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Realization theory for linear and bilinear switched systems: a formal power series approach

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Realization theory for linear and bilinear switched systems: a formal power series approach

ABSTRACT
The paper deals with the realization theory of linear and bilinear switched systems. Necessary and sufficient conditions are formulated for a family of input-output maps to be realizable by a (bi)linear switched system. Characterization of minimal realizations is presented. The paper treats two types of (bi)linear switched systems. The first one is when all switching sequences are allowed. The second one is when only a subset of switching sequences is admissible, but within this restricted set the switching times are arbitrary. The paper uses the theory of formal power series to derive the results on realization theory.

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Keywords and Phrases: Hybrid systems, switched linear systems, switched bilinear systems, realization theory, formal power series, minimal realization

REALIZATION THEORY FOR LINEAR AND BILINEAR SWITCHED SYSTEMS: FORMAL POWER SERIES APPROACH

Mihály Petreczky [1]

Abstract. The paper deals with the realization theory of linear and bilinear switched systems. Necessary and sufficient conditions are formulated for a family of input-output maps to be realizable by a (bi)linear switched system. Characterization of minimal realizations is presented. The paper treats two types of (bi)linear switched systems. The first one is when all switching sequences are allowed. The second one is when only a subset of switching sequences is admissible, but within this restricted set the switching times are arbitrary. The paper uses the theory of formal power series to derive the results on realization theory.

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1. Introduction

Realization theory is one of central topics of systems theory. Apart from its theoretical relevance, realization theory has the potential of being applied for developing control and identification methods, as development of linear systems theory has demonstrated.

Switched systems are one of the best studied subclasses of hybrid systems. A vast literature is available on various issues concerning switched systems, for a comprehensive survey see [12]. The current paper develops realization theory for the following two subclasses of switched systems: linear switched systems and bilinear switched systems.

More specifically, the paper tries to solve the following problems.

(1) *Reduction to a minimal realization*
Consider a linear (bilinear) switched system Σ, and a subset of its input-output maps Φ. Find a minimal linear (bilinear) switched system which realizes Φ.

(2) *Existence of a realization with arbitrary switching*
Find necessary and sufficient condition for the existence of a linear (bilinear) switched system realizing a given set of input-output maps.

(3) *Existence of a realization with constrained switching*
Assume that a set of admissible switching sequences is defined. Assume that the switching times of the

Keywords and phrases: Hybrid systems switched linear systems, switched bilinear systems, realization theory, formal power series, minimal realization

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admissible switching sequences are arbitrary. Consider a set of input-output maps Φ defined only for the admissible sequences. Find sufficient and necessary conditions for the existence of a linear (bilinear) switched system realizing Φ. Give a characterization of the minimal realizations of Φ.

The motivation of the Problem 3 is the following. Assume that the switching is controlled by a finite automaton and the discrete modes are the states of this automaton. Assume that the automaton is driven by external events, which can trigger a discrete-state transition at any time. We impose *no restriction* as to when an external event takes place. Then the traces of this automaton combined with the switching times (which are arbitrary) give us the admissible switching sequences.

If we can solve Problem 3 for such admissible switching sequences that the set of admissible sequences of discrete modes is a regular language, then we can solve the following problem. Construct a realization of a set of input-output maps by a linear (bilinear) switched system, such that switchings of that system are controlled by an automaton which is given in advance. Notice that the set of traces of an automaton is always a regular language.

The following results are proved in the paper.

- A linear (bilinear) switched system is a minimal realization of a set of input-output maps if and only if it is observable and semi-reachable from the set of states which induce the input-output maps of the given set.
- Minimal linear (bilinear) switched systems which realize a given set of input-output maps are unique up to similarity.
- Each linear (bilinear) switched system Σ can be transformed to a minimal realization of any set of input-output maps which are realized by Σ.
- A set of input/output maps is realizable by a linear (bilinear) switched system if and only if it has a *generalized kernel representation* (*generalized Fliess-series expansion*) and the rank of its Hankel-matrix is finite. There is a procedure to construct the realization from the columns of the Hankel-matrix, and this procedure yields a minimal realization.
- Consider a set of input-output maps Φ defined on some subset of switching sequences. Assume that the switching sequences of this subset have arbitrary switching times and that their discrete mode parts form a regular language L. Then Φ has a realization by a linear (bilinear) switched system if and only if the Φ has a generalized kernel representation with constraint L (has a generalized Fliess-series expansion) and its Hankel-matrix is of finite rank. Again, there exists a procedure to construct a realization from the columns of the Hankel-matrix. The procedure yields an observable and semi-reachable realization of Φ. But this realization is not a realization with the smallest state-space dimension possible.

There are some earlier work on the realization theory of switched systems, see [14, 15, 17]. For realization theory for other classes of hybrid systems see [16, 18].

The paper [14] developed realization theory for linear switched systems using elementary techniques. The problem addressed in this paper, even for linear switched systems, is more general than the one dealt with in [14]. There, realization of a single input-output map by a linear switched system was considered. Moreover the input-output map was supposed to be realized from the zero initial state and the input-output map was assumed to be defined on all the switching sequences. If only one input-output map is considered, which is defined for all switching sequences and zero for constant zero input, the results of the paper imply those of [14]. If the set of discrete modes contains only one element, then the results of paper [14] imply the classical ones for linear systems, see [2]

The paper [15] is very similar to the current paper. It approaches realization theory using formal power series, in the same way as it is done in the current paper. However, it develops realization theory only for linear switched systems and does not provide any of the proofs.

The paper [17] sketches realization theory for bilinear switched systems without providing the proofs. The approach taken in [17] and the presented results are very similar to those of the current paper.

The papers [15,17] can be viewed as a short versions of parts of the current paper. The current paper contains all the results of [15, 17] and also provides all the proofs.

The brief overview of the results suggests that there is a remarkable analogy between the realization theories of linear and bilinear switched systems. In fact, this analogy is by no means a coincidence. Both the realization problem for linear and the realization problem for bilinear switched systems are equivalent to finding a (possibly minimal) representation for a set of formal power series. That is, realization theory of both linear and bilinear switched systems can be reformulated in terms of the theory of rational formal power series. This enables us to give a very concise and simple treatment of the realization problem for linear and bilinear switched systems. In fact, if one views switched systems as nonlinear systems and one is familiar with the realization theory of nonlinear systems, then the results of the paper should not be too surprising. Exactly this similarity between realization theory of linear and bilinear switched systems in terms of results and mathematical tools is the motivation to present the realization theory of linear and bilinear switched systems in one paper.

The approach to the realization theory taken in this paper was inspired by works of M.Fliess, B. Jakubczyk and H. Sussman [4, 5, 10, 25]. The main tool used in the paper is the theory of rational formal power series. Rational formal power series were used in systems theory earlier. Realization theory for bilinear systems is one of the major applications of rational formal power series, see [8]. There are a number of definitions for representation of rational formal power series, see [1, 20, 21]. All the cited works deal with representations of a *single formal power series*. In this paper, we will look at representations of *families of formal power* series instead. This requires a slight and straightforward extension of the existing theory.

We will not discuss the algorithmic aspects of realization theory or partial realization theory in this paper. There are some results in this direction, see [17].

The outline of the paper is the following. The first section, Section 2, sets up some notation which will be used throughout the paper. Section 3 describes some properties and concepts related to switched systems which are used in the rest of the paper. Section 4 contains the necessary results on formal power series. The material of Section 4 is an extension of the classical theory of rational formal power series ([1, 11]). The proofs of the statements of Section 4 are given in Appendix A. In Section 5 realization theory of linear switched systems is presented. Section 6 presents realization theory of bilinear systems.

2. Preliminaries

For suitable sets S, B, $S \subseteq \mathbb{R}$ denote by $PC(S, B)$ the class of piecewise-continuous maps from S to B. That is, $f \in PC(S, B)$ if f has finitely many points of discontinuity on each finite interval and at each point of discontinuity the right- and left-hand side limits exist and they are finite. For a set Σ denote by Σ^* the set of finite strings of elements of Σ. For $w = a_1 a_2 \cdots a_k \in \Sigma^*$ the length of w is denoted by $|w|$, i.e. $|w| = k$. The empty sequence is denoted by ϵ. The length of ϵ is zero: $|\epsilon| = 0$. Let $\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$. The concatenation of two strings $v = v_1 \cdots v_k, w = w_1 \cdots w_m \in \Sigma^*$ is the string $vw = v_1 \cdots v_k w_1 \cdots w_m$. If $w \in Q^+$ then w^k denotes the word $\underbrace{ww \cdots w}_{k-times}$. The word w^0 is just the empty word ϵ. Denote by T the set $[0, +\infty) \subseteq \mathbb{R}$. Denote by $\mathbb{N}$ the set of natural number including 0. Denote by $F(A, B)$ the set of all functions from the set A to the set B. By abuse of notation we will denote any constant function $f : T \to A$ by its value. That is, if $f(t) = a \in A$ for all $t \in T$, then f will be denoted by a. For any function f the range of f will be denoted by $\mathrm{Im} f$. If A, B are two sets, then the set $(A \times B)^*$ will be identified with the set $\{(u, w) \in A^* \times B^* \mid |u| = |w|\}$. For any two sets J, X the surjective function $A : J \to X$ is called an *indexed subset* of X or simply and *indexed set*. It will be denoted by $A = \{a_j \in X \mid j \in J\}$. The set J will be called the *index set* of A. The indexed subset $A = \{a_j \in X \mid j \in J\}$ is said to be a *subset* of the indexed subset $B = \{b_i \in X \mid i \in I\}$ if there exists $g : J \to I$ such that $a_j = b_{g(j)}$. The fact that A is a subset of B will be denoted by $A \subseteq B$.

Let $f : A \times (B \times C)^+ \to D$. Then for each $a \in A$, $w \in B^+$ we define the function $f(a, w, .) : C^{|w|} \to D$ by $f(a, w, .)(v) = f(a, (w, v)), v \in C^{|w|}$. By abuse of notation we denote $f(a, w, .)(v)$ by $f(a, w, v)$.

Let $\phi : \mathbb{R}^k \to \mathbb{R}^p$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$. We define $D^\alpha \phi$ as the partial derivative

$$D^\alpha \phi = \frac{d^{\alpha_1}}{dt_1^{\alpha_1}} \frac{d^{\alpha_2}}{dt_2^{\alpha_2}} \cdots \frac{d^{\alpha_k}}{dt_k^{\alpha_k}} \phi(t_1, t_2, \ldots, t_k)|_{t_1 = t_2 = \cdots = t_k = 0}.$$

Let $f, g \in PC(T, A)$ for some suitable set A. Define for any $\tau \in T$ the concatenation $f\#_\tau g \in PC(T, A)$ of f and g by

$$f\#_\tau g(t) = \begin{cases} f(t) & \text{if } t \leq \tau \\ g(t) & \text{if } t > \tau \end{cases}$$

If $f : T \to A$, then for each $\tau \in T$ define $\text{Shift}_\tau(f) : T \to A$ by $\text{Shift}_\tau(f)(t) = f(t + \tau)$. If $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are vector spaces over $\mathbb{R}$, and $F_1 : \mathcal{X} \to \mathcal{Y}$, $F_2 : \mathcal{Y} \to \mathcal{Z}$ are linear maps, then F_1F_2 denotes the composition $F_1 \circ F_2$ of F_1 and F_2. If $x \in \mathcal{X}$, then F_1x denote the value $F_1(x)$ of F_1 at x.

3. Switched Systems

This section contains the definition and elementary properties of switched systems.

Definition 1. *A switched (control) system is a tuple*

$$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\})$$

where

- $\mathcal{X} = \mathbb{R}^n$ *is the state-space*
- $\mathcal{Y} = \mathbb{R}^p$ *is the output-space*
- $\mathcal{U} = \mathbb{R}^m$ *is the input-space*
- Q *is the finite set of discrete modes*
- $f_q : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$*, is a function smooth in both variables* x *and* u*, and globally Lipschitz in* x
- $h_q : \mathcal{X} \to \mathcal{Y}$ *is smooth map for each* $q \in Q$

Elements of the set $(Q \times T)^+$ are called *switching sequences*. The inputs of the switched system Σ are functions from $PC(T, \mathcal{U})$ and sequences from $(Q \times T)^+$. That is, the switching sequences are part of the input, they are specified externally and we allow any switching sequence to occur. In fact, the switching sequences can be considered as discrete inputs.

In the hybrid systems literature the discrete modes are usually viewed as part of the state. One can think of switched systems as hybrid systems without guards, such that the discrete state transitions are triggered by discrete inputs and the discrete state transition rules are trivial. More precisely, there is one-to-one correspondence between discrete states and discrete inputs, and a discrete input changes the discrete state to the discrete state which corresponds to this particular discrete input. That is, the new discrete state of the system *depends only on the discrete input, but not on the previous discrete state.*

Let $u \in PC(T, \mathcal{U})$ and $w = (q_1, t_2)(q_2, t_2) \cdots (q_k, t_l) \in (Q \times T)^+$. The inputs u and w steer the system Σ from state x_0 to the state $x_\Sigma(x_0, u, w)$ given by

$$x_\Sigma(x_0, u, w) = F(q_k, \text{Shift}_{\sum_1^{k-1} t_i}(u), t_k) \circ F(q_{k-1}, \text{Shift}_{\sum_1^{k-2} t_i}(u), t_{k-1}) \circ \cdots\cdots \circ F(q_1, u, t_1)(x_0)$$

where $F(q, u, t) : \mathcal{X} \to \mathcal{X}$ and for each $x \in \mathcal{X}$ the function $F(q, u, t, x) : t \mapsto F(q, u, t)(x)$ is the solution of the differential equation

$$\frac{d}{dt} F(q, u, t, x) = f_q(F(q, u, t, x), u(t)),\ F(q, u, 0, x) = x$$

The empty sequence $\epsilon \in (Q \times T)^*$ leaves the state intact: $x_\Sigma(x_0, u, \epsilon) = x_0$.

The *reachable set* of the system Σ from a set of initial states $\mathcal{X}_0$ is defined by

$$Reach(\Sigma, \mathcal{X}_0) \quad = \quad \{x_\Sigma(x_0, u, w) \in \mathcal{X} \mid u \in PC(T, \mathcal{U}), w \in (Q \times T)^*, x_0 \in \mathcal{X}_0\}$$

Σ is said to be *reachable* from $\mathcal{X}_0$ if $Reach(\Sigma, \mathcal{X}_0) = \mathcal{X}$ holds. Σ is *semi-reachable* from $\mathcal{X}_0$ if $\mathcal{X}$ is the smallest vector space containing $Reach(\Sigma, \mathcal{X}_0)$. In other words, Σ is semi-reachablefrom $\mathcal{X}_0$ if

$$\mathcal{X} = \text{Span}\{x \in \mathcal{X} \mid x \in Reach(\Sigma, \mathcal{X}_0)\}$$

Define the function $y_{\Sigma} : \mathcal{X} \times PC(T,\mathcal{U}) \times (Q \times T)^{+} \to \mathcal{Y}$ by

$$\forall x \in \mathcal{X}, u \in PC(T,U), w = (q_1,t_1)(q_2,t_2)\cdots(q_k,t_k) \in (Q \times T)^{+} : \\ y_{\Sigma}(x,u,w) = h_{q_k}(x_{\Sigma}(x,u,w))$$

By abuse of notation, for each $x \in \mathcal{X}$ define the input-output map $y_{\Sigma}(x,.,.) : PC(T,\mathcal{U}) \times (Q \times T)^{+} \to \mathcal{Y}$ by

$$y_{\Sigma}(x,.,.)(u,w) = y_{\Sigma}(x,u,w)$$

The map $y_{\Sigma}(x,.,.)$ is called the *input-output map of the system* Σ *induced by the state* x. By abuse of notation we will use $y_{\Sigma}(x,u,w)$ for $y_{\Sigma}(x,.,.)(u,w)$.

Two states $x_1 \neq x_2 \in \mathcal{X}$ of the switched system Σ are *indistinguishable* if

$$\forall w \in (Q \times T)^{+}, u \in PC(T,\mathcal{U}) : \qquad y_{\Sigma}(x_1,u,w) = y_{\Sigma}(x_2,u,w)$$

Σ is called *observable* if it has no pair of indistinguishable states.

A set $\Phi \subseteq F(PC(T,\mathcal{U}) \times (Q \times T)^{+}, \mathcal{Y})$ of input-output maps is said to be *realized* by a switched system $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{f_q \mid q \in Q, u \in \mathcal{U}\},\{h_q \mid q \in Q\})$ if there exists $\mu : \Phi \to \mathcal{X}$ such that

$$\forall f \in \Phi\text{:} \quad y_{\Sigma}(\mu(f),.,.) = f$$

or, in other words,

$$\forall f \in \Phi, u \in PC(T,\mathcal{U}), w \in (Q \times T)^{+}\text{: } y_{\Sigma}(\mu(f),u,w) = f(u,w)$$

By abuse of terminology, both Σ and (Σ,μ) will be called a *realization* of Φ. One can think of the map μ as a way to determine the corresponding initial condition for each element of Φ. That is, Σ realizes Φ if and only if for each $f \in \Phi$ there exists a state $x \in \mathcal{X}$ such that $y_{\Sigma}(x,.,.) = f$. Denote by $\dim \Sigma := \dim \mathcal{X}$ the dimension of the state space of the switched system Σ.

A switched system Σ is a *minimal realization of* Φ if Σ is a realization of Φ and for each switched system Σ_1 such that Σ_1 is a realization of Φ it holds that

$$\dim \Sigma \leq \dim \Sigma_1$$

For any $L \subseteq Q^{+}$ define the subset of admissible switching sequences $TL \subseteq (Q \times T)^{+}$ by

$$TL := \{(w,\tau) \in (Q \times T)^{+} \mid w \in L\}$$

That is, TL is the set of all those switching sequences, for which the sequence of discrete modes belongs to L and the sequence of times is arbitrary. Notice that if $L = Q^{+}$ then $TL = (Q \times T)^{+}$. Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ be a set of input-output maps defined only on switching sequences belonging to TL. The system $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{f_q \mid q \in Q, u \in \mathcal{U}\},\{h_q \mid q \in Q\})$ *realizes* Φ *with constraint* L if there exists $\mu : \Phi \to \mathcal{X}$ such that

$$\forall f \in \Phi\text{: } y_{\Sigma}(\mu(f),.,.)|_{PC(T,\mathcal{U}) \times TL} = f$$

or, in other words,

$$\forall w \in \Phi, u \in PC(T,\mathcal{U}), w \in TL\text{: } \; y_{\Sigma}(\mu(f),u,w) = f(u,w)$$

We will call both (Σ,μ) and Σ a *realization* of Φ. Notice that if $L = Q^{+}$ then Σ realizes Φ with constraint L if and only if Σ realizes Φ. If Σ is a switched system, then we say that the realization (Σ,μ) is *semi-reachable* , if Σ is semi-reachable from $\mathrm{Im}\mu$.

4. Formal Power Series

The section presents results on formal power series. The material of this section is based on the classical theory of formal power series, see [1, 11]. However, a number of concepts and results are extensions of the standard ones. In particular, the definition of the rationality is more general than that one occurring in the literature. Consequently, the theorems characterizing minimality are extensions of the well-known results. These generalizations and extensions are rather straightforward and can be easily derived in a manner similar to the classical case. In order to keep the exposition self-contained and complete, the proofs of those theorems which are not part of the classical theory, will be given in Appendix A.

Let X be a finite alphabet. A *formal power series* S with coefficients in $\mathbb{R}^p$ is a map

$$S : X^* \to \mathbb{R}^p$$

We denote by $\mathbb{R}^p \ll X^* \gg$ the set of all formal power series with coefficients in $\mathbb{R}^p$. Let $S \in \mathbb{R}^p \ll X^* \gg$. For each $i = 1, \dots, p$ define the formal power series $S_i \in \mathbb{R} \ll X^* \gg$ by the following equation

$$S_i(w) \quad = \quad (S(w))_i = e_i^T S(w)$$

where e_i is the ith unit vector of $\mathbb{R}^p$. Let J be an arbitrary (possibly infinite) set. An indexed set of formal power series $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg | \; j \in J\}$ with the index set J is called *rational* if there exists a vector space $\mathcal{X}$ over $\mathbb{R}$, $\dim \mathcal{X} < +\infty$ and linear maps

$$C : \mathcal{X} \to \mathbb{R}^p, \quad A_\sigma \in \mathcal{X} \to \mathcal{X} \quad , \sigma \in X$$

and an indexed set with the index set J

$$B = \{B_j \in \mathcal{X} \mid j \in J\}$$

such that for all $j \in J$, $\sigma_1, \dots, \sigma_k \in X$, $k \geq 0$

$$S_j(\sigma_1 \sigma_2 \cdots \sigma_k) = C A_{\sigma_k} A_{\sigma_{k-1}} \cdots A_{\sigma_1} B_j.$$

The 4-tuple $R = (\mathcal{X}, \{A_x\}_{x \in X}, B, C)$ is called a *representation* of S. The number $\dim \mathcal{X}$ is called the *dimension* of the representation R and it is denoted by $\dim R$. We will refer to $\mathcal{X}$ as the *state-space* of the representation R. A formal power series $S \in \mathbb{R}^p \ll X^* \gg$ is called rational if the indexed set $\{S_j \mid j \in \{\emptyset\}\}$, $S_\emptyset = S$, with the singleton index $\{\emptyset\}$, is rational. That is, S is rational is the above sense if and only if it is rational in the classical sense.

In fact, a representation can be viewed as a Moore-automaton with the state-space $\mathcal{X}$, with input space X^*, with output space $\mathbb{R}^p$. The state transition function $\delta : \mathcal{X} \times X \to \mathcal{X}$ is given by the linear map $\delta(x, \sigma) = A_\sigma x$. The output map $\mu : \mathcal{X} \to \mathbb{R}^p$ is given by $\mu(x) := Cx$. The set of initial conditions is given by $\{B_j \mid j \in J\}$. The problem of finding a representation for a set of formal power series Ψ is equivalent to finding a realization of Ψ by a Moore-automaton of the form described above. That is, finding a representation is equivalent to finding a realization by a special class of Moore-automaton. We will not pursue the analogy with automaton theory in this paper. Instead, to keep the presentation self-contained, we will built the theory directly.

A representation R_{min} of Ψ is called *minimal* if for each representation R of Ψ

$$\dim R_{min} \leq \dim R$$

In the sequel the following short-hand notation will be used. Let $A_\sigma : \mathcal{X} \to \mathcal{X}, \sigma \in X$ be linear maps. Then

$$A_w := A_{w_k} A_{w_{k-1}} \cdots A_{w_1} \;, \; w = w_1 w_2 \cdots w_k \in X^*, w_1, \dots, w_k \in X$$

Let $R = (\mathcal{X}, \{A_z\}_{z\in X}, B, C)$, $\widetilde{R} = (\widetilde{\mathcal{X}}, \{\widetilde{A}_z\}_{z\in X}, \widetilde{B}, \widetilde{C})$ be two representations. A linear map $T : \mathcal{X} \to \widetilde{\mathcal{X}}$ is called a *representation morphism* from R to $\widetilde{R}$ and is denoted by $T : R \to \widetilde{R}$ if the following equalities hold

$$TA_z = \widetilde{A}_z T, \forall z \in X, \quad TB_j = \widetilde{B}_j, \forall j \in J, \quad C = \widetilde{C}T$$

Using the automaton-theoretic interpretation discussed one can think of representation morphisms as Moore-automaton morphisms which are linear morphisms between the state-spaces. The representation morphism T is called surjective, injective, isomorphism if T is a surjective, injective or isomorphism respectively if viewed as a linear vector space morphism.

Let $L \subseteq X^*$. If L is a regular language then, by the classical result [1], the power series $\bar{L} \in \mathbb{R} \ll X^* \gg$, $\bar{L}(w) = \begin{cases} 1 & \text{if } w \in L \\ 0 & \text{otherwise} \end{cases}$ is a rational power series. Consider two power series $S, T \in \mathbb{R}^p \ll X^* \gg$. Define the *Hadamard product* $S \odot T \in \mathbb{R}^p \ll X^* \gg$ by

$$(S \odot T)_i(w) = S_i(w)T_i(w), , \ i = 1, \ldots, p$$

Let $w \in X^*$ and $S \in \mathbb{R}^p \ll X^* \gg$. Define $w \circ S \in \mathbb{R}^p \ll X^* \gg$ – the *left shift of S by w* by

$$\forall v \in X^* : w \circ S(v) = S(wv)$$

The following statements are generalizations of the results on rational power series from [1, 21]. The proofs are given in the appendix. Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$. be an indexed set of formal power series with the index set J. Define the set W_Ψ by

$$W_\Psi = \mathrm{Span}\{w \circ S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J, w \in X^*\}$$

Define the Hankel-matrix H_Ψ of Ψ as the infinite matrix $H_\Psi \in \mathbb{R}^{(X^* \times I) \times (X^* \times J)}$, $I = \{1, 2, \ldots, p\}$ and $(H_\Psi)_{(u,i)(v,j)} = (S_j)_i(vu)$.

Theorem 1. *Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$.*

(i) *Assume that $\dim W_\Psi < +\infty$ holds. Then a representation R_Ψ of Ψ is given by*

$$R_\Psi = (W_\Psi, \{A_\sigma\}_{\sigma\in X}, B, C)$$

– $A_\sigma : W_\Psi \to W_\Psi$, $\forall T \in W_\Psi$: $A_\sigma(T) = \sigma \circ T$, $\sigma \in X$.
– $B = \{B_j \in W_\Psi \mid j \in J\}$, $B_j = S_j$ *for each* $j \in J$.
– $C : W_\Psi \to \mathbb{R}^p$, $C(T) = T(\epsilon)$.

(ii) *The following equivalences hold*

$$\Psi \textit{ is rational} \iff \dim W_\Psi < +\infty \iff \mathrm{rank}\ H_\Psi < +\infty$$

Moreover, $\dim W_\Psi = \mathrm{rank}\ H_\Psi$ *holds.*

The proof of the theorem is presented in the appendix. The representation R_Ψ is called *free*. Using the theorem above we can easily show that

Lemma 1. *The indexed set formal power series $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ is rational if and only if the indexed set of formal power series $\Xi = \{S_{(i,j)} \in \mathbb{R}^p \mid (i,j) \in \{1,\ldots,p\} \times J\}$ is rational, where $S_{(i,j)} = (S_j)_i$, $j \in J, i = 1, \ldots, p$.*

Proof. Indeed, define $pr_i : \mathbb{R}^p \to \mathbb{R}$ by $pr_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p) = x_i$ for $i = 1, \ldots, p$. It is easy to see that pr_i is linear and $S_{i,j} = pr_i \circ S_j$. Define the linear maps $P_i : W_\Psi \ni T \mapsto pr_i \circ T$, $i = 1, \ldots, p$. Notice that $\bigcap_{i=1}^p \ker P_i = \{0\}$. It is easy to see that $W_\Xi = \sum_{i=1}^p P_i(W_\Psi)$. That is, $\dim W_\Psi < +\infty \implies \dim W_\Xi < +\infty$.

Conversely, assume that $\dim W_{\Xi} < +\infty$. Define $P : W_{\Psi} \to \bigoplus_{i=1}^{p} Z_i$, $Z_i = W_{\Xi}$, $P(T) = \sum_{i=1}^{p} z_i$, $\forall i = 1, \dots, p :$ $z_i = P_i(T) \in Z_i$. Then $\ker P = \bigcap_{i=1}^{p} \ker P_i = \{0\}$, thus $\dim W_{\Psi} < p \cdot \dim W_{\Xi} < +\infty$. □

Theorem 1 implies the following lemma.

Lemma 2. *Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ and $\Theta = \{T_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ be rational indexed sets. Then $\Psi \odot \Theta := \{S_j \odot T_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ is a rational set. Moreover,* $\text{rank } H_{\Psi \odot \Theta} \leq \text{rank } H_{\Psi} \cdot \text{rank } H_{\Theta}$.

The proof of the lemma can be found in Appendix A. The classical version of the lemma above can be found in [1].

Let $R = (\mathcal{X}, \{A_\sigma\}_{\sigma \in X}, B, C)$ be a representation of $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$. Define the subspaces W_R and O_R of $\mathcal{X}$ by

$$\begin{aligned} W_R &= \text{Span}\{A_w B_j \mid w \in X^*, j \in J\} \\ O_R &= \bigcap_{w \in X^*} \ker C A_w \end{aligned}$$

The sets above have the following automaton-theoretic interpretation. The subspace W_R is the span of states reachable by a $w \in X^*$ from an initial state B_j. Two states x_1, x_2 are indistinguishable, i.e.

$$CA_w x_1 = CA_w x_2 \text{ for all } w \in X^*$$

if and only if $x_1 - x_2 \in O_R$. That is, the automaton corresponding to R is reduced if and only if $O_R = \{0\}$. We will say that the representation R is *reachable* if $\dim W_R = \dim R$, and we will say that R is *observable* if $O_R = \{0\}$.

Lemma 3. *Let $R = (\mathcal{X}, \{A_\sigma\}_{\sigma \in X}, B, C)$ be a representation of Ψ. Then there exists a representation*

$$R_{can} = (\mathcal{X}_{can}, \{A^{can}_\sigma\}_{\sigma \in X}, B^{can}, C^{can})$$

of Ψ such that R_{can} is reachable and observable, and $\mathcal{X}_{can}$ is isomorphic to the quotient $W_R/(O_R \cap W_R)$.

The proof of the lemma is presented in Appendix A.

Theorem 2 (Minimal representation)**.** *Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$. The following are equivalent.*

(i) *$R_{min} = (\mathcal{X}, \{A^{min}_\sigma\}_{\sigma \in X}, B^{min}, C^{min})$ is a minimal representation of Ψ.*
(ii) *R_{min} is reachable and observable.*
(iii) *If R is a reachable representation of Ψ then there exists a surjective representation morphism $T : R \to R_{min}$.*
(iv) $\text{rank } H_{\Psi} = \dim W_{\Psi} = \dim R_{min}$

Corollary 1. (a) *All minimal representations of Ψ are isomorphic.*
(b) *The* free representation *from Theorem 1 is a minimal representation.*

The proof of the theorem and its corollary can be found in Appendix A.

Lemma 4. *Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ and $\Psi' = \{T_{j'} \in \mathbb{R}^p \ll X^* \gg \mid j' \in J'\}$ be two indexed sets of formal power series with index sets J and J' respectively. Assume that there exists a map $f : J' \to J$, such that $\forall j' \in J' : S_{f(j')} = T_{j'}$. Then, if Ψ is rational, then Ψ' is also rational and* $\text{rank } H_{\Psi'} \leq \text{rank } H_{\Psi}$. *If f is surjective, then* $\text{rank } H_{\Psi'} = \text{rank } H_{\Psi}$.

Proof. Indeed, let $R = (\mathcal{X}, \{A_x\}_{x \in X}, B, C)$ be a minimal representation of Ψ. Then it is easy to see that $R' = (\mathcal{X}, \{A_x\}_{x \in X}, B', C)$ is a representation of Ψ', where $B'_{j'} = B_{f(j')}, j' \in J'$. That is, if Ψ is rational, then Ψ' is rational too. By Lemma 3 there exists a reachable and observable representation R'_{can} such that $\dim R'_{can} \leq$

$\dim R^{'} = \dim R$. But $R^{'}_{can}$ is a minimal representation of $\Psi^{'}$. Thus, rank $H_{\Psi'} = \dim R_{can} \leq \dim R = \text{rank } H_{\Psi}$. The representation R is reachable and observable. It is also easy to see that $O_R = O_{R'} = \{0\}$, thus $R^{'}$ is observable too. It is also easy to see that if f is surjective, then $W_{R'} = W_R = \mathcal{X}$, that is, $R^{'}$ is reachable. Thus, if f is surjective, then $R^{'}$ is a minimal representation of $\Psi^{'}$ and rank $H_{\Psi} = \dim R = \dim R^{'} = \text{rank } H_{\Psi'}$. □

Lemma 5. *Let $J_1, \ldots, J_n$ be disjoint sets. Let $\Psi_i = \{S_j \in \mathbb{R}^p \ll Q^* \gg \mid j \in J_i\}$, $i = 1, \ldots, n$ be indexed sets of formal power series. Let $J = J_1 \cup J_2 \cup \cdots \cup J_n$ and let $\Psi = \{S_j \in \mathbb{R}^p \ll Q^* \gg \mid j \in J\}$. Then Ψ is rational if and only if each Ψ_i, $i = 1, \ldots n$ is rational.*

Proof. It is easy to see that $W_{\Psi} = \text{Span}\{S_j \mid j \in J_1 \cup \cdots \cup J_n\} = \sum_{i=1}^{n} \text{Span}\{S_j \mid j \in J_i\} = W_{\Psi_1} + \cdots + W_{\Psi_n}$. For each $i = 1, \ldots, n$, W_{Ψ_i} is a subspace of W_{Ψ}. If Ψ is rational, then by Theorem 1 $\dim W_{\Psi} < +\infty$ and thus $\dim W_{\Psi_i} < +\infty$ for all $i = 1, \ldots, n$. That is, each Ψ_i, $i = 1, \ldots n$ is rational. Conversely, if each Ψ_i, $i = 1, \ldots, n$ is rational, then by Theorem 1, for each $i = 1, \ldots, n$, $\dim W_{\Psi_i} < +\infty$ holds. Thus, $\dim W_{\Psi} = \dim(W_{\Psi_1} + \cdots + W_{\Psi_n}) < +\infty$, that is, Ψ is rational □

Corollary 2. *Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ be an indexed set of formal power series with the index set J. Assume that J is finite. Then Ψ is rational if and only if $S_j \in \mathbb{R}^p \ll X^* \gg$ is rational for each $j \in J$*

Proof. Let $J = \{j_1, \ldots, j_n\}$. Let $\Psi_i = \{S_j \mid j \in \{j_i\}\}$, $i = 1, \ldots, n$. Then $\Psi = \{S_j \mid j \in \{j_1\} \cup \cdots \cup \{j_n\}\}$. Thus, by Lemma 5 Ψ is rational if and only if each Ψ_i, $i = 1, \ldots, n$ is rational. Let $f_i : \{j_i\} \ni j_i \mapsto \emptyset \in \{\emptyset\}$, $i = 1, \ldots, n$. Each f_i is a bijection. For each $i = 1, \ldots, n$ let $Q_i = \{T_j \mid j \in \{\emptyset\}\}$, $T_{\emptyset} = S_{j_i}$. Applying Lemma 4 to Ψ_i, Q_i, f_i and f_i^{-1} we get that Q_i is rational if and only if Ψ_i is rational. Thus, Ψ_i is rational $\iff$ S_{j_i} is rational, for each $i = 1, \ldots, n$. Therefore, Ψ is rational $\iff$ for each $j \in J$, S_j is rational. □

In the classical literature one often finds a procedure for constructing a representation of a rational formal power series from the columns of its Hankel-matrix. A similar construction can be carried out in the setting of this paper too. Indeed, let $\text{Im}H_{\Psi} = \text{Span}\{(H_{\Psi})_{.,(v,j)} \in \mathbb{R}^{X^* \times I} \mid (v,j) \in X^* \times J\}$. Then the map $T : W_{\Psi} \to \text{Im}H_{\Psi}$ defined by $T(w \circ S_j) = (H_{\Psi})_{.,(w,j)}$ is a well defined vector space isomorphism. Moreover, if $R_f = (W_{\Psi}, \{A_{\sigma}\}_{\sigma \in X}, B, C)$ is the free representation of Ψ, then $TB_j = (H_{\Psi})_{.,(\epsilon,j)}$, $CT^{-1}(H_{\Psi})_{.,(v,j)} = \begin{bmatrix} (H_{\Psi})_{(\epsilon,1),(v,j)} & \cdots & (H_{\Psi})_{(\epsilon,p),(v,j)} \end{bmatrix}^T$ and $TA_{\sigma}T^{-1}(H_{\Psi})_{.,(v,j)} = (H_{\Psi})_{(.,(v\sigma,j)}$ for each $\sigma \in X$. Define the representation

$$R_{H,\Psi} = (\text{Im}H_{\Psi}, \{TA_{\sigma}T^{-1}\}_{\sigma \in X}, TB, CT^{-1})$$

Then it is easy to see that $T : R_f \to R_{H,\Psi}$ is a representation isomorphism and $R_{H,\Psi}$ is a representation of Ψ. It is also straightforward to see that the definition of $R_{H,\Psi}$ corresponds to the definition of the representation on the columns of the Hankel-matrix as it is described in the classical literature.

If $R = (\mathcal{X}, \{A_{\sigma}\}_{\sigma \in \Sigma}, B, C)$ is a representation of Ψ, then for any vector space isomorphism $T : \mathcal{X} \to \mathbb{R}^n$, $n = \dim R$, the tuple

$$TR = (\mathbb{R}^n, \{TA_{\sigma}T^{-1}\}_{\sigma \in \Sigma}, TB, CT^{-1})$$

is also a representation of Ψ. It is easy to see that R is minimal if and only if TR is minimal. Moreover, $T : R \to TR$ is a representation isomorphism. That is, when dealing with representations, we can assume without loss of generality that $\mathcal{X} = \mathbb{R}^n$. From now on, we will silently assume that $\mathcal{X} = \mathbb{R}^n$ holds for any representation considered.

So far we have not treated the algorithmic aspects of theory of rational formal power series. One may wonder whether reachability and observability of representations is algorithmically decidable, or whether it is possible to construct a minimal representation algorithmically. One may also wonder whether it is possible to develop some sort of partial realization theory for rational formal power series. These issues fall outside the scope of the article. Nevertheless, we would like to note the following. One can easily design a numerical algorithm for computing the spaces O_R and W_R for a representation R. Subsequently, one can use these spaces for checking observability and reachability or computing a minimal representation. One can also develop partial realization theory. For reference see for instance [7, 16–18]. Moreover, since the classical theory of rational formal power series can be applied to the study of bilinear systems, a number of algorithmic results for bilinear systems theory might be used in the theory of rational formal power series.

5. Realization theory of linear switched systems

This section deals wit the realization theory of linear switched systems. First, definition and elementary properties of linear switched systems are presented. For more on linear switched systems see [6,12–14,23,24,26]. Subsection 5.1 deals with the structure of input/output maps realizable by linear switched systems. Subsection 5.2 presents realization theory of linear switched systems for the case when arbitrary switching is allowed. Subsection 5.3 deals with the case when there is a set of admissible switching sequences, but there is no restriction on the switching times.

Definition 2 (Linear switched systems)**.** *A switched system Σ is called linear, if for each $q \in Q$ there exist linear mappings $A_q : \mathcal{X} \to \mathcal{X}$, $B_q : \mathcal{U} \to \mathcal{X}$ and $C_q : \mathcal{X} \to \mathcal{Y}$ such that*

- $\forall u \in \mathcal{U}, \forall x \in \mathcal{X} : f_q(x,u) = A_q x + B_q u$
- $\forall x \in \mathcal{X} : h_q(x) = C_q x$

To make the notation simpler, linear switched systems will be denoted by $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})$ The term linear switched system will be abbreviated by LSS.

Consider the linear switched systems

$$\Sigma_1 = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\}) \text{ and } \Sigma_2 = (\mathcal{X}_a,\mathcal{U},\mathcal{Y},Q,\{(A^a_q,B^a_q,C^a_q) \mid q \in Q\})$$

A linear map $S : \mathcal{X} \to \mathcal{X}_a$ is said to be a *linear switched system morphism* from Σ_1 to Σ_2 and it is denoted by $S : \Sigma_1 \to \Sigma_2$ if the the following holds

$$A^a_q S = S A_q, \quad B^a_q = S B_q, \quad C^a_q S = C_q \quad \forall q \in Q$$

The map S is called surjective (injective) if it is surjective (injective) as a linear map. The map S is said to be a linear switched system isomorphisms, if it is an isomorphisms as a linear map. By abuse of terminology, if (Σ_i,μ_i), $i = 1,2$ are two linear switched system realizations and $S : \Sigma_1 \to \Sigma_2$ is a linear switched system morphism such that $S \circ \mu_1 = \mu_2$ then we will say that S is *linear switched system morphism* from realization (Σ_1,μ_1) to (Σ_2,μ_2) and we will denote it by $S : (\Sigma_1,\mu_1) \to (\Sigma_2,\mu_2)$. The linear switched systems realizations (Σ_1,μ_1) and (Σ_2,μ_2) are said to be *algebraically similar* or *isomorphic* if there exists an linear switched system isomorphism $S : (\Sigma_1,\mu_1) \to (\Sigma_2,\mu_2)$.

The results presented below can be found in the literature, for references see [13,23].

Proposition 1. *For any LSS $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})$ the following holds*

(1) $\forall u \in PC(T,\mathcal{U}), x_0 \in \mathcal{X}, w = (q_1,t_1)(q_2,t_2)\cdots(q_k,t_k) \in (Q \times T)^*$

$$\begin{aligned}
x_\Sigma(x_0,u,w) = {} & \exp(A_{q_k}t_k)\exp(A_{q_{k-1}}t_{k-1})\cdots\exp(A_{q_1}t_1)x_0 + \\
& \int_0^{t_k} \exp(A_{q_k}(t_k - s))B_{q_k}u(\sum_1^{k-1} t_i + s)ds + \\
& \exp(A_{q_k}t_k)\int_0^{t_{k-1}} \exp(A_{q_{k-1}}(t_{k-1} - s))B_{q_{k-1}}u(\sum_1^{k-2} t_i + s)ds + \\
& \cdots \\
& \exp(A_{q_k}t_k)\exp(A_{q_{k-1}}t_{k-1})\cdots\exp(A_{q_2}t_2)\int_0^{t_1} \exp(A_{q_1}(t_1 - s))B_{q_1}u(s)ds
\end{aligned}$$

and $y_\Sigma(x,u,w) = C_{q_k}x_\Sigma(x,u,w)$.

(2) *Reach*$(\Sigma,\{0\}) = \{A_{q_1}A_{q_2}\cdots A_{q_k}B_{q_{k+1}}u \mid u \in \mathcal{U}, q_1q_2\cdots q_{k+1} \in Q^+, k \geq 0\}$

(3) *Two states $x_1, x_2 \in \mathcal{X}$ are indistinguishable if and only if*

$$x_1 - x_2 \in \bigcap_{q_1,q_2,\ldots,q_{k+1}\in Q, k\geq 0} \ker C_{q_{k+1}}A_{q_k}\cdots A_{q_1}$$

Σ is observable if and only if

$$\bigcap_{q_1,q_2,\ldots,q_{k+1}\in Q, k\geq 0} \ker C_{q_{k+1}}A_{q_k}\cdots A_{q_1} = \{0\}$$

5.1. Input-output maps of linear switched systems

This section deals with properties of input-output maps of linear switched systems. We define the notion of generalized kernel representation of a set of input-output maps, which turns out to be a notion of vital importance for the realization theory of linear switched systems. In fact, the realization problem is equivalent to finding a generalized kernel representation of a particular form for the specified set of input-output maps. The section also contains a number of quite technical statements, which are used in other parts of the paper.

Recall that for any $L \subseteq Q^+$ the set of admissible switching sequences is defined by $TL = \{(w,\tau) \in (Q\times T)^+ \mid w \in L\}$. Let $\Phi \subseteq F(PC(T,\mathcal{U})\times TL, \mathcal{Y})$ be a set of maps of the form $PC(T,\mathcal{U})\times TL \to \mathcal{Y}$. Define the languages $\mathrm{suffix}L = \{u \in Q^* \mid \exists w \in Q^* : wu \in L\}$ and

$$\widetilde{L} = \{u_1^{i_1}\cdots u_k^{i_k} \in Q^* \mid u_1\cdots u_k \in \mathrm{suffix}L, u_j \in Q, i_j \geq 0, j=1,\ldots,k, i_1, i_k > 0\}$$

Definition 3 (Generalized kernel-representation with constraint L)**.** *The set Φ is said to have* generalized kernel representation with constraint L *if for all $f \in \Phi$ and for all $w = w_1w_2\cdots w_k \in \widetilde{L}$, $w_1,\ldots,w_k \in Q$, $k \geq 0$, there exist functions*

$$K^{f,\Phi}_w : \mathbb{R}^k \to \mathbb{R}^p \quad \text{and } G^{\Phi}_w : \mathbb{R}^k \to \mathbb{R}^{p\times m}$$

such that the following holds.

(1) $\forall w \in \widetilde{L}, \forall f \in \Phi$*: $K^{f,\Phi}_w$ is analytic and G^{Φ}_w is analytic*

(2) *For each $f \in \Phi$ and $w, v \in Q^*$ such that $wqqv, wqv \in \widetilde{L}$, it holds that*

$$\begin{aligned} K^{f,\Phi}_{wqqv}(t_1,t_2,\ldots,t_{|w|},t,t^{'},t_{|w|+2},\ldots t_{|w|+|v|+1}) &= K^{f,\Phi}_{wqv}(t_1,t_2,\ldots t_{|w|},t+t^{'},t_{|w|+2}\ldots t_{|w|+|v|+1}) \\ G^{\Phi}_{wqqv}(t_1,t_2,\ldots,t_{|w|},t,t^{'},t_{|w|+2},\ldots t_{|w|+|v|+1}) &= G^{\Phi}_{wqv}(t_1,t_2,\ldots t_{|w|},t+t^{'},t_{|w|+2}\ldots t_{|w|+|v|+1}) \end{aligned}$$

(3) $\forall vw \in \widetilde{L}, w \neq \epsilon, \forall f \in \Phi:$

$$K^{f,\Phi}_{vqw}(t_1,\ldots,t_{|v|},0,t_{|v|+1},\ldots,t_{|wv|}) = K^{f,\Phi}_{vw}(t_1,t_2,\ldots,t_{|vw|})$$

$\forall vw \in \widetilde{L}, v \neq \epsilon, w \neq \epsilon:$

$$G^{\Phi}_{vqw}(t_1,\ldots,t_{|v|},0,t_{|v|+1},\ldots,t_{|wv|}) = G^{\Phi}_{vw}(t_1,\ldots,t_{|vw|})$$

(4) *For each $f \in \Phi$, $(w_1,t_1)(w_2,t_2)\cdots(w_k,t_k) \in TL$, $u \in PC(T,\mathcal{U})$*

$$f(u,w_1w_2\cdots w_k,t_1t_2\cdots t_k) = K^{f,\Phi}_{w_1w_2\cdots w_k}(t_1,t_2,\ldots,t_k) + \sum_{i=1}^{k}\int_0^{t_i} G^{\Phi}_{w_i\cdots w_k}(t_i-s,t_{i+1},\ldots,t_k)u(s+\sum_{j=1}^{i-1}t_j)ds$$

We say that Φ has a *generalized kernel representation* if it has a generalized kernel representation with the constraint $L = Q^+$. The reader may view the functions $K^{f,\Phi}_w$ as the part of the output which depends on the initial condition and the functions G^{Φ}_w as functions determining the dependence of the output on the continuous inputs.

Define the function $y^{\Phi}_0 : PC(T,\mathcal{U}) \times TL \to \mathcal{Y}$ by

$$y^{\Phi}_0(u, w_1 \cdots w_k, t_1 \cdots t_k) := \sum_{i=1}^{k} \int_0^{t_i} G^{\Phi}_{w_i \cdots w_k}(t_i - s, t_{i+1}, \ldots, t_k) u(s + \sum_{j=1}^{i-1} t_j) ds$$

It follows from the fact that Φ has a generalized kernel representation that y^{Φ}_0 can be expressed by $\forall f \in \Phi$: $y^{\Phi}_0(u, w, \tau) = f(u, w, \tau) - f(0, w, \tau)$

Another straightforward consequence of the definition is that the functions $\{K^{f,\Phi}_w, G^{\Phi}_w \mid f \in \Phi, w \in \text{suffix}L\}$ completely determine the functions $\{K^{f,\Phi}_w, G^{\Phi}_w \mid f \in \Phi, w \in \widetilde{L}\}$. Indeed, assume that $\widetilde{L} \ni w = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$ such that $z_1, \ldots, z_k \in Q, \alpha \in \mathbb{N}^k$, $\alpha_k > 0$ and $z_1 \cdots z_k \in \widetilde{L}$. Then by using Part 2 and Part 3 of Definition 3 one gets

$$\begin{array}{rcccl} K^{f,\Phi}_w(t_1, \ldots, t_{|w|}) & = & K^{f,\Phi}_{z_l \cdots z_k}(T_l, \ldots, T_k) & = & K^{f,\Phi}_{z_1 \cdots z_k}(T_1, \ldots, T_k) \\ G^{\Phi}_w(t_1, \ldots, t_{|w|}) & = & G^{\Phi}_{z_l \cdots z_k}(T_l, \ldots, T_k) & & \end{array} \tag{1}$$

where $T_i = \sum_{j=1+\alpha_l+\cdots+\alpha_{i-1}}^{\alpha_l+\cdots+\alpha_i} t_j$, $i = l, \ldots, k$, and $T_i = 0$, $i = 1, \ldots, l-1$, $f \in \Phi$, $l = \min\{z \mid \alpha_z > 0\}$ and $\sum_{j=a}^{b} t_j$ is taken to be 0 if $a > b$. Now, for any $w \in \widetilde{L}$ there exist $d_1, \ldots, d_l \in Q$ and $\xi \in \mathbb{N}^l$ such that $d_1 \cdots d_l \in \text{suffix}L$, $w = d_1^{\xi_1} \cdots d_l^{\xi_l}$ and $\xi_1, \xi_l > 0$. Applying (1) to w, $d_1 \cdots d_l \in \text{suffix}L \subseteq \widetilde{L}$ we get that $K^{\Phi,f}_w$ and G^{Φ}_w are uniquely determined by $K^{\Phi,f}_{d_1 \cdots d_l}$ and $G^{\Phi}_{d_1 \cdots d_l}$.

Using formula (1), the chain rule and induction it is straightforward to show that for each $w \in \widetilde{L}$, $w = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$, $z_1 \cdots z_k \in \widetilde{L}$, $\alpha_k > 0$, $l = \min\{z \mid \alpha_z > 0\}$ the following holds.

$$\begin{array}{rcl} \frac{d^{\beta_1}}{dt_1^{\beta_1}} \cdots \frac{d^{\beta_{|w|}}}{dt_{|w|}^{\beta_{|w|}}} K^{f,\Phi}_w(t_1, \ldots, t_n) & = & \frac{d^{\gamma_1}}{d\tau_l^{\gamma_1}} \cdots \frac{d^{\gamma_{k-l+1}}}{d\tau_k^{\gamma_{k-l+1}}} K^{f,\Phi}_{z_l \cdots z_k}(\tau_l, \ldots, \tau_k)|_{\underline{a}} \\ & = & \frac{d^{\gamma_1}}{d\tau_l^{\gamma_1}} \cdots \frac{d^{\gamma_{k-l+1}}}{d\tau_k^{\gamma_{k-l+1}}} K^{f,\Phi}_{z_1 \cdots z_k}(\tau_1, \ldots, \tau_k)|_{\underline{b}} \\ \frac{d^{\beta_1}}{dt_1^{\beta_1}} \cdots \frac{d^{\beta_{|w|}}}{dt_{|w|}^{\beta_{|w|}}} G^{\Phi}_w(t_1, \ldots, t_n) & = & \frac{d^{\gamma_1}}{d\tau_l^{\gamma_1}} \cdots \frac{d^{\gamma_{k-l+1}}}{d\tau_k^{\gamma_{k-l+1}}} G^{\Phi}_{z_l \cdots z_k}(\tau_l, \ldots, \tau_k)|_{\underline{a}} \end{array} \tag{2}$$

where $\beta \in \mathbb{N}^{|w|}, \gamma \in \mathbb{N}^{k-l+1}$, $\underline{a} \in T^{k-l+1}$, $\underline{b} \in T^k$ and $a_i = \sum_{j=1+\alpha_l+\cdots+\alpha_{l+i-2}}^{\alpha_l+\cdots\alpha_{i+l-1}} t_j$, $\gamma_i = \sum_{j=1+\alpha_l+\cdots+\alpha_{l+i-2}}^{\alpha_l+\cdots+\alpha_{l+i-1}} \beta_j$ for each $i = 1, \ldots, k-l+1$, $b_i = a_{i-l+1}$, for $i = l, \ldots, k$ and $b_i = 0$ for $i = 1, \ldots, l-1$. Substituting 0 for $t_1, \ldots, t_{|w|}$ we get

$$D^{\beta} K^{f,\Phi}_w = D^{\gamma} K^{f,\Phi}_{z_l \cdots z_k} = D^{(\mathbb{O}_{l-1},\gamma)} K^{f,\Phi}_{z_1 \cdots z_k} \text{ and } D^{\beta} G^{\Phi}_w = D^{\gamma} G^{\Phi}_{z_l \cdots z_k} \tag{3}$$

where $\mathbb{O}_{l-1} = (0, 0, \ldots, 0) \in \mathbb{N}^{l-1}$. The discussion above yields the following.

Proposition 2. *Let $z_1, z_2, \ldots, z_k, d_1, d_2, \ldots, d_l \in Q^*$. Let $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ and $\beta = (\beta_1, \ldots, \beta_l) \in \mathbb{N}^l$ Assume that $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_k^{\alpha_k} = d_1^{\beta_1} d_2^{\beta_2} \cdots d_l^{\beta_l}$. If $q_2 z_1 z_2 \cdots z_k q_1 \in \widetilde{L}$ and $q_2 d_1 d_2 \cdots d_l q_1 \in \widetilde{L}$, then*

$$D^{(0,\alpha,0)} G^{\Phi}_{q_2 z_1 z_2 \cdots z_k q_1} = D^{(0,\beta,0)} G^{\Phi}_{q_2 d_1 d_2 \cdots d_l q_1}$$

If $z_1 z_2 \cdots z_k q_1$ and $d_1 d_2 \cdots d_l q_1 \in \widetilde{L}$ then

$$D^{(\alpha,0)} K^{f,\Phi}_{z_1 z_2 \cdots z_k q_1} = D^{(\beta,0)} K^{f,\Phi}_{d_1 d_2 \cdots d_l q_1}$$

Proof. Using (3) one gets that

$$D^{(0,\alpha,0)}G^{\Phi}_{q_2 z q_1} = D^{(0,\mathbb{I},0)}G^{\Phi}_{q_2 z_1^{\alpha_1}\cdots z_k^{\alpha_k} q_1} = D^{(0,\mathbb{I},0)}G^{\Phi}_{q_2 d_1^{\beta_1}\cdots d_l^{\beta_l} q_1} = D^{(0,\beta,0)}G^{\Phi}_{q_2 d q_1}$$

where $\mathbb{I} = (1,1,\ldots,1) \in \mathbb{N}^{\sum_1^k \alpha_i}$, $z = z_1\cdots z_k, d = d_1\cdots d_l$. Similarly $D^{(\alpha,0)}K^{f,\Phi}_{z_1\cdots z_k q_1} = D^{(\alpha^+,0)}K^{f,\Phi}_{z_l\cdots z_k q_1} = D^{(\mathbb{I},0)}K^{f,\Phi}_{z_1^{\alpha_1}\cdots z_k^{\alpha_k} q_1} = D^{(\mathbb{I},0)}K^{f,\Phi}_{d_1^{\beta_1}\cdots d_l^{\beta_l} q_1} = D^{(\beta,0)}K^{f,\Phi}_{d_1\cdots d_l q_1}$, where $l = \min\{z \mid \alpha_z > 0\}$ and $\alpha^+ = (\alpha_l, \ldots, \alpha_k)$. □

If Φ has a realization by a linear switched system, then Φ has a generalized kernel representation

Proposition 3. *For any LSS $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})$, (Σ,μ) is a realization of Φ with constraint L if and only if Φ has a generalized kernel representation defined by*

$$G^{\Phi}_{w_1 w_2\cdots w_k}(t_1,t_2,\ldots,t_k) = C_{w_k}\exp(A_{w_k}t_k)\exp(A_{w_{k-1}}t_{k-1})\cdots\exp(A_{w_1}t_1)B_{w_1}$$

and

$$K^{f,\Phi}_{w_1 w_2\cdots w_k}(t_1,t_2,\ldots,t_k) = C_{w_k}\exp(A_{w_k}t_k)\exp(A_{w_{k-1}}t_{k-1})\cdots\exp(A_{w_1}t_1)\mu(f).$$

where $w_1 w_2\cdots w_k \in \widetilde{L}$. Moreover, if (Σ,μ) is a realization of Φ, then

$$y_0^{\Phi} = y_{\Sigma}(0,.,.)|_{PC(T,\mathcal{U})\times TL}$$

Proof. (Σ,μ) is a realization of Φ if and only if for each $f \in \Phi$, $u \in PC(T,\mathcal{U})$, $w \in TL$ it holds that

$$f(u,w) = y_{\Sigma}(\mu(f),u,w) = C_{q_k}x_{\Sigma}(\mu(f),u,w)$$

where $w = w^{'}(q_k,t_k)$. The statement of proposition follows now directly from from part (1) of Proposition 1. □

If the set Φ has a generalized kernel representation with constraint L, then the collection of analytic functions $\{K^{f,\Phi}_w, G^{\Phi}_w \mid w \in \text{suffix}L, f \in \Phi\}$ determines Φ. Since $K^{f,\Phi}_w$ is analytic, we get that it is determined locally by $\{D^{\alpha}K^{f,\Phi}_w \mid \alpha \in \mathbb{N}^{|w|}\}$. Similarly, G^{Φ}_w is determined locally by $\{D^{\alpha}G^{\Phi}_w \mid \alpha \in \mathbb{N}^{|w|}\}$.

By applying the formula $\frac{d}{dt}\int_0^t f(t,\tau)d\tau = f(t,t) + \int_0^t \frac{d}{dt}f(t,\tau)d\tau$ and Part 4 of Definition 3 one gets

$$D^{\alpha}K^{f,\Phi}_{q_1 q_2\cdots q_k} = D^{\alpha}f(0,q_1 q_2\cdots q_k,.) \tag{4}$$

$$D^{\alpha}G^{\Phi}_{q_l q_{l+1}\cdots q_k}e_z = D^{\beta}y_0^{\Phi}(e_z,q_1 q_2\cdots q_k,.) \tag{5}$$

where $\mathbb{N}^k \ni \beta = (\underbrace{0,0,\ldots,0}_{l-1--times},\alpha_1+1,\alpha_2,\ldots,\alpha_{k-l+1})$. Here e_z is the zth unit vector of $\mathbb{R}^m$, i.e $e_z^T e_j = \delta_{zj}$. Formulas (4) and (5) imply that all the high-order derivatives of the functions $K^{f,\Phi}_w, G^{\Phi}_w$ ($f \in \Phi$, $w \in \text{suffix}L$) at zero can be computed from high-order derivatives with respect to the switching times of the functions from Φ.

Define the set $S = \{(\alpha,w) \in \mathbb{N}^* \times Q^* \mid \alpha \in \mathbb{N}^{|w|}, w \in Q^*\}$. For each $w \in Q^*$, $q_1,q_2 \in Q$ define the sets

$$\begin{aligned} F_{q_1,q_2}(w) &= \{(v,(\alpha,z)) \in Q^* \times S \mid vz \in L, q_2 w q_1 = z_1 z_1^{\alpha_1}\cdots z_k^{\alpha_k}z_k, z_j \in Q, j=1,\ldots,k, z = z_1\cdots z_k\} \\ F_{q_1}(w) &= \{(v,(\alpha,z)) \in Q^* \times S \mid vz \in L, w q_1 = z_1^{\alpha_1}\cdots z_k^{\alpha_k}z_k, z_j \in Q, j=1,\ldots,k, z = z_1\cdots z_k\} \end{aligned}$$

Define $\widetilde{L}_{q_1,q_2} = \{w \in Q^* \mid F_{q_1,q_2}(w) \neq \emptyset\}$ and $\widetilde{L}_q = \{w \in Q^* \mid F_q(w) \neq \emptyset\}$. Denote by $\mathbb{O}_l$ the tuple $(0,0,\ldots,0) \in \mathbb{N}^l$, $l \geq 0$. For any $\alpha \in \mathbb{N}^k$ let $\alpha^+ = (\alpha_1+1,\alpha_2,\ldots,\alpha_k) \in \mathbb{N}^k$, $k \geq 0$.

The intuition behind the definition of the sets $F_{q_1,q_2}(w)$ and $F_{q_1}(w)$ is the following. Let (Σ,μ) be a realization of Φ. Then $(v,(\alpha,z)) \in F_{q_1,q_2}(w)$ if $D^{\alpha^+}y_0^{\Phi}(vz,e_j,.) = D^{(1,1,\ldots,1,0)}y_{\Sigma}(0,q_2 w q_1,e_j,.)$ for each $j = 1,\ldots,m$.

Similarly, $(v,(\alpha,z)) \in F_{q_1}(w)$ if $D^{\alpha} f(vz,0,.) = D^{(1,1,\dots,1,0)} y_{\Sigma}(\mu(f), wq_1, 0)$ for each $f \in \Phi$. That is, $F_{q_1,q_2}(w)$ is non-empty if we can deduce from Φ some information on the output of Σ when the initial condition is 0 and the switching sequence is q_2wq_1. Similarly, $F_{q_1}(w)$ is non-empty, if we can derive from Φ some information on the output of Σ, if the initial condition is $\mu(f)$, the switching sequence is wq_1 and the continuous input is zero.

With the notation above, using the principle of analytic continuation and formulas (4) and (5), one gets the following

Proposition 4. *Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$. For any LSS*

$$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$$

the pair (Σ, μ) is a realization of Φ with constraint L if and only if Φ has a generalized kernel representation with constraint L and the following holds

$$\begin{aligned}
&\forall w \in L, j = 1,2,\dots,m, f \in \Phi, \alpha \in \mathbb{N}^{|w|}: \\
&\quad D^{\alpha} y_0^{\Phi}(e_j, w, .) = D^{\beta} G^{\Phi}_{w_l \cdots w_k} e_j = C_{w_k} A^{\alpha_k}_{w_k} A^{\alpha_{k-1}}_{w_{k-1}} \cdots A^{\alpha_l - 1}_{w_l} B_{w_l} e_j \\
&\quad D^{\alpha} f(0, w, .) = D^{\alpha} K^{f,\Phi}_{w} = C_{w_k} A^{\alpha_k}_{w_k} A^{\alpha_{k-1}}_{w_{k-1}} \cdots A^{\alpha_l}_{w_l} \mu(f)
\end{aligned} \tag{6}$$

where $l = \min\{h \mid \alpha_h > 0\}$, e_z is the zth unit vector of $\mathcal{U}$, $\beta = (\alpha_l - 1, \alpha_{l+1}, \dots, \alpha_k)$ and $w = w_1 \cdots w_k$, $w_1, \dots, w_k \in Q$. Formula (6) is equivalent to

$$\begin{aligned}
&\forall w \in \widetilde{L}, j = 1,2,\dots,m, q_1, q_2 \in Q, (v,(\alpha,z)) \in F_{q_1,q_2}(w): \\
&\quad D^{(\mathbb{O}_{|v|}, \alpha^{+})} y_0^{\Phi}(e_j, vz, .) = D^{(0,\alpha,0)} G^{\Phi}_{q_2 z q_1} e_j = C_{q_1} A^{\alpha_k}_{z_k} \cdots A^{\alpha_1}_{z_1} B_{q_2} e_j \\
&\forall w \in \widetilde{L}, q \in Q, (v,(\alpha,z)) \in F_q(w): \\
&\quad D^{(\mathbb{O}_{|v|}, \alpha)} f(0, vz, .) = D^{(\alpha,0)} K^{f,\Phi}_{zq} = C_q A^{\alpha_k}_{z_k} \cdots A^{\alpha_1}_{z_1} \mu(f)
\end{aligned} \tag{7}$$

Proof. First we show that Φ is realized by (Σ, μ) if and only if Φ has a generalized kernel representation and (6) holds. By Proposition 3 (Σ, μ) is a realization of Φ if and only if Φ has a generalized kernel representation of the form

$$\begin{aligned}
G^{\Phi}_w(t_1,\dots,t_k) &= C_{w_k} \exp(A_{w_k} t_k) \cdots \exp(A_{w_1} t_1) B_{w_1} \\
K^{f,\Phi}_w(t_1,\dots,t_k) &= C_{w_k} \exp(A_{w_k} t_k) \cdots \exp(A_{w_1} t_1) \mu(f)
\end{aligned} \tag{8}$$

for each $w = w_1 \cdots w_k \in \widetilde{L}$, $w_1, \dots, w_k \in Q$. From (1) it follows that it is enough to consider $\{K^{f,\Phi}_w, G^{\Phi}_w \mid w \in \text{suffix} L, f \in \Phi\}$. Since $K^{f,\Phi}_w, G^{\Phi}_w$ are analytic functions, their high-order derivatives at zero determine them uniquely. Using (4), (5) we get that (8) is equivalent to (6).

Next we show that (6) is equivalent to (7). Notice that from (3) it follows that for any $z = z_1 \cdots z_k, z_1 = q_2, z_k = q_1$: $D^{\alpha} G^{\Phi}_{z_1 \cdots z_k} = D^{(0,\alpha,0)} G^{\Phi}_{z_1 z_1 \cdots z_k z_k} = D^{(0,\alpha,0)} G^{\Phi}_{q_2 z q_1}$ and $D^{\alpha} K^{f,\Phi}_z = D^{(\alpha,0)} K^{f,\Phi}_{z q_1}$. First, we will show that (7) implies (6). For any $w \in L$, $\alpha \in \mathbb{N}^{|w|}$, $w = w_1 \cdots, w_k$, $w_1, \dots, w_k \in Q$ define $l = \min\{z \mid \alpha_z > 0\}$, $v = w_1 \cdots w_{l-1}$, $z = w_l \cdots w_{|w|}$ and $x = w_l^{\alpha_l - 1} w_{l+1}^{\alpha_{l+1}} \cdots w_{|w|}^{\alpha_{|w|}}$. Then $(v,(\beta,z)) \in F_{w_l, w_{|w|}}(x)$ where $\beta = (\alpha_l - 1, \dots, \alpha_{|w|})$. Notice that $(\mathbb{O}_{|v|}, \beta^{+}) = \alpha$. From (7) and the remark above we get that $D^{(\mathbb{O}_{|v|}, \beta^{+})} y_0^{\Phi}(e_j, vz, .) = D^{(0,\beta,0)} G^{\Phi}_{w_l z w_{|w|}} e_j = D^{\beta} G^{\Phi}_z e_j = D^{\alpha} y_0^{\Phi}(e_j, w, .) = C_{w_{|w|}} A^{\alpha_{|w|}}_{w_{|w|}} \cdots A^{\alpha_l - 1}_{w_l} B_{w_l} e_j$. Similarly, let $y = w_1^{\alpha_1} \cdots w_{|w|}^{\alpha_{|w|}}$. Then $(\epsilon, (\alpha, w)) \in F_{w_{|w|}}(y)$. Again, from the remark above and (7) we get that $D^{\alpha} f(0, w, .) = D^{(\alpha,0)} K^{f,\Phi}_{w w_{|w|}} = D^{\alpha} K^{f,\Phi}_w = D^{\alpha} f(0, w, .) = C_{w_{|w|}} A^{\alpha_{|w|}}_{w_{|w|}} \cdots A^{\alpha_1}_{w_1} \mu(f)$. That is, (6) holds.

Conversely, (6) $\implies$ (7). Indeed, for any $w \in \widetilde{L}$, $q_1, q_2 \in Q$, $(v,(\alpha,z)) \in F_{q_1,q_2}(w)$ it holds that $vz \in L$, $z = z_1 \cdots z_k$, $z_1 = q_2, z_k = q_1$. Then (6) implies $D^{(\mathbb{O}_{|v|}, \alpha^{+})} y_0^{\Phi}(e_j, vz, .) = D^{(0,\alpha,0)} G^{\Phi}_{q_2 z q_1} e_j = C_{z_k} A^{\alpha_k}_{z_k} \cdots A^{\alpha_1}_{z_1} B_{z_1}$ For any $(v,(\alpha,z)) \in F_q(w)$ it holds that $z = z_1 \cdots z_k$, $z_k = q$ and $vz \in L$. Then (6) implies $D^{(\mathbb{O}_{|v|}, \alpha)} f(0, vz, .) = D^{(\alpha,0)} K^{f,\Phi}_{z q_1} = C_q A^{\alpha_k}_{z_k} \cdots A^{\alpha_1}_{z_1} \mu(f)$. That is, (6) implies (7). $\square$

One may wonder whether a generalized kernel representation is unique, if it exists, and what is the relationship between a generalized kernel representation and such properties of input/output maps as linearity in continuous inputs, causality and etc. Below we will try to answer these questions.

Let $f \in F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$. We will say that f is *causal*, if for any $w = (q_1,t_1)\cdots(q_k,t_k) \in TL$ the following holds

$$\forall u,v \in PC(T,\mathcal{U}) : (\forall t \in [0, \sum_{1}^{k} t_i] : u(t) = v(t)) \implies f(w,u) = f(w,v)$$

That is, the value of $f(w,u)$ depends only on $u|_{[0,\sum_1^k t_i]}$.

Since $\mathcal{Y} = \mathbb{R}^p$, for each $f \in F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ there exist functions $f_j : PC(T,\mathcal{U}) \times TL \to \mathbb{R}$ such that $f(u,w) = (f_1(u,w),\dots,f_p(u,w))^T$. For each $t \in T$ define the map $P_t : PC(T,\mathcal{U}) \to PC(T,\mathcal{U})$ by

$$P_t(u)(s) = \begin{cases} u(s) & \text{if } s \le t \\ 0 & \text{otherwise} \end{cases}$$

For each $w \in TL$ define the map $f_j(w,.) : PC(T,\mathcal{U}) \to \mathbb{R}$ by $f_j(w,.)(u) = f_j(u,w)$. For each $1 \le p \le +\infty$ denote by $L^p([0,t_i],\mathbb{R}^{n\times m})$ the vector space of n by m matrices of functions from $L^p([0,t_i])$. I.e. $f : [0,t_i] \to \mathbb{R}^{n\times m}$ is an element of $L^p([0,t_i],\mathbb{R}^{n\times m})$, if $f = (f_{i,j})_{i=1,\dots,n,j=1,\dots,m}$ and $f_{i,j} \in L^p([0,t_i])$, $i = 1,\dots,n$, $j = 1,\dots,m$. With the notation above we can formulate the following characterization of input/output maps admitting a generalized kernel representation.

Theorem 3. *Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$. Then Φ admits a generalized kernel representation with constraint L if and only if the following conditions hold.*

(1) *Each $f \in \Phi$ is causal and there exists a function $y^\Phi \in F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ such that for each $f \in \Phi$*

$$\forall w \in TL, u \in PC(T,\mathcal{U}) : f(u,w) = f(0,w) + y^\Phi(u,w) \tag{9}$$

(2) *For each $f \in \Phi, w = (q_1,t_1)\cdots(q_k,t_k) \in TL, j = 1,2,\dots,p$ the map $y_j^\Phi(w,.) : PC([0,T_k],\mathcal{U}) \ni u \mapsto y_j^\Phi(w, u\#_{T_k}0) \in \mathbb{R}$ is a continuous linear functional, where $T_k = \sum_{j=1}^k t_j$. Here $PC([0,T_k],\mathcal{U})$ is viewed as a subspace of $L^1([0,T_k],\mathcal{U})$ and the topology considered on $PC([0,T_k],\mathcal{U})$ is the corresponding subspace topology.*

(3) *For each $f \in \Phi$, $s \in (Q \times T)^+$, $w = (w_1,0)\cdots(w_k,0)$, $v = (v_1,0)\cdots(v_l,0) \in (Q \times T)^*$*

$$ws, vs \in TL \implies (\forall u \in PC(T,\mathcal{U}) : f(u,ws) = f(u,vs))$$

(4) *For each $w = (q_1,t_1)\cdots(q_k,t_k) \in TL$, $1 \le l \le k$, $u \in PC(T,\mathcal{U})$*

$$y^\Phi(u,w) = y^\Phi(\mathrm{Shift}_{T_l}(u), v(q_l,t_l)\cdots(q_k,t_k)) + y^\Phi(P_{T_l}(u),w)$$

where $T_l = \sum_1^{l-1} t_i$ and $v = (q_1,0)\dots(q_{l-1},0)$.

(5) *For each $f \in \Phi$, $w,v \in (Q \times T)^*$, $q \in Q$, if $w(q,t_1)(q,t_2)v, w(q,t_1+t_2)v \in TL$, then*

$$\forall u \in PC(T,\mathcal{U}) : f(u, w(q,t_1)(q,t_2)v) = f(u, w(q,t_1+t_2)v)$$

For each $f \in \Phi$, $w,v \in (Q \times T)^*, |v| > 0$, $q \in Q$, if $w(q,0)v, wv \in TL$, then

$$\forall u \in PC(T,\mathcal{U}) : f(u, w(q,0)v) = f(u,wv)$$

(6) *For each $q_1\cdots q_k \in L$, $u_1,\dots u_k, \in \mathcal{U}$, $f \in \Phi$, the maps $f_{q_1\cdots q_k,u_1,\dots,u_k} : T^k \to \mathcal{Y}$ defined below, are analytic.*

$$f_{q_1\cdots q_k,u_1,\dots,u_k}(t_1,\dots,t_k) = f(u,(q_1,t_1)\cdots(q_k,t_k)),$$

where $u(t) = u_i$ if $t \in (\sum_{j=1}^{i-1} t_j, \sum_{j=1}^{i} t_j]$.

If Φ admits a generalized kernel representation, then the Φ admits an unique generalized kernel representation.

The proof of the theorem can be found in Appendix B.

The theorem above gives an important characterization of generalized kernel representation. It states that existence of a generalized kernel representation amounts to i) causality of the input-output maps, ii) switching sequences behaving as discrete inputs, iii) input-output maps being affine and continuous in the continuous inputs iv) input-output maps being analytic for constant inputs. In author's opinion, the theorem above demonstrates that existence of a generalized kernel representation is by no means an unnatural or a very restrictive condition. In particular, if the number of discrete modes is one, then existence of generalized kernel representation is equivalent to the conditions which are usually imposed on the input-output maps of linear (possibly infinite-infinite dimensional) systems. One may also compare the conditions of the above theorem with the so called realizability conditions from [14]. Notice that knowledge of analytic forms of $K^{f,\Phi}_w$ and G^{Φ}_w are not necessary for constructing a realization of Φ. All that is required is the knowledge that the functions $K^{f,\Phi}_w$, G^{Φ}_w exist. Therefore, it hardly makes sense to try to compute the functions $K^{f,\Phi}_w$ and G^{Φ}_w. Note that existence of an algorithm which computes these functions on the basis of Φ would imply the existence of a representation of Φ with finite data. Since elements of Φ are linear maps defined on the infinite-dimensional space $PC(T,\mathcal{U})$, existence of such a finite representation is quite unlikely.

5.2. Realization of input-output maps by linear switched systems with arbitrary switching

In this section the solution to the realization problem will be presented. That is, given a set of input-output maps we will formulate necessary and sufficient conditions for the existence of a linear switched system realizing that set. In addition, characterization of minimal systems realizing the given set of input-output maps will be given. In this section we assume that there are no restrictions on the switching sequences. That is, in this section we study realization with the trivial constraint $L = Q^+$.

The main tool of this section is the theory of rational formal power series. The main idea of the solution is the following. We associate a set of formal power series Ψ_{Φ} with the set of input-output maps Φ . Any representation of Ψ_{Φ} yields a realization of Φ and any realization of Φ yields a representation of Ψ_{Φ}. Moreover, minimal representations give rise to minimal realizations and vice versa. Then we can apply the theory of rational formal power series to characterize minimal realizations.

Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$. Proposition 4 and formula (3) yield the following

Proposition 5. *The LSS $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})$ is a realization of Φ if and only if Φ has a generalized kernel representation and there exists $\mu : \Phi \to \mathcal{X}$ such that*

$$\begin{array}{ccccc} \multicolumn{5}{c}{\forall w = w_1 \cdots w_k \in Q^+, q_1, q_2 \in Q, w_1, \ldots, w_k \in Q, z \in \{1,2,\ldots,m\}, f \in \Phi :} \\ D^{(1,\mathbb{I}_k,0)} y^{\Phi}_0(e_z, q_2 w q_1, .) & = & D^{(0,\mathbb{I}_k,0)} G^{\Phi}_{q_2 w q_1} e_z & = & C_{q_1} A_{w_k} \cdots A_{w_1} B_{q_2} e_z \\ D^{(\mathbb{I}_k,0)} f(0, wq_1, .) & = & D^{(\mathbb{I}_k,0)} K^{f,\Phi}_{wq_1} & = & C_{q_1} A_{w_k} \cdots A_{w_1} \mu(f) \end{array}$$

where $\mathbb{I}_k = (1,1,\ldots,1) \in \mathbb{N}^k$.

Proof. Applying (3) one gets the following equalities.

$$D^{\alpha} K^{f,\Phi}_w = D^{(\alpha,0)} K^{f,\Phi}_{w w_k} = D^{(\mathbb{I}_m,0)} K^{f,\Phi}_{w_1^{\alpha_1} w_2^{\alpha_2} \cdots w_k^{\alpha_k} w_k} \tag{10}$$

$$D^{\alpha} G^{\Phi}_w = D^{(0,\alpha,0)} G^{\Phi}_{w_1 w w_k} = D^{(0,\mathbb{I}_m,0)} G^{\Phi}_{w_1 w_1^{\alpha_1} w_2^{\alpha_2} \cdots w_k^{\alpha_k} w_k} \tag{11}$$

where $m = \sum_1^k \alpha_k$. The statement of the proposition follows now from Proposition 4. □

The proposition above allows us to reformulate the realization problem in terms of rationality of certain power series. Define formal power series $S_{q_1,q_2,z}, S_{f,q_1} \in \mathbb{R}^p \ll Q^* \gg$, ($q_1, q_2 \in Q, f \in \Phi$, $z \in \{1,2,\ldots,m\}$) by

$$S_{q_1,q_2,z}(w) = D^{(1,\mathbb{I}_{|w|},0)} y^{\Phi}_0(e_z, q_2 w q_1, .) \ , \ S_{f,q_1}(w) = D^{(\mathbb{I}_{|w|},0)} f(0, wq_1, .)$$

for each $w \in Q^*$. Notice that the functions $G^{\Phi}_w, K^{f,\Phi}_w$ are not involved in the definition of the series of $S_{q_1,q_2,z}$ and S_{f,q_1}. On the other hand, if Φ has a generalized kernel representation, then

$$S_{q_1,q_2,z}(w) = D^{(0,\mathbb{I}_{|w|},0)} G^{\Phi}_{q_2 w q_1} e_z \text{ and } S_{f,q_1}(w) = D^{(\mathbb{I}_{|w|},0)} K^{f,\Phi}_{wq_1}$$

For each $q \in Q, z = 1,2,\dots,m,\ f \in \Phi$ define the formal power series $S_{q,z}, S_f \in \mathbb{R}^{p|Q|} \ll Q^* \gg$ by

$$S_{q,z} = \begin{bmatrix} S_{q_1,q,z} \\ S_{q_2,q,z} \\ \vdots \\ S_{q_N,q,z} \end{bmatrix}, \quad S_f = \begin{bmatrix} S_{f,q_1} \\ S_{f,q_2} \\ \vdots \\ S_{f,q_N} \end{bmatrix}$$

where $Q = \{q_1, q_2, \dots, q_N\}$.

Define the set $J_{\Phi} = \Phi \cup \{(q,z) \mid q \in Q, z = 1,2,\dots,m\}$. Define the *indexed set of formal power series associated with* Φ by

$$\Psi_{\Phi} = \{S_j \in \mathbb{R}^{p|Q|} \ll Q^* \gg \mid j \in J_{\Phi}\} \tag{12}$$

Define the *Hankel-matrix of* Φ H_{Φ} as the Hankel-matrix of the associated set of formal power series, i.e. $H_{\Phi} := H_{\Psi_{\Phi}}$.

Notice that the only information needed to construct the set of formal power series Ψ_{Φ} are the high-order derivatives at zero of the functions belonging to Φ. The fact that Φ has a generalized kernel representation is needed only to ensure the correctness of the construction. *No knowledge of the analytic forms of the functions* $K^{f,\Phi}_w$, G^{Φ}_w *is required in order to construct* Ψ_{Φ}.

Let $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$ be a LSS, and assume that (Σ, μ) is a realization of Φ. Define the *representation associated with* (Σ, μ) by

$$R_{\Sigma,\mu} = (\mathcal{X}, \{A_q\}_{q \in Q}, \widetilde{B}, \widetilde{C})$$

where $\widetilde{C} : \mathcal{X} \to \mathbb{R}^{p|Q|}$, $\widetilde{C} = \begin{bmatrix} C_{q_1} \\ C_{q_2} \\ \vdots \\ C_{q_N} \end{bmatrix}$ and the indexed set $\widetilde{B} = \{B_j \in \mathcal{X} \mid j \in J_{\Phi}\}$ is defined by $\widetilde{B}_f = \mu(f), f \in \Phi$, and $\widetilde{B}_{q,l} = B_q e_l, l = 1,2,\dots,m, q \in Q$, e_l is the lth unit vector in $\mathcal{U}$.

Conversely, consider a representation of Ψ_{Φ}

$$R = (\mathcal{X}, \{A_q\}_{q \in Q}, \widetilde{B}, \widetilde{C})$$

Then define (Σ_R, μ_R) *the realization associated with* R by

$$\Sigma_R = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\}) \ , \ \mu_R(f) = \widetilde{B}_f$$

where $C_q : \mathcal{X} \to \mathcal{Y}$, $q \in Q$ are such that $\widetilde{C} = \begin{bmatrix} C_{q_1} \\ C_{q_2} \\ \vdots \\ C_{q_N} \end{bmatrix}$, and $B_q e_l = \widetilde{B}_{q,l}$ for each $l = 1,\dots,m$. It is easy to see that C_q, $q \in Q$ are well defined, since

$$C_q = \begin{bmatrix} e^T_{q,1} \widetilde{C} \\ \vdots \\ e^T_{q,p} \widetilde{C} \end{bmatrix}$$

Here for $q = q_z \in Q$ for some $z = 1, \ldots, N$, $i = 1, \ldots, p$ it holds that $e_{q,i} \in \mathbb{R}^{p|Q|}$ and $(e_{q,i})_j = \begin{cases} 1 & \text{if } j = p*(z-1)+i \\ 0 & \text{otherwise} \end{cases}$.
It is easy to see that $\Sigma_{R_{\Sigma,\mu}} = \Sigma$, $\mu_{R_{\Sigma,\mu}} = \mu$ and $R_{\Sigma_R,\mu_R} = R$. In fact, the following theorem holds.

Theorem 4. *Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$. Assume that Φ has a generalized kernel representation.*

(a) *(Σ, μ) is a realization of $\Phi \iff R_{\Sigma,\mu}$ is a representation of Ψ_Φ*

(b) *$R = (\mathcal{X}, \{A_q\}_{q \in Q}, \widetilde{B}, \widetilde{C})$ is a representation of $\Psi_\Phi \iff (\Sigma_R, \mu_R)$ is a realization of Φ*

Proof. First we prove part (a) of the theorem. By Proposition 5 (Σ, μ) is a realization of Φ if and only if for each $q_1, q_2, q \in Q$, $w = w_1 \cdots w_k \in Q^*$, $w_1, \ldots, w_k \in Q$, $k \geq 0$

$$D^{(1,\mathbb{I}_k,0)} y_0(e_z, q_2 w q_1, .) = S_{q_1,q_2,z}(w) = C_{q_1} A_w B_{q_2} e_z$$

$$D^{(\mathbb{I}_k,0)} f(0, wq, .) = S_{f,q}(w) = C_q A_w \mu(f)$$

Here, the notation $A_w = A_{w_k} \cdots A_{w_1}$ introduced in Section 4 is used. That is,

$$\begin{aligned} S_{q_2,z}(w) &= \begin{bmatrix} C_{q_1}^T & C_{q_2}^T & \cdots & C_{q_N}^T \end{bmatrix}^T A_w B_{q_2} e_z = \widetilde{C} A_w \widetilde{B}_{q_2,z} \\ S_f(w) &= \begin{bmatrix} C_{q_1}^T & C_{q_2}^T & \cdots & C_{q_N}^T \end{bmatrix}^T A_w \mu(f) = \widetilde{C} A_w \widetilde{B}_f \end{aligned}$$

That is, $R_{\Sigma,\mu}$ is a representation of Ψ.

Since $R = R_{\Sigma_R,\mu_R}$, part (b) follows from part (a). □

The theorem has the following corollary.

Corollary 3. *Let the assumptions of Theorem 4 hold. If (Σ, μ) is a minimal realization of Φ, then $R_{\Sigma,\mu}$ is a minimal representation of Ψ_Φ. Conversely, if R is a minimal representation of Ψ_Φ, then (Σ_R, μ_R) is a minimal realization of Φ.*

Proof. Notice that $\dim \Sigma = \dim R_{\Sigma,\mu}$ and $\dim \Sigma_R = \dim R$. The statement of the corollary follows now from Theorem 4. □

Theorem 5 (Realization of input/output map)**.** *For any set $\Phi \subseteq F(PC(T,\mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$ the following holds.*

(a) *Φ has a realization by a linear switched system if and only if Φ has a generalized kernel representation and Ψ_Φ is rational.*

(b) *Φ has a realization by a linear switched system if and only if Φ has a generalized kernel representation and* rank $H_\Phi < +\infty$.

Proof. `Part (a)`
If Φ has a realization, then Φ has a generalized kernel representation, moreover, by Theorem 4, Ψ_Φ has a representation, i,e. Ψ_Φ is rational. If Φ has a generalized kernel representation and Ψ_Φ is rational, i.e. it has a representation, then by Theorem 4 Φ has a realization.

`Part (b)`
By Theorem 1 $\dim H_\Phi < +\infty$ is equivalent to Ψ_Φ being rational. The rest of the statement follows now from Part (a) □

The theory of rational power series allows us to formulate necessary and sufficient conditions for a linear switched system to be minimal. Before formulating a characterization of minimal realizations, additional work has to be done. Let $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$ be a linear switched system. Using Proposition 1 it is easy to see that for any $\mu : \Phi \to \mathcal{X}$

$$\begin{aligned} W_{R_{\Sigma,\mu}} &= \text{Span}\{A_w x_0 \mid w \in Q^*, x_0 \in \text{Im}\mu \text{ or } x_0 = B_q u, q \in Q, u \in \mathcal{U}\} \\ &= \text{Span}\{A_{q_1} A_{q_2} \cdots A_{q_k} x_0 \mid q_1, q_2, \ldots, q_k \in Q, x_0 \in \text{Im}\mu\} + \\ &\quad + \mathit{Reach}(\Sigma, \{0\}) \end{aligned}$$

and

$$O_{R_{\Sigma,\mathcal{X}_0}} = O_\Sigma = \bigcap_{q,w_1,w_2,\ldots,w_k \in Q, k \geq 0} \ker C_q A_{w_k} A_{w_{k-1}} \cdots A_{w_1}$$

Moreover, the following is true

Lemma 6. *$W_{R_{\Sigma,\mu}}$ is the smallest vector space containing $Reach(\Sigma, \mathrm{Im}\mu)$.*

Proof. Denote by WR the set $W_{R_{\Sigma,\mu}}$. Denote by $\mathcal{X}_0$ the image of μ.

First, we show that $Reach(\Sigma, \mathcal{X}_0)$ is contained in WR. From Proposition 1 it follows that

$$Reach(\Sigma, \mathcal{X}_0) = \{\exp(A_{q_k} t_k) \exp(A_{q_{k-1}} t_{k-1}) \cdots \exp(A_{q_1} t_1) x_0 + \\ + x_\Sigma(0, u, (q_1, t_1) \cdots (q_k, t_k)) \mid x_0 \in \mathcal{X}_0, (q_1, t_1)(q_2, t_2), \ldots, (q_k, t_k) \in (Q \times T)^*, k \geq 0, u \in PC(T, \mathcal{U})\}$$

But $\exp(A_q t)x = \sum_0^{+\infty} \frac{t^k}{t!} A_q^k x \in \mathrm{Span}\{A_q^j x \mid j \in \mathbb{N}\}$, which implies that

$$\exp(A_{q_k} t_k) \cdots \exp(A_{q_1} t_1) x_0 \in \mathrm{Span}\{A_{w_1} A_{w_2} \cdots A_{w_k} x_0 \mid w_1, w_2, \ldots, w_k \in Q\}$$

Since $x(0, u, (q_1, t_1) \cdots (q_k, t_k)) \in Reach(\Sigma, \{0\})$, we get that $Reach(\Sigma, \mathcal{X}_0) \subseteq WR$.

We will show that WR is the smallest vector space containing $Reach(\Sigma, \mathcal{X}_0)$. Let W be a subspace of $\mathcal{X}$ containing $Reach(\Sigma, \mathcal{X}_0)$. For any $\alpha \in \mathbb{N}^{|w|}$, for any constant input function $u(t) = u \in \mathcal{U}$ $D^\alpha x(x_0, u, w, .) \in W$ must hold. But $x(x_0, u, w, \underline{t}) = x(x_0, 0, w, \underline{t}) + x(0, u, w, \underline{t})$. It is straightforward to show that $\mathrm{Span}\{D^\alpha x(0, u, w, .) \mid w \in Q^+, \alpha \in \mathbb{N}^{|w|}, u \in \mathcal{U}\} = Reach(\Sigma, 0)$. For $w \in Q^+, k := |w|$ define $\exp_w : T^k \to \mathcal{X}$ by

$$\exp_w(t_1, t_2, \ldots, t_k) = \exp(A_{w_k} t_k) \exp(A_{w_{k-1}} t_{k-1}) \cdots \exp(A_{w_1} t_1) x_0$$

It is easy to see that $D^\alpha x(x_0, 0, w, .) = D^\alpha \exp_w = A_{w_k}^{\alpha_k} A_{w_{k-1}}^{\alpha_{k-1}} \cdots A_{w_1}^{\alpha_1} x_0$, and therefore $\mathrm{Span}\{D^\alpha x(x_0, 0, w, .) \mid w \in Q^+, \alpha \in \mathbb{N}^{|w|}, x_0 \in \mathcal{X}_0\} = \mathrm{Span}\{A_w x_0 \mid w \in Q^+\}$. Thus, we get that

$$\mathrm{Span}\{D^\alpha x(x_0, u, w, .) \mid w \in Q^+, \alpha \in \mathbb{N}^{|w|}, u \in \mathcal{U}, x_0 \in \mathcal{X}_0\} = WR$$

which implies that $WR \subseteq W$. $\square$

The results above imply the following

Corollary 4. *Let $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$ and assume that (Σ, μ) is a realization of Φ. Then Σ is observable if and only if R is observable. Σ is semi-reachable from* $\mathrm{Im}\mu$ *if and only if R is reachable.*

It is a natural question to ask what the relationship is between linear switched system morphisms and representation morphisms. The following lemma answers this question.

Lemma 7. *$T : (\Sigma, \mu) \to (\Sigma', \mu')$ is a linear switched system morphism if and only if $T : R_{\Sigma,\mu} \to R_{\Sigma',\mu'}$ is a representation morphism.*

Recall that $T : (\Sigma, \mu) \to (\Sigma', \mu')$ is a linear switched system morphism if T is a linear map from the state-space of Σ to the state-space of Σ' satisfying certain properties. Recall that a representation morphism between two representations is a linear map between the state-spaces of the representations which satisfies certain properties. Since the state spaces of $R_{\Sigma,\mu}$ and $R_{\Sigma',\mu'}$ coincide with the state-space of Σ and Σ' respectively, it is justified to denote both the linear switched system morphism and the representation morphism by the same symbol, indicating that the underlying linear map is the same.

Proof of Lemma 7. Assume that the linear switched systems Σ and Σ' are of the form

$$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\}) \text{ and } \Sigma' = (\mathcal{X}', \mathcal{U}, \mathcal{Y}, Q, \{(A'_q, B'_q, C'_q) \mid q \in Q\})$$

Then T is a switched linear system morphism if and only if $TA_q = A^{'}_qT$, $C_q = C^{'}_qT$, $TB_q = B^{'}_q$ and $T\mu(f) = \mu^{'}(f)$ for each $q \in Q, f \in \Phi$. But this is equivalent to $TA_q = A^{'}_qT, q \in Q$, $T\widetilde{B}_j = \widetilde{B}^{'}_j$ and $\widetilde{C} = \begin{bmatrix}C^T_{q_1} & \cdots & C^T_{q_N}\end{bmatrix}^T = \begin{bmatrix}(C^{'}_{q_1}T)^T & \cdots & (C^{'}_{q_N}T)^T\end{bmatrix}^T = \widetilde{C}'T$, that is, to T being a representation morphism. □

Now we can state the main result of the section.

Theorem 6 (Minimal realizations). *If (Σ, μ) is a realization of Φ, then the following are equivalent.*

(i) (Σ, μ) *is minimal*
(ii) Σ *is semi-reachable from* $\mathrm{Im}\mu$ *and it is observable*
(iii) $\dim \Sigma = \dim H_\Phi$
(iv) *If $(\Sigma^{'}, \mu^{'})$ realizes Φ and $\Sigma^{'}$ is semi-reachable from* $\mathrm{Im}\mu^{'}$*, then there exists a surjective linear switched system morphism $T : (\Sigma^{'}, \mu^{'}) \to (\Sigma, \mu)$. In particular, all minimal realizations of Φ are algebraically similar.*

Proof. *(i) $\Longleftrightarrow$ (ii)*
By Corollary 3 system (Σ, μ) is minimal if and only if $R := R_{\Sigma,\mu}$ is minimal. By Theorem 2 R is minimal if and only if R is reachable and observable. By Corollary 4 the latter is equivalent to Σ being semi-reachable from $\mathrm{Im}\mu$ and observable.

(i) $\Longleftrightarrow$ (iii)
By Corollary 3 (Σ, μ) is minimal $\Longleftrightarrow$ $R_{\Sigma,\mu}$ is minimal. By Theorem 2 $R_{\Sigma,\mu}$ is minimal $\Longleftrightarrow$ $\dim R_{\Sigma,\mu} = \dim \Sigma = \mathrm{rank}\ H_{\Psi_\Phi} = \mathrm{rank}\ H_\Phi$.

(i) $\Longleftrightarrow$ (iv)
Again we are using the fact that (Σ, μ) is minimal if and only if $R_{\Sigma,\mu}$ is minimal. By Theorem 2 R_{min} is minimal if and only if for any reachable representation R there exists a surjective representation morphism $T : R \to R_{min}$. It means that (Σ, μ) is minimal if and only if for any reachable representation R of Ψ_Φ there exists a surjective representation morphism $T : R \to R_{\Sigma,\mu}$. But any reachable representation R gives rise to a semi-reachable realization of Φ and vice versa. That is, we get that (Σ, μ) is minimal if and only if for any semi-reachable realization $(\Sigma^{'}, \mu^{'})$ of Φ there exists a surjective representation morphism $T : R_{\Sigma^{'},\mu^{'}} \to R_{\Sigma,\mu}$. By Lemma 7 we get that the latter is equivalent to $T : (\Sigma^{'}, \mu^{'}) \to (\Sigma, \mu)$ being a surjective linear switched system morphism. From Corollary 1 it follows that if $(\Sigma^{'}, \mu^{'})$ is a minimal realization of Φ, then there exists a representation isomorphism $T : R_{\Sigma^{'},\mu^{'}} \to R_{\Sigma,\mu}$ which means that (Σ, μ) is gives rise to the linear switched system isomorphism $T : (\Sigma^{'}, \mu^{'}) \to (\Sigma, \mu)$, that is, $\Sigma^{'}$ and Σ are algebraically similar. □

5.3. Realization of input-output maps with constraints on the switching

In this section the solution of the realization problem with constraints will be presented. That is, given a set of constraints $L \subseteq Q^+$ and a set of input-output maps with domain $PC(T, \mathcal{U}) \times TL$ we will study linear switched systems realizing this set with constraint L. As in the previous section, the theory of formal power series will be our main tool in solving the realization problem.

Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times TL, \mathcal{Y})$. Recall that (Σ, μ) *realizes* Φ with constraint L if for all $f \in \Phi$ it holds that $f = y_\Sigma(\mu(f), ., .)|_{PC(T,\mathcal{U})\times TL}$. In the sequel, unless stated otherwise, we assume that Φ has a generalized kernel representation with constraint L.

The solution of the realization problem for Φ goes as follows. As in the previous section, we associate a set of formal power series Ψ_Φ with the set of maps Φ. We will show that any representation of Ψ_Φ gives rise to a realization of Φ with constraint L. If L is regular, then any realization of Φ with constraint L gives rise to a representation of Ψ_Φ. Unfortunately minimal representations of Ψ_Φ do not yield minimal realizations of Φ. However, any minimal representation of Ψ_Φ yields an observable and semi-reachable realization of Φ.

Recall from Section 5.1 the definition of the languages $\widetilde{L}$, $\widetilde{L}_{q_1,q_2}$, $\widetilde{L}_q$ and the sets $F_{q_1,q_2}(w)$, $F_q(w)$. Let $E = (1, 1, \dots, 1) \in \mathbb{R}^{1\times p}$. Define the power series $Z_{q_1,q_2} \in \mathbb{R}^p \ll Q^* \gg$ by

$$Z_{q_1,q_2}(w) = \begin{cases} E^T & \text{if } w \in \widetilde{L}_{q_1,q_2} \\ 0 & \text{otherwise} \end{cases}$$

Define the power series $\Gamma_q \in \mathbb{R}^{p|Q|} \ll Q^* \gg$ by

$$\Gamma_q = \begin{bmatrix} Z_{q_1,q} \\ Z_{q_2,q} \\ \vdots \\ Z_{q_N,q} \end{bmatrix}$$

and $\Gamma \in \mathbb{R}^{p|Q|} \ll Q^* \gg$ by

$$\Gamma = \begin{bmatrix} Z_{q_1} \\ Z_{q_2} \\ \vdots \\ Z_{q_N} \end{bmatrix}$$

where $Z_q(w) = \begin{cases} E^T & \text{if } w \in \widetilde{L}_q \\ 0 & \text{otherwise} \end{cases}$ and $Q = \{q_1, \dots, q_N\}$. It is a straightforward exercise in automata theory to show that if L is regular, then the languages $\widetilde{L}_q$ and $\widetilde{L}_{q_1,q_2}$ are regular.

Lemma 8. *With the notation above, if $L \subseteq Q^+$ is a regular language, then $\widetilde{L}$, $\widetilde{L}_{q_1,q_2}$ and $\widetilde{L}_q$ are regular languages for each $q, q_1, q_2 \in Q$.*

Proof. Notice that $\widetilde{L}_{q_1,q_2} = \{w \in Q^* \mid q_1 w q_2 \in \widetilde{L}\}$ and $\widetilde{L}_q = \{w \in Q^* \mid wq \in \widetilde{L}\}$. It is easy to see that if $\widetilde{L}$ is regular, then so are $\widetilde{L}_{q_1,q_2}$ and $\widetilde{L}_q$. It is also easy to see that if L is regular then suffix$L$ is regular. Let $A = (S, Q, \delta, F, s_0)$ be a deterministic automaton accepting suffix$L$. Here S is the state-space, F is the set of accepting states, δ is the state-transition function, s_0 is the set of initial states. Recall, that the extended state-transition function is defined as follows. For each $s_0 \in S, w \in Q^*$, $\delta(s_0, w) = s$ if there exists $s_1, \dots, s_k = s \in Q$ such that $w = w_1 \cdots w_k \in Q^k$ and $s_i = \delta(s_{i-1}, w_i)$ for each $i = 1, \dots, k$.

Define the non-deterministic automaton $B = ((S \times Q) \cup \{s_0^{'}\}, Q, \delta_B, F \times Q, s_0^{'})$ in the following way. Let $\delta_B(s_0^{'}, x) \ni (s, x)$ if $\delta(s_0, wx) = s$ for some $w \in Q^*$. Let $(s^{'}, u) \in \delta_B((s, x), u)$ if either

(i) $u = x$ and $s^{'} = s$, or
(ii) there exists $wu \in Q^*$, such that $\delta(s, wu) = s^{'}$.

We will prove that B accepts $\widetilde{L}$. Denote $s \in \delta_B(z, x)$, $s, z \in (S \times Q) \cup \{s_0^{'}\}$ by $z \xrightarrow{x} s$. Then B accepts $z = z_1 \cdots z_k$ if and only if

$$s_0^{'} \xrightarrow{z_1} (s_1, z_1) \xrightarrow{z_2} \cdots \xrightarrow{z_k} (s_k, z_k)$$

where $s_k \in F$. This is equivalent to the existence of $0 < \alpha_1, \dots, \alpha_l \in \mathbb{N}$ and $w_0, \dots, w_l \in Q^*$ such that $\sum_{j=1}^{l} \alpha_j = k$, $\delta(s_0, w_0 z_1) = s_1$ and $(s_i, z_i) = (s_{i+1}, z_{i+1})$ for each $1 + \sum_1^d \alpha_j \le i < \sum_1^{d+1} \alpha_j$ and $\delta(s_{\sum_1^d \alpha_j}, w_d z_{\sum_1^d \alpha_j}) = s_{1+\sum_1^d \alpha_j}$ for all $0 \le d \le l-1$. Define $u_d = z_{1+\sum_1^d \alpha_j}$. Then it is clear that in the original automaton A it holds that $\delta(s_0, w_0 u_0 w_1 u_1 \cdots w_l u_l) = s_k \in F$. That is, $w_0 u_0 \cdots w_l u_l \in$ suffix$L$ and

$$z = w_{0,1}^0 \cdots w_{0,m_0}^0 u_1^{\alpha_1} w_{1,1}^0 \cdots w_{m_1,1}^0 u_2^{\alpha_2} \cdots w_{l,1}^0 \cdots w_{l,m_l}^0 u_l^{\alpha_l}$$

where $w_i = w_{i,1} \cdots w_{i,m_i}$, $w_{i,1}, \dots, w_{i,m(i)} \in Q$. We get that B accepts exactly the elements of $\widetilde{L}$. □

Corollary 5. *Define the indexed set of formal power series* $\Omega = \{\Lambda_j \in \mathbb{R}^{pN} \ll Q^* \gg \mid j \in Q \times \{\emptyset\}\}$*, where* $\Lambda_q = \Gamma_q$ *and* $\Lambda_\emptyset = \Gamma$*. If* L *regular then the indexed set of formal power series* Ω *is rational.*

Proof. Indeed, if L is regular, then $\widetilde{L}_{q_1,q_2}$ and $\widetilde{L}_q$ are regular languages. Then it is easy to see that for each $l = 1, \dots, pN$, such that $l = p*(z-1)+i$ for some $z = 1, \dots, N$, $i = 1, \dots p$, $(\Gamma)_l(w) = \begin{cases} 1 & \text{if } w \in L_{q_z} \\ 0 & \text{otherwise} \end{cases}$ and $(\Gamma_q)_l(w) = \begin{cases} 1 & \text{if } w \in L_{q_z,q} \\ 0 & \text{otherwise} \end{cases}$. That is, $(\Gamma_q)_l, \Gamma_l \in \mathbb{R} \ll Q^* \gg$ are rational formal power series for each $l = 1, \dots, pN$. Consider the indexed set $\Theta = \{(\Lambda_{(l,j)} \mid (l,j) \in \{1, \dots, pN\} \times (Q \cup \{\emptyset\})\}$, where $\Lambda_{(l,q)} = (\Lambda_q)_l = (\Gamma_q)_l$, $\Lambda_{(l,\emptyset)} = (\Lambda_\emptyset)_l = \Gamma_l$. Then by Corollary 2 from Section 4, Θ is rational. By Lemma 1 from Section 4, it implies that Ω is rational. □

Consider a set of input-output maps $\Phi \subseteq F(PC(T, \mathcal{U}) \times TL, \mathcal{Y})$ with a $L \subseteq Q^*$. Assume that Φ has a generalized kernel representation.

Recall that for any $\alpha \in \mathbb{N}^k$, α^+ denotes $\alpha^+ = (\alpha_1 + 1, \alpha_2, \dots, \alpha_k)$. We define the following formal power series. For $j = 1, 2, \dots, m$ and $f \in \Phi$, $q_1, q_2 \in Q$,

$$S_{q_1,q_2,j}(w) = \begin{cases} D^{(\mathbb{O}_{|v|},\alpha^+)} y_0^\Phi(e_j, vz, .) & \text{if } w \in \widetilde{L}_{q_1,q_2} \text{ and} \\ & (v,(\alpha,z)) \in F_{q_1,q_2}(w) \\ 0 & \text{otherwise} \end{cases}$$

$$S_{q,f}(w) = \begin{cases} D^{(\mathbb{O}_{|v|},\alpha)} f(0, vz, .) & \text{if } w \in \widetilde{L}_q \text{ and } (v,(\alpha,z)) \in F_q(w) \\ 0 & \text{otherwise} \end{cases}$$

We will show that the series $S_{q_1,q_2,z}$ and $S_{q,f}$ are well-defined. Using formulas (4), (5) and (3) from Subsection 5.1 and the fact that $(v,(\alpha,z)) \in F_{q_1,q_2}(w) \implies z_1 = q_2, z_{|z|} = q_1$ and $(v,(\alpha,z)) \in F_q(w) \implies z_{|z|} = q$ we get the following

$$S_{q_1,q_2,j}(w) = \begin{cases} D^\alpha G_z^\Phi = D^{(0,\alpha,0)} G_{q_2 z q_1}^\Phi e_j & \text{if } w \in \widetilde{L}_{q_1,q_2} \text{ and} \\ & (v,(\alpha,z)) \in F_{q_1,q_2}(w) \\ 0 & \text{otherwise} \end{cases}$$

$$S_{q,f}(w) = \begin{cases} D^{(\mathbb{O}_{|v|},\alpha)} K_{vz}^{f,\Phi} = D^\alpha K_z^{f,\Phi} = D^{(\alpha,0)} K_{zq}^{f,\Phi} & \text{if } w \in \widetilde{L}_q \text{ and} \\ & (v,(\alpha,z)) \in F_q(w) \\ 0 & \text{otherwise} \end{cases}$$

That is, $S_{q_1,q_2,j}(w)$ and $S_{q,f}(w)$ do not depend on the choice of v in $(v,(\alpha,z)) \in F_{q_1,q_2}(w)$ or $(v,(\alpha,z)) \in F_q(w)$ respectively. We will argue that the value of $S_{q_1,q_2,z}(w)$ and $S_{q,f}(w)$ do not depend on the choice of (α, z). If $(v,(\alpha,z)),(u,(\beta,x)) \in F_{q_1,q_2}(w)$ then $x_1^{\beta_1} \cdots x_{|x|}^{\beta_{|x|}} = z_1^{\alpha_1} \cdots z_{|z|}^{\alpha_{|z|}} = w$, $z_1 = x_1 = q_2$, $z_{|z|} = x_{|x|} = q_1$ and $q_2 z q_1, q_2 x q_1 \in \widetilde{L}$, so by Proposition 2, $D^{(0,\alpha,0)} G_{q_2 z q_1}^\Phi = D^{(0,\beta,0)} G_{q_2 x q_1}^\Phi$. Similarly, if $(v,(\alpha,z)),(u,(\beta,x)) \in F_q(w)$, then $x_1^{\beta_1} \cdots x_{|x|}^{\beta_{|x|}} = z_1^{\alpha_1} \cdots z_{|z|}^{\alpha_{|z|}} = w$ and $zq, xq \in \widetilde{L}$, so by Proposition 2, $D^{(\alpha,0)} K_{zq}^{f,\Phi} = D^{(\beta,0)} K_{q_2 x q_1}^{f,\Phi}$.

Define the formal power series $S_{q,j}, S_f \in \mathbb{R}^{p|Q|} \ll Q^* \gg$, $j \in \{1, 2, \dots, m\}$, $q \in Q$ and $f \in \Phi$ by

$$S_{q,j} = \begin{bmatrix} S_{q_1,q,j} \\ S_{q_2,q,j} \\ \vdots \\ S_{q_N,q,j} \end{bmatrix}, \quad S_f = \begin{bmatrix} S_{q_1,f} \\ S_{q_2,f} \\ \vdots \\ S_{q_N,f} \end{bmatrix}$$

Define the *indexed set of formal power series associated with* Φ as $\Psi_\Phi = \{S_z \in \mathbb{R}^{p|Q|} \ll Q^* \gg \mid z \in J_\Phi\}$ where $J_\Phi = \Phi \cup (Q \times \{1, 2, \dots, m\})\}$. Define the *Hankel-matrix* H_Φ as the Hankel-matrix of Ψ_Φ.

Consider the map $g : \Phi \cup (Q \times \{1, 2, \dots, m\}) \to Q \times \{\emptyset\}$, where $g(f) = \emptyset, \forall f \in \Phi$ and $g((q, z)) = q$ for all $q \in Q, z = 1, \dots, m$. Recall the indexed set of formal power series Ω from Corollary 5. Define the indexed set of formal power series $\Omega_\Phi = \{\Xi_j \in \mathbb{R}^{pN} \ll Q^* \gg \mid j \in J_\Phi\}$ by $\Xi_j = \Lambda_{g(j)}$, where $\Omega = \{\Lambda_j \mid j \in Q \cup \{\emptyset\}\}$. From Lemma 4 of Section 4 and Corollary 5 it follows that if L is regular, then Ω_Φ is rational. Let (Σ, μ) be a realization of Φ. Define $\Theta_{\Sigma,\mu} = \{y_\Sigma(\mu(f), ., .) \mid f \in \Phi\} \subseteq F(PC(T, \mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$. Define $U(\mu) : \Theta_{\Sigma,\mu} \to \Phi$ by $U(\mu)(y_\Sigma(\mu(f), ., .)) = f$. The map $U(\mu)$ is well defined. Indeed, if $y_\Sigma(\mu(f_1), ., .) = y_\Sigma(\mu(f_2), ., .)$, then $f_1 = y_\Sigma(\mu(f_1), ., .)|_{PC(T,\mathcal{U})\times TL} = y_\Sigma(\mu(f_2), ., .)|_{PC(T,\mathcal{U})\times TL} = f_2$. It is easy to see that $(\Sigma, \mu \circ U(\mu))$ is a realization of $\Theta_{\Sigma,\mu}$. Assume that the set of formal power series associated to $\Theta_{\Sigma,\mu}$ as defined in Section 5.2, (12), is of the form

$$\Psi_{\Theta_{\Sigma,\mu}} = \{T_z \in \mathbb{R}^{p|Q|} \ll Q^* \gg \mid z \in \Theta_{\Sigma,\mu} \cup (Q \times \{1, 2, \dots, m\})\}$$

From Theorem 5 it follows that $\Psi_{\Theta_{\Sigma,\mu}}$ is rational. Define the map $\psi : J_\Phi \to \Theta_{\Sigma,\mu} \cup (Q \times \{1, 2, \dots, m\})$ by $\psi(f) = y_\Sigma(\mu(f), ., .), f \in \Phi$ and $\psi((q, z)) = (q, z), q \in Q, z = 1, \dots, m$. Define $K_{\Sigma,\mu} = \{V_j \in \mathbb{R}^{p|Q|} \ll Q^* \gg \mid j \in J_\Phi\}$, $V_j = T_{\psi(j)}, j \in J_\Phi$. From Lemma 4 of Section 4 it follows that $K_{\Sigma,\mu}$ is rational.

Let $R = (\mathcal{X}, \{A_z\}_{z\in Q}, B, C)$ be a representation of Ψ_Φ. Define (Σ_R, μ_R) the *linear switched system realization associated with* R as in Section 5.2. That is,

$$\Sigma_R = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\}) \text{ and } \mu_R(f) = B_f$$

where $C_q : \mathcal{X} \to \mathcal{Y}$, $q \in Q$ are such that $C = \begin{bmatrix} C_{q_1} \\ \vdots \\ C_{q_N} \end{bmatrix}$ and $B_q e_j = B_{(q,j)}$ for all $q \in Q, j = 1, \dots, m$. Assume that the resulting (Σ_R, μ_R) is a realization of Φ (in fact, this will be shown later). Let $(\Sigma, \mu) = (\Sigma_R, \mu_R \circ U(\mu_R))$. Then (Σ, μ) is a realization of Θ_{Σ_R,μ_R}. Let $\widetilde{R} = R_{\Sigma,\mu}$ – the representation associated to (Σ, μ) as defined in Section 5.2. Then it is easy to see that $\widetilde{R} = (\mathcal{X}, \{A_q\}_{q\in Q}, \widetilde{B}, C)$, where $\widetilde{B}_{y_{\Sigma_R}(\mu_R(f),.,.)} = \mu(y_{\Sigma_R}(\mu_R(f), ., .)) = \mu_R(f) = B_f$, $f \in \Phi$ and $\widetilde{B}_{(q,j)} = B_q e_j = B_{(q,j)}$, $q \in Q, j = 1, \dots, m$. That is, R is observable if and only if $\widetilde{R}$ is observable. R is reachable if and only if $\widetilde{R}$ is reachable. It is also straightforward to see that $\mathrm{Im}\mu_R = \mathrm{Im}\mu_R \circ U(\mu_R) = \mathrm{Im}\mu$. Thus, by Corollary 4, the following holds. Σ_R is observable if and only if R is observable. (Σ_R, μ_R) is semi-reachable if and only if R is reachable.

Using the notation above and combining Proposition 4 and the definition of rational sets of power series one gets the following theorems.

Theorem 7. *Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times TL, \mathcal{Y})$. Then (Σ, μ) is realization of Φ with constraint L if and only if Φ has a general kernel representation with constraint L and*

$$\Psi_\Phi = \Omega_\Phi \odot K_{\Sigma,\mu}$$

or, in other words

$$\begin{aligned} &\forall f \in \Phi, q \in Q, z = 1, 2, \dots, m \\ &\quad S_f = T_{y_\Sigma(\mu(f),.,.)} \odot \Gamma \text{ and } S_{q,z} = T_{q,z} \odot \Gamma_q \end{aligned}$$

Proof. By Proposition 4 (Σ, μ) is a realization of Φ with constraint L, if and only if Φ has a generalized kernel representation with constraint L and

$$\begin{aligned} &\forall w \in \widetilde{L}_{q_1,q_2}, (v, (\alpha, z)) \in F_{q_1,q_2}(w) : \\ &\qquad D^{(0,\alpha,0)} G^{\Phi}_{q_2 z q_1} = C_{q_1} A^{\alpha_k}_{z_k} \cdots A^{\alpha_1}_{z_1} B_{q_2} = C_{q_1} A_w B_{q_2} \\ &\forall w \in \widetilde{L}_q, (v, (\alpha, z)) \in F_q(w) : \\ &\qquad D^{(\alpha,0)} K^{f,\Phi}_{z q_1} = C_{q_1} A^{\alpha_k}_{z_k} \cdots A^{\alpha_1}_{z_1} \mu(f) = C_{q_1} A_w \mu(f) \end{aligned}$$

But $(\Sigma, \mu \circ U(\mu))$ is also a realization of $\Theta = \Theta_{\Sigma,\mu}$ with constraint Q^+, so by Proposition 5 we get that

$$C_{q_1} A_w B_{q_2} = D^{(0,\mathbb{I}_{|w|},0)} G^{\Theta}_{q_2 w q_1} \text{ and } C_q A_w \mu(f) = C_q A_w \mu(U(\mu)(y_\Sigma(\mu(f), ., .))) = D^{(\mathbb{I}_{|w|},0)} K^{y_\Sigma(\mu(f),.,.),\Theta}_{wq}$$

That is, for each $w \in \widetilde{L}_{q_1,q_2}$, $(v,(\alpha,z)) \in F_{q_1,q_2}(w)$, $q_1, q_2 \in Q$, $j = 1, \ldots, m$

$$T_{q_1,q_2,j}(w) = D^{(0,\mathbb{I}_{|w|},0)} G^{\Theta}_{q_2 w q_1} e_j = D^{(0,\alpha,0)} G^{\Phi}_{q_2 z q_1} e_j = S_{q_1,q_2,j}(w)$$

and for each $w \in \widetilde{L}_q$, $(v,(\alpha,z)) \in F_q(w)$

$$T_{q,y_\Sigma(\mu(f),.,.)}(w) = D^{(\mathbb{I}_{|w|},0)} K^{y_\Sigma(\mu(f),.,.),\Theta}_{wq} = D^{(\alpha,0)} K^{f,\Phi}_{zq} = S_{q,f}(w)$$

We get that

$$\begin{array}{rcll} T_{q_1,y_\Sigma(\mu(f),.,.)}(w) & = & S_{q_1,f}(w) & \text{if } w \in \widetilde{L}_{q_1} \\ T_{q_1,z_2,z}(w) & = & S_{q_1,q_2,z}(w) & \text{if } w \in \widetilde{L}_{q_1,q_2} \end{array}$$

Notice that if $w \notin \widetilde{L}_{q_1,q_2}$, then $S_{q_1,q_2,z}(w) = 0$ and $Z_{q_1,q_2}(w) = 0$. Similarly, If $w \notin \widetilde{L}_{q_1}$, then $S_{q_1,f}(w) = 0 = Z_{q_1}(w)$. That is,

$$T_{q,z} \odot \Gamma_q = S_{q,z} \text{ and } T_{y_\Sigma(\mu(f),.,.)} \odot \Gamma = S_f$$

□

Define the language

$$comp(L) = \{w_1 \cdots w_k \in Q^* \mid \widetilde{L}_{w_k} = \emptyset\}$$

Intuitively, the language $comp(L)$ contains those sequences which can never be observed if the switching system is run with constraint L.

Theorem 8. *Assume that Φ has a generalized kernel representation with constraint L. If*

$$R = (\{A_q\}_{q \in Q}, B, C)$$

is a representation of Ψ_Φ, then (Σ_R, μ_R) realizes Φ. Moreover,

$$\forall f \in \Phi, \forall u \in PC(T, \mathcal{U}), w \in T(comp(L)) : y_{\Sigma_R}(\mu_R(f), u, w) = 0$$

Proof. Let $(\Sigma, \mu) = (\Sigma_R, \mu_R)$. If R is a representation of Φ, then

$$\begin{array}{rcl} \forall w \in \widetilde{L}_{q_1,q_2}, (v,(\alpha,z)) \in F_{q_1,q_2}(w) & & \\ D^{(0,\alpha,0)} G^{\Phi}_{q_2 z q_1} e_j & = & S_{q_1,q_2,j}(w) = C_{q_1} A_w B_{q_2,j} \\ & = & C_{q_1} A^{\alpha_k}_{z_k} \cdots A^{\alpha_1}_{z_1} B_{q_2} e_j \\ \forall w \in \widetilde{L}_q, (v,(\alpha,z)) \in F_q(w) & & \\ D^{(\alpha,0)} K^{f,\Phi}_{zq} & = & S_{q,f}(w) = C_q A_w B_f \\ & = & C_q A^{\alpha_1}_{z_1} \cdots A^{\alpha_k}_{z_k} \mu(f) \end{array} \tag{13}$$

Since Φ has a generalized kernel representation, Proposition 4 and (13) yield that (Σ, μ) is a realization of Φ with constraint L.

Let $\Phi' = \Theta_{\Sigma,\mu}$. Then $(\Sigma, \mu \circ U(\mu))$ is a realization of Φ'. It is easy to see that for all $f \in \Phi$, $q_1, q_2 \in Q$, $z = 1, \ldots, m$,

$$\begin{array}{ll} S_{q,f}(w) = C_q A_w \mu(f) = 0 & \text{if } w \notin \widetilde{L}_q \\ S_{q_1,q_2,z}(w) = C_{q_1} A_w B_{q_2} e_z = 0 & \text{if } w \notin \widetilde{L}_{q_1} \supseteq \widetilde{L}_{q_1,q_2} \end{array}$$

As the second step we are going to show that for each $w \in comp(L)$, $y_{\Sigma}(\mu(f),.,.) \in \Phi^{'}$,

$$G_w^{\Phi^{'}} = 0 \text{ and } K_w^{y_{\Sigma}(\mu(f),.,.),\Phi^{'}} = 0 \tag{14}$$

Because of analyticity of these function it is enough to prove that for each $\alpha \in \mathbb{N}^{|w|}$: $D^{\alpha}G_w^{\Phi^{'}} = 0$, $D^{\alpha}K_w^{y_{\Sigma}(\mu(f),.,.),\Phi^{'}} = 0$. But from formulas (4), (5) and Proposition 4 we get that

$$D^{\alpha}G_w^{\Phi^{'}} = C_{w_k}A_vB_{w_1} \text{ and } D^{\alpha}K_w^{y_{\Sigma}(\mu(f),.,.),\Phi^{'}} = C_{w_k}A_v(\mu \circ U(\mu))(y_{\Sigma}(\mu(f),.,.)) = C_{w_k}A_v\mu(f)$$

$w = w_1, \cdots w_k$, $w_1, \ldots, w_k \in Q$, $v = w_1^{\alpha_1} \cdots w_k^{\alpha_k}$. But $w \in comp(L)$ implies $\widetilde{L}_{w_k} = \emptyset$, that is $u \notin \widetilde{L}_{w_k,w_l}$ and $v \notin \widetilde{L}_{w_k}$. Then it follows that $C_{w_k}A_vB_{w_1} = 0$ and $C_{w_k}A_v\mu(f) = 0$. It implies that $D^{\alpha}G_w^{\Phi^{'}} = 0$ and $D^{\alpha}K_w^{f,\Phi^{'}} = 0$.

It is easy to see that if $w_1 \cdots w_k \in comp(L)$, then for any $l \leq k$, $w_l \cdots w_k \in comp(L)$. Then from Definition 3, part 4 it follows that (14) implies $y_{\Sigma}(\mu(f),u,w) = 0$ for all $u \in PC(T,\mathcal{U})$ and $w \in T(comp(L))$. □

If L regular then the power series Γ, Γ_q, $(q \in Q)$ are rational. Then using Theorem 7 and Lemma 2 from Section 4 one gets the following.

Theorem 9. *Consider a language $L \subseteq Q^+$ and a set $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ of input-output maps. Assume that L is regular. Then the following holds.*

(i) *Φ has a realization by a linear switched system with constraint L if and only if Φ has a generalized kernel representation with constraint L and Ψ_{Φ} is rational, or equivalently* $\dim H_{\Phi} < +\infty$.

(ii) *Φ has a realization by a linear switched system with constraint L if and only if there exists a linear switched system realization (Σ,μ) of Φ with constraint L, such that (Σ,μ) is semi-reachable, it is observable, and*

$$\forall f \in \Phi : y_{\Sigma}(\mu(f),.,.)|_{PC(T,\mathcal{U}) \times T(comp(L))} = 0$$

Proof. `Part (i)`
If Φ has a generalized kernel representation with constraint L and Ψ_{Φ} is rational, then there exists a representation R of Ψ_{Φ} and by Theorem 8 (Σ_R,μ_R) is a realization of Φ. Conversely, assume that Φ is realized by (Σ,μ). Then by Theorem 7 Φ has a generalized kernel representation and with the notation of Theorem 7 it holds that $\Psi_{\Phi} = \Omega_{\Phi} \odot K_{\Sigma,\mu}$. Since $(\Sigma, \mu \circ U(\mu))$ is a realization of $\Theta_{\Sigma,\mu}$ without constraint, by Theorem 5 $\Psi_{\Theta_{\Sigma,\mu}}$ is rational. Then by Lemma 4 $K_{\Sigma,\mu}$ is rational too. If L is regular, then by Corollary 5 Ω is rational. Then by Lemma 4 Ω_{Φ} is rational. By Lemma 2 we get that $\Psi_{\Phi} = \Omega_{\Phi} \odot K_{\Sigma,\mu}$ is rational. From Theorem 1 it follows that Ψ_{Φ} is rational if and only if rank $H_{\Psi_{\Phi}} < +\infty$. By definition $H_{\Phi} = H_{\Psi_{\Phi}}$, so we get that Ψ_{Φ} is rational if and only if rank $H_{\Phi} < +\infty$.

`Part(ii)`
Φ has a realization with constraint L if and only if Φ has a generalized kernel representation with constraint L and Ψ_{Φ} is rational. Let $R = (\{A_q\}_{q\in Q}, B, C)$ be a minimal representation of Ψ_{Φ}. Consider $(\Sigma,\mu) = (\Sigma_R,\mu_R)$ – the linear switched system realization associated with R. Then by Theorem 8 (Σ,μ) is a realization of Φ with constraint L such that $\forall f \in \Phi, \forall u \in PC(T,\mathcal{U}), w \in T(comp(L)) : y_{\Sigma}(\mu(f),u,w) = 0$. Since R is reachable and observable, we get that (Σ,μ) is semi-reachableand observable. □

Lemma 2 also yields the following result.

Theorem 10. *Consider a language $L \subseteq Q^+$ and a set $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ of input-output maps. Assume L that is regular and that Φ has a realization by a linear switched system. Let (Σ,μ) be the realization of Φ from part (ii) of Theorem 9. If $(\widetilde{\Sigma},\widetilde{\mu})$ is an arbitrary linear switched system realizing Φ with constraint L, then*

$$\dim \Sigma \leq M \cdot \dim \widetilde{\Sigma}$$

where M depends only on L.

Proof. By Theorem 7 it holds that $\Psi_\Phi = K_{\Sigma,\mu} \odot \Omega_\Phi$. Since $R_{\Sigma,\mu}$ is a minimal representation of Ψ_Φ it holds that $\dim \Sigma = \dim R_{\Sigma,\mu} = \text{rank } H_{\Psi_\Phi}$. But from Lemma 2 one gets that

$$\text{rank } H_{\Psi_\Phi} = \text{rank } H_{K_{\Sigma,\mu} \odot \Omega_\Phi} \leq \text{rank } H_{K_{\Sigma,\mu}} \cdot \text{rank } H_{\Omega_\Phi}$$

Since $\text{rank } H_{K_{\Sigma,\mu}} = \text{rank } H_{\Psi_\Theta} \leq \dim \widetilde{\Sigma}$ and $M := \text{rank } H_\Omega$ depends only on L, we get the statement of the theorem. □

Notice that if L is finite then L is regular. It means that the results of this section in principle allow us to construct a realization of a set of input-output map by examining a finite number of sequences of discrete modes.

Remark

In fact, the result of the Theorem 10 is sharp in the following sense. One can construct an input-output y map and language L and realizations Σ_1 and Σ_2 such that the following holds. Both Σ_1 and Σ_2 realize y from the initial state zero with constraint L and they are both reachable from zero and observable, but $\dim \Sigma_1 = 1$ and $\dim \Sigma_2 = 2$. The construction goes as follows. Let $Q = \{1, 2\}$, $L = \{q_1^k q_2 \mid k > 0\}$, $\mathcal{Y} = \mathcal{U} = \mathbb{R}$. Define $y : PC(T, \mathcal{U}) \times TL \to \mathcal{Y}$ by

$$y(u(.), \underbrace{q_1 \cdots q_1}_{m-times} q_2, t_1 \cdots t_m t_{m+1}) = \int_0^{t_{m+1}} e^{2(t_{m+1}-s)} u(s + \sum_1^m t_i) ds + \int_0^{\sum_1^m t_i} e^{2t_{m+1}} e^{\sum_1^m t_i - s} u(s) ds$$

Define $\Sigma_1 = (\mathbb{R}, \mathbb{R}, \mathbb{R}, Q, \{(A_{1,q}, B_{1,q} C_{1,q}) \mid q \in \{q_1, q_2\}\})$ by

$$\begin{array}{lll} A_{1,q_1} = 1 & B_{1,q_1} = 1 & C_{1,q_1} = 1 \\ A_{1,q_2} = 2 & B_{1,q_2} = 1 & C_{1,q_2} = 1 \end{array}$$

Define $\Sigma_2 = (\mathbb{R}^2, \mathbb{R}, \mathbb{R}, Q\{(A_{2,q}, B_{2,q}, C_{2,q}) \mid q \in Q\})$ by

$$A_{2,q_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B_{2,q_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C_{2,q_1} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$A_{2,q_2} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \quad B_{2,q_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_{2,q_2} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Both Σ_1 and Σ_2 are reachable and observable as linear switched systems, therefore they are the minimal realizations of $y_{\Sigma_1}(0, ., .)$ and $y_{\Sigma_2}(0, ., .)$. Moreover, it is easy to see that

$$y_{\Sigma_1}(0, ., .)|_{PC(T, \mathcal{U}) \times TL} = y = y_{\Sigma_2}(0, ., .)|_{PC(T, \mathcal{U}) \times TL}$$

In fact, Σ_2 can be obtained by constructing the minimal representation of $\Psi_{\{y\}}$, i.e., Σ_2 is a minimal realization of y satisfying part (iii) of Theorem 9.

6. Realization theory for bilinear switched systems

This section deals with the realization theory of bilinear switched systems. First, definition and certain elementary properties of bilinear switched systems will be presented. Then, in Subsection 6.1 the structure of the input/output maps of bilinear switched systems will be discussed. Subsection 6.2 presents the realization theory for bilinear switched systems for the case of arbitrary switching. Subsection 6.3 deals with realization theory for the case of switching with constraints.

Definition 4 (Bilinear switched systems). *A switched system* $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\})$ *is called bilinear if for each* $q \in Q$ *there exist linear mappings* $A_q : \mathcal{X} \to \mathcal{X}$, $B_{q,j} : \mathcal{X} \to \mathcal{X}$, $j = 1, 2, \ldots, m$, $C_q : \mathcal{X} \to \mathcal{Y}$ *such that*

- $\forall x \in \mathcal{X}, u = (u_1, \ldots, u_m)^T \in \mathcal{U} = \mathbb{R}^m : f_q(x, u) = A_q x + \sum_{j=1}^{m} u_j B_{q,j} x$
- $\forall x \in \mathcal{X} : h_q = C_q x.$

We will use the notation $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\ldots,m}, C_q) \mid q \in Q\})$ *to denote bilinear switched systems.*

Recall from [8,9] that the state- and output-trajectory of a bilinear system can be expressed as infinite series of iterated integrals. A similar representation exists for switched bilinear systems. In order to formulate such a representation some notation has to be set up. For each $u = (u_1, \ldots, u_k) \in \mathcal{U}$ denote

$$d\zeta_j[u] = u_j, j = 1, 2, \ldots, m, \quad d\zeta_0[u] = 1$$

Denote the set $\{0, 1, \ldots, m\}$ by Z_m. For each $j_1, \cdots, j_k \in \mathrm{Z}_m$, $k \geq 0, t \in T$, $u \in PC(T, \mathcal{U})$ define $V_{j_1 \cdots j_k}[u](t) \in \mathbb{R}$ as

$$V_{j_1 \cdots j_k}[u](t) = \begin{cases} 1 & \text{if } k = 0 \\ \int_0^t d\zeta_{j_k}[u(\tau)] V_{j_1, \ldots, j_{k-1}}[u](\tau) d\tau & \text{if } k > 1 \end{cases}$$

For each $w_1, \ldots, w_k \in \mathrm{Z}_m^*$, $(t_1, \cdots, t_k) \in T^k$, $u \in PC(T, \mathcal{U})$ define $V_{w_1, \ldots, w_k}[u](t_1, \ldots, t_k) \in \mathbb{R}$ by

$$V_{w_1, \ldots, w_k}[u](t_1, \ldots, t_k) = V_{w_1}(t_1)[u] V_{w_2}(t_2)[\mathrm{Shift}_1(u)] \cdots V(w_k)[\mathrm{Shift}_{k-1}(u)](t_k)$$

where $\mathrm{Shift}_i(u) = \mathrm{Shift}_{\sum_1^i t_i}(u)$, $i = 1, 2, \ldots, k-1$. For each $q \in Q$ and $w = j_1 \cdots j_k, k \geq 0, j_1, \cdots j_k \in \mathrm{Z}_m$ let us introduce the following notation

$$B_{q,0} := A_q,\ B_{q,\epsilon} := Id_{\mathcal{X}},\ ,B_{q,w} := B_{q,j_k} B_{q,j_{k-1}} \cdots B_{q,j_1}$$

where $Id_{\mathcal{X}}$ denotes the identity map on $\mathcal{X}$. With the notation above we can formulate the following result.

Proposition 6. *Using the notation above, for each* $x_0 \in \mathcal{X}$, $u \in PC(T, \mathcal{U})$ *and* $s = (q_1, t_1) \cdots (q_k, t_k) \in (Q \times T)^*$ *the state* $x_\Sigma(x_0, u, s)$ *and the output* $y_\Sigma(x_0, u, s)$ *can be expressed by the following absolutely convergent series.*

$$\begin{aligned} x_\Sigma(x_0, u, s) &= \sum_{w_1, \ldots, w_k \in \mathrm{Z}_m^*} (B_{q_k, w_k} \cdots B_{q_1, w_1} x_0) V_{w_1, \ldots, w_k}[u](t_1, \ldots, t_k) \\ y_\Sigma(x_0, u, s) &= \sum_{w_1, \ldots, w_k \in \mathrm{Z}_m^*} (C_{q_k} B_{q_k, w_k} \cdots B_{q_1, w_1} x_0) V_{w_1, \ldots, w_k}[u](t_1, \ldots, t_k) \end{aligned} \tag{15}$$

Proof. To show absolute convergence of the series we will use the notion of a convergent generating series defined in Section 6.1. Using the notation of Section 6.1 define the series $c_{x_0} : \widetilde{\Gamma}^* \to \mathcal{X}$ by $c_{x_0}((q_1, w_1) \cdots (q_k, w_k)) = B_{q_k, w_k} \cdots B_{q_1, w_1} x_0$. Then $||c_{x_0}|| \leq ||x_0|| M^{\sum_{i=1}^k |w_i|}$, where $M = \max\{||B_{q,j}|| \mid q \in Q, j \in \mathrm{Z}_m\}$. That is, c_{x_0} is a convergent generating series and by Lemma 9 the series

$$F_{c_{x_0}}(u, s) = \sum_{w_1, \ldots, w_k} \in (B_{q_k, w_k} \cdots B_{q_1, w_1} x_0) V_{w_1, \ldots, w_k}[u](t_1, \ldots, t_k)$$

is absolutely convergent, which also implies the absolute convergence of

$$\sum_{w_1, \ldots, w_k \in \mathrm{Z}_m^*} (C_{q_k} B_{q_k, w_k} \cdot \cdots \cdot B_{q_1, w_1} x_0) V_{w_1, \ldots, w_k}[u](t_1, \ldots, t_k)$$

It is left to show that the right-hand sides of (15) equal the respective left-hand sides. We will proceed by induction on k. If $k = 1$, then $x_\Sigma(x_0, u, (q_1, t))$ is the state under input u at time t with initial state x_0 of the bilinear system $\frac{d}{dt}x(t) = A_{q_1}x(t) + \sum_{j=1}^{m}(B_{q_1,j}x)u_j$. By classical results [8] on bilinear systems

$$x_\Sigma(x_0, u, (q_1, t)) = \sum_{w \in Z_m^*} B_{q,w} x_0 V_w[u](t)$$

and the series $\sum_{w\in Z_m^*} B_{q,w}x_0V_w[u](t)$ is absolutely convergent. Assume that the statement of the proposition is true for all $k \leq N$. Notice that for each $s = (q_1, t_1)\cdots(q_N, t_N) \in (Q \times T)^*$ it holds that

$$x_\Sigma(x_0, u, s(q_{N+1}, t_{N+1})) = x_\Sigma(x_\Sigma(x_0, \mathrm{Shift}_{\sum_1^N t_i}(u), s), (q_{N+1}, t_{N+1}))$$

Using the induction hypothesis one gets

$$\begin{aligned} x_\Sigma(x_0, u, s(q_{N+1}, t_{N+1}) &= \sum_{w_{N+1}\in Z_m^*} B_{q_{N+1},w_{N+1}} x_\Sigma(x_0, u, s) V_{w_{N+1}}[u_N](t_{N+1}) \\ &= \sum_{w_{N+1}\in Z_m^*} B_{q_{N+1},w_{N+1}} V_{w_{N+1}}[u_N](t_{N+1}) \times \\ &\times [\sum_{w_1,\dots,w_N \in Z_m^*} B_{q_N,w_N}\cdots B_{q_1,w_1} x_0 V_{w_1,\dots,w_N}[u](t_1,\dots,t_N)] = \\ &= \sum_{w_1,\dots,w_{N+1}\in Z_m^*} B_{q_{N+1},w_{N+1}}\cdots B_{q_1,w_1} x_0 V_{w_1,\dots,w_{N+1}}[u](t_1,\dots,t_{N+1}) \end{aligned}$$

where $u_N = \mathrm{Shift}_{\sum_{i=1}^N t_i}(u)$. The rest of the statement of the proposition follows easily from the fact that

$$y_\Sigma(x_0, u, (q_1, t_1)\cdots(q_k, t_k)) = C_{q_k} x_\Sigma(x_0, u, (q_1, t_1)\cdots(q_k, t_k))$$

□

Reachability and observability properties of bilinear switched systems can be easily derived from the formulas above.

Proposition 7. *Let $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\dots,m}, C_q) \mid q \in Q\})$ be a bilinear switched system. Then the following holds.*

(i) *The linear span $W(\mathcal{X}_0) = \mathrm{Span}\{z \in \mathcal{X} \mid x \in Reach(\mathcal{X}_0, \Sigma)\}$ of the states reachable from $\mathcal{X}_0 \subseteq \mathcal{X}$ is of the following form*

$$W(\mathcal{X}_0) = \mathrm{Span}\{B_{q_k,w_k}\cdots B_{q_1,w_1}x_0 \mid q_k, \dots q_1 \in Q, k \geq 0, w_k, \dots, w_1 \in Z_m^*, x_0 \in \mathcal{X}_0\}$$

(ii) *Define the* observability kernel O_Σ of Σ *by*

$$O_\Sigma = \bigcap_{q_1,\dots,q_k \in Q, k\geq 0, w_1,\dots,w_k \in Z_m^*} C_{q_k} B_{q_k,w_k}\cdots B_{q_1,w_1}$$

$x_1, x_2 \in \mathcal{X}$ *are indistinguishable if and only if*

$$x_1 - x_2 \in O_\Sigma$$

Σ *is observable if and only if*

$$O_\Sigma = \{0\}$$

Proof. `Part (i)`
For each $\mathcal{X}_0 \subseteq \mathcal{X}$, $q_1, \dots, q_k \in Q$ define the set $W_{q_1 \cdots q_k}(\mathcal{X}_0) \subseteq \mathcal{X}$ as

$$\mathrm{Span}\{x_\Sigma(x_0, u, (q_1, t_1) \cdots (q_k, t_k)) \mid u \in PC(T, \mathcal{U}), t_1, \dots, t_k \in T, x_0 \in \mathcal{X}_0\}$$

Notice that $x_\Sigma(x_0, u, (q_1, t_1) \cdots (q_k, t_k)) = x_\Sigma(x_\Sigma(x_0, u, s), \mathrm{Shift}_{T_s}(u), (q_k, t_k))$ where $s = (q_1, t_1) \cdots (q_{k-1}, t_{k-1})$, $T_s = \sum_{i=1}^{k-1} t_i$. Using the fact that in the discrete mode q_k the system Σ behaves like a bilinear system and using the results from [8, 9] one gets that for each fixed $s = (q_1, t_1) \cdots (q_{k-1}, t_{k-1}) \in (Q \times T)^*$ and $u \in PC([0, \sum_1^{k-1} t_j], \mathcal{U})$ it holds that

$$W_{q_k}(\{x_\Sigma(x_0, u, s)\}) = \mathrm{Span}\{B_{q_k, w} x_\Sigma(x_0, u, s) \mid w \in \mathrm{Z}_m^*\}$$

That is,

$$W_{q_1, \dots, q_k}(\mathcal{X}_0) = \mathrm{Span}\{B_{q_k, w} x \mid x \in W_{q_1, \dots, q_{k-1}}(\mathcal{X}_0), w \in \mathrm{Z}_m^*\}$$

Taking into account that by [9] $W_q(\mathcal{X}_0) = \mathrm{Span}\{B_{q,w} x_0 \mid x_0 \in \mathcal{X}_0\}$ and $\mathrm{Span}\{x \mid x \in \mathit{Reach}(\Sigma, \mathcal{X}_0) = \mathrm{Span}\{x \mid x \in W_{q_1, \dots, q_k}(\mathcal{X}_0), q_1, \dots, q_k \in Q, k \geq 0\}$, the statement of the proposition follows.

`Part (ii)`
It is easy to deduce from (15) of Proposition 6 that $y_\Sigma(x, ., .)$ is linear in x, that is, $y_\Sigma(\alpha x_1 + \beta x_2, ., .) = \alpha_1 y_\Sigma(x_1, .,) + \beta y_\Sigma(x_2, ., .)$ That is, $y_\Sigma(x_1, ., .) = y_\Sigma(x_2, ., .)$ is equivalent to $y_\Sigma(x_1 - x_2, ., .) = 0$. Thus, it is enough to show that

$$x \in O_\Sigma \iff y_\Sigma(x, ., .) = 0$$

It is clear from Proposition 6 that $x_1 - x_2 \in O_\Sigma \implies y_\Sigma(x_1 - x_2, ., .) = 0$. It is left to show that $y_\Sigma(x, ., .) = 0 \implies x \in O_\Sigma$. Assume that $y_\Sigma(x, ., .) = 0$. Then for each fixed $w = (q_1, t_1) \cdots (q_k, t_k) \in (Q \times T)^*$, $u \in PC(T, \mathcal{U})$, $q \in Q$ it holds that $y_\Sigma(x_\Sigma(x, u, w), v, (q, t)) = y_\Sigma(x, u \#_{T_w} v, w(q, t)) = 0$ for any $v \in PC(T, \mathcal{U})$, where $T_w = \sum_1^k t_i$. Notice that for any $x_0 \in \mathcal{X}$ the map $PC(T, \mathcal{U}) \times T \ni (v, t) \mapsto y_\Sigma(x_0, v, (q, t))$ is the input-output map of the classical bilinear system $\frac{d}{dt} x(t) = A_q x + \sum_{j=1}^m u_j(t)(B_{q,j} x(t)), y(t) = C_q x(t)$ induced by the inital condition x_0. Thus by the classical result for bilinear systems, see [8], $y_\Sigma(x_\Sigma(x, u, w), v, (q, t)) = 0, \forall v \in PC(T, \mathcal{U})$ implies

$$x_\Sigma(x, u, w) \in \bigcap_{v \in \mathrm{Z}_m^*} \ker C_q B_{q,v}$$

Recall from the proof of part (i) the definition of $W_{q_1, \dots, q_k}(\{x\})$. Since the choice of u and $t_1, \dots, t_k$ are arbitrary, we get that $W_{q_1, \dots, q_k}(\{x\}) \subseteq \bigcap_{v \in \mathrm{Z}_m^*} \ker C_q B_{q,v}$. Using the proof of part (i) we get that $W_{q_1, \dots, q_k}(\{x\}) = \mathrm{Span}\{B_{q_k, w_k} \cdots B_{q_1, w_1} x \mid w_1, \dots, w_k \in \mathrm{Z}_m^*\}$ which implies that

$$x \in \bigcap_{w, w_1, \dots, w_k \in \mathrm{Z}_m^*} \ker C_q B_{q,w} B_{q_k, w_k} \cdots B_{q_1, w_1}$$

Since the choice of q and $q_1, \dots, q_k \in Q$ is arbitrary, we get that $x \in O_\Sigma$. This completes the proof of the proposition. $\square$

Let $\Sigma_1 = (\mathcal{X}_1, \mathcal{U}, \mathcal{Y}, Q, \{(A_q^1, \{B_{q,j}^1\}_{j=1,2,\dots,m}, C_q^1) \mid q \in Q\})$ and $\Sigma_2 = (\mathcal{X}_2, \mathcal{U}, \mathcal{Y}, Q, \{(A_q^2, \{B_{q,j}^2\}_{j=1,2,\dots,m}, C_q^2) \mid q \in Q\})$ be two bilinear switched systems. A linear map $T : \mathcal{X}_1 \to \mathcal{X}_2$ is called a *bilinear switched system morphism* from Σ_1 to Σ_2, denoted by $T : \Sigma_1 \to \Sigma_2$, if the following holds

$$TA_q^1 = A_q^2 T \quad C_q^1 = C_q^2 T \quad TB_{q,j}^1 = B_{q,j}^2$$

By abuse of terminology T is said to be a bilinear switched system morphism from (Σ, μ) to $(\Sigma^{'}, \mu^{'})$, denoted by $T : (\Sigma, \mu) \to (\Sigma^{'}, \mu^{'})$, if $T : \Sigma \to \Sigma^{'}$ is a bilinear switched system morphism in the above sense and $T \circ \mu = \mu^{'}$. If T is a linear isomorphisms then (Σ_1, μ_1) and (Σ_2, μ_2) are said to be *isomorphic* or *algebraically similar*.

Note that switched systems defined above can be viewed as general non-linear systems with discrete inputs. In particular, bilinear switched systems can be viewed as ordinary bilinear systems with particular inputs. Indeed, let $Q = \{q_1, \dots, q_N\}$ and let $\widetilde{\mathcal{U}} = \mathbb{R}^N \oplus (\mathcal{U} \otimes \mathbb{R}^N)$. Denote the standard basis of $\mathbb{R}^N$ by $e_j, j = 1, \dots N$. We will denote e_j by e_{q_j}. Let $b_j, j = 1, \dots, m$ the standard basis of $\mathcal{U}$. Any $\widetilde{u} \in \widetilde{\mathcal{U}}$ has a unique representation $\widetilde{u} = \sum_{q \in Q} \widetilde{u}_q e_q + \sum_{j=1,\dots,m, q \in Q} \widetilde{u}_{j,q} b_j \otimes e_q$,

Consider the bilinear switched system $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\dots,m}, C_q) \mid q \in Q\})$. Define the following bilinear system with input space $\widetilde{\mathcal{U}}$ and output space $\mathcal{Y}$

$$\begin{aligned} \frac{d}{dt}x(t) &= \sum_{q \in Q} \widetilde{u}_q(t)(A_q x) + \sum_{q \in Q, j=1,\dots,m} \widetilde{u}_{q,j}(t)(B_{q,j} x) \\ y(t) &= \sum_{q \in Q} \widetilde{u}_q(t)(C_q x) \end{aligned}$$

Here $\widetilde{u}(t) \in \widetilde{\mathcal{U}}$ denoted the continuous input. The bilinear system above simulates Σ in the following sense. Let $w = (q_1, t_1) \cdots (q_k, t_k) \in (Q \times T)^+$, $u \in PC(T, \mathcal{U})$. Define $U_{u,w} := \widetilde{u} \in PC(T, \widetilde{\mathcal{U}})$ such that for each $i = 0, \dots, k-1$ $\forall \tau \in [\sum_{j=1}^{i} t_j, \sum_{j=1}^{i+1} t_j] : \widetilde{u}_{q_{i+1}}(\tau) = 1, \widetilde{u}_{q_{i+1},j}(\tau) = u_j(\tau)$ and $\widetilde{u}_q(\tau) = 0, \widetilde{u}_{j,q}(\tau) = 0, q \neq q_{i+1}$. Then $y_\Sigma(x, u, w)$ equals the output of the bilinear system above induced by $\widetilde{u}$ and initial state x. Using the correspondence above, one could try to reduce the realization problem for bilinear switched systems to the realization problem for classical bilinear systems and use the existing results on the realization theory of bilinear systems. In this paper we will not pursue this approach. The reason for that is the following. First, dealing with restricted switching would require dealing with the realization problem of bilinear systems with input constraints. The author is not aware of any work on this topic. Second, the author thinks that using bilinear realization theory would not substantially simplify the solution to realization problem for bilinear switched systems. Notice however, that the equivalence of realization problems mentioned above does explain the role of rational formal power series in realization theory of bilinear switched systems.

6.1. Input/output maps of bilinear switched systems

Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times TL, \mathcal{Y})$ be a set of input-output maps defined for sequences of discrete modes belonging to $L \subseteq Q^+$. Let $\widetilde{\Gamma} = Q \times Z_m^*$. Define the set

$$JL = \{(q_1, w_1) \cdots (q_k, w_k) \in \widetilde{\Gamma}^* \mid (q_1, w_1), \dots, (q_k, w_k) \in \widetilde{\Gamma}, k \geq 0, q_1 \cdots q_k \in L\}$$

Define the relation $R \subseteq \widetilde{\Gamma}^* \times \widetilde{\Gamma}^*$ by requiring that $(q, w_1)(q, w_2) R (q, w_1 w_2)$, and $(q, \epsilon)(q', w) R (q', w)$ hold for any $q \in Q, (q', w) \in \widetilde{\Gamma}$ and $(q, w_1), (q, w_2) \in \widetilde{\Gamma}$. Let R^* be smallest congruence relation containing R. That is, R^* is the smallest relation such that $R \subseteq R^*$, R^* is symmetric, reflexive, transitive and $(v, v') \in R^*$ implies $(wvu, wv'u) \in R^*$, for each $w, u \in \widetilde{\Gamma}^*$.

Definition 5 (Generating convergent series on JL). *A $c : JL \to \mathcal{Y}$ is called a generating convergent series on JL if the following conditions hold.*

(1) $(w, v) \in R^*, w, v \in JL \implies c(w) = c(v)$
(2) *There exists $K, M > 0$ such that for each $(q_1, w_1) \cdots (q_k, w_k) \in JL$, $(q_1, w_1) \dots (q_k, w_k) \in \widetilde{\Gamma}$*

$$||c((q_1, w_1) \cdots (q_k, w_k))|| < K M^{|w_1|} \cdots M^{|w_k|}$$

The notion of generating convergent series is an extension of the notion of convergent power series from [8,22]. If $|Q| = 1$ then a generating convergent series in the sense of Definition 5 can be viewed as a convergent formal power series in the sense of [8,22].

Let $c : JL \rightarrow \mathcal{Y}$ be a generating convergent series. For each $u \in PC(T,\mathcal{U})$ and $s = (q_1,t_1)\cdots(q_k,t_k) \in TL$ define the series $F_c(u,s)$ by

$$F_c(u,s) = \sum_{w_1,\dots,w_k \in \mathrm{Z}_m^*} c((q_1,w_1)\cdots(q_k,w_k))V_{w_1,\dots,w_k}[u](t_1,\dots,t_k)$$

We will prove that the series above is absolutely convergent.

Lemma 9. *If $c : JL \rightarrow \mathcal{Y}$ is a convergent generating series, then for each $u \in PC(T,\mathcal{U})$, $s = (q_1,t_1)\cdots(q_k,t_k) \in TL$ the series $F_c(u,s)$ is absolutely convergent.*

Proof. Since u is piecewise-continuous, there exists $R > 1$ such that $\max\{|u_j(t)| \mid j = 1,2,\dots,m, t \in [0,\sum_1^k t_i]\} < R$. Then by induction it is easy to see that for all $w \in \mathrm{Z}_m$ it holds that $|V_w[u](t_i)| \leq \frac{R^{|w|}t^{|w|}}{|w|!}$, consequently

$$|V_{w_1,\dots,w_k}[u](t_1,\dots,t_k)| = \Pi_{i=1}^k |V_{w_i}[u](t_i)| \leq \frac{t_1^{|w_1|}}{|w_1|!}\cdots\frac{t_k^{|w_k|}}{|w_k|!}R^{|w_1|+\cdots+|w_k|}$$

We get that

$$\begin{aligned}\sum_{w_1,\dots,w_k \in \mathrm{Z}_m^*, |w_1|+\ldots+|w_k| \leq N} &||c((q_1,w_1)\cdots(q_k,w_k))V_{w_1,\dots,w_k}[u](t_1,\dots,t_k)|| \leq \\ &\leq \sum_{l_1+\cdots+l_k \leq N} K(MR(m+1))^{l_1+\cdots+l_k}\frac{t_1^{l_1}}{l_1!}\cdots\frac{t_k^{l_k}}{l_k!} \leq \sum_{l=0}^N K(MRk(m+1))^l\frac{T^l}{l!} \leq \\ &\leq K\exp(MRk(m+1)T)\end{aligned}$$

where $T = \sum_1^k t_i$. That is, the series $F_c(u,(q_1,t_1)\cdots(q_k,t_k))$ is absolutely convergent. $\square$

In fact we can define a function $F_c \in F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ by

$$F_c : PC(T,\mathcal{U}) \times TL \ni (u,w) \mapsto F_c(u,w) \in \mathcal{Y}$$

The map F_c has some remarkable properties, listed below.

Lemma 10. *Let $c : JL \rightarrow \mathcal{Y}$ be a generating convergent series. Then the following holds.*

(i) *For each $s = (q_1,t_1)\cdots(q_k,t_k) \in TL$, $u,v \in PC(T,\mathcal{U})$*

$$(\forall t \in [0,\sum_1^k t_i] : u(t) = v(t)) \implies F_c(u,s) = F_c(v,s)$$

(ii) $\forall u \in PC(T,\mathcal{U}), w,s \in (Q \times T)^*, |s| > 0 :$

$$w(q,0)s, ws \in TL \implies F_c(u,w(q,0)s) = F_c(u,ws)$$

(iii) $\forall u \in PC(T,\mathcal{U}), w,v \in (Q \times T)^* :$

$$r = w(q,t_1)(q,t_2)v, \quad p = w(q,t_1+t_2)v \in TL \implies F_c(u,r) = F_c(u,p)$$

(iv) *Let $w = (w_1,0)\cdots(w_k,0), v = (v_1,0)\cdots(v_l,0) \in (Q \times T)^*$ and $s = (q_1,t_1)\cdots(q_h,t_h) \in (Q \times T)^+$*

$$ws, vs \in TL \implies (\forall u \in PC(T,\mathcal{U}) : F_c(u,ws) = F_c(u,vs))$$

Proof. Part (i) and (ii) follow from the obvious facts that $V_w[u](t)$ depends only on $u|_{[0,t]}$ and $V_w[u](0) = 0$ for $|w| > 0$. Part (iv) follows from the fact that $V_w[u](0) = 0$ for $|w| > 0$ and thus $V_{w_1,\dots,w_{k+h}}[u](0,\dots,0,t_1,\dots,t_h) = 0$ if $\exists j \in \{1,\dots,k\} : |w_j| \geq 0$, and

$$V_{w_1,\dots,w_{k+h}}[u](0,\dots,0,t_1,\dots,t_h) = V_{w_{k+1},\dots,w_{k+h}}[u](t_1,\dots,t_h)$$

if $w_{k+1} = \cdots = w_{k+h} = \epsilon$. The proof of Part (iii) is more involved. We will use the following lemma.

Lemma 11. *For each* $w \in \mathrm{Z}_m^*$*:*

$$V_w[u](t_1+t_2) = \sum_{s,z \in \mathrm{Z}_m^*, sz=w} V_s[u](t_1) V_z[\mathrm{Shift}_{t_1}(u)](t_2)$$

Using the lemma above and assuming that $w = (q_1,\tau_1)\cdots(q_i,\tau_i)$, $s = (q_{i+1},\tau_{i+1})\cdots(q_k,\tau_k)$, $k \geq 0$, $\mathrm{T}_z = \sum_{j=1}^{z-1} t_j$ if $z \leq i$, $\hat{\mathrm{T}}_i = \sum_{j=1}^{i} t_i$ and $\mathrm{T}_{l+i} = \hat{\mathrm{T}}_i + t_1 + t_2 + \sum_{j=i+1}^{l+i-1} \tau_j$ we get

$$\begin{aligned}
F_c(u,r) = &\sum_{w_1,\dots,w_k,s,z \in \mathrm{Z}_m^*} c((q_1,w_1)\cdots(q_i,w_i)(q,s)(q,z)(q_{i+1},w_{i+1})\cdots(q_k,w_k))\times \\
&\times V_s[\mathrm{Shift}_{\hat{\mathrm{T}}_i}(u)](t_1) V_z[\mathrm{Shift}_{t+\hat{\mathrm{T}}_i}(u)](t_2) \Pi_{j=1}^k V_{w_j}[\mathrm{Shift}_{\mathrm{T}_j}(u)](\tau_j) = \\
&= \sum_{w_1\dots,w_k \in \mathrm{Z}_m^*} \sum_{w \in \mathrm{Z}_m^*} [c((q_1,w_1)\cdots(q_i,w_i)(q,w)(q_{i+1},w_{i+1})\cdots(q_k,w_k)) \times \\
&\times \Pi_{j=1}^k V_{w_j}[\mathrm{Shift}_{\mathrm{T}_j}(u)](\tau_j)] \sum_{sz=w} V_s[\mathrm{Shift}_{\hat{\mathrm{T}}_i}(u)](t_1) V_z[\mathrm{Shift}_{\hat{\mathrm{T}}_i+t_1}(u)](t_2) \\
&= \sum_{w_1,\dots,w_k,w \in \mathrm{Z}_m^*} \{c((q_1,w_1)\cdots(q_i,w_i)(q,w)(q_{i+1},w_{i+1})\cdots(q_k,w_k)) \times \\
&\Pi_{j=1}^k V_{w_i}[\mathrm{Shift}_{\mathrm{T}_j}(u)](\tau_j)\} V_w[\mathrm{Shift}_{\hat{\mathrm{T}}_i}(u)](t_1+t_2) = F_c(u,p)
\end{aligned}$$

□

Proof of Lemma 11. We proceed by induction on $|w|$. Assume that $|w| = 1$, that is, $w = j \in \mathrm{Z}_m$. Then

$$V_w[u](t_1+t_2) = \int_0^{t_1+t_2} d\zeta_j(\tau)d\tau = \int_0^{t_1} d\zeta_j(\tau)d\tau + \int_0^{t_2} d\zeta_j(t_1+\tau)d\tau = V_j[u](t_1) + V_j[\mathrm{Shift}_{t_1}(u)](t_2)$$

Assume that $w = vj$. Then

$$\begin{aligned}
V_w[u](t_1+t_2) = \int_0^{t_1+t_2} d\zeta_j(\tau)V_v[u](\tau)d\tau = \int_0^{t_1} d\zeta_j(\tau)V_v[u](\tau)d\tau + \int_0^{t_2} d\zeta_j(t_1+\tau) = \\
= V_v[u](t_1+\tau)d\tau V_w[u](t_1) + \int_0^{t_2} d\zeta_j(t_1+\tau)V_v[u](t_1+\tau)d\tau
\end{aligned}$$

By induction hypothesis we get that

$$\begin{aligned}
\int_0^{t_2} d\zeta_j(t_1+\tau)V_v[u](t_1+\tau)d\tau = \sum_{sz=v,s,z \in \mathrm{Z}_m^*} V_s[u](t_1) \int_0^{t_2} d\zeta_j(t_1+\tau)V_z[\mathrm{Shift}_{t_1}(u)](\tau)d\tau = \\
= \sum_{sz=v,s,z \in \mathrm{Z}_m^*} V_s[u](t_1) V_{zj}[\mathrm{Shift}_{t_1}(u)](t_2)
\end{aligned}$$

That is, we get that

$$V_w[u](t_1+t_2) = V_w[u](t_1) + \sum_{sz=v,s,z\in \mathbb{Z}_m^*} V_s[u](t_1)V_{zj}[\mathrm{Shift}_{t_1}(u)](t_2) = \sum_{sz=w,s,z,\in \mathbb{Z}_m^*} V_s[u](t_1)V_z[\mathrm{Shift}_{t_1}(u)](t_2)$$

□

It is a natural to ask whether c determines F_c uniquely. The following result answers this question.

Lemma 12. *Let $L \subseteq Q^*$ and let $d, c : JL \to \mathcal{Y}$ be two convergent generating series. If $F_c = F_d$, then $c = d$.*

Proof. It is enough to show that for any $L \subseteq Q^*$, $d, c : JL \to \mathcal{Y}$, if $F_d = F_c$ then for each $q_1, \dots, q_k \in L$

$$\forall w_1, \dots, w_k \in \mathbb{Z}_m^* : c((q_1, w_1) \cdots (q_k, w_k)) = d((q_1, w_1) \cdots (q_1, w_k)) \tag{16}$$

We proceed by induction on k. If $k = 1$ and $q_1 \in L$, then define the series. $\widetilde{c} : \mathbb{Z}_m^* \ni w \mapsto c((q_1, w))$ and $\widetilde{d} : \mathbb{Z}_m^* \ni w \mapsto d((q_1, w))$. The series $\widetilde{c}$ and $\widetilde{d}$ are convergent series in the sense of [8, 22]. If $F_c = F_d$, then with the notation of [22], $F_{\widetilde{c}}[u](t) = F_c(u, (q_1, t)) = F_d(u, (q_1, t)) = F_{\widetilde{d}}[u](t)$, which by [8] implies that $\widetilde{c} = \widetilde{d}$, that is, $c((q_1, w)) = d((q_1, w))$ for each $w \in \mathbb{Z}_m^*$. Assume that (16) holds for each $k \leq N$. Let $L \subseteq Q^*$ and let $q_1 \cdots q_{N+1} \in L$. Let $w \in \mathbb{Z}_m^*$ and define $c_{(q_1,w)} : JH_{q_1} \to \mathcal{Y}$, $H_{q_1} = \{w \in Q^* \mid q_1 w \in L\}$, by

$$c_{(q_1,w)}(s) = \begin{cases} c((q_1, w)s) & \text{if } s = (q_2, w_2) \cdots (q_{N+1}, w_{N+1}) \text{ for some } w_2, \dots, w_{N+1} \in \mathbb{Z}_m^* \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that for all $(s_1, z_1) \cdots (s_l, z_l) \in JH_{q_1}$

$$||c_{(q_1,w)}((s_1, z_1) \cdots (s_l, z_l))|| < M^{|w|} K M^{|z_1|+\ldots+|z_k|}$$

That is, $c_{q_1,w}$ is a generating convergent series. It is also easy to see that for each $s \in TH_{q_1}$

$$F_{c_{(q_1,w)}}(u, s) = \sum_{w_2,\dots,w_{N+1}\in \mathbb{Z}_m^*} c((q_1, w)(q_2, w_2) \cdots (q_{N+1}, w_{N+1}))V_{w_2,\dots,w_{N+1}}[u](t_2, \dots, t_{N+1})$$

if $s = (q_2, t_2) \cdots (q_{N+1}, t_{N+1})$ for some $t_2, \dots, t_{N+1} \in T$ and $F_{c_{(q_1,w)}}(u, s) = 0$ otherwise. It follows from the proof of Lemma 9 that

$$||F_{c_{q_1,w}}(u, (s_1, \tau_1) \cdots (s_l, \tau_l))|| \leq M^{|w|} K \exp(MRl(m+1) \sum_{1}^{l} \tau_l)$$

where $R \geq \max\{1, \max\{|u_j(t)| \mid j = 1, 2, \dots, m, t \in [0, \sum_1^l \tau_i]\}\}$. Fix an arbitrary $r = (q_2, t_2) \dots (q_{N+1}, t_{N+1})$, $t_2, \dots, t_{N+1} \in T$. Then the map

$$F_{c,q_1}(u, r) : \mathbb{Z}_m^* \ni w \mapsto F_{c_{q_1,w}}(u, r)$$

is a generating convergent series. Moreover, for any $v \in PC(T, \mathcal{U}), t \in T$

$$F_c(v\#_t u, (q_1, t)r) = \sum_{w\in \mathbb{Z}_m^*} F_{c_{q_1},w}(u, r)V_w[v](t)$$

Define $F_{d_{q_1,w_1}}$ and $F_{d,q_1}(u, r)$ is a similar way. Then from $F_c = F_d$ we get that for all $u, v \in PC(T, \mathcal{U})$, $w \in \mathbb{Z}_m^*$, $t \in T$

$$F_c(v\#_t u, (q_1, t)r) = F_d(v\#_t u, (q_1, t)r)$$

For each fixed $u \in PC(T,\mathcal{U})$ by induction hypothesis for $k=1$ we get that

$$\forall w \in \mathbb{Z}_m^* : F_{c_{q_1,w}}(u,r) = F_{d_{q_1,w}}(u,r)$$

Notice that $F_{c_{q_1,w}}(u,s) = 0 = F_{d_{q_1,w}}(u,s)$ for all $s \neq (q_2,\tau_2)\cdots(q_{N+1},\tau_{N+1})$ for some $\tau_2,\ldots,\tau_{N+1}$. That is, $F_{c_{q_1,w}} = F_{d_{q_1,w}}$, and by induction hypothesis for $k=N$ we get that $c_{q_1,w}(s) = d_{q_1,w}(s)$ for all $w \in \mathbb{Z}_m^*, s \in JH_{q_1}, |s| \leq N$. In particular, for each $w_1 \cdots w_{N+1} \in \mathbb{Z}_m^*$

$$c((q_1,w_1)(q_2,w_2)\cdots(q_{N+1},w_{N+1})) = c_{q_1,w_1)}(x) = d_{(q_1,w_1)}(x) = d((q_1,w_1)(q_2,w_2)\cdots(q_{N+1},w_{N+1}))$$

where $x = (q_2,w_2)\cdots(q_{N+1},w_{N+1})$. $\square$

Now we are ready to define the concept of generalized Fliess-series representation of a set of input/output maps.

Definition 6 (Generalized Fliess-series expansion)**.** *The set of input-output maps $\Phi \subseteq F(PC(T,\mathcal{U})\times TL,\mathcal{Y})$ is said to admit* a generalized Fliess-series expansion *if for each $f \in \Phi$ there exist a generating convergent series $c_f : JL \to \mathcal{Y}$ such that $F_{c_f} = f$.*

Notice that if Φ has a generalized kernel representation with constraint L, then Φ has a generalized Fliess-series expansion given as follows. For each $f \in \Phi$, let

$$c_f((q_1,w_1)\cdots(q_k,w_k)) = \\ \begin{cases} D^{|w_k|,\ldots,|w_1|}K^{f,\Phi}_{q_1\cdots q_k} & \text{if } w_1,\ldots,w_k \in \{0\}^* \\ D^{|w_k|,\ldots,|w_l|-1}G^{f,\Phi}_{q_k\cdots q_l}e_j & \text{if } l = \min\{z \mid |w_z|>0\}, w_k,\ldots,w_{l+1} \in \{0\}^*, \\ & w_l = vj, v \in \{0\}^*, j \in \mathbb{Z}_m \setminus \{0\} \\ 0 & \text{otherwise} \end{cases}$$

From Lemma 12 we immediately get the following corollary.

Corollary 6. *Any $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL,\mathcal{Y})$ admits at most one generalized kernel representation with constraint L.*

The following proposition gives a description of the Fliess-series expansion of Φ in the case when Φ is realized by a bilinear switched system.

Proposition 8. *(Σ,μ) is a bilinear switched system realization of Φ with constraint L if and only if Φ has a generalized Fliess-series expansion such that for each $f \in \Phi, (q_1,w_1)\cdots(q_k,w_k) \in JL$*

$$c_f((q_1,w_1)\cdots(q_k,w_k)) = C_{q_k}B_{q_k,w_k}\cdots B_{q_1,w_1}\mu(f) \tag{17}$$

Proof. If (Σ,μ) is a realization of Φ, then by Proposition 6 for each $f \in \Phi$, $w = (q_1,t_1)\cdots(q_k,t_k) \in TL$, $u \in PC(T,\mathcal{U})$

$$f(u,w) = y_\Sigma(\mu(f),u,w) = \\ = \sum_{w_1,\ldots,w_k \in \mathbb{Z}_m^*} C_{q_k}B_{q_k,w_k}\cdots B_{q_1,w_1}V_{w_1,\ldots,w_k}[u](t_1,\ldots,t_k)$$

That is, Φ admits a generalized Fliess-series expansion of the form given in (17). Conversely, if Φ admits a generalized Fliess-series expansion of the form (17), then using Proposition 6 one gets

$$\begin{aligned}
&f(u,(q_1,t_1)\cdots(q_k,t_k)) = \\
&\quad = \sum_{w_1,\dots,w_k\in \mathbb{Z}_m^*} c_f((q_1,w_1)\cdots(q_k,w_k))V_{w_1,\dots,w_k}[u](t_1,\dots,t_k) = \\
&\quad = \sum_{w_1,\dots,w_k\in \mathbb{Z}_m^*} C_{q_k}B_{q_k,w_k}\cdots B_{q_1,w_1}\mu(f)V_{w_1,\dots,w_k}[u](t_1,\dots,t_k) = \\
&\quad = y_\Sigma(\mu(f),u,(q_1,t_1)\cdots(q_k,t_k))
\end{aligned}$$

That is, (Σ,μ) is a realization of Φ with constraint L. $\square$

6.2. Realization of input/output maps by bilinear switched systems with arbitrary switching

In this section realization theory for bilinear switched systems will be developed. We start with the case when the input/output maps are defined for all switching sequences. Let $\Phi \subseteq F(PC(T,\mathcal{U})\times(Q\times T)^+,\mathcal{Y})$ be a set of input/output maps and assume that Φ has a generalized Fliess-series expansion. As in the case of linear switched systems, we will associate with Φ an indexed set of formal power series Ψ_Φ. It turns out that every representation of Ψ_Φ determines a realization of Φ and vice versa. We will be able to use the theory of formal power series to derive the results on realization theory.

Recall that $\widetilde{\Gamma} = Q\times \mathbb{Z}_m^*$. Let $\Gamma = \{(q,j)\mid q\in Q, j\in \mathbb{Z}_m\}$. Define $\phi:\widetilde{\Gamma}\to\Gamma$ by

$$\phi((q,w)) = (q,j_1)\cdots(q,j_k), \quad \phi((q,\epsilon)) = \epsilon$$

where $w = j_1\cdots j_k \in Z_m^*$, $j_1,\dots,j_k\in \mathbb{Z}_m, k\geq 0$. The map ϕ determines a monoid morphisms $\phi:\widetilde{\Gamma}^*\to\Gamma^*$ given by

$$\phi((q_1,w_1)\cdots(q_k,w_k)) = \phi((q_1,w_1))\cdots\phi((q_k,w_k))$$

for each $(q_1,w_1),\dots,(q_k,w_k)\in\widetilde{\Gamma}, k\geq 0$. It is also clear that any element of Γ can be thought of as an element of $\widetilde{\Gamma}$, i.e. we can define the monoid morphism $i:\Gamma^*\to\widetilde{\Gamma}^*$ by $i(\epsilon)=\epsilon$ and $i((q_1,j_1)\cdots(q_k,j_k)) = (q_1,j_1)\cdots(q_k,j_k)$, $(q_1,j_1),\dots,(q_k,j_k)\in\Gamma\subseteq\widetilde{\Gamma}$. It is also easy to see that $\phi(i(w)) = w, \forall w\in\Gamma^*$ and $w(q,\epsilon)R^*i(\phi(w))(q,\epsilon)$, $q\in Q$.

For each $f\in\Phi, q\in Q$ define the formal power series $S_{f,q}\in\mathbb{R}^p \ll \Gamma^* \gg$ as follows

$$S_{f,q}(s) = c_f(i(s)(q,\epsilon))\ ,\ \forall s\in\Gamma^*$$

It is easy to see that in fact $c_f(v(q,\epsilon)) = S_{f,q}(\phi(v)) = c_f(i(\phi(v))(q,\epsilon))$, since $(v(q,\epsilon), i(\phi(v))(q,\epsilon))\in R^*$. Assume that $Q=\{q_1,\dots,q_N\}$. Define the formal power series $S_f\in\mathbb{R}^{Np}\ll\Gamma^*\gg$ by

$$S_f = \begin{bmatrix} S_{f,q_1} \\ S_{f,q_2} \\ \vdots \\ S_{f,q_N}\end{bmatrix}$$

Define the *set of formal power series* Ψ_Φ *associated with* Φ as follows

$$\Psi_\Phi = \{S_f\in\mathbb{R}^{Np}\ll\Gamma^*\gg \mid f\in\Phi\}$$

Define the *Hankel-matrix* H_Φ of Φ as the Hankel-matrix of Ψ_Φ. i.e. $H_\Phi = H_{\Psi_\Phi}$.

Let $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\ldots,m}, C_q) \mid q \in Q\})$ be a bilinear switched system. Define the representation $R_{\Sigma,\mu}$ *associated with the realization* (Σ, μ) of Φ by

$$R_{\Sigma,\mu} = (\mathcal{X}, \{B_{(q,j)}\}_{(q,j)\in\Gamma}, I, \widetilde{C})$$

where $B_{(q,j)} = B_{q,j} : \mathcal{X} \to \mathcal{X}$, $q \in Q, j = 1, \ldots, m$, $B_{q,0} = A_q : \mathcal{X} \to \mathcal{X}$, $q \in Q$, $\widetilde{C} = \begin{bmatrix} C_{q_1} \\ C_{q_2} \\ \vdots \\ C_{q_N} \end{bmatrix} : \mathcal{X} \to \mathbb{R}^{pN}$ and $I_f = \mu(f) \in \mathcal{X}$, $f \in \Phi$.

Let $R = (\mathcal{X}, \{M_{(q,j)}\}_{(q,j)\in\Gamma}, I, \widetilde{C})$ be a representation of Ψ_Φ. Define the realization (Σ_R, μ_R) *associated with* R by

$$\Sigma_R = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\ldots,m}, C_q) \mid q \in Q\})$$

where $\mu_R(f) = I_f \in \mathcal{X}, f \in \Phi$, $B_{q,j} = M_{(q,j)} : \mathcal{X} \to \mathcal{X}$, $q \in Q, j = 1, \ldots, m$, $A_q = M_{(q,0)} : \mathcal{X} \to \mathcal{X}, q \in Q$ and the maps $C_q : \mathcal{X} \to \mathcal{Y}$, $q \in Q$ are such that $\widetilde{C} = \begin{bmatrix} C_{q_1} \\ \vdots \\ C_{q_N} \end{bmatrix}$. It is easy to see that $R_{\Sigma_R,\mu_R} = R$. It turns out that there is a close connection between realizations of Φ and representations of Ψ_Φ.

Proposition 9. *Assume that Φ admits a generalized Fliess-series expansion. Then, (a) (Σ, μ) realization of Φ if and only if $R_{\Sigma,\mu}$ is a representation of Ψ_Φ, (b) Conversely, R is a representation of Ψ_Φ if and only if (Σ_R, μ_R) is a realization of Φ.*

Proof. It is enough prove Part (a). Part (b) follows from Part (a) by using the equality $R_{\Sigma_R,\mu_R} = R$. Assume that $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\ldots,m}, C_q) \mid q \in Q\})$. Notice that the map $\phi : \widetilde{\Gamma}^* \to \Gamma^*$ is surjective and for each $w_1, \ldots, w_k \in \mathrm{Z}_m$ it holds that

$$B_{q,w_1\cdots w_k} = B_{q,w_k} B_{q,w_{k-1}} \cdots B_{q,w_1} = B_{(q,w_k)} \cdots B_{(q,w_1)} = B_{\phi(q,w_1\cdots w_k)}$$

Then it is easy to see that $R_{\Sigma,\mu}$ is a representation of Ψ_Φ if and only if for all $(q_1, w_1), \ldots, (q_k, w_k) \in \widetilde{\Gamma}$

$$\begin{aligned} c_f((q_1, w_1) \cdots (q_k, w_k)) = c_f((q_1, w_1) \cdots (q_k, w_k)(q_k, \epsilon)) = \\ = S_{f,q_k}(\phi((q_1, w_1)) \cdots \phi((q_k, w_k))) = C_{q_k} B_{\phi((q_1,w_1))} \cdots B_{\phi((q_1,w_1))} I_f = \\ = C_{q_k} B_{q_k,w_k} \cdots B_{q_1,w_1} \mu(f) \end{aligned}$$

But by Proposition 8 this is exactly equivalent to (Σ, μ) being a realization of Φ. □

From the discussion above using Theorem 1 one gets the following characterization of realizability.

Theorem 11. *Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$. The following are equivalent*

(i) *Φ has a realization by a bilinear switched system*
(ii) *Φ has a generalized Fliess-series expansion and Ψ_Φ is rational*
(iii) *Φ has a generalized Fliess-series expansion and* $\operatorname{rank} H_\Phi < +\infty$

Proof. First we show that (i) $\iff$ (ii). By Proposition 8 if (Σ, μ) a bilinear switched system realization of Φ, then Φ has a generalized Fliess-series expansion. From Proposition 9 we also get that $R_{\Sigma,\mu}$ is a representation of Ψ_Φ, i.e. Ψ_Φ is rational. Conversely, if Φ has a generalized Fliess-series expansion and R is a representation of Ψ_Φ, then from Proposition 9 it follows that (Σ_R, μ_R) is a realization of Φ. Since by Theorem 1 Ψ_Φ is rational if and only if $\operatorname{rank} H_{\Psi_\Phi} = \operatorname{rank} H_\Phi < +\infty$, we get that (ii) and (iii) are equivalent. □

The next step will be to characterize bilinear switched systems which are minimal realizations of Φ. In order to accomplish this task, we need to the following characterization of observability and semi-reachability of bilinear switched systems.

Lemma 13. *Let Σ be a bilinear switched system. Assume that (Σ,μ) is a realization of Φ. Let $R = R_{\Sigma,\mu}$. (Σ,μ) is observable if and only if R is observable. (Σ,μ) is semi-reachable from* Im μ *if and only if R is reachable.*

Proof. Notice that $B_{q,w} = B_{\phi((q,w))}$ and for each $(q_1,w_1),\ldots,(q_k,w_k) \in \widetilde{\Gamma}$

$$\ker \widetilde{C} B_{\phi((q_1,w_1))} \cdots B_{\phi((q_k,w_k))} = \bigcap_{q\in Q} \ker C_q B_{q_1,w_1} \cdots B_{q_k,w_k}$$

Notice that $\mathrm{Im}\mu = \{\mu(f) \mid f \in \Phi\} = \{I_f \mid f \in \Phi\}$. Then it follows from Proposition 7 that $O_\Sigma = O_R$ and $W_R = \mathrm{Span}\{x \mid x \in \mathit{Reach}(\Sigma,\mathrm{Im}\mu)\}$. Then the lemma follows from Proposition 7 and the definition of observability and reachability for representations. $\square$

It is also easy to see that $\dim\Sigma = \dim R_{\Sigma,\mu}$ and $\dim R = \dim \Sigma_R$. In fact, Proposition 9 implies the following.

Lemma 14. *If R is a minimal representation of Ψ_Φ then (Σ_R,μ_R) is a minimal realization of Φ. Conversely, if (Σ,μ) is a minimal realization of Φ, then $R_{\Sigma,\mu}$ is a minimal representation of Ψ_Φ.*

The following lemma clarifies the relationship between representation morphisms and bilinear switched system morphisms.

Lemma 15. *$T : (\Sigma,\mu) \to (\Sigma^{'},\mu^{'})$ is a bilinear switched system morphism if and only if $T : R_{\Sigma,\mu} \to (\Sigma^{'},\mu^{'})$ is a representation morphism. Moreover, T is injective, surjective, an isomorphism as a bilinear switched system morphism if and only if T is injective, surjective, an isomorphism as a representation morphism.*

Proof. T is a bilinear switched system morphism if and only if

$$TA_q = A_q^{'}T \quad C_q = C_q^{'}T \quad TB_{q,j} = B_{q,j}^{'}T \quad T\mu(f) = \mu^{'}(f)$$

for each $q \in Q, j = 1,2\ldots,m$ and $f \in \Phi$. This is equivalent to $TB_{(q,j)} = B_{(q,j)}^{'}T$ for each $j \in \mathrm{Z}_m$, $TI_f = T\mu(f) = \mu^{'}(f) = I_f^{'}$ and

$$\widetilde{C} = \begin{bmatrix} C_{q_1} \\ \vdots \\ C_{q_N} \end{bmatrix} = \begin{bmatrix} (C_{q_1}^{'}T) \\ \vdots \\ (C_{q_N}^{'}T) \end{bmatrix} = \widetilde{C}^{'}T$$

That is, T is a representation morphism. $\square$

Using the theory of rational formal power series presented in Section 4 we get the following.

Theorem 12. *Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (Q\times T)^{+},\mathcal{Y})$. The following are equivalent*

(i) *(Σ_{min},μ_{min}) is a minimal realization of Φ by a bilinear switched system*
(ii) *(Σ_{min},μ_{min}) is semi-reachable from* $\mathrm{Im}\mu$ *and it is observable*
(iii) $\dim\Sigma_{\mathrm{min}} = \mathrm{rank}\ H_\Phi$
(iv) *For any bilinear switched system realization (Σ,μ) of Φ, such that (Σ,μ) is semi-reachable from* $\mathrm{Im}\mu$, *there exist a surjective homomorphism $T : (\Sigma,\mu) \to (\Sigma_{min},\mu_{min})$. In particular, all minimal realizations of Φ by bilinear switched systems are algebraically similar.*

Proof. (Σ_{min},μ_{min}) is a minimal realization if and only if that $R_{min} = R_{\Sigma_{min},\mu_{min}}$ is minimal representation, that is, by Theorem 2 R_{min} is reachable and observable. By Lemma 13 the latter is equivalent to (Σ_{min},μ_{min}) being semi-reachable from Im μ and observable. That is, we get that (i) $\iff$ (ii). By Theorem 2 a representation R_{min} is minimal if and only if $\dim\Sigma_{min} = \dim R_{min} = \mathrm{rank}\ H_{\Phi_\Psi} = \mathrm{rank}\ H_\Phi$. That is, we showed that (i) $\iff$ (iii). To show that (i) $\iff$ (iv), notice that $(\Sigma_{min}, \mu_{min})$ is a minimal realization if and only if $R_{\Sigma_{min},\mu_{min}}$ is a minimal representation. By Theorem 2 R_{min} is minimal if and only if for any reachable representation R there exists a surjective representation morphism $T : R \to R_{min}$. It means that $(\Sigma_{min}, \mu_{min})$ is minimal if and only if for any reachable representation R of Ψ_Φ there exists a surjective representation morphism $T : R \to R_{\Sigma_{min},\mu_{min}}$. But any reachable representation R gives rise to a semi-reachable realization of Φ and vice versa. That is, we get that $(\Sigma_{min}, \mu_{min})$ is minimal if and only if for any realization $(\Sigma^{'}, \mu^{'})$ of Φ such that $(\Sigma^{'}, \mu^{'})$ is semi-reachable from Imμ there exists a surjective representation morphism $T : R_{\Sigma^{'},\mu^{'}} \to R_{\Sigma_{min},\mu_{min}}$. By Lemma 15 we get that the latter is equivalent to $T : (\Sigma^{'}, \mu^{'}) \to (\Sigma_{min}, \mu_{min})$ being a surjective bilinear switched system morphism. From Corollary 1 it follows that if $(\Sigma^{'}, \mu^{'})$ is a minimal realization of Φ, then there exists a representation isomorphism $T : R_{\Sigma^{'},\mu^{'}} \to R_{\Sigma_{min},\mu_{min}}$ which means that $(\Sigma_{min}, \mu_{min})$ is gives rise to the bilinear switched system isomorphism $T : (\Sigma^{'}, \mu^{'}) \to (\Sigma_{min}, \mu_{min})$, that is, $(\Sigma^{'}, \mu^{'})$ and $(\Sigma_{min}, \mu_{min})$ are algebraically similar. □

6.3. Realization of input/output maps by bilinear switched systems with constraints on the switching

The case of restricted switching is slightly more involved. As in the case of arbitrary switching, we will associate a set Ψ_Φ of formal power series over Γ with the set of input-output maps $\Phi \subseteq F(PC(T,\mathcal{U})\times TL, \mathcal{Y})$. Every representation of Ψ_Φ gives rise to a realization of Φ. If L is a regular language, then existence of a realization of Φ implies existence of a representation of Ψ_Φ. However, the dimension of the minimal representation of Ψ_Φ might be bigger than the dimension of a realization of Φ. Any minimal representation of Ψ_Φ gives rise to an observable and semi-reachable realization of Φ. But this observable and semi-reachable realization need not be a minimal one. Extraction of the right information from Φ and the construction of Ψ_Φ is much more involved in the case of restricted switching than in the case of arbitrary switching.

Recall the definition of the relation $R^* \subseteq \widetilde{\Gamma}^* \times \widetilde{\Gamma}^*$ from Subsection 6.1. Define the set $\widetilde{JL} \subseteq \widetilde{\Gamma}^*$ by

$$\widetilde{JL} = \{s \in \widetilde{\Gamma}^* \mid \exists w \in JL : (w, s) \in R^*\}$$

In fact, $\widetilde{JL}$ contains all those sequences in $\widetilde{\Gamma}^*$ for which we can derive some information based on the values of a convergent generating series for sequences from JL. More precisely, if $c : JL \to \mathcal{Y}$ is a generating convergent sequence, then c can be extended to a generating convergent series $\widetilde{c} : \widetilde{JL} \to \mathcal{Y}$ by defining $\widetilde{c}(s) = c(w)$ for each $s \in \widetilde{JL}, w \in JL, (s, w) \in R^*$. It is clear that for any $s \in \widetilde{JL}$ there exists a $w \in JL$ such that $(s, w) \in R^*$ and if $(s, w), (s, v) \in R^*$, $w, v \in JL$, then $c(w) = c(v) = \widetilde{c}(s)$, since c was assumed to be a generating convergent series. If $(s, x) \in R^*$, then $\widetilde{c}(s) = \widetilde{c}(x)$. Moreover, if $(s, w) \in R^*$ and $s = (z_1, x_1)\cdots(z_l, x_l)$ and $w = (q_1, v_1)\cdots(q_k, v_k)$, then from the definition of R it follows that $\sum_1^k |v_i| = \sum_1^l |x_i|$, that is, $||\widetilde{c}(s)|| = ||c(w)|| \le KM^{|v_1|}\cdots M^{|v_k|} = KM^{\sum_1^k |v_i|} = KM^{\sum_1^l |x_l|}$. That is, $\widetilde{c} : \widetilde{JL} \to \mathcal{Y}$ is indeed a generating convergent series. Moreover, on JL the sequence $\widetilde{c}$ coincides with c, that is, if $w \in JL$, then $\widetilde{c}(w) = c(w)$. By abuse of notation, we will denote $\widetilde{c}$ simply by c in the sequel.

For each $q \in Q$ define $JL_q = \{v(q, w) \in \widetilde{JL} \mid v \in \widetilde{\Gamma}^*, (q, w) \in \widetilde{\Gamma}\}$. Let $L_q = \{w \in \Gamma^* \mid \exists v \in JL_q : \phi(v) = w\}$. Notice that

$$w \in L_q \iff i(w)(q, \epsilon) \in JL_q$$

Indeed, if $i(w)(q, \epsilon) \in JL_q$, then $\phi(i(w)(q, \epsilon)) = \phi(i(w)) = w \in L_q$. Conversely, if $w \in L_q$, then $w = \phi(v)$ for some $v \in JL_q$. But then $v = u(q, z)$ and $(u(q, z)(q, \epsilon), u(q, z\epsilon) = v) \in R^*$ and $(v(q, \epsilon), i(w)(q, \epsilon)) \in R^*$ which implies $(v, i(w)(q, \epsilon)) \in R^*$. Since $v \in \widetilde{JL}$, we know that $i(w)(q, \epsilon) \in \widetilde{JL}$, that is, $i(w)(q, \epsilon) \in JL_q$.

Let $\Phi \subseteq F(PC(T,\mathcal{U})\times TL, \mathcal{Y})$ be a set of input/output maps defined on sequences of discrete modes belonging to L. Assume Φ admits a generalized Fliess-series expansion. For each $q \in Q$, $f \in \Phi$ define the formal power series $T_{f,q} \in \mathbb{R}^p \ll \Gamma^* \gg$ by

$$T_{f,q}(s) = \begin{cases} c_f(i(s)(q, \epsilon)) & \text{if } s \in L_q \\ 0 & \text{otherwise} \end{cases}$$

Notice that for each $s \in L_q$ there exists a $w = u(q,v) \in JL$ such hat $T_{f,q}(s) = c_f(w)$. Indeed, $s \in L_q$ implies that there exists a $w = (q_1,x_1)\cdots(q_l,x_l)(q,x_{l+1}) \in JL$ such that $(w, i(s)(q,\epsilon)) \in R^*$. Thus $T_{f,q}(s) = c_f(i(s)(q,\epsilon)) = c_f(w)$. The intuition behind the definition of $T_{f,q}$ is the following. We store in $T_{f,q}$ the values of all those $c_f(s)$ which show up in the generalized Fliess-series expansion of $f(u,w)$, for some switching sequence $w \in TL$ such that w ends with discrete mode q. For all the other sequences from Γ^* we set the value of $T_{f,q}$ to zero.

Assume that $Q = \{q_1,\ldots,q_N\}$. Define the formal power series $T_f \in \mathbb{R}^{Np} \ll \Gamma^* \gg$ by

$$T_f = \begin{bmatrix} T_{f,q_1} \\ T_{f,q_2} \\ \vdots \\ T_{f,q_N} \end{bmatrix}$$

Define the *set of formal power series* Ψ_Φ *associated with* Φ as follows

$$\Psi_\Phi = \{T_f \in \mathbb{R}^{Np} \ll \Gamma^* \gg \mid f \in \Phi\}$$

Define the *Hankel-matrix* H_Φ *of* Φ as the Hankel-matrix of Ψ_Φ, that is, $H_\Phi = H_{\Psi_\Phi}$.

For each $q \in Q$ define the formal power series $Z_q \in \mathbb{R}^p \ll \Gamma^* \gg$ by $Z_q(w) = \begin{cases} (1,1,\ldots,1)^T & \text{if } w \in L_q \\ 0 & \text{otherwise} \end{cases}$.

Let $Z \in \mathbb{R}^{Np} \ll \Gamma \gg$ be

$$Z = \begin{bmatrix} Z_{q_1} \\ \vdots \\ Z_{q_N} \end{bmatrix}$$

and let Ω be the indexed set $\{Z \mid f \in \Phi\}$, i.e $\Omega : \Phi \to \mathbb{R}^{Np} \ll \Gamma^* \gg$ and $\Omega(f) = Z, f \in \Phi$. With the notation above, the following holds.

Lemma 16. *Let $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,\{B_{q,j}\}_{j=1,2,\ldots,m},C_q) \mid q \in Q\})$ be a bilinear switched system. Assume that (Σ,μ) is a realization of Φ and Φ admits a generalized Fliess-series expansion. Let $\Phi' = \{y_\Sigma(\mu(f),.,.) \in F(PC(T,\mathcal{U}) \times (Q \times T)^+, \mathcal{Y}) \mid f \in \Phi\}$ and let Ψ'_Φ be the set of formal power series associated with Φ' as defined in Subsection 6.2. That is, $\Psi_{\Phi'} = \{S_g \in \mathbb{R}^{Np} \ll \Gamma \gg \mid g \in \Phi'\}$. Let $S_f = S_{y_\Sigma(\mu(f),.,.)}$ and let $\Theta = \{S_f \mid f \in \Phi\}$. Then the following holds*

$$\Psi_\Phi = \Theta \odot \Omega$$

Proof. Define $\mu' : \Phi' \to \mathcal{X}$ by $\mu'(y_\Sigma(\mu(f),.,.)) = \mu(f)$. Since (Σ,μ) is a realization of Φ, if for some $f_1, f_2 \in \Phi$ it holds that $y_\Sigma(\mu(f_1),.,.) = y_\Sigma(\mu(f_2),.,.)$, then $f_1 = y_\Sigma(\mu(f_1),.,.)|_{PC(T,\mathcal{U})\times TL} = y_\Sigma(\mu(f_2),.,.)|_{PC(T,\mathcal{U})\times TL} = f_2$. That is, $f_1 = f_2$ and thus μ' is well-defined. It is also easy to see that (Σ,μ') realizes Φ', therefore Φ' has a generalized Fliess-series expansion. For each $f \in \Phi$, denote by $c_f : \widetilde{JL} \to \mathcal{Y}$ the generating convergent series corresponding to f, i.e. $F_{c_f} = f$. Denote by $d_f : \widetilde{\Gamma}^* \to \mathcal{Y}$ the series corresponding to $y_\Sigma(\mu(f),.,.)$, i.e. $F_{d_f} = y_\Sigma(\mu(f),.,.)$. By Proposition 8 (Σ,μ) is a realization of Φ with constraint L, if and only if $\forall w(q,v) \in JL : c_f(w(q,v)) = C_q B_{q,v} B_{\phi(w)} \mu(f)$. Here we used the fact that if $w = (q_1,z_1)\cdots(q_k,z_k)$, then $B_{q_k,z_k}\cdots B_{q_1,z_1} = B_{\phi(w)}$. But (Σ,μ') realizes Φ', so by Proposition 8 it holds that $\forall s(q,x) \in \widetilde{JL} : d_f(s(q,x)) = C_q B_{q,x} B_{\phi(s)} \mu'(y_\Sigma(\mu(f),.,.))$. Notice that if $(s(q,x), w(q,v)) \in R^*$, then $\phi(s(q,x)) = \phi(w(q,v))$, and therefore $B_{q,v}B_{\phi(w)} = B_{\phi(w(q,v))} = B_{\phi(s(q,x))} = B_{q,x}B_{\phi(s)}$. Notice that $\mu(f) = \mu'(y_\Sigma(\mu(f),.,.))$. Thus for each $s(q,x) \in \widetilde{JL}$, $w(q,v) \in JL$ we get that $c_f(s(q,x)) = c_f(w(q,v)) = d_f(s(q,x))$. Thus, for each $q \in Q$, $f \in \Phi$, $s \in L_q$ we get that $T_{f,q}(s) = c_f(i(s)(q,\epsilon)) = d_f(i(s)(q,\epsilon)) = S_{f,q}(s)$. Notice that for each $s \notin L_q$, $T_{f,q}(s) = 0$ and $Z_q(s) = 0$. That is, $T_{f,q} = S_{f,q} \odot Z_q$ and therefore $T_f = S_f \odot Z$. □

If L is regular, then Ω turns out to be a rational indexed set.

Lemma 17. *If L is regular, then $L_q, q \in Q$ are regular languages and Ω is a rational indexed set of formal power series.*

Proof. It is enough to show that if L is a regular language, then $L_q, q \in Q$ are regular languages. Indeed, if $L_q, q \in Q$ are regular, then $\{e_j^T Z_q\}, q \in Q, j = 1, \dots, p$ are rational sets of formal powers series, since $e_j^T Z_q(w) = 1 \iff w \in L_q$. Therefore, $\{Z = \begin{bmatrix} Z_{q_1}^T & \cdots & Z_{qN}^T \end{bmatrix}^T\}$ is a rational set, therefore Ω is a rational indexed set of formal power series by Lemma 1. Define $pr_Q : \Gamma^* \to Q^*$ by $pr_Q((q_1, j_1) \cdots (q_k, j_k)) = q_1 \cdots q_k$. Recall from Subsection 5.1 the definition of the sets $F_q(w)$ and $\widetilde{L}_q$. Lemma 8 says that if L is regular, then $\widetilde{L}_q$ is regular. We shall prove that $L_q = pr_Q^{-1}(\widetilde{L}_q)$. From this equality it follows that if $\widetilde{L}_q$ is regular, then L_q is regular. Indeed, pr_Q is a monoid morphism, and therefore can be realized by a regular transducer see [3]. Then the regularity of L_q follows from the classical result on regular transducers. Alternatively, if $\mathcal{A} = (S, Q, \delta, F)$ is a finite automaton accepting $\widetilde{L}_q$, then the deterministic finite automaton $\mathcal{A}' = (S, \Gamma, \delta', F)$ defined by $\delta'(s, (q, j)) = \delta(s, q), (q, j) \in \Gamma, s \in S$ accepts L_q.

We now proceed with the proof of the equality $L_q = pr_Q^{-1}(\widetilde{L}_q)$. First we show that $L_q \subseteq pr_Q^{-1}(\widetilde{L}_q)$. If $v = (q_1, j_1) \cdots (q_t, j_t) \in L_q$, then there exists $w(q, z) \in JL_q$, such that $\phi(w(q, p)) = v$. Let $w = (z_1, m_1) \cdots (z_k, m_k)$. Then $z_1 \cdots z_k q \in L$. Let $l = \min\{j \mid |m_j| > 0\}$. Let $s = z_1 \cdots z_{l-1}$, $x = z_l \cdots z_k$. From $\phi(w(q, z)) = v$ it follows that $z_l = q_1 = \cdots = q_{|m_l|}$, $z_{i+1} = q_{|m_i|+1} = \cdots = q_{|m_{i+1}|}$, for $i = l, l+1, \dots, k-1$, $q_{|m_k|+1} = \cdots q_t = q$, and $|p| + \sum_{i=1}^{k} |m_i| = t$. That is, we get that $q_1 \cdots q_t q = z_l^{|m_l|} \cdots z_k^{|m_k|} q^{|p|} q$ and $sxq = z_1 \cdots z_k q \in L$, that is, $(s, ((|m_1|, \dots, |m_k|, |p|), x) \in F_q(q_1 \cdots q_t)$, i.e. $q_1 \cdots q_t = pr_Q((q_1, j_1) \cdots (q_t, j_t)) \in \widetilde{L}_q$. That is, $L_q \subseteq pr_Q^{-1}(\widetilde{L}_q)$. Let $w \in \widetilde{L}_q$ and let $(u, (\alpha, h)) \in F_q(w)$. Assume that $u = q_1 \dots q_{|u|}$ and $h = z_1 \cdots z_k$, $q_1, \dots, q_{|u|}, z_1, \dots z_k \in Q$. Since $w = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$, we get that $v \in pr_Q^{-1}(w)$ if and only if $v = v_1 \cdots v_k$, $v_i = (z_i, j_{1,i}) \cdots (z_i, j_{\alpha_i, i}) \in \Gamma^*$, $|v_i| = \alpha_i$, $j_{i,j} \in \mathrm{Z}_m, i = 1, \dots, \alpha_j, j = 1, \dots, k$. Let $\underline{j_i} = j_{1,i} j_{2,i} \dots j_{\alpha_i, i}$, $s = (q_1, \epsilon) \cdots (q_{|u|}, \epsilon)(z_1, \underline{j_1}) \cdots \cdots (z_k, \underline{j_k})$. Since $uv \in L$, we have that $s \in JL$ and $z_k = q$ implies that $s \in JL_q$. But $\phi(s) = \phi((z_1, \underline{j_1}) \cdots (\phi(z_k, \underline{j_k})) = v_1 \cdots v_k \in L_q$. That is, $pr_Q^{-1}(\widetilde{L}_q) \subseteq L_q$, and consequently $L_q = pr_Q^{-1}(\widetilde{L}_q)$. □

Let $R = (\mathcal{X}, \{M_z\}_{z \in \Gamma}, I, C)$ be a representation of Ψ_Φ. Define the *bilinear switched system realization* (Σ_R, μ_R) *asscociated with* R as in Section 6.2. That is,

$$\Sigma_R = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\dots,m}, C_q) \mid q \in Q\}) \text{ and } \mu_R(f) = I_f$$

where $C_q : \mathcal{X} \to \mathcal{Y}$, $q \in Q$ are such that $C = \begin{bmatrix} C_{q_1} \\ \vdots \\ C_{q_N} \end{bmatrix}$, $B_{q,j} = M_{(q,j)}, A_q = M_{(q,0)}$, $q \in Q, j = 1, \dots, m$. It is easy to see that (Σ_R, μ_R) is semi-reachable (observable) if and only if R is reachable (observable).

Recall from Subsection 5.3 the definition of $comp(L)$:

$$comp(L) = \{w_1 \cdots w_k \in Q^* \mid \widetilde{L}_{w_k} = \emptyset, w_1, \dots, w_k \in Q\}$$

The following statement is an easy consequence of Proposition 8.

Theorem 13. *If Φ has a generalized Fliess-series expansion with constraint L and $R = (\mathcal{X}, \{B_z\}_{z \in \Gamma}, I, \widetilde{C})$ is a representation of Ψ_Φ, then (Σ_R, μ_R) is a realization of Φ. That is, if Ψ_Φ is rational, then Φ has a realization by a bilinear switched system. Moreover, for each $f \in \Phi$, $w \in T(comp(L))$*

$$\forall u \in PC(T, \mathcal{U}) : y_\Sigma(\mu(f), u, w) = 0$$

Proof. Let (Σ_R, μ_R) the realization associated with R. Assume that

$$\Sigma_R = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, \{B_{q,j}\}_{j=1,2,\dots,m}, C_q) \mid q \in Q\})$$

Since R is a representation of Ψ_{Φ}, we get that for each $(q_1,w_1)\cdots(q_k,w_k) \in JL$, $f \in \Phi$

$$\begin{aligned} c_f((q_1,w_1)\cdots(q_k,w_k)) = T_{f,q_k}(\phi((q_1,w_1)\cdots(q_k,w_k))) = \\ = C_{q_k}B_{\phi((q_k,w_k))}\cdots B_{\phi((q_1,w_1))}I_f = C_{q_k}B_{q_k,w_k}\cdots B_{q_1,w_1}\mu(f) \end{aligned} \tag{18}$$

We used the definition of (Σ_R,μ_R) and the fact that $B_{(q,j_1)\cdots(q,j_l)} = B_{\phi((q,j_1\cdots j_l))}$ for each $q \in Q, j_1,\dots,j_l \in \mathbb{Z}_m$. From Proposition 8 we get that (18) implies that (Σ_R,μ_R) is a realization of Φ.

Let $w = (q_1,t_1)\cdots(q_k,t_k) \in T(comp(L))$, that is, $\widetilde{L}_{q_k} = \emptyset$. Then for each $s = (q_1,w_1)\cdots(q_k,w_k) \in \widetilde{\Gamma}^*$ we get that $T_{f,q_k}(\phi(s)) = 0$, since $\phi(s) \notin L_{q_k}$. Indeed, $\widetilde{L}_{q_k} = \emptyset$ and from the proof of Lemma 17 we know that $L_q = pr_Q^{-1}(\widetilde{L}_q)$. If $\phi(s) \in L_{q_k}$, then we get that $pr_Q(\phi(s)) \in \widetilde{L}_{q_k} = \emptyset$, a contradiction. But $g = y_{\Sigma}(\mu(f),.,.)$ has a generalized Fliess-series expansion, and from Proposition 8 it follows that $c_g((q_1,w_1)\cdots(q_k,w_k)) = C_{q_k}B_{q_k,w_k}\cdots B_{q_1,w_1}\mu(f)$. Since R is a representation of Ψ_{Φ}, we also get that $C_{q_k}B_{q_k,w_k}\cdots B_{q_1,w_1}\mu(f) = C_{q_k}B_{\phi((q_k,w_k))}\cdots B_{\phi((q_1,w_1))}I_f = T_{f,q_k}(\phi((q_1,w_1)\cdots\phi(q_k,w_k)) = 0$. That is, if $q_1\cdots q_k \in comp(L)$, then for each $w_1,\dots,w_k \in \mathbb{Z}_m^*$ it holds that

$$c_g((q_1,w_1)\cdots(q_k,w_k)) = 0$$

Then the definition of F_{c_g} implies that $F_{c_g} = g = 0$ for each $q_1\cdots q_k \in T(comp(L))$. □

We see that rationality of Ψ_{Φ}, i.e. the condition that rank $H_{\Phi} < +\infty$, is a sufficient condition for realizability of Φ. It turns out that if L is regular, this is also a necessary condition. From the discussion above, Lemma 16 and Lemma 2 one gets the following.

Theorem 14. *Assume that L is regular. Then the following are equivalent.*

(i) *Φ has a realization by a bilinear switched system*

(ii) *Φ has a generalized Fliess-series expansion and* rank $H_{\Phi} < +\infty$

(iii) *There exists a realization of Φ by a bilinear switched system (Σ,μ) such that Σ is observable and semi-reachable from* Imμ *and*

$$\forall f \in \Phi : y_{\Sigma}(\mu(f),.,.)|_{PC(T,\mathcal{U})\times T(compl(L))} = 0$$

and for any (Σ',μ') bilinear switched system realization of Φ

$$\dim\Sigma \leq \text{rank } H_{\Omega}\cdot\dim\Sigma'$$

Proof. `(i)` $\iff$ `(ii)`

By Lemma 16, if (Σ,μ) is a realization of Φ, then Φ has a generalized Fliess-series expansion and $\Psi_{\Phi} = \Theta\odot\Omega$. Since (Σ,μ) is a realization of $\Phi' = \{y_{\Sigma}(\mu(f),.,.) \mid f \in \Phi\}$ we get that $\Psi_{\Phi'}$ is rational. Define the map $\Phi \ni f \mapsto i(f) = y_{\Sigma}(\mu(f),.,) \in \Phi'$. Since $\Theta = \{S_{i(f)} \mid f \in \Phi\}$, Lemma 4 implies that Θ is rational. Since L is regular, by Lemma 17 Ω is rational, therefore by Lemma 2 $\Psi_{\Phi} = \Theta\odot\Omega$ is rational, that is, rank $H_{\Phi} < +\infty$. Conversely, if Φ admits a generalized Fliess-series expansion and rank $H_{\Phi} < +\infty$, i.e. Ψ_{Φ} is rational, then there exists a representation R of Ψ_{Φ} and by Theorem 13 (Σ_R,μ_R) is a realization of Φ

`(ii)` $\iff$ `(iii)`

It is clear that (iii) implies (i), which implies (ii). We will show that (ii) implies (iii). Assume that Φ admits a generalized Fliess-series expansion and Ψ_{Φ} is rational. Let R be the minimal representation of Ψ_{Φ}. Then (Σ_R,μ_R) is a realization of Φ, moreover Σ_R is observable and semi-reachable from Imμ. From Theorem 13 it follows that

$$y_{\Sigma}(\mu_R(f),.,.)|_{PC(T,\mathcal{U})\times T(comp(L))} = 0$$

Let (Σ',μ') be a realization of Φ. Then $R' = R_{\Sigma',\mu'}$ is a representation of $\Psi_{\Phi'}$, where $\Phi' = \{y_{\Sigma'}(\mu'(f),.,.) \mid f \in \Phi\}$. From Lemma 16 we know that $\Psi_{\Phi} = \Theta\odot\Omega$, where $\Theta = \{S_{y_{\Sigma'}(\mu'(f),.,.)} \mid f \in \Phi\}$. Assume that $R' = (\mathcal{X}',\{B'_z\}_{z\in\Gamma},I',C')$. Then $\widetilde{R} = (\mathcal{X}',\{B'_z\}_{z\in\Gamma},\widetilde{I},C')$, where $\widetilde{I}_f = I_{y_{\Sigma'}(\mu'(f),.,.)}$, $f \in \Phi$, is a representation of Θ. But R is a minimal representation of Ψ_Φ, therefore $\dim R = \dim \Sigma_R = \text{rank } H_{\Psi_\Phi}$. From Lemma 2 it follows that rank $H_{\Psi_\Phi} = \text{rank } H_{\Theta \odot \Omega} \leq (\text{rank } H_\Omega)(\text{rank } H_\Theta)$. Since $\dim \Sigma = \dim R^{'} = \dim \widetilde{R} \geq \text{rank } H_\Theta$, we get that

$$\dim \Sigma_R \leq \text{rank } H_\Omega \cdot \dim \Sigma^{'}$$

Taking (Σ_R, μ_R) for (Σ, μ) completes the proof. □

The following example demonstrates existence of a semi-reachable and observable realization of Φ, which is non-minimal.

Example

Let $Q = \{1, 2\}$, $L = \{q_1^k q_2 \mid k > 0\}$, $\mathcal{Y} = \mathcal{U} = \mathbb{R}$. Define the generating series $c : \widetilde{JL} \to \mathbb{R}$ by $c((q_1, w_1)(q_2, w_2)) = 2^k$, where $w_2 = 0^{j_0} z_1 \cdots z_l 0^{j_l}$, $k = \sum_{i=0}^{l} j_l$, $z_i \in \{1\}^*, i = 1, \ldots, l$. Let $\Phi = \{F_c\}$. Define the system $\Sigma_1 = (\mathbb{R}, \mathbb{R}, \mathbb{R}, Q, \{(A_q, B_{q,1} C_q) \mid q \in \{q_1, q_2\}\})$ by $A_{q_1} = 1, B_{q_1,1} = 1, C_{q_1} = 1$ and $A_{q_2} = 2, B_{q_2,1} = 1, C_{q_2} = 1$. Define the system $\Sigma_2 = (\mathbb{R}^2, \mathbb{R}, \mathbb{R}, Q, \{(\widetilde{A}_q, \widetilde{B}_{q,1},$
$\widetilde{C}_q) \mid q \in Q\})$ by

$$\widetilde{A}_{q_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \widetilde{B}_{q_1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \widetilde{C}_{q_1} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\widetilde{A}_{q_2} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \quad \widetilde{B}_{q_2,1} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \widetilde{C}_{q_2} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Let $\mu_1 : F_c \mapsto 1$ and $\mu_2 : F_c \mapsto (1, 0)^T \in \mathbb{R}^2$. Both (Σ_1, μ_1) and (Σ_2, μ_2) are semi-reachable from Imμ_1 and Imμ_2 respectively and they are observable, therefore they are the minimal realizations of $y_{\Sigma_1}(1, ., .)$ and $y_{\Sigma_2}((1, 0)^T, ., .)$. Moreover, it is easy to see that $(\Sigma_i, \mu_i), i = 1, 2$ are both realizations of Φ with constraint L. Yet, $\dim \Sigma_1 = 1$ and $\dim \Sigma_2 = 2$. In fact, Σ_2 can be obtained by constructing the minimal representation of Ψ_Φ, i.e., Σ_2 is a realization of F_c satisfying part (iii) of Theorem 14.

7. Conclusions

Solution to the realization problem for linear and bilinear switched systems was presented. The realization problem considered is to find a realization of a family of input-output maps. Moreover, it is allowed to restrict the input-output maps to some subsets of switching sequences. Thus, the realization problem covers the case of linear and bilinear switched systems where the switching is controlled by an automaton and the automaton is known in advance. The results of the paper extend those of [14], where a much more restricted realization problem was studied. The paper offers a new technique, the theory of formal power series, to deal with realization problem for switched systems.

Topics of further research include realization theory for piecewise-affine systems, switched systems with switching controlled by an automaton or a timed automaton and non-linear switched systems.

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Appendix A. Proofs for formal power series

Proof of Theorem 1. `Part (i)`

Notice that for any $w \in X^*$, $w = w_1 \cdots w_k$, $w_1, \ldots, w_k \in X$ and for any $T \in \mathbb{R}^p \ll X^* \gg$

$$w \circ T = w_k \circ (w_{k-1} \circ (\cdots (w_1 \circ T) \cdots)))$$

Since $B_j = S_j$, and $A_\sigma T = \sigma \circ T$, we get that for all $w \in X^*$

$$w \circ S_j = A_w S_j = A_w B_j$$

But $S_j(w) = w \circ S_j(\epsilon) = C(w \circ S_j)$, so we get that $S_j(w) = CA_wB_j$, i.e., R_Ψ is indeed a representation of Ψ.

`Part (ii)`

The statement

$$\dim W_\Psi < +\infty \Longrightarrow \Psi \text{ is rational}$$

follows from part (i) of the theorem. We will prove that Ψ rational $\Longrightarrow$ $\dim W_\Psi < +\infty$. Assume $R = (\mathcal{X}, A_{\sigma\sigma\in X}, B, C)$ is a representation of Ψ. Let $\dim \mathcal{X} = n$ and let $e_l \in \mathcal{X}, l = 1, 2, \ldots, n$ be a basis of $\mathcal{X}$. Define $Z_l \in K^p \ll X^* \gg$ by $Z_l(w) = CA_we_l$, $w \in X^*$. For each $j \in J$ there exist $\alpha_{j,1}, \ldots, \alpha_{j,n} \in \mathbb{R}$ such that $B_j = \sum_{l=1}^n \alpha_{j,l}e_l$. We get that

$$S_j(w) = CA_wB = \sum_{l=1}^{n} \alpha_{j,l}CA_we_l = \sum_{l=1} \alpha_{j,l}Z_l(w)$$

On the other hand

$$w \circ Z_l(v) = Z_l(wv) = CA_vA_we_l = \sum_{k=1}^{n} \beta_{k,l}CA_ve_k = \sum_{k=1}^{n} \beta_{k,l}Z_k$$

where $\mathcal{X} \ni A_we_l = \sum_k^n \beta_{k,l}e_k$. Thus, $w \circ S_j, S_j \in \operatorname{Span}\{Z_i \mid i = 1, \ldots, n\}$ holds, which implies that $W_\Psi \subseteq \operatorname{Span}\{Z_i \mid i = 1, \ldots, n\}$. That is, $\dim W_\Psi < +\infty$.

Finally, we show that $\dim W_\Psi < +\infty \Longleftrightarrow \operatorname{rank} H_\Psi < +\infty$. In fact, $\dim W_\Psi = \operatorname{rank} H_\Psi$ and W_Ψ is naturally isomorphic to the span of column vectors of H_Ψ. Indeed, it easy easy to see that $w \circ S_j$ corresponds to $(H_\Psi)_{.,(w,j)}$ and the rest of the statement follows easily from this observation. □

Proof of Lemma 3. Let $R = (\mathcal{X}, \{A_\sigma\}_{\sigma\in X}, B, C)$ be a representation of Ψ. Define $R_r = (W_R, \{A_\sigma^r\}_{\sigma\in X}, B^r, C^r)$ by $A_\sigma^r = A_\sigma|_{W_R}$, $B_j^r = B_j \in W_R$ and $C^r = C|_{W_R}$. Since W_R is invariant w.r.t A_σ, the representation R_r is well defined. It is easy to see that $C^rA_w^rB_j^r = CA_wB_j$, so R_r is a representation of Ψ. It is easy to see that $W_{R_r} = W_R$ and $O_{R_r} = O_R \cap W_R$. Define $R_o = (W_R/O_{R_r}, \{\widetilde{A}_\sigma\}_{\sigma\in X}, \widetilde{B}, \widetilde{C})$ by $\widetilde{A}_\sigma[x] = [A_\sigma^rx]$, $\widetilde{B}_j = [B_j^r]$ and $\widetilde{C}[x] = C^rx$, for each $x \in W_R$. Here $[x]$ denotes the equivalence class of W_R/O_{R_r} represented by $x \in W_R$. The representation R_o is well defined. Indeed, if $x_1 - x_2 \in O_{R_r}$, then $\forall w \in X^* : C^rA_w^r(x_1 - x_2) = 0$, so we get that $\forall w \in X^* : C^rA_w^rA_\sigma^r(x_1 - x_2) = 0$. That is $A_\sigma^rx_1 - A_\sigma^rx_2 \in O_{R_r}$. It implies that $\widetilde{A}_\sigma$ is well defined. It is straightforward to see that $\widetilde{B}_j$ is well defined. Since $x_1 - x_2 \in O_{R_r}$ implies that $x_1 - x_2 \in \ker C^r$, we get that $\widetilde{C}$ is well defined too. Moreover $\widetilde{C}\widetilde{A}_w\widetilde{B}_j = CA_wB_j$, so R_o is a representation of Ψ. It is easy to see that $O_{R_o} = \{0\}$. That is, R_o is observable. Moreover, R_o is reachable, since $\operatorname{Span}\{\widetilde{A}_w\widetilde{B_j} \mid w \in X^*, j \in J\} = \operatorname{Span}\{[A_w^rB_j^r] \mid j \in J, w \in X^*\} = W_R/O_{R_r}$. □

Proof of Theorem 2. (i) $\Longrightarrow$ (ii)

Assume that $W_{R_{min}} \neq \mathcal{X}$ or $O_{R_{min}} \neq \{0\}$. Then by Lemma 3 there exists $R_{can} = (R_{min})_{can}$ representing Ψ such that

$$\dim R_{can} = \dim W_{R_{min}}/(O_{R_{min}} \cap W_{R_{min}}) < \dim R_{min}$$

which implies that R_{min} is not a minimal representation.

(ii) $\Longrightarrow$ (iii)

Let $R = (\mathcal{X}, \{A_z\}_{z\in X}, B, C)$ be a reachable representation of Ψ. Notice that $CA_wB_j = S_j(w) = C^{min}A_w^{min}B_j^{min}$. Define T by $T(A_wB_j) = A_w^{min}B_j^{min}$. We will show that T is well-defined. Assume that $A_uB_j = \sum_{k=1}^l \alpha_kA_{w_k}B_{j_k}$ holds for some $u, w_1, \ldots w_l \in X^*, j_1, \ldots, j_l \in J$, $\alpha_1, \ldots, \alpha_l \in \mathbb{R}$. Then for each $v \in X^*$ it holds that $CA_vA_uB_j = \sum_{k=1}^l \alpha_kCA_vA_{w_k}B_{j_l}$ which implies $C^{min}A_v^{min}A_u^{min}B_j^{min} = \sum_{k=1}^l \alpha_kC^{min}A_v^{min}A_{w_k}^{min}B_{j_l}^{min}$. Thus, $A_u^{min}B_j^{min} - \sum_{k=1}^l \alpha_kA_{w_k}^{min}B_{j_k}^{min} \in O_{R_{min}} = \{0\}$ which means that $A_u^{min}B_j^{min} = \sum_{k=1}^l \alpha_kA_{w_k}^{min}B_{j_k}^{min}$. That is $T(A_uB_j) = \sum_{k=1}^l \alpha_kT(A_{w_k}B_{j_k})$. Thus, T is indeed well-defined and linear. The mapping T is surjective, since the following holds.

$$\mathcal{X}_{min} = \operatorname{Span}\{A_w^{min}B_j^{min} \mid j \in J\} = \operatorname{Span}\{T(A_wB_j) \mid j \in J\} = T(\mathcal{X})$$

We will show that T defines a representation morphism. Equality $TA_\sigma = A_\sigma^{min}T$ holds since $T(A_\sigma A_w B_j) = A_\sigma^{min} A_w^{min} B_j^{min} = A_\sigma^{min} T(A_w B_j)$. Equality $B_j^{min} = TB_j$ holds by definition of T. Equality $C_{min}T = C$ holds because of the fact that $C_{min} A_w^{min} B_j^{min} = CA_w B_j = C_{min} T(A_w B_j)$.

(iii) $\Longrightarrow$ *(i)*

Indeed, if R is a representation of Ψ, then it follows from the proof of Lemma 3 that $R_r = (W_R, \{A_z|_{W_R}\}_{z\in X}, B, C|_{W_R})$ is a reachable representation of Φ and $\dim R_r \leq \dim R$. By part (iii) there exists a surjective map $T : R_r \to R_{min}$. But $\dim R \geq \dim R_r \geq \dim T(W_R) = \dim R_{min}$, so R_{min} is indeed a minimal representation of Ψ.

(iv) $\Longleftrightarrow$ *(i)*

The proof of Corollary 1 doesn't depend on the equivalence to be proved, so we can use it. By Corollary 1 R_Ψ is a minimal representation of Ψ. By construction $\dim R_\Psi = \dim W_\Psi = \text{rank } H_\Psi$. A representation is minimal whenever it has the same dimension as another minimal representation. Thus we get that R_{min} is minimal if and only if $\dim R_{min} = \dim R_\Psi = \text{rank } H_\Psi = \dim W_\Psi$. □

Proof of Corollary 1. `Part (a)`

Let $R_{min} = (\mathcal{X}_{min}, \{A_\sigma^{min}\}_{\sigma\in X}, B^{min}, C^{min})$ be a minimal representation of Ψ. Let $R = (\mathcal{X}, \{A_\sigma\}_{\sigma\in X}, B, C)$ be another minimal representation of Ψ. Then R is reachable and there exists a surjective representation morphism $T : R \to R_{min}$. Since $\dim R \leq \dim R_{min}$ and $\dim R_{min} \leq \dim R$, we get that $\dim R = \dim R_{min}$, which implies that $\dim \mathcal{X}_{min} = \dim \mathcal{X} = \dim T(\mathcal{X})$, which implies that T is a linear isomorphism, that is, T is a representation isomorphism.

`Part (b)`

The equality $W_\Psi = \text{Span}\{w \circ S_j \mid j \in J, w \in X^*\} = \text{Span}\{A_w B_j \mid j \in J, w \in X^*\}$ implies that $W_{R_\Psi} = W_\Psi$. If $T \in W_\Psi$ has the property that for all $w \in X^* : CA_w T = 0$ then it means that for all $w \in X^*$ it holds that $C(w \circ T) = w \circ T(\epsilon) = T(w) = 0$, i.e T=0. So we get that $O_{R_\Psi} = \{0\}$. By Theorem 2 we get that R_Ψ is a minimal representation of Ψ. □

Proof of Lemma 2. By Theorem 2 it is enough to show that $\dim W_{\Psi\odot\Theta} < +\infty$. First, notice that for any $T_1, T_2 \in K^p \ll X^* \gg$ it holds that $w\circ(T_1 \odot T_2) = (w\circ T_1)\odot(w\circ T_2)$. Indeed, $w \circ (T_1 \odot T_2)_l(v) = (T_1)_l \odot (T_2)_l(wv) = (T_1(wv))_l (T_2(wv))_l = (w \circ T_1)_l(v)(w \circ T_2)_l(v) = ((w \circ T_1) \odot (w \circ T_2))_l(v)$. Then we get that

$$\begin{aligned} W_{\Psi\odot\Theta} &= \text{Span}\{(w \circ S_j) \odot (w \circ T_j) \mid j \in J, w \in X^*\} \\ &\subseteq \text{Span}\{(w \circ S_j) \odot (v \circ T_z) \mid z, j \in J, w, v \in X^*\} \end{aligned}$$

Let $w_l \circ T_{z_l}, l = 1, 2, \ldots m, z_l \in J, w_l \in X^*$be a basis of W_Θ. Let $v_k \circ S_{j_k}, v_k \in X^*, k = 1, 2, \ldots n, j_k \in J$ be a basis of W_Ψ. Then it is easy to see that $\text{Span}\{(w\circ S_j)\odot(v\circ T_z) \mid z, j \in J, w, v \in X^*\}$ is spanned by $w_k\circ S_{j_k} \odot v_l\circ T_{z_l}, l = 1, 2, \ldots, m, k = 1, 2, \ldots n, j_k, z_l \in J$. That is, $\dim W_{\Psi\odot\Theta} \leq \dim W_\Psi \cdot \dim W_\Theta$. □

Appendix B. Proof of Theorem 3

Proof of Theorem 3. `only if part`

Assume that Φ has a generalized kernel representation. Then it is clear that for each $f \in \Phi$, f is causal, since for each $w = (q_1, t_1) \cdots (q_k, t_k) \in TL$ we get that $f_i(w, u) = e_i^T K_{q_1\cdots q_k}^{f,\Phi}(t_1, \ldots, t_k) + \sum_{i=1}^k \int_0^{t_i} e_i^T G_{q_i,\ldots,q_k}^{f,\Phi}(t_i - s, \ldots, t_k)u(s + \sum_{j=1}^{i-1} t_j)ds$ $i = 1, \ldots, p$, that is, $f_i(w, u)$ depends only on $u|_{[0,\sum_1^k t_i]}$. It is also clear that the function $y^\Phi = y_0^\Phi$ defined by $y_0^\Phi(u, w) = \sum_{i=1}^k \int_0^{t_i} G_{q_i,\ldots,q_k}^{f,\Phi}(t_i - s, \ldots, t_k)u(s + \sum_{j=1}^{i-1} t_j)ds$ satisfies (9). Moreover, it is easy to see that $y_j^\Phi(w, .), j = 1, \ldots, p$ is a continuous linear map from $PC([0, \sum_{j=1}^k t_j], \mathcal{U})$ to $\mathbb{R}^p$, since it is the sum of maps of the form $\phi_j : u \mapsto \int_0^{t_i} e_j^T G_{q_i\cdots q_k}^\Phi(t_i - s, \ldots, t_k)\text{Shift}_{\sum_{j=1}^{i-1} t_j}(u)(s)ds$ $j = 1, \ldots, p$ and Shift_T is a continuous linear map on $PC(T, \mathcal{U})$, and $g_j(s) = e_j^T G_{q_i\cdots q_k}^\Phi(s, t_{i+1}, \ldots, t_k)$ is analytic, and thus the function $\widetilde{g}_j(s) = g_j(t_i - s)\chi(\{s \in [0, t_i]\})$ is in $L^\infty(T)$. But then $\phi_j(u) = \int_0^{t_i} \widetilde{g}_j(s)\text{Shift}_{\sum_1^{i-1} t_i}(u)(s)ds$ and by [19] if follows that ϕ_j, $j = 1, \ldots, p$ is a a continuous linear map from $PC([0, \sum_1^k t_i], \mathcal{U})$ to $\mathbb{R}^p$ for Thus conditions 2 is satisfied. Let $z = (q_1, t_1) \cdots (q_h, t_h) \in (Q \times T)^+$, $w = (w_1, 0) \cdots (w_k, 0), v = (v_1, 0) \cdots (v_l, 0) \in (Q \times T)^*$. Let $x_1 = q_1 \cdots q_h$, $x_2 = w_1 \cdots w_k$ and $x_3 = v_1 \cdots v_l$. Assume that $wz, vz \in TL$. Then it is easy to see that $x_1 \in \text{suffix}L$. Then $f(0, wz) = K^{f,\Phi}_{x_2 x_1}(0, \ldots, 0, t_1, \ldots, t_h) = K^{f,\Phi}_{x_1}(t_1, \ldots, t_h) = K^{f,\Phi}_{x_3 x_1}(0, \ldots, 0, t_1, \ldots, t_h)$. Notice that

$$
\begin{aligned}
& y_0^{\Phi}(u, wz) = \sum_{i=1}^{k} \int_0^0 G^{\Phi}_{w_i \cdots w_k x_1}(\mathbb{O}_{l-i+1}, \tau) u(s) ds + \\
& \quad + \sum_{i=1}^{h} \int_0^{t_i} G^{\Phi}_{q_i \cdots q_h}(t_i - s, \ldots, t_h) u_i(s) ds = \sum_{i=1}^{h} \int_0^{t_i} G^{\Phi}_{q_i \cdots q_h}(t_i - s, \ldots, t_h) u_i(s) ds = \\
& \quad \sum_{i=1}^{l} \int_0^0 G^{\Phi}_{v_i \cdots v_l x_1}(\mathbb{O}_{l-i+1}, \tau) u(s) ds + \sum_{i=1}^{h} \int_0^{t_i} G^{\Phi}_{q_i \cdots q_h}(t_i - s, \ldots, t_h) u_i(s) ds = \\
& \quad = y_0^{\Phi}(u, vz)
\end{aligned}
$$

where $\tau = (t_1, \ldots, t_h)$, $\mathbb{O}_j = (0, 0, \ldots, 0) \in \mathbb{N}^j, j = 1, \ldots, l$, $u_i = \text{Shift}_{\sum_{j=1}^{i-1} t_i}(u)$. We get that $f(u, wz) = f(0, wz) + y_0^{\Phi}(u, wz) = f(0, vz) + y^{\Phi}(u, vz) = f(u, vz)$. That is, condition 3 is satisfied.

Let $w = (q_1, t_1) \cdots (q_k, t_k) \in TL$. It is also clear that if $z = (q_l, t_l) \cdots (q_k, t_k)$ and $1 \le l \le k$, then

$$
\begin{aligned}
& y_0^{\Phi}(u, w) = \sum_{i=l}^{k} \int_0^{t_i} G^{f,\Phi}_{q_i \cdots q_k}(t_i - s, \ldots, t_k) \text{Shift}_{T_{i-1,l}}(u_l)(s) ds + \\
& \quad + \sum_{i=1}^{l-1} \int_0^{t_i} G^{f,\Phi}_{q_i, \ldots, q_k}(t_i - s, \ldots, t_k) u_{i-1}(s) ds = y_0^{\Phi}(u_l, (q_1, 0) \cdots (q_{l-1}, 0) z) + \\
& \quad + \sum_{i=1}^{k} \int_0^{t_i} G^{f,\Phi}_{q_i, \ldots, q_k}(t_i - s, \ldots, t_k) \text{Shift}_{T_i}(v)(s) ds = y_0^{\Phi}(u_l, z) + y^{\Phi}(v, w)
\end{aligned}
$$

where $T_i = \sum_{j=1}^{i-1} t_j$, $u_i = \text{Shift}_{T_i}(u)$, $i = 1, \ldots, k$, $v = P_{T_l} u$, $T_{i,l} = \sum_{j=l}^{i} t_j$. That is, y^{Φ} satisfies condition 4. Let $w, v \in (Q \times T)^*$, and assume that $w(q, \tau_1)(q, \tau_2) v$, $w(q, \tau_1 + \tau_2) v \in TL$. Assume that $w = (w_1, t_1) \cdots (w_l, t_l)$ and $v = (v_{l+1}, t_{l+1}) \cdots (v_k, t_k)$ where $v_i, w_j \in Q, i = l+1, \ldots, k, j = 1, \ldots, l$. Let $T_i = \sum_{j=1}^{i} t_i$. Then using the properties of the functions $K^{f,\Phi}_z$, $G^{f,\Phi}_z, z \in \text{suffix}L$ one gets.

$$
\begin{aligned}
& f(u, w(q, \tau_1)(q, \tau_2) v) = K^{f,\Phi}_{wqqv}(t_1, \ldots, t_l, \tau_1, \tau_2, \ldots, t_k) + \\
& \quad \sum_{i=1}^{l} \int_0^{t_i} G^{\Phi}_{w_i \cdots w_l qqv}(t_i - s, \ldots, \tau_1, \tau_2, \ldots, t_k) u_i(s) ds + \\
& \quad + \int_0^{\tau_1} G^{\Phi}_{qqv}(\tau_1 - s, \tau_2, \ldots, t_k) u_{l+1}(s) ds + y_0^{\Phi}(\text{Shift}_{T_l + \tau_1 + \tau_2}(u), v) + \\
& \quad + \int_0^{\tau_2} G^{\Phi}_{qv}(\tau_2 - s, \ldots, t_k) u_{l+1}(s + \tau_1) ds = K^{f,\Phi}_{wqv}(t_1, \ldots, t_l, \tau_1 + \tau_2, \ldots, t_k) + \\
& \quad \sum_{i=1}^{l} \int_0^{t_i} G^{\Phi}_{w_i \cdots w_l qv}(t_i - s, \ldots, \tau_1 + \tau_2, \ldots, t_k) u_i(s) ds + \\
& \quad + \int_0^{\tau_1 + \tau_2} G^{\Phi}_{qv}(\tau_1 + \tau_2 - s, \ldots, t_k) u_{l+1}(s) ds + y_0^{\Phi}(\text{Shift}_{T_l + \tau_1 + \tau_2}(u), v) = \\
& \quad = f(u, w(q, \tau_1 + \tau_2) v)
\end{aligned}
$$

That is, Φ satisfies condition 5. If $|v| > 0$, $w(q,0)v, wv \in TL$ and $w = (q_1,t_1)\cdots(q_l,t_l)$, $v = (q_{l+1},t_{l+1})\cdots(q_k,t_k)$, then we get that

$$
\begin{aligned}
& f(u,w(q,0)v) = K^{f,\Phi}_{wv}(t_1,\dots,t_l,\dots,t_k) + \\
& \sum_{i=1}^{l}\int_0^{t_i} G^{\Phi}_{w_i\cdots w_l q v}(t_i - s,\dots,t_l,0,\dots,t_k)\mathrm{Shift}_i(u)(s)ds \\
& + \int_0^{0} G^{\Phi}_{qv}(0-s,\dots,t_k)\mathrm{Shift}_l(u)(s)ds + y^{\Phi}_0(\mathrm{Shift}_{T_l+0}(u),v) = \\
& = K^{f,\Phi}_{wv}(t_1,\dots,t_l,\dots,t_k) + \\
& \sum_{i=1}^{l}\int_0^{t_i} G^{\Phi}_{w_i\cdots w_l v}(t_i - s,\dots,t_k)\mathrm{Shift}_i(u)(s)ds + y^{\Phi}_0(\mathrm{Shift}_{T_l}(u),v) = \\
& f(u,wv)
\end{aligned}
$$

where $T_i = \sum_{j=1}^{i-1} t_j$ and $\mathrm{Shift}_i = \mathrm{Shift}_{T_i}$, $i = 1,\dots,k$. That is, Φ satisfies condition 5. Finally, it is easy to see that Φ satisfies condition 6. Indeed, $f_{q_1\cdots q_k,u_1\cdots u_k}(t_1,\dots,t_k) = K^{f,\Phi}_{q_1\cdots q_k}(t_1,\dots,t_k) + \sum_{i=1}^{k}(\int_0^{t_i} G^{\Phi}_{q_i\cdots q_k}(t_i - s,\dots,t_k)ds)u_i$. But by definition $K^{f,\Phi}_{q_1\cdots g_k}$ and $G^{\Phi}_{q_i\cdots q_k}$ are analytic, and thus $\int_0^{t_i} G^{\Phi}_{q_i\cdots q_k}(t_i - s,\dots,t_k)ds$ are analytic. That is, $f_{q_1\cdots q_k,u_1\cdots u_k}$ has to be analytic too.

`if part`

Assume that the set of maps Φ satisfies the conditions $1-6$. First notice that condition 3 implies that each $f \in \Phi$ can be uniquely extended to a function in $F(PC(T,\mathcal{U})\times T(\mathrm{suffix}L),\mathcal{Y})$. From now on we will assume that $\Phi \subseteq F(PC(T,\mathcal{U})\times T(\mathrm{suffix}L),\mathcal{Y})$. Also notice that all the conditions 1-6 still hold for the extensions of elements of Φ to $F(PC(T,\mathcal{U})\times T(\mathrm{suffix}L),\mathcal{Y})$. Let $w = (q_1,t_1)\cdots(q_k,t_k) \in T(\mathrm{suffix}L)$. We will construct function $K^{f,\Phi}_{q_l\cdots q_k}$ and $G^{f,\Phi}_{q_l\cdots q_k}$ for each $1 \le l \le k$. From condition 6 we get that for each $f \in \Phi$ it holds that $f_{q_1\cdots q_k,0\cdots 0} : T^k \to \mathcal{Y}$ is an analytic function. Let $K^{f,\Phi}_{q_l\cdots q_k}(t_l\cdots,t_k) = f_{q_1\cdots q_k,0\cdots 0}(0,0,\dots,0,t_l,t_{l+1},\dots,t_k)$. Then it is clear that $K^{f,\Phi}_{q_l\cdots q_k}$, $l = 1,\dots,k$ are analytic. Since f satisfies the condition 4 and 5 and $K^{f,\Phi}_{q_l\cdots q_k}(t_l,\dots,t_k = f((q_1,0)\cdots(q_{l-1},0)(q_l,t_l)\cdots(q_k,t_k),0)$ we get that $K^{f,\Phi}_{q_l\cdots q_k}$, $l = 1,\dots,k$ satisfies conditions 3 and 4 of Definition 3.

The definition of $G^{f,\Phi}_{q_l\cdots q_k}$ is a bit more involved. For each $l = 1,\dots,k$ $j = 1,\dots,p$ define the maps

$$y_{(q_l,t_l)\cdots(q_k,t_k),j} : PC([0,t_l],\mathcal{U}) \ni u \mapsto y^{\Phi}_j((q_1,t_1)\cdots(q_k,t_k),\widetilde{u})$$

where $\widetilde{u}(s) = \begin{cases} u(s - T_{l-1}) & \text{if } s \in [T_{l-1},T_l] \\ 0 & \text{otherwise} \end{cases}$ where $T_i = \sum_{j=1}^{i} t_j$. From condition 2 it follows that $y_{(q_l,t_l)\cdots(q_k,t_k),j}$ is a continuous linear functional on $PC([0,t_l],\mathcal{U})$. Since $PC([0,t_l],\mathcal{U})$ is dense in $L^1([0,t_l],\mathcal{U})$, we can extend it a unique way to a continuous linear functional on $L^1([0,t_l],\mathcal{U})$. By abuse of notation we will denote this functional by $y_{(q_l,t_l)\cdots(q_k,t_k),j}$ too. By Theorem 6.16 from [19] we get that there exists an a.s unique $g_{(q_l,t_l)\cdots(q_k,t_k),j} \in L^{\infty}([0,t_l],\mathbb{R}^{1\times m})$ such that

$$y_{(q_l,t_l)\cdots(q_k,t_k),j}(u) = \int_0^{t_l} g_{(q_l,t_l)\cdots(q_k,t_k),j}(s)u(s)ds$$

Let $y_w : u \mapsto \begin{bmatrix} y_{w,1}(u) & \cdots & y_{w,p}(u)\end{bmatrix}^T \in \mathbb{R}^p$ and define the map $g_w : s \mapsto \begin{bmatrix} (g_{w,1}(s))^T & \cdots & (q_{w,p}(s))^T\end{bmatrix}^T \in \mathbb{R}^{p\times m}$. Then

$$y_{(q_l,t_l)\cdots(q_k,t_k)}(u) = \int_0^{t_l} g_{(q_l,t_l)\cdots(q_k,t_k)}(s)u(s)ds$$

Note that if Φ satisfies conditions $1-6$, then y^{Φ} satisfies conditions 3 - 6. We will use this fact to prove certain properties of $g_{(q_1,t_1)\cdots(q_k,t_k)}$.

For any $w,v \in (Q\times T)^*$,$|v|>0$ one gets that if $v(q,\tau_1)(q,\tau_2)w$, $v(q,\tau_1+\tau_2)w \in T(\text{suffix}L)$, then it holds that $y_{v(q,\tau_1)(q,\tau_2)w}(u) = y^{\Phi}(\widetilde{u}, v(q,\tau_1)(q,\tau_2)w)$
$= y^{\Phi}(\widetilde{u}, v(q,\tau_1+\tau_2)w) = y_{v(q,\tau_1+\tau_2)w}(u)$. This implies that

$$g_{v(q,\tau_1)(q,\tau_2)w} = g_{v(q,\tau_1+\tau_2)w} \text{ a.s.} \tag{19}$$

Similarly, if $v(q,0)w, vw \in T(\text{suffix}L)$, $|w|>0$, $|v|>0$, then $y_{v(q,0)w}(u) = y^{\Phi}(\widetilde{u}, v(q,0)w) = y^{\Phi}(\widetilde{u}, vw) = y_{vw}(u)$ which implies

$$g_{v(q,0)w} = g_{vw} \text{ a.s} \tag{20}$$

Moreover, if $(q,t_1)(q,t_2)w \in T(\text{suffix}L)$ and $(q,t_1+t_2)w \in T(\text{suffix}L)$, then for each $u \in PC([0,t_2],\mathcal{U})$ it holds that

$$\begin{gathered} y_{(q,t_1)(q,t_2)w}(u) = y^{\Phi}(\widetilde{u},(q,t_1)(q,t_2)w) = y^{\Phi}(\widetilde{u},(q,t_1+t_2)w) = \\ y_{(q,t_1+t_2)w}(u\#_{t_1}0) = \int_0^{t_1} g_{(q,t_1+t_2)w}(s)u(s)ds \end{gathered}$$

By uniqueness of $g_{(q,t_1)(q,t_2)w}$ we get that

$$g_{(q,t_1)(q,t_2)w}(s) = g_{(q,t_1+t_2)w}(s) \text{ a.s. on } [0,t_1] \tag{21}$$

In addition, from condition 4 one gets for each $(q,t+s)w \in T(\text{suffix}L)$ that for each $u \in PC([0,s],\mathcal{U})$, $v \in PC([0,t+s],\mathcal{U})$, $v = 0\#_t u$,

$$\begin{gathered} y_{(q,t+s)w}(v) = y^{\Phi}(\widetilde{v},(q,t+s)w) = y^{\Phi}(\widetilde{v},(q,t)(q,s)w) = \\ y^{\Phi}(\text{Shift}_t\widetilde{v},(q,s)w) + y^{\Phi}(P_t\widetilde{v},(q,t)(q,s)w) \end{gathered}$$

But $P_t\widetilde{v} = 0$ so $y^{\Phi}(P_t\widetilde{v},(q,t)(q,s)w) = 0$, and in addition $\text{Shift}_t\widetilde{v} = \widetilde{u}$, therefore we get $y_{(q,t+s)w}(v) = y^{\Phi}(\text{Shift}_t(\widetilde{v}),(q,s)w) = y_{(q,s)w}(u)$. That is,

$$y_{(q,s)w}(u) = \int_0^{t+s} g_{(q,t+s)w}(z)v(z)dz = \int_0^{s} g_{(q,t+s)w}(z+t)u(z)dz$$

From uniqueness of $g_{(q,s)w}$ we get

$$g_{(q,s)w}(\tau) = g_{(q,s+t)}(\tau+t) \text{ a.s} \tag{22}$$

From the equalities above we also get that we are free to change each of the maps g_s, $s \in T(\text{suffix}L)$ on some set of measure zero, so in fact we can choose the maps g_s, $s \in T(\text{suffix}L)$ is such a way that the formulas (19),(20), (21) and (22) holds not only almost surely, but exactly on the whole domain. If these equalities hold exactly, then $g_{(q,t)w}(s) = g_{(q,t-s)}(0)$. Let $q_l\cdots q_k \in \text{suffix}L$. Define $G_{q_l\cdots q_k} : T^k \to \mathbb{R}^{p\times m}$ by

$$G_{q_l\cdots q_k}(t_l,\ldots,t_k) = g_{(q_l,t_l)\cdots(q_k,t_k)}(0)$$

Formula (22) implies that $G_{q_l\cdots q_k}(t_l-s,\cdots,t_k) = g_{(q_l,t_l-s)\cdots(q_k,t_k)}(0) =$
$g_{(q_l,t_l-s+s)\cdots(q_k,t_k)}(s) = g_{(q_l,t_l)\cdots(q_k,t_k)}(s)$. We immediately get that

$$y_{(q_l,t_l)\cdots(q_k,t_k)}(u) = \int_0^{t_l} G_{q_l\cdots q_k}(t_l-s,t_{l+1},\ldots,t_k)u(s)ds$$

Now, notice that for each $(q_1,t_1)\cdots(q_k,t_k) \in T(\text{suffix}L)$, by using condition 4 repeatedly, one can derive

$$y^{\Phi}(u,(q_1,t_1)\cdots(q_k,t_k)) = \sum_{i=1}^{k} y^{\Phi}(u_i,(q_i,t_i)\cdots(q_k,t_k))$$

where $u_i = P_{t_i}(\text{Shift}_{\sum_{j=1}^{i-1} t_j} u)$. That is, $u_i(s) = \begin{cases} u(s+\sum_{j=1}^{i-1} t_j) & \text{if } s \in [0,t_i] \\ 0 & \text{otherwise} \end{cases}$ That is, $u_i = \widetilde{v_i}$, $v_i = u_i|_{[0,t_i]} = (\text{Shift}_{\sum_{j=1}^{i-1} t_j} u)|_{[0,t_i]}$. Thus we get that for each $w = (q_1,t_1)\cdots(q_k,t_k) \in T(\text{suffix}L)$ and $u \in PC(T,\mathcal{U})$

$$y^{\Phi}(u,w) = \sum_{i=1}^{k} y_{(q_i,t_i)\cdots(q_k,t_k)}(v_i) = \sum_{i=1}^{k} \int_0^{t_i} G^{\Phi}_{q_i\cdots q_k}(t_i - s, \cdots t_k) u_i(s) ds$$

and

$$f(u,,w) = K^{f,\Phi}_{q_1\cdots q_k}(t_1,\ldots,t_k) + \sum_{i=1}^{k} \int_0^{t_i} G^{\Phi}_{q_i\cdots q_k}(t_i - s, \cdots t_k) u_i(s) ds \tag{23}$$

where $u_i = \text{Shift}_{\sum_{j=1}^{i-1} t_j}(u)$. We already showed that $K^{f,\Phi}_w$ $w \in \text{suffix}L$ satisfies the conditions 1, 2 and 3 of Definition 3. Equalities (19),(20), (21) and (22) imply that G^{Φ}_w satisfies the conditions 2 and 3 too. Equation (23) implies that part 4 of Definition 3 is satisfied too. It is left to show that G^{Φ}_w can be chosen to be analytic for each $f \in \Phi$ and $w \in \text{suffix}L$. Assume that $w = q_1\cdots q_k$. Then condition 6 implies that the function $h_{u_1\cdots u_k} = f_{q_1\cdots q_k, u_1\cdots u_k} - f_{q_1\cdots q_k, 0\cdots 0}$ is analytic for each $u_1,\cdots u_k \in PC(T,\mathcal{U})$ constant functions. But

$$h_{u_1\cdots u_k}(t_1,\ldots,t_k) = f(u,w) - f(0,w) = y^{\Phi}(u,w)$$

where $u(t) = u_i$ if $t \in (T_{i-1},T_i]$, $i = 1,\ldots,k$, $T_i = \sum_{j=1}^{i} t_j$. But then we get that

$$h_{u_1\cdots u_k}(t_1,\ldots,t_k) = \sum_{i=1}^{k} (\int_0^{t_i} G^{\Phi}_{q_i\cdots q_k}(t_i - s, t_{i+1},\ldots,t_k) ds) u_i$$

For each $i = 1\ldots,k$ taking $u_l = 0, j \neq l$ and $u_j = e_z = (0,0,\ldots,1,0,\ldots,0)^T$ we get that $h_{z,q_j\cdots q_k}(t_j,\ldots,t_k) := \int_0^{t_j} G^{\Phi}_{q_j\cdots q_k}(t_j - s, t_{j+1},\ldots,t_k) e_z ds$ is an analytic map. But $h_{z,q_j\cdots q_k}(0,t_{j+1},\ldots,t_k) = 0$, thus

$$h_{z,q_j\cdots q_k}(t_j,\ldots,t_k) = \int_0^{t_j} \frac{d}{ds} h_{z,q_j\cdots q_k}(t_j - s,\ldots,t_k) ds$$

Let $w(s) = G_{q_j\cdots q_k}(s,t_{j+1},\ldots,t_k)e_z - \frac{d}{ds} h_{z,q_j\cdots q_k}(s,t_{j+1},\ldots,t_k)$. That is, for each $t \in T$ we get that $\int_0^t w(t-s)ds = 0$, or equivalently $\int_0^t w(s)ds = 0$. It implies that $\int_E w(s)ds = 0$ for each Borel-set $E \subseteq [0,N]$, $N \in \mathbb{N}$. Then we get that w=0 a.s., that is, $G_{q_j\cdots q_k}(t,t_{j+1},\ldots t_k)e_z = \frac{d}{dt_j} h_{z,q_j\cdots q_k}(s,t_{j+1},\ldots,t_k)$ for almost all s. For each $w \in \text{suffix}L$ let $h_w = (h_{1,w},\ldots,h_{m,w})$. It is easy to see that h_w are analytic and $G^{\Phi}_w(t_1,\ldots,t_{|w|}) = h_w(t_1,\ldots,t_{|w|})$ a.s. in t_1. That is, the set

$$A_w(t_2,\ldots,t_{|w|}) = \{t \in T \mid G^{\Phi}_w(t,t_2,\ldots,t_{|w|}) \neq h_w(t,t_2,\ldots,t_{|w|})\}$$

is of measure zero. Thus, for any $a \in A_w(t_2,\ldots,t_{|w|})$ there exists $x_n \notin A_w(t_2,\ldots,t_{|w|})$, $\lim x_n = a$. Since h_w is continuous, it implies that h_w satisfies the conditions 2, 3, 4 of Definition 3, if G^{Φ}_w does. That is, we can take $G^{\Phi}_w := h_w$ and the resulting functions will satisfy the requirements for generalized kernel representation. We define the functions G^{Φ}_w and $K^{f,\Phi}_v$ only for $w \in \mathrm{suffix}L, v \in L$. But it is easy to see that $\{G^{\Phi}_w, K^{f,\Phi}_w \mid f \in \Phi, w \in \widetilde{L}\}$ is uniquely determined by $\{G^{\Phi}_w, K^{f,\Phi}_v \mid f \in \Phi, w \in \mathrm{suffix}L, v \in L\}$.

It is left to show that generalized kernel representations are unique. Assume that $\{K^{f,\Phi}_w, G^{\Phi}_w\}$ and $\{\widetilde{K}^{f,\Phi}_w, \widetilde{G}^{\Phi}_w\}$ are two different generalize kernel representations of Φ. By the remark above it is enough to show that $K^{f,\Phi}_w = \widetilde{K}^{f,\Phi}_w$ for each $w \in L, f \in \Phi$ and $G^{\Phi}_w = \widetilde{G}^{\Phi}_w$ $w \in \mathrm{suffix}L$. There are two ways to proceed. One can use formula 4 to conclude that $\forall w \in L, \alpha \in \mathbb{N}^{|w|} : D^{\alpha}K^{f,\Phi}_w = D^{\alpha}\widetilde{K}^{f,\Phi}_w = D^{\alpha}f(0,w,.)$, and $\forall w \in \mathrm{suffix}L, \alpha \in \mathbb{N}^{|w|}, j = 1,\dots,m, v \in Q^*, vw \in L : D^{\alpha}G^{\Phi}_w e_j = D^{\alpha}\widetilde{G}^{\Phi}_w e_j = D^{(\mathbb{O}_{|v|},\alpha^+)}y^{f,\Phi}_0(e_j, vw,.)$, where $\mathbb{O}_l = (0,0,\dots,0) \in \mathbb{N}^l, l \geq 0$, $\alpha^+ = (\alpha_1 + 1, \alpha_2, \dots, \alpha_k)$ for each $\alpha \in \mathbb{N}^k, k \geq 0$. That is, we get that the high-order derivatives at zero of $K^{f,\Phi}_w$ and $G^{f,\Phi}_w$ equal the respective high-order derivatives at zero of $\widetilde{K}^{f,\Phi}_w$ and $\widetilde{G}^{\Phi}_w$ respectively. Since $K^{f,\Phi}_w, G^{\Phi}_w, \widetilde{K}^{f,\Phi}_w, \widetilde{G}^{\Phi}_w$ are analytic, we get the required equalities.

Alternatively, we could use the proof of existence of a generalized kernel representation. Notice that $f(0,(q_1,t_1)\cdots(q_k,t_k)) = K^{f,\Phi}_{q_1\cdots q_k}(t_1,\dots,t_k) = \widetilde{K}^{f,\Phi}_{q_1\cdots q_k}(t_1,\dots,t_k)$ for all $(q_1,t_1)\dots(q_k,t_k) \in T(\mathrm{suffix}L)$ and $f \in \Phi$. On the other hand, from the proof above we can easily deduce that for each $w \in \mathrm{suffix}L$. $G^{\Phi}_w = \widetilde{G}^{\Phi}_w$ almost everywhere, that is, $r_w = G^{\Phi}_w - \widetilde{G}^{\Phi}_w = 0$ a.s. But r_w is analytic, and if $r_w \neq 0$, then there exists an open set V such that $\forall v \in V : r_w(v) \neq 0$. But no non-empty open set is of measure zero, so we get that r_w is the constant zero function. But then $G^{\Phi}_w = \widetilde{G}^{\Phi}_w$. $\square$

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