.*

9.3. In case when the difference derivatives $\delta_{\bar{t}} z_{(3.2)}(x,t)$ on the set $\overline{G}_h$, $t > 0$ are sufficiently small as compared to the derivative $(\partial^2/\partial t^2)u(x,t)$ on $\overline{G}$, the use of parallel solvers allows us to accelerate the solution of the basic scheme. In this subsection we determine conditions under which an increase in the number of solvers leads in fact to acceleration of the solution process.

Let the work time of solvers which resolve the discrete boundary value problem on the layer $t = t^1$ of the grid set $\overline{D}_h^0$ from $\overline{D}_{h(4.1)}$, is determined by the quantity $\mu(\overline{D}_h^0)$, i.e., by the number of nodes in the set $\overline{D}_h^0$.

We begin with description of decompositions of the set $\overline{D}$. Assume that the domain D consists of J non-intersecting intervals

$$D^{<j>}, \quad j = 1, \dots, J, \tag{9.10a}$$

where $D^{<i>} \cap D^{<j>} = \emptyset$ for $i \neq j$, $\overline{D} = \bigcup_{j=1}^{J} \overline{D}^{<j>}$; $J \leq M$. On each of the sets $\overline{G}^{<j>} = \overline{D}^{<j>} \times [0,T]$, the grid $\overline{G}_h$ with a given distribution of its nodes generates the grids $\overline{G}_h^{<j>}$

$$\overline{G}_h^{<j>} = \overline{G}^{<j>} \cap \overline{G}_h, \quad j = 1, \dots, J, \qquad \overline{G}_h = \overline{G}_{h(9.1)}. \tag{9.10b}$$

Suppose that the boundaries of the sets $\overline{G}^{<j>}$ pass through the nodes of the grid $\overline{G}_h$. For each of the sets $D^{<j>}$ we construct an interval D^j containing $D^{<j>}$ together with some neighborhood. This set $\overline{D}^j$ satisfies three conditions: (a) $\overline{D}^j$ contains the set of points the distance of which from $\overline{D}^j$ is no greater than δ_0, where

$$\delta_0 = m^1_{(9.10)}\varepsilon \tag{9.10c}$$

with some fixed $m^1_{(9.10)}$; (b) the sides of the set $\overline{G}^j = \overline{D}^j \times [0,T]$ pass through the nodes of the grid $\overline{G}_h$; (c) the number of nodes in each of the grids $\overline{D}_h^j = \overline{D}^j \bigcap \overline{D}_h$ is equal and does not depend on the number j. The sets D^j, $j = 1, \dots, J$, form a covering of the set D, and the sets

$$G^j, \quad j = 1, \dots, J, \tag{9.10d}$$

generated by D^j, form a covering of the set G, that is, $G = \bigcup\limits_{j=1}^{J} G^j$. Assume that

$$\mu(\overline{D}_h^j) = \mu^0, \tag{9.10e}$$

$$\mu(\overline{D}_h^j) = (1 + m^2_{(9.10)})\, \mu(\overline{D}_h^{<j>}), \quad j = 1, \dots, J, \tag{9.10f}$$

where $m^2_{(9.10)}$ is a sufficiently small number.

The sets (9.10d), that is, the decomposition of G, are used for the construction of a difference scheme with P solvers. Further, we construct sets

$$G^{<k>}, \qquad k = 1, \dots, K \tag{9.11a}$$

which cover the set G, where the quantity $K = K(P)$ is chosen from the condition $KP = J$. Each of the sets $G^{<k>}$ is multiply connected (for $P > 1$) and formed by the union of P non-intersecting domains from (9.10d). Thus, for sets G_p^k, which form the sets from (9.11a), the following condition holds:

$$G_p^k \subset \left\{ G^j, \ j = 1, \dots, J \right\}_{(9.10d)}, \quad k = 1, \dots, K, \ \ p = 1, \dots, P, \tag{9.11b}$$

where $\mu(\overline{D}_p^k) = \mu^0$. In such a decomposition the solvers are loaded uniformly, i.e. their load is effective.

Assume that the time, which is necessary for the solution of problem (3.2), (4.1) and (6.5), (9.1), is defined by the relations

$$\vartheta = \vartheta(N_0) \equiv N_0 \mu(\overline{D}_h), \tag{9.12a}$$

$$\vartheta^P = \vartheta^P(N_0^P, P) \equiv N_0^P \sum_{k=1}^{K} \max_p \mu(\overline{D}_{ph}^k). \tag{9.12b}$$

Let us define increase of acting (or acceleration of computations) by the relation

$$C = C(N_0, N_0^P, P) = \vartheta(\vartheta^P)^{-1} \equiv N_0 (N_0^P)^{-1} \mu(\overline{D}_h) \left\{ \sum_{k=1}^{K(P)} \max_p \mu(\overline{D}_{ph}^k) \right\}^{-1}. \tag{9.13}$$

For difference scheme (6.5), (9.11), (9.1), taking into account relations (9.10e), (9.10f), we have

$$\vartheta = (1 + m_{(9.10)}^2)^{-1} N_0 K(P) P \mu^0, \quad \vartheta^P = N_0^P K(P) \mu_0,$$

and, consequently,

$$C = (1 + m_{(9.10)}^2)^{-1} N_0 (N_0^P)^{-1} P.$$

Finally, we give conditions ensuring acceleration for parallel computations.

Let the conditions (9.5) and (4.4) be fulfilled for the derivative $(\partial^2/\partial t^2)u(x,t)$ and the difference derivative $\delta_{\bar{t}} z(x,t)$, respectively.

In case when the steps of the grids $\overline{\omega}_{0(4.1)}^P$ and $\overline{\omega}_{0(9.1)}^P$ satisfy the condition

$$m_{1(9.6)}\, m_{(9.5)}^{(2)}\, \tau \geq \left[M_{2(9.3)} M_{(4.4)}^{(1)} + M_{1(9.3)} M_{(4.3)}^{(2)} \right] \tau^P,$$

which is equivalent to the condition

$$N_0^P \geq (m_{1(9.6)})^{-1} \left(m_{(9.5)}^{(2)} \right)^{-1} \left[M_{2(9.3)} M_{(4.4)}^{(1)} + M_{1(9.3)} M_{(4.3)}^{(2)} \right] N_0 \equiv N_0^{*P}, \tag{9.14}$$

the condition (9.2) is fulfilled for the components $\delta_2(x,t)$, $\delta_2^P(x,t)$, $\delta_3^P(x,t)$. Under the condition $N_0^P = N_{0(9.14)}^{*P}$ we obtain the following relation for the quantity C:

$$C = \left(1 + m_{(9.10)}^2\right)^{-1} m_{1(9.6)}\, m_{(9.5)}^{(2)} \left[M_{2(9.3)} M_{(4.4)}^{(1)} + M_{1(9.3)} M_{(4.3)}^{(2)} \right]^{-1} P.$$

Thus, if the number P of solvers is sufficiently large

$$\begin{aligned} P &> \left(1 + m_{(9.10)}^2\right) \left(m_{1(9.6)}\right)^{-1} \left(m_{(9.5)}^{(2)}\right)^{-1} \left[M_{2(9.3)} M_{(4.4)}^{(1)} + M_{1(9.3)} M_{(4.3)}^{(2)} \right] \\ &\equiv P^* = P^*\left(m_{(9.5)}^{(2)}, M_{(4.4)}^{(1)}, M_{(4.3)}^{(2)}\right), \end{aligned} \tag{9.15}$$

then, under the condition

$$N_0^P = \left(1 + m_{(9.10)}^2\right)^{-1} N_0 P^*, \tag{9.16}$$

the acceleration is achieved for the solution of the boundary value problem; in this case the quantity C is determined by the relation

$$C = P(P^*)^{-1}, \quad P^* = P_{(9.15)}^*. \tag{9.17}$$

Note that the quantity P^* grows infinitely as $m_{(9.5)}^{(2)} \to 0$.

Acceleration of the solution process, generally speaking, is unattainable, if condition (9.15) is essentially violated, for example, under condition (9.9).

Theorem 9.2. *Let conditions* (4.3), (9.5), (4.4) *be fulfilled for the solutions of boundary value problem* (2.1) *and difference scheme* (3.2), (4.1). *Then, in the class of difference schemes* (6.5), (9.1) *for parallel solvers, ε-uniform acceleration of the solution of boundary value problem* (2.1) *is achieved under condition* (9.15); *acceleration of the solution process for fixed values of the parameter ε and N, generally speaking, is unattainable, if condition* (9.15) *is essentially violated. In case of condition* (9.15), *the parameters of schemes* (6.5), (9.11), (9.1), *ensuring acceleration, and the achieved rate of acceleration C are determined by relations* (9.16) *and* (9.17).

Remark. In case of the basic scheme, when one solver is used, the solver works with N unknown quantities. When $P > 1$ solvers are used for parallel computations, one solver works with $N^{KP} \equiv (1+m^2_{(9.10)})K^{-1}P^{-1}N$ unknown quantities. Thus, the application of the domain decomposition (in particular, parallel computations) leads to a decrease in loading for the solvers used.

10. Distribution of scheme (7.4), (4.1) over independent processes

We now compare the basic scheme (7.4), (4.1) with the decomposition scheme (8.4), (4.1) involving P parallel solvers. These schemes have the same order of accuracy. In case of scheme (8.4), (4.1) the error of the grid solution, generally speaking, is greater than that in case of scheme (7.4), (4.1). This fact is caused by the perturbation of the solution of the basic scheme due to its decomposition. The presence of parallel solvers allows one, for the same time of work, to obtain the grid solution in the greatest number of nodes as compared to the basic scheme.

10.1. In this subsection we give conditions under which the use of a scheme with parallel solvers allows us to accelerate the solution of the boundary value problem, as compared to scheme (7.4), (4.1), without loss in accuracy of the numerical solution.

The error of the solution of the basic scheme (7.4), (4.1)

$$\delta_0(x,t) = \delta_0^{(2)}(x,t) = u(x,t) - z_{(7.4)}(x,t), \quad (x,t) \in \overline{G}_h \tag{10.1a}$$

and the error of the component $z^{(1)}_{(7.4)}(x,t)$ of the solution of the basic scheme

$$\delta_0^{(1)}(x,t) = u(x,t) - z^{(1)}_{(7.4)}(x,t), \quad (x,t) \in \overline{G}_h \tag{10.1b}$$

can be represented in the form of a sum of functions

$$\delta_0^{(i)}(x,t) = \delta_1^{(i)}(x,t) + \delta_2^{(i)}(x,t), \quad (x,t) \in \overline{G}_h, \quad i = 1,2, \tag{10.1c}$$

where the component $\delta_2^{(i)}(x,t)$ is caused by the error of the approximation of the time derivative, $\delta_2^{(2)}(x,t) = \delta_2(x,t)$.

The error of the solution of scheme (8.4), (4.1), i.e., the function

$$\delta(x,t) = \delta^{(2)}(x,t) = u(x,t) - z_{(8.4)}(x,t), \quad (x,t) \in \overline{G}_h, \tag{10.2a}$$

and the error of the component $z^{(1)}_{(8.4)}(x,t)$, i.e., the function

$$\delta^{(1)}(x,t) = u(x,t) - z^{(1)}_{(8.4)}(x,t), \quad (x,t) \in \overline{G}_h, \tag{10.2b}$$

can be represented in the form of a sum of components

$$\delta^{(i)}(x,t) = \delta_1^{(i)}(x,t) + \delta_2^{(i)}(x,t) + \delta_3^{(i)}(x,t) = \delta_0^{(i)}(x,t) + \delta_3^{(i)}(x,t), \quad (x,t) \in \overline{G}_h, \quad i = 1,2, \tag{10.2c}$$

where the error $\delta_3^{(i)}(x,t)$ is caused by decomposition of scheme (7.4), (4.1); $\delta^{(2)}(x,t) = \delta(x,t)$, $\delta_3^{(2)}(x,t) = \delta_3(x,t)$.

If the grid (9.1) is used, then the function $\delta^P(x,t)$, i.e., the error of the solution of scheme (8.4), can be written in the form

$$\delta^P(x,t) = \delta_1^P(x,t) + \delta_2^P(x,t) + \delta_3^P(x,t), \quad (x,t) \in \overline{G}^P_{h(9.1)}; \tag{10.3}$$

with $\delta^P_{j(10.3)}(x,t) = \delta_{j(10.2)}(x,t)$ for $\overline{G}_{h(9.1)} = \overline{G}_{h(4.1)}$, $j = 1,2,3$. The function $\delta_3^P(x,t)$ is defined by the relation: $\delta_3^P(x,t) = z_{(7.4;9.1)}(x,t) - z^P_{(8.4;9.1)}(x,t)$, $(x,t) \in \overline{G}^P_h$.

We say that difference scheme (8.4), (9.1) for P parallel solvers accelerates the solution of the boundary value problem, if the duration of computations by scheme (8.4), (9.1) is less than that for the basic scheme (7.4), (9.1), and, besides, the following condition holds:

$$\max_{\overline{G}^P_h} \left| \delta^P_{2(10.3)}(x,t) + \delta^P_{3(10.3)}(x,t) \right| \le \max_{\overline{G}_h} \left| \delta^P_{2(10.1)}(x,t) \right|, \tag{10.4}$$

where $\overline{G}_h = \overline{G}_{h(4.1)}$, $\overline{G}^P_h = \overline{G}_{h(9.1)}$.

10.2. We now give some estimates for the components $\delta_2(x,t)$, $\delta_2^P(x,t)$, $\delta_3^P(x,t)$.

The functions $\delta_2^{(i)}(x,t)$ are the solutions of the problems

$$\Lambda_{(3.2)}\delta_2^{(1)}(x,t) = f_2^{(1)}(x,t), \quad (x,t) \in G_h, \quad \delta_2^{(1)}(x,t) = 0, \quad (x,t) \in S_h;$$

$$\Lambda_{(3.2)}\delta_2^{(2)}(x,t) = f_2^{(2)}(x,t), \quad (x,t) \in G_h, \quad \delta_2^{(2)}(x,t) = 0, \quad (x,t) \in S_h,$$

where

$$f_2^{(1)}(x,t) = p(x,t)\left(\frac{\partial}{\partial t}u(x,t) - \delta_{\bar t}u(x,t)\right),$$

$$f_2^{(2)}(x,t) = p(x,t)\left\{ \frac{\partial}{\partial t}u(x,t) - 2^{-1}\tau\frac{\partial^2}{\partial t^2}u(x,t) - \delta_{\bar t}u(x,t) + 2^{-1}\tau \left\{ \begin{array}{ll} \frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial t^2}u(x,0), & t = \tau, \\ \delta_{2\bar t}\delta_2^{(1)}(x,t), & t \ge 2\tau \end{array} \right\} \right\}, \quad (x,t) \in G_h.$$

For the function $\delta^{(i)}_{2(10.1)}(x,t)$, the following estimates hold:

$$\left| \delta_{l\bar t}\delta_2^{(1)}(x,t) \right| \le M\, M^{(2+l)}_{(4.3)}\, \tau, \quad t \ge l\tau, \tag{10.5}$$

$$\left|\delta_{r\bar t}\,\delta_2^{(2)}(x,t)\right| \le M\, \left(M^{(r+3)}_{(4.3)} + M^{(r+4)}_{(4.3)}\right)\tau^2, \quad t \ge r\tau, \quad (x,t) \in \overline{G}_h, \quad l \le K,$$

with $r \le K - 1$, where K is the quantity from Theorem 7.1.

We give some estimates from below for the function $\delta_2(x,t)$.

Let on a certain set,

$$\overline{G}^0 = \left\{(x,t) : x_0^1 \le x \le x_0^2,\ t_0^1 \le t \le t_0^2\right\}, \qquad \overline{G}^0 \subseteq \overline{G}, \tag{10.6a}$$

the following condition hold:

$$\left| \frac{\partial^3}{\partial t^3} u(x,t) \right| \ge m^{(3)}, \quad (x,t) \in \overline{G}^0. \tag{10.6b}$$

Then the following estimate is valid for the $\delta_2(x,t)$:

$$\max_{\overline{G}_h} |\, \delta_2(x,t) \,| \ge m_1\, m^{(3)}\, \tau^2 = m_1\, m^{(3)}\, T^2\, N_0^{-2}, \tag{10.7}$$

where $\overline{G}_h = \overline{G}_{h(4.1)}$, $\tau = \tau_{(4.1)}$, $m^{(3)} = m^{(3)}_{(5.1)}$, $m_1 = m_{1(9.6)}$.

Lemma 10.1. *Let the conditions of Theorem 7.1 be fulfilled for the data of boundary value problem* (2.1). *Then estimates* (10.5) *hold for the function* $\delta_2(x,t)$ *and the component* $\delta_2^{(1)}(x,t)$. *If, besides, condition* (10.6) *is fulfilled, then estimate* (10.7) *is valid.*

Let us estimate $\delta^P_{3(10.3)}$. The functions $\delta_3^{(i)P}(x,t) = z^{(i)}_{(7.4;9.1)}(x,t) - z^{(i)P}_{(8.4;9.1)}(x,t)$, $(x,t) \in \overline{G}_{h(9.1)}$ are the solutions of the problems

$$Q_{(6.5)}\left(\delta_3^{(1)P}(x,t);\ f^{(1)}(\cdot),\ \varphi^{(1)}(\cdot),\ \psi^{(1)}(\cdot)\right) = 0,$$
$$Q_{(6.5)}\left(\delta_3^{(2)P}(x,t);\ f^{(2)}(\cdot),\ \varphi^{(2)}(\cdot),\ \psi^{(2)}(\cdot)\right) = 0, \quad (x,t) \in \overline{G}_{h(9.1)},$$

where

$$f^{(1)}(x,t) \equiv 0, \quad \varphi^{(1)}(x,t) = \varphi^{(2)}(x,t) \equiv 0,$$
$$\psi^{(1)}(x,t;\, t^n) = \tau^P \delta_{\bar t}\, z^{(1)}(x,t), \quad t = t^{n+1},$$
$$f^{(2)}(x,t) = 2^{-1}\tau^P p(x,t)\left(\delta_{2\bar t}\, \delta_2^{(1)}(x,t) - \delta_{2\bar t}\, \delta_3^{(1)P}(x,t)\right),$$
$$\psi^{(2)}(x,t;\, t^n) = \tau^P\left(\delta_{\bar t}\, \delta_3^{(1)P}(x,t) - \delta_{\bar t}\, \delta_2^{(2)}(x,t)\right), \quad t = t^{n+1}.$$

Taking into account estimates (4.4), (10.5), we find

$$\left|\delta_{l\bar t}\, \delta_3^{(1)P}(x,t)\right| \le M\, M^{(1)}_{(4.3)} \tau^P, \quad t \le l\tau^P,$$
$$\left|\delta_{r\bar t}\, \delta_3^{(2)P}(x,t)\right| \le M\, M^{(r+4)}_{(4.3)} (\tau^P)^2, \quad (x,t) \in \overline{G}_{h(9.1)}, \quad l \le K+2, \tag{10.8}$$

with $r \le K - 2$, where $K \ge 2$ is the quantity from the hypothesis of Theorem 7.1.

For the component $\delta^P_{2(10.3)}$ we have the estimate

$$|\delta_{r\bar t}\delta_2^P(x,t)| \le M\left(M^{(r+3)}_{(4.3)} + M^{(r+4)}_{(4.3)} t\right)(\tau^P)^2, \quad (x,t) \in \overline{G}_{h(9.1)}, \quad r \le K-1. \tag{10.9}$$

Lemma 10.2. *Let the conditions of Theorem 7.1, where* $K = 2$, *be fulfilled for the data of boundary value problem* (2.1). *Then estimates* (10.8), (10.9) *are valid for the functions* $\delta_2^P(x,t)$, $\delta_3^P(x,t)$ *and the component* $\delta_3^{(1)P}(x,t)$.

10.3. In case when the derivative $(\partial^3/\partial t^3)u(x,t)$ on the set $\overline{G}$ is not too small, the use of parallel solvers allows us to accelerate the solution of the basic scheme (7.4), (4.1).

Let the duration of the solution of problems (7.4), (4.1) and (8.4), (9.1) be defined by relations (9.12a) and (9.12b), respectively, and acceleration of the solution of the boundary value problem is defined by relation (9.13). For the decomposition scheme we use the grid construction (9.11).

In case when the steps of the grids $\overline{\omega}^P_{0(4.1)}$ and $\overline{\omega}^P_{0(9.1)}$ satisfy the condition

$$m_{1(10.7)}\, m^{(3)}_{(10.6)}\, \tau^2 \ge \left[M_{(10.8)} M^{(5)} + M_{(10.9)} \left(M^{(3)} + M^{(4)} T\right)\right] \left(\tau^P\right)^2,$$

which is equivalent to the condition

$$\begin{aligned} N_0^P &\ge \left\{ (m_{1(10.7)})^{-1} (m^{(3)}_{(10.6)})^{-1} \left[M_{(10.8)} M^{(5)} + M_{(10.9)} \left(M^{(3)} + M^{(4)}\, T\right)\right] \right\} N_0 \\ &\equiv M^* N_0 \equiv N_0^{*P}, \end{aligned} \tag{10.10}$$

$$M^* = M^*\left(m^{(3)},\, M^{(3)},\, M^{(4)},\, M^{(5)}\right) \tag{10.11}$$

where $M^{(k_0)} = M^{(k_0)}_{(4.3)}$, the estimate (10.4) holds for the components $\delta_2(x,t)$, $\delta_2^P(x,t)$, $\delta_3^P(x,t)$. Under the condition $N_0^P = N^{*P}_{0(10.10)}$ we obtain the following expression for the quantity C:

$$C = \left(1 + m^2_{(9.10)}\right)^{-1} \left(M^*\right)^{-1} P.$$

In case when the number P of solvers is sufficiently large

$$P > (1 + m^2_{(9.10)})\, M^* \equiv P^*, \tag{10.12}$$

acceleration can be achieved for the solution of the boundary value problem. In fact, acceleration is achieved under the condition

$$N_0^P = (1 + m^2_{(9.10)})^{-1} N_0\, P^*\,. \tag{10.13a}$$

The quantity C, which characterizes the achieved acceleration of the solution, is defined by

$$C = P(P^*)^{-1}, \qquad P^* = P^*_{(10.12)}. \tag{10.13b}$$

Theorem 10.1. *Let the conditions of Theorem 7.1, where $K = 3$, hold for the data of boundary value problem* (2.1) *and let condition* (10.6) *be satisfied for the solution of the problem. Then in the class of difference scheme* (8.4), (9.1) *for parallel solvers ε-uniform acceleration of the solution of the boundary value problem, as compared to the basic scheme* (7.4), (4.1), *can be achieved under condition* (10.12), *where M^* is given by* (10.11). *In case of condition* (10.12), (10.11), *the parameters of scheme* (8.4), (9.11), (9.1), *ensuring the acceleration, and the achieved rate of acceleration, i.e., the quantity C, are defined by relations* (10.13).

Remark 1. In case when condition (10.6) is violated, that is, $(\partial^3/\partial t^3)u(x,t) \equiv 0$ on the whole domain $\overline{G}$, we can consider problem (2.1) as a problem with constant coefficients (and with $c(x,t) \equiv 0$); the function $u(x,t) = x(1-x)t^2$ is the solution of this problem. The function $z^{(2)}_{(7.4)}(x,t)$ is almost $u(x,t)$ up to terms $O(\varepsilon^2)$:

$$\max_{\overline{G}_h} \left| \delta_j^{(2)}(x,t) \right| \le M\,\varepsilon^2, \quad j = 0,1,2. \tag{10.14a}$$

For the function $z^{(1)}_{(7.4)}$, we have the estimate

$$| z^{(1)}_{(7.4)} - (t^2 + \tau\, t) | \le M\, \varepsilon^2, \quad (x,t) \in \overline{G}_h. \tag{10.14b}$$

The functions $\delta^{(i)}_3(x,t)$ are the solutions of the problem

$$Q_{(6.5)}\left(\delta^{(1)P}_3(x,t);\, 0,\, 0,\, \psi^{(1)}(\cdot)\right) = 0, \quad Q_{(6.5)}\left(\delta^{(2)P}_3(x,t);\, 0,\, 0,\, \psi^{(2)}(\cdot)\right) = 0, \quad (x,t) \in \overline{G}^P_h,$$

where

$$\begin{aligned} \psi^{(1)}(x,t;t^n) &= -z^{(1)}_{(7.4)}(x,t^{n+1}) + z^{(1)}_{(7.4)}(x,t^n), \\ \psi^{(2)}(x,t;t^n) &= \delta^{(1)P}_3(x,t^{n+1}) - \delta^{(1)P}_3(x,t^n) + \delta^{(2)}_0(x,t^{n+1}) - \delta^{(2)}_0(x,t^n), \quad t = t^{n+1}. \end{aligned}$$

From (10.14), it follows that in case (5.6) we have the estimate

$$\max_{\overline{G}^P_h} \left| \delta^{(2)P}_3(x,t) \right| \ge m\, (\tau^P)^2\,.$$

Thus, ε-uniform acceleration of the solution of the basic scheme (7.4), (4.1) is unattainable for any large number P of solvers ($P \le M_0$, where M_0 is a sufficiently large number).

Remark 2. In case when condition (10.6) holds, but condition (10.12) is essentially violated, the acceleration of the solution process by using parallel solvers is, generally speaking, unattainable (for fixed values of the parameter ε and N).

10.4. Let us consider the basic scheme (7.4), (4.1) and its decomposition, i.e., scheme (8.6), (9.1) with parallel solvers. We assume that the condition

$$\left| \frac{\partial^4}{\partial t^4}\, u(x,t) \right| \ge m^{(4)}, \quad (x,t) \in \overline{G}^0 \tag{10.15}$$

holds on a certain set $\overline{G}^0_{(10.6a)}$.

For scheme (8.6), (9.1) we use decomposition (9.11). Then, under condition (10.10), where

$$M^* = M^*\left(m^{(4)},\, M^{(4)},\, M^{(5)},\, M^{(6)},\, M^{(7)}\right), \quad M^{(k_0)} = M^{(k_0)}_{(4.3)}, \tag{10.16}$$

the estimate (10.4) is fulfilled for the components $\delta_2(x,t)$, $\delta^P_2(x,t)$, $\delta^P_3(x,t)$, corresponding to schemes (7.4), (4.1) and (8.6), (9.11), (9.1).

Acceleration of the solution of the boundary value problem, if we use scheme (8.6), (9.1) instead of scheme (7.7), (4.1), can be achieved when the number P of solvers is sufficiently large, namely, under condition (10.12), (10.16). Acceleration of the solution process is achieved under condition (10.13a); for the quantity C relation (10.13b) is fulfilled.

Theorem 10.2. *Let the conditions of Theorem 7.2, where $K = 5$, hold for the data of boundary value problem* (2.1) *and let condition* (10.15) *be fulfilled for the solution of the problem. Then in the class of difference schemes* (8.4), (9.1) *for parallel solvers ε-uniform acceleration of the solution of the boundary value problem, as compared to the basic scheme* (7.4), (4.1), *can be achieved under condition* (10.12), *where M^* is given by* (10.16). *In case of condition* (10.12), (10.16) *the parameters of scheme* (8.6), (9.11), (9.1), *ensuring* *the acceleration, and the achieved rate of acceleration, i.e., the quantity C, are defined by relations* (10.13), (10.16).

Remark. If condition (10.15) is violated, then acceleration of the solution of the boundary value problem is not achieved even if a large number P of solvers is used. If condition (10.12), (10.16) is violated, i.e., the number of solvers used is not sufficiently large, then acceleration of the solution of the problem cannot be achieved.

Conclusion

In order to efficiently solve a singularly perturbed parabolic PDE by an ε-uniform discretization procedure, 2nd order accurate in space and high-order in time, we studied a defect correction procedure. To reduce the computation time, we splitted the procedure into P independent processes, preserving ε-uniform convergence. We gave a precise description of conditions under which the splitting does not affect the accuracy of the method.

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11. Appendix: Estimates of the solution and its derivatives

Here we consider a-priori estimates for the solution of problem (2.1) and its derivatives derived for elliptic and parabolic equations in [11, 13].

We denote by $H^{(\alpha)}(\overline{G}) = H^{\alpha,\alpha/2}(\overline{G})$ a Hölder space, where α is an arbitrary positive number [6]. We suppose that functions $f(x,t)$ and $\varphi(x,t)$ satisfy compatibility conditions at the corner points, so that the solution of the boundary value problem is smooth for every fixed value of the parameter ε.

For simplicity, we assume that at the corner points $S_0 \cap \overline{S}_1$ the following conditions hold:

$$\frac{\partial^k}{\partial x^k}\varphi(x,t) = \frac{\partial^{k_0}}{\partial t^{k_0}}\varphi(x,t) = 0, \quad k + 2k_0 \le [\,\alpha\,] + 2n, \tag{11.1}$$
$$\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}} f(x,t) = 0, \quad k + 2k_0 \le [\,\alpha\,] + 2n - 2,$$

where $[\,\alpha\,]$ is the integer part of a number α, $\alpha > 0$, $n \ge 0$ is an integer number. We also suppose that $[\,\alpha\,] + 2n \ge 2$.

Using interior a-priori estimates and estimates up to the boundary for a regular function $\widetilde{u}(\xi,t)$, [6], where $\widetilde{u}(\xi,t) = u(x(\xi),t)$, $\xi = x/\varepsilon$, we find for $(x,t) \in \overline{G}$ the estimate

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}} u(x,t)\right| \le M\,\varepsilon^{-k}, \quad k + 2k_0 \le 2n+4, \quad n \ge 0. \tag{11.2}$$

This estimate holds, for example, for

$$u \in H^{(2n+4+\nu)}(\overline{G}), \quad \nu > 0, \tag{11.3}$$

where ν is some small number.

For example, (11.3) is guaranteed for the solution of (2.1) if the coefficients a, c, p, $f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n \ge 0$ and condition (11.1) is fulfilled.

In fact we need a more accurate estimate than (11.2). Therefore, we represent the solution of the boundary value problem (2.1) in the form of a sum,

$$u(x,t) = U(x,t) + W(x,t), \quad (x,t) \in \overline{G}, \tag{11.4}$$

where $U(x,t)$ represents the regular part, and $W(x,t)$ the singular part, i.e. the parabolic boundary layer. The function $U(x,t)$ is a smooth solution of equation (2.1a) satisfying condition (2.1b) for $t = 0$. For example, under suitable assumptions for the data of the problem, we can consider the solution of the Dirichlet boundary value problem for equation (2.1a) smoothly extended to the domain $\overline{G}^*$ ($\overline{G}^*$ is a sufficiently large neighborhood of $\overline{G}$).

On the domain $\overline{G}$, the coefficients and the initial value of the extended problem are the same as for (2.1). Then the function $U(x,t)$ is a restriction (on $\overline{G}$) of the solution to the extended problem, and $U \in H^{(2n+4+\nu)}(\overline{G})$, $\nu > 0$. The function $W(x,t)$ is the solution of a boundary value problem for the parabolic equation

$$L_{(2.1)} W(x,t) = 0, \quad (x,t) \in G, \qquad W(x,t) = u(x,t) - U(x,t), \quad (x,t) \in S. \tag{11.5}$$

If (11.3) is true then $W \in H^{(4+2n+\nu)}(\overline{G})$. We assume that a, c, p, $f \in H^{(\alpha+4n)}(\overline{G})$, $\varphi \in H^{(\alpha+4n)}(\overline{G})$, $\alpha > 4$, $n \geq 0$. Now, for the functions $U(x,t)$ and $W(x,t)$ we derive the following estimates:

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \, \partial t^{k_0}} U(x,t) \right| \leq M \, [1 + \varepsilon^{2n+2-k}], \tag{11.6}$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \, \partial t^{k_0}} W(x,t) \right| \leq M \, \varepsilon^{-k} \exp(-m_{(11.7)} \varepsilon^{-1} r(x,\gamma)\,), \tag{11.7}$$

$$(x,t) \in \overline{G}, \quad k + 2k_0 \leq 2n + 4,$$

where $r(x,\gamma)$ is the distance between a point $x \in \overline{D}$ and the set $\gamma = \overline{D} \setminus D$, $m_{(11.7)}$ is a sufficiently small positive number. We summarize these results in the following theorem (see [13]):

Theorem 11.1. *Assume in equation* (2.1) *that* a, c, p, $f \in H^{(\alpha+2n)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n \geq 0$ *and let condition* (11.3) *be fulfilled. Then, for the solution* $u(x,t)$ *of problem* (2.1), *and for its components in representation* (11.4), *it follows that* u, U, $W \in H^{(4+2n)}(\overline{G})$ *and that the estimates* (11.2), (11.6), (11.7) *hold.*

P.W. Hemker
CWI, Amsterdam, The Netherlands
E-mail: pieth@cwi.nl

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G.I. Shishkin, L.P. Shishkina
Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
GSP-384, 620219 Ekaterinburg, Russia
E-mails: Grigorii@shishkin.ural.ru (G.I. Shishkin)
Lida@shishkin.ural.ru (L.P. Shishkina)