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Realization theory for linear and bilinear hybrid systems

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ABSTRACT
The paper deals with the realization theory of linear and bilinear hybrid systems, i.e. hybrid systems with continuous dynamics determined by linear (respectively bilinear) control systems. We will formulate necessary and sufficient conditions for the existence of a linear (bilinear) hybrid system realizing the specified input/output maps. We will also present a characterization of a minimal linear (bilinear) hybrid realization and a procedure to convert a linear (bilinear) hybrid system to a minimal one. Partial realization of linear (bilinear) hybrid systems will be discussed too.

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Realization Theory For Linear and Bilinear Hybrid Systems

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Abstract

The paper deals with the realization theory of linear and bilinear hybrid systems, i.e. hybrid systems with continuous dynamics determined by linear (respectively bilinear) control systems. We will formulate necessary and sufficient conditions for the existence of a linear (bilinear) hybrid system realizing the specified input/output maps. We will also present a characterization of a minimal linear (bilinear) hybrid realization and a procedure to convert a linear (bilinear) hybrid system to a minimal one. Partial realization of linear (bilinear) hybrid systems will be discussed too.

1 Introduction

Realization theory is one of central topics of systems theory. Apart from its theoretical relevance, realization theory has the potential of being applied for developing control and identification methods, as development of linear systems theory has demonstrated.

The current paper develops realization theory for two special classes of hybrid systems called *linear hybrid systems* and *bilinear hybrid systems*. A linear (bilinear) hybrid system is a hybrid system such that the continuous dynamics at each location is determined by a continuous time linear (bilinear) control system and the system switches from one discrete location to another whenever an external discrete input event takes place. The automaton specifying the discrete-state transition is assumed to be *deterministic*. Discrete events act as discrete inputs, one can specify arbitrary sequence of them arriving at any time instant. There are no guards and the reset maps are assumed to be linear. The inputs of a linear (bilinear) hybrid system are of two types. Piecewise-continuous inputs are fed to the linear (bilinear) system belonging to the current discrete location. Timed sequences of discrete events determine the relative arrival times and relative order of external events which trigger transition of discrete states. The outputs of the linear (bilinear) hybrid system consist of the continuous outputs of the underlying linear systems and the discrete outputs of the discrete states. The class of linear hybrid systems studied here is completely different from linear hybrid systems (linear hybrid automaton) from [6]. The class of hybrid systems studied in this paper bears a certain resemblance to linear switched systems [15], except that in [15] the external discrete events are viewed as disturbances not as inputs and the finite state automaton is non-deterministic.

The paper presents a solution to the following problems.

1. *Reduction to a minimal realization* Consider a linear (bilinear) hybrid system H, and a subset of its input-output maps Φ. Find a minimal linear (bilinear) hybrid system

which realizes Φ.

2. *Existence of a realization* Find necessary and sufficient condition for existence of a linear (bilinear) hybrid system realizing a specified set of input-output maps.

3. *Partial realization* Find a procedure for constructing a linear (bilinear) hybrid system realization of a set of input-output maps from finite data.

The following results are presented in the paper.

- A linear (bilinear) hybrid system is a minimal realization of a set of input-output maps if and only if it is observable and semi-reachable. Minimal linear (bilinear) hybrid systems which realize a given set of input-output maps are unique up to isomorphism. Each linear (bilinear) hybrid system H realizing a set of input-output maps Φ can be transformed to a minimal realization of Φ.

- A set of input/output maps is realizable by a linear hybrid system if and only if it has a *hybrid kernel representation*, the rank of its Hankel-matrix is finite, the discrete parts of the input/output maps are realizable by a finite Moore-automaton and certain other finiteness conditions hold. A set of input/output maps is realizable by a bilinear hybrid system if and only if it has a *hybrid Fliess-series expansion*, the rank of its Hankel-matrix is finite and the discrete parts of the input/output maps are realizable by a finite Moore-automaton. There is a procedure to construct the linear (bilinear) hybrid system realization from the columns of the Hankel-matrix, and this procedure yields a minimal realization.

- There exists a procedure which constructs a linear (bilinear) hybrid system realization from finite data. Under certain conditions, similar to those for linear and bilinear systems, this realization is a minimal realization of the specified input-output maps.

Earlier works on realization theory dealt with realization theory of linear switched systems (hybrid systems of the type described in [11]) , see [12, 13]. There is a strong link between the notion of minimal realization and the notion of biggest bisimulation. In fact, for deterministic systems the biggest bisimulation coincides with the indistinguishability relation. For more on bisimulation for hybrid systems see [15, 17, 10]. The main tool used in the paper is the theory of formal power series. The connection between realization theory and formal power series has been explored in several paper, see [8, 16, 7]. The theory of partial realization is analogous to that of for linear and bilinear systems, see [5, 9].

The outline of the paper is the following. The first section, Section 2, sets up some notation which will be used throughout the paper. Section 4 contains the necessary results on formal power series. Section 3 presents realization theory of finite Moore-automata. Section 5 describes the notion of linear hybrid systems. Section 5.1 presents certain properties of the input-output maps generated by linear hybrid systems. Finally, Subsection 5.2 develops realization theory for linear hybrid systems. Subsection 5.3 presents partial realization theory for linear hybrid systems and algorithms for checking observability and reachability as well as computing minimal realization. Section 6 introduces the notion of bilinear hybrid systems. Subsection 6.1 presents certain properties of the input-output maps generated by bilinear hybrid systems. Subsection 6.2 develops realization theory for bilinear hybrid systems. The last section, Subsection 6.3, discusses the relationship between bilinear hybrid systems and linear hybrid systems. It turns out that the class of input-output maps generated by linear hybrid systems is contained in the class of input-output maps generated by bilinear hybrid systems.

2 Preliminaries

For an interval $A \subseteq \mathbb{R}$ and for a suitable set X denote by $PC(A, X)$ the set of piecewise-continuous maps from A to X, i.e., maps which have at most finitely many points of discontinuity on any bounded interval and at any point of discontinuity the left-hand and the right-hand side limits exist and are finite. For a set Σ denote by Σ^* the set of finite strings of elements of Σ. For $w = a_1 a_2 \cdots a_k \in \Sigma^*$, $a_1, a_2, \ldots, a_k \in \Sigma$ the length of w is denoted by $|w|$, i.e. $|w| = k$. The empty sequence is denoted by ϵ. The length of ϵ is zero: $|\epsilon| = 0$. Let $\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$. The concatenation of two strings $v = v_1 \cdots v_k, w = w_1 \cdots w_m \in \Sigma^*$ is the string $vw = v_1 \cdots v_k w_1 \cdots w_m$. We denote by w^k the string $\underbrace{w \cdots w}_{k-times}$. The word w^0 is just the empty word ϵ. Denote by T the set $[0, +\infty) \subseteq \mathbb{R}$. Denote by $\mathbb{N}$ the set of natural numbers including 0. Denote by $F(A, B)$ the set of all functions from the set A to the set B. For any two sets A, B, define the functions $\Pi_A : A \times B \to A$ and $\Pi_B : A \times B \to B$ by $\Pi_A(a, b) = a$ and $\Pi_B(a, b) = b$. By abuse of notation we will denote any constant function $f : T \to A$ by its value. That is, if $f(t) = a \in A$ for all $t \in T$, then f will be denoted by a. For any function f the range of f will be denoted by $\mathrm{Im} f$. If A, B are two sets, then the set $(A \times B)^*$ will be identified with the set $\{(u, w) \in A^* \times B^* \mid |u| = |w|\}$. For any set A we will denote by $card(A)$ the cardinality of A. For any two sets J, X an *indexed subset* of X with the *index set* J is simply a map $Z : J \to X$, denoted by $Z = \{a_j \in X \mid j \in J\}$, where $a_j = Z(j), j \in J$. Let $f : A \times (B \times C)^+ \to D$. Then for each $a \in A$, $w \in B^+$ we define the function $f(a, w, .) : C^{|w|} \to D$ by $f(a, w, .)(v) = f(a, (w, v)), v \in C^{|w|}$. By abuse of notation we denote $f(a, w, .)(v)$ by $f(a, w, v)$. Denote by $\mathbb{N}^k$ the set of k tuples of non-negative integers. If $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ and $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}^m$, then $(\alpha, \beta) = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m) \in \mathbb{N}^{k+m}$. Let $\phi : \mathbb{R}^k \to \mathbb{R}^p$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$. We define $D^\alpha \phi$ by

$$D^\alpha \phi = \frac{d^{\alpha_1}}{dt_1^{\alpha_1}} \frac{d^{\alpha_2}}{dt_2^{\alpha_2}} \cdots \frac{d^{\alpha_k}}{dt_k^{\alpha_k}} \phi(t_1, t_2, \ldots, t_k)|_{t_1 = t_2 = \cdots = t_k = 0}.$$

For each $f : T \to A$, A an arbitrary set, and for each $\tau \in T$ denote by $\mathrm{Shift}_\tau(f)$ the map

$$\mathrm{Shift}_\tau(f) : T \ni t \mapsto f(\tau + t)$$

3 Finite Moore-automaton

A *finite Moore-automaton* is a tuple $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ where Q, Γ are finite sets, $\delta : Q \times \Gamma \to Q$, $\lambda : Q \to O$. The set Q is called the *state-space*, O is called the *output space* and Γ is called the *input space*. The function δ is the *state-transition map*, λ is the *readout map*. Denote by $card(\mathcal{A})$ the cardinality of the state-space Q of $\mathcal{A}$, i.e. $card(\mathcal{A}) = card(Q)$.

Define the functions $\widetilde{\delta} : Q \times \Gamma^* \to Q$ and $\widetilde{\lambda} : Q \times \Gamma^* \to O$ as follows. Let $\widetilde{\delta}(q, \epsilon) = q$ and

$$\widetilde{\delta}(q, w\gamma) = \delta(\widetilde{\delta}(q, w), \gamma), w \in \Gamma^*, \gamma \in \Gamma$$

Let $\widetilde{\lambda}(q, w) = \lambda(\widetilde{\delta}(q, w)), w \in \Gamma^*$. By abuse of notation we will denote $\widetilde{\delta}$ and $\widetilde{\lambda}$ simply by δ and λ respectively.

Let $\mathcal{D} = \{\phi_j \in F(\Gamma^*, O) \mid j \in J\}$ be an indexed set of functions. A pair $(\mathcal{A}, \zeta)$ is said to be *an automaton realization* of $\mathcal{D}$ if $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$, $\zeta : J \to Q$ and

$$\lambda(\zeta(j), w) = \phi_j(w), \forall w \in \Gamma^*, j \in J$$

An automaton $\mathcal{A}$ is said to be a realization of $\mathcal{D}$ if there exists a $\zeta : J \to Q$ such that $(\mathcal{A}, \zeta)$ is a realization of $\mathcal{D}$.

Let $(\mathcal{A}, \zeta)$ and $(\mathcal{A}^{'}, \zeta^{'})$ be two automaton realizations. Assume that $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ and $\mathcal{A}^{'} = (Q^{'}, \Gamma, O, \delta^{'}, \lambda^{'})$. A map $\phi : Q \to Q^{'}$ is said to be an *automaton morphism* from $(\mathcal{A}, \zeta)$ to $(\mathcal{A}^{'}, \zeta^{'})$, denoted by $\phi : (\mathcal{A}, \zeta) \to (\mathcal{A}^{'}, \zeta^{'})$ if $\phi(\delta(q, \gamma)) = \delta^{'}(\phi(q), \gamma), \forall q \in Q, \gamma \in \Gamma$, $\lambda(q) = \lambda^{'}(\phi(q)), \forall q \in Q$, $\phi(\zeta(j)) = \zeta^{'}(j), j \in J$. An automaton realization $(\mathcal{A}, \zeta)$ of $\mathcal{D}$ is called *minimal* if for each automaton realization $(\mathcal{A}^{'}, \zeta^{'})$ of $\mathcal{D}$ $card(\mathcal{A}) \leq card(\mathcal{A}^{'})$. Let $\phi : \Gamma^* \to O$. For every $w \in \Gamma^*$ define $w \circ \phi : \Gamma^* \to O$*–the left shift of* ϕ *by* w as $w \circ \phi(v) = \phi(wv)$. For $\mathcal{D} = \{\phi_j \in F(\Gamma^*, O) \mid j \in J\}$ define the set $W_{\mathcal{D}} \subseteq F(\Gamma^*, O)$ by

$$W_{\mathcal{D}} = \{w \circ \phi_j : \Gamma^* \to O \mid w \in \Gamma^*, j \in J\}$$

An automaton $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ is called *reachable from* $Q_0 \subseteq Q$, if

$$\forall q \in Q : \exists w \in \Gamma^*, q_0 \in Q_0 : q = \delta(q_0, w)$$

A realization $(\mathcal{A}, \zeta)$ is called reachable if $\mathcal{A}$ is reachable from $\mathrm{Im}\zeta$. A realization $(\mathcal{A}, \zeta)$ is called *observable* or *reduced*, if

$$\forall q_1, q_2 \in Q : [\forall w \in \Gamma^* : \lambda(q_1, w) = \lambda(q_2, w)] \implies q_1 = q_2$$

The following result is a simple reformulation of the well-known properties of realizations by automaton. For references see [4].

Theorem 1. *Let* $\mathcal{D} = \{\phi_j \in F(\Gamma^*, O) \mid j \in J\}$. $\mathcal{D}$ *has a realization by a finite Moore-automaton if and only if* $W_{\mathcal{D}}$ *is finite. In this case a realization of* $\mathcal{D}$ *is given by* $(\mathcal{A}_{can}, \zeta_{can})$ *where* $\mathcal{A} = (W_{\mathcal{D}}, \Gamma, O, L, T)$, $\zeta_{can}(j) = \phi_j$ *and*

$$L(\phi, \gamma) = \gamma \circ \phi,\ T(\phi) = \phi(\epsilon),\ \phi \in W_{\mathcal{D}}, \gamma \in \Gamma$$

The realization $(\mathcal{A}_{can}, \zeta_{can})$ *is reachable and observable.*

Theorem 2. *Let* $(\mathcal{A}, \zeta)$ *be a finite Moore-automaton realization of* $\mathcal{D} = \{\phi_j \in F(\Gamma^*, O) \mid j \in J\}$. *The following are equivalent:*

- $(\mathcal{A}, \zeta)$ *is minimal,*
- $(\mathcal{A}, \zeta)$ *is reachable and observable,*
- $card(\mathcal{A}) = card(W_{\mathcal{D}})$,
- *For each reachable realization* $(\mathcal{A}^{'}, \zeta^{'})$ *of* $\mathcal{D}$ *there exists a surjective automaton morphism* $T : (\mathcal{A}^{'}, \zeta^{'}) \to (\mathcal{A}, \zeta)$. *In particular, all minimal realizations of* $\mathcal{D}$ *are isomorphic*

The realization $(\mathcal{A}_{can}, \zeta_{can})$ is minimal.

For each map $\phi : \Gamma^* \to O$ and for each $N \in \mathbb{N}$ define

$$\phi_N = \phi|_{\{w \in \Gamma^* \mid |w| \leq N\}}$$

Let $\mathcal{D} = \{\phi_j \in F(\Gamma^*, O) \mid j \in J\}$. Let $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda), \zeta : J \to Q$. The pair $(\mathcal{A}, \zeta)$ is said to be *N-partial realization of* $\mathcal{D}$ if

$$\forall w \in \Gamma^*, |w| \leq N : \lambda(\zeta(j), w) = \phi_j(w)$$

For each $N, M > 0$ define

$$W_{\mathcal{D},N,M} = \{(w \circ \phi_j)_M \mid j \in J, w \in \Gamma^*, |w| \leq N\}$$

Theorem 3 (Partial realization by automaton). *With the notation above the following holds.*

- *If* $(\mathcal{A}, \zeta)$ *is a realization of* Φ *and* $card(\mathcal{A}) \leq N$ *, then*

$$card(W_{\mathcal{D},N,N}) = card(W_{\mathcal{D}})$$

- *If* $card(W_{\mathcal{D},N,N+1}) = card(W_{\mathcal{D},N+1,N}) = card(W_{\mathcal{D},N,N})$*, then* $(\mathcal{A}_N, \zeta_N)$ *is an* N*-partial realization of* $\mathcal{D}$*, where*

$$\mathcal{A}_N = (W_{\mathcal{D},N,N}, \Gamma, O, \delta, \lambda)$$

where for each $w \in \Gamma^*, |w| \leq N, j \in J$ $\delta((w \circ \phi_j)_N, x) = (wx \circ \phi_j)_N$, $\forall f \in W_{\mathcal{D},N,N}$: $\lambda(f) = f(\epsilon)$, $\forall j \in J, \zeta(j) = \phi_j|_N$,

- *If* $\mathcal{D}$ *has a realization* $(\mathcal{A}, \zeta)$ *such that* $N \geq card(\mathcal{A})$*, then* $(\mathcal{A}_N, \zeta_N)$ *is a minimal realization of* $\mathcal{D}$*.*

4 Formal Power Series

The section presents the necessary results on formal power series. For more on the classical theory of rational formal power series, see [1, 16]. The results of the current section are extensions of the classical ones. Most of material of the current section can be found in [13, 14].

Let X be a finite alphabet. A *formal power series* S with coefficients in $\mathbb{R}^p$ is a map

$$S : X^* \to \mathbb{R}^p$$

We denote by $\mathbb{R}^p \ll X^* \gg$ the set of all formal power series with coefficients in $\mathbb{R}^p$. An indexed set of formal power series $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ is called *rational* if there exists a vector space $\mathcal{X}$ over $\mathbb{R}$, $\dim \mathcal{X} < +\infty$, linear maps

$$C : \mathcal{X} \to \mathbb{R}^p,\ A_\sigma : \mathcal{X} \to \mathcal{X},\ \sigma \in X$$

and an indexed set $B = \{B_j \in \mathcal{X} \mid j \in J\}$ of elements of $\mathcal{X}$ such that for all $\sigma_1, \ldots, \sigma_k \in X, k \geq 0$,

$$S_j(\sigma_1 \sigma_2 \cdots \sigma_k) = C A_{\sigma_k} A_{\sigma_{k-1}} \cdots A_{\sigma_1} B_j.$$

The 4-tuple $R = (\mathcal{X}, \{A_x\}_{x \in X}, B, C)$ is called a *representation* of S. The number $\dim \mathcal{X}$ is called the *dimension* of the representation R and it is denoted by $\dim R$. In the sequel the following short-hand notation will be used $A_w := A_{w_k} A_{w_{k-1}} \cdots A_{w_1}$ for $w = w_1 \cdots w_k$. A_ϵ is the identity map. A representation R_{min} of Ψ is called *minimal* if for each representation R of Ψ it holds that $\dim R_{min} \leq \dim R$. It is easy to see that if Ψ rational and $\Psi^{'} \subseteq \Psi$, then $\Psi^{'}$ is rational.

Define $w \circ S \in \mathbb{R}^p \ll X^* \gg$ – the *left shift of* S *by* w by

$$\forall v \in X^* : w \circ S(v) = S(wv)$$

The following statements are generalizations of the results on rational power series from [1]. Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$. Define$W_\Psi$ by

$$W_\Psi = \mathrm{Span}\{w \circ S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J, w \in X^*\}$$

Define the Hankel-matrix H_Ψ of Ψ as $H_\Psi \in \mathbb{R}^{(X^* \times I) \times (X^* \times J)}$, $I = \{1, 2, \ldots, p\}$ and

$$(H_\Psi)_{(u,i)(v,j)} = (S_j)_i(vu)$$

Notice that $\dim W_\Psi = \mathrm{rank}\ H_\Psi$.

Theorem 4. *Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$. The following are equivalent.*

(i) Ψ *is rational.*

(ii) $\dim W_\Psi = \text{rank } H_\Psi < +\infty$,

(iii) *The tuple* $R_\Psi = (W_\Psi, \{A_\sigma\}_{\sigma \in X}, B, C)$, *where* $A_\sigma : W_\Psi \to W_\Psi$, $A_\sigma(T) = \sigma \circ T$, $B = \{B_j \in W_\Psi \mid j \in J\}$, $B_j = S_j$ *for each* $j \in J$, $C : W_\Psi \to \mathbb{R}^p$, $C(T) = T(\epsilon)$, *defines a representation of* Ψ.

The representation R_Ψ is called *free*. Since the linear space spanned by the column vectors of H_Ψ and the space W_Ψ are isomorphic, one can construct a representation of Ψ over the space of column vectors of H_Ψ in a way similar to the construction of R_Ψ.

Let $R = (\mathcal{X}, \{A_\sigma\}_{\sigma \in X}, B, C)$ be a representation of Ψ. Define the subspaces W_R and O_R of $\mathcal{X}$ by

$$W_R = \text{Span}\{A_w B_j \mid w \in X^*, j \in J\} \text{ and } O_R = \bigcap_{w \in X^*} \ker C A_w$$

A representation R is called *observable*, if $O_R = \{0\}$. A representation R is called *reachable*, if $\dim R = \dim W_R$.

It is easy to see that if $n = \dim \mathcal{X}$, then

$$O_R = \bigcap_{w \in X^*, |w| \le n} \ker C A_w \text{ and } W_R = \text{Span}\{A_w B_j \mid j \in J, |w| \le n\}$$

That is, if J is a finite set, then observability and reachability of representations can be checked by checking whether certain finite matrices are of full rank. Moreover, if R is a representation of Ψ, then R can be transformed to a reachable representation of Ψ:

$$R_r = (W_R, \{A_\sigma|_{W_R}\}_{\sigma \in X}, B, C|_{W_R})$$

It can also be transformed to an observable representation of Ψ:

$$R_o = (\mathcal{X}/O_R, \{A_x^{obs}\}_{x \in X}, B^{obs}, C^{obs})$$

where $C^{obs}(x + O_R) = Cx$, $B_j^{obs} = B_j + O_R$, $A_\sigma(x + O_R) = A_\sigma x + O_R$. The constructions above are computable from R if J is finite.

Let $R = (\mathcal{X}, \{A_x\}_{x \in X}, B, C)$ and $\widetilde{R} = (\widetilde{\mathcal{X}}, \{\widetilde{A}_x\}_{x \in X}, \widetilde{B}, \widetilde{C})$ be two representations of Ψ. Then a linear map $T : \widetilde{\mathcal{X}} \to \mathcal{X}$ is called a *representation morphism* from $\widetilde{R}$ to R, denoted by $T : \widetilde{R} \to R$, if

$$T\widetilde{A}_x = A_x T, (x \in X) \quad T\widetilde{B}_j = B_j, (j \in J), \quad \widetilde{C} = CT$$

The representation morphism T is said to be injective (surjective), if it is an injective (surjective) linear map. A representation isomorphism is simply a bijective representation morphism. Two representations are said to be isomorphic, if there exists a representation isomorphism between them.

Let $R = (\mathcal{X}, \{A_x\}_{x \in X}, B, C)$ be a representation and let $W \subseteq \mathcal{X}$ be a linear subspace of $\mathcal{X}$. R is said to be W-*observable*, if $W \cap O_R = \{0\}$. It is clear that if R is observable, then R is W-observable for any subspace W. It is also easy to see that if R is W-observable and $T : R \to R^{'}$ is a representation morphism then $T|_W$ is an injective linear map.

Theorem 5 (Minimal representation). *Let* $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$. *The following are equivalent.*

(i) R_{min} *is a minimal representation of* Ψ,

(ii) R_{min} *is reachable and observable,*

(iii) $\text{rank } H_\Psi = \dim W_\Psi = \dim R_{min}$,

(iv) If R *is a reachable representation of* Ψ, *then there exists a surjective representation morphism* $T : R \to R_{min}$.

In particular, if R *is a minimal representation, then* T *is a representation isomorphism.*

Using the theorem above it is easy to check that the free representation R_Ψ is minimal. One can also give a procedure, similar to reachability and observability reduction for linear systems, such that the procedure transforms any representation of Ψ to a minimal representation of Ψ. If $R = (\mathcal{X}, \{A_\sigma\}_{\sigma\in\Sigma}, B, C)$ is a representation of Ψ, then for any vector space isomorphism $T : \mathcal{X} \to \mathbb{R}^n$, $n = \dim R$, the tuple $R^{'} = (\mathbb{R}^n, \{TA_\sigma T^{-1}\}_{\sigma\in\Sigma}, TB, CT^{-1})$ is also a representation of Ψ. It is easy to see that R is minimal if and only if $R^{'}$ is minimal. From now on, we will silently assume that $\mathcal{X} = \mathbb{R}^n$ holds for any representation considered.

For each formal power series $S \in \mathbb{R}^p \ll X^* \gg$ and for each $N \in \mathbb{N}$ define $S_N = S_{\{w\in X^*, |w|\le N\}}$. Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ and let $R = (\mathcal{X}, \{A_x\}_{x\in X}, C, B)$, $B = \{B_j \in \mathcal{X} \mid j \in J\}$ be a representation. The representation R is said to be an N-*partial representation of* Ψ if

$$\forall j \in J, \forall w \in X^*, |w| \le N : S_j(w) = CA_w B_j$$

Define the sets $I_M = \{(v, i) \mid v \in X^*, |v| \le M, i = 1, \dots, p\}$, $J_N = \{(u, j) \mid j \in J, u \in X^*, |u| \le N\}$. Define $H_{\Psi,N,M} \in \mathbb{R}^{I_M \times J_N}$ by

$$(H_{\Psi,N,M})_{(v,i),(u,j)} = ((S_j(uv))_i)$$

Notice that $H_{\Psi,N,M}$ is a finite matrix, if J is finite. Define

$$W_{\Psi,N,M} = \text{Span}\{(w \circ S_j)|_M \mid w \in X^*, |w| \le N, j \in J\}$$

Notice that $\text{rank } H_{\Psi,N,M} = \dim W_{\Psi,N,M}$.

Theorem 6 (Partial representation). *(i) If* R *is a representation of* Ψ, $\dim R \le N$, *then* $\text{rank } H_\Psi = \text{rank } H_{\Psi,N,N}$

(ii) If $\text{rank } H_{\Psi,N,N} = \text{rank } H_{\Psi,N,N+1} = \text{rank } H_{\Psi,N+1,N}$, *then there exists an* N-*representation* R_N *of* Ψ, *such that*

$$R_N = (W_{\Psi,N,N}, \{A_x\}_{x\in X}, C, B)$$

and it holds that $A_x((w \circ S_j)_N) = (wx \circ S_j)_N$, $C(T) = T(\epsilon)$, $B_j = (S_j)_N$ *for each* $j \in J$ *and* $x \in X, w \in X^*, |w| \le N$.

(iii) If Ψ *has a representation* R *such that* $N \ge \dim R$, *then* R_N *is a minimal representation of* Ψ.

The theorem above implies that if J is finite and we know that Ψ has a representation of dimension at most N, then a minimal representation of Ψ can be computed from finite data.

5 Linear Hybrid Systems

This section contains the definition and elementary properties of linear hybrid system. The notation and notions described in this section are largely based on [12].

Definition 1 (Linear hybrid systems). *A linear hybrid system (abbreviated as* HLS *) is a tuple* $H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, B_q, C_q)_{q \in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})$ *where* $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ *is a finite Moore-automaton,* $\mathcal{X}_q = \mathbb{R}^{n_q}$*,* $\mathcal{U} = \mathbb{R}^m$*,* $\mathcal{Y} = \mathbb{R}^p$ *for some* $n_q, p, m > 0, q \in Q$ *and* $A_q : \mathcal{X}_q \to \mathcal{X}_q$*,* $B_q : \mathcal{U} \to \mathcal{X}_q$*,* $C_q : \mathcal{X}_q \to \mathcal{Y}$ *and* $M_{q_1,\gamma,q_2} : \mathcal{X}_{q_2} \to \mathcal{X}_{q_1}$ *are linear maps.*

Let $\mathcal{H} = \bigcup_{q \in Q} \{q\} \times \mathcal{X}_q$. Let $\mathcal{X} = \bigoplus_{q \in Q} \mathcal{X}_q$, $\mathcal{A}_H = \mathcal{A}$. The inputs of the linear hybrid system H are functions from $PC(T, \mathcal{U})$ and sequences from $(\Gamma \times T)^*$.

The interpretation of a sequence $(\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*$ is the following. The event γ_i took place *after* the event γ_{i-1} and t_{i-1} is the *elapsed time between the arrival of* γ_{i-1} *and the arrival of* γ_i. That is, t_i is the difference of the arrival times of γ_i and γ_{i-1}. Consequently, $t_i \geq 0$ but we allow $t_i = 0$, that is, we allow γ_i to arrive instantly after γ_{i-1}. If $i = 1$, then t_1 is simply the time when the event γ_1 arrived.

The *state trajectory* of the system H is a map

$$\xi_H : \mathcal{H} \times PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T \to \mathcal{H}$$

of the following form. For each $u \in PC(T, \mathcal{U})$, $w = (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*$, $t_{k+1} \in T$, $h_0 = (q_0, x_0) \in H$ it holds that

$$\xi_H(h_0, u, w, t_{k+1}) = (\delta(q_0, \gamma_1 \cdots \gamma_k), x_H(h_0, u, w, t_{k+1}))$$

where the map $x : T \ni t \mapsto x_H(h_0, u, w, t)$ is the solution of the differential equation

$$\frac{d}{dt} x(t) = A_{q_k} x(t) + B_{q_k} u(t + \sum_1^k t_j)$$

where $q_i = \delta(q_0, \gamma_1 \cdots \gamma_i)$, $i = 1, \ldots, k$ and

$$x(0) = x_H(h_0, u, w, 0) = M_{q_k, \gamma_k, q_{k-1}} x_H(x_0, u, (\gamma_1, t_1) \ldots (\gamma_{k-1}, t_{k-1}), t_k)$$

if $k > 0$ and $x(0) = x_0$ if $k = 0$. In fact,

$$\begin{aligned} x_H(h_0, u, w, t_{k+1}) = & \exp(A_{q_k} t_{k+1}) M_{q_k, \gamma_k, q_{k-1}} \exp(A_{q_{k-1}} t_k) \cdots \\ & \cdots M_{q_1, \gamma_1, q_0} \exp(A_{q_0} t_1) x_0 + \\ & + \sum_{i=0}^{k} \exp(A_{q_k} t_{k+1}) M_{q_k, \gamma_k, q_{k-1}} \exp(A_{q_{k-1}} t_k) \cdots M_{q_{i+1}, \gamma_i, q_i} \times \\ & \times \int_0^{t_{i+1}} \exp(A_{q_i}(t_{i+1} - s)) B_{q_i} u_i(s) ds \end{aligned}$$

where $q_{i+1} = \delta(q_i, \gamma_{i+1})$, $u_i(s) = u(\sum_{j=1}^{i} t_j + s)$, $0 \leq i \leq k$ Define the set

$$Reach(\Sigma, \mathcal{H}_0) = \{x_H(h_0, u, w, t) \in \mathcal{X} \mid u \in PC(T, \mathcal{U}), w \in (\Gamma \times T)^*, t \in T, h_0 \in \mathcal{H}_0\}$$

H is *semi-reachable* from $\mathcal{H}_0$ if $\mathcal{X}$ is the vector space of the smallest dimension containing $Reach(H, \mathcal{H}_0)$ and the automaton $\mathcal{A}_H$ is reachable from $\Pi_Q(\mathcal{H}_0)$. Define the function $\upsilon_H : \mathcal{H} \times PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T \to \mathcal{Y} \times O$ by

$$\upsilon_H((q_0, x_0), u, (w, \tau), t) = (\lambda(q_0, w), C_q x_H((q_0, x_0), u, (w, \tau), t))$$

where $q = \delta(q_0, w)$. For each $h \in \mathcal{H}$ the *input-output map of the system H induced by h* is the function

$$\upsilon_H(h,.) : PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T \ni (u,(w,\tau),t) \mapsto \upsilon_H(h,u,(w,\tau),t) \in \mathcal{Y} \times O$$

Two states $h_1 \neq h_2 \in \mathcal{H}$ of the linear hybrid system H are *indistinguishable* if $\upsilon_H(h_1,.) = \upsilon_H(h_2,.)$. H is called *observable* if it has no pair of indistinguishable states. A set $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ is said to be *realized* by a linear hybrid system H if there exists $\mu : \Phi \to \mathcal{H}$ such that

$$\forall f \in \Phi : \ \upsilon_H(\mu(f),.,.) = f$$

Both H and (H,μ) are called a *realization* of Φ. Thus, H realizes Φ if and only if for each $f \in \Phi$ there exists a state $h \in \mathcal{H}$ such that $\upsilon_H(h,.) = f$. We say that a realization (H,μ) is observable if H is observable and we say that (H,μ) is semi-reachable if H is semi-reachable from $\mathrm{Im}\mu$.

For a linear hybrid system H from Definition 3 the dimension of H is defined as

$$\dim H = (card(Q), \sum_{q \in Q} \dim \mathcal{X}_q) \in \mathbb{N} \times \mathbb{N}$$

The first component of $\dim H$ is the cardinality of the discrete state-space, the second component is the sum of dimensions of the continuous state-spaces. For each $(m,n),(p,q) \in \mathbb{N} \times \mathbb{N}$ define the partial order relation $(m,n) \leq (p,q)$, if $m \leq p$ and $n \leq q$. A realization H of Φ is called a *minimal realization* of Φ, if for any realization $H^{'}$ of Φ:

$$\dim H \leq \dim H^{'}$$

The reason for defining the dimension of a linear hybrid system as above is that there is a trade-off between the number of discrete states and dimensionality of each continuous state-space component. That is, one can have two realizations of the same input/output maps, such that one of the realizations has more discrete states than the other, but its continuous state components are of smaller dimension than those of the other system.

Let (H,μ) and $(H^{'},\mu^{'})$ be two realizations

$$\begin{aligned} H &= (\mathcal{A},\mathcal{U},\mathcal{Y},(\mathcal{X}_q,A_q,B_q,C_q)_{q\in Q},\{M_{q_1,\gamma,q_2} \mid q_1,q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2,\gamma)\}) \\ H^{'} &= (\mathcal{A}^{'},\mathcal{U},\mathcal{Y},(\mathcal{X}^{'}_q,A^{'}_q,B^{'}_q,C^{'}_q)_{q\in Q^{'}},\{M^{'}_{q_1,\gamma,q_2} \mid q_1,q_2 \in Q^{'}, \gamma \in \Gamma, q_1 = \delta(q_2,\gamma)\}) \end{aligned}$$

where $\mathcal{A} = (Q,\Gamma,O,\delta,\lambda)$ and $\mathcal{A}^{'} = (Q^{'},\Gamma,O,\delta^{'},\lambda^{'})$. A pair $T = (T_D,T_C)$ is called an *O-morphism* from (H,μ) to $(H^{'},\mu^{'})$, denoted by $T : (H,\mu) \to (H^{'},\mu^{'})$, if the the following holds. $T_D : (\mathcal{A},\mu_D) \to (\mathcal{A}^{'},\mu^{'}_D)$, where $\mu_D(f) = \Pi_Q(\mu_D(f))$, $\mu^{'}_D(f) = \Pi_Q(\mu^{'}_D(f))$, is an automaton morphism and $T_C : \bigoplus_{q\in Q} \mathcal{X}_q \to \bigoplus_{q\in Q^{'}} \mathcal{X}^{'}_q$ is a linear morphism, such that

- $\forall q \in Q : T_C(\mathcal{X}_q) \subseteq \mathcal{X}^{'}_{T_D(q)}$,
- $T_C A_q = A^{'}_{T_D(q)} T_C \quad T_C B_q = B^{'}_{T_D(q)} \quad C_q = C^{'}_{T_D(q)} T_C$ for each $q \in Q$,
- $T_C M_{q_1,\gamma,q_2} = M^{'}_{T_D(q_1),\gamma,T_D(q_2)} T_C$, $\forall q_1,q_2 \in Q, \gamma \in \Gamma, \delta(q_2,\gamma) = q_1$,
- $T_C(\Pi_{\mathcal{X}_q}(\mu(f))) = \Pi_{\mathcal{X}^{'}_{T_D(q)}}(\mu^{'}(f))$ for each $q = \mu_D(f), f \in \Phi$.

The O-morphism T is said to be injective, surjective or bijective if both T_D and T_C are respectively injective, surjective and bijective. Bijective O-morphisms are called O-isomorphisms. Two linear hybrid system realizations are *isomorphic* if there exists an O-isomorphisms between them.

5.1 Input-output maps of linear hybrid systems

This section deals with properties of input-output maps of linear hybrid systems. Let $f \in F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ be an input-output map. Define $f_C = \Pi_{\mathcal{Y}} \circ f : PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T \to \mathcal{Y}$ and $f_D = \Pi_O \circ f : PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T \to O$.

Definition 2 (hybrid kernel representation). *A set $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ is said to have* hybrid kernel representation *if there exist functions $K^{f,\Phi}_w : \mathbb{R}^{k+1} \to \mathbb{R}^p$ and $G^{f,\Phi}_{w,j} : \mathbb{R}^j \to \mathbb{R}^{p\times m}$ for each $f \in \Phi, w \in \Gamma^*, |w| = k, j = 1, 2, \ldots, k+1$, such that*

1. *$\forall w \in \Gamma^*, \forall f \in \Phi, j = 1, 2, \ldots, |w|+1$: $K^{f,\Phi}_w$ is analytic and $G^{\Phi,f}_{w,j}$ is analytic*
2. *For each $f \in \Phi$, the function f_D depends only on Γ^*, i.e.*

$$\forall u_1, u_2 \in PC(T,\mathcal{U}), w \in \Gamma^*, \tau_1, \tau_2 \in T^{|w|}, t_1, t_2 \in T :$$
$$f_D(u_1, (w, \tau_1), t_1) = f_D(u_2, (w, \tau_2), t_2)$$

   The function f_D will be regarded as a function $f_D : \Gamma^* \to O$.

3. *For each $f \in \Phi$, $w = \gamma_1\gamma_2 \cdots \gamma_k \in \Gamma^*$, $t_{k+1} \in T$, $\gamma_1, \ldots, \gamma_k \in \Gamma$, $\underline{t} = (t_1, \ldots, t_k) \in T^k$:*

$$f_C(u, (w, \underline{t}), t_{k+1})) = K^{f,\Phi}_w(t_1, \ldots, t_k, t_{k+1}) + \\ + \sum_{i=0}^{k} \int_0^{t_{i+1}} G^{f,\Phi}_{w,k+1-i}(t_{i+1} - s, t_{i+2}, \ldots, t_{k+1})\sigma_i u(s) ds$$

   where $\sigma_j u(s) = u(s + \sum_{i=1}^{j} t_i)$.

Using the notation above, define for each $f \in \Phi$ the function $y^{f,\Phi}_0 : PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T \to \mathcal{Y}$ by

$$y^{f,\Phi}_0(u, (w, \underline{t}), t_{k+1}) = \sum_{i=0}^{k} \int_0^{t_{i+1}} G^{f,\Phi}_{w,k+1-i}(t_{i+1} - s, t_{i+2}, \ldots, t_{k+1})\sigma_i u(s) ds$$

where $\underline{t} = (t_1, \ldots, t_k)$. It follows that $y^{f,\Phi}_0(u, (w, \tau), t) = f_C(u, (w, \tau), t) - f_C(0, (w, \tau), t)$. If (H, μ) is a realization of Φ, then for each $f \in \Phi$, $y^{f,\Phi}_0 = \Pi_{\mathcal{Y}} \circ \upsilon_H((\Pi_Q(\mu(f)), 0), .)$. If the set Φ has a hybrid kernel representation, then the collection of analytic functions $\{K^{f,\Phi}_w, G^{f,\Phi}_{w,j} \mid w \in \Gamma^*, j = 1, 2, \ldots, |w|+1, f \in \Phi\}$ determines $\{f_C \mid f \in \Phi\}$. Since $K^{f,\Phi}_w$ is analytic, we get that the collection $\{D^\alpha K^{f,\Phi}_w, D^\beta G^{f,\Phi}_{w,j} \mid \alpha \in \mathbb{N}^{|w|}, \beta \in \mathbb{N}^j\}$ determines $K^{f,\Phi}_w$ and $G^{f,\Phi}_{w,j}$ locally.

For each $f \in \Phi$, $u \in PC(T,\mathcal{U}), w \in \Gamma^*$ define the maps

$$f_C(u, w, .) : T^{|w|+1} \ni (t_1, \ldots, t_{|w|+1}) \mapsto f_C(u, (w, t_1 \cdots t_{|w|}), t_{|w|+1})$$

and

$$y^{f,\Phi}_0(u, w, .) : T^{|w|+1} \ni (t_1, \ldots, t_{|w|+1}) \mapsto y^{f,\Phi}_0(u, (w, t_1 \cdots t_{|w|}), t_{|w|+1})$$

By applying the formula $\frac{d}{dt}\int_0^t f(t,\tau)d\tau = f(t,t) + \int_0^t \frac{d}{dt}f(t,\tau)d\tau$ and Definition 2 one gets

$$D^\alpha K^{f,\Phi}_w = D^\alpha f_C(0, w, .) \quad , D^\xi G^{f,\Phi}_{w,l} e_z = D^\beta y^{f,\Phi}_0(e_z, w, .)$$

where $w = \gamma_1 \cdots \gamma_k$, $l \le k+1$, $\mathbb{N}^{k+1} \ni \beta = (\underbrace{0, 0, \ldots, 0}_{k-l+1-times}, \xi_1 + 1, \xi_2, \ldots, \xi_l)$, and e_z is the zth unit vector of $\mathbb{R}^m$, i.e $e_z^T e_j = \delta_{zj}$. The formula above implies that all the high-order derivatives of the functions $K_w^{f,\Phi}, G_{w,j}^{f,\Phi}$ ($f \in \Phi$, $w \in \Gamma^*$, $j = 1, 2, \dots |w|+1$) at zero can be computed from high-order derivatives with respect to the relative arrival times of discrete events of the functions from Φ. With the notation above, using the principle of analytic continuation , from the discussion above one gets the following

Proposition 1. *Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$. The pair (H,μ), where $H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, B_q, C_q)_{q\in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2,\gamma)\})$, $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ is a realization of Φ if and only if Φ has a hybrid kernel representation and for each $w \in \Gamma^*$, $f \in \Phi$, $j = 1, 2, \dots, m$ and $\alpha \in \mathbb{N}^{|w|+1}$ the following holds*

$$\begin{aligned} D^\alpha y_0^{f,\Phi}(e_j, w, .) &= D^\beta G_{w,k+2-l}^{f,\Phi} e_j = \\ & \quad C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_l,\gamma_l,q_{l-1}} A_{q_{l-1}}^{\alpha_l - 1} B_{q_{l-1}} e_j \\ D^\alpha f_C(0, w, .) &= D^\alpha K_w^{f,\Phi} = C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} A_{q_0}^{\alpha_l} x_0 \\ f_D(w) &= \lambda(q_0, w) \end{aligned}$$

where $l = \min\{h \mid \alpha_h > 0\}$, e_z is the zth unit vector of $\mathcal{U}$, $\beta = (\alpha_l - 1, \dots, \alpha_{|w|+1})$ and $w = \gamma_1 \cdots \gamma_k$, $\gamma_1, \dots, \gamma_k \in \Gamma$, $q_j = \delta(q_0, \gamma_1 \cdots \gamma_j)$ and $\mu(f) = (q_0, x_0)$.

5.2 Realization of input-output maps by linear hybrid systems

In this section the solution to the realization problem will be presented. That is, given a set of input-output maps we will formulate necessary and sufficient conditions for the existence of a linear hybrid system realizing that set. In addition, characterization of minimal systems realizing the specified set of input-output maps will be given.

The following two theorems characterize observability and semi-reachability of linear hybrid systems. Observability of related classes of hybrid systems was investigated in [18, 2, 3]. Let

$$H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, B_q, C_q)_{q\in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2,\gamma)\})$$

be a linear hybrid system. The following theorems hold.

Theorem 7. *H is observable if and only if*

(i) For each $s_1, s_2 \in Q$, $s_1 = s_2$ if and only if for all $\gamma_1, \dots \gamma_k \in \Gamma, j_1, \dots, j_{k+1} \geq 0, 0 \leq l \leq k, k \geq 0$:

$$\begin{aligned} &\lambda(s_1, \gamma_1 \cdots \gamma_k) = \lambda(s_2, \gamma_1 \cdots \gamma_k) \text{ and} \\ &C_{q_k} A_{q_k}^{j_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_{l+1},\gamma_{l+1},q_l} A_{q_l}^{j_l} B_{q_l} = \\ &\quad C_{v_k} A_{v_k}^{j_{k+1}} M_{v_k,\gamma_k,v_{k-1}} \cdots M_{v_{l+1},\gamma_{l+1},v_l} A_{q_l}^{j_l} B_{v_l} \end{aligned}$$

where $q_j = \delta(s_1, \gamma_1 \cdots \gamma_j)$ and $v_j = \delta(s_2, \gamma_1 \cdots \gamma_j), j = 0, 1, \dots, k$.

(ii) For each $q \in Q$ it holds that $O_{H,q} := \bigcap_{w\in\Gamma^*} O_{q,w} = \{0\} \subseteq \mathcal{X}_q$ where $\forall w = \gamma_1 \cdots \gamma_k \in \Gamma^*, \gamma_1, \dots, \gamma_k \in \Gamma, k \geq 0$:

$$O_{q_0,w} = \bigcap_{j_1,\dots,j_k \geq 0} \ker C_{q_k} A_{q_k}^{j_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} A_{q_0}^{j_1}$$

where $q_0 \in Q, q_l = \delta(q_0, \gamma_1 \cdots \gamma_l), 1 \leq l \leq k, k \geq 0$.

Notice that part (i) of the theorem above is equivalent to

$$\upsilon_H((q_1, 0), .) = \upsilon_H((q_2, 0), .) \iff q_1 = q_2, \forall q_1, q_2 \in Q$$

Part (ii) of the theorem says that for each $q_1 = q_2 \in Q$,

$$\upsilon_H((q_1,x_1),.) = \upsilon_H((q_2,x_2),.) \iff x_1 = x_2, , \forall x_1, x_2 \in \mathcal{X}_{q_1} = \mathcal{X}_{q_2}$$

The proof of the theorem above relies on the following observation. Due to the linearity of continuous outputs in continuous inputs, if $\upsilon((q_1,x_1),.) = \upsilon((q_2,x_2),.)$ for some $(q_1,x_1),(q_2,x_2) \in \mathcal{H}$, then $\upsilon((q_1,0),.) = \upsilon((q_2,0),.)$.

Theorem 8. *(H,μ) is semi-reachable if and only if (A_H,μ_D), $\mu_D = \Pi_Q \circ \mu$, is reachable and* $\dim W_H = \sum_{q\in Q} \dim \mathcal{X}_q$*, where*

$$\begin{aligned} W_H = \mathrm{Span}\{ & A_{q_k}^{j_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_{l+1},\gamma_{l+1},q_l} A_{q_l}^{j_l} B_{q_l} u, \\ & A_{q_k}^{j_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} A_{q_1}^{j_1} x_f, \mid j_1,\ldots,j_{k+1} \geq 0, u \in \mathcal{U}, \\ & \gamma_1,\ldots,\gamma_k \in \Gamma, (q_f,x_f) = \mu(f), f \in \Phi, q_j = \delta(q_0,\gamma_1\cdots\gamma_j) \\ & ,1 \leq l,j \leq k, k \geq 0\} \subseteq \bigoplus_{q\in Q} \mathcal{X}_q \end{aligned}$$

Using the results above, we can give a procedure, which transforms any realization (H,μ) of Φ to a semi-reachable realization (H_r,μ_r) such that $\dim H_r \leq \dim H$. The procedure goes as follows. Let $\mathcal{A}_r = (Q^r,\Gamma,O,\delta^r,\lambda^r)$ be the sub automaton of $\mathcal{A}_H$ reachable from $\Pi_Q(\mathrm{Im}\mu)$ and for each $q \in Q_r$ let

$$\mathcal{X}_q^r = W_H \cap \mathcal{X}_q \ , \ A_q^r = A_q|_{\mathcal{X}_q^r} \ , \ C_q^r = C_q|_{\mathcal{X}_q^r} \ , \ B_q^r = B_q \ , \ M_{q_1,\gamma,q_e}^r = M_{q_1,\gamma,q_e}|_{\mathcal{X}_q^r}$$

Let $(H_r,\mu_r) = (\mathcal{A}^r,\mathcal{U},\mathcal{Y},(\mathcal{X}_q^r,A_q^r,B_q^r,C_q^r)_{q\in Q^r},\{M_{q_1,\gamma,q_2}^r \mid q_1,q_2 \in Q^r,\gamma\in\Gamma,q_1 = \delta(q_2,\gamma)\})$. It is clear that $\dim H_r \leq \dim H$. At the end of Section 5.3 we will give a procedure for transforming the realization (H,μ) to a reachable and observable one and we will outline a procedure for checking observability and reachability.

Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ be a set of input-output maps. Assume that Φ has a hybrid kernel representation. Then Proposition 2 allows us to reformulate the realization problem in terms of rationality of certain power series. Let $\widetilde{\Gamma} = \Gamma \cup \{e\}, e \notin \Gamma$. Every $w \in \widetilde{\Gamma}$ can be written as $w = e^{\alpha_1}\gamma_1 e^{\alpha_2}\gamma_2 \cdots \gamma_k e^{\alpha_{k+1}}$ for some $\gamma_1,\ldots,\gamma_k \in \Gamma$, $\alpha_1,\ldots,\alpha_{k+1} \geq 0$. For each $f \in \Phi$ define the formal power series $Z_f, Z_{f,j} \in \mathbb{R}^p \ll \widetilde{\Gamma}^* \gg$, $j = 1,\ldots,m$ as follows.

$$\begin{aligned} Z_f(e^{\alpha_1}\gamma_1 e^{\alpha_2} \cdots \gamma_k e^{\alpha_{k+1}}) &= D^\alpha f_C(0,w,.) \\ Z_{f,j}(e^{\alpha_1}\gamma_1 e^{\alpha_2} \cdots \gamma_k e^{\alpha_{k+1}}) &= D^\alpha y_0^{f,\Phi}(e_j,w,.) \end{aligned}$$

where $w = \gamma_1 \cdots \gamma_k$ and $\alpha = (\alpha_1,\ldots,\alpha_{k+1})$. Notice that $Z_{f,j}(v) = 0$ for all $v \in \Gamma^*$. Notice that the complete knowledge of the functions $K_w^{f,\Phi}$ and $G_{w,l}^{f,\Phi}$ is not needed in order to construct the formal power series $Z_f, Z_{f,j}$. In fact, one can think of Z_f as an object containing all the information on the behavior of f with the zero continuous input. The series $Z_{f,j}$ contains all the information on the behavior of the pair $(q,0)$, where q is the discrete part of the hybrid state inducing f in some realization of Φ (if there is any). Let $I_\Phi = \Phi \cup (\Phi \times \{1,2,\ldots,m\})$. Define the *set of formal power series associated with* Φ by

$$\Psi_\Phi = \{Z_j \in \mathbb{R}^p \ll \widetilde{\Gamma}^* \gg \mid j \in I_\Phi\}$$

Define the *Hankel-matrix* H_Φ of Φ as $H_\Phi = H_{\Psi_\Phi}$. Notice that if Φ is finite, then Ψ_Φ has finitely many elements.

Consider the linear hybrid system

$$H = (\mathcal{A},\mathcal{U},\mathcal{Y},(\mathcal{X}_q,A_q,B_q,C_q)_{q\in Q},\{M_{q_1,\gamma,q_2} \mid q_1,q_2\in Q,\gamma\in\Gamma,q_1=\delta(q_2,\gamma)\})$$

Assume that (H,μ) is a realization of Φ. Assume that $Q=\{q_1,\dots,q_N\}$. Fix a basis $\{e_{q,j}\mid q\in Q, j=1,\dots,m\}$ in $\mathbb{R}^{Nm}$. Define the *representation associated with* (H,μ) by

$$R_{H,\mu} = (\mathcal{X},\{M_z\}_{z\in\widetilde{\Gamma}},\widetilde{B},\widetilde{C})$$

where

- $\mathcal{X} = (\bigoplus_{q\in Q}\mathcal{X}_q)\oplus\mathbb{R}^{Nm}$,
- $\widetilde{C}:\mathcal{X}\to\mathbb{R}^p$, $\widetilde{C}x = C_q x$ if $x\in\mathcal{X}_q$ and $\widetilde{C}e_{q,j}=0$ for each $q\in Q$,$j=1,\dots,m$,
- $\widetilde{B}=\{\widetilde{B}_j\in\mathcal{X}\mid j\in I_\Phi\}$ is defined by $\widetilde{B}_f = x_f\in\mathcal{X}_{q_f}$ and $\widetilde{B}_{f,l}=e_{q_f,l}$, $f\in\Phi$, $\mu(f)=(q_f,x_f)$ and $l=1,2,\dots,m$,
- $M_e:\mathcal{X}\to\mathcal{X}$, such that $\forall x\in\mathcal{X}_q: M_e x = A_q x$, $M_e e_{q,j} = B_q e_j\in\mathcal{X}_q$, e_j is the jth unit vector in $\mathcal{U}$,
- $M_\gamma:\mathcal{X}\to\mathcal{X}$, $\gamma\in\Gamma$ such that $\forall x\in\mathcal{X}_q: M_\gamma x = M_{\delta(q,\gamma),\gamma,q}x$ and $M_\gamma e_{q,j} = e_{\delta(q,\gamma),j}$, $\forall q\in Q, j\in\{1,\dots,m\}$.

Let $\bar{O} = \underbrace{\mathbb{R}^p \ll\widetilde{\Gamma}\gg \times\cdots\times \mathbb{R}^p\ll\widetilde{\Gamma}\gg}_{m-times}$. For each $f\in F(PC(T,\mathcal{U})\times(\Gamma\times T)^*\times T,\mathcal{Y}\times O)$ such that f belongs to a set which admits a hybrid kernel representation, define $S_{f,w}\in\bar{O}$, $w\in\Gamma^*$ by

$$S_{f,w} = (w\circ Z_{f,1}\dots,w\circ Z_{f,m})$$

Define the maps

$$\kappa_f:\Gamma^*\ni w\mapsto S_{f,w}\in\bar{O} \text{ and } \psi_f:\Gamma^*\ni w\mapsto (f_D(w),\kappa_f(w))\in O\times\bar{O}$$

In fact, for each $f\in\Phi$ there is one-to-one correspondence between $y_0^{f,\Phi}$ and κ_f. Define the set of maps

$$\mathcal{D}_\Phi = \{\psi_f:\Gamma^*\to O\times\bar{O}\mid f\in\Phi\}$$

Let (H,μ) be a linear hybrid system realization and assume that $\mathcal{A}_H=(Q,\Gamma,O,\delta,\lambda)$. Define

$$\bar{\mathcal{A}}_H = (Q,\Gamma,O\times\bar{O},\delta,\bar{\lambda})$$

where $\bar{\lambda}(q) = (\lambda(q),S_{y_q,\epsilon})$ and $y_q = \upsilon_H((q,0),.,.,)$. If $\mu:\Phi\to\mathcal{H}$, then let $\mu_D: f\mapsto \Pi_Q(\mu(f))\in Q$. Proposition 1 implies the following.

Theorem 9. *(H,μ) is a realization of $\Phi\subseteq F(PC(T,\mathcal{U})\times(\Gamma\times T)^*\times T,\mathcal{Y}\times O)$ if and only if $R_{(H,\mu)}$ is a representation of Ψ_Φ and $(\bar{\mathcal{A}}_H,\mu_D)$ is a realization of $\mathcal{D}_\Phi$.*

Proof. Notice that

$$CM_e^{j_{k+1}}M_{\gamma_k}M_e^{j_k}\cdots M_{\gamma_1}M_e^{j_1}x = C_{q_k}A_{q_k}^{j_{k+1}}M_{q_k,\gamma_k,q_{k-1}}A_{q_{k-1}}^{j_k}\cdots M_{q_1,\gamma_1,q_0}A_{q_0}^{j_1}x$$

for all $x\in\mathcal{X}_{q_0}$, $q_0\in Q$ and $B_f\in\mathcal{X}_{\mu_D(f)}$ and $M_eM_wB_{f,j} = B_{\delta(\mu_D(f),w)}e_j$, $w\in\Gamma^*$ by construction. From Proposition 2 we get that (H,μ) is a realization of $\Phi \iff R_H$ is a representation of Ψ_Φ and $(\mathcal{A}_H, \mu_D)$ is a realization of $\Phi_D = \{f_D \in F(\Gamma^*, O) \mid f \in \Phi\}$. But if $(\bar{\mathcal{A}}_H, \mu_D)$ is a realization of $\mathcal{D}_\Phi$, then $(\mathcal{A}_H, \mu_D)$ is a realization of Φ_D.

Conversely, if (H, μ) is a realization of Φ, then $y_0^f = \Pi_{\mathcal{Y}} \circ \upsilon((\mu_D(f), 0), .)$ and

$$\Pi_{\mathcal{Y}} \circ \upsilon_H((\mu_D(f), 0), u, (wv, \tau_1\tau_2), t) = \Pi_{\mathcal{Y}} \circ \upsilon_H((\delta(\mu_D(f), w), 0), u^{'}, (v, \tau_2), t)$$

where $\tau_1 = (t_1, \dots, t_{|w|})$, $T = \sum_1^{|w|} t_i$, $u^{'}(s) = u(s+T)$, $u(s) = 0, 0 \le s \le T$. Thus,

$$S_{(y_{\mu_D(f), w})} = S_{f,w} = S_{(y_{\delta(\mu_D(f), w)}), \epsilon}$$

for all $w \in \Gamma^*$, which implies that $(\bar{\mathcal{A}}_H, \mu_D)$ is a realization of Φ_D. $\square$

In fact, if $R_{H,\mu}$ is a representation of Ψ_Φ, then $(\mathcal{A}_H, \mu_D)$ is a realization of $\{f_D \mid f \in \Phi\}$ $\iff$ $(\bar{\mathcal{A}}_H, \mu_D)$ is a realization of $\mathcal{D}_\Phi$. Above we associated a representation and a finite Moore-automaton to each linear hybrid system realization. Conversely, to each representation of Ψ_Φ and finite Moore-automaton realization of $\mathcal{D}_\Phi$ satisfying certain conditions we can associate a realization of Φ. Let $R = (\mathcal{X}, \{M_z\}_{z \in \widetilde{\Gamma}}, \widetilde{B}, \widetilde{C})$ be an observable representation of Ψ_Φ and let $(\bar{\mathcal{A}}, \zeta), \bar{\mathcal{A}} = (Q, \Gamma, O \times \bar{O}, \delta, \bar{\lambda})$ be a realization of $\mathcal{D}_\Phi$, which is reachable from Imζ. Then define $(H_{R,\bar{\mathcal{A}},\zeta}, \mu_{R,\bar{\mathcal{A}},\zeta})$ – *the linear hybrid realization associated with* R *and* $(\bar{\mathcal{A}}, \zeta)$ as $H_{R,\bar{\mathcal{A}},\zeta} = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, B_q, C_q)_{q \in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})$ where

- $\mathcal{A} = (Q, \Gamma, O, \delta, \Pi_O \circ \bar{\lambda})$,
- $\mathcal{X}_q = \text{Span}\{z \mid z \in W_q\} \quad \forall q \in Q$,

$$\begin{aligned} W_q = \{ & M_e^{j_{k+1}} M_{\gamma_k} M_e^{j_k} \cdots M_{\gamma_l} M_e^{j_l} M_e M_{\gamma_{l-1}} \cdots M_{\gamma_2} M_{\gamma_1} \widetilde{B}_{f,j} \mid \\ & \gamma_1, \dots, \gamma_k \in \Gamma, f \in \Phi, k \ge 0, q = \delta(\zeta(f), \gamma_1 \cdots \gamma_k), \\ & 1 \le l \le k, j_{k+1}, \dots, j_l \ge 0\} \\ & \cup \{M_e^{j_{k+1}} M_{\gamma_k} M_e^{j_{k-1}} \cdots M_{\gamma_1} M_e^{j_1} \widetilde{B}_f \mid \\ & \gamma_k, \dots, \gamma_k \in \Gamma, j_{k+1}, \dots, j_1 \ge 0, k \ge 0, \\ & q = \delta(\zeta(f), \gamma_1 \cdots \gamma_k)\} \end{aligned}$$

- $A_q = M_e|_{\mathcal{X}_q}, C_q = \widetilde{C}|_{\mathcal{X}_q}$ and $B_q e_j = M_e M_w \widetilde{B}_{f,j} \in \mathcal{X}_q$ such that $\delta(\zeta(f), w) = q$,
- $M_{q_1,\gamma,q_2} x = M_\gamma x, x \in \mathcal{X}_{q_2}, \gamma \in \Gamma, q_1, q_2 \in Q$, $q_1 = \delta(q_2, \gamma)$
- $\mu_{R,\bar{\mathcal{A}},\zeta}(f) = (\zeta(f), \widetilde{B}_f)$.

Notice that B_q is indeed well-defined for each $q \in Q$. If $q = \delta(\zeta(f), w) = \delta(\zeta(g), v)$, then $\psi_g(v) = \psi_f(w)$, since $\bar{\mathcal{A}}$ is a realization of $\mathcal{D}_\Phi$. But then $\psi_g(v) = (g_D(v), S_{g,v}) = (f_D(w), S_{f,w}) = \psi_f(w)$, i.e, $v \circ Z_{g,j} = w \circ Z_{f,j}$. Since R is a representation of Ψ_Φ we get that $v \circ Z_{g,j}(es) = Z_{g,j}(ves) = Z_{f,j}(wes) = \widetilde{C} M_s M_e M_w \widetilde{B}_{f,j} = \widetilde{C} M_s M_e M_v \widetilde{B}_{g,j}$ for each $s \in \widetilde{\Gamma}^*$. Observability of R implies that $M_e M_w \widetilde{B}_{f,j} = M_e M_v \widetilde{B}_{g,j}$,

It is easy to see that $(H_{R,\bar{\mathcal{A}},\zeta}, \mu_{R,\widetilde{\mathcal{A}},\zeta})$ is semi-reachable.

Theorem 10. *If R is an observable representation of Ψ_Φ and $(\bar{\mathcal{A}}, \zeta)$ is a reachable realization of $\mathcal{D}_\Phi$, then $H_{R,\bar{\mathcal{A}},\zeta}$ is a realization of Φ.*

Proof. Notice that

$$\widetilde{C} M_e^{j_{k+1}} M_{\gamma_k} M_e^{j_k} \cdots M_{\gamma_1} M_e^{j_1} x = C_{q_k} A_{q_k}^{j_{k+1}} M_{q_k,\gamma_k,q_{k-1}} A_{q_{k-1}}^{j_k} \cdots M_{q_1,\gamma_1,q_0} A_{q_0}^{j_1} x$$

for all $x \in \mathcal{X}_{q_0}$, $q_0 \in Q$. Moreover, $\widetilde{B}_f \in \mathcal{X}_{\zeta(f)}$ and for each $w \in \Gamma^*$,

$$M_e M_w \widetilde{B}_{f,j} = B_{\delta(\zeta(f),w)} e_j$$

Since $(\bar{\mathcal{A}}, \zeta)$ is a realization of $\mathcal{D}_\Phi$, we get that $(\mathcal{A}, \zeta)$ is a realization of $\{f_D \mid f \in \Phi\}$. From Proposition 2 it follows that $(H_{R,\bar{\mathcal{A}},\zeta}, \mu_{R,\bar{\mathcal{A}},\zeta})$ is a realization of Φ. $\square$

Notice that $\mathcal{D}_\Phi$ has a realization by a finite Moore-automaton if and only if both the indexed set $\Phi_D = \{f_D \mid f \in \Phi\}$ and the indexed set $\{\kappa_f \mid f \in \Phi\}$ have a realization by a finite Moore-automaton. By Theorem 1 this is equivalent to $card(W_{\Phi_D}) < +\infty$ and $K_\Phi = \{w \circ \kappa_f \mid w \in \Gamma^*, f \in \Phi\}$ being a finite set, i.e. $card(K_\Phi) < +\infty$. Notice that $w \circ \kappa_f(v) = S_{f,wv} = (wv \circ Z_{f,1} \ldots, wv \circ Z_{f,m})$. That is, K_Φ is finite if and only if $\{w \circ Z_{f,j} \mid w \in \Gamma^*, f \in \Phi, j = 1, \ldots, m\}$ is finite. Consider the following set

$$H_{\Phi,O} = \{((H_\Phi)_{(u,i),(v,f,j)})_{(u,i) \in \widetilde{\Gamma}^* \times \{1,\ldots,p\}} \mid f \in \Phi, j = 1, \ldots, m, v \in \Gamma^*\}$$

It is easy to see that $H_{\Phi,O}$ is the set of columns of H_Φ indexed by $(f,j,v), f \in \Phi, j = 1, \ldots, m, v \in \Gamma^*$. Notice that there is one-to-one correspondence between the columns of H_Φ indexed by (f,j,v) and the power series $v \circ Z_{f,j}$. Thus, $card(K_\Phi) < +\infty \iff card(H_{\Phi,0}) < +\infty$.

From the discussion above, using the results on theory of formal power series and automata theory, we can derive the following.

Theorem 11 (Realization of input/output map). *Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$. The following are equivalent.*

(i) Φ has a realization by a linear hybrid system,

(ii) Φ has a hybrid kernel representation, Ψ_Φ is rational and $\mathcal{D}_\Phi$ has a realization by finite Moore-automaton,

(iii) Φ has a hybrid kernel representation, $\operatorname{rank} H_\Phi < +\infty$*, $card(W_{\Phi_D}) < +\infty$ and $card(H_{\Phi,O}) < +\infty$.*

Proof. (i) $\implies$ (ii) follows from Proposition 2 and Theorem 9. (ii) $\implies$ (i) follows from Theorem 10 together with the following facts. If $\mathcal{D}_\Phi$ has a realization by finite Moore-automaton, then it has a reachable finite Moore-automaton realization. If Ψ_Φ has a representation, it has an observable representation. These facts follow from Theorem 5, Theorem 2. Finally, (ii) $\iff$ (iii) by Theorem 4, Theorem 1 and the discussion before the current theorem $\square$

Notice that if $(H,\mu) = (H_{R,\bar{\mathcal{A}},\zeta}, \mu_{R,\bar{\mathcal{A}},\zeta})$, then $\bar{\mathcal{A}}_H = \bar{\mathcal{A}}$ but $R_{H,\mu} = R$ need not hold. However, in this case there exists a representation morphism $i_R : R_{H,\mu} \to R$, such that $i_R(x) = x, \quad \forall x \in \mathcal{X}_q, q \in Q$. Consider two linear hybrid systems

$$\begin{aligned} H_1 &= (\mathcal{A}^1, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q^1, A_q^1, B_q^1, C_q^1)_{q \in Q^1}, \{M^1_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q^1, \gamma \in \Gamma, q_1 = \delta(q_2,\gamma)\}) \\ H_2 &= (\mathcal{A}^2, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q^2, A_q^2, B_q^2, C_q^2)_{q \in Q^2}, \{M^2_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q^2, \gamma \in \Gamma, q_1 = \delta(q_2,\gamma)\}) \end{aligned}$$

If $T = (T_D, T_C) : (H_1, \mu_1) \to (H_2, \mu_2)$ is an O-morphism, then we can define a representation morphism

$$\widetilde{T} : R_{H_1,\mu_1} \to R_{H_2,\mu_2}$$

such that $\widetilde{T}x = T_C x, x \in \mathcal{X}_q^1, q \in Q^1$ and $\widetilde{T}e_{q,j} = e_{T_D(q),j}$, $q \in Q^1, j = 1, \ldots, m$. Notice that the map T_D is an automaton morphism

$$T_D : (\bar{\mathcal{A}}_{H_1}, (\mu_1)_D) \to (\bar{\mathcal{A}}_{H_2}, (\mu_2)_D)$$

where $(\mu_i)_D = \Pi_{Q_i} \circ \mu_i$, $i = 1, 2$,

Assume that (H, μ) is a semi-reachable realization, R is observable and $(\bar{\mathcal{A}}, \zeta)$ is reachable. If $T : R_{H,\mu} \to R$ is a representation morphism and $\phi : (\bar{\mathcal{A}}_H, \mu_D) \to (\bar{\mathcal{A}}, \zeta)$ is an automaton morphism, then there exists a surjective O-morphism $H(T) = (\phi, T_C) : (H, \mu) \to (H_{R,\bar{\mathcal{A}},\zeta}, \mu_{R,\bar{\mathcal{A}},\zeta})$ such that $T_C x = Tx$ for all $x \in \mathcal{X}_q$, $q \in Q$.

It is easy to see that (H, μ) is semi-reachable $\iff R_{H,\mu}$ is reachable and $(\bar{\mathcal{A}}_H, \mu_D)$ is reachable. (H, μ) is observable $\iff \bar{\mathcal{A}}_H$ is observable and $R_{H,\mu}$ is $\mathcal{X}_q$ observable for all $q \in Q$.

If $R = (\mathcal{X}, \{M_z\}_{z \in \widetilde{\Gamma}}, \widetilde{B}, \widetilde{C})$ is an observable representation of Ψ_Φ and $(\bar{\mathcal{A}}, \zeta)$ is a minimal realization of $\mathcal{D}_\Phi$, then $(H, \mu) = (H_{R,\bar{\mathcal{A}},\zeta}, \mu_{R,\bar{\mathcal{A}},\zeta})$ is an observable and semi-reachable realization of Φ. Indeed, (H, μ) is semi-reachable and $(\bar{\mathcal{A}}_H, \mu_D) = (\bar{\mathcal{A}}, \zeta)$ is observable. Consider the representation

$$R_{H,\mu} = (\bigoplus_{q \in Q} \mathcal{X}_q \oplus \mathbb{R}^{Nm}, \{M_z^{'}\}_{z \in \widetilde{\Gamma}}, \widetilde{B}^{'}, \widetilde{C}^{'})$$

Then there exists $i_R : R_{H,\mu} \to R$ such that for each $x \in \mathcal{X}_q, q \in Q$: $i_R(x) = x$ and thus

$$\widetilde{C}^{'} M_w^{'} x = \widetilde{C} M_w i_R(x) = \widetilde{C} M_w x$$

If $x \in O_{R_{H,\mu}}$, then $x \in O_R = \{0\}$, so we get that $\mathcal{X}_q \cap O_{R_{H,\mu}} = \{0\}$, that is $R_{H,\mu}$ is $\mathcal{X}_q$ observable for each $q \in Q$.

The theory of rational power series allows us to formulate necessary and sufficient conditions for a linear hybrid system to be minimal.

Theorem 12 (Minimal realization). *If (H, μ) is a realization of Φ, then the following are equivalent.*

(i) (H, μ) *is minimal,*

(ii) (H, μ) *is semi-reachable and it is observable,*

(iii) For each $(H^{'}, \mu^{'})$ *semi-reachable realization of* Φ *there exists a surjective O morphims* $T : (H^{'}, \mu^{'}) \to (H, \mu)$. *In particular, all minimal hybrid linear systems realizing* Φ *are O-isomorphic.*

Proof. It is clear that (iii) implies (i), since any linear hybrid system realization of Φ can be converted to a semi-reachable realization of Φ with dimension not bigger than that of the original realization, according to the remark after Theorem 8.

Let R be a minimal representation of Ψ_Φ and $(\bar{\mathcal{A}}, \zeta)$ a minimal realization of $\mathcal{D}_\Phi$. Then $(H, \mu) = (H_{R,\bar{\mathcal{A}},\zeta}, \mu_{R,\bar{\mathcal{A}},\zeta})$ is an observable and semi-reachable realization of Φ.

We will show that (iii) holds for (H, μ). Indeed, if $(H^{'}, \mu^{'})$ is a semi-reachable realization of Φ, then $R_{H^{'},\mu^{'}}$ is reachable and $(\bar{\mathcal{A}}_{H^{'}}, \mu_D^{'})$ is reachable. By Theorem 2 and Theorem 5 there exists surjective morphisms $T : R_{H^{'},\mu^{'}} \to R$ and $\phi : (\bar{\mathcal{A}}_{H^{'}}, \mu_D^{'}) \to (\bar{\mathcal{A}}, \zeta)$. Then by the discussion before the theorem, there exists a surjective O-morphism $(\phi, T_C) : (H^{'}, \mu^{'}) \to (H, \mu)$ such that $T_C x = Tx$ for all $x \in \mathcal{X}_q$, $q \in Q$. Moreover, if $(H^{'}, \mu^{'})$ is observable, then $(\bar{\mathcal{A}}_{H^{'}}, \mu_D^{'})$ is observable and $R_{H^{'},\mu^{'}}$ is $\mathcal{X}_q^{'}, q \in Q^{'}$ observable, which implies that ϕ is bijective and $T|_{\mathcal{X}_q^{'}}$ is injective for all $q \in Q^{'}$. Since $T_C|_{\mathcal{X}_q^{'}} = T|_{\mathcal{X}_q^{'}}$ and $T_C x \in \mathcal{X}_q$ if and only if $x \in \mathcal{X}_{\phi^{-1}(q)}$ we get that T_C is an isomorphism. That is, each realization of Φ which is semi-reachable and observable is isomorphic to (H, μ).

That is, we get that (H, μ) is a minimal realization of Φ, and any semi-reachable and observable realization of Φ is isomorphic to $(H, , \mu)$. Since (H, μ) satisfies (iii), we get that (ii) $\implies$ (iii). That is, (ii) $\implies$ (iii) $\implies$ (i). It can be shown that (iii) $\implies$ (ii). $\square$

Notice that if R is a minimal representation of Ψ_Φ and $(\bar{\mathcal{A}}, \zeta)$ is a minimal realization of $\mathcal{D}_\Phi$, then $H_{R,\bar{\mathcal{A}},\zeta}$ is a minimal realization of Φ. That is, the classical constructions of the minimal automaton realization of $\mathcal{D}_\Phi$ and the minimal representation of Ψ_Φ yield a minimal realization of Φ.

5.3 Partial realization of linear hybrid systems, computational issues

In this section partial realization theory for linear hybrid systems will be discussed. At the end of the section a procedure for transforming a linear hybrid system realization to a minimal one will be presented. Procedure for checking observability and reachability will be formulated too. In the sequel the notation of Section 4 will be used.

Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$. Recall the results on partial realization by a Moore automaton from Section 3. Recall the results on partial representation from Section 4. Let $N \in \mathbb{N}$ and define

$$\kappa_{f,N} : \Gamma^* \ni w \mapsto ((w \circ Z_{f,1})_N, \ldots, (w \circ Z_{f,m})_N)$$

Consider the map $\eta_N : W_{\Psi_\Phi} \ni T \mapsto T_N$. Notice that if $N \geq \text{rank } H_{\Psi_\Phi}$, then η_N is a bijection. Define the map $\psi_{f,N} : \Gamma^* \ni w \mapsto (f_D(w), \kappa_{f,N}(w))$ and define

$$\mathcal{D}_{\Phi,N} = \{\psi_{f,N} \mid f \in \Phi\}$$

The discussion above yields the following. Define the set

$$\bar{O}_N = \{((S_1)_N, \ldots, (S_m)_N) \mid S_i \in \mathbb{R}^p \ll \widetilde{\Gamma}^* \gg, i = 1, \ldots, m\}$$

Assume that $(\mathcal{A}, \zeta)$ is a realization of $\mathcal{D}_{\Phi,N}$, where

$$\mathcal{A} = (Q, \Gamma, O \times \bar{O}_N, \delta, \lambda)$$

If $N \geq \text{rank } H_\Phi$, then $(\bar{\mathcal{A}}, \zeta)$ is a realization of $\mathcal{D}_\Phi$, where $\bar{\mathcal{A}} = (Q, \Gamma, O \times \bar{O}, \delta, \bar{\lambda})$ and

$$\bar{\lambda}(q) = (o, (\eta_N^{-1}(T_1), \ldots, \eta_N^{-1}(T_m))) \iff \lambda(q) = (o, (T_1, \ldots, T_m))$$

The following theorem is an easy consequence of Theorem 3 and Theorem 6.

Theorem 13. *Assume that* $\text{rank } H_{\Psi_\Phi,N,N} = \text{rank } H_{\Psi_\Phi,N+1,N} = \text{rank } H_{\Psi_\Phi,N,N+1}$ *and* $card(W_{\mathcal{D}_{\Phi,N},N,N}) = card(W_{\mathcal{D}_{\Phi,N},N+1,N}) = card(W_{\mathcal{D}_{\Phi,N},N,N+1})$*. Let* R_N *be the* N*-partial representation of* Ψ_Φ *from Theorem 6. Let* $(\mathcal{A}_N, \zeta_N)$ *be the* N*-partial realization of* $\mathcal{D}_{\Phi,N}$ *from Theorem 3. Consider the linear hybrid system*

$$(H_N, \mu_N) = (H_{R_N,\bar{\mathcal{A}}_N,\zeta_N}, \mu_{R_N,\bar{\mathcal{A}}_N,\zeta_N})$$

If Φ *has a realization* (H, μ) *such that* $\dim H = (q,p)$ *and* $mq + p < N$*, then* (H_N, μ_N) *is a minimal realization of* Φ*.*

Notice that in order to compute (H_N, μ_N), only the knowledge of $R_N, \mathcal{A}_N$ and $Q, \delta, \Pi_O \circ \lambda, \zeta_N$ is required, where $\mathcal{A}_N = (Q, \Gamma, O \times \bar{O}_N, \delta, , \lambda)$. That is, there is no need to compute $\bar{\mathcal{A}}_N$. In particular, if Φ is a finite collection of input-output functions and it is known that Φ has a realization of dimension at most (p,q), then a minimal linear hybrid system realization of Φ can be constructed from finite data. The results above also enable us to formulate an algorithm for constructing a minimal linear hybrid system realization.

Let $H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, B_q, C_q)_{q \in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})$ be a realization of Φ and assume that Φ is a finite collection of input-output maps. Based on (H, μ), we will compute a minimal realization of Φ as follows:

(i) Construct $R_{H,\mu}$ and $(\mathcal{A}_N, \mu_D)$, where

$$\mathcal{A}_N = (Q, \Gamma, O \times \bar{O}_N, \delta, \widetilde{\lambda}),\ \widetilde{\lambda}(q) = (\lambda(q), ((Z_{y_q,1})_N, \ldots, (Z_{y_q,m})_N))$$

$\dim H = (q,p)$, $qm + p < N$. Since (H, μ) is a realization of $v((q,0),.)$, $q \in Q$, we get that

$$Z_{y_q,j}(e^{\alpha_1}\gamma_1 \cdots \gamma_k e^{\alpha_{k+1}}) = C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_l,\gamma_l,q_{l-1}} A_{q_{l-1}}^{\alpha_l - 1} B_{q_{l-1}} e_j$$

where $q_j = \delta(q, \gamma_1 \cdots \gamma_j)$, $l = \min\{h \mid \alpha_h > 0\}$. It is easy to see that $(\mathcal{A}_N, \mu_D)$ is a realization of $\mathcal{D}_{\Phi,N}$, it can be represented by finite data and can be computed from (H, μ).

(ii) Compute a minimal representation R of Ψ_Φ from $R_{H,\mu}$. Compute a a minimal realization $(\mathcal{B}_N, \zeta)$ of $\mathcal{D}_{\Phi,N}$ from $(\mathcal{A}_N, \mu_D)$ This can be done algorithmically, provided that the representation of $(Z_{y_q,j})_N$ we are using allows us to decide whether $(Z_{y_{q_1},j})_N = (Z_{y_{q_2},j})_N$. It follows that $(\bar{\mathcal{B}}_N, \zeta)$ is a minimal realization of $\mathcal{D}_\Phi$.

(iii) Compute $(H_{min}, \mu_{min}) = (H_{R,\bar{\mathcal{B}}_N,\zeta}, \mu_{R,\bar{\mathcal{B}}_N,\zeta})$. Notice that this can be done without explicitly computing $\bar{\mathcal{B}}_N$. Then (H_{min}, μ_{min}) is a minimal realization of Φ.

Checking semi-reachability and observability of (H, μ) can be done as follows. (H, μ) is semi-reachable $\iff$ both are $(\mathcal{A}_H, \mu_D)$ and $R_{H,\mu}$ are reachable. It is easy to see that $(\bar{A}_H, \mu_D)$ is observable $\iff$ $(\mathcal{A}_N, \mu_D)$ is observable. (H, μ) is observable $\iff$ both $R_{H,\mu}$ is $\mathcal{X}_q$ observable, $q \in Q$ and $(\mathcal{A}_N, \mu_D)$ is observable. Reachability of $R_{H,\mu}$, and $\mathcal{X}_q$ $q \in Q$ observability of $R_{H,\mu}$ can be checked by checking whether certain finite matrices are of full rank. Reachability of $(\mathcal{A}_H, \mu_D)$ can be checked algorithmically. Observability of $(\mathcal{A}_N, \mu_D)$ can be checked algorithmically if we can decide whether $(Z_{y_{q_1},j})|_N = (Z_{y_{q_2},j})|_N$.

6 Bilinear Hybrid Systems

This section contains the definition and elementary properties of bilinear hybrid system.

Definition 3 (Bilinear hybrid systems). *A bilinear hybrid system (abbreviated as* BHS *) is a tuple*

$$H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, \{B_{q,j}\}_{j=1,\ldots,m}, C_q)_{q \in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, \delta(q_2, \gamma) = q_1\})$$

where $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ *is a finite-Moore-automaton,* $\mathcal{X}_q = \mathbb{R}^{n_q}$, $\mathcal{U} = \mathbb{R}^m$, $\mathcal{Y} = \mathbb{R}^p$, $\mathbb{N} \ni n_q, p, m > 0, q \in Q$ *and*

$$A_q : \mathcal{X}_q \to \mathcal{X}_q, B_{q,j} : \mathcal{X}_q \to \mathcal{X}_q, C_q : \mathcal{X}_q \to \mathcal{Y}, M_{q_1,\gamma,q_2} : \mathcal{X}_{q_2} \to \mathcal{X}_{q_1}$$

are linear maps.

Let $\mathcal{H} = \bigcup_{q \in Q} \{q\} \times \mathcal{X}_q$. Let $\mathcal{X} = \bigoplus_{q \in Q} \mathcal{X}_q$, $\mathcal{A}_H = \mathcal{A}$. The inputs of the bilinear hybrid system H are functions from $PC(T, \mathcal{U})$ and sequences from $(\Gamma \times T)^*$.

The interpretation of a sequence $(\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*$ is the following. The event γ_i took place *after* the event γ_{i-1} and t_{i-1} is the *time elapsed between the arrival of* γ_{i-1} *and the arrival of* γ_i. That is, t_i is the difference of the arrival times of γ_i and γ_{i-1}. Consequently, $t_i \geq 0$ but we allow $t_i = 0$, that is, we allow γ_i to arrive instantly after γ_{i-1}. If $i = 1$, then t_1 is simply the time when the event γ_1 arrived. The interpretation above also implies an ordering of discrete input events which arrive at the same time.

The *state trajectory* of the system H is a map

$$\xi_H : \mathcal{H} \times PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T \to \mathcal{H}$$

of the following form. For each $u \in PC(T,\mathcal{U})$, $w = (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*$, $t_{k+1} \in T$, $h_0 = (q_0, x_0) \in H$ it holds that

$$\xi_H(h_0, u, w, t_{k+1}) = (\delta(q_0, \gamma_1 \cdots \gamma_k), x_H(h_0, u, w, t_{k+1}))$$

where $x : T \ni t \mapsto x_H(h_0, u, w, t)$ is the solution of the differential equation

$$\frac{d}{dt}x(t) = A_{q_k}x(t) + \sum_{j=1}^{m} u_j(t + \sum_{1}^{k} t_j)B_{q_k,j}x(t)$$

with $u(s) = (u_1(s), \ldots, u_m(s))^T \in \mathcal{U}, s \in T$ with the initial condition

$$x(0) = M_{q_k,\gamma_k,q_{k-1}}x_H(x_0, u, (\gamma_1, t_1) \ldots (\gamma_{k-1}, t_{k-1}), t_k)$$

if $k > 0$ and $x(0) = x_0$ if $k = 0$. Here $q_i = \delta(q_0, \gamma_1 \cdots \gamma_i), i = 0, \ldots, k$.

That is, $A_q x + \sum_{j=1}^{m} u_j B_{q,j} x$ is the bilinear control system associated with the discrete state $q \in Q$ and M_{q_1,γ,q_2} is the reset map associated with input event $\gamma \in \Gamma$ and discrete states $q_1, q_2 \in Q$. Similarly to ordinary bilinear systems, the trajectory of a hybrid bilinear system admits a representation by an absolutely convergent series of iterated integrals.

For each $u = (u_1, \ldots, u_k) \in \mathcal{U}$ denote

$$d\zeta_j[u] = u_j, j = 1, 2, \ldots, m, \quad d\zeta_0[u] = 1$$

Denote the set $\{0, 1, \ldots, m\}$ by Z_m. For each $j_1 \cdots j_k \in \mathrm{Z}_m^*, j_1, \cdots, j_k \in \mathrm{Z}_m$, $k \geq 0, t \in T$, $u \in PC(T,\mathcal{U})$ define

$$V_{j_1 \cdots j_k}[u](t) = 1 \text{ if } k = 0 \quad V_{j_1 \cdots j_k}[u](t) = \int_0^t d\zeta_{j_k}[u(\tau)]V_{j_1,\ldots,j_{k-1}}[u](\tau)d\tau \text{ if } k > 1$$

For each $w_1, \ldots, w_k \in \mathrm{Z}_m^*$, $(t_1, \cdots, t_k) \in T^k$, $u \in PC(T,\mathcal{U})$ define

$$V_{w_1,\ldots,w_k}[u](t_1, \ldots, t_k) = V_{w_1}(t_1)[u]V_{w_2}(t_2)[\mathrm{Shift}_1(u)] \cdot \cdots \cdot V_{w_k}[\mathrm{Shift}_{k-1}(u)](t_k)$$

where $\mathrm{Shift}_i(u) = \mathrm{Shift}_{\sum_1^i t_i}(u)$, $i = 1, 2, \ldots, k-1$. For each $q \in Q$ and $w = j_1 \cdots j_k, k \geq 0, j_1, \cdots j_k \in \mathrm{Z}_m$ let us introduce the following notation $B_{q,0} := A_q$, $B_{q,\epsilon} := Id_{\mathcal{X}_q}$, , $B_{q,w} := B_{q,j_k}B_{q,j_{k-1}} \cdots B_{q,j_1}$. Using induction and the well-known result on the iterated integral series expansion of state trajectories of bilinear systems one can easily derive

$$x_H(h_0, u, s, t) = \sum_{w_1,\ldots,w_{k+1} \in \mathrm{Z}_m^*} (B_{q_k,w_{k+1}}M_{q_k,\gamma_k,q_{k-1}}B_{q_{k-1},w_k} \cdots \\ \cdots M_{q_1,\gamma_1,q_0}B_{q_0,w_1}x_0)V_{w_1,\ldots,w_{k+1}}[u](t_1, \ldots, t_{k+1})$$

where $t_{k+1} = t$, $q_{i+1} = \delta(q_i, \gamma_{i+1})$, $h_0 = (q_0, x_0)$ and $s = (\gamma_1, t_1) \cdots (\gamma_k, t_k)$, $u \in PC(T,\mathcal{U})$, $0 \leq i \leq k$. Define the set

$$Reach(\Sigma, \mathcal{H}_0) = \{x_H(h_0, u, w, t) \in \mathcal{X} \mid u \in PC(T,\mathcal{U}), w \in (\Gamma \times T)^*, t \in T, h_0 \in \mathcal{H}_0\}$$

H is *semi-reachable* from $\mathcal{H}_0$ if $\mathcal{X}$ is the vector space of the smallest dimension containing $Reach(H, \mathcal{H}_0)$ and the automaton $\mathcal{A}_H$ is reachable from $\Pi_Q(\mathcal{H}_0)$.

Define the function $\upsilon_H : \mathcal{H} \times PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T \to O \times \mathcal{Y}$ by

$$\upsilon_H((q_0, x_0), u, (w, \tau), t) = (\lambda(q_0, w), C_q x_H((q_0, x_0), u, (w, \tau), t))$$

where $q = \delta(q_0, w)$. For each $h \in \mathcal{H}$ the *input-output map of the system H induced by h* is the function

$$\upsilon_H(h, .) : PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T \ni (u, (w, \tau), t) \mapsto \upsilon_H(h, u, (w, \tau), t) \in \mathcal{Y} \times O$$

Two states $h_1 \neq h_2 \in \mathcal{H}$ of the bilinear hybrid system H are *indistinguishable* if

$$\upsilon_H(h_1, .) = \upsilon_H(h_2, .)$$

H is called *observable* if it has no pair of distinct indistinguishable states.

A set of input-output maps $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ is said to be *realized* by a bilinear hybrid system H if there exists a map $\mu : \Phi \to \mathcal{H}$ such that

$$\forall f \in \Phi : \ \upsilon_H(\mu(f), ., .) = f$$

Both H and (H, μ) are called a *realization* of Φ. Thus, H realizes Φ if and only if for each $f \in \Phi$ there exists a state $h \in \mathcal{H}$ such that $\upsilon_H(h, .) = f$. We say that the realization (H, μ) is observable if H is observable and we say that (H, μ) is semi-reachable if H is semi-reachable from $\mathrm{Im}\mu$. For a bilinear hybrid system H from Definition 3 the dimension of H is defined as

$$\dim H = (card(Q), \sum_{q \in Q} \dim \mathcal{X}_q) \in \mathbb{N} \times \mathbb{N}$$

The first component of $\dim H$ is the cardinality of the discrete state-space, the second component is the sum of dimensions of the continuous state-spaces. For each $(m, n), (p, q) \in \mathbb{N} \times \mathbb{N}$ we will write $(m, n) \leq (p, q)$, if $m \leq p$ and $n \leq q$. A realization H of Φ is called a *minimal realization* of Φ, if for any realization $H^{'}$ of Φ: $\dim H \leq \dim H^{'}$.

Consider two hybrid bilinear system realizations (H, μ) and $(H^{'}, \mu^{'})$, where

$$\begin{aligned} H &= (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, \{B_{q,j}\}_{j=1,\ldots,m}, C_q)_{q \in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, \delta(q_2, \gamma) = q_1\}) \\ H^{'} &= (\mathcal{A}^{'}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}^{'}_q, A^{'}_q, \{B^{'}_{q,j}\}_{j=1,\ldots,m}, C^{'}_q)_{q \in Q^{'}}, \{M^{'}_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q^{'}, \gamma \in \Gamma, \delta^{'}(q_2, \gamma) = q_1\}) \end{aligned}$$

$\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ and $\mathcal{A}^{'} = (Q^{'}, \Gamma, O, \delta^{'}, \lambda^{'})$. A pair $T = (T_D, T_C)$ is called an *O-morphism* from (H, μ) to $(H^{'}, \mu^{'})$, denoted by $T : (H, \mu) \to (H^{'}, \mu^{'})$ if the the following holds.

$$T_D : (\mathcal{A}, \mu_D) \to (\mathcal{A}^{'}, \mu^{'}_D)$$

where $\mu_D(f) = \Pi_Q(\mu_D(f))$, $\mu^{'}_D(f) = \Pi_{Q'}(\mu^{'}_D(f))$, is an automaton morphism and

$$T_C : \bigoplus_{q \in Q} \mathcal{X}_q \to \bigoplus_{q \in Q'} \mathcal{X}^{'}_q$$

is a linear morphism, such that

(a) $\forall q \in Q : T_C(\mathcal{X}_q) \subseteq \mathcal{X}^{'}_{T_D(q)}$,

(b) $T_C A_q = A^{'}_{T_D(q)} T_C$, $T_C B_{q,j} = B^{'}_{T_D(q)} T_C$, $C_q = C^{'}_{T_D(q)} T_C$, for all $q \in Q, j = 1, \ldots, m$,

(c) $T_C M_{q_1,\gamma,q_2} = M^{'}_{T_D(q_1),\gamma,T_D(q_2)} T_C$, $\forall q_1, q_2 \in Q, \gamma \in \Gamma, \delta(q_2, \gamma) = q_1$,

(d) $T_C(\Pi_{\mathcal{X}_q}(\mu(f))) = \Pi_{\mathcal{X}'_{T_D(q)}}(\mu^{'}(f))$ for each $q = \mu_D(f), f \in \Phi$.

The O-morphism T is said to be injective, surjective, or bijective if both T_D and T_C are respectively injective, surjective, or bijective. Bijective O-morphisms are called O-isomorphisms. Two bilinear hybrid system realizations are *isomorphic* if there exists an O-isomorphism between them.

6.1 Input-output maps of bilinear hybrid systems

Let $\widetilde{\Gamma} = \Gamma \cup \mathbb{Z}_m$. Then any $w \in \widetilde{\Gamma}$ is of the form $w = w_1\gamma_1 \cdots w_k\gamma_k w_{k+1}$, $\gamma_1, \ldots, \gamma_k \in \Gamma$, $w_1, \ldots, w_{k+1} \in \mathbb{Z}_m^*$, $k \geq 0$. A map $c : \widetilde{\Gamma}^* \to \mathcal{Y}$ is called a *generating convergent series on* $\widetilde{\Gamma}^*$ if there exists $K, M > 0$, $K, M \in \mathbb{R}$ such that for each $w \in \widetilde{\Gamma}^*$,

$$||c(w)|| < KM^{|w|}$$

where $||.||$ is some norm in $\mathcal{Y} = \mathbb{R}^p$. The notion of generating convergent series is related to the notion of convergent power series from [7]. Let $c : \widetilde{\Gamma}^* \to \mathcal{Y}$ be a generating convergent series. For each $u \in PC(T,\mathcal{U})$ and $s = (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*, t_{k+1} \in T$ define the series

$$F_c(u, s, t_{k+1}) = \sum_{w_1,\ldots,w_{k+1} \in \mathbb{Z}_m^*} c(w_1\gamma_1 \cdots \gamma_k w_{k+1}) V_{w_1,\ldots,w_{k+1}}[u](t_1, \ldots, t_{k+1})$$

It can be shown that the series above are absolutely convergent. In fact we can define a function $F_c \in F(PC(T,\mathcal{U}) \times (\Gamma \times T)^*, \mathcal{Y})$ by $F_c : (u, w, t) \mapsto F_c(u, w, t)$. It can be shown that F_c is uniquely determined by c. That is, if $d, c : \widetilde{\Gamma}^* \to \mathcal{Y}$ are two convergent generating series, then $F_c = F_d \iff c = d$.

Now we are ready to define the concept of hybrid Fliess-series representation of a set of input/output maps, which is related to the concept of Fliess-series expansion in [7]. For any map $f \in F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$, define $f_C = \Pi_{\mathcal{Y}} \circ f$, $f_D = \Pi_O \circ f$. Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$.

Definition 4 (Hybrid Fliess-series expansion). Φ *is said to admit* a hybrid Fliess-series expansion *if*

(1) For each $f \in \Phi$ there exists a generating convergent series $c_f : \widetilde{\Gamma}^* \to \mathcal{Y}$ such that $F_{c_f} = f_C$

(2) For each $f \in \Phi$ the map f_D depends only on Γ^*, that is, for each $w \in \Gamma^*$,

$$\forall u_1, u_2 \in PC(T,\mathcal{U}), \tau_1, \tau_2 \in T^{|w|}, t_1, t_2 \in T :$$
$$f_D(u_1, (w, \tau_1), t_1) = f_D(u_2, (w, \tau_2), t_2)$$

We will regard f_D as a function $f_D : \Gamma^* \to O$.

The notion of hybrid Fliess-series representation is an extension of the notion of Fliess-series for input-output maps of non-linear systems, see [7]. The following proposition gives a description of the hybrid Fliess-series expansion of Φ in the case when Φ is realized by a bilinear hybrid system.

Proposition 2. *(H, μ) is a bilinear hybrid system realization of Φ if and only if Φ has a hybrid Fliess-series expansion such that for each $f \in \Phi, w_1\gamma_1 \cdots \gamma_k w_{k+1} \in \widetilde{\Gamma}^*$, $\gamma_1, \ldots, \gamma_k \in \Gamma$, $w_1, \ldots, w_{k+1} \in Z_m^*$, $k \geq 0$*

$$\begin{aligned} c_f(w_1\gamma_1 \cdots \gamma_k w_{k+1}) &= C_{q_k} B_{q_k,w_{k+1}} M_{q_k,\gamma_k,q_{k-1}} B_{q_{k-1},w_k} \cdots M_{q_1,\gamma_1,q_0} B_{q_0,w_1} \mu_C(f) \\ f_D(\gamma_1 \cdots \gamma_k) &= \lambda(q_0, \gamma_1 \cdots \gamma_k) \end{aligned}$$

where $\mu(f) = (q_0, \mu_C(f))$ and $q_i = \delta(q_0, \gamma_1 \cdots \gamma_i)$, $i = 0, \ldots, k$.

6.2 Realization of input-output maps by bilinear hybrid systems

In this section the solution to the realization problem will be presented. In addition, characterization of minimal systems realizing the specified set of input-output maps will be given. The following two theorems characterize observability and semi-reachability of bilinear hybrid systems. Using the notation of Definition 3, the following holds.

Theorem 14. *The bilinear hybrid system H is observable if and only if*

(i) $\mathcal{A}_H = \mathcal{A}$ is observable, and

(ii) For each $q \in Q$,

$$O_{H,q} = \bigcap_{\gamma_1,\ldots,\gamma_k \in \Gamma, k \geq 0} \; \bigcap_{w_1,\ldots,w_{k+1} \in Z_m^*} \ker C_{q_k} B_{q_k,w_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} B_{q_0,w_1} = \{0\}$$

where $q_l = \delta(q, \gamma_1 \cdots \gamma_l), 0 \leq l \leq k, k \geq 0, q = q_0$.

Notice that part (i) of the theorem above is equivalent to

$$\upsilon_H((q_1, 0), .) = \upsilon_H(q_2, 0), .) \iff q_1 = q_1, \, \forall q_1, q_2 \in Q$$

Part (ii) of the theorem says that for each $q \in Q$:

$$\upsilon_H((q, x_1), .) = \upsilon_H((q, x_2), .) \iff x_1 = x_2, , \forall x_1, x_2 \in \mathcal{X}_q$$

The proof relies on the observation that $\upsilon_H((q, 0), .) = (\lambda(q, .), 0)$, and thus $\upsilon_H((q_1, 0), .) = \upsilon_H((q_2, 0), .) \iff \lambda(q_1, .) = \lambda(q_2, .)$.

Theorem 15. *(H, μ) is semi-reachable if and only if (A_H, μ_D), $\mu_D = \Pi_Q \circ \mu$, is reachable and $\dim W_H = \sum_{q \in Q} \dim \mathcal{X}_q$, where*

$$\begin{aligned} W_H = \operatorname{Span}\{ & B_{q_k,w_{k+1}} M_{q_k,\gamma_k,q_{k-1}} \cdots M_{q_1,\gamma_1,q_0} B_{q_0,w_1} x_f, \mid (q_f, x_f) = \mu(f), \\ & f \in \Phi, w_1, \ldots, w_{k+1} \in Z_m^*, q_j = \delta(q_0, \gamma_1 \cdots \gamma_j), 1 \leq j \leq k, k \geq 0\} \end{aligned}$$

Using the results above, we can give a procedure, which transforms any realization (H, μ) of Φ to an observable and semi-reachable realization $(H^{'}, \mu^{'})$ of Φ such that $\dim H^{'} \leq \dim H$.

Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ be a set of input-output maps. Assume that Φ has a hybrid Fliess-series expansion. Then Proposition 2 allows us to reformulate the realization problem in terms of rationality of certain power series. Define the *set of formal power series associated with* Φ by

$$\Psi_\Phi = \{ c_f \in \mathbb{R}^p \ll \widetilde{\Gamma}^* \gg \mid f \in \Phi \}$$

Define the *Hankel-matrix* H_Φ of Φ as $H_\Phi = H_{\Psi_\Phi}$. Notice that if Φ is finite, then Ψ_Φ is a finite set.

Let $H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, \{B_{q,j}\}_{j=1,\ldots,m}, C_q)_{q \in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, \delta(q_2, \gamma) = q_1\})$ be a HBS, $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ and assume that (H, μ) is a realization of Φ. Define the *representation associated with* (H, μ) by $R_{H,\mu} = (\mathcal{X}, \{M_z\}_{z \in \widetilde{\Gamma}}, \widetilde{I}, \widetilde{C})$ where

- $\mathcal{X} = \bigoplus_{q \in Q} \mathcal{X}_q$,
- $\widetilde{C} : \mathcal{X} \to \mathbb{R}^p$, such that $\forall x \in \mathcal{X}_q : \widetilde{C}x = C_q x$,
- $\widetilde{I} = \{\mu_C(f) \mid f \in \Phi\}$ where $\mu(f) = (\mu_D(f), \mu_C(f))$,
- $M_0 : \mathcal{X} \to \mathcal{X}$, such that $\forall x \in \mathcal{X}_q : M_0 x = A_q x$ and $M_j : \mathcal{X} \to \mathcal{X}$, such that $\forall x \in \mathcal{X}_q : M_j x = B_{q,j} x$, $j = 1, \ldots, m$,
- $M_\gamma : \mathcal{X} \to \mathcal{X}$, $\gamma \in \Gamma$ such that $\forall x \in \mathcal{X}_q : M_\gamma x = M_{\delta(q,\gamma),\gamma,q} x$

Define the indexed set of maps $\mathcal{D}_\Phi = \{f_D : \Gamma^* \to O \mid f \in \Phi\}$.

Theorem 16. *(H, μ) is a realization of $\Phi \iff R_{(H,\mu)}$ is a representation of Ψ_Φ and $(\mathcal{A}_H, \mu_D)$ is a realization of $\mathcal{D}_\Phi$.*

Let $R = (\mathcal{X}, \{M_z\}_{z \in \widetilde{\Gamma}}, \widetilde{I}, \widetilde{C})$ be a representation of Ψ_Φ and let $(\mathcal{A}, \zeta)$, $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ be a realization of $\mathcal{D}_\Phi$, which is reachable from Imζ. Then define $(H_{R,\mathcal{A},\zeta}, \mu_{R,\mathcal{A},\zeta})$ – *the bilinear hybrid realization associated with R and $(\mathcal{A}, \zeta)$* as

$$H_{R,\mathcal{A},\zeta} = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, \{B_{q,j}\}_{j=1,\ldots,m}, C_q)_{q \in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, \delta(q_2, \gamma) = q_1\})$$

where

- $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$,
- $\forall q \in Q : \mathcal{X}_q = \text{Span}\{M_{w_{k+1}} M_{\gamma_k} M_{w_k} \cdots M_{\gamma_1} M_{w_1} \widetilde{I}_f \mid \gamma_1, \ldots, \gamma_k \in \Gamma, f \in \Phi, k \geq 0, q = \delta(\zeta(f), \gamma_1 \cdots \gamma_k), w_1, \ldots, w_{k+1} \in Z_m^*\}$,
- $A_q x = M_0 x, C_q x = \widetilde{C} x$ and $B_{q,j} x = M_j x$, $\forall x \in \mathcal{X}_q$,
- $M_{q_1,\gamma,q_2} x = M_\gamma x, \forall x \in \mathcal{X}_{q_2}, \gamma \in \Gamma, q_1, q_2 \in Q$ if $q_1 = \delta(q_2, \gamma)$,
- $\mu_{R,\mathcal{A},\zeta}(f) = (\zeta(f), \widetilde{I}_f)$.

It is easy to see that $(H_{R,\mathcal{A},\zeta}, \mu_{R,\mathcal{A},\zeta})$ is semi-reachable. Note that $\mathcal{X}_q \cong \mathbb{R}^{n_q}, n_q = \dim \mathcal{X}_q$, $q \in Q$.

Theorem 17. *If R is a representation of Ψ_Φ and $(\mathcal{A}, \zeta)$ is a reachable realization of $\mathcal{D}_\Phi$, then $H_{R,\mathcal{A},\zeta}$ is a realization of Φ.*

From the discussion above, using the results on theory of formal power series and automata theory, we can derive the following.

Theorem 18 (Realization of input/output map). *Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ be a set of input-output maps. The following are equivalent.*

(i) *Φ has a realization by a bilinear hybrid system,*

(ii) *Φ has a hybrid Fliess-series expansion, Ψ_Φ is rational and $\mathcal{D}_\Phi$ has a realization by finite Moore-automaton,*

(iii) rank $H_\Phi < +\infty$ *and $\mathcal{D}_\Phi$ has a realization by a finite Moore-automaton, i.e. $card(W_{\mathcal{D}_\Phi}) < +\infty$.*

Notice that if $(H,\mu) = (H_{R,\mathcal{A},\zeta}, \mu_{R,\mathcal{A},\zeta})$, then $\mathcal{A}_H = \mathcal{A}$ but $R_{H,\mu} = R$ need not hold. However, in this case there exists a representation morphism $i_R : R_{H,\mu} \to R$, such that $i_R(x) = x \quad \forall x \in \mathcal{X}_q, q \in Q$. If $T = (T_D, T_C) : (H_1, \mu_1) \to (H_2, \mu_2)$ is an O-morphism, then $T_C : R_{H_1,\mu_1} \to R_{H_2,\mu_2}$ is a representation morphism and $T_D : (\mathcal{A}_{H_1}, (\mu_1)_D) \to (\mathcal{A}_{H_2}, (\mu_2)_D)$ is an automaton morphism, where $(\mu_i)_D = \Pi_{Q_i} \circ \mu_i$ and Q_i is the state space of $\mathcal{A}_{H_i}$, $i = 1, 2$. Assume that (H,μ) is a semi-reachable realization, R is a representation of Φ, and $(\mathcal{A},\zeta)$ is reachable. If $T : R_{H,\mu} \to R$ is a representation morphism and $\phi : (\mathcal{A}_H, \mu_D) \to (\mathcal{A},\zeta)$ is a surjective automaton morphism, then there exists a surjective O-morphism $H(T) = (\phi, T_C) : (H,\mu) \to (H_{R,\mathcal{A},\zeta}, \mu_{R,\mathcal{A},\zeta})$ such that $T_C x = Tx$ for all $x \in \mathcal{X}_q$.

For any realization (H,μ) the following holds. (H,μ) is semi-reachable if and only if $R_{H,\mu}$ is reachable and $(\mathcal{A}_H, \mu_D)$ is reachable. (H,μ) is observable if and only if $\mathcal{A}_H$ is observable and $R_{H,\mu}$ is $\mathcal{X}_q$ observable for all $q \in Q$. The theory of rational power series allows us to formulate necessary and sufficient conditions for a bilinear hybrid system to be minimal.

Theorem 19 (Minimal realization). *If (H,μ) is a realization of Φ, then the following are equivalent.*

(i) (H,μ) *is minimal,*

(ii) (H,μ) *is semi-reachable and it is observable,*

(iii) For each $(H^{'}, \mu^{'})$ *semi-reachable realization of* Φ *there exists a surjective O morphism* $T : (H^{'}, \mu^{'}) \to (H,\mu)$. *In particular, all minimal hybrid bilinear systems realizing* Φ *are O-isomorphic.*

Notice that if R is a minimal representation of Ψ_Φ and $(\mathcal{A},\zeta)$ is a minimal realization of $\mathcal{D}_\Phi$, then $H_{R,\mathcal{A},\zeta}$ is a minimal realization of Φ. That is, a minimal realization of Φ can be constructed on the column space of H_Φ. We can also formulate a partial realization theorem for bilinear hybrid systems.

Theorem 20. *Assume that* $\operatorname{rank} H_{\Psi_\Phi,N,N} = \operatorname{rank} H_{\Psi_\Phi,N+1,N} = \operatorname{rank} H_{\Psi_\Phi,N,N+1}$ *and* $card W_{\mathcal{D}_\Phi,N,N} = card W_{\mathcal{D}_\Phi,N+1,N} = card W_{\mathcal{D}_\Phi,N,N+1}$. *Let*

$$(H_N, \mu_N) = (H_{R_N,\mathcal{A}_N,\zeta_N}, \mu_{R_N,\mathcal{A}_N,\zeta_N})$$

Assume Φ *has a realization* (H,μ) *such that* $(N,N) \geq \dim H$. *Then* (H_N,μ_N) *is a minimal realization of* Φ.

In particular, if Φ is a finite collection of input-output functions and it is known that Φ has a realization of dimension at most (N,N), then a minimal bilinear hybrid system realization of Φ can be computed from finite data.

6.3 Linear hybrid systems versus bilinear hybrid systems

Recall the definition of linear hybrid systems. In this section it will be shown that the input-output behavior of linear hybrid systems can be realized by bilinear hybrid systems. Moreover, we will give a characterization of those input/output maps which can be realized both by a bilinear hybrid system and a linear hybrid system.

Let $H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\mathcal{X}_q, A_q, B_q, C_q)_{q\in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2,\gamma)\})$ be a linear hybrid system. Define the bilinear hybrid system

$$H_b = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (\widetilde{\mathcal{X}}_q, \widetilde{A}_q, \{\widetilde{B}_{q,j}\}_{j=1,\dots,m}, \widetilde{C}_q)_{q\in Q}, \{\widetilde{M}_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma\})$$

as follows.

- $\widetilde{\mathcal{X}}_q = \mathcal{X}_q \oplus \mathrm{Z}$, $\mathrm{Z} \cong \mathbb{R}$. Let $0 \neq e \in Z, Z = \{se \mid s \in \mathbb{R}\}$.
- $\widetilde{A}_q x = A_q x$, $x \in \mathcal{X}_q$, $\widetilde{A}_q e = 0$
- $\widetilde{B}_{q,j} x = 0$, $x \in \mathcal{X}_q$, $\widetilde{B}_{q,j} e = B_q e_j$, where e_j is the jth unit vector of $\mathcal{U}$,
- $\widetilde{C}_q x = C_q x, x \in \mathcal{X}_q$, $\widetilde{C}_q e = 0$,
- $\widetilde{M}_{q_1,\gamma,q_2} x = M_{q_1,\gamma,q_2} x, x \in \mathcal{X}_{q_2}$ and $\widetilde{M}_{q_1,\gamma,q_2} e = e$.

Then for every $(q,x) \in \{q\} \times \mathcal{X}_q$ it holds that

$$\upsilon_H((q,x),.) = \upsilon_{H_b}((q,x+e),.)$$

Indeed, for each $\widetilde{x} = x + e$, $x \in \mathcal{X}_q$:

$$\widetilde{A}_q \widetilde{x} + \sum_{j=1}^{m} u_j \widetilde{B}_{q,j} \widetilde{x} = A_q x + B_q u \in \mathcal{X}_q$$

and $\widetilde{M}_{q_1,\gamma,q} \widetilde{x} = M_{q_1,\gamma,q} x + e$, thus

$$x_{H_b}((q,\widetilde{x}),u,(w,\tau),t) = x_H((q,x),u,(w,\tau),t) + e$$

Thus, every linear hybrid system can be viewed as a bilinear hybrid system of a special type. Moreover, if (H,μ) is a linear hybrid system realization of Φ, then (H_b,μ_b), where $\mu_b(f) = (q,x+e) \iff \mu(f) = (q,x)$, is a bilinear hybrid system realization of Φ. It is also easy to see that if Φ has hybrid kernel representation,then it has a hybrid Fliess-series expansion, defined as follows. For each $f \in \Phi$, $\gamma_1,\ldots,\gamma_k \in \Gamma$, $k \geq 0$ let

$$c_f(w_1\gamma_1\cdots\gamma_k w_{k+1}) = D^{\alpha} K^{f,\Phi}_{\gamma_1\cdots\gamma_{k+1}}$$

if $w_i \in \{0\}^*, i = 1,\ldots,k+1$, $\alpha = (|w_1|,\ldots,|w_{k+1}|)$, and

$$c_f(\gamma_1\ldots\gamma_l j w_{l+1}\gamma_{l+1}\cdots\gamma_k w_{k+1}) = D^{\beta} G^{f,\Phi}_{\gamma_1\cdots\gamma_k,k+1-l} e_j$$

where $0 \leq l \leq k, j \in \{1,\ldots,m\}, w_i \in \{0\}^*, i = l+1,\ldots,k+1$, $\beta = (|w_{l+1}|,\ldots,|w_{k+1}|)$ and let $c_f(s) = 0$ for all other $s \in \widetilde{\Gamma}^*$.

Theorem 21. *If Φ has a realization by a linear hybrid system, then it has a realization by a bilinear hybrid system. Moreover, if Φ has a hybrid kernel representation, then Φ has a hybrid Fliess-series expansion.*

Theorem 22. *Assume that Φ has a hybrid Fliess-series expansion. Then Φ has a realization by a linear hybrid system if and only if*

(i) *Φ has a realization by a bilinear system,*

(ii) *For each $f \in \Phi$, $s = w_1\gamma_1\cdots\gamma_k w_{k+1} \in \widetilde{\Gamma}^*$ if $c_f(s) \neq 0$, then either $w_{l+1} = jv, v, w_i \in \{0\}^*, i = l+2,\ldots,k+1, w_1 = \cdots = w_l = \epsilon$ or $w_1,\ldots,w_{k+1} \in \{0\}^*$,*

(iv) *The set $\{w \circ \kappa_f \mid f \in \Phi, w \in \Gamma^*\}$ is finite, where $\kappa_f : \Gamma^* \ni v \mapsto (v1 \circ c_f,\ldots,vm \circ c_f)$.*

If (H,μ) is a minimal linear hybrid system realization of Φ, $(\widetilde{H},\widetilde{\mu})$ is a minimal bilinear hybrid system realization of Φ, and $\dim H = (q,p)$, then $\dim \widetilde{H} \leq (q,p+q)$. Notice that the conditions of realizability by a bilinear hybrid system are much easier to check than the conditions for existence of a linear hybrid system realization. It is also easier to construct the minimal bilinear hybrid system realization.

7 Conclusions

Solution to the realization problem for linear and bilinear hybrid systems has been presented. The realization problem considered was to find a realization of a family of input-output maps. The paper combines the theory of formal power series with the classical automata theory to derive the results. The paper also discusses partial realization theory for linear and bilinear hybrid systems. Topics of further research include realization theory for piecewise-affine systems on polytopes, and general non-linear hybrid systems without guards.

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References

[1] J. Berstel and C. Reutenauer. In *Rational series and Their Languages*, EATCS Monographs on Theoretical Computer Science. Springer-Verlag, 1984.

[2] P. Collins and J. H. van Schuppen. Observability of piecewise-affine hybrid systems. In *Hybrid Systems: Computation and Control*, pages 265–280, 2004.

[3] E. De Santis, M.D. Di Benedetto, and Pola G. On observability and detectability of continuous-time linear switching systems. In *CDC2003*, 2003.

[4] Samuel Eilenberg. *Automata, Languages and Machines.* Academic Press, New York, London, 1974.

[5] Dieter Gollmann. Partial realization by discrete-time internally bilinear systems: An algorithm. In *Mathematical theory of networks and systems, Proc. int. Symp., Beer Sheva/Isr. 1983*, 1983.

[6] T.A. Henzinger and etc. Alur, R. The algorithmic analysis of hybrid systems. *Theoretical Computer Science*, 138, 1995.

[7] Alberto Isidori. *Nonlinear Control Systems.* Springer Verlag, 1989.

[8] Bronisław Jakubczyk. Realization theory for nonlinear systems, three approaches. In M Fliess and M Hazewinkel, editors, *Algebraic and Geometric Methods in Nonlinear Control Theory*, pages 3–32. D.Reidel Publishing Company, 1986.

[9] R.E. Kalman. On minimal partial realization of a linear input-output map. In *Topics in Mathematical Systems Theory*, pages 151–192. McGraw-Hill, New York, 1969.

[10] Gerardo Lafferriere, George J. Pappas, and Shankar Sastry. Hybrid systems with finite bisimulations. In *Hybrid Systems V*, volume 1567 of *Lecture Notes in Computer Science.* Springer-Verlag, 1997.

[11] Daniel Liberzon. *Switching in Systems and Control.* Birkhäuser, Boston, 2003.

[12] Mihaly Petreczky. Realization theory for linear switched systems. In *Mathematical Theory of Networks and Systems*, 2004.

[13] Mihaly Petreczky. Realization theory for linear switched systems: Formal power series approach. Technical Report MAS-R0403, CWI, 2004. Available at `ftp.cwi.nl/CWIreports/MAS/MAS-R0403.pdf`.

[14] Mihaly Petreczky. Realization theory of linear and bilinear switched systems: A formal power series approach. 2005. Submitted to ESAIM Control and Optimization , to appear as a CWI research report.

[15] Gordano Pola, Arjan J. van der Schaft, and Maria D. Di Bendetto. Bisimulation theory fo switching linear systems. In *Proceedings IEEE Conference on Decision and Control*, 2004.

[16] Eduardo D. Sontag. Realization theory of discrete-time nonlinear systems: Part I – the bounded case. *IEEE Transaction on Circuits and Systems*, CAS-26(4), April 1979.

[17] Arjan J. van der Schaft. Bisimulation of dynamical systems. In *Hybrid Systems: Computation and Control*, number 2993 in Lecture Notes in Computer Science, 2004.

[18] R. Vidal, S. Sastry, and A. Chiuso. Observability of linear hybrid systems. In *Hybrid Systems: Computation and Control*, pages 526–540, 2003.