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Minkowski decomposition of convex polygons into their symmetric and asymmetric parts

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Abstract

This paper discusses Minkowski decomposition of convex polygons into their symmetric and totally asymmetric parts. Two different types of symmetries are considered: finite-order rotations and line reflections. The approach is based on the representation of convex polygons through their perimetric measure. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Convex polygon; Minkowski addition; Perimetric measure; Rotation; Reflection; Symmetry; (Strongly) cyclic transformation; Polygon decomposition

1. Introduction

In this paper, the following problem will be addressed: given a compact, convex set $P \subseteq \mathbb{R}^2$, find a decomposition of the form

$$P = P_s \oplus P_a, \qquad (1)$$

where P_s is *symmetric* in a sense to be specified, and where P_a is *totally asymmetric* (i.e., P_a does not contain any symmetric parts). Here $\oplus$ denotes *Minkowski addition*. Matheron and Serra (1988), who considered this problem for the case of central symmetry, used a perimetric representation to obtain such decompositions. In the work of Jourlin and Laget (1988) and Schneider (1989) one can find related material concerning the Minkowski decomposition of convex sets.

In this paper we show how the approach by Matheron and Serra (1988) can also be used to deal with rotation as well as (line) reflection symmetry. It turns out, however, that these two cases are essentially different. In the rotation-symmetric case, the perimetric measure of the symmetric part equals the minimum of the corresponding rotations of the perimetric measure of the original shape. In the reflection-symmetric case, such a result does not hold, but we are able to present an algorithm which finds the symmetric part with largest area. Although our arguments apply to arbitrary compact, convex sets, we shall restrict ourselves to convex polygons in order to obtain efficient algorithms.

Minkowski addition is one of the basic operations in mathematical morphology (e.g. (Heijmans, 1994; Serra, 1982)), where it is used to define *dilation*.

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Mathematical morphology is a powerful toolbox for (nonlinear) image processing with a solid mathematical foundation. In most cases, specific hardware for morphological image processing allows only neighborhood operations. Therefore, the problem of decomposing shapes (structuring elements) into smaller parts is relevant with respect to the efficient implementation of morphological routines.

Several authors have been concerned with the problem of decomposing (convex) shapes into simpler ones, both in the continuous (Ghosh, 1990, 1993; Grünbaum, 1963; Kanungo and Haralick, 1992) and the discrete case (Ghosh, 1996; Xu, 1991; Zhuang and Haralick, 1986). Note in particular that every convex polygon in $\mathbb{R}^2$ can be decomposed into a Minkowski sum of segments and triangles (Yaglom and Boltyansky, 1951).

The paper is organized as follows. Basic notations and definitions are given in Section 2. In Sections 3 and 4 we present algorithms for polygon decompositions: in Section 3 for rotation symmetry, and in Section 4 for reflection symmetry. Finally, in Section 5 we illustrate our theoretical findings with some concrete examples and we end with some concluding remarks.

2. Preliminaries

In this section we present some basic notation and terminology which we use in the sequel of the paper. $\mathscr{P}(\mathbb{R}^2)$, or $\mathscr{P}$ for short, denotes the family of convex polygons in $\mathbb{R}^2$. As in this paper the exact location of a polygon is irrelevant, we define an equivalence relation "$\equiv$" on $\mathscr{P}$: two polygons P and Q are said to be *equivalent*, $P \equiv Q$, if they differ only by translation.

A convex polygon $P \subseteq \mathbb{R}^2$ can be represented uniquely by specifying the position of one of its vertices and the lengths and directions of all its edges. By p_i we denote the length of edge i and by u_i the vector orthogonal to this edge: see Fig. 1. The angle between the positive x-axis and u_i is denoted by $\angle u_i$. Since the location of P is not important, it is sufficient to give the set $\{(u_1,p_1),(u_2,p_2),\dots,(u_n,p_n)\}$, where $n = n_P$ is the number of vertices of P. This set, denoted by $\boldsymbol{M}$, is called the *perimetric representation* of P. In Fig. 1 we give an illustration.

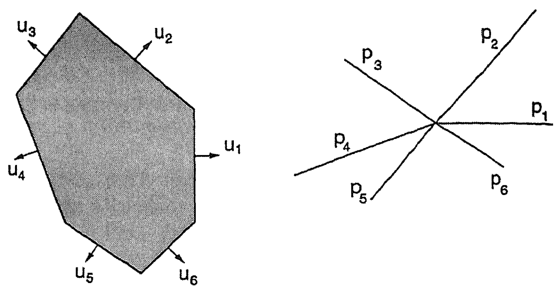

Fig. 1. Perimetric representation of a convex polygon.

A closely related representation of a convex polygon is the so-called *perimetric measure* $M(P,\cdot)$ (see e.g. (Matheron and Serra, 1988)):

$$M(P,u) = \begin{cases} p_i & \text{if } u = u_i, \\ 0 & \text{otherwise.} \end{cases}$$

We point out that the perimetric measure is a special case of the concept of *area measure* (see (Schneider, 1993)). It is easy to see that, for every convex polygon P, the identity

$$\sum M(P,u)\, u = 0 \tag{2}$$

holds; here the sum is taken over all u for which $M(P,u) \neq 0$. Moreover, this equality is sufficient for every discrete positive function defined on the unit circle to be the perimetric measure of a convex polygon. In fact, this relation expresses that the contour of P is closed.

The operation which plays a major role in this paper is *Minkowski addition* "$\oplus$" given by

$$A \oplus B = \{a + b \mid a \in A, b \in B\},$$

for two arbitrary sets $A, B \subseteq \mathbb{R}^2$. It is a well-known fact that the set of convex polygons $\mathscr{P}$ is closed with respect to Minkowski addition (see Chapter 1 of (Hadwiger, 1957)), and, moreover, that the Minkowski sum of two convex polygons can be computed by merging their perimetric representations; see e.g. (Ghosh, 1993; Grünbaum, 1967). Mathematically, this amounts to the following relation:

$$M(P \oplus Q,u) = M(P,u) + M(Q,u), \quad \text{for } P,Q \in \mathscr{P} \text{ and } u \in S^1. \tag{3}$$

Here S^1 denotes the unit circle in $\mathbb{R}^2$.

By G we shall denote the group of linear transformations on $\mathbb{R}^2$. Two important subsets of G are

R, the rotations around the origin (which forms a subgroup), and L, the reflections with respect to lines through the origin. This latter collection is not a subgroup. Denote by r_θ the rotation around the origin over angle θ. If $\theta = 2\pi/m$ then we speak of a *rotation of order m*. Denote by l_α the reflection with respect to the line through the origin which makes an angle α with the positive x-axis. We denote this line of reflection by L_α. Finally, we denote by $I \subseteq G$ the collection of *isometries* consisting of all rotations as well as all line reflections.

Definition 1. A transformation e in G is called a *symmetry* of a polygon P if $e(P) \equiv P$; we also say that P is *e-symmetric*.

A polygon $P \subseteq \mathbb{R}^2$ is called *rotation-symmetric of order m* if P is $r_{2\pi/m}$-symmetric. If $m = 2$, then we say that P is *central symmetric*. Central symmetry has been investigated in detail by Grünbaum (1963) and Matheron and Serra (1988). A polygon P is called *reflection-symmetric* with respect to axis L_α if P is l_α-symmetric.

Definition 2. (a) A transformation e is called a *cyclic transformation of order m* if

$$e^m(x) = x, \quad \text{for every } x \in \mathbb{R}^2. \tag{4}$$

(b) A transformation e is called a *strongly cyclic transformation of order m* if it has the property

$$x + e(x) + \cdots + e^{m-1}(x) = 0, \quad \text{for every } x \in \mathbb{R}^2. \tag{5}$$

It is easy to see that every strongly cyclic transformation of order m is also cyclic. The converse is not true, however. Finite-order rotations are strongly cyclic, whereas line reflections are cyclic (of order 2), but not strong. Note, furthermore, that $g^{-1}eg$ is (strongly) cyclic if e is (strongly) cyclic and $g \in G$.

3. Rotation decomposition

Recall that $M(P,\cdot)$ is the perimetric measure of P. If $P \in \mathcal{P}$ and $g \in I$ (i.e., g is a rotation or a reflection), then

$$M(g(P),u) = M(P,g^{-1}(u)). \tag{6}$$

Definition 3. Let $e \in I$ be given. A convex polygon P is said to be *totally e-asymmetric* if there does not exist a nontrivial e-symmetric polygon Q and a polygon R such that $P = Q \oplus R$.

Proposition 1. *Let e be a cyclic transformation of order m. If*

$$\min_{k=0,1,\ldots,m-1} M(P,e^k(u)) = 0, \quad \text{for every } u \in S^1, \tag{7}$$

then the polygon P is totally e-asymmetric.

Proof. Assume that e is cyclic and that Eq. (7) holds. Suppose that $P = Q \oplus R$, where Q is e-symmetric. Since $M(P,u) = M(Q,u) + M(R,u)$, we get that

$$M(P,e^k(u)) \geqslant M(Q,e^k(u)) = M(Q,u).$$

But this contradicts Eq. (7); we conclude that P is totally e-asymmetric. □

Assume now that e is a strongly cyclic transformation of order m, and that P is a convex polygon which is not totally e-asymmetric. Define

$$M(u) := \min_{k=0,1,\ldots,m-1} M(P,e^k(u)), \quad \text{for every } u \in S^1. \tag{8}$$

Let u be such that $M(u) \neq 0$; as P is not totally e-asymmetric, such a u does exist. Now $M(e^k(u)) = M(u)$, and

$$\sum_{k=0}^{m-1} M(e^k(u))e^k(u) = M(u)\sum_{k=0}^{m-1} e^k(u) = 0,$$

since e is strongly cyclic. This yields that

$$\sum_{u \in S^1} M(u)u = 0,$$

thus M is the perimetric measure of an e-symmetric polygon P_s^e. It is obvious that $M(P,u) - M(u) \geqslant 0$, with equality everywhere iff $P = P_s^e$. Suppose $M(\cdot) \neq M(P,\cdot)$; we get that $M(P,\cdot) - M(\cdot)$ is the perimetric measure of a convex polygon, which we denote by Q. Now, for every $u \in S^1$,

$$\begin{aligned}\min_{k=0,1,\ldots,m-1} M(Q,e^k(u)) &= \min_{k=0,1,\ldots,m-1} \left[M(P,e^k(u)) - M(e^k(u))\right] \\ &= \min_{k=0,1,\ldots,m-1} M(P,e^k(u)) - M(u) = 0.\end{aligned}$$

This yields that Q is totally e-asymmetric. We write $P_a^e := Q$. Observe that $P_a^e = P$ if P is totally e-asymmetric. The following result has been established.

Proposition 2. *If e is a strongly cyclic transformation of order m and if P is an arbitrary convex polygon, then P can be decomposed as*

$$P \equiv P_s^e \oplus P_a^e, \tag{9}$$

where P_s^e is e-symmetric and P_a^e is totally e-asymmetric. The perimetric measures of P_s^e and P_a^e are respectively given by

$$M(P_s^e,u) = \min_{k=0,1,\ldots,m-1} M(P,e^k(u)),$$

$$M(P_a^e,u) = M(P,u) - M(P_s^e,u). \tag{10}$$

The polygon P is totally e-asymmetric (i.e., $P_a^e \equiv P$) if and only if Eq. (7) *holds. Note that in the latter case $P_s^e \equiv \{0\}$.*

See Fig. 7 for an illustration.

The decomposition in Eq. (9) is a generalization of a result by Matheron and Serra (1988) where they consider the central symmetric case.

4. Reflection decomposition

When we consider line reflections, the decomposition problem is more difficult. Namely, in this case, the function M given by Eq. (8) is not the perimetric measure of a convex polygon, in general, since Eq. (2) is not satisfied. Here we shall describe an algorithm which, for a given line reflection $e = l_\alpha$, yields a unique decomposition

$$P \equiv P_s^e \oplus P_a^e, \tag{11}$$

such that P_s^e is l_α-symmetric and has the largest possible area. The basic idea is captured by Fig. 2. The line $L_\alpha^\perp$ which is orthogonal to L_α separates the plane in a left part H_α^- and a right part H_α^+; see Fig. 2(a). Furthermore, we put $H_\alpha^0 = L_\alpha^\perp$. We are interested in all directions $u \in S^1$ in the support of $M(P,\cdot)$ for which $l_\alpha(u) = u'$ lies in the support as well. In Fig. 2(b) we have drawn all these vectors. The vectors u_{+i} and $u'_{+i} = l_\alpha(u_{+i})$ $(i = 1,2,\ldots,k)$ lie in H_α^+, and the vectors u_{-i} and $u'_{-i} = l_\alpha(u_{-i})$ $(i = 1,2,\ldots,l)$ lie in H_α^-. If there exist vectors in H_α^0 with the given properties, they will be denoted by u_0 and u'_0. The vector u_{+1} is the vector in H_α^+ which makes the largest angle with the line L_α. Furthermore, it is possible that $\angle u_{+k} = \angle u'_{+k} = \alpha$ and that $\angle u_{-l} = \angle u'_{-l} = \alpha + \pi$.

Let us suppose that the set comprising u_{+i}, u'_{+i}, u_0, u'_0, u_{-j}, u'_{-j}, with $i = 1,\ldots,k$ and $j = 1,\ldots,l$, contains at least three different vectors and $k,l > 0$. Then the decomposition in Eq. (11) has a solution which can be found by the following algorithm (see Proposition 3). The basic idea is to choose pairs

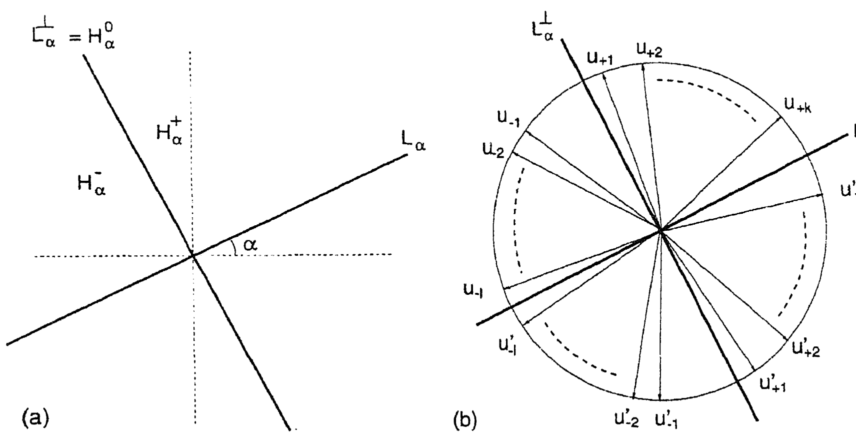

Fig. 2. L_α is the line of reflection. The line $L_\alpha^\perp$ separates vectors u in the perimetric representation for which also $u' = l_\alpha(u)$ is present, into two subclasses: u_{+i},u'_{+i} at the right and u_{-j},u'_{-j} at the left. More details can be found in the text.

u_{+i}, u'_{+i} and lengths $p_{+i} = p'_{+i}$ as well as pairs u_{-i}, u'_{-i} and lengths $p_{-i} = p'_{-i}$ such that

$$\sum p_{+i}(u_{+i} + u'_{+i}) = -\sum p_{-i}(u_{-i} + u'_{-i}). \quad (12)$$

Notice that $u_{+i} + u'_{+i}$ is directed along L_α in the positive direction, whereas $u_{-i} + u'_{-i}$ is directed along L_α in the negative direction. Eq. (12) expresses that the collection consisting of the pairs (u_{+i}, p_{+i}), (u'_{+i}, p_{+i}), (u_{-j}, p_{-j}), (u'_{-j}, p_{-j}), along with (u_0, p_0), (u'_0, p_0) (if present), defines a perimetric measure. In general, there will be more than one solution to Eq. (12). We have to find the solution for which the area of the resulting polygon P_s^e is maximal.

We define q_{+i}, $1 \leqslant i \leqslant k$, as follows:

$$q_{+i} = \begin{cases} \min\{M(P,u_{+i}), M(P,u'_{+i})\} & \text{if } u_{+i} \neq u'_{+i}, \\ \frac{1}{2}M(P,u_{+i}), & \text{if } u_{+i} = u'_{+i} \end{cases} \quad (13)$$

Observe that $u_{+i} = u'_{+i}$ implies $i = k$. The values q_{-i}, $1 \leqslant i \leqslant l$, are defined similarly, and $q_0 = \min\{M(P,u_0), M(P,u'_0)\}$, if u_0 does occur. Furthermore, we define

$$S_+ = \sum_{i=1}^{k} q_{+i} \cdot \|u_{+i} + u'_{+i}\| \quad \text{and}$$

$$S_- = \sum_{i=1}^{l} q_{-i} \cdot \|u_{-i} + u'_{-i}\|.$$

As we noted earlier, to construct the perimetric measure of a reflection-symmetric polygon, we have to satisfy Eq. (12).

There are three possibilities:

- $S_+ = S_-$: put $p_{-i} = q_{-i}$ and $p_{+i} = q_{+i}$; now the two vectors in Eq. (12), both directed along L_α, have the same length and therefore Eq. (12) holds;
- $S_+ > S_-$: we take $p_{-i} = q_{-i}$ and choose $p_{+i} \leqslant q_{+i}$ such that Eq. (12) holds and the area of P_s^e is maximal;
- $S_+ < S_-$: analogous ($p_{+i} = q_{+i}$).

We present our algorithm for the case $S_+ > S_-$; the case $S_+ < S_-$ is analogous. Since

$$\|u_{+i} + u'_{+i}\| < \|u_{+(i+1)} + u'_{+(i+1)}\|$$

for every i (the same holds for u_{-i}), the algorithm starts with the smallest indices in order to obtain a maximal area for P_s^e.

Define the set $\boldsymbol{M}$ to contain the pairs (u_{-i}, p_{-i}), (u'_{-i}, p_{-i}), where $p_{-i} = q_{-i}$. This set corresponds with the left part of a convex polygon (see Fig. 3). The idea is to "complete" this polygon to the right by adding vectors with direction u_{+i}, u'_{+i} and length $p_{+i} \leqslant q_{+i}$ until Eq. (12) is satisfied.

Algorithm 1. (Case that $S_+ > S_-$.)
$\boldsymbol{M} := \{(u_{-i}, p_{-i}), (u'_{-i}, p_{-i}) \mid i = 1,2,\ldots,l\}$;
 add (u_0, p_0), (u'_0, p_0) to $\boldsymbol{M}$ if present;
$S := 0$;
$i := 1$;
$p_{+1} := q_{+1}$;
$\triangle S := p_{+1} \cdot \|u_{+1} + u'_{+1}\|$;
while $(S + \triangle S < S_-)$ {
 $\boldsymbol{M} := \boldsymbol{M} \cup \{(u_{+i}, p_{+i}), (u'_{+i}, p_{+i})\}$;
 $S := S + \triangle S$;
 $i := i + 1$;
 $p_{+i} := q_{+i}$;
 $\triangle S := p_{+i} \cdot \|u_{+i} + u'_{+i}\|$;
}
$p := (S_- - S)/\cos(\angle u_{+i} - \alpha)$;
 (compute the remainder)
$\boldsymbol{M} := \boldsymbol{M} \cup \{(u_{+i}, p), (u'_{+i}, p)\}$

It has been explained that the equality $S_- = S_+$ must hold to have a perimetric measure corresponding with a reflection-symmetric polygon. This explains our stopping criterion $S + \triangle S < S_-$. Now the last two lines in the algorithm are necessary to obtain $S_+ = S_-$.

The perimetric measure M associated with the resulting perimetric set $\boldsymbol{M}$ represents an l_α-symmetric convex polygon P_s^e, and the difference $M(P,\cdot) - M(\cdot)$ represents the asymmetric part P_a^e.

It remains to be shown that Algorithm 1 yields the unique decomposition with P_s^e having maximal area. This is demonstrated by the following two observations. First we explain that, starting with a perimetric set $\boldsymbol{M}$ (first line of Algorithm 1) the algorithm yields the polygon with maximal area whose perimetric set contains $\boldsymbol{M}$. The set $\boldsymbol{M}$ yields a left part of an l_α-symmetric polygon. Our algorithm extends this polygon rightwards in a symmetric fashion, but it does so by choosing a path from point A on $L_\alpha^\perp$ (see Fig. 3) to the line L_α which has smallest descent, thus maximizing the area. This means that our algorithm is optimal if we can show that the initial choice for $\boldsymbol{M}$ is optimal; see Fig. 3.

However, any other choice for $\boldsymbol{M}$ in combination with our algorithm leads to a perimetric measure M'

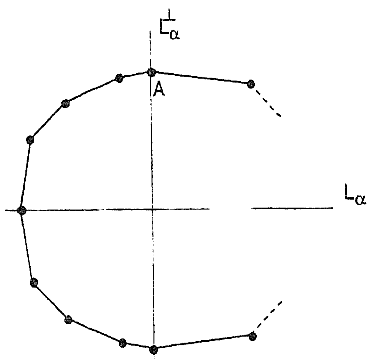

Fig. 3. Construction of reflection-symmetric part with maximal area.

which is smaller than the perimetric measure M obtained from Algorithm 1: $M'(u) \leqslant M(u)$ for every $u \in S^1$. This implies, however, that the area of the corresponding polygon is smaller, too.

Thus we have shown that Algorithm 1 yields the decomposition in Eq. (11) where P_s^e has maximal area. See Fig. 7 for an illustration.

Proposition 3. *Given a line reflection $e = l_\alpha$ and a convex polygon P, there exists a solution of* Eq. (11) *such that P_s^e is l_α-symmetric and has largest possible area, if and only if $k \geqslant 1$, $l \geqslant 1$, and the set S comprising u_{+i}, u'_{+i}, u_0, u'_0, u_{-j}, u'_{-j}, with $i = 1,\dots,k$ and $j = 1,\dots,l$, contains at least three different vectors, and in this case Algorithm 1 yields a solution.*

Observe that, when the assumptions above are not satisfied, then the algorithm yields an M which is empty or $M = \{(u_0, p_0), (u'_0, p_0)\}$ depending on whether the vectors u_0, u'_0 exist or not. In the first case, there does not exist a decomposition, in the second case we find that P_s^e is a line segment, which has zero area.

For most angles α the condition in Proposition 3 will not be satisfied. To find an upper estimate for the number of angles which have to be checked if P contains $n = n_P$ vertices, we have to consider the angles $\alpha_{i,j} = \frac{1}{2}(\angle u_i + \angle u_j) \bmod \pi$, with $1 \leqslant j \leqslant i \leqslant n$. Namely, u_j is the reflection of u_i with respect to the line that makes an angle $\alpha_{i,j}$ with the positive x-axis. An angle α is a candidate solution if there exists at least two pairs i_1, j_1 and i_2, j_2 such that $\alpha_{i_1,j_1} = \alpha_{i_2,j_2} = \alpha$. Furthermore, it is not allowed that both $i_1 = j_1$ and $i_2 = j_2$. An upper bound for the number of candidates is $\frac{1}{2}\sum_{i=1}^{n} i = \frac{1}{4}n(n+1)$, where $n = n_P$.

5. Examples and concluding remarks

In this final section we present some concrete examples to illustrate our results.

Let us consider first an example which illustrates the decomposition according to Algorithm 1. Consider the polygon P depicted in Fig. 4(a). Suppose that the reflection line coincides with the OX axis. The perimetric representations of original and reflected polygons are given in Fig. 4(b) and (c), respectively. The minimum of these perimetric representations computed according to Eq. (13) is presented in Fig. 4(d). Since this set contains 5 vectors, Proposition 3 says that there exists a Minkowski decomposition of the original polygon.

The resulting vectors S_+ and S_- computed for the right and left half-planes H^+ and H^- are shown in Fig. 5(a). The sum of S_+ and S_- does not equal 0 and therefore the set shown in Fig. 4(d) is not a

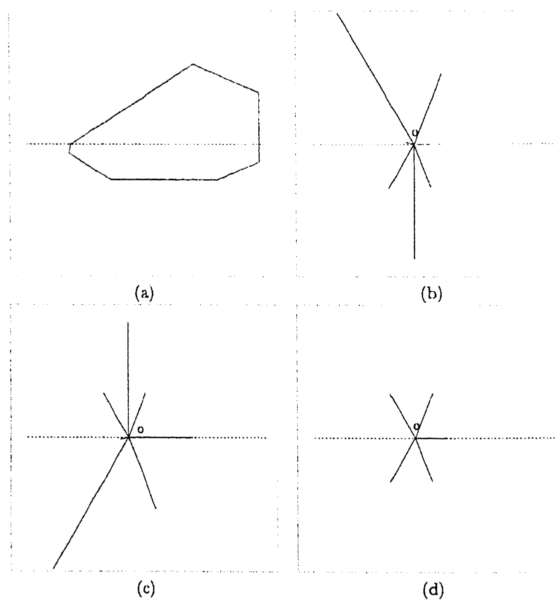

Fig. 4. (a) Polygon P; (b) perimetric representation of P; (c) perimetric representation of the polygon $l_0(P)$; (d) resulting set of vectors.

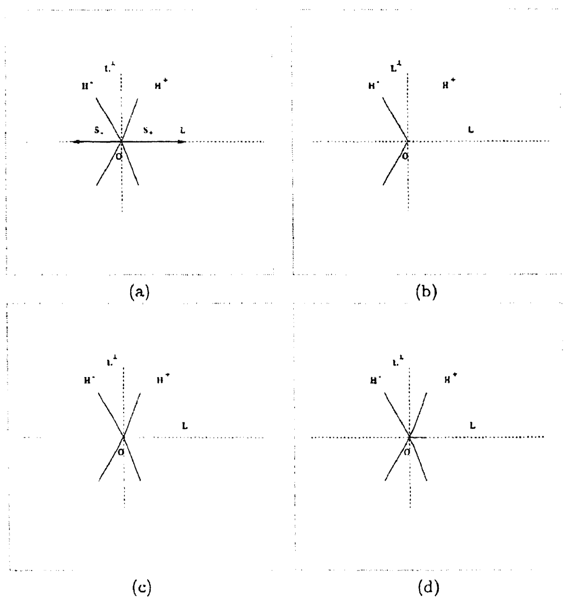

Fig. 5. Demonstration of Algorithm 1. (a) Vectors S_+ and S_-; (b) initial step of the algorithm; (c) first step of the algorithm; (d) second step of the algorithm.

perimetric representation. To extract the perimetric representation from this set we apply Algorithm 1.

Since vector S_- is shorter than S_+, the initial set M contains only vectors from the left half-plane; see Fig. 5(b).

At the first step of the algorithm we add two vectors from the right half-plane (see Fig. 5(c)).

At the second and last step we add a part of the third vector from the right half-plane. The resulting perimetric measure is shown in Fig. 5(d). This perimetric measure defines a reflection-symmetric polygon (Fig. 6) which is the solution of the decomposition problem with respect to the fixed reflection plane.

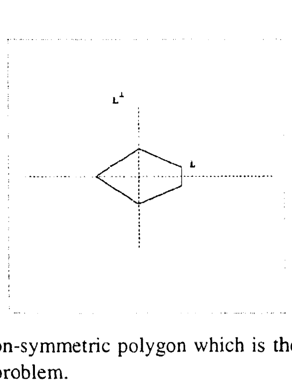

Fig. 6. Reflection-symmetric polygon which is the solution of the decomposition problem.

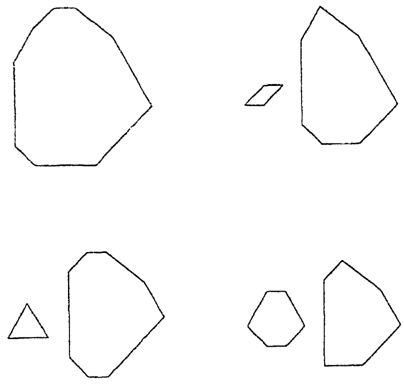

Fig. 7. From left to right and top to bottom: the original polygon P and its decomposition with respect to rotation over 180°, rotation over 120°, and reflection in the vertical axis.

In Fig. 7 we depict a convex polygon and three decompositions associated with three different symmetries: rotation over 180° (central symmetry), rotation over 120°, and reflection with respect to the vertical axis.

At first glance, one might expect that the triangle which represents the symmetric part with respect to rotation over 120°, should be contained in the symmetric part with respect to the line reflection. However, as we explained, our algorithm corresponding with line reflections yields the decomposition in which the symmetric part has maximal area: as a result, not the entire triangle is included in the symmetrical part but only part of it.

Having achieved a decomposition like in Eq. (9) or Eq. (11), we can define a functional $\mu : \mathscr{P} \times E \to [0,1]$ by

$$\mu(P,e) = \frac{V(P_s^e)}{V(P)}, \tag{14}$$

where $V(P)$ denotes the area of P, and where E consists of all finite-order rotations (in Eq. (9)) or line reflections (in Eq. (11)). Heijmans and Tuzikov (1996) have shown that μ has the following properties for $P \in \mathscr{P}$, $e \in E$:

1. $\mu(P,e) = \mu(P',e)$ if $P \equiv P'$;
2. $\mu(P,e) = \mu(e^k(P),e)$, $k \geqslant 1$;
3. $\mu(P,e) = 1$ iff P is e-symmetric;
4. $\mu(P,e) = \mu(h(P),heh^{-1})$, $h \in I$.

We call functionals μ which satisfy these properties and which, in addition, are continuous in the first variable with respect to the Hausdorff metric, I-invariant E-symmetry measures (Heijmans and Tuzikov, 1996). Note, however, that μ defined in Eq. (14) is not continuous. We present a systematic treatment of symmetry measures for convex sets based on Minkowski addition and the Brunn–Minkowski inequality (Heijmans and Tuzikov, 1996).

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