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Forcing relations for homoclinic orbits of the Smale horseshoe map

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ABSTRACT

An important problem in the dynamics of surface homeomorphisms is determining the forcing relation between orbits. The forcing relation between periodic orbits can be computed using existing algorithms. Here we consider forcing relations between homoclinic orbits. We outline a general procedure for computing the forcing relation, and apply this to compute the equivalence and forcing relations for homoclinic orbits of the Smale horseshoe map.

Forcing relations for homoclinic orbits
of the Smale horseshoe map

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12 November 2004

Abstract

An important problem in the dynamics of surface homeomorphisms is determining the forcing relation between orbits. The forcing relation between periodic orbits can be computed using existing algorithms. Here we consider forcing relations between homoclinic orbits. We outline a general procedure for computing the forcing relation, and apply this to compute the equivalence and forcing relations for homoclinic orbits of the Smale horseshoe map.

Keywords: Homoclinic orbit; Forcing relation; Smale horseshoe; Braid type.

AMS Subject Classifications: 37E30; 37C27; 37B10; 37E25.

1 Introduction

In this paper, we study the braid equivalence and forcing relation for homoclinic orbits of the Smale horseshoe map. This problem is important in obtaining a detailed understanding of the chaotic behaviour of general surface diffeomorphisms. The braid type of an orbit gives a combinatorial way of specifying the orbit, and the forcing relation places restrictions of the coexistence of orbits of different braid types. For systems exhibiting full-blown chaotic behaviour, we can then deduce the existence of infinitely many orbits from knowledge of just a single orbit, and also obtain a lower bound on the topological entropy. For parameterised families of systems exhibiting a transition to chaos, the forcing relation gives information on the order in which different orbits are created. The forcing relation for homoclinic orbits can also be used to give information about the periodic orbits of the system. We restrict to the Smale horseshoe map since this provides a model of chaotic dynamics in two dimensions which is universal (in a combinatorial sense), for which a natural symbolic description of orbits is available, and which models other important systems, such as the Hénon map.

Chaotic behaviour and the forcing relation is well-understood for one-dimensional (non-invertible) maps. Sharkovskii's theorem [23] describes the forcing relation of periodic orbits of interval maps in terms of the period. For unimodal interval maps, the kneading theory of Milnor and Thurston [21] shows that periodic orbits are created in a linear ordering. The kneading theory has been partially generalised to non-unimodal maps [1] to give a fairly complete picture of the bifurcation structure.

For systems in two-dimensions, the situation is more complicated and still not well-understood. It is still possible to obtain considerable information from a single orbit, but periodic orbits are only partially ordered [3] by the forcing relation. This makes the study of surface diffeomorphisms particularly

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interesting; the dynamics is much richer than that of one-dimensional maps, but there is enough structure that a fairly complete understanding of chaotic dynamics is feasible.

Considerable attention in the literature has been paid to the Hénon family, a two-dimensional analogue of the one-dimensional quadratic family, and to the Smale horseshoe map, an analogue of the tent map, and a theoretically tractable chaotic system. It has been shown that the Hénon map has a full Smale horseshoe for certain parameter values, and a partially-formed horseshoe for others [12]. In fact, any $C^{1+\epsilon}$ surface diffeomorphism with positive topological entropy must have a horseshoe [17], so the Smale horseshoe map is pivotal in studying chaos in two-dimensions. Further, the third iterate of the Smale horseshoe map contains periodic orbits of all braid types [18], so is "combinatorially universal".

The forcing relation for periodic orbits of surface diffeomorphisms can be studied using the Nielsen-Thurston theory of surface homeomorphisms [24]. This gives a classification of surface homeomorphisms up to isotopy, and provides a canonical diffeomorphism in each isotopy class which minimises the topological entropy and has minimal orbit structure (see [14, 4]). A number of algorithms exist to compute the forcing relation [2, 13, 19].

A conjecture of Hall and de Carvalho [10] suggests that the periodic orbits of the Smale horseshoe map are partitioned into families on which the forcing relation restricts to a linear order, and that each family has an associated homoclinic orbit which forces all periodic orbits of the family. The forcing relation between families can be determined from a knowledge of the forcing relation between the associated homoclinic orbits. If the conjecture is true, being able to compute the forcing relation between homoclinic orbits is an important problem in determining the complete forcing relation. Partial results on the conjecture are given in [11].

In this paper we compute the braid equivalence and forcing relations for homoclinic orbits of the Smale horseshoe map using the trellis theory developed in [7, 8]. The braid type of a homoclinic orbit to a saddle fixed point can be described by the *trellis* formed by finite pieces of the stable and unstable manifolds of the saddle fixed point. The trellis can be represented combinatorially by giving the relative ordering of the intersection points along the stable and unstable manifolds. Since approximations to trellises can be computed numerically for a given diffeomorphism, we can, in theory, extract the combinatorics of a numerically computed trellis to find the braid type. The results of this paper, however, use a purely combinatorial construction of the trellis from the symbolic coding of the homoclinic orbit.

The fundamental operation on trellises is that of *pruning*, related to the pruning isotopies of de Carvalho [9]. However, the pruning theory for trellises is much simpler than the general theory, and has a straightforward formulation in terms of the combinatorics of the trellis. We compute the trellis associated with a given homoclinic horseshoe orbit by pruning a trellis for the Smale horseshoe map relative to the given orbit. The braid equivalence relation is given by comparing the combinatorics of the resulting trellis, and the forcing relation can be given by following the other orbits through the pruning. By computing the *graph representative* of the trellis, a lower bound for the topological entropy forced by the homoclinic orbit can be found.

A number of other authors have considered forcing relations for homoclinic orbits and tangles. Rom-Kedar [22] gave forcing relations and entropy bounds for some infinite families of trellises. McRobie and Thompson [20] discussed the possible bifurcation sequences associated with the break-up of horseshoes in a driven oscillator using methods similar to our prunings. Methods for the computation of the forcing relation for homoclinic orbits have been given by Handel [15] and Hulme [16]. The methods outlined in this paper have the advantage of being applicable to general trellises and homoclinic/heteroclinic orbits, and the computations are straightforward to implement. Further, this is the first paper to present numerical computations of forcing relations between homoclinic orbits.

The paper is organised as follows. In Section 2, we describe the combinatorics of periodic and homoclinic orbits of the Smale horseshoe map, and introduce the braid type of a homoclinic orbit. In Section 3 we introduce the results of trellis theory which are necessary for the calculations. In Section 4 we describe the algorithms used to compute the braid equivalence and forcing relations, and discuss the implementation of the pruning procedure. In Section 5 we present the results of numerical computations of the equivalence and forcing relations for homoclinic orbits of the Smale horseshoe map with short symbolic codings. Finally, we give some conclusions and open problems in Section 6.

The computations of this paper were performed using the C++ library "`trellis`", which is freely available for download [6]. This package allows computations on combinatorial trellises, including the computation of the trellis associated with a horseshoe homoclinic orbit, and the computation of the *graph representative* of the trellis.
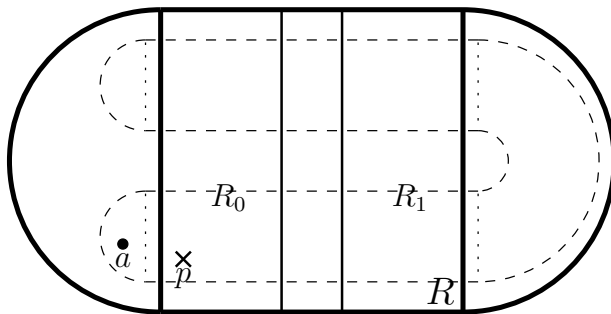
2

2 Horseshoe Orbits

Figure 1: The Smale horseshoe map

In this paper, the model of the Smale horseshoe map $f\colon S^2 \to S^2$ depicted in Figure 1 will be used. The stadium-shaped domain shown, consisting of two half-discs and a square R, is mapped into itself as in an orientation-preserving way as indicated by the dotted lines, with a stable fixed point a in the left hemisphere. The map is then extended to a homeomorphism of S^2 with a repelling fixed point at ∞ whose basin includes the complement of the stadium domain. The saddle fixed point of negative index (i.e. positive eigenvalues) is denoted p.

The non-wandering set $\Omega(f)$ consists of the fixed points a and ∞, together with a Cantor set

$$\Lambda = \{x \in S^2 \;:\; f^n(x) \in R \text{ for all } n \in \mathbb{Z}\}.$$

Since Λ is contained in the union of the rectangles R_0 and R_1, symbolic dynamics can be introduced in the usual way, providing an *itinerary* homeomorphism

$$k\colon \Lambda \to \Sigma_2 = \{0,1\}^{\mathbb{Z}},$$

with the property that $\sigma(k(x)) = k(f(x))$ for all $x \in \Lambda$ (where $\sigma\colon \Sigma_2 \to \Sigma_2$ is the shift map). The *code* of an orbit H in Λ is the bi-infinite sequence given by the itinerary of any of its points.

The itinerary of a point x is periodic of period n if and only if x is a period n point of F. This paper is concerned with orbits which are homoclinic to p, and the term *homoclinic orbit* will be used exclusively to mean such orbits. A point $x \in \Lambda$ is homoclinic if and only if its itinerary consists of finitely many 1's.

Definition 2.1 (Core) Let H be a homoclinic orbit of the horseshoe. The *core* of H is the longest word in the code of H which begins and ends with 1.

The *signature* of H is equal to the length of the core minus one. Thus, for example, the point with itinerary $\overline{0}110 \cdot 0101\overline{0}$ is a point of the homoclinic orbit $\overline{0}1100101\overline{0}$ with core 1100101 and signature 6. The *primary* homoclinic orbits are those with cores 1 and 11; these two orbits have the same homoclinic braid type, are forced by every other homoclinic orbit, but do not force any other periodic or homoclinic orbit. By contrast, the orbits with cores 111 and 101 will be shown to force all periodic and homoclinic orbits of the horseshoe (cf. [15]).

Definition 2.2 (Decoration) The *decoration* of H is defined to be $*$ if H has code $\overline{0}1_1^0 1\overline{0}$, $\cdot$ if H has code $\overline{0}1_{11}^{00} 1\overline{0}$ and c if H has code $\overline{0}1_1^0 c_1^0 1\overline{0}$.

Orbits of the Smale horseshoe map with the same decoration have the same braid type, as defined below.

Definition 2.3 (Braid type) The *braid type* of a homoclinic orbit H of a surface diffeomorphism f, denoted $\mathrm{BT}[H; f]$ is the conjugacy class of the isotopy class of f relative to H.

In other words, orbits H of f and H' of f' have the same braid type if there is an isotopy (f_t) and a diffeomorphism h such that $f = f_0$, H is a homoclinic orbit of f_t for all t, $H' = h(H)$ and $f' = h \circ f_1 \circ h^{-1}$.

Definition 2.4 (Forcing) A braid type BT forces BT' if every homeomorphism f with a homoclinic orbit H with $\mathrm{BT}(H, f) = \mathrm{BT}$ has a homoclinic orbit H' with $\mathrm{BT}(H', f) = \mathrm{BT}'$.

3

3   Horseshoe trellises

In this section those aspects of trellis theory which will be used later are reviewed. Trellis theory is applicable in a much more general setting (see [8] for full details), but here the key definitions and results are presented in a manner tailored for the study of horseshoe trellises. All results stated in this section can be found in [8].

The key ideas presented are as follows. A *trellis* is a finite portion of the tangle of stable and unstable manifolds of a saddle fixed point. Starting with the familiar tangle of the full horseshoe, the full horseshoe trellis of signature n can be defined for each integer $n \geq 2$: it has longer and longer stable and unstable branches as n increases. Given a horseshoe homoclinic orbit, the full horseshoe trellis of appropriate signature can be *pruned*, by removing as many intersections as possible without disturbing the given homoclinic orbit. This pruned trellis is a complete invariant of homoclinic braid type, and so the technique can be used to determine whether or not two given homoclinic orbits have the same braid type.

Given a trellis (and the action of a diffeomorphism on it), there is a lower bound on the dynamics of any diffeomorphism which has such a trellis. This minimal dynamics can be computed as the dynamics of a tree map, using techniques similar to those developed for computing train tracks for periodic orbits by Bestvina and Handel [2] and Franks and Misiurewicz [13].

The dynamics forced by a given horseshoe homoclinic orbit can thus be determined by finding the appropriate pruned horseshoe trellis, and calculating the associated tree map. In this paper, we are only concerned with forcing relations between homoclinic orbits.

3.1   The full horseshoe trellis

Definition 3.1 (Trellis) Let $f: S^2 \to S^2$ be a diffeomorphism, and p be a hyperbolic saddle fixed point of f. Then a *trellis for f (at p)* is a pair $T = (T^U, T^S)$, where T^U and T^S are intervals in $W^U(f; p)$ and $W^S(f; p)$ respectively containing p. (Here, $W^U(f; p)$ and $W^S(f; p)$ denote the unstable and stable manifolds, respectively, of f at p.) Given a trellis $T = (T^U, T^S)$, denote by T^V the set of intersections of T^U and T^S. The trellis is *transverse* if all of its intersection points are transverse.

Since all trellises considered in this paper will be transverse, and hence the word *trellis* will be understood to mean *transverse trellis*.

Definition 3.2 (Segment) Let $T = (T^U, T^S)$ be a trellis. A *segment* of T is a closed subinterval of either T^U or T^S with endpoints in T^V but interior disjoint from T^V. The segment is called *unstable* or *stable* according as it is a subinterval of T^U or of T^S.

Definition 3.3 (Region, bigon) Let $T = (T^U, T^S)$ be a trellis. Then a *region* of T is the closure of a component of $S^2 \setminus (T^U \cup T^S)$. A *bigon* of T is a region bounded by two segments (one unstable and one stable).

Definition 3.4 (Full horseshoe trellis) Let f be the Smale horseshoe map, and p be the fixed point with code $\overline{0}$. Let q_0 be the homoclinic point with itinerary $\overline{0} \cdot 1\overline{0}$, and $q_k = f^k(q_0)$. Given $i \geq 0$ and $j \leq 0$, denote by T_i^U an interval in $W^U(f; p)$ with end intersections p and q_i, and by T_j^S the interval in $W^S(f; p)$ with end points p and q_j. For $n \geq 2$, a *full horseshoe trellis of signature n* is a trellis $T = (T_i^U, T_j^S)$ such that $i - j = n$.

It is clear that all full horseshoe trellises as defined above are differentiably conjugate.

Example 3.5 The full horseshoe trellis of signature 2 is depicted in Figure 2(a). The chaotic dynamics is supported in the regions labelled R_0 and R_1. All points in R_U are in the basin of the attracting fixed point a, and all points in the interior of R_U are in the basin of the repelling point at infinity. The point r_0 has itinerary $\overline{01} \cdot 01\overline{0}$, and the point r_1 has itinerary $\overline{01} \cdot 11\overline{0}$. The full horseshoe trellis of signatures 3 is depicted in Figure 2(b).
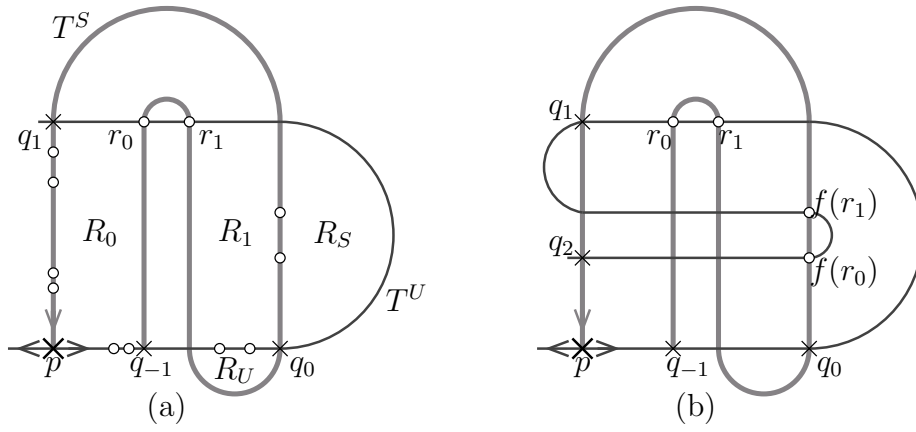
4

Figure 2: (a) The full horseshoe trellis with signature 2. (b) The full horseshoe trellis with signature 3.

3.2 Pruning isotopies

Given a trellis T for a diffeomorphism f, a pruning isotopy is an isotopy which removes the intersections on the boundary of one or more bigons of f. To be more precise, it is an isotopy from f to a diffeomorphism f' which has a trellis T' obtained from T by removing such intersections. There are two possibilities; we can either remove both intersections of a single bigon, as depicted in Figure 3(a), or remove intersections from two neighbouring bigons, changing the orientation of the crossing at the remaining intersection, as depicted in Figure 3(b).

Figure 3: The local effect of pruning isotopies on a trellis

However, an isotopy of the diffeomorphism f supported in some open set U will also change the trellis outside of U. If we are trying to reduce the number of intersections of T, we need to ensure that no other intersections are created when we remove intersections locally. This gives rise to the notion of an *inner bigon*

Definition 3.6 (Inner bigon) A bigon B is *inner* if $B \cap \bigcup_{n \in \mathbb{Z}} f^n(T^V) = B \cap T^V$. i.e. A bigon is B inner if the only intersections of B with the orbits of the intersection points of the trellis are the vertices of B.

The following result follows from the proof of Theorem 3.5 of [8].

Theorem 3.7 (Pruning away an inner bigon) *Let T be a trellis of a diffeomorphism f. Suppose either that B is an inner bigon with vertices v_0 and v_1, or that B_0 and B_1 are inner bigons with a common vertex v and other vertices v_0 and v_1 on different orbits. Then there is a diffeomorphism $h : S^2 \to S^2$, which we can take to be supported on a neighbourhood U of B or $B_0 \cup B_1$, such that $f' = f \circ h$ has a trellis T' with the same intersections apart from those on the orbits of v_0 and v_1 under f.*

The trellis T' is obtained by removing all the intersections of T contained in the orbit of U. The diffeomorphism f' is isotopic to f, and a general pruning isotopy can be constructed as a sequence of isotopies pruning away an inner bigon.

5

Example 3.8 Figure 4a) depicts the full horseshoe trellis of signature 3, and a shaded neighbourhood U of an inner bigon B, together with its image. Pruning away the bigon B yields a diffeomorphism f' with trellis T' as shown in b). Pruning away the bigon B' yields a diffeomorphism f'' with the trellis T'' depicted in c).
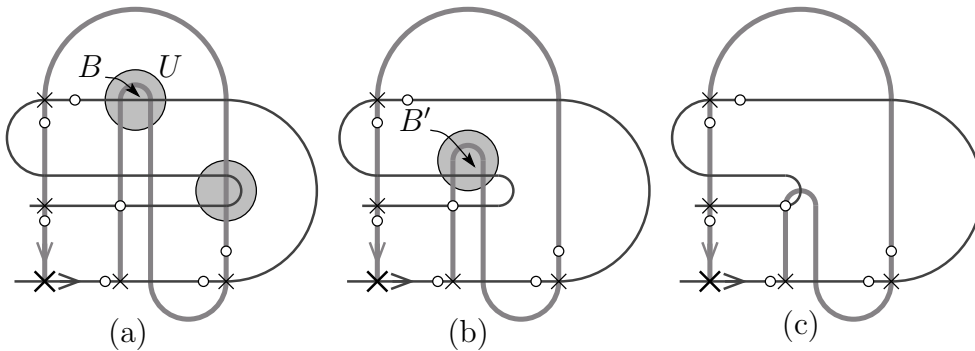

Figure 4: Pruning away inner bigons in the horseshoe trellis

3.3 Horseshoe trellises

A trellis obtained by pruning the full horseshoe trellis as in Example 3.8 is called a horseshoe trellis:

Definition 3.9 (Horseshoe trellis) A *horseshoe trellis* is a trellis T obtained from the full horseshoe trellis by a pruning isotopy (i.e. by pruning away a sequence of inner bigons).

A horseshoe trellis can be associated to each homoclinic orbit of the horseshoe, by pruning away as many inner bigons as possible without touching the homoclinic orbit. It is trivial that the signature of a horseshoe homoclinic orbit H is equal to the least integer n such that H is an intersection of the full horseshoe trellis of signature n.

Definition 3.10 (Trellis forced by a homoclinic orbit) Let H be a horseshoe homoclinic orbit of signature n, and let $m \geq n$. The *trellis of signature m forced by H* is the trellis T obtained from the full horseshoe trellis of signature m by pruning away inner bigons as much as possible without removing points of H.

Example 3.11 The white circles in Figure 4 represent points of the homoclinic orbit H with code $\overline{0}10010\overline{0}$ (which thus has signature 3). The trellis of Figure 4c) is thus the trellis forced by this homoclinic orbit. Note that every bigon has a point of H on its boundary.

This method makes it possible to determine whether or not two horseshoe homoclinic orbits have the same homoclinic braid type:

Definition 3.12 (Trellis type) Let T and T' be horseshoe trellises for diffeomorphism f and f' respectively. We say that $(f; T)$ and $(f'; T')$ have the same *trellis type* if there is a diffeomorphism g isotopic to f relative to T, and a homeomorphism $h : S^2 \to S^2$ such that $h(T) = T'$ and $h^{-1} \circ f' \circ h = g$.

We denote the trellis type containing $(f; T)$ by $[f; T]$.

For horseshoe trellises, the trellis type is determined by the geometry of the trellis:

Theorem 3.13 *Let T and T' be horseshoe trellises for diffeomorphism f and f' respectively. Then $(f; T)$ and $f'; T')$ have the same trellis type if and only if T and T' are diffeomorphic.*

PROOF: It suffices to consider the case $T = T'$. Since the points with itinerary $\overline{0}10\overline{0}$ all lie on a single homoclinic orbit, we can deduce the action of f and f' on all the vertices of T from the action of this orbit simply by counting vertices in each fundamental domain. The result follows since all regions of T are simply-connected, so the isotopy class is determined by the action on the segments. □

6

Since a horseshoe trellis type is fully determined by the geometry of the trellis, we define the type of a horseshoe trellis T to be the type of $(f;T)$ for any diffeomorphism f with trellis T which can be obtained by pruning away inner bigons.

Horseshoe trellises $(f;T)$ and $(f';T')$ have the same type if and only if the trellises T and T' are homeomorphic, and this occurs if and only if the orderings of the intersections on the stable and unstable manifolds are the same.

Definition 3.14 (Intersection ordering) Let $(f;T)$ be a horseshoe trellis, with intersections $T^V = \{v_i : i = 0 \ldots n - 1\}$ such that $v_i <_u v_{i+1}$ (i.e. v_i is closer to p along the unstable manifold). Then the *relative ordering* of the unstable and stable manifolds is the permutation π_T such that $v_{\pi_T(i)} <_s v_{\pi_T(j)}$ if and only if $\pi_T(i) < \pi_T(j)$.

The following result gives a computable criterion for the equivalence of horseshoe trellises.

Theorem 3.15 *Let T and T' be horseshoe trellises. Then T and T' have the same trellis type if and only if $\pi_T = \pi_{T'}$.*

The following result is immediate from Theorem 2 of [8], and shows that a homoclinic braid type is determined by the geometry of the trellis obtained by pruning up to the given orbit. In particular, homoclinic orbits can only have the same braid type if they have the same signature (i.e. their cores have the same length).

Theorem 3.16 *Let H and H' be horseshoe homoclinic orbits of signatures n and n', and let T and T' be the trellises of signature $m \geq \max\{n, n'\}$ forced by them. Then H and H' have the same homoclinic braid type if and only if T and T' have the same trellis type.*

The following result follows from the main theorem (Theorem 1) of [5] which shows that the dynamics forced by a trellis is minimal in the isotopy class.

Theorem 3.17 *Let H and H' be horseshoe homoclinic orbits of signatures n and n', and let T and T' be the trellises of signature $m \geq \max\{n, n'\}$ forced by H. The homoclinic braid type of H forces the braid type of H' if and only if T has an intersection on an orbit with the same braid type as H'.*

4 Computation of the equivalence and forcing relations

We now give a brief outline of the algorithm used to compute the equivalence and forcing relations for horseshoe homoclinic orbits, and then discuss the implementation details of the pruning procedure.

4.1 Algorithm

To compute the trellis forced by the orbit H, a full horseshoe trellis of the same signature as H is constructed, and then pruned relative to the orbit H and the orbit with code $\overline{0}1\overline{0}$, the latter being used to fix the end intersections. The pruning process sweeps through the trellis, pruning whenever an inner bigon is found whose vertices are not points of H, or a pair of inner bigons is found whose outer vertices are not points of H.

The equivalence relation for orbits of signature n is determined by first computing all the trellises of signature n associated with the homoclinic orbits. The homoclinic orbits of the same decoration are known to have the same braid type, so only one orbit with each decoration is needed. Since the trellis is a braid type invariant by Theorem 3.16, and the relative intersection ordering characterises horseshoe orbits by Theorem 3.15, the relative intersection ordering is computed for each orbit, and the orbits partitioned into equivalence classes accordingly. To compute the equivalence relation between homoclinic horseshoe orbits, it suffices to compute the trellis of signature n forced by each.

Since the (horseshoe) itineraries of intersections can be continued through the pruning, the codes of the remaining orbits can be recovered from the pruned trellis. Note, though, that in order to show that the homoclinic braid type of a homoclinic orbit H does *not* force that of a homoclinic orbit H', it is necessary to show that *none* of the homoclinic orbits of the same type as H' persist through the pruning.

Thus it is necessary to compute equivalences in order to be able to compute the forcing relation. Hence, to compute whether $BT(H)$ forces $BT(H')$ where H has signature n and H' has signature n', we need to compute all trellises of signature n' to determine the equivalence relation on braid types of signature n'. We then compute the trellis of signature $\max\{n, n'\}$ forced by H, and determine if it contains an orbit of braid type $BT(n')$.

The topological entropy forced by the homoclinic orbit of a given braid type was computed by finding the graph representative, as detailed in [7, 8].

4.2   Implementation of pruning

The most natural implementation of a trellis type is as a list of vertices with image information. Hence an intersection can be represented as

```
class Intersection is
  Orientation orientation;
  Intersection previous_unstable;
  Intersection next_unstable;
  Intersection previous_stable;
  Intersection next_stable;
  Intersection preimage;
  Intersection image;
end;
```

Here, each `Intersection` data element is a *reference* to another intersection, which may be `void`, and the `orientation` is either positive or negative, depending on the orientation of the intersections of the periodic orbit. We remark that this contains more information than is needed to specify the trellis structure; it suffices to store the `next_unstable`, `next_stable image` intersections. However, at least one of `previous_unstable` or `previous_stable` is required for the algorithm, and `preimage` is useful for general trellises. Note that the orientations for horseshoe trellises always alternate between positive and negative.

The main procedure needs to detect and prune inner bigons, as shown in Figure 3(a). Detecting bigons and inner bigons can be accomplished by the predicates `is_bigon` and `is_inner_bigon`. We assume throughout that the function arguments are not `void`.

```
bool is_bigon(Intersection a, Intersection b) is
  return( (a.next_unstable==b or b.next_unstable==a)
    and (a.next_stable==b or b.next_stable==a) );
end;
```

```
bool is_inner_bigon(Intersection a, Intersection b) is
  /* assume a.preimage==void and b.preimage==void */
  do
    if(a==void and b==void) then return true; end;
    if not is_bigon(a,b)) then return false; end;
    a:=a.image;
    b:=b.image;
  end;
end;
```

The pruning is performed by the procedures `prune_bigon` and `prune_inner_bigon` described below. The vertices `a` and `b` are deleted following a `prune_bigon` function.

```
prune_bigon(Intersection a, Intersection b) is
  /* assume is_bigon(a,b) */
  if(a.next_unstable!=b) then swap(a,b) end;
  a.previous_unstable.next_unstable:=b.next_unstable;
```

```
      b.next_unstable.previous_unstable:=a.previous_unstable;

    if(a.next_stable!=b) then swap(a,b) end;
    a.previous_stable.next_stable:=b.next_stable;
    b.next_stable.previous_stable:=a.previous_stable;
end;

prune_inner_bigon(Intersection a, Intersection b) is
  /* assume is_inner_bigon(a,b) */
  while(a!=void) do
    prune_bigon(a,b);
    a:=a.image;
    b:=b.image;
  end;
end;
```

The procedure `prune_inner_bigon` can be easily constructed from `prune_bigon`. Detecting and removing the intersections of two adjacent bigons as shown in Figure 3(b) can be carried out as follows: The vertices `a` and `c` may be deleted after a `prune_bigon_pair` procedure.

```
bool is_bigon_pair(Intersection a, Intersection b, Intersection c) is
  return( ((a.next_unstable==b and b.next_unstable==c)
        or (c.next_unstable==b and b.next_unstable==a))
      and ((a.next_stable==b and b.next_stable==c)
        or (c.next_stable==b and b.next_stable==a)) );
end;

prune_bigon_pair(Intersection a, Intersection b, Intersection c) is
  /* assume is_bigon_pair(a,b,c) */
  if(a.next_unstable!=b) then swap(a,c) end;
  a.previous_unstable.next_unstable:=b;
  c.next_unstable.previous_unstable:=b;

  if(a.next_stable!=b) then swap(a,c) end;
  a.previous_stable.next_stable:=b;
  c.next_stable.previous_stable:=b;

  b.orientation := -b.orientation;
end;
```

When dealing with trellises which are not horseshoe trellises, extra code is needed to take care of degenerate cases involving endpoints. In particular, in the function `is_inner_bigon`, an extra line

```
    if(a==void or b==void) then return false; end;
```

is needed in case an endpoint occurs between `a` and `b`. These procedures can be further optimised; we present them in the form given here to show explicitly the steps required, omitting only the details of garbage collection.

5   Numerical results

In this section we compute, for horseshoe homoclinic orbits with short cores, the equivalence classes under the relation of having the same homoclinic braid type, and the forcing relation.

Sig	Dec	Ordering
2	$*$	**0**, *7*, 4, *3*, 2, **5**, **6**, *1*
3	$\cdot$	**0**, *9*, 6, *5*, 4, *3*, 2, **7**, **8**, *1*
4	0	**0**, *11*, 8, *7*, 6, *5*, 4, *3*, 2, **9**, **10**, *1*
	1	**0**, *23*, 12, *11*, 6, *5*, 4, 19, 16, 15, 20, *3*, 2, 21, 14, 9, 8, **17**, **18**, 7, 10, 13, 22, *1*

Table 1: Relative orderings for trellis with signature up to 4. The intersections corresponding to the fixed point and the forcing orbits are indicated in bold type, and the intersections corresponding to the primary homoclinic orbit with code $\overline{0}1\overline{0}$ are indicated in italic type.

5.1 Horseshoe homoclinic orbits with the same homoclinic braid type

We have computed the trellis types forced by all horseshoe homoclinic orbits of signature 12 or less, and determine which pairs have the same homoclinic braid type. Table 1 presents the relative orderings of trellises of signature up to 4, and Table 3 presents the equivalence relation for orbits of signatures up to 9.

For orbits of signature up to 4, the only equivalences are trivial; two homoclinic orbits have the same homoclinic braid type if and only if they have the same decoration. However the orbits of signature 5 with decorations 01 and 10, and codes $\overline{0}1_1^001_1^01\overline{0}$ and $\overline{0}1_1^010_1^01\overline{0}$ respectively, are equivalent (see Figure 5).
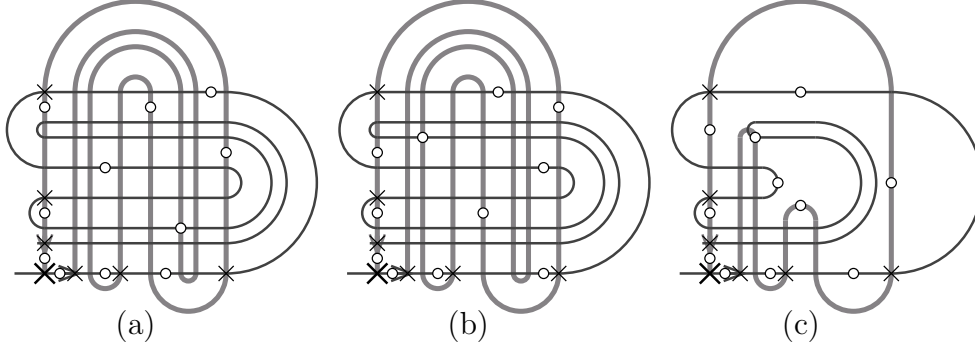
Figure 5: The horseshoe trellis and homoclinic orbits with codes (a) $\overline{0}1101111\overline{0}$ and (b) $\overline{0}1110111\overline{0}$, and (c) the trellis they force.

For homoclinic orbits of signature at most 7, all orbits have the same homoclinic braid type as their time reversal. However, for orbits of signature 8, there are two pairs of orbits whose homoclinic braid type differs from that of their time reversal. This is a counterexample to the conjecture that horseshoe orbits which are time-reversals have the same braid type.
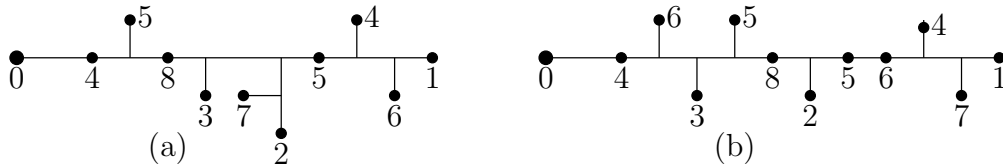

Figure 6: Topological graph representatives for the homoclinic orbits with codes (a) $\overline{0}1_1^001001_1^01\overline{0}$ and (b) $\overline{0}1_1^010010_1^01\overline{0}$.

The trellises forced by the orbits with codes $\overline{0}1_1^001001_1^01\overline{0}$ and $\overline{0}1_1^010010_1^01\overline{0}$ are not equivalent, even though these words are reverses of each other. Each trellis has 176 intersections. The graph representatives

Sig	$\#T^V$	h_{top}	Decorations
2	8	0.693147	$*$
3	10	0.528049	$\cdot$
4	12	0.434175	0
	24	0.637160	1
5	14	0.372312	00
	26	0.585210	01, 10
	38	0.589844	11
6	16	0.327931	000
	28	0.621811	001, 100
	39	0.493726	011, 010, 110
	76	0.618251	111
	92	0.655290	101
7	18	0.294274	0000
	29	0.658093	0001, 1000
	42	0.450806	0011, 0010, 0100, 1100
	58	0.471989	0110
	70	0.549761	0111, 0101, 1110, 1010
	122	0.591665	1101, 1011
	134	0.604018	1111
	218	0.684471	1001

Table 2(a): Equivalences for homoclinic orbits with signature up to 7.

Sig	$\#T^V$	h_{top}	Decorations
8	19	0.267727	00000
	31	0.676320	00001, 10000
	43	0.458765	00011, 00010, 01000, 11000
	60	0.410427	00110, 00100, 01100
	72	0.563961	00111, 00101, 11100, 10100
	100	0.535252	01101, 01110, 01010, 10110
	112	0.518203	01111, 01011, 11010, 11110
	176	0.612495	01001
	252	0.646515	11001
	176	0.612495	10010
	252	0.646515	10011
	208	0.576220	11011
	252	0.612102	11111
	268	0.625290	11101, 10111
	336	0.650772	10101
	472	0.690998	10001

Table 3(b): Equivalences for homoclinic orbits with signature 8.

Sig	$\#T^V$	h_{top}	Decorations
9	22	0.246163	000000
	34	0.684979	000001, 100000
	46	0.485379	000011, 000010, 010000, 110000
	62	0.376307	000110, 000100, 001000, 011000
	74	0.597015	000111, 000101, 111000, 101000
	82	0.398273	001100
	94	0.537060	001101, 001110, 001010, 001001, 011100, 010100, 101100, 100100
	106	0.504368	001111, 001011, 110100, 111100
	150	0.604226	011001, 100110
	162	0.484357	011011, 011010, 011110, 010110, 110110
	198	0.537481	011111, 010111, 111110, 111010
	214	0.563076	011101, 010101, 101010, 101110
	182	0.502700	010010, 010011, 110010
	430	0.659895	010001
	510	0.671444	110001
	310	0.548079	110011
	378	0.583547	110111, 110101, 111011, 101011
	446	0.603536	111101, 101111
	458	0.607837	111111
	650	0.666417	111001
	722	0.668611	101001
	725	0.670332	101101
	722	0.668611	100101
	650	0.666417	100111
	430	0.659895	100010
	510	0.671444	100011
	982	0.692607	100001

Table 3(c): Equivalences for homoclinic orbits with signature 9.

are shown in Figure 6. The topological entropy of both these orbits is $\log \lambda$, where λ is the largest root of the polynomial

$$\lambda^{13} - 2\lambda^{12} + 2\lambda^8 + \lambda^7 - 4\lambda^5 - 2\lambda^4 + 2\lambda^2 + 2\lambda - 2.$$

Numerically, $\lambda_{\max} \approx 1.845$, giving $h_{\text{top}} > 0.612$.

Another counterexample is given by the orbits with codes $\overline{0}1^0_1 11001^0_1 1\overline{0}$ and $\overline{0}1^0_1 10011^0_1 1\overline{0}$. These trellises each have 252 intersections. The topological entropy of both these orbits is $\log \lambda$, where λ is the largest root of the polynomial

$$\lambda^{13} - 2\lambda^{12} + 3\lambda^7 - 4\lambda^6 + 4\lambda^5 + 2\lambda^4 + 2\lambda - 2.$$

Numerically, $\lambda_{\max} \approx 1.909$, giving $h_{\text{top}} > 0.646$.

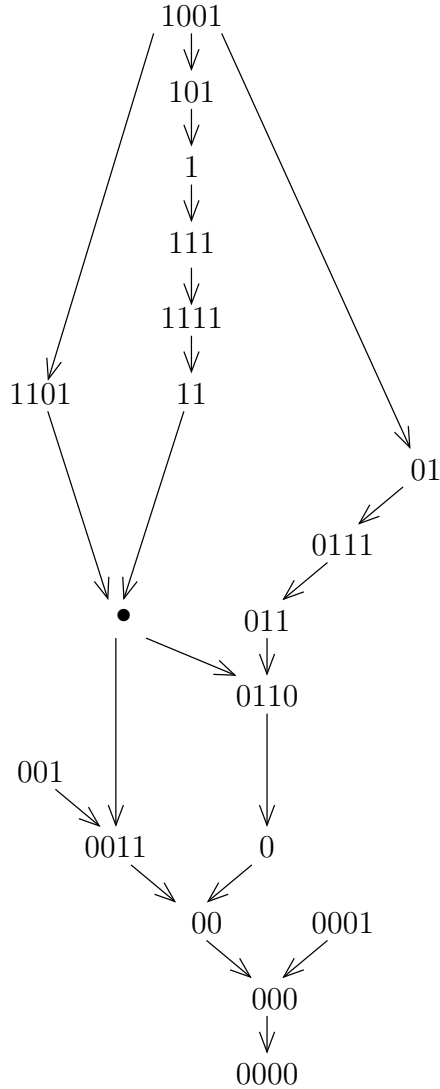## 5.2 The forcing relation

Figure 7: Decorations and scopes of homoclinic braid types of signature up to 7 and the forcing relation between them.

Figure 7 shows the forcing relation between horseshoe homoclinic orbits of signature 7 or less. Each homoclinic orbit is specified by its decoration, and the scope of the decoration is also given. Only one

decoration is given for each equivalence class of homoclinic braid types (so, for example, the decoration 10 is not included, since it is equivalent to 01).

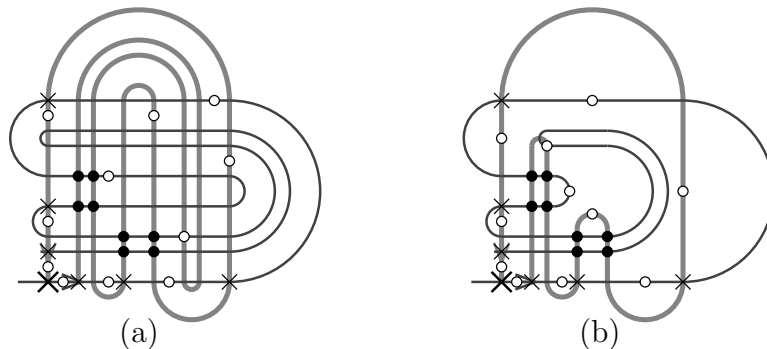We give two examples illustrating the computation of Figure 7 the interpretation of the results.

Figure 8: The homoclinic orbit $\overline{0}1101110\overline{0}$ forces the orbits $\overline{0}1_1^0 0_1^0 1\overline{0}$.

Example 5.1 Figure 8 shows the horseshoe trellis, on which the orbit H with code $\overline{0}1101110\overline{0}$ (decoration 01) has been marked with white dots, and the intersections on the orbits with codes $\overline{0}1_1^0 0_1^0 1\overline{0}$ with black dots. On performing a pruning isotopy to obtain the trellis forced by the orbit P, it can be seen that the marked intersections persist. Therefore the homoclinic braid type with decoration 01 forces the homoclinic braid type with decoration 0. From Figure 8 it can also be seen that the only other forced homoclinic braid type with signature less than 5 is that with decoration 00.
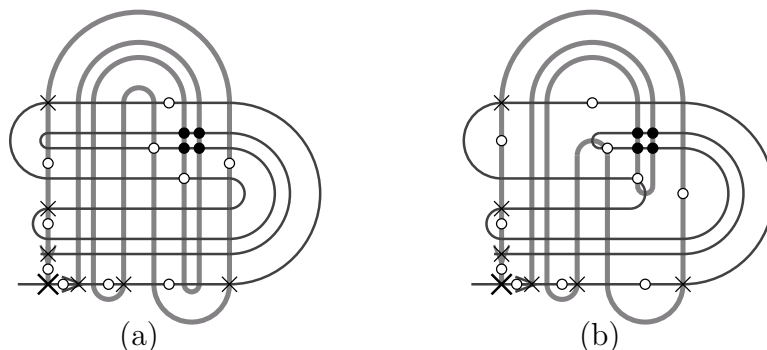

Figure 9: The homoclinic orbit $\overline{0}111111\overline{0}$ forces the orbits $\overline{0}1_1^0 11_1^0 1\overline{0}$.

Example 5.2 It is possible to compute braid types of arbitrary signature forced by any homoclinic orbit. In Figure 9, we show the trellis of signature 5 forced by the homoclinic orbit $\overline{0}111111\overline{0}$ which has decoration 1 and signature 4, allowing us to compute the braid types of signature up to 5 forced by the orbit. In particular, we see that the orbit $\overline{0}111111\overline{0}$ forces the orbits $\overline{0}1_1^0 11_1^0 1\overline{0}$, so the homoclinic braid type with decoration 1 forces the homoclinic braid type with decoration 11. The other homoclinic braid types with signature less than 5 forced by the braid type with decoration 1 have decorations ·, 0 and 00.

6   Conclusions and further research

We have described a numerical method for determining the forcing relation for homoclinic orbits of the Smale horseshoe map using trellises. The method is exact, in that is considers orbits given in terms of their symbolic codes, and The numerical techniques can also be used to determine forcing relations between homoclinic and heteroclinic orbits of general surface homeomorphisms.

14

One of the motivations for this work was to help in determining the forcing relation between periodic orbits of the Smale horseshoe map, and between homoclinic and periodic orbits. Unfortunately, while it is fairly straightforward to compute a train track for a pseudo-Anosov periodic orbit, it is less straightforward to determine the codes of the periodic orbits forced by a given homoclinic orbit. The main difficulties is that there may be many periodic orbits of the Smale horseshoe map with the same braid type, and that pruning the trellis destroys the partition of the regions R_0 and R_1, making it difficult to recover the braid type from a trellis or its graph representative.

There are also number of important related computational problems for which no methods are currently available. On the combinatorial side, it is also important to develop methods for computing the topological graph representative of a trellis type; this has been implemented for horseshoe trellises, but not for trellises on general surfaces. It is also important to have an automatic procedure for generating a combinatorial trellis from numerically-computed stable and unstable manifolds. Unfortunately, naïve methods fail since a numerically computed trellis may have orbits not present in the actual system, and may even be inconsistent. An extension of the `trellis` package is planned which will allow the dynamics forced by a numerically computed trellis to be found.

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