Determining Local Singularity Strength and its Spectrum with the Wavelet Transform

or

What Has Your Heartbeat Rate Got To Do With Your Financial Stock Record

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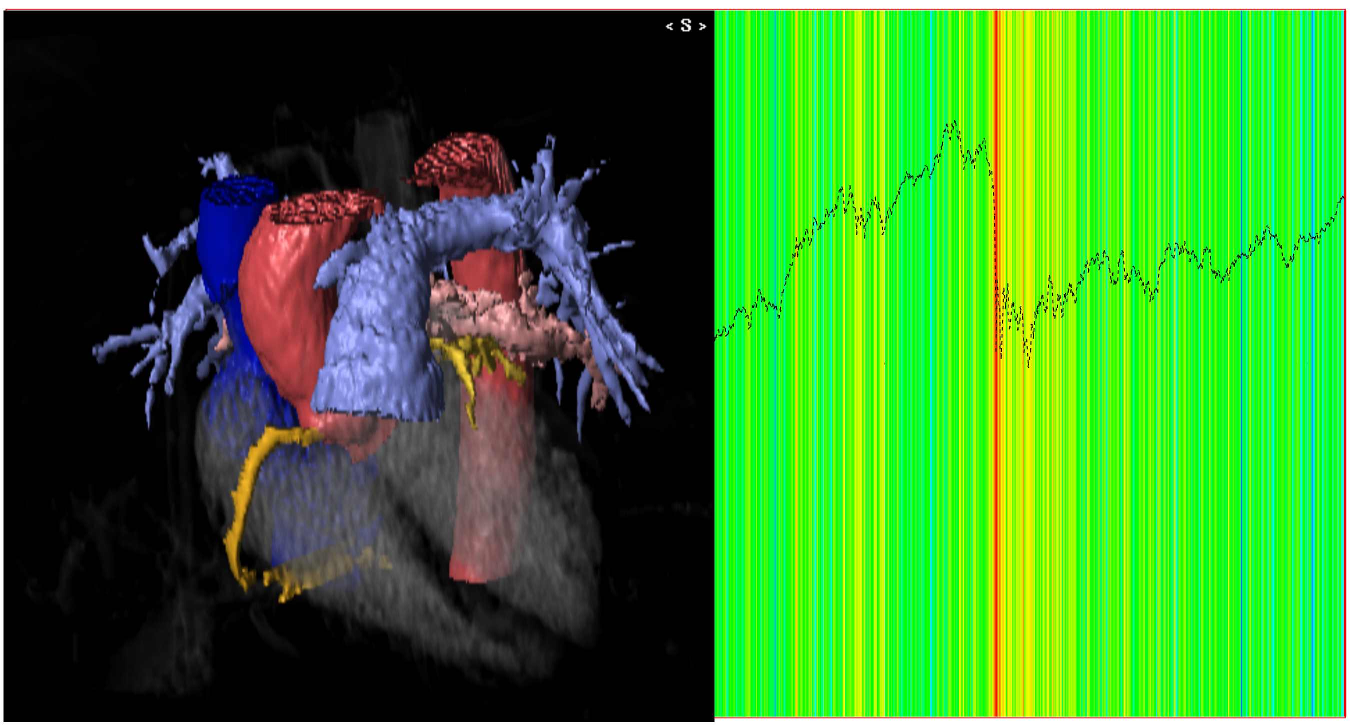


Heart MRI scan courtesy Philips Medical Systems, Best, The Netherlands

Contents

- The Problem - Global (Legendre) MF formalism
- The Proposition - Local MF formalism
- The Tool - Wavelet Transform (WTMM)
- The Idea - Local Effective Hölder Exponent
- Examples
- Conclusions

The Problem

- The Multifractal Formalism (MF) has a well established place in research and engineering,

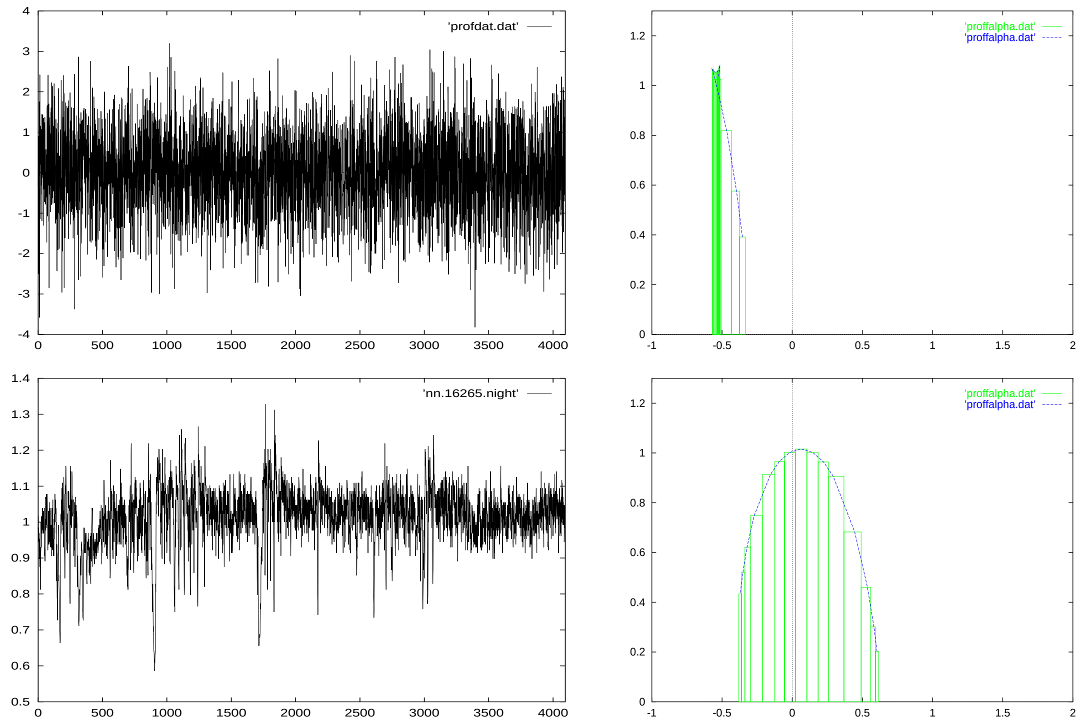
  BUT the Multifractal Spectra are:

  1. difficult to obtain
  2. difficult to interpret.

- Why?

  1. Because the methodology is sensitive to spikes, boundary effects and other noise.
  2. Because the methodology is intrinsically statistical and global, thus lacking any local information.

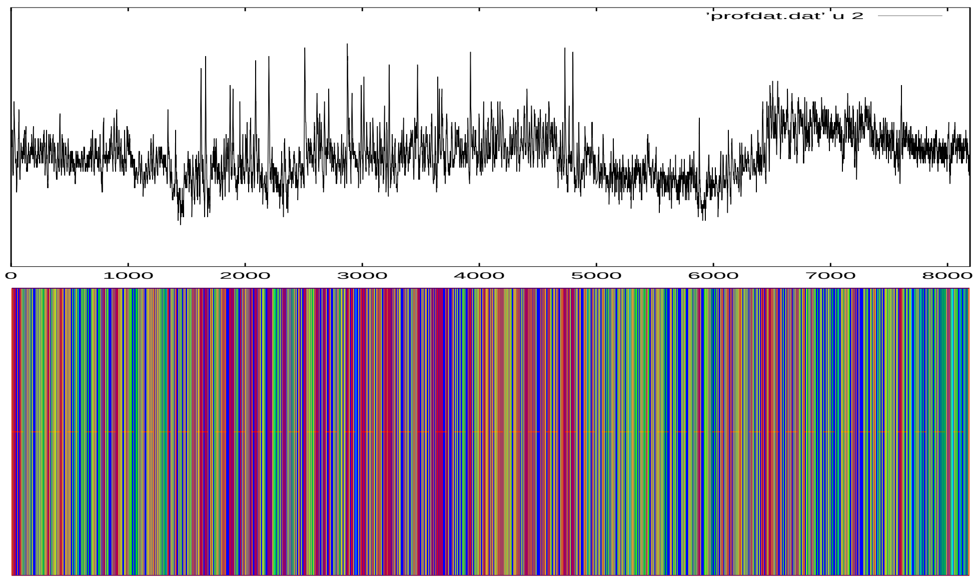
• So what is it we get from multifractal formalism?



• Is there more MF-related information out there?

The Proposition:

- Show the *local* contributions to the Multifractal Spectra!



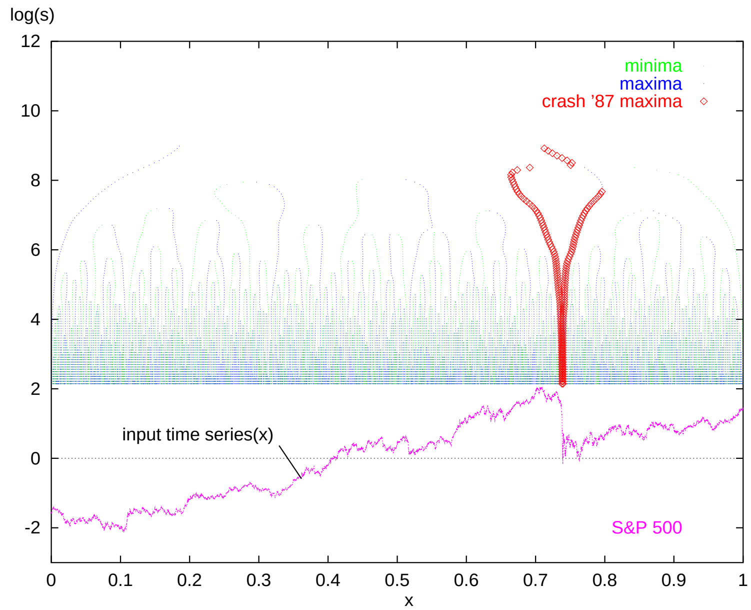
- We all know that within state of the art MF formalism technology this is *impossible*!
- So in the following I will show how to do it...

What is the State of the Art Technology in Multifractal Formalism?

- It is the Wavelet Transform Modulus Maxima based multifractal formalism introduced by Arneodo, Muzy and Bacry.
- This multifractal methodology is intrinsically statistical and global.
- It is also a very robust methodology which has proved to be effective in many fields.
- Largely, this is due to the use of the Wavelet Transform Modulus Maxima decomposition.

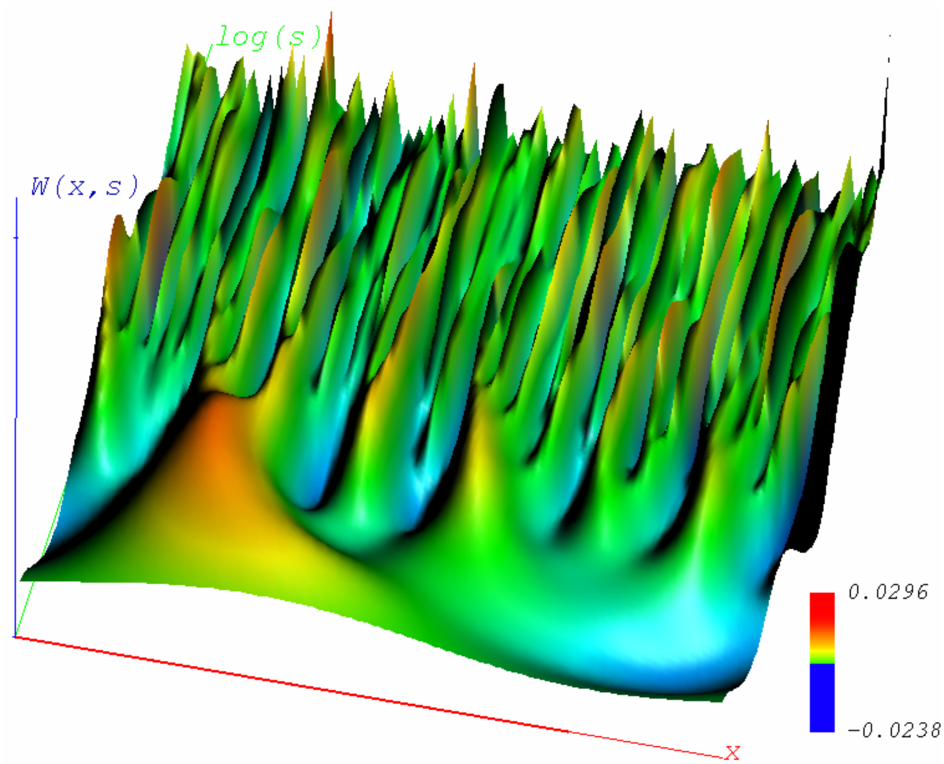
Why Wavelets?

- Reveal the *hierarchy* of (singular) features including the scaling behaviour.
- Provide means of *local* analysis.
- In particular in the presence of *non-stationarities* like global or local trends or biases.



The wavelet transform is a convolution product of the signal with the scaled and translated kernel - the wavelet $\psi(x)$.

$$(Wf)(s,b) = \frac{1}{s} \int dx\ f(x)\ \psi(\frac{x-b}{s})\ .$$



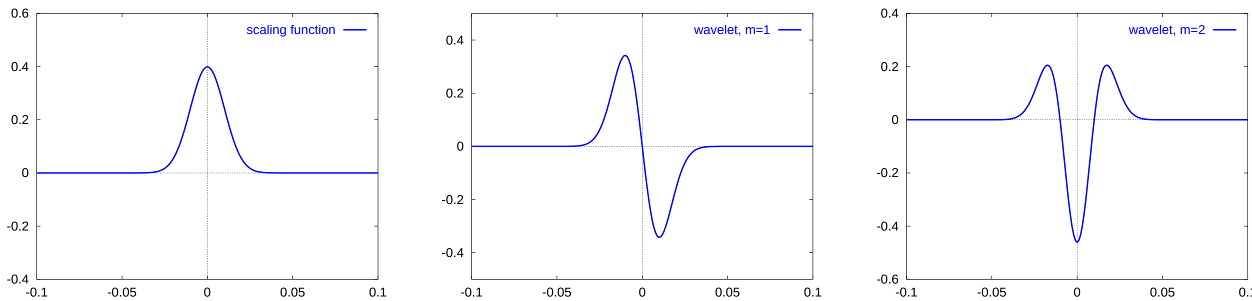
The scale parameter s 'adapts' the width of the wavelet kernel to the *microscopic resolution* required, thus changing its frequency contents; the location of the analysing wavelet is determined by the parameter b.

The Wavelet ψ

- The only requirement for the wavelet ψ is orthogonality to polynomial of some degree n.

- This corresponds to the requirement that the wavelet has zero mean - it is a wave function, hence the name *wavelet*.

$$\int_{-\infty}^{\infty} x^n \, \psi(x) \, dx = 0$$



- This *admissibity* requirement also results in filtering polynomials of degree P_n.

The Hölder Exponent

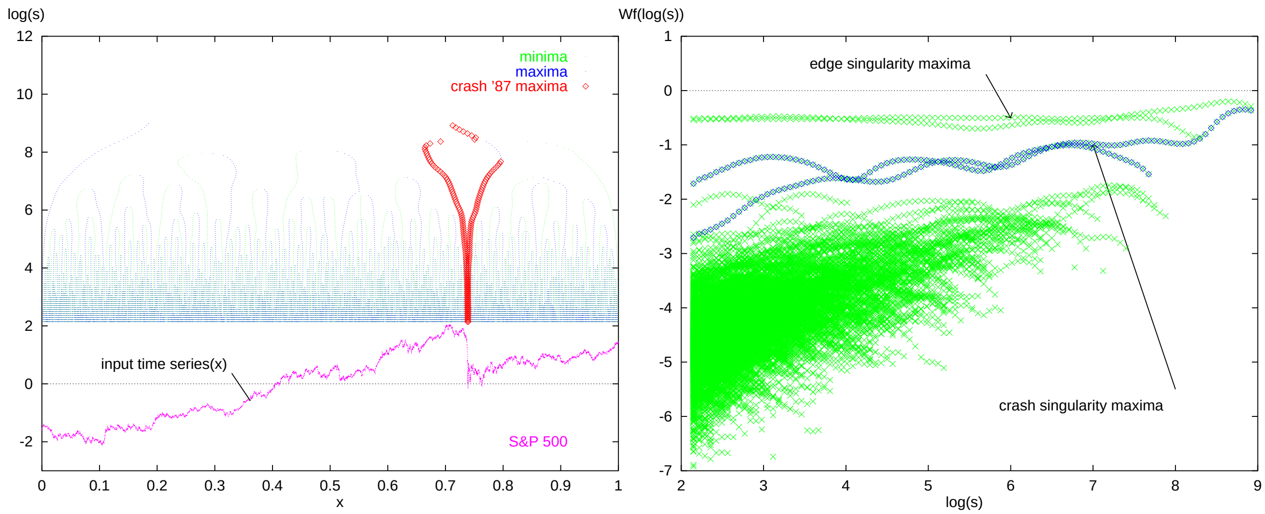
- The ability of Wavelet Transform to filter polynomials of degree P_n is particularly useful for us since it allows the assessment of local scaling behaviour.

- This scaling behaviour is represented by the so-called Hölder exponent $h(x_0)$ of the function f(x):

$$|f(x) - P_n(x - x_0)| \leq C|x - x_0|^{h(x_0)}.$$

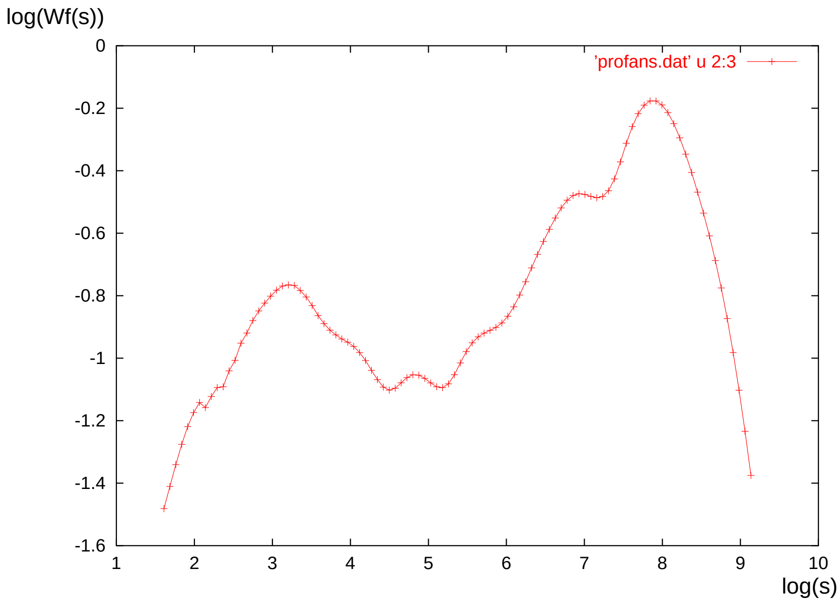
- We can be tempted to extract $h(x_0)$ from the scaling of the wavelet transform $Wf(x_0, s)$.

- Extracting the local scaling behaviour seems possible following the *modulus maxima* lines of the WT:



1. The maxima converge towards the singularities.
2. For *isolated* singularities of the *cusp* type, the Hölder exponent can be easily extracted.

- However, for *densely* packed singularities as in the case of fractal signals this is not possible!

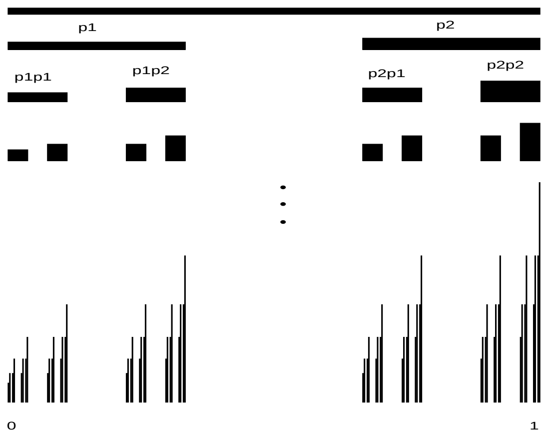


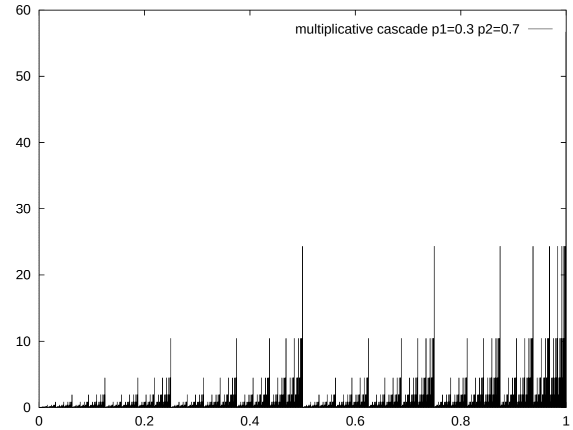
- This is why in the Legendre MF formalism of Arneodo et al, the global quantity - the *partition function* is used:

$$\mathcal{Z}(s,q) = \sum_{\Omega(s)} (Wf\omega_i(s))^q \,.$$

The Multiplicative Cascade

- We can assume that the singularities are generated in some collective process of a generic class.





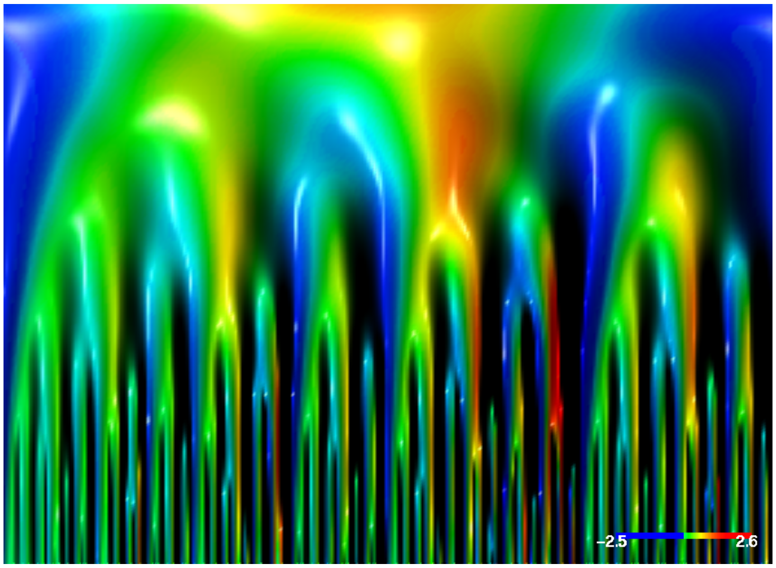
- Suppose that we denote the density of the cascade at the generation level F_i (i running from 0 to max) by $\kappa(F_i)$, we then have

$$\kappa(F_{max}) = p_{s_1} \, ... \, p_{s_n} \; \kappa(F_0) = P_{F_0}^{F_{max}} \; \kappa(F_0)$$

- and the local exponent is related to the product $P_{F_0}^{F_{max}}$ of these weights:

$$h_{F_{max}}^{F_0} = \frac{\log(P_{F_0}^{F_{max}})}{\log((1/2)^{max}) - \log((1/2)^0)} \; .$$

- In any experimental situation, the weights p_i are not known and have to be estimated.

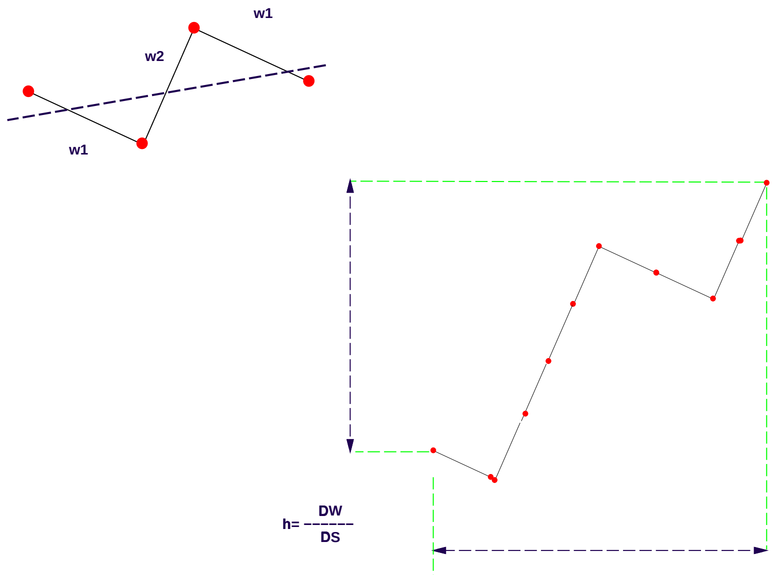


- The densities along the process branch can be estimated with the wavelet transform, using its remarkable ability to reveal the entire process tree of a multiplicative process.

- The estimate of the effective Hölder exponent becomes:

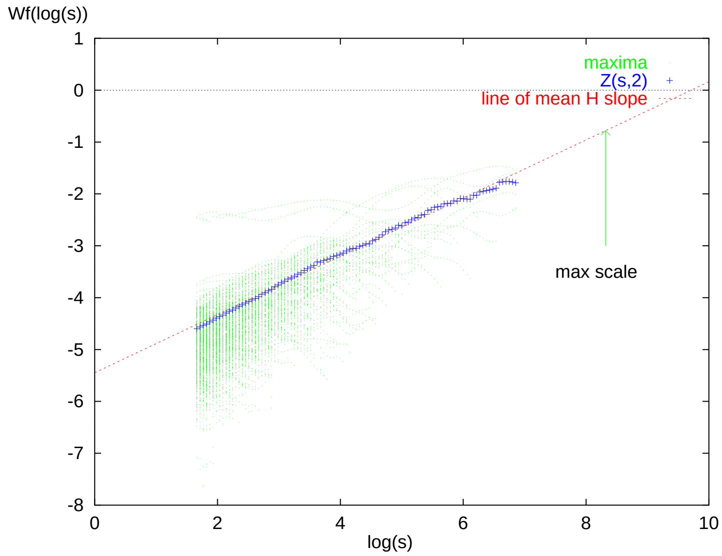
$$\hat{h}_{s_{lo}}^{s_{hi}} = \frac{\log(Wf\omega_{pb}(s_{lo})) - \log(Wf\omega_{pb}(s_{hi}))}{\log(s_{lo}) - \log(s_{hi})} ,$$

- where $Wf\omega_{pb}(s)$ is the value of the wavelet transform at the scale s, along the maximum line ω_{pb} corresponding to the given process branch.



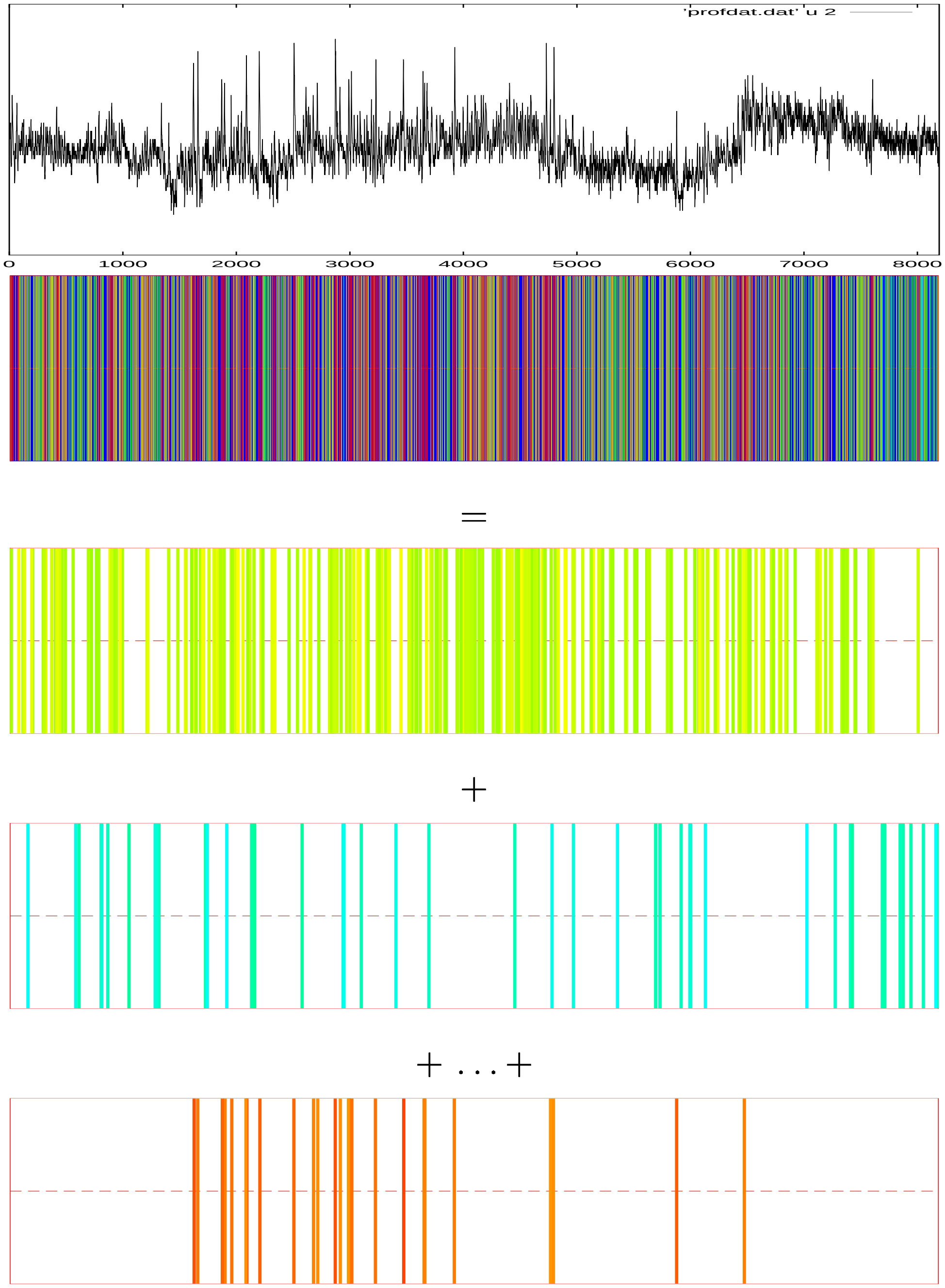
- Scale s_{lo} corresponds with generation F_{max}, while s_{hi} corresponds with generation F_0.

- Unfortunately, the wavelet transform coefficients at the largest scale are heavily distorted by finite size effects.



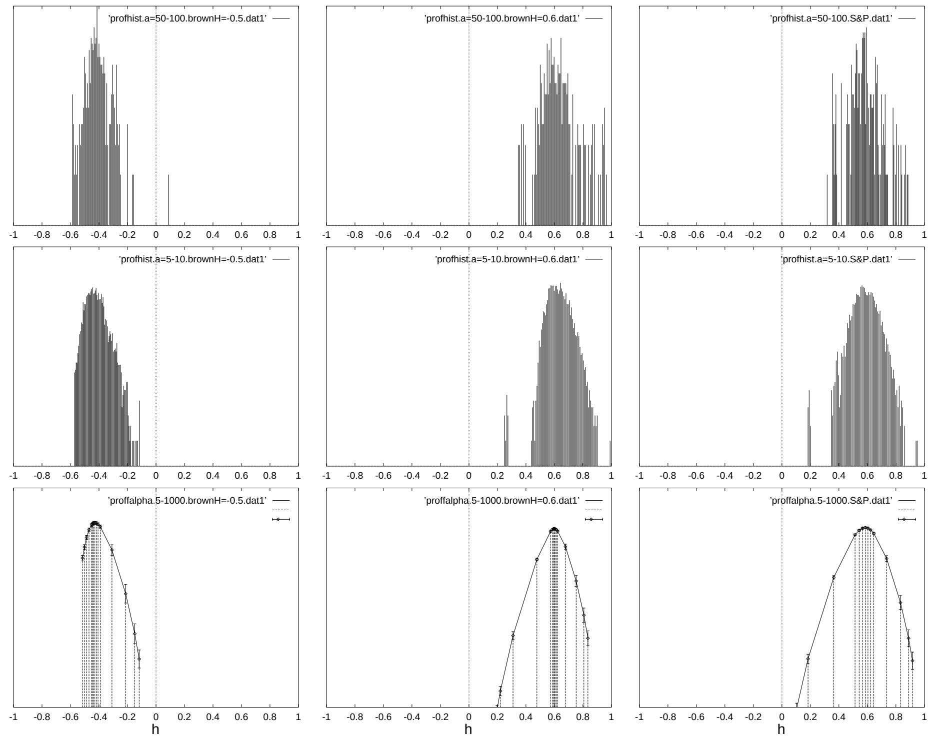
- This is why we estimate the value of $Wf\omega_{pb}(s_{hi})$ using the mean h exponent.

- Finally we can plot the *local effective Hölder exponent*:



\+ . . .

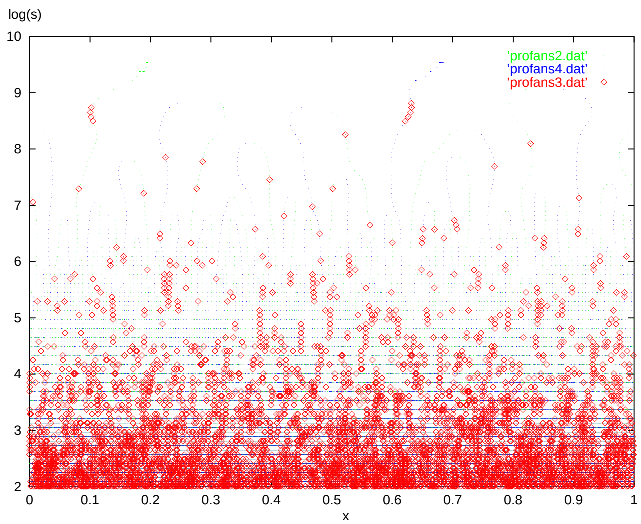
- with all its monochromatic *i.e mono-fractal* components separable!

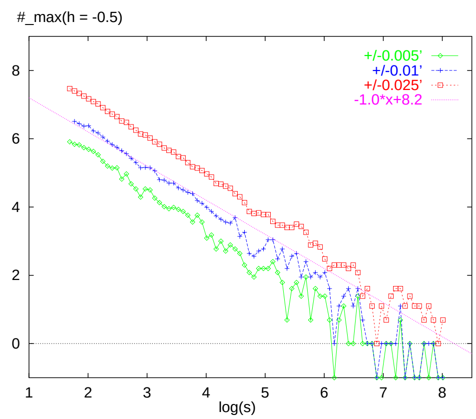


- Log-histograms of the exponent $\hat{h}$ at different scales for three example time series.
- Compare them to the $D_m(h)$ spectrum obtained by the partition functions method.

Scale-wise Evolution of the Effective Hölder Exponent

- The number of locations that fall within the band range visibly grows with scale, and this growth determines the dimension $D(h)$ which can be associated with the particular $\hat{h}$, at the band resolution ϵ.





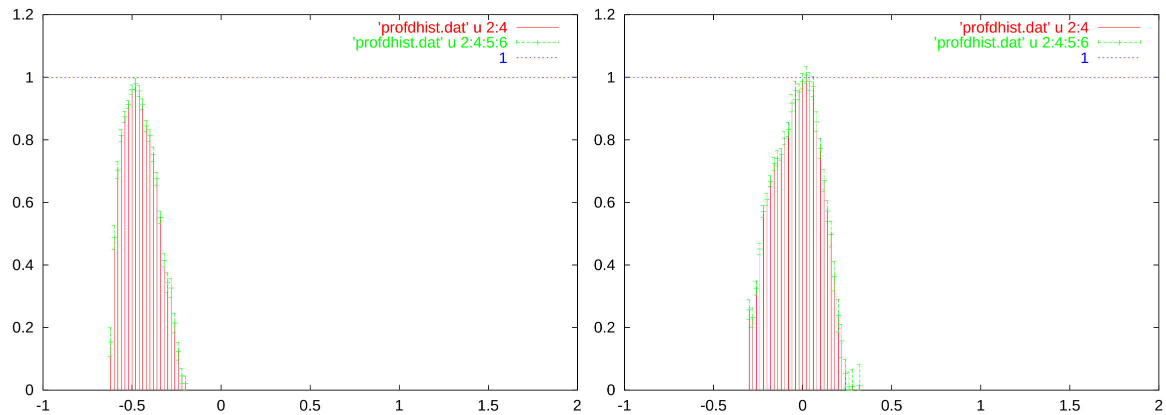
- Such $D(h)$ can be estimated for the entire range of h, resulting in the so-called *spectrum of singularities*.

$$D(\hat{h}) \sim \lim_{\epsilon \to 0} \lim_{s_{lo} \to 0} \frac{\log(\mu_\epsilon(\hat{h}(s_{lo})))}{\log(s_{lo})} ,$$

  where μ_ϵ is the measure of the total number of locations (selected maxima) that fall within the band of size ϵ at a particular scale location s_{lo}.

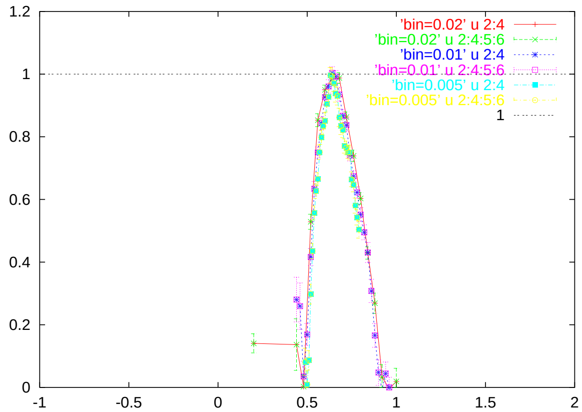
Direct Singularity Spectra from the WTMM Tree

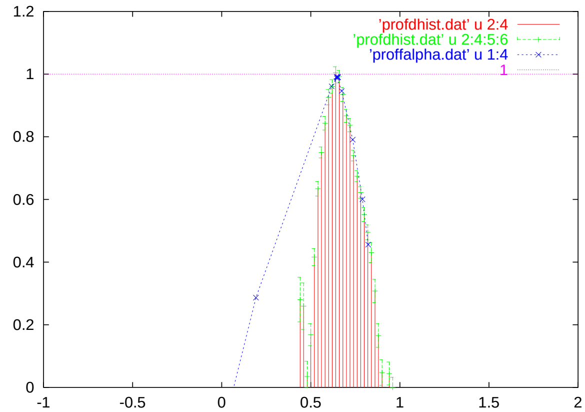
- The evaluation of direct spectra from the ϵ bands of the Hölder exponent simply requires covering the entire range of the local effective Hölder exponents detected on the maxima tree.



- The width of the spectrum of white noise is non-zero, as is inevitable for the finite length sample.
- Still, the heartbeat sample clearly shows a considerably wider spectrum, confirming the recently reported finding.

- Due to the fact that it relies on selecting a very narrow band of exponents, this procedure is, however, inherently sensitive to the choice of parameters such as the band width.
- The experiments indicate that the spectrum obtained remains stable for a wide choice of ϵ without loss of quality.

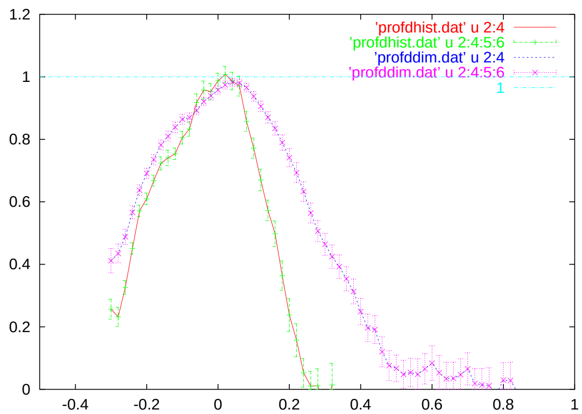


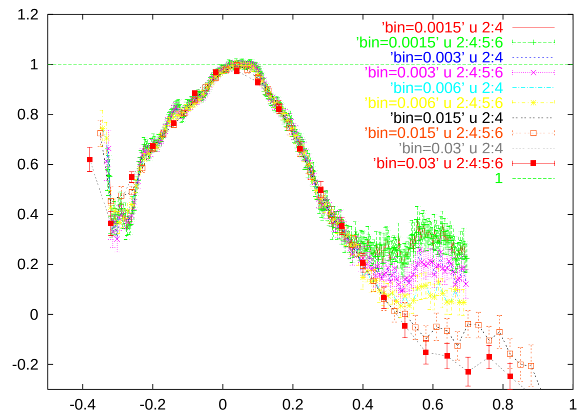


- At the cost of a slightly lower stability, we obtain the advantages of the direct spectrum calculation.
- The spectrum better captures local variations in the scaling of the h bands, where the partition function method provides only rough, 'outline' information about the $D(h)$ spectrum.

Direct Singularity Spectra from the Entire CWT

- Making use of the redundant information contained in the original CWT (as opposed to the WTMM used thus far).
- The comparison of the direct spectra obtained with both WTMM and the CWT suggest that the CWT may contain some information lacking in the WTMM.





- The CWT direct spectra show excellent stability with respect to the ϵ band width variation. We went down to the spectacular $\epsilon = 0.0015$ resolution.

Conclusions

- From: "H. E. Stanley" <hes@argento.bu.edu>
  Date: Wed, 28 Apr 1999 08:11:42 -0400

  [...] his panels showing diff color for each hurst expt
  are the CLEAREST exposition i know of what is a mf.

- Due to the local character of our effective Hölder exponent panels, additional information can be perceived such as non-stationarity.
- Large deviations from consistent statistical behaviour (e.g. boundary effects) can be directly assessed, in the panels or in the histogram.
- (All the above are hidden in the Legendre Transform, global approach.)
- (Stable) direct evaluation of the MF spectrum seems possible.