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On some conjectural inequalities and their consequences

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On some conjectural inequalities and their consequences

ABSTRACT
We discuss some conjectural inequalities involving the sums sig_n(s) := 1^s + 2^s + ... + n^s. Two of our Conjectures assert that both a(s):=sig_(n+1)(s)/sig_n(s) and a(s)*sig_(n+1)(s)/sig_(n+2)(s) are strictly log-convex in s on all of the real axis. We will also present a common generalization of these two Conjectures. Various applications are described, to existing theorems as well as to some other unproven conjectures.

2000 Mathematics Subject Classification: Primary 26D15, Secondary 26A51, 65D32.
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On some conjectural inequalities and their consequences

(on the occasion of Dr N. M. Temme's departure from CWI on May 27, 2005)

by

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ABSTRACT. We discuss some conjectural inequalities involving the sums $\sigma_n(s) := 1^s + 2^s + \ldots + n^s$.
Two of our Conjectures assert that both $\frac{\sigma_{n+1}(s)}{\sigma_n(s)}$ and $\frac{\sigma_{n+1}^2(s)}{\sigma_n(s)\,\sigma_{n+2}(s)}$ are *strictly log−convex* in s on all of $\mathbb{R}$. We will also present a common generalization of these two Conjectures.
Various applications are described, to existing theorems as well as to some other unproven conjectures.

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Part I.

1. Statement of the first Conjecture

For $s \in \mathbb{R}$ and $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ we define

(1.1) $\sigma_n(s) := 1^s + 2^s + 3^s + \ldots + n^s$

and

(1.2) $Q(s) := Q_n(s) := \dfrac{\sigma_{n+1}(s)}{\sigma_n(s)}$.

Our first CONJECTURE, to be discussed in this section, may be formulated in various (obviously) equivalent ways:

(A) For every (fixed) $n \in \mathbb{N}$ the function $Q(s)$ is *strictly log−convex* on all of $\mathbb{R}$.

(B) For every (fixed) $n \in \mathbb{N}$ the function $\log Q(s)$ is *strictly convex* on all of $\mathbb{R}$.

(C) $Q(s)\,Q''(s) > (Q'(s))^2$.

(D) $\dfrac{Q(s)}{Q'(s)} > \dfrac{Q'(s)}{Q''(s)}$.

(E) $$\frac{\sigma_{n+1}(s)\,\sigma''_{n+1}(s) - (\sigma'_{n+1}(s))^2}{\sigma^2_{n+1}(s)} > \frac{\sigma_n(s)\,\sigma''_n(s) - (\sigma'_n(s))^2}{\sigma^2_n(s)}.$$

(F) $$\frac{\sigma''_{n+1}(s)}{\sigma_{n+1}(s)} - \left(\frac{\sigma'_{n+1}(s)}{\sigma_{n+1}(s)}\right)^2 > \frac{\sigma''_n(s)}{\sigma_n(s)} - \left(\frac{\sigma'_n(s)}{\sigma_n(s)}\right)^2.$$

It seems that (even for small $n \geq 2$) this is by no means trivial. The reader is invited to give it a try.

2.0 The genesis of the first Conjecture

In [1], [2], [3] and [10] we considered the *monotonicity* of the canonical Riemann Upper and Lower sums corresponding to the elementary integral $\int_0^1 x^s\,dx$, (s fixed and > 0)

(2.0.1) $$U_n(s) := \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s = \frac{\sigma_n(s)}{n^{s+1}} \quad \text{and} \quad L_n(s) := \frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^s = \frac{\sigma_{n-1}(s)}{n^{s+1}}.$$

It was shown there that (for fixed $s > 0$) the sequences $(U_n(s))_{n\in\mathbb{N}}$ and $(L_n(s))_{n\in\mathbb{N}}$ are *strictly monotonic* in *n,* ($U_n(s)$ being *strictly decreasing* and $L_n(s)$ being *strictly increasing*):

(2.0.2) $$U_{n+1}(s) < U_n(s) \quad \text{or} \quad \frac{\sigma_{n+1}(s)}{(n+1)^{s+1}} < \frac{\sigma_n(s)}{n^{s+1}}, \; (s > 0)$$

and

(2.0.3) $$L_{n+1}(s) > L_n(s) \quad \text{or} \quad \frac{\sigma_n(s)}{(n+1)^{s+1}} > \frac{\sigma_{n-1}(s)}{n^{s+1}}, \; (s > 0).$$

It is easily seen that these inequalities may be put together as (note that $\sigma_{n+1}(s) = \sigma_n(s) + (n+1)^s$)

(2.0.4) $$\frac{n^{s+1}(n+1)^s}{(n+1)^{s+1} - n^{s+1}} < \sigma_n(s) < \frac{n^s(n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}}, \; (s > 0).$$

PROOF of (2.0.2). (Assertion (2.0.3) may be shown in a similar manner.)
Inequality (2.0.2) is easily seen to be true for $n = 1$ and all $s > 0$.
Assume that it is still true for $n = 1, ..., \mathrm{N}$ and all $s > 0$. Then we have

(2.0.5) $$\sigma_{N+1}(s) = (N+1)^s + \sigma_n(s) > (N+1)^s + \frac{N^{s+1}(N+1)^s}{(N+1)^{s+1} - N^{s+1}}$$

so that it suffices to show that

(2.0.6) $$(N+1)^s + \frac{N^{s+1}(N+1)^s}{(N+1)^{s+1} - N^{s+1}} \geq \frac{(N+1)^{s+1}(N+2)^s}{(N+2)^{s+1} - (N+1)^{s+1}}$$

or

(2.0.7) $$(N+1)^s(N+2)^{s+1} - (N+1)^{2s+1} \geq (N+1)^{s+1}(N+2)^s - N^{s+1}(N+2)^s.$$

Writing $x := 1/(N+1)$ we will be through if we can show that

(2.0.8) $$(1+x)^{s+1} - 1 \geq (1+x)^s - (1-x)^{s+1}(1+x)^s, \; \left(0 < x \leq \tfrac{1}{2}\right)$$

or

(2.0.9) $$\frac{(1+x)^{s+1} - 1}{x} \geq \frac{1 - (1-x^2)^{s+1}}{x^2}, \; \left(0 < x \leq \tfrac{1}{2}\right).$$

Since for every (fixed) $s > 0$ the function t^{s+1} is convex in t on $\mathbb{R}^+$, it follows that (2.0.9) is true for all $x \in (0,1)$, completing the proof of (2.0.2). ■

Various other proofs of (2.0.2) have been given. See [2] and NAvW [1], especially the generalization by van Lint, who proved that if $f : [a,b] \to \mathbb{R}$ is either convex or concave on $[a,b]$, then the sequence of canonical Riemann Upper / Lower sums corresponding to $\int_a^b f(x)\, dx$ is decreasing / increasing (in n).

This should suffice, one would say. Enough is enough !

However, while studying inequality (2.0.2) it was observed that, with

(2.0.10) $$h(s) := h_n(s) := \frac{\sigma_{n+1}(s)}{\sigma_n(s)} \left(\frac{n}{n+1} \right)^s, \quad (s \in \mathbb{R})$$

we may write (2.0.2) as

(2.0.11) $$h(s) < \frac{n+1}{n}, \quad (s > 0).$$

Since $h(0) = \frac{n+1}{n}$, the truth of (2.0.2) would also follow from the assertion: h(s) is *strictly decreasing* (in s) on all of $\mathbb{R}$. In 1995 we succeeded in proving this assertion. Although fully elementary, the proof is *not very simple,* and since it was only published in [4], we will present the full original version here. (The proof in [4] contains some printing errors.) Actually we will prove the following slightly stronger

THEOREM.

(2.0.12) $$h'(s) < 0 \text{ for all } s \in \mathbb{R}.$$

PROOF of (2.0.12). After rewriting h(s) as

(2.0.13) $$h(s) = \left(\frac{n}{n+1} \right)^s + \frac{1}{\sum_{k=1}^{n} \left(\frac{k}{n}\right)^s}$$

and differentiating with respect to s, it is easily seen that our claim is equivalent to

(2.0.14) $$\sum_{k=1}^{n} k^s \log k > \sigma_n(s) \log n - \frac{\sigma_n^2(s)}{(n+1)^s} \log\left(1 + \frac{1}{n}\right), \quad (n \in \mathbb{N}, s \in \mathbb{R}).$$

We consider a number of cases.
(i) The case $s \leq -1$.
The LHS of (2.0.14) is ≥ 0 so that we are done if we can show that the RHS is < 0.
Since $\sigma_n(s) > 0$ it suffices to show that

(2.0.15) $$\frac{\sigma_n(s)}{(n+1)^s} \log\left(1 + \frac{1}{n}\right) > \log n \quad \text{or} \quad \frac{\sigma_n(s)}{(n+1)^s} \log\left(1 + \frac{1}{n}\right)^{n+1} > (n+1) \log n.$$

Since $\left(1 + \frac{1}{n}\right)^{n+1} > e$ it is enough to show the sharper inequality

(2.0.16) $$\frac{\sigma_n(s)}{(n+1)^s} > (n+1) \log n.$$

For $s \leq -1$ we have

(2.0.17) $$\frac{\sigma_n(s)}{(n+1)^s} = \sum_{k=1}^{n} \left(\frac{k}{n+1}\right)^s = \sum_{k=1}^{n} \left(\frac{n+1}{k}\right)^{-s} \geq \sum_{k=1}^{n} \frac{n+1}{k} > (n+1)\log n$$

so that we are done with the case $s \leq -1$. ■

(ii) The case $s > -1$.
In this case we use mathematical induction with respect to n.
It is easily verified that (2.0.14) is true for $n = 1$ (and all $s \in \mathbb{R}$).
Suppose that (2.0.14) still holds for some $n \in \mathbb{N}$ and all $s \in \mathbb{R}$. Then

(2.0.18) $$\sum_{k=1}^{n+1} k^s \log k = \sum_{k=1}^{n} k^s \log k + (n+1)^s \log(n+1) >$$

$$> (n+1)^s \log(n+1) + \sigma_n(s)\log n - \frac{\sigma_n^2(s)}{(n+1)^s} \log\left(1+\frac{1}{n}\right)$$

and we will be through if the RHS of (2.0.18) is

(2.0.19) $$\geq \sigma_{n+1}(s)\log(n+1) - \frac{\sigma_{n+1}^2(s)}{(n+2)^s} \log\left(1+\frac{1}{n+1}\right).$$

Observing that

(2.0.20) $$\sigma_{n+1}(s) = \sigma_n(s) + (n+1)^s$$

and performing some routine calculations, it is easily seen that we need to show that

(2.0.21) $$\frac{\sigma_{n+1}(s)}{(n+2)^s} \log\left(1+\frac{1}{n+1}\right) \geq \frac{\sigma_n(s)}{(n+1)^s} \log\left(1+\frac{1}{n}\right)$$

or that $\frac{\sigma_n(s)}{(n+1)^s} \log\left(1+\frac{1}{n}\right)$ is increasing in n.

(iiA) The sub–cases $-1 < s \leq 0$ and $s \geq 1$.
We multiply (2.0.21) by $n(n+1)$ and observe that $\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$, so that it suffices to prove the sharper inequality

(2.0.22) $$n\,\frac{\sigma_{n+1}(s)}{(n+2)^s} \geq (n+1)\,\frac{\sigma_n(s)}{(n+1)^s}$$

which may also be written as

(2.0.23) $$\frac{n(n+1)^s}{(n+2)^s} \geq \sigma_n(s)\left(\frac{n+1}{(n+1)^s} - \frac{n}{(n+2)^s}\right).$$

It is easily verified that the factor of $\sigma_n(s)$ in the RHS is > 0 for $s \geq -1$, so that (2.0.23) may also be written as

(2.0.24) $$\sigma_n(s) \leq \frac{n(n+1)^{2s}}{(n+1)(n+2)^s - n(n+1)^s}.$$

Similarly as with (2.0.6), this may be shown by mathematical induction, the ultimate x–inequality (compare (2.0.9)) here being

(2.0.25) $$\frac{(1+2x)^s-(1+x)^s}{x} \leq \frac{(1+2x+x^2)^s-(1+2x)^s}{x^2}.$$

Since, for any fixed $s \geq 1$ (as well as for $s \leq 0$), the function t^s is convex in t on $\mathbb{R}^+$, our claim follows. ■

(iiB) The (more difficult) case $0 < s < 1$.
Condition (2.0.21) is equivalent to

(2.0.26) $$\sigma_n(s)\left\{\frac{\log(1+\frac{1}{n})}{(n+1)^s} - \frac{\log(1+\frac{1}{n+1})}{(n+2)^s}\right\} \leq \left(\frac{n+1}{n+2}\right)^s \log\left(1+\frac{1}{n+1}\right).$$

For $s > 0$ (even for $s \geq -1$) the LHS is > 0, so that we may just as well show that

(2.0.27) $$\sigma_n(s) \leq \frac{(n+1)^{2s}\log(1+\frac{1}{n+1})}{(n+2)^s\log(1+\frac{1}{n})-(n+1)^s\log(1+\frac{1}{n+1})}.$$

We will prove this by mathematical induction. For $n = 1$ this reads

(2.0.28) $$1 \leq \frac{4^s \log\frac{4}{3}}{3^s\log 2 - 2^s\log\frac{3}{2}}$$

or

(2.0.29) $$\frac{\log 2}{\log\frac{3}{2}} \leq \left(\frac{2}{3}\right)^s + \left(\frac{4}{3}\right)^s := f(s).$$

The LHS $\simeq 1.709$ and the RHS is (log–convex and) minimal at $s_0 \simeq 0.495$ with minimal value $f(s_0) \simeq 1.971$. So, our claim holds for $n = 1$.
Mathematical induction will be successful if we can show that

(2.0.30) $$(n+1)^s + \frac{(n+1)^{2s}\log(1+\frac{1}{n+1})}{(n+2)^s\log(1+\frac{1}{n})-(n+1)^s\log(1+\frac{1}{n+1})} \leq \frac{(n+2)^{2s}\log(1+\frac{1}{n+2})}{(n+3)^s\log(1+\frac{1}{n+1})-(n+2)^s\log(1+\frac{1}{n+2})}$$

which may also be written as

(2.0.31) $$\frac{\left(\frac{n+2}{n+1}\right)^s}{\log\frac{n+2}{n+1}} - \frac{1}{\log\frac{n+1}{n}} \geq \frac{\left(\frac{n+3}{n+2}\right)^s}{\log\frac{n+3}{n+2}} - \frac{1}{\log\frac{n+2}{n+1}}.$$

Now observe that $\phi(s) :=$ LHS – RHS of (2.0.31) is increasing in s for $s \geq 0$. Indeed, its derivative

(2.0.32) $$\phi'(s) = \left(\frac{n+2}{n+1}\right)^s - \left(\frac{n+3}{n+2}\right)^s > 0 \quad \text{for } s > 0.$$

Hence, it suffices to show (2.0.31) for $s = 0$, or

(2.0.33) $$\frac{1}{\log\frac{n+2}{n+1}} - \frac{1}{\log\frac{n+1}{n}} \geq \frac{1}{\log\frac{n+3}{n+2}} - \frac{1}{\log\frac{n+2}{n+1}}.$$

Setting $n = \frac{1}{t}$ it suffices to show that

(2.0.34) $$\frac{1}{\log \frac{1+2t}{1+t}} - \frac{1}{\log(1+t)}$$

is increasing for $0 < t \leq 1$. Writing $u = \frac{t}{t+1}$, it remains to show that

(2.0.35) $$\psi(u) := \frac{1}{\log(1+u)} + \frac{1}{\log(1-u)}$$

is increasing for $0 < u \leq \frac{1}{2}$. So, we will be done if we can show that

(2.0.36) $$\psi\,'(s) = -\frac{\frac{1}{1+u}}{\log^2(1+u)} - \frac{\frac{-1}{1-u}}{\log^2(1-u)} > 0$$

or

(2.0.37) $$\frac{\frac{1}{1-u}}{\log^2(1-u)} > \frac{\frac{1}{1+u}}{\log^2(1+u)}$$

or

(2.0.38) $$\chi(u) := (1+u)\log^2(1+u) - (1-u)\log^2(1-u) > 0.$$

Since $\chi(0) = 0$, we will be through if

(2.0.39) $$\chi\,'(u) = \log^2(1+u) + 2\log(1+u) + \log^2(1-u) + 2\log(1-u) > 0.$$

Since $\chi\,'(0) = 0$, it suffices to show that

(2.0.40) $$\chi\,''(u) = \frac{2\log(1+u)+2}{1+u} - \frac{2\log(1-u)+2}{1-u} > 0$$

or

(2.0.41) $$\lambda(u) := (1-u)\log(1+u) - (1+u)\log(1-u) - 2u > 0.$$

Since $\lambda(0) = 0$ it is enough to show that

(2.0.42) $$\lambda'(u) = -\log(1+u) + \frac{1-u}{1+u} - \log(1-u) + \frac{1+u}{1-u} - 2 > 0$$

or

(2.0.43) $$-\log(1-u^2) + \left(\frac{1-u}{1+u} + \frac{1+u}{1-u} - 2\right) > 0.$$

But this is trivially true, completing the proof of our claim that $h'(s) < 0$ for all $s \in \mathbb{R}$. ■

Clearly, it would be desirable to have an essentially simpler proof.
While constructing the above proof, several plots of $h(s)$ were made and these suggested that $h(s)$ is *strictly convex* in s (i.e. $h''(s) > 0$) on all of $\mathbb{R}$. Till now, no proof for this has been found (to our knowledge). Moreover, these plots suggest that $h(s)$ is even *strictly log−convex* in s on all of $\mathbb{R}$.
Since $\left(\frac{n+1}{n}\right)^s$ is log−convex in s and since the product of log−convex functions is log−convex again, it would follow that $Q(s) = h(s)\left(\frac{n+1}{n}\right)^s$ is also log−convex in s on all of $\mathbb{R}$ (and conversely).
We add to say that various (numerical) tests have been performed, especially on (D) and (E).
Although these tests strongly suggest that our Conjecture is true, no proof for arbitrary $n \in \mathbb{N}$ has been found (yet).

We add the

CONJECTURE. For every (fixed) $n \in \mathbb{N}$ the function $\frac{\omega_{n+1}(s)}{\omega_n(s)} \left(\frac{2n-1}{2n+1}\right)^s$ is strictly decreasing in s on all of $\mathbb{R}$.
Here
(2.0.44) $\omega_n(s) := 1^s + 3^s + ... + (2n-1)^s$.

A related QUESTION: What is the smallest constant a for which $\left(\frac{a+n-1}{a+n}\right)^s \frac{a^s+(a+1)^s+...+(a+n)^s}{a^s+(a+1)^s+...+(a+n-1)^s}$ is strictly decreasing in s on all of $\mathbb{R}$? (Note that the truth of this for $a \leq \frac{1}{2}$ would yield the previous Conjecture.)

Similarly one may also investigate (for $\alpha > 0$) the quotients $q(s) := \frac{t_{n+1,\alpha}(s)}{t_{n,\alpha}(s)}$ of the approximations $t_{n,\alpha}(s) := \frac{1}{n^{s+1}} \sum_{k=0}^{n-1} (k+\alpha)^s$.
It seems that this $q(s)$ is still (*monotonic*, *convex*) *log–convex* in s for certain $\alpha < 1$.
Some estimates for α_{Mono}, α_{Conv} and $\alpha_{\text{Log Conv}}$ are: $\alpha_M > 0.462, \alpha_C > 0.785, \alpha_{LC} > 0.803$.

Note: Similarly, defining (for $s > 0$) the canonical trapezoidal approximations $T_n(s)$ of $\int_0^1 x^s \, dx$ by $T_n(s) := \frac{1}{2}(L_n(s) + U_n(s))$, we found experimentally that $q_n(s) := \frac{T_{n+1}(s)}{T_n(s)}$ is *strictly decreasing* in s on all of $\mathbb{R}$. No proof of this is available at present.

Similar observations can be made for the various integral approximations (on page 94, for example) in Hildebrand's Introduction to Numerical Analysis.

2.1. A conjecture of J. P. Lambert

In 1985, J. P. Lambert [8], proposed the following (unsolved) problem (in the Amer. Math. Monthly, 1985, Problem E 3102*):
For all integers $n, s \geq 1$

(2.1.1) $$\frac{\sum_{k=1}^{n} (2k-1)^{2s}}{\sum_{k=1}^{n} (2k)^{2s}} < \left(\frac{2n}{2n+1}\right)^{2s+1}.$$

Also see Diamond [9].

First we try to get some idea about the (possible) origin of this inequality.
We consider the canonical midpoint approximations of the integral $\int_{-1}^{1} |x|^s \, dx$, ($s > 0$)

(2.1.2) $$t_n^*(s) := \frac{2}{n} \sum_{k=1}^{n} \left|-1 + \frac{2k-1}{n}\right|^s = \frac{2}{n^{s+1}} \sum_{k=1}^{[\frac{n}{2}]} (n - 2k + 1)^s.$$

For s not too small, $|x|^s$ will be convex in x on $[-1,1]$, and it stands to reason that $t_n^*(s)$ might then be *increasing* in n, i. e. $t_n^*(s) < t_{n+1}^*(s)$.
Since

(2.1.3a) $$t_{2n}^*(s) = \frac{2}{(2n)^{s+1}} \sum_{k=1}^{n} (2n - 2k + 1)^s$$
and
(2.1.3b) $$t_{2n+1}^*(s) = \frac{2}{(2n+1)^{s+1}} \sum_{k=1}^{n} (2n - 2k + 2)^s$$

the suggested monotonicity may also be written as

(2.1.4) $$t^*_{2n}(s) \;<\; t^*_{2n+1}(s) \;<\; t^*_{2n+2}(s)$$

or, in full, (with ω_n (s) as above in (2.0.44))

(2.1.5) $$\frac{\omega_n(s)}{(2n)^{s+1}} \;<\; \frac{2^s\,\sigma_n(s)}{(2n+1)^{s+1}} \;<\; \frac{\omega_{n+1}(s)}{(2n+2)^{s+1}}\,.$$

The first inequality here is Lambert's, while the second one seems to be its natural companion (compare (2.0.4)).
In order to deal with these inequalities we consider the functions (with $s \in \mathbb{R}$)

(2.1.6) $$f(s) := f_n(s) := \frac{\omega_n(s)}{2^s\,\sigma_n(s)}\left(\frac{2n+1}{2n}\right)^{s+1} \quad \text{and} \quad g(s) := g_n(s) := \frac{2^s\,\sigma_n(s)}{\omega_{n+1}(s)}\left(\frac{2n+2}{2n+1}\right)^{s+1}.$$

After extensive testing we arrived at the following

CONJECTURE. The function $f(s)$ is *strictly decreasing* in s on all of $\mathbb{R}$ (with $f'(s) < 0$) whereas $g(s)$ is *strictly decreasing* in s for $s \geq 2$ (with $g'(s) < 0$).

It would follow that if $s > s_0$ then $f(s) < f(s_0)$ and $g(s) < g(s_0)$.
For $s_0 = 2$ this would read

(2.1.7a) $$\frac{\omega_n(s)}{2^s\,\sigma_n(s)}\left(\frac{2n+1}{2n}\right)^{s+1} < f(2) = 1 - \frac{1}{4}\,\frac{n+\frac{1}{4}}{(n+1)\,n^3}$$

and

(2.1.7b) $$\frac{2^s\,\sigma_n(s)}{\omega_{n+1}(s)}\left(\frac{2n+2}{2n+1}\right)^{s} < 1 - \frac{1}{4}\,\frac{n+\frac{3}{4}}{(n+\frac{3}{2})\,(n+\frac{1}{2})^3}$$

the first inequality being an improvement of the original Lambert–inequality.
We expect that the proofs (if possible) will be quite demanding. Compare the proof of $h'(s) < 0$ in Section 2.0.

In relation to Lambert's inequality it was observed (numerically) that the function

(2.1.8) $$\phi(s) := \frac{\omega_n(s)}{\sigma_n(s)}\left(\frac{2n+1}{4n}\right)^{s}, \quad (s \in \mathbb{R})$$

is *strictly decreasing* in s on all of $\mathbb{R}$ (with $\phi'(s) < 0$).
In order to prove this we might just as well show that $\psi(s) := \log\phi(s)$ is strictly decreasing, or that

(2.1.9) $$L := \frac{\omega'}{\omega} + \log\frac{2n+1}{4n} < \frac{\sigma'}{\sigma} =: R.$$

Since

(2.1.10) $$\lim_{s\to\infty} L = \log(2n-1) + \log\frac{2n+1}{4n} \quad \text{and} \quad \lim_{s\to\infty} R = \log n$$

it is clear that (2.1.9) holds if s is large enough. It is trivial that (2.1.9) holds true for $s \downarrow -\infty$.
More precisely, since (for all $s \in \mathbb{R}$)

(2.1.11) $$\frac{\omega'}{\omega} = \frac{3^s\log 3 + 5^s\log 5 + \ldots + (2n-1)^s\log(2n-1)}{1 + 3^s + 5^s + \ldots + (2n-1)^s} \leq \log(2n-1)$$

and (for $s > 0$)

(2.1.12) $$\frac{\sigma'}{\sigma} = \frac{2^s \log 2 + 3^s \log 3 + \ldots + n^s \log n}{1^s + 2^s + \ldots + n^s} \geq \frac{n^s \log n}{1^s + 2^s + \ldots + n^s} = \frac{\log n}{\sum_{k=1}^{n} \left(\frac{k}{n}\right)^s} =$$

$$= \frac{\frac{1}{n} \log n}{\frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^s} = \frac{\frac{1}{n} \log n}{\int_0^1 x^s \, dx + \left(\frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^s - \int_0^1 x^s \, dx\right)} > \frac{\frac{1}{n} \log n}{\frac{1}{s+1} + \frac{1}{n}}$$

it follows easily that (2.1.9) holds for $s \geq n / \left(\frac{\log n}{\log n + \log(1 - \frac{1}{4 n^2})} - 1 \right)$.
In a similar fashion one may show that (2.1.9) is true for

(2.1.13) $$s \leq \log \frac{\frac{1}{n} \log 2}{\log (2n-1)!!} \Big/ \log \frac{3}{2}$$

where (2n−1)!! := 1 * 3 * 5 * ... * (2n−3) * (2n−1).

Numerically we observe that (2.1.9) is " most critical " for $2 < s < 3$, where it seems to have a (positive) local minimum $> \frac{1}{n^3 \log n}$.
Even the case $n = 2$ is not entirely trivial. In this case (2.1.9) is equivalent to

(2.1.14) $$L := 3^s \log \frac{15}{8} < \log \frac{8}{5} + 2^s \log \frac{16}{15} + 6^s \log \frac{16}{15} =: R.$$

Clearly this is true if $\log \frac{15}{8} \leq 2^s \log \frac{16}{15}$ or $s \geq \log \frac{\log 15 - \log 8}{\log 16 - \log 15} / \log 2 \simeq 3.284$.
This inequality is also true if $3^s \log \frac{15}{8} \leq \log \frac{8}{5}$ or $s \leq \log \frac{\log 8 - \log 5}{\log 15 - \log 8} \Big/ \log 3 \simeq -0.265$.
The rest may be completed by a numerical Newton−type procedure (see Section 5). Indeed, note that L is increasing and R is increasing and convex.

A slight generalization.
We consider (for $0 \leq a \leq \frac{1}{2}$)

(2.1.15) $$\phi := \phi_{n,a}(s) := \frac{\sum_{k=1}^{n} (2k - 1 + a)^s}{\sum_{k=1}^{n} (2k - a)^s} \left(\frac{2n + 1 - a}{2n + a}\right)^s.$$

We observe numerically that ϕ is strictly decreasing in s (for $0 \leq a < \frac{1}{2}$).
Let $\psi := \psi(s) := -\frac{\partial}{\partial s} \log \phi(s) = \frac{\sigma'}{\sigma} - \frac{\omega'}{\omega} - \log \frac{2n+1-a}{2n+a}$. We would like to show that this is > 0.
In support of this we observe numerically that

(2.1.16) $$\frac{\partial \psi}{\partial a} < 0 \quad \text{for all } s \in \mathbb{R} \text{ and all } a \in \left[0, \tfrac{1}{2}\right].$$

We venture to TRY again and prove (2.1.9) on the basis of our first Conjecture: $\frac{\sigma_{n+1}(s)}{\sigma_n(s)}$ is log−convex (in s) on $\mathbb{R}$.
As an easy consequence we have that $\frac{\sigma_{2n}(s)}{\sigma_n(s)}$ is log−convex on $\mathbb{R}$. Hence, also $\frac{\sigma_{2n}(s)}{2^s \sigma_n(s)}$ is log−convex on $\mathbb{R}$.
It follows that $\frac{\sigma_{2n}(s)}{2^s \sigma_n(s)}$ is convex, so that also $\frac{\sigma_{2n}(s)}{2^s \sigma_n(s)} - 1 = \frac{\omega_n(s)}{2^s \sigma_n(s)} =: h(s)$ is convex.
Note that $\lim_{s \to \infty} h(s) = 0$. So, we must have $h'(s) < 0$. In full this reads

(2.1.17) $$h'(s) = \frac{2^s \sigma \omega' - (2^s \sigma \log 2 + 2^s \sigma') \omega}{2^{2s} \sigma^2(s)} < 0$$

which is equivalent to (the non−trivial) $\frac{\omega'}{\omega} - \frac{\sigma'}{\sigma} < \log 2$.
This falls just short of our goal: To prove that $\frac{\omega'}{\omega} - \frac{\sigma'}{\sigma} < \log(\frac{4n}{2n+1}) = \log(2 - \frac{1}{n+\frac{1}{2}})$.
So, it may very well be that even our (strong) Conjecture 1 is *not strong enough* to prove (2.1.9).

3. Relations with some other conjectures

3.1. A conjecture of H. Alzer and A. A. Jagers.

In [5], Alzer and Jagers proposed the following (unsolved) problem.
Let

$$(3.1.1) \qquad f_n(s) := \left(\frac{\frac{1}{n+1} \sum_{i=1}^{n+1} i^s}{\frac{1}{n} \sum_{i=1}^{n} i^s} \right)^{\frac{1}{s}}.$$

Prove (or disprove) that $f_n(s)$ is *strictly increasing* in s on all of $\mathbb{R}^+$. (No full proof was submitted.)

Here we will show that the monotonicity of $f_n(s)$ on $\mathbb{R}^+$ (even on all of $\mathbb{R}$) is a simple consequence of our (first) Conjecture in Section 1.

PROOF. We sketch the details only for $s > 0$. The remaining case(s) may be treated similarly.
It suffices to show that the function

$$(3.1.2) \qquad \log f_n(s) = \frac{1}{s} \log\left(\frac{n}{n+1} \frac{\sigma_{n+1}(s)}{\sigma_n(s)} \right) = \frac{1}{s} \log\left(\frac{n}{n+1} Q(s) \right)$$

is strictly increasing (in s) on all of $\mathbb{R}^+$. Sufficient for this is that

$$(3.1.3) \qquad \frac{d}{ds} \log f_n(s) = -\frac{1}{s^2} \log \frac{n}{n+1} - \frac{1}{s^2} \log Q(s) + \frac{1}{s} \frac{Q'(s)}{Q(s)} > 0$$

or

$$(3.1.4) \qquad \phi(s) := -\log \frac{n}{n+1} - \log Q(s) + s \frac{Q'(s)}{Q(s)} > 0.$$

Since $\phi(0) = 0$, it suffices to show that (for $s > 0$)

$$(3.1.5) \qquad \phi'(s) = s \frac{d}{ds} \frac{Q'(s)}{Q(s)} > 0 \quad \text{or} \quad (\log Q(s))'' > 0.$$

But this is equivalent with our (first) Conjecture in Section 1, proving our claim. ■

3.2. Martins' theorem.

For a given positive sequence $\{u_n\}_{n\in\mathbb{N}}$, Martins [6] considered / compared the quotients

$$(3.2.1) \qquad \frac{A_{n+1}}{A_n} \quad \text{and} \quad \frac{G_{n+1}}{G_n}$$

where

$$(3.2.2) \qquad A_n := \frac{1}{n} \sum_{k=1}^{n} u_k \quad \text{and} \quad G_n := \left(\prod_{k=1}^{n} u_k \right)^{\frac{1}{n}}$$

are the progressive Arithmetical and Geometrical means of $\{ u_n \}_{n \in \mathbb{N}}$.
For the special sequence $u_n := u_n(s) := n^s$ Martins proved that

(3.2.3) $$\frac{G_{n+1}}{G_n} \leq \frac{A_{n+1}}{A_n}, \quad \forall\ s \in \mathbb{R}.$$

We will show that this inequality is also a consequence of our (first) Conjecture in Section 1.

PROOF. Martins' inequality (3.2.3) may be written as

(3.2.4) $$\frac{((n+1)!)^{\frac{s}{n+1}}}{(n!)^{\frac{s}{n}}} \leq \frac{n}{n+1} \frac{\sigma_{n+1}(s)}{\sigma_n(s)}$$

or

(3.2.5) $$\phi(\mathrm{s}) := \frac{n}{n+1} \frac{\sigma_{n+1}(s)}{\sigma_n(s)} (n!)^{\frac{s}{n}} ((n+1)!)^{-\frac{s}{n+1}} \geq 1.$$

Note that $\phi(0) = 1$.
Since, in view of Section 1, the function $\phi(s)$ " is " log–convex on $\mathbb{R}$, we will be through if we can show that

(3.2.6) $$\phi'(0) = 0 \quad \text{or} \quad (\log \phi(s))' = \frac{\phi'(s)}{\phi(s)} = 0 \ \text{ at } s = 0.$$

But, this is a simple exercise, proving our claim. ■

Note that it even follows that

(3.2.7) $$\frac{G_{n+1}(s)}{G_n(s)} < \frac{A_{n+1}(s)}{A_n(s)} \quad \text{for all } s \neq 1.$$

3.3. Alzer's conjecture.

Alzer [7] has conjectured, as a supplement to Martins' theorem, that

(3.3.1) $$\frac{A_{n+1}(s)}{A_n(s)} \leq \frac{G_{n+1}(s)}{G_n(s)} \frac{2^{\frac{s}{2}} + 2^{-\frac{s}{2}}}{2}, \ (s > 0).$$

It seems that this inequality has no direct relation to our first Conjecture. We will deal with it in Part II.

4. An attempt to prove our first Conjecture.

Some simple observations:
From

(4.1) $$Q_n(s) = \frac{\sigma_{n+1}(s)}{\sigma_n(s)} = \frac{\sigma_n(s) + (n+1)^s}{\sigma_n(s)} = 1 + \frac{(n+1)^s}{\sigma_n(s)} = 1 + \frac{1}{\left(\frac{1}{n+1}\right)^s + \left(\frac{2}{n+1}\right)^s + \ldots + \left(\frac{n-1}{n+1}\right)^s + \left(\frac{n}{n+1}\right)^s}$$

it is clear that $\frac{\sigma_{n+1}(s)}{\sigma_n(s)}$ is strictly increasing in s on $\mathbb{R}$.
For large s we have $Q_n(s) \sim \left(1 + \frac{1}{n}\right)^s$, which happens to be log–convex in s.

The first derivative of $\log Q(s)$ has the following fairly simple form (easy to show):

$$(4.2)\qquad \frac{d}{ds}\log Q_n(s) \;=\; \frac{\sum_{k=0}^{n-1}((n-k)(n+1))^s \log\frac{n+1}{n-k}}{\sigma_n(s)\,\sigma_{n+1}(s)} \;=\; \frac{\sum_{k=1}^{n}(k(n+1))^s \log\frac{n+1}{k}}{\sigma_n(s)\,\sigma_{n+1}(s)}$$

which is clearly >0, as it should be.
We write this as

$$(4.3)\qquad \frac{d}{ds}\log Q_n(s) \;=\; \frac{\sum_{k=1}^{n}(k(n+1))^s \log\frac{n+1}{k}}{\sigma_n(s)\,\sigma_{n+1}(s)} \;=$$

$$=\; \frac{n^s(n+1)^s\log(\frac{n+1}{n}) + \sum_{k=1}^{n-1}(k(n+1))^s \log\frac{n+1}{k}}{\sigma_n(s)\,\sigma_{n+1}(s)} \;=\; \frac{n^s(n+1)^s\log\frac{n+1}{n} + \text{increasing}}{\sigma_n(s)\,\sigma_{n+1}(s)}\;.$$

Now observe that

$$(4.4)\qquad \frac{n^s(n+1)^s\log\frac{n+1}{n}}{\sigma_n(s)\,\sigma_{n+1}(s)} \;=\; \frac{\log\frac{n+1}{n}}{\left((\frac{1}{n})^s+(\frac{2}{n})^s+\ldots+(\frac{n}{n})^s\right) * \left((\frac{1}{n+1})^s+(\frac{2}{n+1})^s+\ldots+(\frac{n}{n+1})^s+(\frac{n+1}{n+1})^s\right)}$$

which is increasing in s indeed (with limit $=\log\frac{n+1}{n}$ as $s\uparrow\infty$).
But, this argument is, surprisingly enough, not sufficient for (4.3) to be increasing.

Some brief notes:
(1) We were able to prove Conjecture 1 for $s=0$.
We computed $(\log Q_n(s))''(0)$. This seems to be >0 and *decreasing* in n.
We were able to prove the >0 part here.
The proof by mathematical induction may be reduced to showing that

$$n\log^2 n + 2\log(n+1) + \frac{\log 2\pi n}{n}\;\log n! + \frac{1}{6n^3}\log n! \;\le$$

$$\le\; \text{n}\log^2(n+1) + \frac{\log 2\pi(n+1)}{n+1}\;\log(n+1)!.$$

We omit the quite tedious details.
(2) Numerically we observe that Conjecture 1 is " most critical " for $2<s<4$.

A related Open Problem. For any fixed $n\in\mathbb{N}$ we define

$$(4.5)\qquad q(s) := q_n(s) \;=\; \frac{n^s}{1^s+2^s+3^s+\ldots+n^s} = \frac{n^s}{\sigma_n(s)}\,,\quad (s\in\mathbb{R}).$$

For $n\ge 2$ we then have

$$(4.6)\qquad \log\frac{\sigma_n(s)}{\sigma_{n-1}(s)} \;=\; -\log\frac{\sigma_{n-1}(s)}{\sigma_n(s)} \;=\; -\log\left(1-\frac{n^s}{\sigma_n(s)}\right) \;=\; \sum_{k=1}^{\infty}\frac{1}{k}\,q^k(s).$$

We would be immediately through with our Conjecture 1 if all $q^k(s)$ would be convex on all of $\mathbb{R}$. But, this is not the case ! However, there seems to exist an $s=s_n$, such that $q''(s)>0$ for all $s<s_n$.
Defining (for $m\in\mathbb{N}$)

$$(4.7)\qquad f_m(s) := \sum_{k=1}^{m}\frac{1}{k}\,q^k(s)$$

let $s = r_m$ be the smallest zero of $f_m''(s)$. If non extant then we are through !
Prove (or disprove) that $r_1 < r_2 < r_3 < \ldots$ and that $\lim_{m\to\infty} r_m = +\infty$.

In view of (4.6) it would follow from this that $f_\infty(s) := \log \frac{\sigma_n(s)}{\sigma_{n-1}(s)}$ is convex, proving our first Conjecture.
Note: It is clear that $\lim_{s\downarrow -\infty} q(s) = 0$, and writing $q(s) = \frac{1}{\left(\frac{1}{n}\right)^s + \left(\frac{2}{n}\right)^s + \ldots \left(\frac{n-1}{n}\right)^s + \left(\frac{n}{n}\right)^s}$ it follows that $\lim_{s\uparrow\infty} q(s) = 1$. It thus stands to reason that $q(s)$ is convex " at $s = -\infty$ " but not " at $s = +\infty$ ".
Hence: s_n exists. Similarly: r_m exists.

Still another unsolved Problem.

Let
(4.8) $\phi(s) := \phi_n(s) := (2n)^s - (2n-1)^s + (2n-2)^s - + \ldots + 2^s - 1^s$.

(a) Prove that $\phi(s)$ is *strictly increasing* in s on all of $\mathbb{R}$.
(b) Determine (or estimate) the largest α such that

(4.9) $(2n)^s \log^\alpha(2n) - (2n-1)^s \log^\alpha(2n-1) + (2n-2)^s \log^\alpha(2n-2) - + \ldots + 2^s \log^\alpha(2) > 0$

for all $s \in \mathbb{R}$.

Numerical work suggests that $\alpha > \pi$.

5. Numerical verification of our first Conjecture.

When trying to verify our first Conjecture numerically, one may want to use the following simple (but useful)

NUMERICAL LEMMA (of Newton–type).
If $L(s) < R(s)$ for $s = a$, and $L(s)$ is increasing, and $R(s)$ is increasing and convex, then $L(s) < R(s)$ also holds for $a - \frac{R(a) - L(a)}{R'(a)} \leq s \leq a$.

Our log–convexity criteria (D), (E) and (F) may be written as $L < R$ with

(5.1) $L := \sigma_n \, \sigma_n'' \, \sigma_{n+1}^2 + \sigma_n^2 \, (\sigma_{n+1}')^2$ and $R := \sigma_n^2 \, \sigma_{n+1} \, \sigma_{n+1}'' + (\sigma_n')^2 \, \sigma_{n+1}^2$.

and these L and R satisfy the conditions of the above Lemma.
Now just start with an arbitrary $a > 0$ (from above) and iterate as suggested above.
One may carry out these computations using *Mathematica*, for example.
Quite a time consuming job, though. But it seems to work !

It seems that ($\log(1 + \frac{(n+1)^s}{\sigma_n(s)}))''$ is critical only for $0 \leq s \leq 3$ with a positive local minimum.

QUESTION: Is there any point in verifying / proving that $Q(s) := \frac{\sigma_{n+1}(s)}{\sigma_n(s)}$ is (just) *convex* in s ?
Try some examples for small n. $Q''(s)$ seems to have a positive local minimum for $2 < s < 4$.
Mathematica experiments indicate that (for n not too small) this local minimum is $> \frac{1}{n^2}$.

Part II

1. Statement of our second Conjecture.

Our second CONJECTURE reads: For any fixed n $\in \mathbb{N}$, the function $\frac{\sigma_{n+1}^2(s)}{\sigma_n(s)\,\sigma_{n+2}(s)}$ is *log−convex* in s on all of $\mathbb{R}$.

2. The genesis of our second Conjecture.

While studying Alzer's inequality (see (3.1) in Part I)

(2.1) $$\frac{A_{n+1}(s)}{A_n(s)} \le \frac{G_{n+1}(s)}{G_n(s)}\ \frac{2^{\frac{s}{2}} + 2^{-\frac{s}{2}}}{2}, \quad (s > 0)$$

or

(2.2) $$\frac{A_{n+1}(s)}{A_n(s)}\ \frac{G_n(s)}{G_{n+1}(s)} \le \frac{2^{\frac{s}{2}} + 2^{-\frac{s}{2}}}{2}, \quad (s > 0)$$

it was (numerically) observed that for any fixed $s \in \mathbb{R}$, the LHS := $\frac{A_{n+1}(s)}{A_n(s)}\ \frac{G_n(s)}{G_{n+1}(s)}$ is *decreasing* in n. Clearly, this would entail that Alzer's inequality is true for all $n \in \mathbb{N}$ if it holds for $n = 3$. Its validity for $n = 3$ will be shown below in Section 5.

```
(* Testing the monotonicity just mentioned *)
prec = 25 ; s = 1/4 ;    (* Pick your choice *)
σn = Sum[k^s , {k, n}] ;
An = σn/n ; Anp1 = (σn + (n + 1)^s)/(n + 1) ; Gn = (n!)^(s/n) ; Gnp1 = ((n + 1)!)^(s/(n+1)) ; f = Anp1/An * Gn/Gnp1 ;
For[n = 2 , n ≤ 10 , n++ ,
Print["n= ", n , "  s= ", N[s] , "          f(s)= " , N[f , prec]]]
```

```
n= 2  s= 0.25          f(s)= 1.002599668283378350809485
n= 3  s= 0.25          f(s)= 1.001939826338575098526561
n= 4  s= 0.25          f(s)= 1.001520157589413043434971
n= 5  s= 0.25          f(s)= 1.001233120792851125950798
n= 6  s= 0.25          f(s)= 1.001026302603682735994488
n= 7  s= 0.25          f(s)= 1.000871326388842402466966
n= 8  s= 0.25          f(s)= 1.000751587656873653937862
n= 9  s= 0.25          f(s)= 1.000656771536769794830343
n= 10  s= 0.25          f(s)= 1.000580158605126596481757
```

So, we get the *impression* that for any fixed $s \in \mathbb{R}$

(2.3) $$\frac{A_{n+1}(s)}{A_n(s)}\ \frac{G_n(s)}{G_{n+1}(s)} < \frac{A_n(s)}{A_{n-1}(s)}\ \frac{G_{n-1}(s)}{G_n(s)}$$

or

(2.4) $$1 < \frac{A_n^2(s)}{A_{n-1}(s)\,A_{n+1}(s)}\ \frac{G_{n-1}(s)\,G_{n+1}(s)}{G_n^2(s)} = Q_A(s)\,Q_G(s),\ \text{say.}$$

It is clear that $Q_G(s)$, being an exponential function of the simple form a^s, is log−convex in s on all of $\mathbb{R}$. Further, it is a matter of routine to show that $(Q_A(s)\,Q_G(s))' = 0$ at $s = 0$.
Now observe that if $Q_A(s)$ would be log−convex in s then the whole RHS of (2.4) would be log−convex (and hence convex) in s, with RHS'(0) = 0, proving Alzer's inequality.
In view of

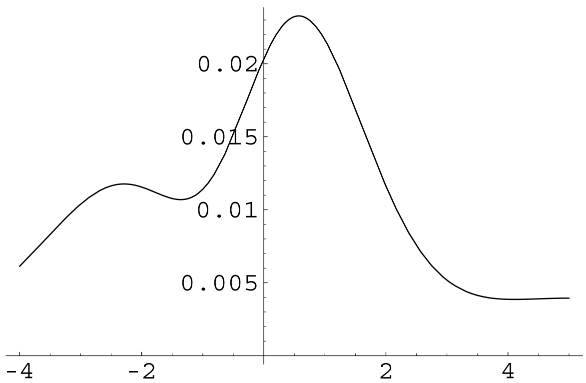
(2.5) $$\frac{A_n^2(s)}{A_{n-1}(s)\,A_{n+1}(s)} = \frac{n^2-1}{n^2}\ \frac{\sigma_n^2(s)}{\sigma_{n-1}(s)\,\sigma_{n+1}(s)}$$

where $n \in \mathbb{N}$ is constant, the above observations led us to our second Conjecture.

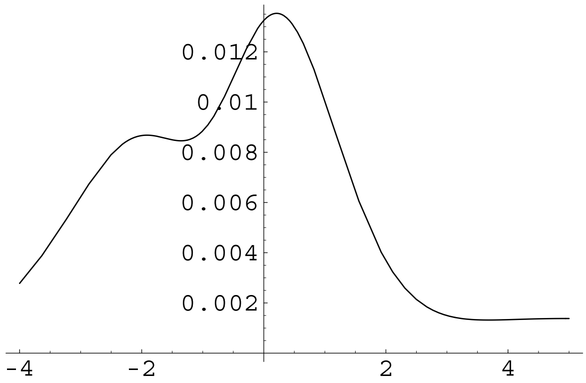
3. Verification of the second Conjecture.

Using *Mathematica*, we performed various checks (similar to those in Section 4 of Part I) on our second Conjecture. Needless to say, all these numerical / graphical experiments suggest that the Conjecture is true indeed.
Especially the graphs of the second derivative of $\log \frac{\sigma_n^2(s)}{\sigma_{n-1}(s)\,\sigma_{n+1}(s)}$ are very intriguing.
Also compare Section 5 below.

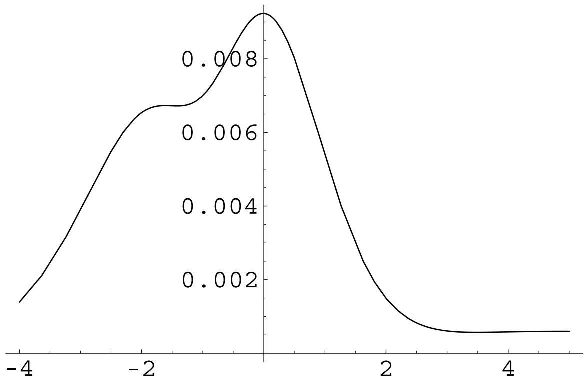
```
Clear[n, s]; σn = Sum[k^s, {k, n}]; σnm1 = σn − n^s; σnp1 = σn + (n + 1)^s;
f = σn^2/(σnm1 * σnp1); Lf = Log[f]; Lf2 = ∂s ∂s Lf;
nBound = 6; (* say *)
For[n = 3, n ≤ nBound, n++, Plot[Lf2, {s, −4, 5}];
Print["↑ This was n= ", n, "   Local Minimum → ", FindMinimum[Lf2, {s, 2, 5}]]]
```



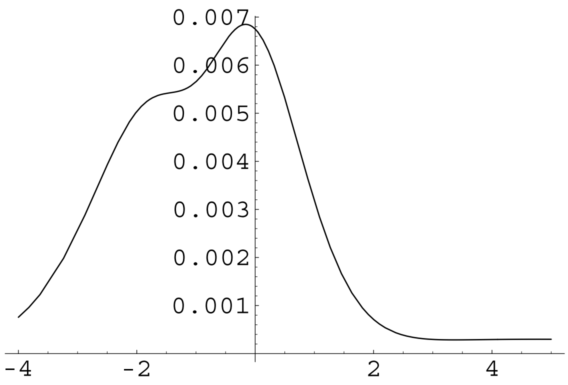
↑ This was n= 3   Local Minimum → {0.00385861, {s → 4.11215}}



↑ This was n= 4 Local Minimum → {0.00131938, {s → 3.68589}}



↑ This was n= 5 Local Minimum → {0.000569114, {s → 3.49106}}



↑ This was n= 6 Local Minimum → {0.00028568, {s → 3.38689}}

4. A consequence of Conjecture 2.

Writing (2.1) as

(4.1) $$\frac{A_{n+1}(s)}{G_{n+1}(s)} \le \frac{A_n(s)}{G_n(s)} \, \frac{2^{-s}+2^{s}}{2}, \quad (s>0)$$

a repeated application of this to itself would result in

(4.2) $$\frac{A_n(s)}{G_n(s)} \le \left(\frac{2^{-s}+2^{s}}{2}\right)^{n-1}$$ for all $n \in \mathbb{N}$ and all $s \in \mathbb{R}$.

For $n=1$ this is trivial and for $n=2$ it is easily verified.
More generally

(4.3) $$f(s) := (n!)^{\frac{s}{n}} \left(\frac{2^{s/2}+2^{-s/2}}{2}\right)^{n-1} - \frac{1}{n} * (1^s + 2^s + \ldots n^s)$$

seems to be convex at $s=0$. Moreover, $f_n''(0)$ (>0) seems to be increasing in n.
So, (4.2) seems to be " rather easy " at $s \simeq 0$. However, no full proof of (4.2) is known to us.

5. Verification of Alzer's inequality.

It seems that (for small n) Alzer's inequality is much easier to verify than our own two Conjectures.
We will perform such a verification for $s>0$ (the cases $s=0$ and $n=2$ are trivial).
So, from now on we assume $n \ge 3$.
We start in a very simplistic manner: Alzer claims that

(5.1) $$\frac{n-1}{n}(1^s+2^s+\ldots+n^s) \le (1^s+2^s+\ldots+(n-1)^s)\, n^{\frac{s}{n-1}}\, (n!)^{-\frac{s}{n(n-1)}} \, \frac{2^{\frac{s}{2}}+2^{-\frac{s}{2}}}{2}.$$

Sharper than this is

(5.2) $$2\,\frac{n-1}{n}\, n\, n^s \le (n-1)^s\, n^{\frac{s}{n-1}}\, (n!)^{-\frac{s}{n(n-1)}}\, 2^{\frac{s}{2}}$$

or

(5.3) $$2(n-1) \le \left(\frac{n-1}{n}\, n^{\frac{1}{n-1}}\, (n!)^{-\frac{1}{n(n-1)}} \sqrt{2}\right)^s$$

or

(5.4) $$2(n-1) \le a^s \quad \text{with} \quad a := \frac{n-1}{n}\, n^{\frac{1}{n-1}}\, (n!)^{-\frac{1}{n(n-1)}} \sqrt{2}\,.$$

LEMMA 1. For $n \ge 3$ we have $a>1$ in (5.4).

PROOF. We have to show that

(5.5) $$n^{\frac{1}{n-1}} \sqrt{2} > \frac{n}{n-1}\, (n!)^{\frac{1}{n(n-1)}}$$

or

(5.6) $$n^n\, 2^{\frac{n(n-1)}{2}} > \left(1+\frac{1}{n-1}\right)^{n(n-1)} n!.$$

Sharper than this is (using Stirling)

(5.7) $n^n\, 2^{\frac{n(n-1)}{2}} > e^n\, n^n\, e^{-n} \sqrt{2\pi n}\, e^{\frac{1}{12n}}$

or

(5.8) $2^{n(n-1)} > 2\pi n\, e^{\frac{1}{6n}}$ or $2^{n(n-1)} > 6.65\, n$

which is true indeed for $n \geq 3$. ■

Consequence: In view of (5.4), Alzer's inequality holds true for $s \geq s_0 := \frac{\log(2(n-1))}{\log a}$. Later we will improve on this.
Here is a simple *Mathematica* program computing some a and s_0.

```
prec = 30; Clear[n]; a = (n - 1)/n * n^(1/(n-1)) * (n!)^(-1/(n*(n-1))) * Sqrt[2]; s0 = Log[2*(n - 1)]/Log[a];
For[n = 3, n ≤ 10, n++, Print["n= ", n, "  a= ", N[a, prec], "  s0= ", N[s0, prec]]]
```

n= 3 a= 1.21141372855475977259414117088 s_0 = 7.22826251895962701374721299244

n= 4 a= 1.29194885544219791385434161069 s_0 = 6.99491214304942877773858189137

n= 5 a= 1.33164487995932388137446564092 s_0 = 7.26024143416826576160088643782

n= 6 a= 1.35431467787292939662956751449 s_0 = 7.59188542554862154894929685960

n= 7 a= 1.36856293427842082319837273196 s_0 = 7.91973756766943950825265923669

n= 8 a= 1.37814358608082555195971619981 s_0 = 8.22809441266003199851301550326

n= 9 a= 1.38491735633824125943949924900 s_0 = 8.51426342937290467947249668226

n= 10 a= 1.38989618369636924703886291198 s_0 = 8.77921222934515474946743418353

Now we write Alzer's inequality as

(5.9) $L(s) := 2\,\frac{n-1}{n}\,\sigma_n(s)\,(n!)^{\frac{s}{n(n-1)}}\,2^{\frac{s}{2}} \leq \sigma_{n-1}(s)\,n^{\frac{s}{n-1}}\,(2^s+1) =: R(s).$

Note that $L(s)$ is increasing and that $R(s)$ is increasing and (log−) convex on all of $\mathbb{R}$.
By our Newton−type argument we thus find that (5.9) also holds true for all

(5.10) $s \geq s_0 - \frac{R(s) - L(s)}{R'(s)}$.

One may iterate this procedure a number of times to extend the domain of validity of (5.9).
However, since $R'(0) - L'(0) = 0$, one will experience that only very modest progress is being made.
We improve on this as follows: We want to show that $f(s) := R(s) - L(s) > 0$ for $0 < s < s_0$.
Since $f(0) = 0$, it suffices to show that $f'(s) > 0$ for $0 < s < s_0$.
Also $f'(0) = 0$, so that it is enough to show that $f''(s) > 0$ or $R''(s) - L''(s) > 0$ for $0 < s < s_0$.
One may actually verify (numerically) that $R''(s_0) > L''(s_0)$ for $3 \leq n \leq 10$, say. Or, depending on time and patience, for $3 \leq n \leq 1000$, say.
Also, $L(s)$ and $R(s)$ satisfy the conditions of our numerical Lemma. Iterating, we find, for example, the validity of (5.9) for $n = 3$ in some 67 steps.
Here is a simple *Mathematica* program.

Clear[n, s]; σ = Sum[k^s, {k, n}]; $L = 2 * \frac{n-1}{n} * \sigma * (n!)^{\frac{s}{n*(n-1)}} * 2^{\frac{s}{2}}$; L2 = $\partial_s \partial_s L$;

$R = (\sigma - n^s) * n^{\frac{s}{n-1}} * (2^s + 1)$; R2 = $\partial_s \partial_s R$; R3 = ∂_s R2; T = Simplify$\left[\frac{\text{R2} - \text{L2}}{\text{R3}}\right]$;

For$\left[n = 3, n \le 10, n{+}{+}, a = N\left[\frac{n-1}{n} * n^{\frac{1}{n-1}} * (n!)^{-\frac{1}{n*(n-1)}} * \sqrt{2}\right]; \text{s0} = \frac{\text{Log}[2*(n-1)]}{\text{Log}[a]}\right.$;
k = 0;
While[s0 > 0, Δ = T /. s → s0; s0 = N[s0 − Δ]; k += 1];
Print["n= ", n, "   s0= ", s0, "   niter= ", k]]

n= 3   s0= −0.00180437   niter= 67

n= 4   s0= −0.00367959   niter= 62

n= 5   s0= −0.00442469   niter= 65

n= 6   s0= −0.00974961   niter= 70

n= 7   s0= −0.0100881   niter= 75

n= 8   s0= −0.000357881   niter= 79

n= 9   s0= −0.0019707   niter= 84

n= 10   s0= −0.00454609   niter= 89

We may improve on this as follows: We sharpen (5.9) by means of the trivial estimates

(5.11) $\sigma_n(s) < \frac{(n+1)^{s+1}}{s+1}$ and $\frac{(n-1)^{s+1}}{s+1} < \sigma_{n-1}(s)$
to

(5.12) $2(n-1)\,\frac{(n+1)^{s+1}}{s+1} \le n\; n^{\frac{s}{n-1}}\,(n!)^{-\frac{s}{n(n-1)}}\,2^{\frac{s}{2}}\,\frac{(n-1)^{s+1}}{s+1}$
or

(5.13) $2\,\frac{n+1}{n} \le a^s$ with $a := \frac{n^{\frac{1}{n-1}}\,\sqrt{2}\,(n-1)}{(n+1)\,(n!)^{\frac{1}{n(n-1)}}}$.

For $n \ge 6$ this is better than above. Here are some results:

$$\textbf{Clear}[n]\,;\ \textbf{prec} = 25\,;\ a = \frac{n^{\frac{1}{n-1}} * \sqrt{2} * (n-1)}{(n+1) * (n!)^{\frac{1}{n*(n-1)}}}\,;\ s_0 = \frac{\textbf{Log}\left[2 * \frac{n+1}{n}\right]}{\textbf{Log}[a]}\,;$$

For[*n* = 4, *n* ≤ 10 , *n*++ , Print["n= ", *n* , "   a= ", *N*[*a*, prec] , " s_0 = ", *N*[s_0 , prec]]]

n= 4 a= 1.033559084353758331083473 s_0 = 27.75943107450229630271386

n= 5 a= 1.109704066632769901145388 s_0 = 8.410417598496762008725611

n= 6 a= 1.160841152462510911396772 s_0 = 5.681039106244187474052304

n= 7 a= 1.197492567493618220298576 s_0 = 4.586801811965574200150807

n= 8 a= 1.225016520960733823964192 s_0 = 3.995629039305886521226762

n= 9 a= 1.246425620704417133495549 s_0 = 3.624967638233412415065291

n= 10 a= 1.263541985178517497308057 s_0 = 3.370644438604987862615722

Starting at these new improved s_0 we find the following:

```
Clear[n , s ];
σ = Sum[k^s, {k, n}]; L = 2 * (n - 1)/n * σ * (n!)^(s/(n*(n-1))) * 2^(s/2); L1 = ∂_s L; L2 = ∂_s L1;
R = (σ - n^s) * n^(s/(n-1)) * (2^s + 1); R2 = ∂_s ∂_s R; R3 = ∂_s R2; T = Simplify[(R2 - L2)/R3];
For[n = 6, n ≤ 10, n++, a = N[(n^(1/(n-1)) * √2 * (n - 1))/((n + 1) * (n!)^(1/(n*(n-1))))]; s0 = Log[2 * (n+1)/n]/Log[a];
k = 0;
While[s0 > 0, Δ = T /. s → s0; s0 = N[s0 - Δ]; k += 1];
Print["n= ", n, "    a= ", a, "    s0= ", s0, "    # iter= ", k]]
```

n= 6 a= 1.16084 s0= −0.0109249 # iter= 64

n= 7 a= 1.19749 s0= −0.00524811 # iter= 63

n= 8 a= 1.22502 s0= −0.00585335 # iter= 64

n= 9 a= 1.24643 s0= −0.000650036 # iter= 65

n= 10 a= 1.26354 s0= −0.00195887 # iter= 67

The upshot of all this is that we need not worry about small n.

Still better is the following: For $s > 0$ the Riemann sums $U_n(s)$ and $L_n(s)$ are monotonic in n

(5.14) $\quad U_n(s) < U_{n-1}(s)$ and $L_{n-1}(s) < L_n(s)$.

Hence

(5.15) $$\frac{A_n}{A_{n-1}} < \left(\frac{n}{n-1}\right)^s$$

and we use this to sharpen (5.9) to

(5.16) $$2 \leq a^s \quad \text{with} \quad a := \frac{(n-1)\, n^{\frac{1}{n-1}} \sqrt{2}}{n\,(n!)^{\frac{1}{n(n-1)}}}.$$

So, $s \geq \frac{\log 2}{\log a}$ is sufficient for (5.9). We need $a > 1$, but this should not be too difficult to prove.
We even observe that a is increasing in n. Also this is a matter of routine.
Clearly, $\lim_{n \to \infty} a = \sqrt{2}$ so that $\lim_{n \to \infty} s_0 = 2$. Compare the next program.

```
For[n = 3, n ≤ 10, n++, a = N[((n - 1) * n^(1/(n-1)) * √2) / (n * (n!)^(1/(n*(n-1))))];
Print["n= ", n, "    a= ", a, "    s0= ", Log[2]/Log[a]]]
```

n= 3 a= 1.21141 s0= 3.61413

n= 4 a= 1.29195 s0= 2.706

n= 5 a= 1.33164 s0= 2.42008

n= 6 a= 1.35431 s0= 2.28539

n= 7 a= 1.36856 s0= 2.20915

n= 8 a= 1.37814 s0= 2.16111

n= 9 a= 1.38492 s0= 2.12857

n= 10 a= 1.3899 s0= 2.10536

From the above it is clear that Alzer's inequality is " only critical " for small s ($\simeq 0$).
We therefore restrict ourselves to $0 < s < 1$.
One more attempt to improve on the above endeavors:

LEMMA 2. For $0 < s < 1$ we have $\frac{A_{n+1}(s)}{A_n(s)} < \left(\frac{n+2}{n+1}\right)^s$.

PROOF.
(a) We might apply mathematical induction.
(b) We *venture* to apply our first log–convexity Conjecture.
We have to show that

$$(5.17) \qquad \frac{n}{n+1}\,\frac{\sigma_{n+1}(s)}{\sigma_n(s)} < \left(\frac{n+2}{n+1}\right)^s \quad \text{or} \quad \phi(s) := \frac{n}{n+1}\,\frac{\sigma_{n+1}(s)}{\sigma_n(s)}\left(\frac{n+1}{n+2}\right)^s < 1.$$

Since $\phi(0) = \phi(1) = 1$, and $\phi(s)$ " is " (log–) convex on $0 \le s \le 1$, we find that $\phi(s) < 1$ for all $s \in (0,1)$. ■

By means of this Lemma Alzer's inequality may be sharpened (for $0 < s < 1$) to

$$(5.18) \qquad \psi(s) := \tfrac{1}{2}\,a^s\left(2^{\frac{s}{2}} + 2^{-\frac{s}{2}}\right) > 1 \quad \text{with} \quad a := \frac{n}{n+1}\, n^{\frac{1}{n-1}}\,(n!)^{-\frac{1}{n(n-1)}}.$$

Note that $\psi(0) = 1$ and that $\psi(s)$ is (log–) convex. Also, a appears to be increasing (↑ 1).

The corresponding s_0 may be found by the following *Mathematica* program.

```
For[n = 3, n ≤ 10, n++, a = N[n/(n+1) * n^(1/(n-1)) * (n!)^(-1/(n*(n-1)))];
Print["n= ", n, "      a= ", a, "      s0= ", FindRoot[a^s * (2^(s/2) + 2^(-s/2)) == 2, {s, 1}]]]
```

n= 3   a= 0.963674   s0= {$s \to 0.62086$}

n= 4   a= 0.974449   s0= {$s \to 0.432592$}

n= 5   a= 0.980849   s0= {$s \to 0.322645$}

n= 6   a= 0.985006   s0= {$s \to 0.251892$}

n= 7   a= 0.987881   s0= {$s \to 0.203194$}

n= 8   a= 0.989963   s0= {$s \to 0.168069$}

n= 9   a= 0.991526   s0= {$s \to 0.141775$}

n= 10   a= 0.992732   s0= {$s \to 0.121524$}

```
(*  The next program relates to the log-convexity in Conjecture 1  *)
$MinPrecision = 25; (*  Higher precision ?!  *)
Clear[s];
n = 4000; dn = 1000;
While[0 == 0, n += dn;
 σ = Sum[k^s , {k, n}]; q = 1 + (n + 1)^s/σ; Lq = Log[q]; Lq1 = ∂s q; Lq2 = ∂s Lq1;
 (*  Plot[Lq2,{s,0,3}];  *)
 m = FindMinimum[Lq2, {s, 0, 3}, AccuracyGoal → 25, WorkingPrecision → 35];
 Print["n= ", n, "  n^2 * min= ", n^2 * m[[1]]]]
```

```
n= 5000  min= 1.000171304137127467738765

n= 6000  min= 1.000141578311702395389296

n= 7000  min= 1.000120511212464176935035

n= 8000  min= 1.000104821169033464591794

n= 9000  min= 1.000092692619840969988359

n= 10000  min= 1.000083043161272667688070
```

6. On the " higher σ–quotients ".

So far we discussed the log–convexity in s of the functions $\frac{\sigma_{n+1}(s)}{\sigma_n(s)}$ and $\frac{\sigma_{n+1}{}^2(s)}{\sigma_n(s)\,\sigma_{n+2}(s)}$.
We will now have a look at the " higher σ–quotients ".

With $\sigma_n := \sigma_n(s) := 1^s + 2^s + 3^s + \ldots + n^s$ we found experimentally that
all functions $q_1 := q_1(s) := \frac{\sigma_{n+1}(s)}{\sigma_n(s)}$ (with $n \in \mathbb{N}$) are log–convex in s
on the entire real s–axis. This is the case $m = 1$ below.
The next case ($m = 2$) will be $q_2 := q_2(s) := \frac{\sigma_{n+1}^2(s)}{\sigma_n(s)\,\sigma_{n+2}(s)}$.
It seems that also *all* these functions are log–convex in s on all of $\mathbb{R}$.

The next case ($m = 3$) is $q_3 := q_3(s) := \frac{\sigma_{n+1}^3(s)\,\sigma_{n+3}}{\sigma_n(s)\,\sigma_{n+2}^3(s)}$, and after this we get
($m = 4$) $q_4 := q_4(s) := \frac{\sigma_{n+1}^4(s)\,\sigma_{n+3}^4(s)}{\sigma_n(s)\,\sigma_{n+2}^6(s)\,\sigma_{n+4}(s)}$.
One will easily recognize the general pattern with binomial coefficients.
In general we get in the *m*–th case (after taking logarithms)

(6.1) $$f := f_m := \log q_m := \log q_m(s) := -\sum_{i=0}^{m} (-1)^i \binom{m}{i} \log \sigma_{n+i}(s).$$

After performing some experiments we found that all functions $f := \log q_m(s)$ are
convex in s on all of $\mathbb{R}$ indeed, *save* for possibly a " few " exceptions for small n.

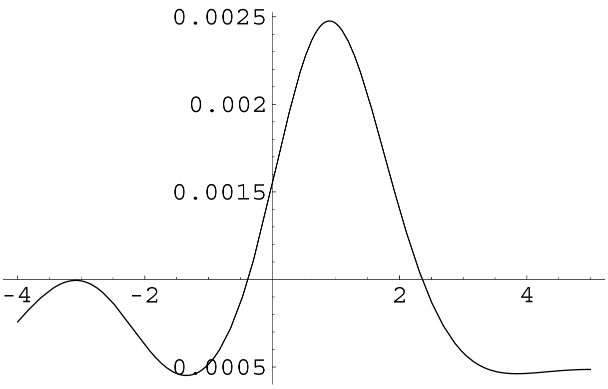
Things seem to be OK, though, for
$m = 1$ $n \geq 1$ (= our first Conjecture)
$m = 2$ $n \geq 1$ (= our second Conjecture)

$m = 3$ $n \geq 1$ (hence, no exceptions here either)
$m = 4$ $n \geq 2$
$m = 5$ $n \geq 2$
$m = 6$ $n \geq 3$
$m = 7$ $n \geq 4$
$m = 8$ $n \geq 4$
$m = 9$ $n \geq 5$
$m = 10$ $n \geq 6$
$m = 11$ $n \geq 6$
$m = 12$ $n \geq 7$
$m = 13$ $n \geq 7$
$m = 14$ $n \geq 8$
$m = 15$ $n \geq 9$
$m = 16$ $n \geq 9$
$m = 17$ $n \geq 10$
$m = 18$ $n \geq 10$

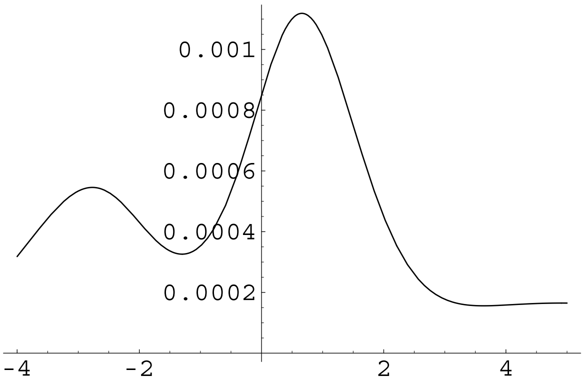
For larger values of m the computations (and plots) become rather fuzzy. So, we stopped the verification here.
One may want to experiment further with the following *Mathematica* program.
The resulting plots will suggest / reveal / indicate the most critical s–intervals.
One should pay special attention to the interval $2 < s < 4$ (for n not too small).

```
m = 4; (*  Here one may take  m = 1, 2, 3, 4, ...   *)
For[n = 3, n ≤ 6, n++,
 f = - Sum[(-1)^i * Binomial[m, i] * Log[Sum[ k^s , {k, n + i}]], {i, 0, m}];
f1 = ∂s f; f2 = ∂s f1;
 μ = FindMinimum[ f2, {s, -2, -1}, AccuracyGoal → 20, WorkingPrecision → 30];
 Print["↓  m= ", m, "     n= ", n, "     μ= ", μ[[1]]]
  Plot[ f2, {s, -4, 5}];
(*  One should also have a look at other s-intervals !  *)]
```

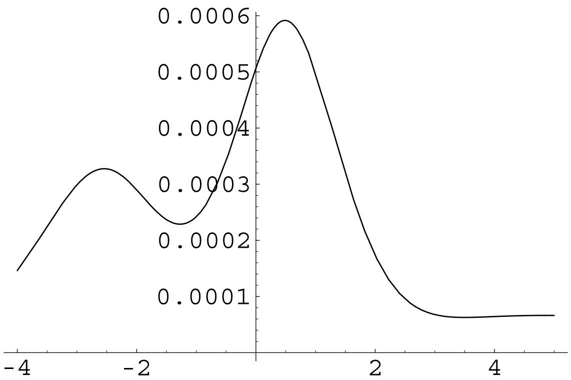
↓ m= 4 n= 3 μ= 0.0004519846914273



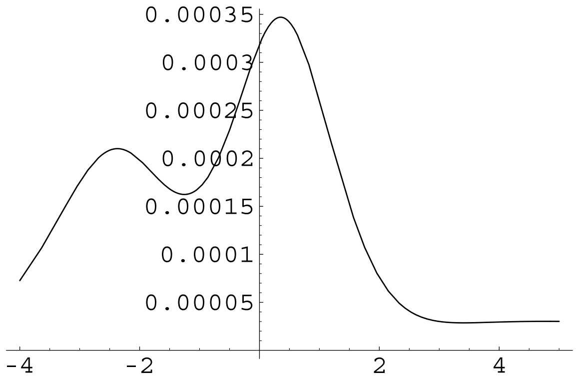
↓ m= 4 n= 4 μ= 0.0003259991033577898142726933



↓ m= 4 n= 5 μ= 0.000228835561796304719453869 5



↓ m= 4 n= 6 μ= 0.000162378658911971447310003



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