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ABSTRACT
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A Lévy Process Reflected at a Poisson Age Process

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Abstract

We consider a Lévy process with no negative jumps, reflected at a stochastic boundary which is a positive constant multiple of an age process associated with a Poisson process. We show that the stability condition for this process is identical to the one for the case of reflection at the origin. In particular, there exists a unique stationary distribution which is independent of initial conditions. We identify the Laplace-Stieltjes transform of the stationary distribution and observe that it satisfies a decomposition property. In fact, it is a sum of two independent random variables, one of which has the stationary distribution of the process reflected at the origin, and the other has the stationary distribution of a certain clearing process. The latter is itself distributed like an infinite sum of independent random variables. Finally, we discuss the tail behavior of the stationary distribution and in particular observe that the second distribution in the decomposition always has a light tail.

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AMS Subject Classification: Primary 60K05; Secondary 60K25

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1 Introduction

In various fields, like queueing and risk theory, it is natural to study a Lévy process that is reflected at the origin. In this paper we study a Lévy process with no negative jumps, that is reflected at a *stochastic* lower barrier. This barrier is a straight line, which drops to zero after exponentially distributed time periods, and then increases again linearly at rate a. Put differently: This barrier is a positive multiple of the age process of an independent Poisson process. One can also view it as a clearing process that increases linearly at some fixed rate a and at event epochs of the Poisson process drops to zero (clears all the content from the system).

Applications where such a reflected Lévy process is natural are models where in addition to the input and output mechanism there is a constant input which is not available on liquid basis, but can only be used after some maturity date has been reached. In our case this maturity date is exponentially distributed. For example, one considers the combined behavior of two financial accounts, viz., a daily account and a savings account. The content of the daily account behaves like a Lévy process. The content of the savings account grows linearly at rate a, and is moved to the daily account at exponentially distributed intervals. It is not allowed to let the daily account become negative. The combined level of the two accounts now behaves like a Lévy process reflected at a stochastic lower barrier. As will be shown, already for such a "simple" model the analysis is not trivial.

We compute the stationary distribution of such a reflected Lévy process (when it exists) and observe that it satisfies a decomposition property. That is, it is the distribution of a sum of two independent random variables. The first has the stationary distribution of a Lévy process with no negative jumps reflected at the *origin* and the second is an independent infinite sum of random variables which also corresponds to the stationary distribution of a certain clearing process associated with the local time of the process.

The paper is organized in the following way. In Section 2 we study the process Z, a Lévy process reflected at the origin, and its local time process L. Using martingale methods, we derive the joint distribution of $Z(T)$, $L(T)$ and T, where T (the clearing time) is exponentially distributed (Theorem 1). In Section 3 we turn to the Lévy process W reflected at the above-described stochastic lower barrier, or age process A: A straight line, which drops to zero at event epochs of a Poisson process and then increases again linearly at rate a. We determine necessary and sufficient conditions for the two-dimensional process (W, A) to have a unique stationary distribution and a limiting distribution which is independent of initial conditions (Theorem 2). Using Theorem 1, the stationary distribution of W is derived in Section 4 (Theorem 3); we show in particular that it satisfies a decomposition property, which is discussed at length in Section 5. The tail behavior of the stationary distribution of W is analyzed in Section 6 (Theorem 4), exploiting detailed knowledge of the two parts of the decomposition.

2 Preliminaries

Let X be a right continuous Lévy process which is not almost surely nondecreasing (i.e.,not a subordinator) starting at an arbitrary initial value with no negative jumps and Laplace exponent $\varphi(\alpha) = \log Ee^{-\alpha X(1)}$. Let $L(t) = -\inf_{0\le s\le t} X^-(s)$ and $Z(s) = X(s) + L(s)$. Letting $T \sim \exp(\gamma)$, independently of X, we would like to identify the joint distribution of $(Z(T), L(T), T)$ for any given initial value $z \ge 0$. Let P_z and E_z denote the probability and expected value when X starts from z, that is $P_z[X(\cdot) \in A] = P_0[z + X(\cdot) \in A]$.

Applying [6] to $X+Y$ where $Y(t) = z + (1+\beta/\alpha)L(t)$, noting that $Z(t) = 0$ for points of increase of L, and simplifying, the following is a martingale:

$$\begin{aligned} M(t) &= \varphi(\alpha)\int_0^t e^{-(\alpha Z(s)+\beta L(s))}\mathrm{d}s \\ &+e^{-\alpha Z(0)} - e^{-(\alpha Z(t)+\beta L(t))} - (\alpha+\beta)\int_0^t e^{-\beta L(s)}\mathrm{d}L(s)\ . \end{aligned} \tag{1}$$

Clearly,

$$\int_0^t e^{-\beta L(s)}\mathrm{d}L(s) = \frac{1-e^{-\beta L(t)}}{\beta}. \tag{2}$$

For a bounded jointly measurable process $U = \{U(t)|\ t \ge 0\}$ and $T \sim \exp(\lambda)$ which is independent of U,

$$\lambda E\int_0^T U(s)\mathrm{d}s = \lambda E\int_0^\infty U(s)1_{\{T>s\}}\mathrm{d}s = \int_0^\infty EU(s)\lambda e^{-\lambda s}\mathrm{d}s = EU(T)\ . \tag{3}$$

In particular, if T is independent of X (and thus of (Z, L)), it follows that

$$\lambda E_z \int_0^T e^{-(\alpha Z(s)+\beta L(s))} \mathrm{d}s = E_z e^{-(\alpha Z(T)+\beta L(T))} . \tag{4}$$

Since $E_z M(T \wedge t) = 0$, where $a \wedge b = \min(a, b)$, applying bounded and monotone convergence theorems in the appropriate places, it follows that $E_z M(T) = 0$. Thus, applying (4) to $E_z M(T)$ and simplifying leads to the following identity:

$$\frac{\varphi(\alpha) - \lambda}{\lambda} E_z e^{-(\alpha Z(T)+\beta L(T))} = -e^{-\alpha z} + \frac{\alpha + \beta}{\beta} \left(1 - E_z e^{-\beta L(T)}\right) . \tag{5}$$

Noting that φ is convex (thus continuous) with $\varphi(0) = 0$, and recalling that X is not a subordinator, $\varphi(\alpha) \to \infty$ when $\alpha \to \infty$, we have that for every positive number x there is a unique α for which $\varphi(\alpha) = x$. Let us denote this α by $\psi(x)$. In particular, for $x = \lambda$ we obtain that

$$0 = -e^{-\psi(\lambda)z} + \frac{\psi(\lambda) + \beta}{\beta} \left(1 - E_z e^{-\beta L(T)}\right) . \tag{6}$$

Thus

$$\frac{\varphi(\alpha) - \lambda}{\lambda} E_z e^{-(\alpha Z(T)+\beta L(T))} = -e^{-\alpha z} + \frac{\alpha + \beta}{\psi(\lambda) + \beta} e^{-\psi(\lambda)z}. \tag{7}$$

As a consequence we can write

$$E_z e^{-(\alpha Z(T)+\beta L(T))} = e^{-\psi(\lambda)z} \frac{1 - \frac{\alpha}{\psi(\lambda)}}{1 - \frac{\varphi(\alpha)}{\lambda}} \frac{\psi(\lambda)}{\psi(\lambda) + \beta} + \frac{e^{-\alpha z} - e^{-\psi(\lambda)z}}{1 - \frac{\varphi(\alpha)}{\lambda}}. \tag{8}$$

Setting $z = 0$ we have that

$$E_0 e^{-(\alpha Z(T)+\beta L(T))} = E_0 e^{-\alpha Z(T)} E_0 e^{-\beta L(T)} = \frac{1 - \frac{\alpha}{\psi(\lambda)}}{1 - \frac{\varphi(\alpha)}{\lambda}} \frac{\psi(\lambda)}{\psi(\lambda) + \beta}. \tag{9}$$

Note that the distribution of $L(T)$ and the Laplace-Stieltjes transform (LST) of $Z(T)$ for $z = 0$ can be deduced from Corollary 2 on page 190 and Equation (3) on page 192 of [3]. We also note that if $\tau^0 = -\inf\{t|X(t) = 0\}$, then $P_z[T > \tau^0] = E_z e^{-\lambda\tau^0} = e^{-\psi(\lambda)z}$. Recall that when $\varphi'(0) < 0$, $\alpha^* = \psi(0+) > 0$, in which case $P_z[\tau^0 = \infty] = 1 - e^{-\alpha^* z}$. It is clear from the memoryless property of T and the Markovian structure of (X, Z, L) that

$$E_z e^{-(\alpha Z(T)+\beta L(T))} 1_{\{T>\tau^0\}} = P_z[T > \tau^0] E_0 e^{-(\alpha Z(T)+\beta L(T))} . \tag{10}$$

Thus the first term on the right of (8) is the right side of (10).

Now, when $T \leq \tau^0$, $L(T) = 0$, so

$$\begin{aligned} E_z e^{-(\alpha Z(T)+\beta L(T))} 1_{\{T\leq\tau^0\}} &= E_0 e^{-(\alpha(z+X(T))} 1_{\{T\leq\tau^0\}} \\ &= e^{-\alpha z} E_0 e^{-\alpha X(T)} - E_0 e^{-\alpha(z+X(T))} 1_{\{T>\tau^0\}} \\ &= e^{-\alpha z} E_0 e^{-\alpha X(T)} - P_0[T > \tau^0] E_0 e^{-\alpha X(T)} \\ &= (e^{-\alpha z} - e^{-\psi(\lambda) z}) E_0 e^{-\alpha X(T)} \ . \end{aligned} \tag{11}$$

For values of α for which $\varphi(\alpha) < \lambda$,

$$E_0 e^{-\alpha X(T)} = \int_0^\infty e^{\varphi(\alpha)t} \lambda e^{-\lambda t} dt = \frac{1}{1 - \frac{\varphi(\alpha)}{\lambda}} \ . \tag{12}$$

Thus, for small values of α, the rightmost expression of (11) is the second term on the right of (8). However, we emphasize that this second term also holds for large values of α. In particular, for $\alpha = \psi(\lambda)$ we have that

$$\begin{aligned} E_z e^{-(\psi(\lambda)Z(T)+\beta L(T))} &= e^{-\psi(\lambda)z} \frac{\frac{1}{\psi(\lambda)}}{\frac{\varphi'(\psi(\lambda))}{\lambda}} \frac{\psi(\lambda)}{\psi(\lambda)+\beta} + \frac{z e^{-\psi(\lambda)z}}{\frac{\varphi'(\psi(\lambda))}{\lambda}} \\ &= \lambda \psi'(\lambda) e^{-\psi(\lambda)z} \left(\frac{1}{\psi(\lambda)+\beta} + z \right) \ . \end{aligned} \tag{13}$$

Now, consider the probability measure $Q_\gamma(A) = E_z e^{-\gamma T} 1_{\{A\}} / E_z e^{-\gamma T}$, noting that $E_z e^{-\gamma T} = \lambda/(\lambda+\gamma)$. It is easy to check that under Q_γ, T and X are independent, X is a Lévy process with the same exponent φ and the same initial value as under the original measure and $T \sim \exp(\lambda+\gamma)$. Thus $E_z e^{-(\alpha Z(T)+\beta L(T)+\gamma T)} / E_z e^{-\gamma T}$ is given by (8), only that λ is replaced by $\lambda+\gamma$ throughout. As a consequence, we have the following result.

Theorem 1 *Let X be a Lévy process with no negative jumps, which is not a subordinator, with exponent φ and $\psi(\alpha_0) = \inf\{\alpha | \varphi(\alpha) > \alpha_0\}$ for $\alpha_0 \geq 0$. Then for*

$T \sim \exp(\lambda)$ *which is independent of* X*,*

$$E_z e^{-(\alpha Z(T)+\beta L(T)+\gamma T)}$$

$$= \left(e^{-\psi(\lambda+\gamma)z} \frac{1 - \frac{\alpha}{\psi(\lambda+\gamma)}}{1 - \frac{\varphi(\alpha)}{\lambda+\gamma}} \frac{\psi(\lambda+\gamma)}{\psi(\lambda+\gamma)+\beta} + \frac{e^{-\alpha z} - e^{-\psi(\lambda+\gamma)z}}{1 - \frac{\varphi(\alpha)}{\lambda+\gamma}} \right) \frac{\lambda}{\lambda+\gamma} \tag{14}$$

for all nonnegative z, α, β, γ*.*

3 Stability condition

Consider now our Lévy process which is reflected along a boundary of the form $A(t) = A(0)1_{\{T_1>t\}} + a(t - T_{N(t)})$, where $a > 0$, N is a Poisson process with rate λ and arrival epochs T_n and is independent of the Lévy process and $A(0)$ is some nonnegative random variable (possibly zero) which is independent of everything else. That is, at each point the boundary increases linearly at a rate of a and at the arrival epochs of the Poisson process it jumps back to zero, but at time zero it may start at some arbitrary nonnegative value.

The appropriate stability condition for such a Lévy process is that $\varphi'(0) > 0$. To see this we observe that the process is of the form $W(t) = W(0) + X(t) + L(t)$ where $L(0) = 0$, L is nondecreasing, $W(t) \geq A(t)$ and L can increase only at points t for which $W(t) = A(t)$. Thus $W(t) - A(t) = W(0) + X(t) - A(t) + L(t)$ is a reflected process with driver $X(t) - A(t)$. Therefore

$$W(t) - A(t) = W(0) + X(t) - A(t) - \inf_{0\leq s\leq t}(W(0) + X(s) - A(s)) \wedge 0 \ . \tag{15}$$

Since $X(t)/t \to -\varphi'(0)$ and $A(t)/t \to 0$, both almost surely, it follows that there is some finite $T > T_1$ for which $W(0) + X(T) - A(T) \leq 0$. Hence for $t \geq T$ the minimization with 0 can be omitted. After cancelling $W(0)$ on the right side and $A(t)$ from both sides we have for $t > T$ that

$$W(t) = X(t) - \inf_{0\leq s\leq t}(X(s) - A(s)) \ . \tag{16}$$

Also, since $X(t) - A(t) \to -\infty$, then for sufficiently large t the infimum depends on values of s which are larger than T_1 and thus for such values of t, $(W(t), A(t))$

does not depend on the value of $A(0)$. This means that for any initial $W(0)$ and $A(0)$ the process $(W(t), A(t))$ can be coupled with the process which starts with $W(0) = A(0) = 0$ and thus if a limiting distribution exists, then it does not depend on initial conditions.

To show that a limiting distribution exists we start the process with $W(0) = A(0)$ where $A(0) \sim \exp(\lambda/a)$. With this choice, $A(t)$ becomes a stationary process. In fact, $A(t)/a$ is a stationary version of the age process associated with the Poisson process. Let us first extend both the Lévy process X and A to the whole real line. Since $W(0) - A(0) = 0$ we have that

$$W(t) = X(t) - \inf_{0 \le s \le t} (X(s) - A(s)) = \sup_{0 \le s \le t} (X(t) - X(t-s) + A(t-s)) \tag{17}$$

and upon shifting by t and recalling the stationarity of A and the (strong) stationary increments property of X

$$W(t) \sim S(t) \equiv \sup_{0 \le s \le t} (-X(-s) + A(-s)) \ , \tag{18}$$

and in fact $(W(t), A(t)) \sim (S(t), A(0))$. In particular, $W(t)$ is stochastically increasing.

It is well known that $\{-X(-s) | s \ge 0\}$ is a (left-continuous) Lévy process with the same exponent as X and $A(-s)$ is a left-continuous version of the process $R(t) = a(T_{N(t)+1} - t)$, where we note that R/a is the residual lifetime process associated with N. As the supremum does not depend on whether we take the right or left continuous versions (in particular the end points are a.s. points of continuity), we see that $(W(t), A(t))$ has the same distribution as

$$\left(\sup_{0 \le s \le t} (X(s) + R(s)), R(0) \right) \tag{19}$$

where X and R are independent processes. Recalling that $X(t)/t \to -\varphi'(0) < 0$ and noting that $R(t)/t \to 0$ both almost surely, it follows that $X(t)+R(t) \to -\infty$ almost surely and thus $W(t)$ converges in distribution to $M(\infty) = \sup_{s \ge 0}(X(s) + R(s))$, where $M(\infty)$ is an almost surely finite random variable.

In fact, we observe that the process (W^*, A), in which

$$\begin{aligned} W^*(t) &= \sup_{s \le 0}(X(t) - X(t+s) + A(t+s)) \\ &= X(t) - \inf_{s \le t}(X(s) - A(s)) \qquad (20) \\ &= W^*(0) + X(t) - \inf_{0 \le s \le t}(W^*(0) + X(s))^-, \end{aligned}$$

is a stationary version of W. Also we observe that if $\varphi'(0) \le 0$, then W is above the process $W(0) + X(t) - \inf_{0 \le s \le t}(W(0) + X(s))^-$, which is not positive recurrent and thus neither is W. Thus, we have shown the following.

Theorem 2 *The Markov process $\{(W(t), A(t)) |\ t \ge 0\}$ is positive Harris recurrent (that is, it has a unique stationary distribution and a limiting distribution which is independent of initial conditions) if and only if $\varphi'(0) > 0$.*

Now let $Z(t) = W(t) - at = W(0) + X(t) - at + L(t)$. Until T_1, (Z, L) is a reflected Lévy process with driving process $X(t) - at$, having the exponent $\tilde{\varphi}(\alpha) = \varphi(\alpha) + a\alpha$ which satisfies $\tilde{\varphi}'(0) > a$. Thus $W(T)$ is distributed like $W(0)$ if and only if $Z(T) + aT$ is distributed like $Z(0)$ and due to Poisson Arrivals See Time Averages (PASTA) this would also be the time stationary distribution of W. In the following section we will find the distribution of $Z(0)$ for which $Z(T) + aT \sim Z(0)$, relying on Theorem 1.

4 When does $Z(0) \sim Z(T) + aT$?

As described in the previous section, the key to computing the stationary distribution of W is finding an initial distribution such that $Z(0)$ and $Z(T) + aT$ are identically distributed. For ease of notation we will use the notation φ rather than $\tilde{\varphi}$, where $\tilde{\varphi}(\alpha) = \varphi(\alpha) + a\alpha$ throughout, hence we assume that $\varphi'(0) > a$. Also here we let E denote the expectation associated with the initial distribution that we are seeking.

Set $\gamma = \alpha a$ and $\beta = 0$ in (14) and simplify to obtain the following:

$$(\varphi(\alpha) - \alpha a)\frac{Ee^{-\alpha Z(0)}}{\alpha} = \lambda \frac{Ee^{-\psi(\lambda + \alpha a)Z(0)}}{\psi(\lambda + \alpha a)} \ , \qquad (21)$$

and in particular, by taking $\alpha \downarrow 0$,

$$\varphi'(0) - a = \lambda \frac{Ee^{-\psi(\lambda)Z(0)}}{\psi(\lambda)} \ . \tag{22}$$

Since the right side is positive, we see that a necessary condition for the existence of a distribution of $Z(0)$ for which $Z(T) + aT \sim Z(0)$ is indeed that $\varphi'(0) > a$. Letting $f(\lambda, a, \alpha) = Ee^{-\alpha Z(0)}$, where $Z(0)$ has the stationary distribution that we seek and noting that $\psi'(0) = 1/\varphi'(0)$, we obtain that

$$\lim_{\lambda \downarrow 0} f(\lambda, a, \psi(\lambda)) = 1 - \frac{a}{\varphi'(0)} < 1 \ . \tag{23}$$

It may seem like (23) implies that $P[Z(0) < \infty] < 1$, but it does not as the distribution of $Z(0)$ depends on λ.

If we let $a = 0$ then we obtain

$$\varphi'(0) = \frac{\lambda}{\psi(\lambda)} Ee^{-\psi(\lambda)Z(0)} \ , \tag{24}$$

and thus $Ee^{-\alpha Z(0)} = \varphi'(0)\alpha/\varphi(\alpha)$ (generalized Pollaczek-Khinchin formula) as expected, since then Z is reflected at the origin and in this case the continuous-time process has the same stationary distribution as the process sampled at Poisson epochs.

Now, let $h(\alpha) = \psi(\lambda + a\alpha)$ and assume that $\varphi'(0) > a$. Then, since ψ is concave, $h'(0) = a\psi'(\lambda) < a\psi'(0) = a/\varphi'(0) < 1$ and since h is concave, it is Lipshitz continuous with $|h(\alpha) - h(\beta)| \le h'(0)|\alpha - \beta|$ and is thus a contraction. Denoting $h^0(\alpha) = \alpha$ and $h^{i+1}(\alpha) = h(h^i(\alpha))$, we thus have that for any $\alpha \ge 0$, $h^i(\alpha) \to \alpha^*$ as $i \to \infty$, where $\alpha^* = h(\alpha^*)$. That is, if we denote by $\varphi_a(\alpha) = \varphi(\alpha) - a\alpha$ and $\psi_a(\beta)$ is its inverse, then it is easy to check that $\alpha^* = \psi_a(\lambda)$.

With this in mind, (21) may be rewritten as follows

$$Ee^{-\alpha Z(0)} = \frac{\lambda}{\varphi_a(\alpha)} \frac{\alpha E e^{-h(\alpha)Z(0)}}{h(\alpha)} \ , \tag{25}$$

which, for $n \ge 1$, immediately leads to

$$Ee^{-\alpha Z(0)} = \frac{\lambda \alpha E e^{-h^{n+1}(\alpha)Z(0)}}{\varphi_a(\alpha) h^{n+1}(\alpha)} \prod_{i=1}^{n} \frac{\lambda}{\varphi_a(h^i(\alpha))} \ . \tag{26}$$

Upon substituting $\alpha = 0$, we have that

$$1 = \frac{\lambda E e^{-h^{n+1}(0)Z(0)}}{\varphi_a'(0) h^{n+1}(0)} \prod_{i=1}^{n} \frac{\lambda}{\varphi_a(h^i(0))} \; . \tag{27}$$

Therefore

$$Ee^{-\alpha Z(0)} = \frac{\alpha \varphi_a'(0)}{\varphi_a(\alpha)} \frac{h^{n+1}(0) E e^{-h^{n+1}(\alpha)Z(0)}}{h^{n+1}(\alpha) E e^{-h^{n+1}(0)Z(0)}} \prod_{i=1}^{n} \frac{\varphi_a(h^i(0))}{\varphi_a(h^i(\alpha))} \; . \tag{28}$$

Observe that $h^{n+1}(\alpha)$ and $h^{n+1}(0)$ both converge to $\alpha^* = \psi_a(\lambda)$ as $n \to \infty$. Also, since $\varphi_a'(0) = \varphi'(0) - a > 0$ it holds that φ_a is an increasing function on $[0, \infty)$. By induction, the h^i are also increasing and thus, for every i, $\varphi_a(h^i(0))/\varphi_a(h^i(\alpha)) < 1$ for all $\alpha > 0$. Thus the product converges to a proper limit and we obtain the following decomposition result.

Theorem 3 *For all $\alpha \geq 0$,*

$$Ee^{-\alpha Z(0)} = \frac{\varphi_a'(0)\alpha}{\varphi_a(\alpha)} \prod_{i=1}^{\infty} \frac{\varphi_a(h^i(0))}{\varphi_a(h^i(\alpha))} \; . \tag{29}$$

5 Interpretation of the decomposition result

This section is devoted to understanding the right side of the decomposition (29) in Theorem 3, and in particular to showing that it is the LST of a proper random variable.

It is easy to check that $\varphi_a(h^i(\alpha)) = \lambda + (h^{i-1}(\alpha) - h^i(\alpha))a$ and thus, we may rewrite (29) as

$$Ee^{-\alpha Z(0)} = \frac{\varphi_a'(0)\alpha}{\varphi_a(\alpha)} \prod_{i=1}^{\infty} \frac{\lambda + (h^{i-1}(0) - h^i(0))a}{\lambda + (h^{i-1}(\alpha) - h^i(\alpha))a} \; . \tag{30}$$

In particular, recalling that $\varphi_a(\alpha^*) = \lambda$ and $h^i(\alpha^*) = \alpha^*$,

$$Ee^{-\alpha^* Z(0)} = \frac{\varphi_a'(0)\alpha^*}{\lambda} \prod_{i=1}^{\infty} \left(1 + (h^{i-1}(0) - h^i(0))\frac{a}{\lambda}\right) . \tag{31}$$

Since $\alpha < \alpha^* = \psi_a(\lambda)$ is equivalent to $\varphi(\alpha) - a\alpha < \lambda$, which is in turn equivalent to $\alpha < \psi(\lambda + a\alpha) = h(\alpha)$, we have that $h^i(\alpha)$ is an increasing sequence. Similarly when $\alpha > \alpha^*$ it is decreasing.

To show that the right side of (30) is the LST of a proper distribution it suffices to show that the second term (the product term) converges to one as $\alpha \downarrow 0$. To see this we first give the following simple lemma.

Lemma 1 *Let $a_i : [0,\infty) \to [0,\infty)$ be such that $a_i(\alpha) \to 0$ as $\alpha \downarrow 0$, $a_i(\alpha) \leq a_i b(\alpha)$, where $b(\alpha)$ is bounded in some neighborhood of zero, and $\sum_{i=1}^{\infty} a_i < \infty$. Then*

$$\lim_{\alpha \downarrow 0} \prod_{i=1}^{\infty} \frac{1}{1 + a_i(\alpha)} = 1 \ . \tag{32}$$

Proof: For a given $\epsilon > 0$ choose N such that $\sum_{i=N+1}^{\infty} a_i < \epsilon$. Since $1 + a_i(\alpha) \leq 1 + a_i b(\alpha) \leq e^{a_i b(\alpha)}$ we have that

$$\prod_{i=N+1}^{\infty} \frac{1}{1 + a_i(\alpha)} \geq e^{-b(\alpha)\sum_{i=N+1}^{\infty} a_i} \geq e^{-b(\alpha)\epsilon} \geq 1 - b(\alpha)\epsilon \ . \tag{33}$$

Clearly

$$\lim_{\alpha \downarrow 0} \prod_{i=1}^{N} \frac{1}{1 + a_i(\alpha)} = 1 \tag{34}$$

for each N and thus, as ϵ is arbitrary and b is bounded in some neighborhood of zero, the proof is complete. ∎

Now, observe that if we let

$$\frac{\lambda + (h^{i-1}(0) - h^i(0))a}{\lambda + (h^{i-1}(\alpha) - h^i(\alpha))a} = \frac{1}{1 + a_i(\alpha)} \ , \tag{35}$$

then

$$a_i(\alpha) = \frac{\frac{a}{\lambda}(h^{i-1}(\alpha) - h^{i-1}(0) + h^i(0) - h^i(\alpha))}{1 + \frac{a}{\lambda}(h^{i-1}(0) - h^i(0))} \ . \tag{36}$$

Recall that h is contracting, so that $|h^n(\alpha) - h^n(0)| \leq \rho^n \alpha$, where $\rho = h'(0) = a\psi'(\lambda) < 1$. In particular $|h^{i-1}(0) - h^i(0)| \leq \rho^{i-1} h(0)$. Therefore

$$a_i(\alpha) \leq \frac{\frac{a}{\lambda}(1+\rho)\rho^{i-1}\alpha}{1 - \frac{a}{\lambda}\rho^{i-1}h(0)} \ . \tag{37}$$

Let i_0 be such that $\frac{a}{\lambda}\rho^{i_0-1}h(0) < 1/2$, $a_i = \rho^{i-1}$ and

$$b(\alpha) = 2\left(\frac{a}{\lambda}(1+\rho)\alpha\right)\max_{1\le i<i_0}\frac{a_i(\alpha)}{\rho^{i-1}} \ . \tag{38}$$

Then the conditions of Lemma 1 are satisfied and thus if the right side of (30) is an LST, it is the LST of a proper distribution.

Let us now argue that it is indeed an LST.

Lemma 2 *$e^{-h^n(\alpha)z} = Ee^{-(\lambda\xi_n(z)+\alpha\eta_n(z))}$ where (ξ_n, η_n) is a two dimensional nondecreasing Lévy process with exponent f_n satisfying $f_n(x,y) = \psi(x + af_{n-1}(x,y))$, so that $f_n(\lambda,\alpha) = h^n(\alpha)$. Thus, for every nonnegative random variable ζ,*

$$\frac{Ee^{-h^n(\alpha)\zeta}}{Ee^{-h^n(0)\zeta}} = E[e^{-\alpha\eta_n(\zeta)}|\xi_n(\zeta) < T] \tag{39}$$

where T is an independent exponential random variable with rate λ. Therefore, the left side is an LST for any distribution of ζ.

Proof: Assume that X_i are i.i.d. Lévy processes all distributed like X. For $z \ge 0$ let $\tau_i(z) = \inf\{t|X_i(t) + z = 0\}$. Then it is well known that τ_i are (independent) subordinators (nondecreasing Lévy processes) with $Ee^{-\alpha\tau_i(z)} = e^{-\psi(\alpha)z}$. In particular,

$$e^{-h^1(\alpha)z} = e^{-\psi(\lambda+a\alpha)z} = Ee^{-(\lambda+a\alpha)\tau_1(z)}, \tag{40}$$

$$e^{-h^2(\alpha)z} = e^{-\psi(\lambda+ah^1(\alpha))z} = Ee^{-(\lambda+ah^1(\alpha))\tau_2(z)} = Ee^{-(\lambda\tau_2(z)+(\lambda+a\alpha)\tau_1(a\tau_2(z)))}; \tag{41}$$

by induction it is easy to check that

$$e^{-h^n(\alpha)z} = Ee^{-\left(\frac{\lambda}{a}\xi_n(z)+\alpha\eta_n(z)\right)} \ , \tag{42}$$

where $\eta_n(z) = \eta_{n-1}(a\tau_n(z))$ with $\eta_0(z) = z$ and $\xi_n(z) = \tau_n(z) + \xi_{n-1}(a\tau_n(z))$ with $\xi_0(z) = 0$. It is easy to check that $(\eta_1, \ldots, \eta_n)$ is a multidimensional Lévy process and thus (ξ_n, η_n) is a two-dimensional one. Obtaining the form of the exponent is straightforward upon observing that $f_n(\lambda,\alpha) = h^n(\alpha)$. ∎

Lemma 3 *Let $g(\alpha) = \alpha - a\psi(\alpha)$ and $\tau(z) = \inf\{t|\ X(t) + z = 0\}$. Set $J(t) = a\tau(t) - t$. Then J is a Lévy process having no negative jumps with Laplace exponent g. If $\varphi'(0) > a$ then $g'(0) > 0$ and thus $Ee^{-\alpha\zeta} = g'(0)\alpha/g(\alpha)$ is the LST of the stationary distribution associated with J reflected at the origin.*

Proof: It is well known that $\tau(\cdot)$ is a subordinator with exponent $-\psi$. Thus the exponent of J is clearly $\alpha - a\psi(\alpha) = g(\alpha)$. Now note that $g'(0) = 1 - a\psi'(0) = 1 - a/\varphi'(0) > 0$ which is the condition under which the generalized Pollaczek-Khinchin formula for the stationary distribution is valid. ■

It is now easy to check that $\varphi_a(h^n(\alpha)) = g(\lambda + ah^{n-1}(\alpha))$ for $n \geq 1$, where we recall that $h^0(\alpha) = \alpha$. Thus, we can write

$$\frac{\varphi_a(h^n(0))}{\varphi_a(h^n(\alpha))} = \frac{g(\lambda + ah^{n-1}(0))(\lambda + ah^{n-1}(\alpha))}{g(\lambda + ah^{n-1}(\alpha))(\lambda + ah^{n-1}(0))} \cdot \frac{\lambda + ah^{n-1}(0)}{\lambda + ah^{n-1}(\alpha)}. \tag{43}$$

Applying Lemma 2 with $T = T_1$ and $\zeta = aT_2$ where $T_1, T_2 \sim \exp(\lambda)$ are independent, we have that

$$\frac{\lambda + ah^{n-1}(0)}{\lambda + ah^{n-1}(\alpha)} = \frac{Ee^{-h^{n-1}(\alpha)aT_2}}{Ee^{-h^{n-1}(0)aT_2}} = E[e^{-\alpha\eta_{n-1}(aT_2)}|\xi_{n-1}(aT_2) < T_1]. \tag{44}$$

In a similar way it is easy to check that if ζ has the stationary distribution of the reflected version of J from Lemma 3, then the first term on the right side of (43) can be written as

$$\begin{aligned}\frac{Ee^{-(\lambda+ah^{n-1}(\alpha))\zeta}}{Ee^{-(\lambda+ah^{n-1}(0))\zeta}} &= \frac{Ee^{-(\lambda(\zeta+\xi_{n-1}(a\zeta))+\alpha\eta_{n-1}(a\zeta))}}{Ee^{-\lambda(\zeta+\xi_{n-1}(a\zeta))}} \\ &= E[e^{-\alpha\eta_{n-1}(a\zeta)}|\zeta + \xi_{n-1}(a\zeta) < T].\end{aligned} \tag{45}$$

Therefore, we have the following.

Corollary 1

$$\frac{\varphi_a(h^n(0))}{\varphi_a(h^n(\alpha))} = E[e^{-\alpha\eta_{n-1}(a\zeta)}|\zeta + \xi_{n-1}(a\zeta) < T]E[e^{-\alpha\eta_{n-1}(aT_2)}|\xi_{n-1}(aT_2) < T_1] \tag{46}$$

and is thus the LST of a proper random variable.

There is also a different way to show that $\varphi_a(h^n(0))/\varphi_a(h^n(\alpha))$ is an LST, which is given as follows.

Lemma 4 *If $\varphi_a'(0) = \varphi'(0) - a > 0$, then*

$$\frac{\varphi_a(h^n(0))h^n(\alpha)}{\varphi_a(h^n(\alpha))h^n(0)} \tag{47}$$

is an LST of a proper distribution on $[0, \infty)$.

Proof: Let Z_a^* be a random variable having the stationary distribution of the reflected process $Z_a(t) = X(t) - at - \inf_{0 \le s \le t}(X(s) - as)$. Then it is well known (e.g., Corollary 3.4 on page 257 of [2]) that

$$Ee^{-\alpha Z_a^*} = \frac{\varphi_a'(0)\alpha}{\varphi_a(\alpha)} \ . \tag{48}$$

Therefore, with the notation from the proof of Lemma 2, assuming that Z_a^* is independent of $X_1, X_2, \ldots$, we have from (42):

$$Ee^{-\left(\frac{\lambda}{a}\xi_n(Z_a^*)+\alpha\eta_n(Z_a^*)\right)} = Ee^{-h^n(\alpha)Z_a^*} = \frac{\varphi_a'(0)h^n(\alpha)}{\varphi_a(h^n(\alpha))} \ , \tag{49}$$

implying that the right side is completely monotone and upon normalization the result follows. ∎

Lemma 5 *$h^n(0)/h^n(\alpha)$ is the LST of a proper distribution for every $n \ge 1$.*

Proof: This follows by observing that if $T_\mu \sim \exp(\mu)$ then

$$\frac{\mu + h^n(0)}{\mu + h^n(\alpha)} = \frac{Ee^{-h^n(\alpha)T_\mu}}{Ee^{-h^n(0)T_\mu}} \tag{50}$$

where the right side is the LST of a proper distribution. Thus letting $\mu \downarrow 0$ the result is immediate. ∎

The following is now evident.

Corollary 2 *For every $n \ge 1$, $\varphi_a(h^n(0))/\varphi_a(h^n(\alpha))$ is the LST of a proper distribution on $[0, \infty)$ and thus, so is*

$$\prod_{n=1}^{\infty} \frac{\varphi_a(h^n(0))}{\varphi_a(h^n(\alpha))} \ . \tag{51}$$

Since $\varphi_a'(0)\alpha/\varphi_a(\alpha)$ is the LST of the stationary distribution of $Z_a(t) = X(t) - at - \inf_{0 \le s \le t}(X(s) - as)$ (recalling that $\varphi'(0) > a$) it is now clear that the right side of (29) is indeed the LST of a proper distribution on $[0, \infty)$.

There are, however, alternative ways to interpret the decomposition of Theorem 3. One that adds insight is the following. Apply once again the martingale of [6], and observe that $W(t) = at$ at points of increase of $L(t)$ for $0 \le t \le T$. As a consequence, with E denoting the expectation associated with the stationary version,

$$\varphi(\alpha)E\int_0^T e^{-\alpha W(s)}\mathrm{d}s = Ee^{-\alpha W(T)} - Ee^{-\alpha W(0)} + \alpha E\int_0^T e^{-\alpha a s}\mathrm{d}L(s)\ . \tag{52}$$

Since $W(0) \sim W(T)$ and $\lambda E\int_0^T e^{-\alpha W(s)}\mathrm{d}s = Ee^{-\alpha W(T)} = Ee^{-\alpha W(0)}$ (see (3)) then

$$Ee^{-\alpha W(0)} = \frac{\lambda\alpha}{\varphi(\alpha)}E\int_0^T e^{-\alpha a t}\mathrm{d}L(t) \tag{53}$$

and upon setting $\alpha = 0$ we obtain that $EL(T) = \varphi'(0)/\lambda$ and in particular we have that

$$Ee^{-\alpha W(0)} = \frac{\varphi'(0)\alpha}{\varphi(\alpha)}\frac{E\int_0^T e^{-\alpha a t}\mathrm{d}L(t)}{EL(T)} \tag{54}$$

The first term in this decomposition is the generalized Pollaczek-Khinchin formula associated with the stationary distribution of X reflected at the origin. It is easy to check that the second term in the decomposition on the right is the LST of the following distribution:

$$F(x) = \frac{EL(T \wedge \frac{x}{a})}{EL(T)}\ , \tag{55}$$

where the expectation is taken when W is initiated with its stationary distribution. That is, if one thinks of a clearing process with cycles distributed like T and during a clearing cycle the process behaves like $L(t)$ where $W(0)$ has the stationary distribution, then $F(ax) = EL(T \wedge x)/EL(T)$ is well known to be the stationary distribution of this process.

We mention that in [7] related decompositions have been studied. Also in the framework of that paper (a reflected Lévy process with additional jumps), the stationary distribution is a convolution of two or more distributions, one of which is the stationary distribution of the process reflected at the origin.

6 Tail Behavior

In this section we analyze the tail behavior of the distribution with LST (51). More specifically, we show that this tail is exponential with decay rate λ/a, and we compute the prefactor.

To obtain the full tail behavior of $Z(0)$, this needs to be combined with the tail behavior associated with the LST $\varphi_a'(0)\alpha/\varphi_a(\alpha)$, cf. Theorem 3. These asymptotics have been studied in detail before. For the Cramér case, where the tail is exponential, see in particular [4]. The subexponential case, where the Lévy process is an independent sum of a Brownian motion and a subordinator, has been studied in e.g. [1]. Recently [5] produced a more complete study of the tail behavior of the supremum of a Lévy process. The latter also contains an overview of the related literature.

Consider the infinite product (51). Since $h^n(\alpha) = h^{n-1}(h(\alpha)) = h^{n-1}(\psi(\lambda+a\alpha))$, then $h^n(\alpha) \geq h^{n-1}(0)$ for $\alpha \geq -\lambda/a$. Thus,

$$\prod_{n=2}^{\infty} \frac{\varphi_a(h^n(0))}{\varphi_a(h^n(\alpha))} \leq \prod_{n=2}^{\infty} \frac{\varphi_a(h^n(0))}{\varphi_a(h^{n-1}(0))} = \frac{\varphi_a(\alpha^*)}{\varphi_a(\psi(\lambda))} = \frac{\lambda}{\lambda - a\psi(\lambda)} \ , \tag{56}$$

with equality when $\alpha = -\lambda/a$. As for the first term,

$$\frac{\varphi_a(h(0))}{\varphi_a(h(\alpha))} = \frac{g(\lambda)}{g(\lambda + a\alpha)} = \frac{\lambda}{\lambda + a\alpha} \frac{g(\lambda)(\lambda + a\alpha)}{\lambda g(\lambda + a\alpha)} \ , \tag{57}$$

where due to Lemma 3, the second factor of the rightmost expression is the LST of a proper random variable. Let $V_1 \sim \exp(\lambda/a)$ and let V_2 be independent of V_1 with

$$Ee^{-\alpha V_2} = \frac{g(\lambda)(\lambda + a\alpha)}{\lambda g(\lambda + a\alpha)} \prod_{n=2}^{\infty} \frac{\varphi_a(h^n(0))}{\varphi_a(h^n(\alpha))} \ . \tag{58}$$

Then it is easy to check, recalling the right side of (56), that

$$Ee^{(\lambda/a)V_2} = \frac{g(\lambda)}{\lambda g'(0)} \frac{\lambda}{\lambda - a\psi(\lambda)} = \frac{1}{1 - \frac{a}{\varphi'(0)}} \ . \tag{59}$$

Therefore, (51) is the LST of $V = V_1 + V_2$, where $Ee^{\beta V_2} < \infty$ and $V_1 \sim \exp(\beta)$, with $\beta = \lambda/a$. Since

$$P[V_2 > t] \leq e^{-\beta t} E e^{\beta V_2} 1_{\{V_2 > t\}} \ , \tag{60}$$

it follows by dominated convergence that $e^{\beta t} P[V_2 > t] \to 0$ as $t \to \infty$. Now,

$$e^{\beta t} P[V > t] = e^{\beta t} E e^{-\beta(t - V_2)^+} = E e^{\beta V_2} 1_{\{V_2 \leq t\}} + e^{\beta t} P[V_2 > t] \tag{61}$$

and taking $t \to \infty$ gives

$$e^{\beta t} P[V > t] \to E e^{\beta V_2} \; . \tag{62}$$

Thus, in view of (59), as $t \to \infty$,

$$P[V > t] \sim \frac{e^{-(\lambda/a)t}}{1 - \frac{a}{\varphi'(0)}} \; . \tag{63}$$

Because $V_1 \sim \exp(\beta)$ and $P[V_2 > t]/P[V_1 > t] = e^{\beta t} P[V_2 > t] \to 0$, (62) is in fact a special case of Lemma 2.1 of [8].

The above can be summarized as follows.

Theorem 4 *Let V be a random variable with LST (51). Then, as $t \to \infty$,*

$$P[V > t] \sim \frac{e^{-(\lambda/a)t}}{1 - \frac{a}{\varphi'(0)}} \; . \tag{64}$$

Denote by $\bar{V}$ a random variable with LST $\varphi_a'(0)\alpha/\varphi_a(\alpha)$, with $\bar{V}$ independent of V, such that $Z(0)$ is distributed as $\bar{V} + V$, cf. Theorem 3. Theorem 4 entails that if $\bar{V}$ is subexponential, then so is $Z(0)$, with the same tail behavior as $\bar{V}$. If the tail of $\bar{V}$ is exponential (as in the Cramér case mentioned above), with a decay rate that is different from λ/a, then the heavier tail dominates, and determines the decay rate of $Z(0)$; the prefactor can be computed in a similar way as above.

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