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MAS A difference scheme of improved accuracy for a quasilinear singularly perturbed elliptic convection-diffusion equation

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ABSTRACT

A Dirichlet boundary value problem for a quasilinear singularly perturbed elliptic convection-diffusion equation on a strip is considered. For such a problem, a difference scheme based on classical approximations of the problem on piecewise uniform meshes condensing in the layer converges epsilon-uniformly with an order of accuracy not more than 1. We construct a linearized iterative scheme based on the nonlinear Richardson scheme, where the nonlinear term is computed using the sought function taken at the previous iteration; the solution of the iterative scheme converges to the solution of the nonlinear Richardson scheme at the rate of a geometry progression epsilon-uniformly with respect to the number of iterations. The use of lower and upper solutions of the linearized iterative Richardson scheme as a stopping criterion allows us during the computational process to define a current iteration under which the same epsilon-uniform accuracy of the solution is achieved as for the nonlinear Richardson scheme.

A Difference Scheme of Improved Accuracy
for a Quasilinear Singularly Perturbed
Elliptic Convection-Diffusion Equation

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ABSTRACT

A Dirichlet boundary value problem for a quasilinear singularly perturbed elliptic convection-diffusion equation on a strip is considered. For such a problem, a difference scheme based on classical approximations of the problem on piecewise uniform meshes condensing in the layer converges ε-uniformly with an order of accuracy not more than 1. Using the Richardson technique, we construct a scheme (nonlinear) that converges ε-uniformly at the rate $\mathcal{O}\left(N_1^{-2}\ln^2 N_1 + N_2^{-2}\right)$, where $N_1 + 1$ is the number of nodes in the mesh with respect to x_1 and $N_2 + 1$ is the number of mesh points along the x_2-axis on a unit segment. We construct a linearized iterative scheme based on the nonlinear Richardson scheme, where the nonlinear term is computed using the sought function taken at the previous iteration; the solution of the iterative scheme converges to the solution of the nonlinear Richardson scheme at the rate of a geometry progression ε-uniformly with respect to the number of iterations. The use of lower and upper solutions of the linearized iterative Richardson scheme as a stopping criterion allows us during the computational process to define a current iteration under which the same ε-uniform accuracy of the solution is achieved as for the nonlinear Richardson scheme.

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1. INTRODUCTION

At the present time, for sufficiently wide classes of singularly perturbed boundary value problems special numerical methods are constructed that allow us to obtain ε-uniformly convergent solutions. In the case of boundary value problems for elliptic reaction-diffusion equations, the order of the ε-uniform convergence of well known special methods does not exceed 2 and for convection-diffusion equations it does not exceed 1 (see, e.g., [1, 3, 7, 11, 12, 15] and also the bibliography therein). Due to this, for elliptic equations an interest arises to construct special schemes for reaction-diffusion and convection-diffusion problems, the ε-uniform order of which is more than 2 and 1 respectively.

To improve accuracy of discrete solutions to *regular* boundary value problem, the defect correction method and Richardson method (see, e.g., [2, 9, 10] and also the bibliography therein) were applied. The same methods turn out to be effective also for improvement of ε-uniform accuracy to *linear* singularly perturbed problems. Finite difference schemes with improved accuracy for such problems were constructed using the defect correction technique for parabolic reaction-diffusion and convection diffusion equations (see, e.g., [4, 5, 17] and also the bibliography therein).

Application of the Richardson extrapolation technique for parabolic and elliptic equations in the case of *linear* singularly perturbed boundary value problems one can find in [6, 14, 16]. The Richardson extrapolation method is based on a representation of the main part of the solution error for a base difference scheme in the form of an expansion with respect to effective step-sizes of meshes used. This representation allows us to construct a suitable linear combination of discrete solutions obtained on

embedded meshes which has increased accuracy (on the intersection of the meshes) as compared with the base scheme. The advantage of this method is so that one and the same discrete problem is solved but on one-type embedded meshes.

Last time, applying new approaches for the use of a Richardson technique, ε-uniformly convergent difference schemes with improved accuracy were constructed for *quasilinear* singularly perturbed equations of parabolic type (reaction-diffusion on a strip [18], convection-diffusion on a segment [20]) and elliptic type (reaction-diffusion on a strip [19]).

In the present paper we consider a boundary value problem for the quasilinear singularly perturbed elliptic convection-diffusion equation on a vertical strip (the problem formulation is given in section 2).

For the quasilinear problem under consideration, we construct a special nonlinear base scheme on a piecewise uniform mesh condensing in a boundary layer that converges ε-uniformly at the rate $\mathcal{O}\left(N_1^{-1}\ln N_1 + N_2^{-1}\right)$, where $N_1 + 1$ and $N_2 + 1$ are the number of nodes in meshes with respect to the segment $[0,\,d]$ on the x_1-axis and on a unit segment on the x_2-axis respectively (see section 4). It is known that for nonlinear (in particular, quasilinear) singularly perturbed problems (as it is in the case of linear problems in the presence of parabolic layers), when constructing ε-uniformly convergent methods the use of meshes condensing in a boundary layer is necessary (see, e.g., [11]).

This base scheme is applied for construction of

- a linearized iterative scheme, where the nonlinear term is computed using the sought function from the previous iteration (see section 5) and

- a scheme with increasing accuracy using a Richardson technique (see section 6); the improved nonlinear scheme converges ε-uniformly at the rate $\mathcal{O}\left(N_1^{-2}\ln^2 N_1 + N_2^{-2}\right)$.

Using the linearized iterative scheme (from section 5), we construct a linearized iterative Richardson scheme (similar to the one in section 6), which converges to the solution of the boundary value problem with improved convergence order. Solutions of this iterative scheme converge to the solution of the nonlinear scheme at the rate of a geometric progression ε-uniformly with respect to the number of iterations. The use of upper and lower solutions of the linearized iterative Richardson scheme as indicators permits one to define termination of computations, i.e. the current iteration when ε-uniform accuracy of the solution of the nonlinear noniterative base Richardson scheme has been attained (see section 7).

2. Problem formulation
On a vertical strip $\overline{D}$

$$D = \{x : 0 < x_1 < d, \quad x_2 \in I\!R\}, \tag{2.1}$$

we consider the boundary value problem for the quasilinear elliptic convection-diffusion equation[1]

$$L_{(2.2)}\left(u(x)\right) \;\equiv\; L^2_{(2.2)}\,u(x) - f\left(x, t,\, u(x)\right) = 0, \quad x \in D, \tag{2.2}$$

$$u(x) \;=\; \varphi(x), \quad x \in \Gamma.$$

Here $\Gamma = \overline{D} \setminus D$,

$$L^2_{(2.2)} = L^{(2)}_{(2.2)} + L^{(1)}_{(2.2)}, \quad L^{(2)}_{(2.2)} \equiv \varepsilon \sum_{s=1,2} a_s(x)\frac{\partial^2}{\partial x_s^2}, \quad L^{(1)}_{(2.2)} \equiv \sum_{s=1,2} b_s(x)\frac{\partial}{\partial x_s} - c(x), \quad x \in D,$$

the functions $a_s(x)$, $b_s(x)$, $c(x)$, $f(x,\,u)$ and $\varphi(x)$ are assumed to be sufficiently smooth on $\overline{D}$, $\overline{D} \times I\!R$

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[1] Throughout the paper, the notation $L_{(j.k)}$ ($M_{(j.k)}$, $G_{h(j.k)}$) means that these operators (constants, grids) are introduced in formula $(j.k)$.

and Γ, respectively, moreover [2]

$$a_0 \leq a_s(x) \leq a^0, \quad b_0 \leq b_s(x) \leq b^0, \quad s = 1, 2, \quad |c(x)| \leq c^0, \quad x \in \overline{D}; \tag{2.3}$$

$$|f(x, u)| \leq M, \quad c_1 \leq c(x) + \frac{\partial}{\partial u} f(x, u) \leq c^1, \quad (x, u) \in \overline{D} \times \mathbb{R};$$

$$|\varphi(x)| \leq M, \quad x \in \Gamma; \quad a_0, b_0, c_1 > 0;$$

the perturbation parameter ε takes arbitrary values in the half-open interval $(0, 1]$.

Remark 2.1 In that case when in (2.3) the condition

$$c(x) + \frac{\partial}{\partial u} f(x, u) \geq c_1 > 0, \quad (x, u) \in \overline{D} \times \mathbb{R} \tag{2.4}$$

is violated, we pass to a new variable $v(x)$, $x \in \overline{D}$, $u(x) = v(x) \exp(-\alpha x)$, $x \in \overline{D}$. Under the condition $\varepsilon \leq \varepsilon_0$, where ε_0 is sufficiently small, $m \leq \varepsilon_0 \leq 1$, we choose the value α so that in a new differential equation (the function $v(x)$ satisfies this equation), a condition similar to (2.4) is valid.

We will denote by Γ^- (by Γ^+) that part of the boundary Γ, through which characteristics of the reduced equation passing through points $x \in D$, leave (enter in) the set D, $\Gamma = \Gamma^- \bigcup \Gamma^+$, $\Gamma^- = \{x : x_1 = 0, x_2 \in \mathbb{R}\}$.

When ε tends to zero, a boundary layer appears in a neighbourhood of the set Γ^-.

Our aim is for the boundary value problem (2.2), (2.1) with using a Richardson technique, to construct a difference scheme convergent ε-uniformly with the accuracy order more than one.

3. *A priori* ESTIMATES OF SOLUTIONS AND DERIVATIVES

We give *a-priori* estimates of solutions and derivatives for boundary value problem (2.2), (2.1); derivation of the estimates is similar to those in [15].

Using the majorant function technique (see, e.g., [8]), we find the estimate

$$|u(x)| \leq M, \quad x \in \overline{D}. \tag{3.1}$$

We represent the solution of the problem as the sum of functions

$$u(x) = U(x) + V(x), \quad x \in \overline{D}, \tag{3.2a}$$

where $U(x)$ and $V(x)$ are the regular and singular parts of the solution. The function $U(x)$, $x \in \overline{D}$ is the restriction on $\overline{D}$ of the function $U^0(x)$, $x \in \overline{D}^0$, $U(x) = U^0(x)$, $x \in \overline{D}$. The function $U^0(x)$, $x \in \overline{D}^0$ is the solution of the boundary value problem

$$L^0(U^0(x)) \equiv L^{20} U^0(x) - f^0\left(x, U^0(x)\right) = 0, \quad x \in D^0, \tag{3.3}$$

$$U^0(x) = \varphi(x), \quad x \in \Gamma^+.$$

Here $\overline{D}^0$ is the half-plane, which is an extension D beyond the side Γ^-; the data of problem (3.3) are smooth continuations of the data to problem (2.2), (2.1), preserving properties (2.3) on $\overline{D}^0$; $L^{20} = L^{(2)0} + L^{(1)0}$. Assume that function $f^0(x, u)$, $x \in \overline{D}^0$ outside the m_1-neighbourhood of the set $\overline{D}$ is equal to zero. The function $V(x)$ is the solution of the problem

$$L^2 V(x) - [f(x, U(x) + V(x)) - f(x, U(x))] = 0, \quad x \in D, \tag{3.4}$$

$$V(x) = \varphi(x) - U(x), \quad x \in \Gamma.$$

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[2] Throughout this paper, M, M_i (or m) denote sufficiently large (small) positive constants that do not depend on ε and on the discretization parameters.

We represent the function $U(x)$ as the sum of functions

$$U(x) = U_0(x) + \varepsilon U_1(x) + \cdots + \varepsilon^n U_n(x) + v_U(x), \quad x \in \overline{D}, \tag{3.2b}$$

corresponding to the representation of the function $U^0(x)$

$$U^0(x) = U_0^0(x) + \varepsilon U_1^0(x) + \cdots + \varepsilon^n U_n^0(x) + v_U^0(x), \quad x \in \overline{D}^0 \tag{3.5a}$$

are solutions of the boundary value problem (3.3). Here $v_U^0(x)$ is the remainder term;

$$U(x) = U^0(x), \ldots, v_U(x) = v_U^0(x), \quad (x) \in \overline{D}.$$

In (3.5a) the functions $U_0^0(x)$, $U_i^0(x)$, $i = 1, \ldots, n$ are solutions of the problems

$$L^{(1)0} U_0^0(x) \ - \ f^0(x, U_0^0(x)) = 0, \quad x \in D^0, \tag{3.5b}$$

$$U_0^0(x) \ = \ \varphi(x), \quad x \in \Gamma^+;$$

$$L^{(1)0} U_i^0(x) \ - \ \varepsilon^{-i} \left[f^0\left(x, \sum_{j=0}^{i} \varepsilon^j U_j^0(x)\right) - f\left(x, \sum_{j=0}^{i-1} \varepsilon^j U_j^0(x)\right)\right] = L^{(2)0} U_{i-1}^0(x), \quad x \in D^0,$$

$$U_i^0(x) \ = \ 0, \quad x \in \Gamma^+, \quad i = 1, \ldots, n.$$

Here $L^{(1)0}$ is the operator L^{20} for $\varepsilon = 0$

$$L^{(1)0} \equiv \sum_{s=1,2} b_s^0(x) \frac{\partial}{\partial x_s} - c^0(x), \quad x \in \overline{D}^0.$$

Under sufficient smoothness of coefficients to the operator $L_{(2.2)}^2$ on $\overline{D}$, the functions $\varphi(x)$ on Γ and $f(x, u)$ on $\overline{D} \times \mathbb{R}$ (for $f \in C^{l_0}(\overline{D} \times \mathbb{R})$, $\varphi \in C^{l_0}(\Gamma)$, $l_0 = 3n + 2 + \alpha$), the following inclusions are fulfilled

$$u, U \in C^{l^1}(\overline{D}), \quad U_i \in C^{l^2}(\overline{D}), \quad i = 0, 1, \ldots, n,$$

where $l^1 = n + 2 + \alpha$, $l^2 = 3n + 2 - 2i$, $n \geq 1$, $\alpha > 0$.

In that case for $U(x)$, $V(x)$, we obtain the estimates

$$\left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2}} U(x) \right| \leq M \left[1 + \varepsilon^{n+1-k}\right], \tag{3.6a}$$

$$\left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2}} V(x) \right| \leq M \left[-k_1 + \varepsilon^{1-k}\right] \exp\left(-m\varepsilon^{-1} r(x, \Gamma^-)\right), \quad x \in \overline{D}, \tag{3.6b}$$

where $r(x, \Gamma^-)$ is the distance from the point x to the set Γ^-, m is an arbitrary number in the interval $(0, m_0)$, $m_0 = \min_{s, \overline{D}} \left[a_s^{-1}(x) b_s(x)\right]$; $K = n + 2$ for sufficient smoothness of the data of boundary value problem (2.2), (2.1).

For the function $u(x)$ also the following estimate is valid

$$\left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2}} u(x) \right| \leq M \varepsilon^{-k}, \quad x \in \overline{D}, \quad k \leq K. \tag{3.7}$$

Theorem 3.1 *Let the data of the boundary value problem (2.2), (2.1) satisfy the condition a_s, b_s, $c \in C^{3n+2+\alpha}(\overline{D})$, $f \in C^{3n+2+\alpha}(\overline{D} \times \mathbb{R})$, $\varphi \in C^{3n+2+\alpha}(\overline{D})$, $s = 1, 2$, $n \geq 1$, $\alpha > 0$. Then for the solution of the boundary value problem and its component in the representation (3.2) the estimates (3.1), (3.6), (3.7), where $K = n + 2$, are satisfied.*

4. BASE SCHEME FOR PROBLEM (2.2), (2.1)

First we give ε-uniformly convergent finite difference scheme constructing on the base of classical approximation of problem (2.2), (2.1). We will use the solutions of the base scheme for construction of discrete solutions with improved accuracy order.

On the set $\overline{D}$ we introduce the rectangular mesh

$$\overline{D}_h = \overline{\omega}_1 \times \omega_2, \tag{4.1}$$

where $\overline{\omega}_1$ and ω_2 are arbitrary, in general, nonuniform meshes on the segment $[0, d]$ and at the x_2-axis respectively. Let $h_s^i = x_s^{i+1} - x_s^i$, $x_s^i, x_s^{i+1} \in \overline{\omega}_1$ for $s = 1$ and $x_s^i, x_s^{i+1} \in \omega_2$ for $s = 2$; let $h_s = \max_i h_s^i$, $h = \max_s h_s$. Assume that $h \leq M\,N^{-1}$, where $N = \min[N_1, N_2]$; $N_1 + 1$ and $N_2 + 1$ are the number of nodes in the mesh $\overline{\omega}_1$ and the minimal number of nodes in the mesh ω_2 on a unit segment.

Problem (2.2), (2.1) is approximated by the finite difference scheme [13]

$$\Lambda\left(z(x)\right) \equiv \Lambda^2 z(x) - f(x, z(x)) = 0, \quad x \in D_h, \tag{4.2}$$
$$z(x) = \varphi(x), \quad x \in \Gamma_h.$$

Here $D_h = D \bigcap \overline{D}_h$, $\Gamma_h = \Gamma \bigcap \overline{D}_h$,

$$\Lambda z(x) \equiv \left\{ \varepsilon \sum_{s=1,2} a_s(x)\,\delta_{\overline{x_s x_s}} + \sum_{s=1,2} b_s(x)\,\delta_{x_s} - c(x) \right\} z(x),$$

$\delta_{x_s} z(x)$ and $\delta_{\overline{x_s x_s}}$ are the first (forward) and second difference derivatives; for example, $\delta_{\overline{x_1} \widehat{x_1}} z(x) = 2\left(h_1^i + h_1^{i-1}\right)^{-1} [\delta_{x_1} z(x) - \delta_{\overline{x_1}} z(x)]$, $x = (x_1^i, x_2)$.

Scheme (4.2), (4.1) is monotone [13] ε-uniformly.

The following variant of the comparison theorem is valid.

Theorem 4.1 *Let the functions $z^1(x)$, $z^2(x)$, $x \in \overline{D}_h$ satisfy the conditions*

$$\Lambda\left(z^1(x)\right) < \Lambda\left(z^2(x)\right), \quad x \in D_h, \quad z^1(x) > z^2(x), \quad x \in \Gamma_h.$$

Then $z^1(x) > z^2(x)$, $x \in \overline{D}_h$.

In the case of uniform meshes

$$\overline{D}_h = \overline{\omega}_1 \times \omega_2, \tag{4.3}$$

where $\overline{\omega}_1$ and ω_2 are uniform meshes, for solutions of the difference scheme, taking into account *a priori* estimates, we establish convergence under the condition $h = o(\varepsilon)$

$$|u(x) - z(x)| \leq M\left[N_1^{-1}(\varepsilon + N_1^{-1})^{-1} + N_2^{-1}\right], \quad x \in \overline{D}_{h(4.3)}. \tag{4.4}$$

Let us consider a scheme on *piecewise uniform* meshes
On the set $\overline{D}$ we construct the mesh

$$\overline{D}_h = \overline{\omega}_1^* \times \omega_2. \tag{4.5a}$$

Here $\omega_2 = \omega_{2(4.3)}$ is a *uniform* mesh, $\overline{\omega}_1^*$ is a mesh with piecewise constant step-size. When constructing the mesh $\overline{\omega}_1^*$, the segment $[0, d]$ is divided into two parts $[0, \sigma]$, $[\sigma, d]$, σ is a parameter in the interval $(0, d)$. In each interval the step-sizes are constant and equal to $h_1^{(1)} = 2\,\sigma\,N_1^{-1}$ in $[0, \sigma]$ and $h_1^{(2)} = 2\,(d - \sigma)\,N_1^{-1}$ in $[\sigma, d]$. The parameter σ is defined by

$$\sigma = \sigma(\varepsilon, N_1, d;\, l, m) = \min[2^{-1} d,\, l\,m^{-1}\,\varepsilon \ln N_1], \tag{4.5b}$$

where m is an arbitrary number from $(0, m_0)$, $m_0 = \min_{s, \overline{D}} [a_s^{-1}(x) b_s(x)]$, $l > 0$ is a mesh parameter; $N = \min[N_1, N_2]$. The mesh $\overline{\omega}_1^*$, and hence the mesh $\overline{D}_h = \overline{D}_h(l)$ are constructed.

For solutions of boundary value problem (2.2), (2.1) we use the scheme (4.2) on the mesh

$$\overline{D}_h = \overline{D}_{h(4.5)}(l = 1). \tag{4.6}$$

For solutions of difference scheme (4.2), (4.6), i.e, a nonlinear base scheme, we obtain the ε-uniform estimate

$$|u(x) - z(x)| \le M \left[N_1^{-1} \ln N_1 + N_2^{-1} \right], \quad x \in \overline{D}_h. \tag{4.7}$$

Theorem 4.2 *Let solutions of boundary value problem (2.2), (2.1) satisfy a priori estimates (3.6), (3.7) for $K = 3$. Then the solution of nonlinear base difference scheme (4.2), (4.6) for $N \to \infty$ converges ε-uniformly to the solution of the boundary value problem at the rate $\mathcal{O}\left(N_1^{-1} \ln N_1 + N_2^{-1}\right)$. For discrete solutions the estimates (4.4), (4.7) are valid.*

5. LINEARIZED ITERATIVE BASE SCHEME

On mesh (4.1) we consider an iterative monotone two-level difference scheme in which the nonlinear term of the differential equation is computed using the sought function from the previous iterative level. To the boundary value problem (2.2), (2.1) corresponds the (linearized) difference scheme (see [13])

$$\begin{aligned}
\Lambda_{(5.1)}(z(x, t)) &\equiv \Lambda^2_{(4.2)} z(x, t) - p\,\delta_{\bar{t}}\, z(x, t) - f(x, \breve{z}(x, t)) = 0, \quad (x, t) \in G_h, \\
z(x, t) &= \psi(x, t), \quad (x, t) \in S_h.
\end{aligned} \tag{5.1a}$$

Here

$$\overline{G}_h = G_h \bigcup S_h, \quad \overline{G}_h = \overline{D}_h \times \overline{\omega}_0, \quad G_h = D_h \times \omega_0, \tag{5.1b}$$

$\overline{\omega}_0$ is a *uniform* mesh on the semi-axis $t \ge 0$ with the step-size $h_t = 1$, the variable $t \in \overline{\omega}_0$ defines the number of iteration; $S_h = S_h^L \cup S_{h0}$, $S_h^L = \Gamma_h \times \omega_0$ is the lateral part of the boundary;

$$\delta_{\bar{t}}\, z(x, t) = h_t^{-1} \left[z(x, t) - \breve{z}(x, t) \right], \quad \breve{z}(x, t) = z(x, t - h_t), \quad (x, t) \in G_h;$$

the coefficient p satisfies the condition

$$p - \frac{\partial}{\partial u} f(x, u) \ge p_0, \quad (x, u) \in \overline{D} \times \mathbb{R}, \quad p_0 > 0, \tag{5.1c}$$

ensuring the monotonicity of the difference scheme. The function $\psi(x, t)$, $(x, t) \in S_h$ on the lateral boundary S_h^L satisfies the condition

$$\psi(x, t) = \varphi(x), \quad (x, t) \in S_h^L,$$

moreover, $\psi(x, 0)$, $x \in \overline{D}_h$ is an arbitrary sufficiently bounded function. We call the function $z(x, t)$, $(x, t) \in \overline{G}_h$, where $\overline{G}_h$ is generated by the mesh $\overline{D}_{h(4.1)}$, the solution of the linearized iterative difference scheme (5.1), (4.1).

On each iteration $t \in \omega_0$ the function $z(x, t)$, $x \in \overline{D}_h$ is the solution of a linear problem.

The difference scheme (5.1), (4.1) is monotone.

The difference operator $\Lambda_{(5.1)}(\cdot)$ can be written in the form

$$\begin{aligned}
\Lambda_{(5.1)}(z(x, t)) &\equiv \Lambda^2_{(4.2)} z(x, t) - f(x, z(x, t)) - p\,\delta_{\bar{t}}\, z(x, t) + \\
&\quad + \left[f\big(x, z(x, t)\big) - f\big(x, \breve{z}(x, t)\big) \right] = 0, \quad (x, t) \in G_h.
\end{aligned}$$

Using the majorant function technique, we find the estimate

$$|z(x) - z(x, t)| \le M\, q^t, \quad (x, t) \in \overline{G}_h, \tag{5.2}$$

where $z(x)$, $x \in \overline{D}_h$ are $z(x, t)$, $(x, t) \in \overline{G}_h$ are solutions of difference schemes (4.2), (4.1) and (5.1), (4.1) respectively,

$$q \leq q_0 \equiv p^0(c_{10} + p^0)^{-1}, \quad p^0 = \max\left(p - \frac{\partial}{\partial u} f(x, u)\right),$$

$$c_{10} = \min\left(c(x) + \frac{\partial}{\partial u} f(x, u)\right), \quad (x, u) \in \overline{D} \times I\!R.$$

Thus, the solution of the linearized iterative difference scheme (5.1), (4.1) converges to the solution of the base nonlinear difference scheme (4.2), (4.1) at the rate of a geometry progression.

In the case of the mesh (4.6) we obtain the estimate

$$|u(x) - z(x, t)| \leq M\left[N_1^{-1} \ln N_1 + N_2^{-1} + q^t\right], \quad (x, t) \in \overline{G}_h, \tag{5.3}$$

where $q \leq q_{0(5.2)}$.

Note that the function $z(x)$, $x \in \overline{D}_h$ satisfies the estimate

$$|z(x)| \leq c_{10}^{-1} \max_{\overline{D}} |f(x, 0)|, \quad x \in \overline{D}_h.$$

Let us give a definition.

Let the functions $z^{(1)}(x, t)$, $z^{(2)}(x, t)$, $(x, t) \in \overline{G}_h$ be solutions of some difference scheme on the mesh $\overline{G}_h$, satisfying in an "initial moment" the condition

$$z^{(1)}(x, 0) \leq z(x) \leq z^{(2)}(x, 0), \quad x \in \overline{D}_h.$$

We call the functions $z^{(1)}(x, t)$ and $z^{(2)}(x, t)$, $(x, t) \in \overline{G}_h$ respectively, *lower* and *upper* solutions of the discrete problem (4.2) on the mesh $\overline{D}_h$, which generates the mesh $\overline{G}_h$, if the following inequality holds:

$$z^{(1)}(x, t) \leq z(x) \leq z^{(2)}(x, t), \quad (x, t) \in \overline{G}_h,$$

moreover,

$$\max_x |z^{(i)}(x, t) - z(x)| \downarrow 0, \quad x \in \overline{D}_h \quad for \quad t \to \infty, \quad i = 1, 2.$$

Let $z^{(1)}(x, t)$, $z^{(2)}(x, t)$, $(x, t) \in \overline{G}_h$ be solutions of difference scheme (5.1), (4.1). By virtue of the monotonicity of scheme (5.1), (4.1), the functions $z^{(1)}(x, t)$ and $z^{(2)}(x, t)$ are *the lower* and *upper* solutions, moreover,

$$0 \leq z^{(2)}(x, t) - z^{(1)}(x, t) \leq M q^t, \quad (x, t) \in \overline{G}_h,$$

where $q \leq q_{0(5.2)}$.

It is suitable to use the lower and upper solutions in order to estimate a current iteration under which the accuracy of the iterative scheme is the same as it is for the base scheme.

For the linearized iterative base difference scheme (5.1), (4.6), we have the estimate

$$|u(x) - z^{(j)}(x, t)| \leq M[N_1^{-1} \ln N_1 + N_2^{-1}] + z^{(2)}(x, t) - z^{(1)}(x, t), \tag{5.4}$$

$$(x, t) \in \overline{G}_h, \quad j = 1, 2.$$

Also the two-sided estimate holds

$$z^{(1)}(x, t) - M[N_1^{-1} \ln N_1 + N_2^{-1}] \leq u(x), z(x) \leq z^{(2)}(x, t) + M[N_1^{-1} \ln N_1 + N_2^{-1}], \tag{5.5}$$

$$(x, t) \in \overline{G}_h.$$

The error of the solution of the iterative scheme (5.1), (4.6) can be represented as the sum

$$z_{(5.1;4.6)}^{(j)}(x,\,t) - u(x) = \left(z_{(4.2;4.6)}(x) - u(x)\right) + \left(z_{(5.1;4.6)}^{(j)}(x,\,t) - z_{(4.2;4.6)}(x)\right),$$

$$(x,\,t) \in \overline{G}_h, \quad j = 1, 2.$$

It seems appropriate to choose the value T, i.e. the number of iterations in the scheme (5.1), (4.6), so that the error of the solution of the base difference scheme (4.2), (4.6) and a difference between the solution of the iterative scheme (5.1), (4.6) and the solution of the base scheme (4.2), (4.6) are commensurable.

We call the function $z_{(5.1;4.6)}^{(j)}(x,\,T)$, $x \in \overline{D}_h$, the (*upper* for $j = 2$ and *lower* for $j = 1$) solution of scheme (5.1), (4.6), consistent with respect to accuracy of the scheme (4.2), (4.6) and to the number of iterations of the scheme (5.1), (4.6). The value T is defined by the relations

$$\max_{\overline{D}_h}[\,z^{(2)}(x,\,t) - z^{(1)}(x,\,t)\,] > M_1\,[N_1^{-1}\ln N_1 + N_2^{-1}], \tag{5.6}$$

$$\max_{\overline{D}_h}[z^{(2)}(x,\,T) - z^{(1)}(x,\,T)] \le M_1\,[N_1^{-1}\ln N_1 + N_2^{-1}], \quad x \in \overline{D}_h, \quad t < T.$$

For the consistent solution of the linearized iterative base difference scheme (5.1), (4.6) we obtain the estimate

$$|u(x) - z^{(j)}(x,\,T)| \le M_2\,[N_1^{-1}\ln N_1 + N_2^{-1}], \quad x \in \overline{D}_h, \quad j = 1, 2, \tag{5.7a}$$

and also, for the number of iterations T the estimate is valid

$$T \le M_3\left(\ln q_0^{-1}\right)^{-1}\ln N, \tag{5.7b}$$

where $q_0 = q_{0(5.2)}$, constants $M_{1(5.6)}$, $M_{2(5.7)}$, $M_{3(5.7)}$ are independent of q_0; the value T is defined in the computational process according to the relations (5.6).

Theorem 5.1 *Let hypothesis of Theorem 4.2 be fulfilled. Then the solution of the linearized iterative difference scheme (5.1), (4.6) for N_1, N_2, $t \to \infty$ converges to the solution of the boundary value problem (2.2), (2.1) ε-uniformly at the rate $\mathcal{O}\left(N_1^{-1}\ln N_1 + N_2^{-1} + q_0^t\right)$, where $q_0 = q_{0(5.2)}$. For discrete solutions the estimates (5.2), (5.3), (5.4), (5.5), (5.7) are valid.*

6. RICHARDSON METHOD FOR PROBLEM (2.2), (2.1)

The Richardson method (extrapolation) for improvement of accuracy of discrete solutions for regular problems is sufficiently well developed in the case of difference schemes on uniform meshes (see, e.g., [10]). In this method, it is applied an expansion of the solution of a discrete problem with respect to the step-size of the mesh set. Coefficients of the expansion are independent of the step-size.

A linear combination (extrapolation) of discrete solutions on embedded meshes with different step-sizes applied in this method allows us to increase the order of accuracy of the approximate solution. We note the Richardson method on piecewise uniform meshes applied in [10, Ch. 3, §3.3] for solving ordinary differential equations with discontinuous coefficients. Stepsizes of such meshes on regions of smoothness of the coefficients are commensurable.

Now we describe the Richardson method used to improve accuracy of discrete solutions on the base of special difference scheme (4.2), (4.5).

In the case of scheme (4.2), (4.5) the mesh set $\overline{D}_h^{*}$ and the discrete solution $z(x)$, $x \in \overline{D}_h^{*}$ are defined by scheme parameters N_1, N_2 and the perturbation parameter ε. It is required, on the base of this scheme to construct difference schemes whose solutions have main terms in expansions with respect to an effective mesh step-size the same as the solution of the base scheme. It is suitable to use the value N^{-1}, $N = \min[N_1, N_2]$ as the effective mesh step-size.

On the set $\overline{D}$ we construct meshes

$$\overline{D}_h^{i} = \overline{\omega}_1^{*i} \times \omega_2^{i}, \quad i = 1, 2, \tag{6.1a}$$

uniform in x_2 and *piecewise-uniform* in x_1. Here $\overline{D}_h^2$ is $\overline{D}_{h(4.5a)}$, where

$$\sigma = \sigma_{(4.5b)}(\varepsilon, N_1, l) \quad \text{for} \quad l \geq 2; \tag{6.2}$$

$\overline{D}_h^1$ is a *coarsened* mesh. For the parameters σ^i, which define piecewise uniform meshes $\overline{\omega}_1^{*i} = \overline{\omega}_1^{*i}(\sigma^i)$, we impose the condition $\sigma^1 = \sigma^2$, where $\sigma^2 = \sigma_{(6.2)}$, that is, segments on which the meshes $\overline{\omega}_1^{*1}$ and $\overline{\omega}_1^{*2}$ have a constant stepsize, are the same. Stepsizes in the mesh $\overline{\omega}_1^{*1}$ on the segments $[0, \sigma]$, $[\sigma, d - \sigma]$, $[d - \sigma, d]$ are k times larger than stepsizes in the mesh $\overline{\omega}_1^{*2}$, and stepsizes in the mesh ω_2^1 are k times larger than stepsizes in the mesh ω_2^2; ($k^{-1} N_1 + 1$ and $k^{-1} N_2 + 1$ are the number of nodes in the mesh $\overline{\omega}_1^{*1}$ and in the mesh ω_2^1 on a unit segment respectively). Let

$$\overline{D}_h^0 = \overline{D}_h^1 \bigcap \overline{D}_h^2. \tag{6.1b}$$

$\overline{D}_h^0 = \overline{D}_h^1$ if k is integer, $(k \geq 2)$; $\overline{D}_h^0 \neq \overline{D}_h^1$ if k is noninteger.

Let $z^i(x)$, $x \in \overline{D}_h^i$, $i = 1, 2$ be solutions of the difference schemes

$$\Lambda_{(4.2)}(z^i(x)) = 0, \qquad x \in D_h^i, \tag{6.3a}$$

$$z^i(x) = \varphi(x), \qquad x \in \Gamma_h^i, \quad i = 1, 2.$$

Assume

$$z^0(x) = \gamma \, z^1(x) + (1 - \gamma) \, z^2(x), \quad x \in \overline{D}_h^0, \tag{6.3b}$$

where

$$\gamma = \gamma(k) = -(k - 1)^{-1}. \tag{6.3c}$$

We call the function $z^0(x)$, $x \in \overline{D}_h^0$ the solution of the difference scheme (6.3), (6.1), i.e. the scheme based on the Richardson method on two embedded meshes; the functions $z^1(x)$, $x \in \overline{D}_h^1$ and $z^2(x)$, $x \in \overline{D}_h^2$ are called the components generating the solution of the difference scheme (6.3), (6.1).

The value γ in (6.3c) is defined by an expansion of the functions $z^1(x)$ and $z^2(x)$ (two first terms) with respect to N_s^{-1}, where $N_s = N_{s(4.5)}$, $s = 1, 2$. The expansions of the functions are constructed assuming that the value σ for meshes $\overline{D}_h^1$ and $\overline{D}_h^2$ is one and the same, $\sigma = \sigma_{(6.2)}(l = 2)$. Note, that the main first term in the expansions of the functions $z^1(x)$ and $z^2(x)$ is the function $u(x)$, i.e. the solution of the boundary value problem (2.2), (2.1).

For justification of convergence to Richardson scheme (6.3), (6.1) under condition (6.1), we apply a technique similar to the one used in [6]. It is suitable to consider a problem solution in the form of a decomposition.

Let us construct expansions for solutions of the difference scheme (4.2), (4.5) under the condition (6.2).

To the decomposition

$$u(x) = U(x) + V(x), \quad x \in \overline{D} \tag{6.4a}$$

of the solution of the boundary value problem (2.2), (2.1) (see, e.g., representation (3.2)) corresponds the discrete decomposition

$$z(x) = z_U(x) + z_V(x), \quad x \in \overline{D}_h \tag{6.4b}$$

of the solution of the difference scheme (4.2), (4.5), (6.2). The functions $z_U(x)$, $z_V(x)$ in the representation (6.4b) are solutions of the problems

$$\Lambda_{(4.2)}(z_U(x)) = 0, \qquad x \in D_h,$$

$$z_U(x) = U(x), \qquad x \in \Gamma_h;$$

$$\Lambda_{(4.2)}^2 z_V(x) - \left[f\big(x, z_U(x) + z_V(x)\big) - f\big(x, z_U(x)\big) \right] = 0, \qquad x \in D_h,$$

$$z_V(x) = V(x), \qquad x \in \Gamma_h.$$

We represent the function $z_V(x)$ as the sum of functions

$$z_V(x) = V(x) + N_1^{-1} V_1(x) + N_2^{-1} V_2(x) + \rho_V(x), \quad x \in \overline{D}, \tag{6.5a}$$

where $\rho_V(x)$ is a remainder term. The functions $V_1(x)$, $V_2(x)$, $x \in \overline{D}$ are founded as solutions of the boundary value problems

$$\widetilde{L}_{(6.6)} V_1(x) = -\sigma\, b_1(x) \frac{\partial^2}{\partial x_1^2} V(x), \qquad x \in D_{(1)}, \tag{6.6}$$

$$V_1(x) = 0, \quad x \in \Gamma;$$

$$\widetilde{L}_{(6.6)} V_2(x) = -2^{-1} d\, b_2(x) \frac{\partial^2}{\partial x_2^2} V(x), \quad x \in D_{(1)},$$

$$V_2(x) = 0, \quad x \in \Gamma,$$

where $\widetilde{L}_{(6.6)} \equiv L^2_{(2.2)} - f_u\big(x,\, u(x)\big)$.

Derivation of *a priori* estimates for the components $V_i(x)$, $x \in \overline{D}$, $i = 1, 2$ is similar to that for estimates (3.6b). These bounds are used for estimation of the function $\rho_V(x)$, $x \in \overline{D}_h$, i.e. the remainder term in the expansion (6.5a). For components in this expansion, we obtain the estimates

$$|V_1(x)| \le M \ln N_1, \quad |V_2(x)| \le M\,\varepsilon, \quad x \in \overline{D}; \tag{6.7a}$$

$$|\rho_V(x)| \le M \left[N_1^{-2} \ln^2 N_1 + N_2^{-2} \right], \quad x \in \overline{D}_h.$$

The function $z_U(x)$, $x \in \overline{D}_h$ can be represented in such a form

$$z_U(x) = U(x) + N_1^{-1} U_1(x) + N_2^{-1} U_2(x) + \rho_U(x), \quad x \in \overline{D}_h, \tag{6.5b}$$

where

$$U_1(x) = \sum_{k=1,2} U_1^k(x), \quad x \in \overline{D}. \tag{6.5c}$$

The functions $U_1^k(x)$, $x \in \overline{D}$ are solutions of the problems

$$\widetilde{L}_{(6.6)} U_1^1(x) = -(d - \sigma)\, b_1(x) \frac{\partial^2}{\partial x_1^2} U(x), \quad x \in D,$$

$$U_1^1(x) = 0, \quad x \in \Gamma;$$

$$\widetilde{L}_{(6.6)} U_1^2(x) = \left\{ \begin{array}{ll} (d - 2\,\sigma)\, b_1(x) \dfrac{\partial^2}{\partial x_1^2} U(x), & x_1 < \sigma \\[2mm] 0, & x_1 > \sigma \end{array} \right\}, \quad x \in \overline{D},$$

$$U_1^2(x) = 0, \quad x \in \Gamma;$$

$$\widetilde{L}_{(6.6)} U_2(x) = -2^{-1} b_2(x) \frac{\partial^2}{\partial x_2^2} U(x), \quad x \in D,$$

$$U_2(x) = 0, \quad x \in \Gamma.$$

For components in the representation (6.5c), we obtain *a priori* estimates which are used for estimation of the function $\rho_U(x)$, $x \in \overline{D}_h$. For components in the expansion $\{(6.5b), (6.5c)\}$, we obtain the estimates

$$\left|U_1^1(x)\right|,\, |U_2(x)| \le M, \qquad \left|U_1^2(x)\right| \le M\,\varepsilon \ln N_1, \tag{6.7b}$$

$$|\rho_U(x)| \le M \left[N_1^{-2} \ln^2 N_1 + N_2^{-2} \right], \quad x \in \overline{D}_h.$$

From the representation (6.4) and expansion (6.5) it follows the expansion for the function $z(x)$

$$z(x) = u(x) + N_1^{-1} \left[u_1^0(x) + u_1^1(x) \right] + N_2^{-1} \left[u_2^0(x) + u_2^1(x) \right] + \rho_u(x), \quad x \in \overline{D}_h, \tag{6.8a}$$

where

$$u_1^0(x) = U_1^1(x) + V_1(x), \quad u_1^1(x) = U_1^2(x), \quad u_2^0(x) = U_2(x), \tag{6.8b}$$

$$u_2^1(x) = V_2(x), \quad x \in \overline{D}; \quad \rho_u(x) = \rho_U(x) + \rho_V(x), \quad x \in \overline{D}_h.$$

For the components in (6.8) the following estimates hold

$$|u_i^0(x)| \leq M \ln N_1, \quad |u_i^1(x)| \leq M \varepsilon \ln N_1, \quad x \in \overline{D}, \quad i = 1, 2,$$

$$|\rho_u(x)| \leq M \left[N_1^{-2} \ln^2 N_1 + N_2^{-2} \right], \quad x \in \overline{D}_h.$$

Thus, for the function $z_{(6.3b)}^0(x)$, $x \in \overline{D}_h^0$ for $\gamma = \gamma_{(6.3c)}$ we obtain the estimate

$$|u(x) - z^0(x)| \leq M \left[N_1^{-2} \ln^2 N_1 + N_2^{-2} \right], \quad x \in \overline{D}_h^0. \tag{6.9}$$

Theorem 6.1 *Let solutions of the boundary value problem (2.2), (2.1) satisfy a priori estimates (3.6), (3.7) for $K = 7$. Then the function $z_{(6.3b)}^0(x)$, $x \in \overline{D}_h^0$, i.e. the approximation of the Richardson method on the base of solutions of difference scheme (4.2) on meshes $\overline{D}_{h(6.1)}^i$, under the conditions (6.2), (6.3c) converges for $N \to \infty$ to the boundary value problem (2.2), (2.1) ε-uniformly at the rate $\mathcal{O}\left(N_1^{-2} \ln^2 N_1 + N_2^{-2}\right)$; for the function $z(x)$, $x \in \overline{D}_h$ the expansion (6.8) is valid, and for the function $z^0(x)$, $x \in \overline{D}_h^0$ the estimate (6.9) holds.*

Remark 6.1 If the condition

$$\varepsilon \leq M N^{-1},$$

is fulfilled, then the expansion (6.8) is essentially simplified. For the function $z(x)$ the following expansion is valid

$$z(x) = u(x) + N_1^{-1} u_1^0(x) + N_2^{-1} u_2^0(x) + \rho_u(x), \quad x \in \overline{D}_h,$$

where $u_i^0(x) = u_{i(6.8)}^0(x)$, $i = 1, 2$, moreover,

$$|\rho_u(x)| \leq M \left[N_1^{-2} \ln^2 N_1 + N_2^{-2} \right], \quad x \in \overline{D}_h.$$

7. LINEARIZED ITERATIVE SCHEME OF IMPROVED ACCURACY

We now give a linearized iterative difference scheme of improved accuracy which is constructed using a Richardson technique.

On the meshes

$$\overline{G}_h^i = \overline{D}_h^i \times \overline{\omega}_0, \quad i = 1, 2, \tag{7.1a}$$

where $\overline{D}_h^i = \overline{D}_{h(6.1)}^i$, $\overline{\omega}_0 = \overline{\omega}_{0(5.1)}$, we consider the functions $z^i(x, t)$, $(x, t) \in \overline{G}_h^i$, $i = 1, 2$, i.e. solutions of the iterative schemes

$$\Lambda_{(5.1)}(z^i(x, t)) = 0, \quad (x, t) \in G_h^i \tag{7.1b}$$

$$z^i(x, t) = \psi(x, t), \quad (x, t) \in S_h^i, \quad i = 1, 2;$$

here $\psi(x, t) = \psi_{(5.1)}(x, t)$, $(x, t) \in S_h^i$.

On the set

$$\overline{G}_h^0 \equiv \overline{G}_h^1 \cap \overline{G}_h^2 = \overline{D}_h^0 \times \overline{\omega}_0, \tag{7.1c}$$

where $\overline{D}_h^0 = \overline{D}_{h(6.1)}^0$, we define the function

$$z^0(x, t) = \gamma z^1(x, t) + (1 - \gamma) z^2(x, t), \quad (x, t) \in \overline{G}_h^0, \tag{7.1d}$$

where $\gamma = \gamma_{(6.3)}$. We call the function $z^0(x, t)$, $(x, t) \in \overline{G}_h^0$, $\overline{G}_h^0 = \overline{G}_h^0(\overline{D}_{h(4.1)}^0)$ the solution of the linearized difference scheme (7.1), (6.1), i.e. linearized iterative scheme on the base of the Richardson method on two embedded meshes (meshes $\overline{D}_h^1$ and $\overline{D}_h^2$).

For the function $z^0(x, t)$, by virtue of estimate (5.2), we have

$$|z^0(x) - z^0(x, t)| \le M \, q^t, \quad (x, t) \in \overline{G}_h^0, \tag{7.2}$$

where $z^0(x)$, $x \in \overline{D}_h^0$ is the solution of nonlinear improved Richardson difference scheme (6.3), (6.1), $q \le q_{0(5.2)}$. Taking into account estimate (6.9), we obtain

$$|u(x) - z^0(x, t)| \le M \, [N_1^{-2} \ln^2 N_1 + N_2^{-2} + q^t], \quad (x, t) \in \overline{G}_h^0, \tag{7.3}$$

where $q \le q_{0(5.2)}$.

We consider how to use the upper and lower solutions for estimation of solutions of the nonlinear Richardson difference scheme.

Note that for the functions $z^i(x)$, $x \in \overline{D}_h^i$, $i = 1, 2$, i.e. components generating the solution of difference scheme (6.3), (6.1), the estimate is valid

$$|z^i(x)| \le c_{10}^{-1} \max_{\overline{D}} |f(x, 0)|, \quad x \in \overline{D}_h^i, \quad i = 1, 2,$$

where $c_{10} = c_{10(5.2)}$. We will denote by $z^{(1)i}(x, t)$, $z^{(2)i}(x, t)$, $(x, t) \in \overline{G}_h^i$, $i = 1, 2$ the solution of problem (5.1) on the mesh $\overline{D}_{h(6.1)}^i$, satisfying at the "initial moment" the condition

$$z^{(1)i}(x, 0) \le z^i(x) \le z^{(2)i}(x, 0), \quad x \in \overline{D}_h^i, \quad i = 1, 2,$$

where $z^i(x)$, $x \in \overline{D}_h^i$ is the solution of nonlinear base difference scheme (4.2) on the meshes $\overline{D}_h^i$, $i = 1, 2$. For the functions $z^i(x)$, $x \in \overline{D}_h^i$, the estimate holds true

$$z^{(1)i}(x, t) \le z^i(x) \le z^{(2)i}(x, t), \quad (x, t) \in \overline{G}_h^i, \quad i = 1, 2.$$

We call the functions $z^{(1)i}(x, t)$ and $z^{(2)i}(x, t)$, $(x, t) \in \overline{G}_h^i$ the lower and upper solutions (more precisely, sequence of solutions) of nonlinear base difference scheme (4.2) on the meshes $\overline{D}_h^i$, $i = 1, 2$ from (6.1).

We introduce the functions $z^{[1]0}(x, t)$, $z^{[2]0}(x, t)$, $(x, t) \in \overline{G}_h^0$, where

$$z^{[1]0}(x, t) = \gamma \, z^{(2)1}(x, t) + (1 - \gamma) \, z^{(1)2}(x, t),$$

$$z^{[2]0}(x, t) = \gamma \, z^{(1)1}(x, t) + (1 - \gamma) \, z^{(2)2}(x, t), \quad (x, t) \in \overline{G}_h^0, \quad \gamma = \gamma_{(6.3)}.$$

For the functions $z^{[1]0}(x, t)$, $z^{[2]0}(x, t)$ the estimates are valid

$$z^{[1]0}(x, t) \le z^0(x) \le z^{[2]0}(x, t), \quad (x, t) \in \overline{G}_h^0.$$

Thus, the functions $z^{[1]0}(x, t)$ and $z^{[2]0}(x, t)$, $(x, t) \in \overline{G}_h^0$ are lower and upper, respectively, solutions of the scheme (6.3), (6.1), i.e. nonlinear Richardson difference scheme of improved accuracy.

Note that

$$0 \le z^{[2]0}(x, t) - z^{[1]0}(x, t) \le M \, q^t, \quad (x, t) \in \overline{G}_h^0,$$

where $q \le q_{0(5.2)}$.

We will use the upper and lower solutions of improved nonlinear scheme (6.3), (6.1) in order to define the number of iterations ensuring the same accuracy of linearized iterative solutions as it is for the scheme (6.3), (6.1).

For solutions of the linearized iterative difference scheme (7.1), (6.1) we obtain the estimate

$$|u(x) - z^{[j]0}(x, t)| \leq M \left[N_1^{-2} \ln^2 N_1 + N_2^{-2} \right] + z^{[2]0}(x, t) - z^{[1]0}(x, t), \tag{7.4}$$

$$(x, t) \in \overline{G}_h^0, \quad j = 1, 2.$$

Also we have the two-sided estimate

$$z^{[1]0}(x, t) - M \left[N_1^{-2} \ln^2 N_1 + N_2^{-2} \right] \leq u(x), z^0(x) \leq \tag{7.5}$$

$$\leq z^{[2]0}(x, t) + M \left[N_1^{-2} \ln^2 N_1 + N_2^{-2} \right], \quad (x, t) \in \overline{G}_h^0.$$

The error of the solution of the iterative scheme (7.1), (6.1) can be represented as the sum

$$z^{[j]0}_{(7.1;6.1)}(x, t) - u(x) = \left(z^0_{(6.3;6.1)}(x) - u(x) \right) + \left(z^{[j]0}_{(7.1;6.1)}(x, t) - z^0_{(6.3;6.1)}(x) \right),$$

$$(x, t) \in \overline{G}_h^0, \quad j = 1, 2.$$

We choose the value T, i.e. the number of iterations in the scheme (7.1), (6.1), so that the error of the solution of the scheme (6.3), (6.1) and a difference between the solution of the iterative scheme (7.1), (6.1) and the solution of the nonlinear scheme (6.3), (6.1) were commensurable.

We call the functions $z^{[j]0}_{(7.1;6.1)}(x, T)$, $x \in \overline{D}_h^0$ the (upper for $j = 2$ and lower for $j = 1$) solution of scheme (7.1), (6.1), consistent with respect to accuracy (of the improved nonlinear scheme (6.3), (6.1)) and to the number of iterations (of the improved linearized scheme (7.1), (6.1)).

The value T is defined by the relations

$$\max_{\overline{D}_h^0} \left[z^{[2]0}(x, t) - z^{[1]0}(x, t) \right] > M_1 \left[N_1^{-2} \ln^2 N_1 + N_2^{-2} \right], \tag{7.6}$$

$$\max_{\overline{D}_h^0} \left[z^{[2]0}(x, T) - z^{[1]0}(x, T) \right] \leq M_1 \left[N_1^{-2} \ln^2 N_1 + N_2^{-2} \right], \quad x \in \overline{D}_h^0, \quad t < T.$$

For the consistent solution of the linearized iterative difference scheme (7.1), (6.1) the estimate is valid

$$|u(x) - z^{[j]0}(x, T)| \leq M_2 \left[N_1^{-2} \ln^2 N_1 + N_2^{-2} \right], \quad x \in \overline{D}_h^0, \quad j = 1, 2; \tag{7.7a}$$

and also, for the number of iterations T the following estimate holds

$$T \leq M_3 \left(\ln q_0^{-1} \right)^{-1} \ln N, \tag{7.7b}$$

where $q_0 = q_{0(5.2)}$, constants $M_{1(7.6)}$, $M_{2(7.7)}$, $M_{3(7.7)}$ are independent of q_0; the value T is defined in the computational process according to the relations (7.6).

Theorem 7.1 *Let hypothesis of Theorem* 6.1 *be fulfilled. Then the solution of the linearized iterative difference scheme* (7.1), (6.1) *for* $N_1, N_2, t \to \infty$ *converges to the solution of the boundary value problem* (2.2), (2.1) ε-*uniformly at the rate* $\mathcal{O} \left(N_1^{-2} \ln^2 N_1 + N_2^{-2} + q_0^t \right)$, *where* $q_0 = q_{0(5.2)}$. *For discrete solutions the estimates* (7.2), (7.3), (7.4), (7.5), (7.7) *are valid.*

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