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Generalizations of Gronwall's integral inequality and their discrete analogies

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ABSTRACT
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2000 Mathematics Subject Classification: 26D10, 45K05
Keywords and Phrases: generalization; Gronwall's integral inequality; discrete analogy; integro-differential equations

GENERALIZATIONS OF GRONWALL'S INTEGRAL INEQUALITY AND THEIR DISCRETE ANALOGIES

MAKSAT ASHYRALIYEV

ABSTRACT. In this paper the generalizations of Gronwall's integral inequality and their discrete analogies are obtained. In applications these results are used to obtain the stability estimates of solutions for the initial value problems for integral-differential equations of the parabolic type.

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1. INTRODUCTION

Integral inequalities play a significant role in the theory of integro-differential equations. Mostly it is difficult to solve these equations. However, it is useful to investigate some properties of the solutions of equations, such as existence, uniqueness and stability. There are many classical methods to investigate existence and uniqueness of the solutions of integro-differential equations, see for instance [8, 9]. In order to establish stability of the solutions of these equations integral inequalities are widely used, see for instance [2, 3].

Gronwall in 1919 showed the following lemma [1].

Lemma 1.1. *If* $M = \text{const} > 0$, $\delta = \text{const} > 0$ *and continuous function* $x(t) \geq 0$ *satisfies the inequalities*

$$x(t) \leq M \int_0^t x(s)ds + \delta,$$

for $t \in [0, T]$, *then*

$$x(t) \leq \delta \exp\left[Mt\right], \quad 0 \leq t \leq T.$$

The primary contribution of Lemma 1.1 is that it can be applied in the study of stability of the solutions of various types of integral equations and boundary value problems for ordinary or partial differential equations. In this work we will derive some generalizations of Gronwall's integral inequality.

Most scientific and technical problems can be solved by using mathematical modelling and new numerical methods. This is based on the mathematical description of real processes and the subsequent solving of the appropriate mathematical problems on the computer. The mathematical models of many scientific and technical problems follow to already known or new problems of Partial Differential Equations. Indeed, in most of the cases it is difficult to find the exact solutions of the problems for the Partial Differential Equations. This is usually so due to the fact that required solution cannot be expressed by using only known elementary functions. For this reason discrete methods play a significant role, especially due to increasing the role of mathematical methods of the solving problems in various areas of science and engineering, and also with the appearance of

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highly-efficient computers. A well-known and widely applied method of approximate solutions for problems of differential equations is the method of difference schemes. Modern computers allow us to implement highly accurate difference schemes. Hence, the task is to construct and investigate highly accurate difference schemes for various types of boundary value of partial differential equations. The investigation of stability and convergence of these difference schemes is based on the discrete analogies of integral inequalities.

In numerical analysis literature, see for instance [5, 6], one can find the following discrete analogy of Lemma 1.1.

Lemma 1.2. *If x_j, $j=\overline{0,N}$ is a sequence of real numbers with*

$$|x_i| \le \delta + hM\sum_{j=0}^{i-1}|x_j|, \quad i=\overline{1,N},$$

where $M=\text{const}>0$ and $\delta=\text{const}>0$, then

$$|x_i| \le (hM|x_0|+\delta)\exp\left[Mih\right], \quad i=\overline{1,N}.$$

The primary contribution of Lemma 1.2 is that it can be used in order to demonstrate the convergence of the discrete solutions of difference schemes to the solution of corresponding differential equations in case of some discretization procedure. In this work we will derive the discrete analogies of generalizations of the Gronwall's integral inequality which can be used as a powerful tool in the analysis of finite difference equations.

2. Gronwall's type integral inequalities with two dependent limits and their discrete analogies

Theorem 2.1. *Assume that $v_i\ge 0$, $a_i\ge 0$, $\delta_i\ge 0$ $(i=\overline{-N,N+2M})$ are the sequences of real numbers and the inequalities*

$$v_i \le \delta_i + h\left(\sum_{j=-|i-M|+M+1}^{|i-M|+M-1} a_j v_j - a_M v_M\right), \quad i=\overline{-N,N+2M} \tag{2.1}$$

hold. Then for v_i the inequalities

$$v_{M-1}\le\delta_{M-1}, \quad v_{M+1}\le\delta_{M+1}, \quad v_M\le\delta_M+h(a_{M-1}\delta_{M-1}+a_{M+1}\delta_{M+1}), \tag{2.2}$$

$$v_i\le\delta_i+h\sum_{j=M+1}^{|i-M|+M-1}(a_j\delta_j+a_{2M-j}\delta_{2M-j})B_{ij}, \quad i=\overline{M+2,N+2M}, i=\overline{-N,M-2} \tag{2.3}$$

are satisfied, where

$$B_{ij}=\begin{cases}\exp\left[h\sum\limits_{n=j+1}^{|i-M|+M-1}(a_n+a_{2M-n})\right], & \text{if } j=\overline{M+1,|i-M|+M-2},\\ 1, & \text{if } j=|i-M|+M-1.\end{cases} \tag{2.4}$$

Proof. Bu putting $i=M-1,M+1,M$ directly in (2.1), we obtain the inequalities (2.2), correspondingly. Let us prove (2.3). We denote

$$y_i = h\left(\sum_{j=-|i-M|+M+1}^{|i-M|+M-1} a_j v_j - a_M v_M\right), \quad i=\overline{-N,N+2M}. \tag{2.5}$$

Then (2.1) gets the form

$$v_i \le \delta_i + y_i, \quad i = \overline{-N, N+2M}. \tag{2.6}$$

For $i > M$ from (2.5) we have

$$y_{i+1} - y_i = h\left(\sum_{j=2M-i}^{i} a_j v_j - a_M v_M\right) - h\left(\sum_{j=2M-i+1}^{i-1} a_j v_j - a_M v_M\right) = h\left(a_i v_i + a_{2M-i} v_{2M-i}\right) \ge 0,$$

and

$$y_{2M-i} = h\left(\sum_{j=-|M-i|+M+1}^{|M-i|+M-1} a_j v_j - a_M v_M\right) = h\left(\sum_{j=2M-i+1}^{i-1} a_j v_j - a_M v_M\right) = y_i.$$

So, $y_i \le y_{i+1}$ and $y_{2M-i} = y_i$ for $i > M$. Moreover, by using (2.6), we obtain

$$y_{i+1} - y_i = h\left(a_i v_i + a_{2M-i} v_{2M-i}\right) \le h a_i (y_i + \delta_i) + h a_{2M-i}(y_{2M-i} + \delta_{2M-i}) \le h(a_i + a_{2M-i}) y_i + h(a_i \delta_i + a_{2M-i}\delta_{2M-i}),$$

or

$$y_{i+1} \le [1 + h(a_i + a_{2M-i})] y_i + h(a_i \delta_i + a_{2M-i}\delta_{2M-i}), \quad i > M.$$

Then by induction we can prove that

$$y_i \le \prod_{n=1}^{i-M-1} (1 + h(a_{M+n} + a_{M-n})) y_{M+1} + \sum_{j=1}^{i-M-2} h(a_{M+j}\delta_{M+j} + a_{M-j}\delta_{M-j}) \prod_{n=j+1}^{i-M-1} (1 + h(a_{M+n} + a_{M-n})) + h(a_{i-1}\delta_{i-1} + a_{2M-i+1}\delta_{2M-i+1}), \quad i = \overline{M+2, N+2M}.$$

Since $y_{M+1} = 0$, we have

$$y_i \le h \sum_{j=1}^{i-M-2} (a_{M+j}\delta_{M+j} + a_{M-j}\delta_{M-j}) \prod_{n=j+1}^{i-M-1} (1 + h(a_{M+n} + a_{M-n})) + h(a_{i-1}\delta_{i-1} + a_{2M-i+1}\delta_{2M-i+1}) = h \sum_{j=M+1}^{i-2} (a_j \delta_j + a_{2M-j}\delta_{2M-j}) \prod_{n=j+1}^{i-1} (1 + h(a_n + a_{2M-n})) + h(a_{i-1}\delta_{i-1} + a_{2M-i+1}\delta_{2M-i+1}) \le h \sum_{j=M+1}^{i-2} (a_j \delta_j + a_{2M-j}\delta_{2M-j}) \exp\left[h \sum_{n=j+1}^{i-1} (a_n + a_{2M-n})\right] + h(a_{i-1}\delta_{i-1} + a_{2M-i+1}\delta_{2M-i+1}), \quad i = \overline{M+2, N+2M}.$$

Now, by using (2.6) and the notation (2.4), we obtain (2.3) for $i = \overline{M+2, N+2M}$.

Let us prove (2.3) for $i = \overline{-N, M-2}$. From (2.1) we have

$$v_i \le \delta_i + h\left(\sum_{j=i+1}^{2M-i-1} a_j v_j - a_M v_M\right), \quad i = \overline{-N, M-2}.$$

Since $\sum\limits_{j=i+1}^{2M-i-1} a_j v_j = \sum\limits_{j=i+1}^{2M-i-1} a_{2M-j} v_{2M-j}$, we have

$$v_i \le \delta_i + h\left(\sum_{j=i+1}^{2M-i-1} a_{2M-j}v_{2M-j} - a_M v_M\right), \quad i = \overline{-N, M-2}.$$

We denote $i = 2M - i_1$. So, $i_1 = 2M - i > M$ and

$$v_{2M-i_1} \le \delta_{2M-i_1} + h\left(\sum_{j=2M-i_1+1}^{i_1-1} a_{2M-j}v_{2M-j} - a_M v_M\right), \quad i_1 = \overline{M+2, N+2M}.$$

By denoting $v_{2M-k} = w_k, a_{2M-k} = b_k, \delta_{2M-k} = \lambda_k$, we rewrite it as

$$w_{i_1} \le \lambda_{i_1} + h\left(\sum_{j=2M-i_1+1}^{i_1-1} b_j w_j - b_M w_M\right), \quad i_1 = \overline{M+2, N+2M}.$$

By using the proved part of the theorem, namely the inequalities (2.3) for $i = \overline{M+2, N+2M}$, we obtain

$$w_{i_1} \le \lambda_{i_1} + h\sum_{j=M+1}^{i_1-2}(b_j\lambda_j + b_{2M-j}\lambda_{2M-j})\exp\left[h\sum_{n=j+1}^{i_1-1}(b_n + b_{2M-n})\right] + h(b_{i_1-1}\lambda_{i_1-1} + b_{2M-i_1+1}\lambda_{2M-i_1+1}), \quad i_1 = \overline{M+2, N+2M},$$

which can be rewritten as

$$v_{2M-i_1} \le \delta_{2M-i_1} + h\sum_{j=M+1}^{i_1-2}(a_{2M-j}\delta_{2M-j} + a_j\delta_j)\exp\left[h\sum_{n=j+1}^{i_1-1}(a_{2M-n} + a_n)\right] + h(a_{2M-i_1+1}\delta_{2M-i_1+1} + a_{i_1-1}\delta_{i_1-1}), \quad i_1 = \overline{M+2, N+2M},$$

or

$$v_i \le \delta_i + h\sum_{j=M+1}^{2M-i-2}(a_{2M-j}\delta_{2M-j} + a_j\delta_j)\exp\left[h\sum_{n=j+1}^{2M-i-1}(a_{2M-n} + a_n)\right] + h(a_{i+1}\delta_{i+1} + a_{2M-i-1}\delta_{2M-i-1}), \quad i = \overline{-N, M-2}.$$

The Theorem 2.1 is proved. □

By putting $Nh = 1$, $2Mh = T$ and passing to limit $h \to 0$ in the Theorem 2.1, we obtain the following generalization of Gronwall's integral inequality.

Theorem 2.2. *Assume that* $v(t) \ge 0$, $\delta(t) \ge 0$ *are the continuous functions on* $[-1, 1+T]$ *and* $a(t) \ge 0$ *is an integrable function on* $[-1, 1+T]$ *and the inequalities*

$$v(t) \le \delta(t) + sgn\left(t - \frac{T}{2}\right)\int_{T-t}^{t} a(s)v(s)ds, \quad -1 \le t \le 1+T$$

hold. Then for $v(t)$ *the inequalities*

$$v(t) \le \delta(t) + \int_{\frac{T}{2}}^{t}(a(s)\delta(s) + a(T-s)\delta(T-s))\exp\left[\int_s^t (a(\tau) + a(T-\tau))d\tau\right]ds, \quad \frac{T}{2} \le t \le 1+T,$$

$$v(t) \leq \delta(t) + \int_{\frac{T}{2}}^{T-t} (a(s)\delta(s) + a(T-s)\delta(T-s)) \exp\left[\int_{s}^{T-t} (a(\tau) + a(T-\tau))d\tau\right] ds,$$
$$-1 \leq t < \frac{T}{2}$$

are satisfied.

By putting $\delta(t) \equiv const$ and $a(t) \equiv const$ in the Theorem 2.2, we get the following theorem, which was early obtained in [2].

Theorem 2.3. *Assume that $v(t) \geq 0$ is a continuous function on $[-1, 1+T]$ and the inequalities*

$$v(t) \leq C + L \int_{T-t}^{t} sgn\,(t - T/2)\, v(s)ds, \quad -1 \leq t \leq 1+T$$

hold, where $C = const \geq 0, L = const \geq 0$. Then for $v(t)$ the inequalities

$$v(t) \leq C \exp\left(2L|t - T/2|\right), \quad -1 \leq t \leq 1+T$$

are satisfied.

By putting $T = 0$ in the Theorem 2.2, we obtain the following generalization of Gronwall's integral inequality with two dependent limits.

Theorem 2.4. *Assume that $v(t) \geq 0$, $\delta(t) \geq 0$ are the continuous functions on $[-1, 1]$ and $a(t) \geq 0$ is an integrable function on $[-1, 1]$ and the inequalities*

$$v(t) \leq \delta(t) + sgn(t) \int_{-t}^{t} a(s)v(s)ds, \quad -1 \leq t \leq 1$$

hold. Then for $v(t)$ the inequalities

$$v(t) \leq \delta(t) + sgn(t) \int_{0}^{t} (a(s)\delta(s) + a(-s)\delta(-s)) \exp\left[sgn(t) \int_{s}^{t} (a(\tau) + a(-\tau))d\tau\right] ds,$$
$$-1 \leq t \leq 1$$

are satisfied.

By putting $\delta(t) \equiv const$ and $a(t) \equiv const$ in the Theorem 2.4, we get the following theorem, which was early obtained in [3].

Theorem 2.5. *Assume that $v(t) \geq 0$ is a continuous function on $[-1, 1]$ and the inequalities*

$$v(t) \leq C + L \int_{-t}^{t} sgn(t)v(s)ds, \quad -1 \leq t \leq 1$$

hold, where $C = const \geq 0, L = const \geq 0$. Then for $v(t)$ the inequalities

$$v(t) \leq C \exp\left[2L|t|\right], \quad -1 \leq t \leq 1$$

are satisfied.

By putting $M = 0$ and $a_i = 0$, $i = \overline{-N, 0}$ in the Theorem 2.1, we obtain the following generalization of Lemma 1.2.

Theorem 2.6. *Assume that* $v_i \geq 0$, $a_i \geq 0$ $(i = \overline{0,N})$, $\delta_i \geq 0$ $(i = \overline{1,N})$ *are the sequences of real numbers and the inequalities*

$$v_i \leq \delta_i + h\left(\sum_{j=0}^{i-1} a_j v_j - a_0 v_0\right), \quad i = \overline{1,N}$$

hold. Then for v_i *the inequalities*

$$v_1 \leq \delta_1, \quad v_2 \leq \delta_2 + h a_1 \delta_1, \quad v_i \leq \delta_i + h\left(\sum_{j=1}^{i-2} a_j \delta_j \exp\left[h \sum_{n=j+1}^{i-1} a_n\right] + a_{i-1}\delta_{i-1}\right), \quad i = \overline{3,N}$$

are satisfied.

By putting $Nh = 1$ and passing to limit $h \to 0$ in the Theorem 2.6, we get the following generalization of Gronwall's integral inequality with one dependent limit, which was early obtained in [4].

Theorem 2.7. *Assume that* $v(t) \geq 0$, $\delta(t) \geq 0$ *are the continuous functions on* $[0,1]$ *and* $a(t) \geq 0$ *is an integrable function on* $[0,1]$ *and the inequalities*

$$v(t) \leq \delta(t) + \int_0^t a(s)v(s)ds, \quad 0 \leq t \leq 1$$

hold. Then for $v(t)$ *the inequalities*

$$v(t) \leq \delta(t) + \int_0^t a(s)\delta(s)\exp\left[\int_s^t a(\tau)d\tau\right]ds, \quad 0 \leq t \leq 1$$

are satisfied.

Now, we consider the generalizations of Gronwall's type integral inequalities with the singular kernel.

Theorem 2.8. *Assume that* $v(t) \geq 0$, $a(t) \geq 0$ *are the integrable functions on* $[-1,1]$ *and the inequalities*

$$v(t) \leq a(t) + Lsgn(t)\int_{-t}^t (|t| - |s|)^{\beta-1} v(s)ds, \quad -1 \leq t \leq 1 \tag{2.7}$$

hold, where $L \geq 0$, $\beta \geq 0$. *Then for* $v(t)$ *the inequalities*

$$v(t) \leq a(t) + \sum_{n=1}^{\infty} 2^{n-1}\frac{(LG(\beta))^n}{G(n\beta)} sgn(t)\int_{-t}^{t} (|t| - |s|)^{n\beta-1} a(s)ds, \quad -1 \leq t \leq 1 \tag{2.8}$$

are satisfied, where $G(\beta)$ *is the Gamma function of* β.

Proof. We denote

$$Bv(t) = Lsgn(t)\int_{-t}^t (|t| - |s|)^{\beta-1} v(s)ds, \quad -1 \leq t \leq 1. \tag{2.9}$$

Then by using (2.7), we have

$$v(t) \leq \sum_{k=0}^{n-1} B^k a(t) + B^n v(t), \quad -1 \leq t \leq 1. \tag{2.10}$$

Let us prove that

$$B^n v(t) = 2^{n-1}\frac{(LG(\beta))^n}{G(n\beta)} sgn(t)\int_{-t}^{t}(|t|-|s|)^{n\beta-1}v(s)ds, \quad -1\le t\le 1 \tag{2.11}$$

holds for any $n\in N$. (2.11) for $n=1$ follows from (2.9) directly. Assume that (2.11) holds for some $n\in N$.

Then for $0\le t\le 1$ we have

$$B^{n+1}v(t) = L\int_{-t}^{t}(t-|s|)^{\beta-1}B^n v(s)ds = L2^{n-1}\frac{(LG(\beta))^n}{G(n\beta)}$$

$$\times\int_{-t}^{t}(t-|s|)^{\beta-1}sgn(s)\int_{-s}^{s}(|s|-|\tau|)^{n\beta-1}v(\tau)d\tau ds = L2^{n-1}\frac{(LG(\beta))^n}{G(n\beta)}$$

$$\times\left[\int_0^t(t-s)^{\beta-1}\int_{-s}^{s}(s-|\tau|)^{n\beta-1}v(\tau)d\tau ds+\int_{-t}^{0}(t+s)^{\beta-1}\int_{s}^{-s}(-s-|\tau|)^{n\beta-1}v(\tau)d\tau ds\right]$$

$$= L2^{n-1}\frac{(LG(\beta))^n}{G(n\beta)}\left[\int_0^t v(\tau)\int_{\tau}^{t}(t-s)^{\beta-1}(s-\tau)^{n\beta-1}dsd\tau\right.$$

$$+\int_{-t}^{0}v(\tau)\int_{-\tau}^{t}(t-s)^{\beta-1}(s+\tau)^{n\beta-1}dsd\tau+\int_0^t v(\tau)\int_{-t}^{-\tau}(t+s)^{\beta-1}(-s-\tau)^{n\beta-1}dsd\tau$$

$$\left.+\int_{-t}^{0}v(\tau)\int_{-t}^{\tau}(t+s)^{\beta-1}(-s+\tau)^{n\beta-1}dsd\tau\right] = L2^n\frac{(LG(\beta))^n}{G(n\beta)}$$

$$\times\left[\int_0^t v(\tau)\int_0^{t-\tau}z^{\beta-1}(t-\tau-z)^{n\beta-1}dzd\tau+\int_{-t}^{0}v(\tau)\int_0^{t+\tau}z^{\beta-1}(t+\tau-z)^{n\beta-1}dzd\tau\right]$$

$$= L2^n\frac{(LG(\beta))^n}{G(n\beta)}\left[\int_0^t v(\tau)(t-\tau)^{(n+1)\beta-1}d\tau+\int_{-t}^{0}v(\tau)(t+\tau)^{(n+1)\beta-1}d\tau\right]$$

$$\times\int_0^1\rho^{\beta-1}(1-\rho)^{n\beta-1}d\rho = L2^n\frac{(LG(\beta))^n}{G(n\beta)}\int_{-t}^{t}v(\tau)(t-|\tau|)^{(n+1)\beta-1}d\tau\frac{G(\beta)G(n\beta)}{G(n\beta+\beta)}.$$

So,

$$B^{n+1}v(t) = 2^n\frac{(LG(\beta))^{n+1}}{G((n+1)\beta)}\int_{-t}^{t}(t-|s|)^{(n+1)\beta-1}v(s)ds, \quad 0\le t\le 1. \tag{2.12}$$

In the similar way for $-1\le t<0$ we have

$$B^{n+1}v(t) = L\int_t^{-t}(-t-|s|)^{\beta-1}B^n v(s)ds = L2^{n-1}\frac{(LG(\beta))^n}{G(n\beta)}$$

$$\times\int_{t}^{-t}(-t-|s|)^{\beta-1}sgn(s)\int_{-s}^{s}(|s|-|\tau|)^{n\beta-1}v(\tau)d\tau ds=L2^{n-1}\frac{(LG(\beta))^{n}}{G(n\beta)}$$

$$\times\left[\int_{0}^{-t}(-t-s)^{\beta-1}\int_{-s}^{s}(s-|\tau|)^{n\beta-1}v(\tau)d\tau ds+\int_{t}^{0}(-t+s)^{\beta-1}\int_{s}^{-s}(-s-|\tau|)^{n\beta-1}v(\tau)d\tau ds\right]$$

$$=L2^{n-1}\frac{(LG(\beta))^{n}}{G(n\beta)}\left[\int_{0}^{-t}v(\tau)\int_{\tau}^{-t}(-t-s)^{\beta-1}(s-\tau)^{n\beta-1}dsd\tau\right.$$

$$+\int_{t}^{0}v(\tau)\int_{-\tau}^{-t}(-t-s)^{\beta-1}(s+\tau)^{n\beta-1}dsd\tau+\int_{0}^{-t}v(\tau)\int_{t}^{-\tau}(-t+s)^{\beta-1}(-s-\tau)^{n\beta-1}dsd\tau$$

$$\left.+\int_{t}^{0}v(\tau)\int_{t}^{\tau}(-t+s)^{\beta-1}(-s+\tau)^{n\beta-1}dsd\tau\right]=L2^{n}\frac{(LG(\beta))^{n}}{G(n\beta)}$$

$$\times\left[\int_{0}^{-t}v(\tau)\int_{0}^{-t-\tau}z^{\beta-1}(-t-\tau-z)^{n\beta-1}dzd\tau+\int_{t}^{0}v(\tau)\int_{0}^{-t+\tau}z^{\beta-1}(-t+\tau-z)^{n\beta-1}dzd\tau\right]$$

$$=L2^{n}\frac{(LG(\beta))^{n}}{G(n\beta)}\left[\int_{0}^{-t}v(\tau)(-t-\tau)^{(n+1)\beta-1}d\tau+\int_{t}^{0}v(\tau)(-t+\tau)^{(n+1)\beta-1}d\tau\right]$$

$$\times\int_{0}^{1}\rho^{\beta-1}(1-\rho)^{n\beta-1}d\rho=L2^{n}\frac{(LG(\beta))^{n}}{G(n\beta)}\int_{t}^{-t}v(\tau)(-t-|\tau|)^{(n+1)\beta-1}d\tau\frac{G(\beta)G(n\beta)}{G(n\beta+\beta)}.$$

So,

$$B^{n+1}v(t)=2^{n}\frac{(LG(\beta))^{n+1}}{G((n+1)\beta)}\int_{t}^{-t}(-t-|s|)^{(n+1)\beta-1}v(s)ds,\quad -1\leq t<0. \tag{2.13}$$

Combining (2.12) and (2.13), we prove by induction that (2.11) holds for any $n\in N$. Since $B^{n}v(t)\geq 0$ for any $n\in N$ and

$$\frac{B^{n+1}v(t)}{B^{n}v(t)}=\frac{2LG(\beta)G(n\beta)}{G((n+1)\beta)}\frac{\int_{-|t|}^{|t|}(|t|-|s|)^{(n+1)\beta-1}v(s)ds}{\int_{-|t|}^{|t|}(|t|-|s|)^{n\beta-1}v(s)ds}\leq\frac{2LG(\beta)G(n\beta)}{G((n+1)\beta)}\to 0,\quad n\to\infty,$$

we have $\lim_{n\to\infty}B^{n}v(t)=0$. Then, by letting $n\to\infty$ in (2.10) and using (2.11), we obtain the inequalities (2.8). □

Note that by putting $\beta=1$ and $a(t)\equiv$ *const* in the Theorem 2.8, we obtain the Theorem 2.5.

Theorem 2.9. *Assume that* $L \geq 0$, $\beta > 0$, $1 \geq j > 0$, $\beta + j > 1$, $a \geq 0$ *and* $v(t) \geq 0$ *is an integrable function on* $[-1, 1]$ *and the inequalities*

$$v(t) \leq a + Lsgn(t) \int_{-t}^{t} (|t| - |s|)^{\beta-1} |s|^{j-1} v(s) ds, \quad -1 \leq t \leq 1 \tag{2.14}$$

hold. Then for $v(t)$ *the following inequalities hold*

$$v(t) \leq a \sum_{n=0}^{\infty} \frac{G(j)}{G(j + n\beta)} \left(2LG(\beta)|t|^{\beta+j-1}\right)^n, \quad -1 \leq t \leq 1. \tag{2.15}$$

Proof. We denote

$$Bv(t) = Lsgn(t) \int_{-t}^{t} (|t| - |s|)^{\beta-1} |s|^{j-1} v(s) ds, \quad -1 \leq t \leq 1. \tag{2.16}$$

Then by using (2.14), we have

$$v(t) \leq \sum_{k=0}^{n-1} B^k a + B^n v(t), \quad -1 \leq t \leq 1. \tag{2.17}$$

Let us prove that

$$B^n v(t) \leq \left(2|t|^{j-1}\right)^{n-1} \frac{(LG(\beta))^n}{G(n\beta)} sgn(t) \int_{-t}^{t} (|t| - |s|)^{n\beta-1} |s|^{j-1} v(s) ds, \quad -1 \leq t \leq 1 \tag{2.18}$$

holds for any $n \in N$. (2.18) for $n = 1$ follows from (2.16) directly. Assume that (2.18) holds for some $n \in N$.

Then for $0 \leq t \leq 1$ we have

$$B^{n+1} v(t) = L \int_{-t}^{t} (t - |s|)^{\beta-1} |s|^{j-1} B^n v(s) ds \leq \frac{2^{n-1} L^{n+1} (G(\beta))^n}{G(n\beta)}$$

$$\times \int_{-t}^{t} (t - |s|)^{\beta-1} |s|^{n(j-1)} sgn(s) \int_{-s}^{s} (|s| - |\tau|)^{n\beta-1} |\tau|^{j-1} v(\tau) d\tau ds$$

$$= \frac{2^{n-1} L^{n+1} (G(\beta))^n}{G(n\beta)} \left[\int_{0}^{t} (t - s)^{\beta-1} s^{n(j-1)} \int_{-s}^{s} (s - |\tau|)^{n\beta-1} |\tau|^{j-1} v(\tau) d\tau ds \right.$$

$$\left. + \int_{-t}^{0} (t + s)^{\beta-1} (-s)^{n(j-1)} \int_{s}^{-s} (-s - |\tau|)^{n\beta-1} |\tau|^{j-1} v(\tau) d\tau ds \right] = \frac{2^{n-1} L^{n+1} (G(\beta))^n}{G(n\beta)}$$

$$\times \left[\int_{0}^{t} \tau^{j-1} v(\tau) \int_{\tau}^{t} (t - s)^{\beta-1} (s - \tau)^{n\beta-1} s^{n(j-1)} ds d\tau \right.$$

$$+ \int_{-t}^{0} (-\tau)^{j-1} v(\tau) \int_{-\tau}^{t} (t - s)^{\beta-1} (s + \tau)^{n\beta-1} s^{n(j-1)} ds d\tau$$

$$+ \int_{0}^{t} \tau^{j-1} v(\tau) \int_{-t}^{-\tau} (t + s)^{\beta-1} (-s - \tau)^{n\beta-1} (-s)^{n(j-1)} ds d\tau$$

$$+\int_{-t}^{0}(-\tau)^{j-1}v(\tau)\int_{-t}^{\tau}(t+s)^{\beta-1}(-s+\tau)^{n\beta-1}(-s)^{n(j-1)}dsd\tau\Bigg]$$

$$\leq \frac{2^nL^{n+1}(G(\beta))^n}{G(n\beta)}t^{n(j-1)}\left[\int_0^t \tau^{j-1}v(\tau)\int_0^{t-\tau} z^{\beta-1}(t-\tau-z)^{n\beta-1}dzd\tau\right.$$

$$\left.+\int_{-t}^{0}(-\tau)^{j-1}v(\tau)\int_0^{t+\tau} z^{\beta-1}(t+\tau-z)^{n\beta-1}dzd\tau\right]=\frac{2^nL^{n+1}(G(\beta))^n}{G(n\beta)}t^{n(j-1)}$$

$$\times\left[\int_0^t \tau^{j-1}(t-\tau)^{(n+1)\beta-1}v(\tau)d\tau+\int_{-t}^{0}(-\tau)^{j-1}(t+\tau)^{(n+1)\beta-1}v(\tau)d\tau\right]\int_0^1\rho^{\beta-1}(1-\rho)^{n\beta-1}d\rho$$

$$=\frac{2^nL^{n+1}(G(\beta))^n}{G(n\beta)}t^{n(j-1)}\int_{-t}^{t}|\tau|^{j-1}(t-|\tau|)^{(n+1)\beta-1}v(\tau)d\tau\frac{G(\beta)G(n\beta)}{G(n\beta+\beta)}.$$

So,

$$B^{n+1}v(t)\leq\left(2t^{j-1}\right)^n\frac{(LG(\beta))^{n+1}}{G((n+1)\beta)}\int_{-t}^{t}(t-|s|)^{(n+1)\beta-1}|s|^{j-1}v(s)ds,\quad 0\leq t\leq 1. \tag{2.19}$$

In the similar way for $-1\leq t<0$ we have

$$B^{n+1}v(t)=L\int_t^{-t}(-t-|s|)^{\beta-1}|s|^{j-1}B^nv(s)ds\leq\frac{2^{n-1}L^{n+1}(G(\beta))^n}{G(n\beta)}$$

$$\times\int_t^{-t}(-t-|s|)^{\beta-1}|s|^{n(j-1)}sgn(s)\int_{-s}^{s}(|s|-|\tau|)^{n\beta-1}|\tau|^{j-1}v(\tau)d\tau ds$$

$$=\frac{2^{n-1}L^{n+1}(G(\beta))^n}{G(n\beta)}\left[\int_0^{-t}(-t-s)^{\beta-1}s^{n(j-1)}\int_{-s}^{s}(s-|\tau|)^{n\beta-1}|\tau|^{j-1}v(\tau)d\tau ds\right.$$

$$\left.+\int_t^0(-t+s)^{\beta-1}(-s)^{n(j-1)}\int_s^{-s}(-s-|\tau|)^{n\beta-1}|\tau|^{j-1}v(\tau)d\tau ds\right]=\frac{2^{n-1}L^{n+1}(G(\beta))^n}{G(n\beta)}$$

$$\times\left[\int_0^{-t}\tau^{j-1}v(\tau)\int_\tau^{-t}(-t-s)^{\beta-1}(s-\tau)^{n\beta-1}s^{n(j-1)}dsd\tau\right.$$

$$+\int_t^0(-\tau)^{j-1}v(\tau)\int_{-\tau}^{-t}(-t-s)^{\beta-1}(s+\tau)^{n\beta-1}s^{n(j-1)}dsd\tau$$

$$+\int_0^{-t}\tau^{j-1}v(\tau)\int_t^{-\tau}(-t+s)^{\beta-1}(-s-\tau)^{n\beta-1}(-s)^{n(j-1)}dsd\tau$$

$$
\begin{aligned}
&+\int_t^0(-\tau)^{j-1}v(\tau)\int_t^{\tau}(-t+s)^{\beta-1}(-s+\tau)^{n\beta-1}(-s)^{n(j-1)}dsd\tau\Bigg] \\
&\le \frac{2^nL^{n+1}(G(\beta))^n}{G(n\beta)}(-t)^{n(j-1)}\left[\int_0^{-t}\tau^{j-1}v(\tau)\int_0^{-t-\tau}z^{\beta-1}(-t-\tau-z)^{n\beta-1}dzd\tau\right. \\
&\qquad\left.+\int_t^0(-\tau)^{j-1}v(\tau)\int_0^{-t+\tau}z^{\beta-1}(-t+\tau-z)^{n\beta-1}dzd\tau\right] \\
&= \frac{2^nL^{n+1}(G(\beta))^n}{G(n\beta)}(-t)^{n(j-1)}\left[\int_0^{-t}\tau^{j-1}(-t-\tau)^{(n+1)\beta-1}v(\tau)d\tau\right. \\
&\qquad\left.+\int_t^0(-\tau)^{j-1}(-t+\tau)^{(n+1)\beta-1}v(\tau)d\tau\right]\int_0^1\rho^{\beta-1}(1-\rho)^{n\beta-1}d\rho \\
&= \frac{2^nL^{n+1}(G(\beta))^n}{G(n\beta)}(-t)^{n(j-1)}\int_t^{-t}|\tau|^{j-1}(-t-|\tau|)^{(n+1)\beta-1}v(\tau)d\tau\frac{G(\beta)G(n\beta)}{G(n\beta+\beta)}.
\end{aligned}
$$

So,

(2.20)

$$
B^{n+1}v(t)\le\left(2(-t)^{j-1}\right)^n\frac{(LG(\beta))^{n+1}}{G((n+1)\beta)}\int_t^{-t}(-t-|s|)^{(n+1)\beta-1}|s|^{j-1}v(s)ds,\quad -1\le t<0.
$$

Combining (2.19) and (2.20), by induction we prove that (2.18) holds for any $n\in N$. Right-hand side of (2.18), which we denote by $Q^nv(t)$, goes to 0 when $n\to\infty$. Indeed, $Q^nv(t)\ge 0$ for any $n\in N$ and

$$
\frac{Q^{n+1}v(t)}{Q^nv(t)}=\frac{2LG(\beta)G(n\beta)|t|^{j-1}}{G((n+1)\beta)}\frac{\int_{-|t|}^{|t|}(|t|-|s|)^{(n+1)\beta-1}|s|^{j-1}v(s)ds}{\int_{-|t|}^{|t|}(|t|-|s|)^{n\beta-1}|s|^{j-1}v(s)ds}\le\frac{2LG(\beta)G(n\beta)}{G((n+1)\beta)}\to 0
$$

when $n\to\infty$. Then, by letting $n\to\infty$ in (2.17) and using (2.18), we obtain

$$
\begin{aligned}
v(t)&\le\sum_{n=0}^{\infty}B^na=a\sum_{n=0}^{\infty}\left(2|t|^{j-1}\right)^{n-1}\frac{(LG(\beta))^n}{G(n\beta)}sgn(t)\int_{-t}^{t}(|t|-|s|)^{n\beta-1}|s|^{j-1}ds \\
&=a\sum_{n=0}^{\infty}\left(2|t|^{j-1}\right)^{n-1}\frac{(LG(\beta))^n}{G(n\beta)}2|t|^{n\beta+j-1}\int_0^1\rho^{j-1}(1-\rho)^{n\beta-1}d\rho \\
&=a\sum_{n=0}^{\infty}\frac{\left(2LG(\beta)|t|^{\beta+j-1}\right)^n}{G(n\beta)}\frac{G(j)G(n\beta)}{G(j+n\beta)}=a\sum_{n=0}^{\infty}\frac{G(j)}{G(j+n\beta)}\left(2LG(\beta)|t|^{\beta+j-1}\right)^n.
\end{aligned}
$$

□

Note that by putting $j=\beta=1$ in the Theorem 2.9, we obtain again the Theorem 2.5.

3. Applications

In applications, we consider the initial value problem

$$\begin{cases} \frac{du(t)}{dt} + sgn(t)Au(t) = \int_{-t}^{t} B(s)u(s)ds + f(t), & -1 \le t \le 1, \\ u(0) = u_0 \end{cases} \tag{3.1}$$

in an arbitrary Banach space E with unbounded linear operators A and $B(t)$ in E with dense domain $D(A) \subset D(B(t))$ and

$$\|B(t)A^{-1}\|_{E\to E} \le M, \quad -1 \le t \le 1.$$

Suppose that the following estimates hold:

$$\|e^{-tA}\|_{E\to E} \le M, \quad \|tAe^{-tA}\|_{E\to E} \le M, \quad 0 \le t \le 1. \tag{3.2}$$

A function $u(t)$ is called a solution of the problem (3.1) if the following conditions are satisfied:

i) $u(t)$ is continuously differentiable on $[-1, 1]$. The derivatives at the endpoints are understood as the appropriate unilateral derivatives.
ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [-1, 1]$, and the functions $Au(t)$ and $B(t)u(t)$ are continuous on $[-1, 1]$.
iii) $u(t)$ satisfies the equation and the initial condition (3.1).

Theorem 3.1. *Suppose that $F(t) \in C([-1, 1], E)$, $K(t, s) \in C([-1, 1], E)$, where $C([-1, 1], E)$ is the Banach space of all continuous abstarct functions $v(t)$ defined on $[-1, 1]$ with norm $\|v\|_{C([-1,1],E)} = \max_{-1\le t\le 1} \|v(t)\|_E$. Then there is a unique solution of the integral equation*

$$z(t) = F(t) + sgn(t) \int_{-t}^{t} K(t, s)z(s)ds, \quad -1 \le t \le 1.$$

Proof. The proof of this theorem is based on a fixed point theorem. It is easy to see that the operator

$$Az(t) = F(t) + sgn(t) \int_{-t}^{t} K(t, s)z(s)ds, \quad -1 \le t \le 1$$

maps $C([-1, 1], E)$ into $C([-1, 1], E)$. By using a special value of λ in the norm

$$\|v\|_{C^*([-1,1],E)} = \max_{-1\le t\le 1} e^{-\lambda|t|}\|v(t)\|_E,$$

we can prove that A is the contracting operator on $C^*([-1, 1], E)$. Indeed, we have

$$\begin{aligned} e^{-\lambda|t|}\|Az(t) - Au(t)\|_E &\le \int_{-|t|}^{|t|} \|K(t, s)\|_{E\to E} e^{-\lambda(|t|-|s|)} e^{-\lambda|s|}\|z(s) - u(s)\|_E ds \\ &\le \max_{-1\le s,t\le 1} \|K(t, s)\|_{E\to E} \int_{-|t|}^{|t|} e^{-\lambda(|t|-|s|)}\|z - u\|_{C^*([-1,1],E)} ds \\ &= 2 \max_{-1\le s,t\le 1} \|K(t, s)\|_{E\to E} \int_{0}^{|t|} e^{-\lambda(|t|-s)} ds \|z - u\|_{C^*([-1,1],E)} \\ &= 2 \max_{-1\le s,t\le 1} \|K(t, s)\|_{E\to E} \|z - u\|_{C^*([-1,1],E)} \frac{1 - e^{-\lambda|t|}}{\lambda} \\ &\le \|z - u\|_{C^*([-1,1],E)} \frac{2 \max_{-1\le s,t\le 1} \|K(t, s)\|_{E\to E}(1 - e^{-\lambda})}{\lambda} \end{aligned}$$

for any $t \in [-1,1]$. So,

$$\|Az - Au\|_{C^*([-1,1],E)} \leq \|z-u\|_{C^*([-1,1],E)}\alpha_\lambda,$$

where $\alpha_\lambda = \frac{2\max\limits_{-1\leq s,t\leq 1}\|K(t,s)\|_{E\to E}(1-e^{-\lambda})}{\lambda}$ and $\alpha_\lambda \to 0$ when $\lambda \to \infty$. Finally, we note that the norms

$$\|v\|_{C^*([-1,1],E)} = \max_{-1\leq t\leq 1} e^{-\lambda|t|}\|v(t)\|_E$$

and

$$\|v\|_{C([-1,1],E)} = \max_{-1\leq t\leq 1} \|v(t)\|_E$$

are equivalent in $C([-1,1],E)$. □

Theorem 3.2. *Suppose that $u_0 \in D(A)$ and $f(t)$ is a continuously differentiable on $[-1,1]$ function. Then there is a unique solution of the problem (3.1) and stability inequality*

$$\max_{-1\leq t\leq 1}\left\|\frac{du(t)}{dt}\right\|_E + \max_{-1\leq t\leq 1}\|Au(t)\|_E \leq M^*\left[\|Au_0\|_E + \|f(0)\|_E + \int_{-1}^{1}\|f'(s)\|_E ds\right] \tag{3.3}$$

holds, where M^* does not depend on $f(t)$, t and u_0.

Proof. The proof of the existence and uniqueness of the solution of problem (3.1) is based on the following formula

$$u(t) = e^{-|t|A}u_0 + sgn(t)A^{-1}f(t) - sgn(t)e^{-|t|A}A^{-1}f(0) - sgn(t)\int_0^t e^{-(|t|-|s|)A}A^{-1}f'(s)ds$$

$$+sgn(t)\int_{-t}^{t}\left[I - e^{-(|t|-|s|)A}\right]A^{-1}B(s)u(s)ds, \quad -1\leq t\leq 1 \tag{3.4}$$

and the Theorem 3.1. Let us prove (3.4). First, we consider the case when $0\leq t\leq 1$. It is well-known that the Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = F(t), \quad 0\leq t\leq 1, \\ u(0) = u_0 \end{cases}$$

for differential equations in an arbitrary Banach space E with positive operator A has a unique solution

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}F(s)ds, \quad 0\leq t\leq 1$$

for smooth $F(t)$. By putting $F(t) = \int_{-t}^{t} B(s)u(s)ds + f(t)$, we obtain

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}f(s)ds + \int_0^t e^{-(t-s)A}\int_{-s}^{s}B(\tau)u(\tau)d\tau ds = e^{-tA}u_0 + A^{-1}f(t)$$

$$= -e^{-tA}A^{-1}f(0) - \int_0^t e^{-(t-s)A}A^{-1}f'(s)ds + \int_0^t\int_\tau^t e^{-(t-s)A}B(\tau)u(\tau)dsd\tau$$

$$+\int_{-t}^{0}\int_{-\tau}^{t} e^{-(t-s)A}B(\tau)u(\tau)dsd\tau = e^{-tA}u_0 + A^{-1}f(t) - e^{-tA}A^{-1}f(0)$$

$$-\int_0^t e^{-(t-s)A}A^{-1}f'(s)ds + \int_0^t\left(I - e^{-(t-\tau)A}\right)A^{-1}B(\tau)u(\tau)d\tau$$

$$+\int_{-t}^{0}\left(I - e^{-(t+\tau)A}\right)A^{-1}B(\tau)u(\tau)d\tau = e^{-tA}u_0 + A^{-1}f(t) - e^{-tA}A^{-1}f(0)$$

$$-\int_0^t e^{-(t-s)A}A^{-1}f'(s)ds+\int_{-t}^t \left(I-e^{-(t-|s|)A}\right)A^{-1}B(s)u(s)ds$$

So, we proved (3.4) for $0\le t\le 1$.

Now, let $-1\le t\le 0$. Then we consider the problem

$$\begin{cases} \frac{du(t)}{dt}-Au(t)=F(t), \quad -1\le t\le 0,\\ u(0)=u_0\end{cases}$$

for differential equations in an arbitrary Banach space E with positive operator A. We denote $t=-\tau$ and $u(-\tau)=v(\tau)$. Then the problem

$$\begin{cases} \frac{dv(\tau)}{d\tau}+Av(\tau)=-F(-\tau), \quad 0\le \tau\le 1,\\ v(0)=u_0\end{cases}$$

has a unique solution

$$v(\tau)=e^{-\tau A}u_0-\int_0^\tau e^{-(\tau-s)A}F(-s)ds=e^{-\tau A}u_0-\int_{-\tau}^0 e^{-(\tau+s)A}F(s)ds, \quad 0\le\tau\le 1.$$

So,

$$u(t)=e^{tA}u_0-\int_t^0 e^{-(-t+s)A}F(s)ds, \quad -1\le t\le 0.$$

By putting $F(t)=\int_{-t}^t B(s)u(s)ds+f(t)$, we obtain

$$u(t)=e^{tA}u_0-\int_t^0 e^{-(-t+s)A}f(s)ds+\int_t^0 e^{-(-t+s)A}\int_s^{-s}B(\tau)u(\tau)d\tau ds=e^{tA}u_0-A^{-1}f(t)$$
$$+e^{tA}A^{-1}f(0)-\int_t^0 e^{-(-t+s)A}A^{-1}f'(s)ds+\int_0^{-t}\int_t^{-\tau}e^{-(-t+s)A}B(\tau)u(\tau)dsd\tau$$
$$+\int_t^0\int_t^\tau e^{-(-t+s)A}B(\tau)u(\tau)dsd\tau=e^{tA}u_0-A^{-1}f(t)+e^{tA}A^{-1}f(0)-\int_t^0 e^{-(-t+s)A}A^{-1}f'(s)ds$$
$$+\int_0^{-t}\left(I-e^{-(-t-\tau)A}\right)A^{-1}B(\tau)u(\tau)d\tau+\int_t^0\left(I-e^{-(-t+\tau)A}\right)A^{-1}B(\tau)u(\tau)d\tau=e^{tA}u_0$$
$$-A^{-1}f(t)+e^{tA}A^{-1}f(0)-\int_t^0 e^{-(-t+s)A}A^{-1}f'(s)ds+\int_t^{-t}\left(I-e^{-(-t-|s|)A}\right)A^{-1}B(s)u(s)ds.$$

So, we proved (3.4) for $-1\le t\le 0$. From (3.4) we have

$$Au(t)=e^{-|t|A}Au_0+sgn(t)f(t)-sgn(t)e^{-|t|A}f(0)-sgn(t)\int_0^t e^{-(|t|-|s|)A}f'(s)ds$$
$$+sgn(t)\int_{-t}^t\left[I-e^{-(|t|-|s|)A}\right]B(s)u(s)ds, \quad -1\le t\le 1.$$

Applying the triangle inequality and estimates (3.2), we obtain

$$\|Au(t)\|_E\le\|e^{-|t|A}\|_{E\to E}\|Au_0\|_E+\|f(t)\|_E+\|e^{-|t|A}\|_{E\to E}\|f(0)\|_E$$
$$+\int_{-|t|}^{|t|}\|e^{-(|t|-|s|)A}\|_{E\to E}\|f'(s)\|_E ds$$
$$+\int_{-|t|}^{|t|}\left[1+\|e^{-(|t|-|s|)A}\|_{E\to E}\right]\|B(s)A^{-1}\|_{E\to E}\|Au(s)\|_E ds$$

$$\leq (M+1)\left[\|Au_0\|_E + \|f(0)\|_E + \int_{-1}^{1} \|f'(s)\|_E ds\right] + sgn(t)M(M+1)\int_{-t}^{t} \|Au(s)\|_E ds.$$

So,

$$\|Au(t)\|_E \leq (M+1)\left[\|Au_0\|_E + \|f(0)\|_E + \int_{-1}^{1} \|f'(s)\|_E ds\right] + sgn(t)M(M+1)\int_{-t}^{t} \|Au(s)\|_E ds, \quad -1 \leq t \leq 1.$$

Then from the Theorem 2.5 we have

$$\|Au(t)\|_E \leq (M+1)\left[\|Au_0\|_E + \|f(0)\|_E + \int_{-1}^{1} \|f'(s)\|_E ds\right] e^{2M(M+1)|t|}$$
$$\leq (M+1)e^{2M(M+1)}\left[\|Au_0\|_E + \|f(0)\|_E + \int_{-1}^{1} \|f'(s)\|_E ds\right], \quad -1 \leq t \leq 1.$$

So,

$$\max_{-1\leq t\leq 1} \|Au(t)\|_E \leq (M+1)e^{2M(M+1)}\left[\|Au_0\|_E + \|f(0)\|_E + \int_{-1}^{1} \|f'(s)\|_E ds\right]. \tag{3.5}$$

By applying the triangle inequality in (3.1), we obtain

$$\left\|\frac{du(t)}{dt}\right\|_E \leq \|Au(t)\|_E + \int_{-|t|}^{|t|} \|B(s)A^{-1}\|_{E\to E}\|Au(s)\|_E ds + \|f(t)\|_E$$
$$\leq (2M+1)\max_{-1\leq t\leq 1} \|Au(t)\|_E + \|f(0)\|_E + \int_{-1}^{1} \|f'(s)\|_E ds, \quad -1 \leq t \leq 1.$$

So,

$$\max_{-1\leq t\leq 1}\left\|\frac{du(t)}{dt}\right\|_E \leq (2M+1)\max_{-1\leq t\leq 1} \|Au(t)\|_E + \|f(0)\|_E + \int_{-1}^{1} \|f'(s)\|_E ds.$$

Then using (3.5), we have

$$\max_{-1\leq t\leq 1}\left\|\frac{du(t)}{dt}\right\|_E + \max_{-1\leq t\leq 1} \|Au(t)\|_E \leq 2(M+1)\max_{-1\leq t\leq 1} \|Au(t)\|_E + \|f(0)\|_E$$
$$+\int_{-1}^{1} \|f'(s)\|_E ds \leq \left(2(M+1)^2 e^{2M(M+1)} + 1\right)\left[\|Au_0\|_E + \|f(0)\|_E + \int_{-1}^{1} \|f'(s)\|_E ds\right].$$

So, stability inequality (3.3) holds with $M^* = 2(M+1)^2 e^{2M(M+1)} + 1$. □

Now, we consider the Rothe difference scheme for approximate solutions of problem (3.1).

$$\begin{cases} \frac{u_k - u_{k-1}}{\tau} + Au_k = \sum\limits_{i=-k+1}^{k-1} B_i u_i \tau + \varphi_k, \quad k = \overline{1, N}, \\ \frac{u_k - u_{k-1}}{\tau} - Au_{k-1} = -\sum\limits_{i=k}^{-k} B_i u_i \tau + \varphi_k, \quad k = \overline{-N+1, 0}, \\ B_k = B(t_k), \quad t_k = k\tau, \quad k = \overline{-N, N}, \\ u_0 = u(0). \end{cases} \tag{3.6}$$

Theorem 3.3. *Suppose that the requirements of the Theorem 3.2 are satisfied. Then for the solution of difference scheme (3.6) the following stability inequalities*

$$\max_{k=\overline{-N+1,N}} \left\| \frac{u_k - u_{k-1}}{\tau} \right\|_E + \max_{k=\overline{-N,N}} \|Au_k\|_E \leq M^* \left[\|Au_0\|_E + \|\varphi_0\|_E + \sum_{k=-N+1}^{N} \|\varphi_k - \varphi_{k-1}\|_E \right]$$

hold, where M^* does not depend on φ_k, $k = \overline{-N, N}$ and u_0.

Proof. By induction we can prove that the initial value problem

$$\frac{u_k - u_{k-1}}{\tau} + Au_k = \psi_k, \quad k = \overline{1, N}, \quad u_0 = u(0)$$

for difference equations in an arbitrary Banach space E with positive operator A has a unique solution u_k:

$$u_k = R^k u_0 + \sum_{i=1}^{k} R^{k-i+1} \psi_i \tau, \quad k = \overline{1, N},$$

where $R = (I + \tau A)^{-1}$. By putting $\psi_k = \sum_{i=-k+1}^{k-1} B_i u_i \tau + \varphi_k$, we obtain

$$Au_k = R^k A u_0 + A \sum_{i=1}^{k} R^{k-i+1} \varphi_i \tau + A \sum_{i=1}^{k} R^{k-i+1} \sum_{j=-i+1}^{i-1} B_j u_j \tau^2, \quad k = \overline{1, N}. \tag{3.7}$$

Since

$$\tau \sum_{i=\pm j+1}^{k} R^{k-i+1} = \tau(R + R^2 + \ldots + R^{k\mp j}) = \tau R(I + R + \ldots + R^{k\mp j-1})$$
$$= \tau R(I - R)^{-1}(I - R^{k\mp j}) = A^{-1}(I - R^{k\mp j}),$$

we have

$$A \sum_{i=1}^{k} R^{k-i+1} \sum_{j=-i+1}^{i-1} B_j u_j \tau^2 = A \sum_{j=-k+1}^{0} \tau \sum_{i=-j+1}^{k} R^{k-i+1} B_j u_j \tau + A \sum_{j=1}^{k-1} \tau \sum_{i=j+1}^{k} R^{k-i+1} B_j u_j \tau$$
$$= A \sum_{j=-k+1}^{0} A^{-1}(I - R^{k+j}) B_j u_j \tau + A \sum_{j=1}^{k-1} A^{-1}(I - R^{k-j}) B_j u_j \tau = \sum_{i=-k+1}^{k-1} [I - R^{k-|i|}] B_i u_i \tau.$$

Putting it in (3.7), we get

$$Au_k = AR^k u_0 + A \sum_{i=1}^{k} R^{k-i+1} \varphi_i \tau + \sum_{i=-k+1}^{k-1} [I - R^{k-|i|}] B_i u_i \tau, \quad k = \overline{1, N}.$$

Since $\tau AR = I - R$, we obtain

$$A \sum_{i=1}^{k} R^{k-i+1} \varphi_i \tau = \sum_{i=1}^{k} (I - R) R^{k-i} \varphi_i = \sum_{i=1}^{k} R^{k-i} \varphi_i - \sum_{i=1}^{k} R^{k-i+1} \varphi_i$$
$$= \sum_{i=2}^{k+1} R^{k-i+1} \varphi_{i-1} - \sum_{i=1}^{k} R^{k-i+1} \varphi_i = \sum_{i=1}^{k} R^{k-i+1} \varphi_{i-1} - R^k \varphi_0 + \varphi_k - \sum_{i=1}^{k} R^{k-i+1} \varphi_i$$

$$= \varphi_k - R^k\varphi_0 - \sum_{i=1}^{k} R^{k-i+1}(\varphi_i - \varphi_{i-1}).$$

Therefore, we have

$$Au_k = R^k Au_0 + \varphi_k - R^k\varphi_0 - \sum_{i=1}^{k} R^{k-i+1}(\varphi_i - \varphi_{i-1})$$
$$+ \sum_{\substack{i=-k+1 \\ i\neq 0}}^{k-1} [I - R^{k-|i|}]B_i u_i \tau + (I - R^k)B_0 u_0 \tau, \quad k = \overline{1, N}.$$

Applying the triangle inequality and the estimates (3.2), we obtain

$$\|Au_k\|_E \le \|R^k\|_{E\to E}\|Au_0\|_E + \|\varphi_k\|_E + \|R^k\|_{E\to E}\|\varphi_0\|_E + \sum_{i=1}^{k} \|R^{k-i+1}\|_{E\to E}\|\varphi_i - \varphi_{i-1}\|_E$$
$$+ \sum_{\substack{i=-k+1 \\ i\neq 0}}^{k-1} (1 + \|R^{k-|i|}\|_{E\to E})\|B_i A^{-1}\|_{E\to E}\|Au_i\|_E\tau + (1 + \|R^k\|_{E\to E})\|B_0 A^{-1}\|_{E\to E}\|Au_0\|_E\tau$$
$$\le M\|Au_0\|_E + \|\sum_{i=1}^{k}(\varphi_i - \varphi_{i-1}) + \varphi_0\|_E + M\|\varphi_0\|_E + M\sum_{i=1}^{k}\|\varphi_i - \varphi_{i-1}\|_E$$
$$+ M(M+1)\sum_{\substack{i=-k+1 \\ i\neq 0}}^{k-1} \|Au_i\|_E\tau + M(M+1)\tau\|Au_0\|_E \le M(1 + \tau(M+1))\|Au_0\|_E$$
$$+ (M+1)\sum_{i=-N+1}^{N}\|\varphi_i - \varphi_{i-1}\|_E + (M+1)\|\varphi_0\|_E + M(M+1)\sum_{\substack{i=-k+1 \\ i\neq 0}}^{k-1}\|Au_i\|_E\tau, \quad k = \overline{1, N}.$$

So,

$$\|Au_k\|_E \le M(1 + \tau(M+1))\|Au_0\|_E + (M+1)\left(\sum_{i=-N+1}^{N}\|\varphi_i - \varphi_{i-1}\|_E + \|\varphi_0\|_E\right)$$

$$+M(M+1)\sum_{\substack{i=-k+1 \\ i\neq 0}}^{k-1}\|Au_i\|_E\tau, \quad k = \overline{1, N}. \tag{3.8}$$

In the similar way by induction we can prove that the initial value problem

$$\frac{u_k - u_{k-1}}{\tau} - Au_{k-1} = \psi_k, \quad k = \overline{-N+1, 0}, \quad u_0 = u(0)$$

for difference equations in an arbitrary Banach space E with positive operator A has a unique solution u_k:

$$u_k = R^{-k}u_0 - \sum_{i=k+1}^{0} R^{-k+i}\psi_i\tau, \quad k = \overline{-N, 0}.$$

By putting $\psi_k = -\sum_{i=k}^{-k} B_i u_i \tau + \varphi_k$, we obtain

$$Au_k = R^{-k}Au_0 - A\sum_{i=k+1}^{0} R^{-k+i}\varphi_i\tau + A\sum_{i=k+1}^{0} R^{-k+i}\sum_{j=i}^{-i} B_j u_j \tau^2, \quad k=\overline{-N,0}. \tag{3.9}$$

Since

$$\tau\sum_{i=k+1}^{\pm j} R^{-k+i} = \tau(R+R^2+\ldots+R^{-k\pm j}) = \tau R(I+R+\ldots+R^{-k\pm j-1})$$
$$= \tau R(I-R)^{-1}(I-R^{-k\pm j}) = A^{-1}(I-R^{-k\pm j}),$$

we have

$$A\sum_{i=k+1}^{0} R^{-k+i}\sum_{j=i}^{-i} B_j u_j\tau^2$$
$$= A\sum_{j=k+1}^{0}\tau\sum_{i=k+1}^{j} R^{-k+i}B_j u_j\tau + A\sum_{j=1}^{-k-1}\tau\sum_{i=k+1}^{-j} R^{-k+i}B_j u_j\tau$$
$$= A\sum_{j=k+1}^{0} A^{-1}(I-R^{-k+j})B_j u_j\tau + A\sum_{j=1}^{-k-1} A^{-1}(I-R^{-k-j})B_j u_j\tau = \sum_{i=k+1}^{-k-1}[I-R^{-k-|i|}]B_i u_i\tau.$$

Putting it in (3.9), we obtain

$$Au_k = R^{-k}Au_0 - A\sum_{i=k+1}^{0} R^{-k+i}\varphi_i\tau + \sum_{i=k+1}^{-k-1}[I-R^{-k-|i|}]B_i u_i\tau, \quad k=\overline{-N,0}.$$

Since

$$A\sum_{i=k+1}^{0} R^{-k+i}\varphi_i\tau = \sum_{i=k+1}^{0}(I-R)R^{-k+i-1}\varphi_i = \sum_{i=k+1}^{0} R^{-k+i-1}\varphi_i - \sum_{i=k+1}^{0} R^{-k+i}\varphi_i$$
$$= \sum_{i=k+1}^{0} R^{-k+i-1}\varphi_i - \sum_{i=k+2}^{1} R^{-k+i-1}\varphi_{i-1} = \sum_{i=k+1}^{0} R^{-k+i-1}\varphi_i - R^{-k}\varphi_0 + \varphi_k - \sum_{i=k+1}^{0} R^{-k+i-1}\varphi_{i-1}$$
$$= \varphi_k - R^{-k}\varphi_0 + \sum_{i=k+1}^{0} R^{-k+i-1}(\varphi_i - \varphi_{i-1}),$$

we obtain

$$Au_k = R^{-k}Au_0 - \varphi_k + R^{-k}\varphi_0 - \sum_{i=k+1}^{0} R^{-k+i-1}(\varphi_i-\varphi_{i-1})$$
$$+ \sum_{\substack{i=k+1\\ i\neq 0}}^{-k-1}[I-R^{-k-|i|}]B_i u_i\tau + (I-R^{-k})B_0 u_0\tau, \quad k=\overline{-N,0}.$$

Applying the triangle inequality and the estimates (3.2), we obtain

$$\|Au_k\|_E \le \|R^{-k}\|_{E\to E}\|Au_0\|_E + \|\varphi_k\|_E + \|R^{-k}\|_{E\to E}\|\varphi_0\|_E$$

$$+\sum_{i=k+1}^{0}\|R^{-k+i-1}\|_{E\to E}\|\varphi_i-\varphi_{i-1}\|_E+\sum_{\substack{i=k+1\\ i\neq 0}}^{-k-1}(1+\|R^{-k-|i|}\|_{E\to E})\|B_iA^{-1}\|_{E\to E}\|Au_i\|_E\tau$$

$$+(1+\|R^{-k}\|_{E\to E})\|B_0A^{-1}\|_{E\to E}\|Au_0\|_E\tau\leq M\|Au_0\|_E+\|-\sum_{i=k+1}^{0}(\varphi_i-\varphi_{i-1})+\varphi_0\|_E$$

$$+M\|\varphi_0\|_E+M\sum_{i=k+1}^{0}\|\varphi_i-\varphi_{i-1}\|_E+M(M+1)\sum_{\substack{i=k+1\\ i\neq 0}}^{-k-1}\|Au_i\|_E\tau+M(M+1)\tau\|Au_0\|_E$$

$$\leq M(1+\tau(M+1))\|Au_0\|_E+(M+1)\sum_{i=-N+1}^{N}\|\varphi_i-\varphi_{i-1}\|_E+(M+1)\|\varphi_0\|_E$$

$$+M(M+1)\sum_{\substack{i=k+1\\ i\neq 0}}^{-k-1}\|Au_i\|_E\tau,\quad k=\overline{-N,0}.$$

So,

$$\|Au_k\|_E\leq M(1+\tau(M+1))\|Au_0\|_E+(M+1)\left(\sum_{i=-N+1}^{N}\|\varphi_i-\varphi_{i-1}\|_E+\|\varphi_0\|_E\right)$$

$$+M(M+1)\sum_{\substack{i=k+1\\ i\neq 0}}^{-k-1}\|Au_i\|_E\tau,\quad k=\overline{-N,0}. \tag{3.10}$$

Now, the proof of this theorem is based on the Theorem 2.1 in the case $M=0$ and the inequalities (3.8) and (3.10). □

Now, we consider the initial value problem

$$\begin{cases}\frac{d^2u(t)}{dt^2}+Au(t)=\int_{-t}^{t}B(\rho)u(\rho)d\rho+f(t), & -1\leq t\leq 1,\\ u(0)=u_0,\quad u'(0)=u'_0\end{cases} \tag{3.11}$$

for differential equation in a Hilbert space H, where $A=A^*\geq \delta I$ $(\delta>0)$ is a positive definite and self-adjoint operator with dense domain $\bar{D}(A)=H$, and $B(t)$ in H with domain $D(B(t))$, $D(A)\subset D(B(t))$ and

$$\|B(t)A^{-1}\|_{H\to H}\leq M,\quad -1\leq t\leq 1.$$

A function $u(t)$ is called a *solution* of the problem (3.11) if the following conditions are satisfied:

i) $u(t)$ is twice continuously differentiable on $[-1,1]$. The derivative at the endpoints of the segment are understood as the appropriate unilateral derivatives.
ii) The element $u(t)$ belongs to $D(A)$ for all $t\in[-1,1]$, and the function $Au(t)$ is continuous on $[-1,1]$.
iii) $u(t)$ satisfies the equations and the initial conditions (3.11).

Theorem 3.4. *Suppose that $u_0\in D(A)$, $u'_0\in D(A^{1/2})$ and $f(t)$ is a continuously differentiable function on [-1,1]. Then there is a unique solution of the problem (3.11) and stability inequalities*

$$\max_{-1\leq t\leq 1}\left\|\frac{d^2u(t)}{dt^2}\right\|_H+\max_{-1\leq t\leq 1}\|Au(t)\|_H$$

$$\leq M^*\left[\|Au_0\|_H + \|A^{1/2}u_0'\|_H + \|f(0)\|_H + \int_{-1}^{1}\|f'(s)\|_H ds\right] \tag{3.12}$$

hold, where M^* does not depend on u_0, u_0' and $f(t)$, $t \in [-1,1]$.

Proof. The proof of the existence and uniqueness of the solution of problem (3.11) is based on the following formula

$$u(t) = c(tA)u_0 + s(tA)u_0' + A^{-1}f(t) - A^{-1}c(tA)f(0) - \int_0^t A^{-1}c((|t|-|\tau|)A)f'(\tau)d\tau$$

$$+sgn(t)\int_{-t}^{t} A^{-1}(I - c((|t|-|\tau|)A))B(\tau)u(\tau)d\tau, \quad -1 \leq t \leq 1 \tag{3.13}$$

and the Theorem 3.1. Here

$$c(tA) = \frac{e^{itA^{1/2}} + e^{-itA^{1/2}}}{2}, \quad s(tA) = A^{-1/2}\frac{e^{itA^{1/2}} - e^{-itA^{1/2}}}{2i}.$$

Let us prove (3.13). First, we consider the case when $0 \leq t \leq 1$. It is well-known that the Cauchy problem

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = F(t), \quad 0 \leq t \leq 1, \\ u(0) = u_0, \quad u'(0) = u_0' \end{cases}$$

for differential equations in an arbitrary Hilbert space H with positive and self-adjoint operator A has a unique solution

$$u(t) = c(tA)u_0 + s(tA)u_0' + \int_0^t s((t-\tau)A)F(\tau)d\tau, \quad 0 \leq t \leq 1$$

for smooth $F(t)$. By putting $F(t) = \int_{-t}^{t} B(\rho)u(\rho)d\rho + f(t)$, we obtain

$$u(t) = c(tA)u_0 + s(tA)u_0' + \int_0^t s((t-\tau)A)f(\tau)d\tau + \int_0^t s((t-\tau)A)\int_{-\tau}^{\tau} B(\rho)u(\rho)d\rho d\tau$$

$$= c(tA)u_0 + s(tA)u_0' + A^{-1}f(t) - A^{-1}c(tA)f(0) - \int_0^t A^{-1}c((t-\tau)A)f'(\tau)d\tau$$

$$+ \int_0^t\int_\rho^t s((t-\tau)A)B(\rho)u(\rho)d\tau d\rho + \int_{-t}^{0}\int_{-\rho}^{t} s((t-\tau)A)B(\rho)u(\rho)d\tau d\rho$$

$$= c(tA)u_0 + s(tA)u_0' + A^{-1}f(t) - A^{-1}c(tA)f(0) - \int_0^t A^{-1}c((t-\tau)A)f'(\tau)d\tau$$

$$+ \int_0^t A^{-1}\left(I - c((t-\rho)A)\right)B(\rho)u(\rho)d\rho + \int_{-t}^{0} A^{-1}\left(I - c((t+\rho)A)\right)B(\rho)u(\rho)d\rho$$

$$= c(tA)u_0 + s(tA)u_0' + A^{-1}f(t) - A^{-1}c(tA)f(0) - \int_0^t A^{-1}c((t-\tau)A)f'(\tau)d\tau$$

$$+ \int_{-t}^{t} A^{-1}\left(I - c((t-|\tau|)A)\right)B(\tau)u(\tau)d\tau$$

So, we proved (3.13) for $0 \leq t \leq 1$. Now, let $-1 \leq t \leq 0$. Then we consider the problem

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = F(t), \quad -1 \leq t \leq 0, \\ u(0) = u_0, \quad u'(0) = u_0' \end{cases}$$

for differential equations in an arbitrary Banach space E with positive operator A. We denote $t = -\tau$ and $u(-\tau) = v(\tau)$. Then the problem

$$\begin{cases} \frac{d^2v(\tau)}{d\tau^2} + Av(\tau) = F(-\tau), \quad 0 \le \tau \le 1, \\ v(0) = u_0, \quad v'(0) = -u_0' \end{cases}$$

has a unique solution

$$v(\tau) = c(\tau A)u_0 - s(\tau A)u_0' + \int_0^\tau s((\tau-\rho)A)F(-\rho)d\rho$$
$$= c(\tau A)u_0 - s(\tau A)u_0' + \int_{-\tau}^0 s((\tau+\rho)A)F(\rho)d\rho, \quad 0 \le \tau \le 1.$$

So,

$$u(t) = c(tA)u_0 + s(tA)u_0' + \int_t^0 s((-t+\tau)A)F(\tau)d\tau, \quad -1 \le t \le 0.$$

By putting $F(t) = \int_{-t}^t B(\rho)u(\rho)d\rho + f(t)$, we obtain

$$u(t) = c(tA)u_0 + s(tA)u_0' + \int_t^0 s((-t+\tau)A)f(\tau)d\tau - \int_t^0 s((-t+\tau)A)\int_\tau^{-\tau} B(\rho)u(\rho)d\rho d\tau$$
$$= c(tA)u_0 + s(tA)u_0' + A^{-1}f(t) - A^{-1}c(-tA)f(0) + \int_t^0 A^{-1}c((-t+\tau)A)f'(\tau)d\tau$$
$$+ \int_0^{-t}\int_t^{-\rho} s((-t+\tau)A)B(\rho)u(\rho)d\tau d\rho + \int_t^0\int_t^\rho s((-t+\tau)A)B(\rho)u(\rho)d\tau d\rho$$
$$= c(tA)u_0 + s(tA)u_0' + A^{-1}f(t) - A^{-1}c(tA)f(0) + \int_t^0 A^{-1}c((-t+\tau)A)f'(\tau)d\tau$$
$$+ \int_0^{-t} A^{-1}\left(I - c((-t-\rho)A)\right)B(\rho)u(\rho)d\rho + \int_t^0 A^{-1}\left(I - c((-t+\rho)A)\right)B(\rho)u(\rho)d\rho$$
$$= c(tA)u_0 + s(tA)u_0' + A^{-1}f(t) - A^{-1}c(tA)f(0) + \int_t^0 A^{-1}c((-t+\tau)A)f'(\tau)d\tau$$
$$+ \int_t^{-t} A^{-1}\left(I - c((-t-|\tau|)A)\right)B(\tau)u(\tau)d\tau$$

So, we proved (3.13) for $-1 \le t \le 0$. From (3.13) we have

$$Au(t) = c(tA)Au_0 + s(tA)Au_0' + f(t) - c(tA)f(0) - \int_0^t c((|t|-|\tau|)A)f'(\tau)d\tau$$
$$+ sgn(t)\int_{-t}^t (I - c((|t|-|\tau|)A))B(\tau)u(\tau)d\tau, \quad -1 \le t \le 1.$$

Applying the triangle inequality and the following estimates

$$\|c(tA)\|_{H\mapsto H} \le 1, \quad \|A^{1/2}s(tA)\|_{H\mapsto H} \le 1,$$

we get

$$\|Au(t)\|_H \le \|c(tA)\|_{H\to H}\|Au_0\|_H + \|A^{1/2}s(tA)\|_{H\to H}\|A^{1/2}u_0'\|_H + \|f(t)\|_H$$
$$+ \|c(tA)\|_{H\to H}\|f(0)\|_H + \int_{-|t|}^{|t|} \|c((|t|-|\tau|)A)\|_{H\to H}\|f'(\tau)\|_H d\tau$$
$$+ \int_{-|t|}^{|t|} \left[1 + \|c((|t|-|\tau|)A)\|_{H\to H}\right]\|B(\tau)A^{-1}\|_{H\to H}\|Au(\tau)\|_H d\tau$$

$$\leq 2\left[\|Au_0\|_H + \|A^{1/2}u_0'\|_H + \|f(0)\|_H + \int_{-1}^{1}\|f'(s)\|_H ds\right] + 2M sgn(t)\int_{-t}^{t}\|Au(s)\|_H ds.$$

Then, by using the Theorem 2.5, we obtain

$$\|Au(t)\|_H \leq 2\left[\|Au_0\|_H + \|A^{1/2}u_0'\|_H + \|f(0)\|_H + \int_{-1}^{1}\|f'(s)\|_H ds\right] e^{4M|t|}$$

$$\leq 2e^{4M}\left[\|Au_0\|_H + \|A^{1/2}u_0'\|_H + \|f(0)\|_H + \int_{-1}^{1}\|f'(s)\|_H ds\right], \quad -1 \leq t \leq 1.$$

So,

$$\max_{-1\leq t\leq 1}\|Au(t)\|_H \leq 2e^{4M}\left[\|Au_0\|_H + \|A^{1/2}u_0'\|_H + \|f(0)\|_H + \int_{-1}^{1}\|f'(s)\|_H ds\right]. \tag{3.14}$$

By applying the triangle inequality in (3.11), we obtain

$$\left\|\frac{d^2u(t)}{dt^2}\right\|_H \leq \|Au(t)\|_H + \int_{-|t|}^{|t|}\|B(\rho)A^{-1}\|_{H\to H}\|Au(\rho)\|_H d\rho + \|f(t)\|_H$$

$$\leq (2M+1)\max_{-1\leq t\leq 1}\|Au(t)\|_H + \|f(0)\|_H + \int_{-1}^{1}\|f'(s)\|_H ds, \quad -1 \leq t \leq 1.$$

So,

$$\max_{-1\leq t\leq 1}\left\|\frac{d^2u(t)}{dt^2}\right\|_H \leq (2M+1)\max_{-1\leq t\leq 1}\|Au(t)\|_H + \|f(0)\|_H + \int_{-1}^{1}\|f'(s)\|_H ds.$$

Then, using (3.14), we have

$$\max_{-1\leq t\leq 1}\left\|\frac{d^2u(t)}{dt^2}\right\|_H + \max_{-1\leq t\leq 1}\|Au(t)\|_H \leq 2(M+1)\max_{-1\leq t\leq 1}\|Au(t)\|_H + \int_{-1}^{1}\|f'(s)\|_H ds$$

$$+\|f(0)\|_H \leq \left(4(M+1)e^{4M}+1\right)\left[\|Au_0\|_H + \|A^{1/2}u_0'\|_H + \|f(0)\|_H + \int_{-1}^{1}\|f'(s)\|_H ds\right].$$

So, stability inequality (3.12) holds with $M^* = 4(M+1)e^{4M}+1$. □

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References

[1] T.N.Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, *Ann. of Math.* **II.**, Ser.20, (1919), 292–296.

[2] S.Ashirov and N.Kurbanmamedov, Investigation of solution one type of Volterra's integral equations, *Izv. Vyssh. Uchebn. Zav., Matematika* **9**, (1987), 3–8.

[3] S.Ashirov and Ya.D.Mamedov, A Volterra-type integral equation, *UMJ* **40**, No 4, (1988), 510–515.

[4] A.Cordumeamu, A note on the Gronwall Inequality in two independent variables, *J.Int.Eq.* **4**, No 3, (1982), 271–276.

[5] P.Linz, Numerical methods for Volterra integral equations of the first kind, *Comput.J.* **12**, No 3, (1969), 393–397.

[6] S.Mckee, Generalised Discrete Gronwall Lemmas, *ZAMM* **62**, No 3, (1982), 429–434.

[7] E.F.Beckenbach AND R.Bellman, *Inequalities*, Berlin, Springer–Verlag, 1961.

[8] S.G.Krein *Linear Differential Equations in a Banach Space*, Nauka, Moscow 1966.

[9] A.N.Tihonov AND A.A.Samarskii, *Equations of Mathematical Physics*, Nauka, Moscow 1977.

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