Determining Local Singularity Strength and its Spectrum with the Wavelet Transform

OR

What Has Your Heartbeat Rate Got To Do
With Your Financial Stock Record

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Heart MRI scan courtesy Philips Medical Systems, Best, The Netherlands
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The Problem

- The Multifractal Formalism (MF) has a well established place in research and engineering,

BUT the Multifractal Spectra are:

1. difficult to obtain
2. difficult to interpret.

- Why?

1. Because the methodology is sensitive to spikes, boundary effects and other noise.
2. Because the methodology is intrinsically statistical and global, thus lacking any local information.
• So what is it we get from multifractal formalism?

- Is there more MF-related information out there?
The Proposition:

- Show the *local* contributions to the Multifractal Spectra!

- We all know that within state of the art MF formalism technology this is *impossible*!

- So in the following I will show how to do it...
What is the State of the Art Technology in Multifractal Formalism?

- It is the Wavelet Transform Modulus Maxima based multifractal formalism introduced by Arneodo, Muzy and Bacry.

- This multifractal methodology is intrinsically statistical and global.

- It is also a very robust methodology which has proved to be effective in many fields.

- Largely, this is due to the use of the Wavelet Transform Modulus Maxima decomposition.
Why Wavelets?

- Reveal the hierarchy of (singular) features including the scaling behaviour.
- Provide means of local analysis.
- In particular in the presence of non-stationarities like global or local trends or biases.
The wavelet transform is a convolution product of the signal with the scaled and translated kernel - the wavelet $\psi(x)$.

$$(Wf)(s,b) = \frac{1}{s} \int dx \ f(x) \ \psi\left(\frac{x - b}{s}\right).$$

The scale parameter $s$ ‘adapts’ the width of the wavelet kernel to the \textit{microscopic resolution} required, thus changing its frequency contents; the location of the analysing wavelet is determined by the parameter $b$. 
The Wavelet $\psi$

- The only requirement for the wavelet $\psi$ is orthogonality to polynomial of some degree $n$.

- This corresponds to the requirement that the wavelet has zero mean - it is a wave function, hence the name wavelet.

$$\int_{-\infty}^{\infty} x^n \psi(x) \, dx = 0$$

- This admissibility requirement also results in filtering polynomials of degree $P_n$. 

![Graphs of scaling function, wavelet m=1, wavelet m=2](image-url)
The Hölder Exponent

- The ability of Wavelet Transform to filter polynomials of degree $P_n$ is particularly useful for us since it allows the assessment of local scaling behaviour.

- This scaling behaviour is represented by the so-called Hölder exponent $h(x_0)$ of the function $f(x)$:

  $$|f(x) - P_n(x - x_0)| \leq C|x - x_0|^{h(x_0)}.$$ 

- We can be tempted to extract $h(x_0)$ from the scaling of the wavelet transform $Wf(x_0, s)$. 
Extracting the local scaling behaviour seems possible following the *modulus maxima* lines of the WT:

1. The maxima converge towards the singularities.
2. For *isolated* singularities of the *cusp* type, the Hölder exponent can be easily extracted.
• However, for densely packed singularities as in the case of fractal signals this is not possible!

\[
\text{log}(Wf(s)) \quad \text{vs.} \quad \text{log}(s)
\]

• This is why in the Legendre MF formalism of Arneodo et al, the global quantity - the partition function is used:

\[
Z(s, q) = \sum_{\Omega(s)} (Wf \omega_i(s))^q.
\]
The Multiplicative Cascade

- We can assume that the singularities are generated in some collective process of a generic class.

- Suppose that we denote the density of the cascade at the generation level $F_i$ ($i$ running from 0 to $\text{max}$) by $\kappa(F_i)$, we then have

\[
\kappa(F_{\text{max}}) = p_{s_1} \cdots p_{s_n}, \quad \kappa(F_0) = P_{F_{0}}^{F_{\text{max}}} \kappa(F_0)
\]

- and the local exponent is related to the product $P_{F_{0}}^{F_{\text{max}}}$ of these weights:

\[
h_{F_{\text{max}}}^{F_{0}} = \frac{\log(P_{F_{0}}^{F_{\text{max}}})}{\log((1/2)^{\text{max}}) - \log((1/2)^{0})}.
\]
• In any experimental situation, the weights $p_i$ are not known and have to be estimated.

• The densities along the process branch can be estimated with the wavelet transform, using its remarkable ability to reveal the entire process tree of a multiplicative process.
• The estimate of the effective Hölder exponent becomes:

\[
\hat{h}_{s_{lo}}^{s_{hi}} = \frac{\log(W f \omega_{pb}(s_{lo})) - \log(W f \omega_{pb}(s_{hi}))}{\log(s_{lo}) - \log(s_{hi})},
\]

• where \( W f \omega_{pb}(s) \) is the value of the wavelet transform at the scale \( s \), along the maximum line \( \omega_{pb} \) corresponding to the given process branch.

• Scale \( s_{lo} \) corresponds with generation \( F_{max} \), while \( s_{hi} \) corresponds with generation \( F_0 \).
• Unfortunately, the wavelet transform coefficients at the largest scale are heavily distorted by finite size effects.

• This is why we estimate the value of $W f\omega_{pb}(s_{hi})$ using the mean $h$ exponent.
Finally we can plot the local effective Hölder exponent:

with all its monochromatic i.e mono-fractal components separable!
- Log-histograms of the exponent $\hat{h}$ at different scales for three example time series.

- Compare them to the $D_m(h)$ spectrum obtained by the partition functions method.
Scale-wise Evolution of the Effective Hölder Exponent

- The number of locations that fall within the band range visibly grows with scale, and this growth determines the dimension $D(h)$ which can be associated with the particular $\hat{h}$, at the band resolution $\epsilon$.

- Such $D(h)$ can be estimated for the entire range of $h$, resulting in the so-called spectrum of singularities.

$$D(h) \sim \lim_{\epsilon \to 0} \lim_{s_{lo} \to 0} \frac{\log(\mu_{\epsilon}(\hat{h}(s_{lo})))}{\log(s_{lo})},$$

where $\mu_{\epsilon}$ is the measure of the total number of locations (selected maxima) that fall within the band of size $\epsilon$ at a particular scale location $s_{lo}$.
Direct Singularity Spectra from the WTMM Tree

- The evaluation of direct spectra from the $\epsilon$ bands of the Hölder exponent simply requires covering the entire range of the local effective Hölder exponents detected on the maxima tree.

- The width of the spectrum of white noise is non-zero, as is inevitable for the finite length sample.

- Still, the heartbeat sample clearly shows a considerably wider spectrum, confirming the recently reported finding.
• Due to the fact that it relies on selecting a very narrow band of exponents, this procedure is, however, inherently sensitive to the choice of parameters such as the band width.

• The experiments indicate that the spectrum obtained remains stable for a wide choice of $\epsilon$ without loss of quality.

• At the cost of a slightly lower stability, we obtain the advantages of the direct spectrum calculation.

• The spectrum better captures local variations in the scaling of the $h$ bands, where the partition function method provides only rough, ‘outline’ information about the $D(h)$ spectrum.
Direct Singularity Spectra from the Entire CWT

- Making use of the redundant information contained in the original CWT (as opposed to the WTMM used thus far).

- The comparison of the direct spectra obtained with both WTMM and the CWT suggest that the CWT may contain some information lacking in the WTMM.

- The CWT direct spectra show excellent stability with respect to the $\epsilon$ band width variation. We went down to the spectacular $\epsilon = 0.0015$ resolution.
Conclusions

- From: ”H. E. Stanley” <hes@argentobu.edu>
  Date: Wed, 28 Apr 1999 08:11:42 -0400
  [...] his panels showing diff color for each hurst expt
  are the CLEAREST exposition i know of what is a mf.

- Due to the local character of our effective Hölder exponent panels, additional information can be perceived such as non-stationarity.

- Large deviations from consistent statistical behaviour (e.g. boundary effects) can be directly assessed, in the panels or in the histogram.

- (All the above are hidden in the Legendre Transform, global approach.)

- (Stable) direct evaluation of the MF spectrum seems possible.