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Reachability of affine systems on polytopes in the plane

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ABSTRACT

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Keywords and Phrases: Affine system, polytope, co-reachability, exit set, domain of attraction.

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Reachability of Affine Systems on Polytopes in the Plane

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Abstract

In this paper a problem related to reachability analysis of piecewise-affine hybrid systems is considered. We focus on one discrete mode of a hybrid system, and study the continuous dynamics in this mode, described by an affine autonomous system on a polytope. As soon as the continuous state leaves the polytope, a discrete event is triggered, transferring the hybrid system to a different mode. This discrete event depends on the facet through which the polytope is left. In this paper, the domains of attraction of these so-called exit facets are determined. This result describes the interplay between the continuous and discrete dynamics of a hybrid system, and may be useful in some approaches to reachability analysis for hybrid systems, proposed in the literature. The method presented in this paper is restricted to affine systems on two-dimensional polytopes. Since some results are based on arguments from planar geometry, generalization of the approach to higher dimensions requires a rather extensive and detailed mathematical study that does not at all seem promising.

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1 Introduction

In the last decade the study of hybrid systems has received considerable attention. Reasons for this growing interest are manifold. On the one hand an increasing number of engineering systems is controlled by computers, thus creating an interaction between the continuous dynamics of a physical system and the discrete dynamics of a computer. On the other hand, also the modeling of complex dynamical systems may be facilitated by the use of hybrid system models. Examples of this hybrid modeling can be found in a large spectrum, ranging from the modeling of car engines to the description of biomolecular networks.

Recently, a specific subclass of hybrid systems, so-called piecewise-affine hybrid systems, introduced by Sontag in [15], [16], [18], has been studied quite extensively (see e.g. [2], [4], [5], and several papers in the conference proceedings [12], [6], and [13]). A piecewise-affine hybrid system consists of an automaton, with at each discrete mode of the automaton an affine system on a polyhedral set, evolving in continuous time. As soon as the continuous state reaches the boundary of the polyhedral set, a discrete event occurs, and the automaton switches to a new discrete mode. There the continuous state is restarted and will evolve according to the system dynamics of the affine system corresponding to the new discrete mode. In every discrete mode, the dynamics of the corresponding continuous-time affine system, and the polyhedral set on which this system is defined, may be different. In this paper we will assume that at each discrete mode the corresponding affine dynamics are defined on a polytope, i.e. on a *bounded* polyhedral set, and that the discrete event that occurs upon reaching the boundary of a polytope, depends on the facet through which the polytope is left.

In the literature, several results on control of piecewise-affine hybrid systems are available (see e.g. [5], [14], [7]). In most of these studies, the problem of reachability plays an important role: is it possible to steer the system from a given point or set of initial states to a given set of final states? A similar question occurs in the problem of safety verification, where one has to guarantee that some undesirable (unsafe) states are never reached while the system is in operation. Also for other system theoretic properties, like for example reduction of a realization, it turns out that reachability is an important property in order to obtain useful results (see [8]). In full generality, the reachability problem is not always decidable, or may have a large complexity (see e.g. [1], [10], and [17] for some results in this direction). However, for interesting special cases, solutions do exist; in this respect the class of O-minimal systems introduced in [11] seems very important. One of the goals of this study is to obtain a better understanding of the difficulties in solving the reachability problem.

One of the problems in reachability analysis of piecewise-affine hybrid systems is the determination of the domain of attraction of an exit facet. Given an affine system on a polytope, the exit facets are the facets through which the state may try to leave the polytope. Since leaving through a different facet corresponds to a different transition in the automaton, reachability analysis requires that for every element in the state polytope, the corresponding exit facet is determined. This problem of characterization of the domains of attraction of the exit facets (and in particular their geometric structure) is studied in this paper. In the literature this question is sometimes called the co-reachability problem for the exit facets, but in this paper we will not use this terminology. In full generality this problem is difficult to solve. Therefore we confine ourselves to affine systems on polytopes of dimension 2, i.e. to the planar case. It turns out that this situation is simple enough to obtain useful results, and rich enough to gain some understanding why the problem becomes so difficult in higher dimensions. In the 2-dimensional case arguments from planar geometry can be used, that are not longer valid in higher dimensions.

The paper is organized as follows. In Section 2 we start with a precise description of the problem treated in this paper. The parts of the boundary of the state polytope through which the state may leave the polytope, called the exit sets, are studied in Section 3. In Section 4 the partitioning of the state polytope in different domains of attraction is described. Some examples of subdivisions of the state polytope are given in Section 5, and we end the paper with some concluding remarks and an outlook for future research in Section 6. A few proofs are omitted because of limitations of space.

2 Problem description

We consider one discrete mode of a piecewise-affine hybrid system. The continuous dynamics in this mode are described by a two-dimensional affine autonomous system on a two-dimensional polytope (polygon) P. I.e. the differential equation describing the continuous dynamics is given by

$$\dot{x} = Ax + a, \qquad x(0) = x_0, \tag{1}$$

with $A \in \mathbb{R}^{2\times 2}$ and $a \in \mathbb{R}^2$, and remains valid as long as the state x is contained in the state polytope P. As soon as the state leaves the polytope P through one of its facets, a discrete event occurs, and the hybrid system will switch to another discrete mode with possibly different continuous dynamics. Since this discrete event depends on the facet through which P is left, we want to determine for every initial state $x_0 \in P$, through which facet of P the corresponding solution trajectory $x(t, x_0)$ will leave P, or whether this trajectory will remain in P forever.

Let $F_1, \dots, F_k$ denote the facets of P. For $i = 1, \dots, k$, the normal vector of facet F_i is denoted by n_i, and without loss of generality we assume that n_i is of unit length and points out of the polytope P. So there exist numbers $\alpha_1, \dots, \alpha_k$ such that for $i = 1, \dots, k$, the facet F_i is given by

$$F_i = \{x \in P \mid n_i^T x = \alpha_i\}.$$

In the same fashion, the polytope P itself is described as the intersection of k half spaces:

$$P = \bigcap_{i=1}^{k} \{x \in \mathbb{R}^2 \mid n_i^T x \leq \alpha_i\}.$$

Definition 2.1 A facet F_i of polytope P is called an *exit facet* if there exists a point $x_0 \in F_i$ and an $\varepsilon > 0$ such that the solution trajectory of the system $\dot{x} = Ax + a$ on $\mathbb{R}^2$ (i.e. the dynamics are not assumed to be restricted to the polytope P), and starting in x_0 at time $t = 0$ satisfies

$$\forall t \in (0, \varepsilon) : n_i^T x(t, x_0) > \alpha_i.$$

Problem 2.2 Determine for every exit facet of the polytope P the corresponding *domain of attraction*. I.e. if the facet F_i is an exit facet, determine all points $x_0 \in \text{int}(P)$ for which there exists a time $T > 0$ such that the solution trajectory $x(t, x_0)$ of system (1) on $\mathbb{R}^2$, and starting in x_0, satisfies

(i) $\forall t \in [0, T) : x(t, x_0) \in \text{int}(P)$,

(ii) $x(T, x_0) \in F_i$,

(iii) $\exists \varepsilon > 0$ such that $\forall t \in (T, T + \varepsilon) :\ n_i^T x(t, x_0) > \alpha_i$.

As soon as Problem 2.2 is solved in every discrete mode of a piecewise-affine hybrid system, essential information is obtained on the switching behavior between the discrete modes of this hybrid system. This information can be used in the reachability analysis for piecewise-affine hybrid systems, e.g. in the approach described in [14].

3 Exit sets

In order to determine the domains of attraction of all exit facets of an affine system on a polytope P, we first concentrate on the boundary ∂P of P. For every point $x_0 \in \partial P$ we have to determine whether it is possible to leave the polytope P through x_0.

Definition 3.1 Consider the affine system $\dot{x} = Ax + a$ on the polytope P. The *total exit set* U_{tot} consists of those points x_0 on the boundary of P, such that the solution of the differential equation $\dot{x} = Ax + a$, valid on the whole space $\mathbb{R}^2$ and with initial state $x(0) = x_0$, immediately leaves the polytope P:

$$U_{\text{tot}} = \{x_0 \in \partial P \mid \exists \varepsilon > 0\ \forall t \in (0, \varepsilon) : x(t, x_0) \notin P\}. \tag{2}$$

For the explicit determination of the total exit set U_{tot}, Definition 3.1 is not very helpful. Instead it is more appropriate to use the direction of the vector field of the differential equation $\dot{x} = Ax + a$ on the facets of the polytope P.

Let F_i be a facet of P with normal vector n_i of unit length and pointing out of the polytope P. Consider the function

$$g_i : F_i \longrightarrow \mathbb{R} : g_i(x) = n_i^T(Ax + a). \tag{3}$$

It is obvious that the sign of the function g_i in a point $x \in F_i$ determines whether $x \in U_{\text{tot}}$. Furthermore it is important to note that the function g_i is affine, so if g_i is not identically zero, it will change sign at most once.

Lemma 3.2 *Let $i \in \{1, \dots, k\}$, and let F_i be a facet of polytope P with normal vector n_i pointing out of P. Let v_1 and v_2 denote the vertices of F_i, and consider the system $\dot{x} = Ax+a$ on P. Let g_i be the function defined in (3).*

1. *If $g_i(v_1) > 0$ and $g_i(v_2) > 0$ then $F_i \subset U_{\text{tot}}$.*
2. *If $g_i(v_1) \leq 0$ and $g_i(v_2) \leq 0$, then, except for the vertices v_1 and v_2, none of the points of F_i belongs to U_{tot}. (Whether the vertices v_1 and v_2 belong to U_{tot} depends on the direction of the vector field in each vertex with respect to the other facet that meets F_i in this vertex).*
3. *If $g_i(v_1) > 0$ and $g_i(v_2) \leq 0$, then there exists a unique point $v \in F_i$ such that $g_i(v) = 0$. In this case, v_1 and all points on F_i located between v_1 and v belong to U_{tot}, and all points on F_i located between v and v_2 do not belong to U_{tot}. Furthermore, $Av + a$, the vector of the derivative in the point v, points along the facet F_i. If $Av + a$ points in the direction of v_1, then $v \in U_{\text{tot}}$. If on the other hand $Av + a = 0$ or if $v \neq v_2$ and $Av + a$ points in the direction of v_2, then $v \notin U_{\text{tot}}$.*

Using Lemma 3.2, every point x on the boundary ∂P of the polytope P may be classified whether or not to belong to the total exit set U_{tot}. Furthermore, since on every facet F_i the corresponding function g_i is affine, the set U_{tot} consists of a finite number of connected components. Hence it is possible to divide U_{tot} in a unique way into a *minimal* number $K \in \mathbb{N} \cup \{0\}$ of connected subsets:

$$U_{\text{tot}} = U_1 \cup U_2 \cup \cdots \cup U_K.$$

Each of the maximal connected components $U_1, \dots, U_K$ of U_{tot} is called an *exit set.* Note that each component U_i may consist of (parts of) different facets of P. Instead of determining the domains of attraction of all exit facets, we will start by computing the domains of attraction of the exit sets $U_1, \dots, U_K$. A finer decomposition into domains of attraction for separate facets may be carried out afterward.

4 Partitioning the interior of the polytope P

In this section, we concentrate on the interior $\text{int}(P)$ of the polytope P. We will partition the interior $\text{int}(P)$ into sets of different type. This subdivision is a useful tool to determine the domain of attraction of each exit facet.

Definition 4.1 Consider an affine system $\dot{x} = Ax + a$ on a polytope P, with exit sets $U_1, \dots, U_K$. We define the following subsets of the interior $\text{int}(P)$:

1. For $i = 1, \ldots, K$ let G_i denote the *domain of attraction of exit set* U_i, i.e.

$$G_i := \{x_0 \in \text{int}(P) \mid \exists T > 0 \text{ such that } x(t, x_0) \in \text{int}(P) \text{ for } t \in [0, T) \text{ and } x(T, x_0) \in U_i\}.$$

2. Let $\mathcal{O}$ denote the *maximal invariant set in* $\text{int}(P)$, consisting of all solutions that remain in the interior of the polytope P forever:

$$\mathcal{O} := \{x_0 \in \text{int}(P) \mid x(t, x_0) \in \text{int}(P) \text{ for } t \in [0, \infty)\}.$$

3. Let R denote the set of all trajectories that reach ∂P in finite time, without leaving P on that occasion:

$$R := \{x_0 \in \text{int}(P) \mid \exists T > 0 \text{ such that } x(t, x_0) \in \text{int}(P) \text{ for } t \in [0, T) \text{ and } x(T, x_0) \in \partial P \backslash U_{\text{tot}}\}.$$

After touching the boundary ∂P, trajectories in R do not leave the polytope P, but continue to evolve inside the polytope P. From there on, the solution trajectory may either remain in $\text{int}(P)$ forever, or will leave the polytope P through an exit set at a later time instant (see e.g. Example 5.2).

Lemma 4.2 *Consider an affine system $\dot{x} = Ax + a$ on a polytope P, and let $G_1, \ldots, G_K$, $\mathcal{O}$ and R be the subsets of* $\text{int}(P)$ *described in Definition 4.1. Then*

(i) $\text{int}(P) = G_1 \cup G_2 \cup \cdots G_K \cup \mathcal{O} \cup R$,

(ii) The intersection of any pair from the collection $G_1, \ldots, G_K, \mathcal{O}, R$ is empty.

Proof: Let $x_0 \in \text{int}(P)$. Then the corresponding solution $x(t, x_0)$ either remains in $\text{int}(P)$ forever (i.e. $x_0 \in \mathcal{O}$), or it leaves $\text{int}(P)$ in finite time. In the second case, let $T > 0$ be the smallest time instant that the solution $x(t, x_0)$ reaches the boundary ∂P. If $x(T, x_0) \notin U_{\text{tot}}$, then $x_0 \in R$; otherwise, if $x(T, x_0) \in U_{\text{tot}}$, then x_0 will belong to exactly one of the sets $G_1, \ldots, G_K$. Note that for $i \neq j$ we have $G_i \cap G_j = \varnothing$ because $U_i \cap U_j = \varnothing$. ■

Lemma 4.2 states that Definition 4.1 describes a decomposition of the interior of the polytope P. The goal of the remainder of this section is to determine this decomposition explicitly.

4.1 A property of the domain of attraction of an exit set

The following property of the domains of attraction of exit sets turns out to be important for their further characterization.

Proposition 4.3 *Consider an affine system $\dot{x} = Ax + a$ on a two-dimensional polytope, with exit sets $U_1, \ldots, U_K$. For every $i = 1, \ldots, K$ the domain of attraction G_i of exit set U_i is open.*

Proof: First consider an exit set U_i. Then either $U_i = \partial P$, or U_i is a strict subset of ∂P that is connected. Since we only consider 2-dimensional systems, the relative boundary of U_i (i.e. the boundary of U_i when restricted to ∂P), consists of two points $b_1, b_2 \in \partial P$, that may or may not belong to U_i. A solution of the system $\dot{x} = Ax + a$ through one of these boundary points touches the exit set U_i either from the inside or from the outside; a point of reflection cannot occur. This implies that a solution trajectory of the system cannot escape from $\text{int}(P)$ through a relative boundary point of an exit set.

Next, let $x_0 \in G_i$. Then there exists a $T > 0$ such that $x(t, x_0) \in \text{int}(P)$ for $t \in [0, T)$ and $x(T, x_0) \in U_i$. Furthermore, $x(T, x_0)$ lies in the relative interior of U_i. Since the solution of the differential equation $\dot{x} = Ax + a$ on a finite time-interval depends continuously on the initial value, there exists a neighborhood of x_0 such that all solution trajectories with initial value within this neighborhood, leave the polytope P in finite time through exit set U_i. Hence a neighborhood of x_0 is contained in G_i, and thus G_i is open. ■

Proposition 4.3 implies that the boundary of a domain of attraction of one exit set cannot belong to the domain of attraction of an other exit set:

Corollary 4.4 *If $v \in \partial G_i \cap \text{int}(P)$, then $v \in \mathcal{O} \cup R$.*

Proof: If $v \in \partial G_i$, then $v \notin G_i$. Suppose that $v \in G_j$ for some $j \neq i$. Then also a neighborhood of v belongs to G_j. Since $v \in \partial G_i$, there exists a sequence in G_i converging to v. Hence $G_i \cap G_j \neq \varnothing$, which yields a contradiction with Lemma 4.2 *(ii)*. So, according to Lemma 4.2 *(i)*, $v \in \mathcal{O} \cup R$. ■

4.2 Computation of the set R

If a point $x_0 \in \text{int}(P)$ belongs to the set R, then the trajectory of the system with initial value $x(0) = x_0$ will reach the boundary ∂P of the polytope P in finite time T, but will not leave the polytope P at that time. The point $\hat{x} := x(T, x_0) \in \partial P$, where the solution hits the boundary of the polytope P has some specific properties. These may be used to obtain an explicit characterization of all points in $\text{int}(P)$ that belong to the set R.

Proposition 4.5 *Consider a system $\dot{x} = Ax + a$ on the polytope P, with facets $F_1, \ldots, F_k$. Let $\hat{x}_i \in \partial P$ be an element of facet F_i, satisfying the following properties:*

(1) $\hat{x}_i$ is not a vertex of P,

(2) $n_i^T(A\hat{x}_i + a) = 0$, and the function $x \mapsto n_i^T(Ax + a)$ on facet F_i changes sign in $\hat{x}_i$.

(3) $A\hat{x}_i + a \neq 0$ and points in the direction of that part of the facet F_i where $n_i^T(Ax + a) < 0$.

Then the solution of $\dot{x} = Ax + a$ with initial value $x(0) = \hat{x}_i$ and solved backwards *in time belongs to the set R as long as this solution is contained in* $\text{int}(P)$*, i.e.*

(i) if $x(t, \hat{x}_i) \in \text{int}(P)$ for all $t \in (-\infty, 0)$, then

$$R_i := \{x(t, \hat{x}_i) \in \mathbb{R}^2 \mid t \in (-\infty, 0)\} \subset R.$$

(ii) if there exists $t < 0$ such that $x(t, \hat{x}_i) \notin \text{int}(P)$, then $T := \sup_{t<0} x(t, \hat{x}_i) \notin \text{int}(P)$ is well-defined and

$$R_i := \{x(t, \hat{x}_i) \in \mathbb{R}^2 \mid t \in (T, 0)\} \subset R.$$

Furthermore, every facet F_i, $i = 1, \ldots, k$ contains at most one point $\hat{x}_i \in F_i$ satisfying properties (1), (2), and (3) above. If F_i does not contain a point $\hat{x}_i$ satisfying (1), (2), and (3), define $R_i = \varnothing$. Otherwise, define R_i either as in (i) or as in (ii) (depending on which condition is satisfied). Then the set R is given by

$$R = \bigcup_{i=1}^{k} R_i.$$

The first condition in Proposition 4.5 is based on the fact that a trajectory belonging to R cannot end in a vertex, because a solution through a vertex will either enter or leave the polytope. The other two conditions guarantee that there is a solution trajectory of the autonomous system that touches the boundary of P in $\hat{x}_i$ from inside, i.e. without leaving the polytope on that occasion. The points on this solution trajectory that are passed before reaching $\hat{x}_i \in \partial P$ belong to the set R.

Remark 4.6 For every $i = 1, \ldots, k$ the point $\hat{x}_i \in F_i$ satisfying conditions *(1)*, *(2)*, and *(3)* of Proposition 4.5 is (if it exists) a relative boundary point of an exit set U_j, but it is not an element of this exit set.

In the context of hybrid systems, the solution trajectories that are contained in R play a special role. Depending on the exact definition, these trajectories may belong to different domains of attraction. In this paper, the set R_i consisting of the solution trajectory that reaches facet F_i without leaving the polytope there, is not considered as a trajectory in the domain of attraction G_i of F_i. Instead, the solution is continued, and even if it leaves the polytope P somewhat later through facet F_ℓ, the definition of the domain of attraction G_ℓ of F_ℓ implies that R_i does not belong to G_ℓ either. In this situation it seems legitimate to consider the trajectory R_i to be contained either in the domain of attraction of F_i or in the domain of attraction of F_ℓ. The specific choice that has to be made depends on the definition of the hybrid system. If in the hybrid system under consideration it is sufficient to reach facet F_i without leaving the polytope P, in order to enforce a discrete transition, then R_i belongs to the domain of attraction of F_i. On the other hand, if the hybrid system continues to evolve in the same discrete mode until facet F_ℓ is reached, then R_i belongs to the domain of attraction of facet F_ℓ. In any case, the trajectory contained in R_i plays an important role: it is located on the boundary of the domains of attraction of the facets F_i and F_ℓ. To emphasize this role, we will not classify the points in the set R to belong to a specific domain of attraction. Instead the set R is used to separate the domains of attraction of the exit sets $U_1, \ldots, U_K$, and, if appropriate, the invariant set $\mathcal{O}$.

4.3 Determination of the invariant set $\mathcal{O}$

The most involved part in the partitioning of the interior of the polytope P is the explicit description of the invariant set $\mathcal{O}$. The characterization of this set requires the study of a number of particular cases, that depend on the dynamics of the system under consideration. Especially the location of fixed points with respect to the polytope P, and the location of the eigenvalues of the matrix A in the complex plane turn out be relevant.

If all points of $\mathbb{R}^2$ are fixed points of the system $\dot{x} = Ax + a$, then the system remains at its initial value forever, and there is no dynamics at all. This case is trivial. Also the situation where there is a line of fixed points in $\mathbb{R}^2$ is relatively simple. In this case A has an eigenvalue 0, with eigenvector v. If A has a second eigenvalue λ with eigenvector w, all solutions will move along lines parallel to the vector w. If $\lambda > 0$ solutions will move away from the line of fixed points, and if $\lambda < 0$ they will move towards the line of fixed points. If on the other hand there is a line of fixed points, but 0 is the only eigenvalue of A, then the solutions will move along lines parallel to the eigenvector v. In any case, if there is a line of fixed points, all solutions evolve along parallel straight lines, and partitioning of the state polytope according to Definition 4.1 is straightforward. Therefore we confine ourselves in the rest of the paper to the only non-trivial situation in which the system $\dot{x} = Ax + a$ has either one or no fixed point at all.

In Corollary 4.4 it was shown that the boundary of the domains of attraction of the exit sets either belong to R or to $\mathcal{O}$. This result was completely based on the fact that the sets $G_1, \ldots, G_K$ are open. The same argument is also applicable to the maximal invariant set $\mathcal{O}$, provided that $\mathcal{O}$ is open.

Corollary 4.7 *Let $\dot{x} = Ax + a$ be an affine system on a two-dimensional polytope P. Consider the decomposition of $\mathrm{int}(P)$ described in Definition 4.1. Assume that the maximal invariant set $\mathcal{O}$ is nonempty and open. Then*

(i) $v \in \partial G_i \cap \mathrm{int}(P)$ implies that $v \in R$,

(ii) $v \in \partial \mathcal{O} \cap \mathrm{int}(P)$ implies that $v \in R$.

Corollary 4.7 states that, if $\mathcal{O}$ is open, all boundaries between domains of attraction $G_1, \ldots, G_K$ and invariant set $\mathcal{O}$ consist of trajectories belonging to the set R. So, in this situation it is not necessary to obtain a more explicit description of the maximal invariant set $\mathcal{O}$, in order to derive the partitioning of $\mathrm{int}(P)$ described in Definition 4.1.

Proposition 4.8 *Consider a system $\dot{x} = Ax + a$ on a two-dimensional polytope P, and let $\mathcal{O}$ be the corresponding maximal invariant set as introduced in Definition 4.1 (2). Assume that the system has at most one fixed point in $\mathbb{R}^2$. Then, depending on the affine dynamics and the location of the polytope P, the invariant set $\mathcal{O}$ satisfies the following claims:*

(1) *If the system has no fixed point in P, then $\mathcal{O} = \varnothing$.*

(2) *If the system has a fixed point $x_f \in \mathrm{int}(P)$:*

(2a) *if the fixed point x_f is stable (all eigenvalues of A have negative real part) or is marginally stable (all eigenvalues of A are located on the imaginary axis), then $\mathcal{O}$ is open,*

(2b) *if the fixed point x_f is totally unstable (all eigenvalues of A have strictly positive real part), then $\mathcal{O} = \{x_f\}$,*

(2c) *if the fixed point x_f is a saddle, i.e. A has eigenvalues $\lambda_1 > 0$ and $\lambda_2 < 0$, then*

$$\mathcal{O} = (x_f + V_{\lambda_2}) \cap \mathrm{int}(P),$$

where $V_{\lambda_2} = \ker(A - \lambda_2 I)$ is the eigenspace corresponding to λ_2.

(3) *If the system has a fixed point $x_f \in \partial P$:*

(3a) *if all eigenvalues of A are complex (i.e. their imaginary part is nonzero), then $\mathcal{O} = \varnothing$,*

(3b) *if all eigenvalues of A are real and positive, then $\mathcal{O} = \varnothing$,*

(3c) *if the fixed point x_f is a saddle, i.e. A has eigenvalues $\lambda_1 > 0$ and $\lambda_2 < 0$, then*

$$\mathcal{O} = (x_f + V_{\lambda_2}) \cap \mathrm{int}(P),$$

where $V_{\lambda_2} = \ker(A - \lambda_2 I)$ is the eigenspace corresponding to λ_2.

(3d) *if all eigenvalues of A are real and negative:*

(3d1) *if A has two different negative eigenvalues $\lambda_2 < \lambda_1 < 0$, define $S := (x_f + V_{\lambda_2}) \cap \mathrm{int}(P)$, with $V_{\lambda_2} = \ker(A - \lambda_2 I)$. Then $S \subset \mathcal{O}$ and $\mathcal{O} \backslash S$ is open,* (3d2) *if A has one negative eigenvalue λ_1 and rank$(A-\lambda_1 I)=0$, then $\mathcal{O}=\text{int}(P)$,*

(3d3) *if $\lambda_1<0$ is the only eigenvalue of A, and rank$(A-\lambda_1 I)=1$, define $S:=(x_f+V_{\lambda_1})\cap\text{int}(P)$, with $V_{\lambda_1}=\ker(A-\lambda_1 I)$. Then $S\subset\mathcal{O}$ and $\mathcal{O}\backslash S$ is open.*

Note that in Proposition 4.8 the set S may be empty, and if $\mathcal{O}$ or $\mathcal{O}\backslash S$ are claimed to be open, also these sets are possibly empty.

If in Proposition 4.8 $x_f\in\text{int}(P)$ is a stable fixed point, the continuous dependence of solutions of $\dot{x}=Ax+a$ on finite time intervals implies that $\mathcal{O}$ is an open set. Most other conditions (except (3d)) are relatively straightforward.

4.4 A subdivision of the interior of P

Procedure 4.9 Consider an affine system $\dot{x}=Ax+a$ on a two-dimensional polytope P, and assume that this system has at most one fixed point. Initialize the set $\mathcal{B}$ of boundaries between the different domains of attraction as $\mathcal{B}:=\varnothing$. Then the domains of attraction of the exit facets may be determined as follows.

1. Determine the total exit set U_{tot} using Lemma 3.2, and partition this set in maximally connected components, the exit sets $U_1,\dots,U_K$.

2. Determine the set R introduced in Definition 4.1, using the method described in Proposition 4.5. Define $\mathcal{B}:=R$.

3. Compute the fixed point x_f of the system (if it exists) and the eigenvalues of A. Determine which case from Proposition 4.8 is valid.

    - in cases (2b), (2c), and (3c), $\mathcal{B}:=\mathcal{B}\cup\mathcal{O}$,
    - in cases (3d1) and (3d3), $\mathcal{B}:=\mathcal{B}\cup S$.

4. The set of boundaries $\mathcal{B}$ divides int(P) in separate regions. Every region contains at most one exit set U_i on its boundary. This region is the domain of attraction of exit set U_i.

5. If an exit set U_i contains only one exit facet, the domain of attraction of the exit set is the domain of attraction of this exit facet. Otherwise, the exit set U_i contains some vertices $v_1,\dots,v_m$ of P in its relative interior. Solve the differential equation $\dot{x}=Ax+a$ with initial values $x(0)=v_j$, $(j=1,\dots,m)$, backward in time, i.e. after defining $T_j=\sup_{t<0}x(t,v_j)\notin\text{int}(P)$ (in particular $T_j=-\infty$ if $x(t,v_j)\in\text{int}(P)$ for all $t<0$), determine

    $$Q_j:=\{x(t,v_j)\in\mathbb{R}^2\mid t\in(T_j,0)\}.$$

    Then $Q_1,\dots,Q_m$ divide the domain of attraction G_i of U_i into separate domains of attraction for all exit facets that are (partially) contained in U_i.

Proof of correctness: The set of boundaries $\mathcal{B}$ consists of solution trajectories of the affine system. Therefore, solutions with initial value in int$(P)\backslash\mathcal{B}$ cannot cross $\mathcal{B}$. Furthermore, all points in int$(P)\backslash\mathcal{B}$ belong to one of the open sets $G_1,\dots,G_k$, and if appropriate $\mathcal{O}$ or $\mathcal{O}\backslash S$. These sets do not intersect each other, and their boundaries are contained in $\mathcal{B}\cup\partial P$. So $\mathcal{B}$ divides int(P) in separate regions, and the region containing the exit set U_i on its boundary must be the domain of attraction of this exit set. Note that this region is unique, because the set of boundaries $\mathcal{B}$ does not divide any exit set. Finally, the subdivision of a domain of attraction G_i into domains of attraction of the separate exit facets is based on the same principle. ■

If a domain of attraction G_i of an exit set U_i is separated by a trajectory Q_j ending in vertex v_j, it is unclear to which domain of attraction of an exit facet this trajectory belongs. In fact, the trajectory is on the boundary of two domains of attraction, and it is a matter of definition how this set itself is classified. In a hybrid systems context, this choice will be based on the discrete event that is triggered upon reaching the vertex v_j.

5 Examples

Example 5.1 Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \text{ on the rectangle } 0 \leq x_1 \leq 6,\ 0 \leq x_2 \leq 4.$$

The vector field of the derivative in the vertices of the rectangle (depicted in Figure 1) shows that the rectangle has two exit facets, $U_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0 \wedge 0 \leq x_2 \leq 4\}$ and $U_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 6 \wedge 0 \leq x_2 \leq 4\}$, which at the same time are exit sets. Since there are no points on the boundary of the polytope, where the solution touches the boundary, $R = \varnothing$. Therefore, the two domains of attraction G_1 and G_2 are separated by the maximal invariant set $\mathcal{O}$. The system has one fixed point, $x_f = (3, 2)$, located in the interior of the

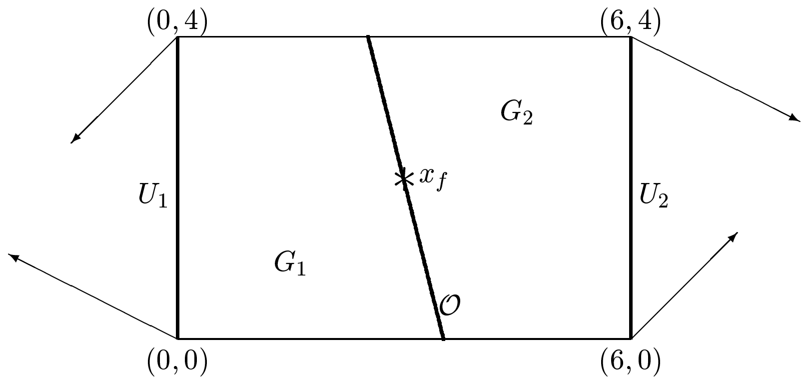

Figure 1: Partitioning of the state rectangle in Example 5.1

rectangle. The matrix A has eigenvalues $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{2}$, and $v = (1, -4)^T$ is the eigenvector corresponding to eigenvalue $-\frac{1}{2}$. So $\mathcal{O} = (x_f + \ker(A - \lambda_2 I)) \cap \operatorname{int}(P)$ consists of the line between $(\frac{7}{2}, 0)$ and $(\frac{5}{2}, 4)$ but does not include the begin- and endpoint. Points on this line converge to x_f. At the right hand side of the line, every solution reaches exit facet U_2 in finite time. On the left hand side of this line, exit facet U_1 is reached in finite time (see Figure 1).

Example 5.2 (Inward spiraling). Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{16} & \frac{3}{16} \\ -\frac{3}{16} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{2} \end{pmatrix}, \text{ on the square } 0 \leq x_1 \leq 4,\ 0 \leq x_2 \leq 4.$$

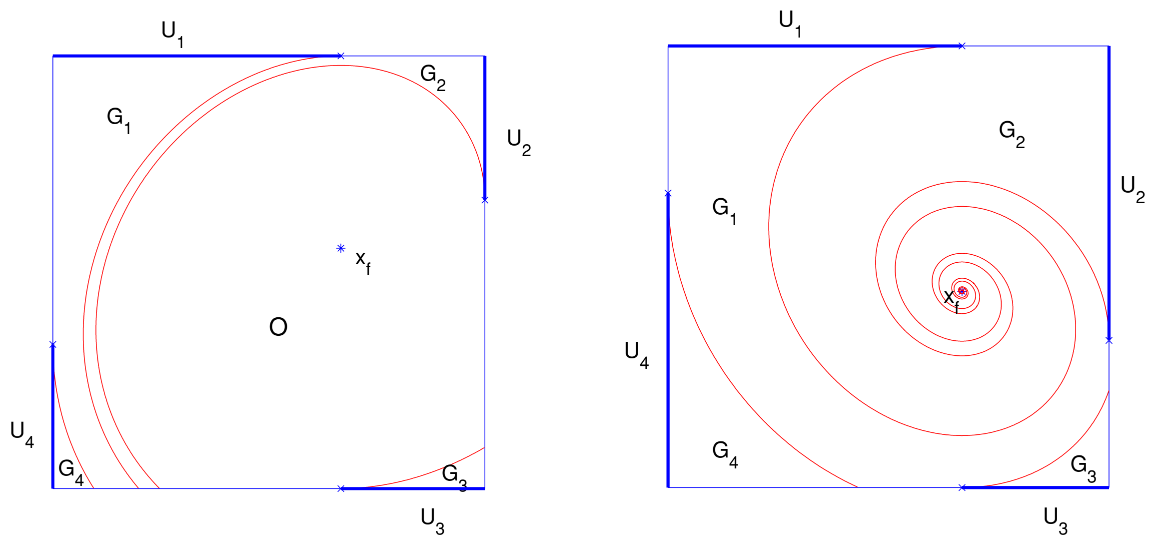

Figure 2: Domains of attraction of Example 5.2 (left) and Example 5.3 (right)

The vector field of the derivative in the vertices of the square indicates that all facets are exit facets. In fact, the system has four exit sets, given by

$$\begin{aligned}
U_1 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 < \frac{8}{3} \wedge x_2 = 4\}, \\
U_2 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 4 \wedge \frac{8}{3} < x_2 \leq 4\}, \\
U_3 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid \frac{8}{3} < x_1 \leq 4 \wedge x_2 = 0\}, \\
U_4 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0 \wedge 0 \leq x_2 < \frac{4}{3}\},
\end{aligned}$$

depicted in the left diagram of Figure 2. Every exit set has a boundary point, not belonging to the exit set, (the points $(\frac{8}{3}, 4)$, $(4, \frac{8}{3})$, $(\frac{8}{3}, 0)$, and $(0, \frac{4}{3})$), and the solutions through these points touch the boundary of the square without leaving it on that occasion. By taking these four points as initial values, and solving the differential equation backward in time as long as this backward solution is contained in the square, Figure 2 (left) is obtained. To make a decomposition of the state set in domains of attraction, we first have to study the maximal invariant set $\mathcal{O}$. The fixed point of the system is $x_f = (\frac{8}{3}, \frac{20}{9})$, which is clearly contained in the square. Furthermore A has eigenvalues $-\frac{1}{32} \pm \frac{1}{32}\sqrt{35}i$, so according to Proposition 4.8 (2a), $\mathcal{O}$ is open. Therefore the set R contains all boundaries between the domains of attraction G_1, G_2, G_3, G_4 and the set $\mathcal{O}$. In Figure 2 (left) the resulting decomposition is depicted. The area containing the fixed point x_f is the maximal invariant set $\mathcal{O}$. The boundary of this set does not intersect any exit set. All other regions in the square have exactly one exit set on their boundary. Hence each of these regions is the domain of attraction of the corresponding exit set.

Note that, although the dynamics of this system describe an inward spiraling process, the system restricted to the square still has exit sets, because there exist solutions that leave the square before converging to the fixed point x_f.

Example 5.3 (Outward spiraling) Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{3}{4} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \text{ on the square } 0 \le x_1 \le 4,\ 0 \le x_2 \le 4.$$

Again, the vector field of the derivative at the vertices of the square shows that all facets are exit facets. The corresponding four exit sets are given by

$$\begin{aligned}
U_1 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1 < \frac{8}{3} \wedge x_2 = 4\}, \\
U_2 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 4 \wedge \frac{4}{3} < x_2 \le 4\}, \\
U_3 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid \frac{8}{3} < x_1 \le 4 \wedge x_2 = 0\}, \\
U_4 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0 \wedge 0 \le x_2 < \frac{8}{3}\},
\end{aligned}$$

(see Figure 2, right). The boundary points $(\frac{8}{3}, 4)$, $(4, \frac{4}{3})$, $(\frac{8}{3}, 0)$, and $(0, \frac{8}{3})$ do not belong to these exit sets, but are the endpoints of solution trajectories contained in the set R. After solving the differential equation backwards in time, these trajectories have been plotted in Figure 2 (right). The system has one fixed point $x_f = (\frac{8}{3}, \frac{16}{9})$ which is unstable because the eigenvalues of the matrix A are $\frac{1}{8} \pm \frac{1}{8}\sqrt{35}i$. Hence $\mathcal{O} = \{x_f\}$ and the set R divides the square in four different regions. Every region has exactly one exit set on its boundary and is therefore the domain of attraction of this exit set (see Figure 2 (right)). However, the configuration is rather complicated because the domains G_1 and G_2 spiral around each other towards the fixed point x_f.

6 Discussion of the results and an outlook for future developments

In this paper, a problem related to reachability analysis of piecewise affine hybrid systems was studied. For an autonomous affine dynamical system on a two-dimensional polytope a method was derived for the computation of the domains of attraction of all exit facets. Although this decomposition is always possible, the examples in Section 5 show that the complexity of the solution depends on the dynamics of the problem under consideration. In Example 5.1, where the matrix A has one positive and one negative eigenvalue, the decomposition is realized by a straight line that is straightforward to compute. In Examples 5.2 and 5.3, where the eigenvalues of the matrix A are complex, the boundaries between domains of attraction are obtained by solving a differential equation backward in time. Although an analytic solution exists, Figure 2 is obtained with numerical computations. Furthermore, Example 5.3 shows that in case of outward spiraling it is difficult to decide to which domain of attraction a point in the neighborhood of the fixed point belongs. Different domains of attraction spiral around each other towards the fixed point. Unlike the maximal invariant set $\mathcal{O}$, domains of attraction are not necessarily convex.

In a hybrid systems context, one may consider the situation, where the continuous dynamics after a discrete transition to a new discrete mode, always start on one of the facets of the new polytope. In this situation, the method presented in this paper may be used in the reachability analysis of piecewise-affine hybrid systems. For example, in the approach described in [14] and [7] reachability is studied in terms of so-called departure and arrival sets. For systems without continuous inputs the results in this paper can be applied to check whether a departure set is reachable from an arrival set. This observation can be used in the backward recursion algorithm proposed in [7]. This will result in a sequence of finer partitionings of the state polytope and of its facets. If the procedure ends after finitely many steps, then it is possible to decide on reachability questions for the hybrid system under consideration. Additional research is necessary to judge on the merits of this approach. In particular it would be interesting to know under what conditions the partitioning of a polytope in the backward recursion stops after finitely many steps. However, it is not to be expected that this property will hold in general (although [3] contains a proof of the decidability of two-dimensional hybrid systems with piecewise-constant derivatives). The results in the present paper suggest that for piecewise-affine hybrid systems the interaction between the continuous dynamics and the discrete switching is the main source for decidability problems. Nevertheless, the domains of attraction of the different exit facets, that can be seen as an important coupling mechanism between continuous and discrete dynamics, have been characterized in great detail. In this respect, handling the switching between different modes seems more problematic than the study of the continuous dynamics. However, the continuous dynamics do lead to an additional problem: the numerical computation of system trajectories, for example in case of inward or outward spiraling. In these cases the geometric complexity of the domains of attraction of the exit sets, and in particular their lack of convexity may be other complicating factors, as was illustrated in Examples 5.2 and 5.3.

The results of this paper may also be useful for the control of piecewise affine hybrid systems with continuous input. In [7] an approach to this problem was presented, based on the idea of control to one particular facet in each discrete mode of a hybrid system. The idea of control to facet was further developed in [9]. The results in the present paper show that most autonomous dynamical systems allow more than one exit facet. If continuous control is available, this observation could be used to design feedbacks that realize more subtle control objectives than just steering a system on a polytope to one particular facet. Indeed, closed-loop systems with more than one exit facet are possible. To allow easy determination of the domains of attraction of exit sets, a control law can be selected such that the closed-loop system has only real eigenvalues and a fixed point $x_f \in \text{int}(P)$. In order to apply this kind of ideas in a hybrid systems context, more research is required on the interplay between feedback control for affine systems and the structure of the polytope on which this system is defined.

The results presented in this paper were specially developed for two-dimensional systems. Nevertheless, the approach that was proposed by splitting up the state polytope in an invariant set $\mathcal{O}$, domains of attraction $G_1, \ldots, G_K$ of the exit sets, and the set R of solutions that touch the boundary, also remains valid in higher dimensions. The main problem in higher dimensions is the explicit analytic determination of these sets. In a two-dimensional setting, planar geometry was used to obtain a characterization of the different sets in the partitioning. For the authors it is unclear how this type of results can be generalized to higher dimensions. Especially the computation of the set R seems difficult, because in higher dimensions also the dimension of this set will grow. It is even questionable, whether it is worthwhile to pursue an extension to higher dimensions. The examples in Section 5 show that already for two-dimensional systems the geometric structure of the partitioning becomes rather complex. In higher dimensions it is to be expected that even more complicated behavior may occur. Therefore it is unclear whether such results would be useful for application in a hybrid systems context.

Although the analytic approach of this paper fails in higher dimensions, the computational methods for approximate reachability developed in [2] may still be applicable.

Despite the two-dimensional character of the obtained results, it remains interesting to study classes of piecewise affine 2-dimensional hybrid systems, on the basis of partitioning of the state polytope as described in this paper. Although this class of systems seems rather restricted, it may exhibit interesting hybrid dynamics, of a more complicated nature than, for example, timed-automata. Further research is necessary to investigate this class of systems in more detail, and to find out which analysis and control problems may be solvable for systems within this restricted class.

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