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Representations of isotropic random fields with homogeneous increments

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ABSTRACT
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Representations of isotropic random fields with homogeneous increments

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Abstract

We present the series expansion and the moving average representation of the isotropic Gaussian random field with homogeneous increments, making use of concepts of the theory of vibrating strings. We illustrate our results using the example of Lévy's fractional Brownian motion on $\mathbb{R}^N$.

1 Introduction

Let X_t, $t \in \mathbb{R}^N$ be the zero mean, mean-square continuous Gaussian random field with homogenous increments, starting from the origin so that $X_t - X_s$

and X_{t-s} have the same finite dimensional distributions. Moreover, we assume that this field is isotropic, i.e. for any $\mathbf{A}$ from the group of orthogonal matrices on $\mathbb{R}^N$ it holds that random fields $X_{\mathbf{A}t}$ and X_t have the same finite dimensional distributions. According to [33], section 25, the covariance function of this field has the following spectral representation

$$\mathbb{E}X_tX_s = \int_{\mathbb{R}^N} \left(e^{i\langle v,t\rangle} - 1\right)\left(e^{-i\langle v,s\rangle} - 1\right) d\varrho(v) \tag{1.1}$$

where $\langle \cdot\,,\cdot\rangle$ is the usual inner product in $L^2(\mathbb{R}^N)$. The spectral measure ϱ satisfies the condition

$$\int_{\mathbb{R}^N} \frac{\|v\|^2}{1+\|v\|^2} d\varrho(v) < \infty. \tag{1.2}$$

In this paper we obtain the series expansion of a general isotropic random field with homogenous increments. This work is inspired by the recent paper by Malyarenko [21] and intends to extend the results contained in two papers of Dzhaparidze and van Zanten [6], [8] where only the one-dimensional fractional Brownian motion $(B_t)_{t\in[0,1]}$ has been treated and the following series expansion has been obtained:

$$B_t = \sum_{n=1}^{\infty} \frac{\sin x_n t}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1-\cos y_n t}{y_n} Y_n$$

where $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are independent, zero mean Gaussian random variables with certain variances and the numbers x_n and y_n are positive real zeros of two Bessel function of the first kind of certain order. The expansion in the multivariate case is much more complicated. As is shown in [21], instead of two sums we will encounter countable number of terms, each having the form of series of products of certain deterministic functions and independent Gaussian random variables (see the representation (7.16) below and compare with that of Theorem 2 in [21]). As in [21], the series expansion of the isotropic fractional Brownian field is derived as a special case of the general expansion (5.3) but unlike [21], Theorem 1, we require only some mild conditions on the spectral measure. This is achieved by evoking the powerful spectral theory of vibrating strings. Even though Krein's seminal papers come from 1950's it seems these methods never has been applied in the present context until the recent paper of Dzhaparidze, van Zanten and Zareba [9]. The theory is based on the one-to-one relation between spectral measure and a differential operator associated with the vibrating string. The latter brings about the notions gathered in section 2 that are of fundamental importance for analyzing the random processes and, as we will see below, of isotropic fields as well.

Another subject of this paper is a moving average representation of a general homogenous isotropic random field. The idea behind such representation is to express the random field that in general is of complicated structure, in terms of certain basic processes, like in our moving average representation (6.2) that involves a counting number of elementary integrals with respect to mutually independent martingales. This result is obtained by the straightforward generalization of the method used in [9]. Like in [21], special attention is devoted to the applications to the fractional Brownian motion.

The paper is organized as follows. In section 2 we give a short introduction to the theory of vibrating strings. Section 3 explains the concepts of spherical harmonics and spherical Bessel functions which later occur in our series and moving average representations. Section 4 provides representation of the covariance function which is used in section 5 to obtain series expansion and in section 6 to derive moving average representation. In section 7 we apply the theory to particular examples of Lévy's ordinary and fractional Brownian motion. Section 8 comments on the present results and discusses the possibilities for further developments.

2 Introduction to theory of strings

In this section we present a short account of the spectral theory of vibrating strings. This theory is initiated by Krein in a series of papers from 1950's, however it took the present shape in the fundamental work by Dym and McKean [5] (see also [17]; the proofs of all statements in the present section can be found in these works). We only mention strictly necessary facts to afford direct understanding of our arguments.

2.1 The vibrating string

A *string* is described by the pair $\mathbf{lm}$. The number $\mathbf{l} \in (0, \infty]$ is called the *length* of the string and the nonnegative, right-continuous, nondecreasing function $\mathbf{m} = \mathbf{m}(x)$ defined on the interval $[0, \mathbf{l}]$ is called the *mass* of the string. Values $x \in [0, \mathbf{l}]$ are interpreted as a locations on the string between left endpoint $x = 0$ and the right endpoint $x = \mathbf{l}$, value of the function $\mathbf{m}(x)$ is thought of as a total mass of the $[0, x]$-part of the string. The jump of

$\mathbf{m}$ at the point x is denoted by $\Delta\mathbf{m}(x) = \mathbf{m}(x) - \mathbf{m}(x-)$. We assume that $\Delta m(0) = m(0)$.

It is said that the string is *long* if $\mathbf{l}+\mathbf{m}(\mathbf{l}-) = \infty$ and *short* if $\mathbf{l}+\mathbf{m}(\mathbf{l}-) < \infty$. In the case of a short string we need another constant in order to describe the string, that is the so-called *tying constant* $k \in [0,\infty]$. We define also the Hilbert space $L^2(\mathbf{m}) = L^2([0,\mathbf{l}], d\mathbf{m})$.

With the general string (not necessarily smooth) we can associate the differential operator

$$\mathscr{G} f := \frac{df^+}{d\mathbf{m}},$$

where $f^+(f^-)$ denotes the right(left)-hand side derivative of the function f. It can be proved (cf. [5], [9]) that in both cases of long and short string there exists a dense subset $\mathscr{D}(\mathscr{G})$ of $L^2(\mathbf{m})$ such that every $f \in \mathscr{D}(\mathscr{G})$ has left and right derivatives, satisfies $f^-(0) = 0$ (and $f(\mathbf{l}) + kf^+(\mathbf{l}) = 0$ in case of short string) and the operator $\mathscr{G} : \mathscr{D}(\mathscr{G}) \to L^2(\mathbf{m})$ is well defined, self-adjoint and negative definite. Let us just remark that the domain $\mathscr{D}(\mathscr{G})$ consists of functions defined on the real line and satisfying $f(x) = f(0) + xf^-(0)$ for $x \le 0$, $f(x) = f(\mathbf{l}) + (x - \mathbf{l})f^+(\mathbf{l})$ for $x \ge \mathbf{l}$ if $\mathbf{l} < \infty$ and

$$f(x) = f(0) + f^-(0)x + \int_0^x \Big(\int_{[0,y]} \mathscr{G} f(z)\, d\mathbf{m}(z) \Big)\, dy$$

for $0 \le x < \mathbf{l}$.

We consider the differential equation $\mathscr{G} A = -\lambda^2 A$. Since the spectrum of the operator $\mathscr{G}$ is a subset of a half-line $(-\infty, 0]$ (self-adjoint and negative definite) this equation cannot have a solution in $\mathscr{D}(\mathscr{G})$ if λ^2 is not real, nonnegative number. However, this equation has solutions for any complex λ^2. We define the function $x \mapsto A(x,\lambda)$ as a solution of

$$\mathscr{G} A(\cdot,\lambda) = -\lambda^2 A(\cdot,\lambda), \quad A(0,\lambda) = 1, \quad A^-(0,\lambda) = 0.$$

The function A can be represented (cf. [5], p. 162, 171; [17], p. 29) as follows

$$A(x,\lambda) = \sum_{n=0}^{\infty} (-1)^n \lambda^{2n} p_n(x), \tag{2.1}$$

where p_n's are defined recurrently according to $p_n(x) = \int_0^x \int_0^y p_{n-1}(z) d\mathbf{m}(z) dy$ and $p_0(x) = 1$. Thus the function $A(x,\lambda)$ (and $A^+(x,\lambda)$) for any fixed $x \in [0,\mathbf{l}]$ is an entire function of variable λ taking real values for real λ.

If λ^2 is not a positive real number we can construct a complementary solution $D(x,\lambda)$ of

$$\mathscr{G}D(\cdot,\lambda) = -\lambda^2 D(\cdot,\lambda), \quad D^-(0,\lambda) = -1,$$

by putting

$$D(x,\lambda) = A(x,\lambda)\int_x^{\mathbf{l}+k} \frac{1}{A^2(y,\lambda)}\,dy.$$

Yet another function which will appear in the reminder of this paper is the function

$$B(x,\lambda) = -\frac{1}{\lambda}A^+(x,\lambda).$$

2.2 Spectral measure of the string

We define the so-called *resolvent kernel*

$$r_\lambda(x,y) = \begin{cases} A(x,\lambda)D(y,\lambda), & \text{if } x \le y, \\ A(y,\lambda)D(x,\lambda), & \text{if } x \ge y. \end{cases}$$

This term comes from the fact that for any λ^2 outside $[0,\infty)$ we can define the *resolvent* $R_\lambda := (-\lambda^2 I - \mathscr{G})^{-1}$ which can be represented as the integral

$$R_\lambda f(x) = \int_{[0,\mathbf{l}]} r_\lambda(x,y) f(y)\,d\mathbf{m}(y).$$

Having at hand all required notions, we can now formulate the fundamental theorem.

Theorem 2.1. *For every given string, there exists a unique symmetric measure μ on $\mathbb{R}$ such that*

$$r_\lambda(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{A(x,\omega)A(y,\omega)}{\omega^2-\lambda^2}\,\mu(d\omega) \tag{2.2}$$

holds. We call such a measure the principal spectral function. Conversely, given a symmetric measure μ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \frac{\mu(d\lambda)}{1+\lambda^2} < \infty, \tag{2.3}$$

there exists a unique string for which (2.2) holds true.

To make this assertion less abstract, we will now give the reader some idea of the construction of the principal spectral measure. In case of the short string the spectrum of the operator $\mathscr{G}$ is $\{-\omega_n^2 : \ n = 1, 2, ...\}$ where ω_n's are nonnegative roots of the equation

$$kA^+(\mathbf{l}, \lambda) + A(\mathbf{l}, \lambda) = 0$$

(or $A^+(\mathbf{l}, \lambda) = 0$ if $k = \infty$). Since $\mathscr{G}A(\cdot, \lambda) = -\lambda^2 A(\cdot, \lambda)$ for every λ, the corresponding eigenfunctions are $A(\cdot, \omega_n)$. Now, we define the symmetric measure μ on the real line which jumps by the amount

$$\frac{\pi}{2\|A(\cdot, \omega_n)\|^2_{L^2(\mathbf{m})}}$$

at the points $\pm\omega_n$. It is not difficult to show that such a measure, indeed, satisfy (2.2) (we use the fact that eigenvalues of the operator $\mathscr{G}$ coincide with eigenvalues of R_λ which is compact operator on $L^2(\mathbf{m})$, hence $A(\cdot, \omega_n)$ form a complete system in which we can expand the resolvent kernel).

If the string is long we first cut it to make it short. Then construct the measure for the short string according to the procedure described above and let the cutting point tend to infinity.

2.3 The transforms

In this section we will introduce the key concept of odd and even transform. Let μ be the principal spectral function of the string $\mathbf{m}$ and let A and B be the functions associated with $\mathbf{m}$. If we denote $L^2_{\text{even}}(\mu)$ and $L^2_{\text{odd}}(\mu)$ as the spaces of all even, respectively, odd functions in $L^2(\mu)$ we have the following

Theorem 2.2. *The map* $\wedge : L^2(\mathbf{m}) \to L^2_{\text{even}}(\mu)$ *defined by*

$$\wedge : f \longrightarrow \hat{f}_{\text{even}}(\lambda) = \int_{[0,\mathbf{l}]} A(x, \lambda) f(x) \, d\mathbf{m}(x)$$

is one to one and onto. Its inverse is given by

$$\vee : \psi \longrightarrow \check{\psi}_{\text{even}}(x) = \frac{1}{\pi} \int_{\mathbb{R}} A(x, \lambda) \psi(\lambda) \, \mu(d\lambda).$$

It holds that $\|\hat{f}_{\text{even}}\|^2_{L^2(\mu)} = \pi \|f\|^2_{L^2(\mathbf{m})}$.

Before introducing the odd analogue of the above, we need to define the space $\mathscr{X}$, which will be the subspace of $L^2([0, \mathbf{l} + k], dx)$ of all functions which are constant on a mass-free intervals. Note that $k = 0$ if the string is long. If $k = \infty$ we require also that the functions vanish on $[\mathbf{l}, \infty]$.

Theorem 2.3. *The map* $\wedge : \mathscr{X} \to L^2_{\text{odd}}(\mu)$ *defined by*

$$\wedge : f \longrightarrow \hat{f}_{\text{odd}}(\lambda) = \int_0^{\mathbf{l}+k} B(x,\lambda) f(x)\, dx$$

is one to one and onto. Its inverse is given by

$$\vee : \psi \longrightarrow \check{\psi}_{\text{odd}}(x) = \frac{1}{\pi} \int_{\mathbb{R}} B(x,\lambda)\psi(\lambda)\, \mu(d\lambda).$$

It holds that $\|\hat{f}_{\text{odd}}\|^2_{L^2(\mu)} = \pi \|f\|^2_{L^2(\mathbb{R})}$.

Define

$$T(x) = \int_0^x \sqrt{\mathbf{m}'(y)} dy$$

where $\mathbf{m}'$ is the derivative of the absolute continuous part of $\mathbf{m}$. Let $x(T+)$ and $x(T-)$ denote the biggest and the smallest root $x \in [0, \mathbf{l}]$ of

$$T = \int_0^x \sqrt{\mathbf{m}'(y)} dy.$$

Now we will describe the concept of the *Krein* space. If $x \in (0, \mathbf{l})$ is a growth point of the string $\mathbf{m}$ then we define the class $\mathbf{K}^x$ of all functions $f \in L^2(\mu)$ that satisfy

$$\check{f}_{\text{even}}(y) = \check{f}_{\text{odd}}(y) = 0 \quad \text{for} \quad y > x.$$

Let us introduce one more notion. The entire function $f(z)$ is said to be of exponential type τ if

$$\limsup_{R\to\infty} R^{-1} \max_{|z|=R} \log |f(z)| = \tau$$

(cf. [2], [5]).

Denoting by I^T the set of all entire functions $f \in L^2(\mu)$ of finite exponential type less or equal T, we can formulate the following identification theorem for this set.

Theorem 2.4. *Either* $T < T(\mathbf{l})$ *and* I^T *coincides with the Krein space* $\mathbf{K}^{x(T+)}$ *or else* $T \geq T(\mathbf{l})$ *and* I^T *spans* $L^2(\mu)$.

In other words, this theorem states that if the function is of a finite exponential type its inverse transforms are supported on the finite interval.

2.4 The orthogonal basis

Let us deal for awhile with the short sting, assuming $\mathbf{l}+\mathbf{m}(\mathbf{l}-)<\infty$ with the tying constant $k=0$. Consider the family of functions

$$x \mapsto A(x,\omega_n), \quad n=1,2,\ldots \tag{2.4}$$

where ω_n's are positive, real zeros of $A(\mathbf{l},\cdot)$ (we skip the dependence of ω_n's on $\mathbf{l}$, but the reader should keep it in mind).

Using definition of A and integration by parts we have the following

$$-\omega^2\int_0^{\mathbf{l}} A(x,\lambda)A(x,\omega)d\mathbf{m}(x) = \int_0^{\mathbf{l}} A(x,\lambda)dA^+(x,\omega)$$
$$= \left[A(x,\lambda)A^+(x,\omega)\right]_0^{\mathbf{l}} - \int_0^{\mathbf{l}} A^+(x,\omega)A^+(x,\lambda)dx.$$

Reversing the roles of ω and λ gives

$$-\lambda^2\int_0^{\mathbf{l}} A(x,\lambda)A(x,\omega)d\mathbf{m}(x)= \left[A(x,\omega)A^+(x,\lambda)\right]_0^{\mathbf{l}} -\int_0^{\mathbf{l}} A^+(x,\lambda)A^+(x,\omega)dx.$$

Taking the difference of two above equalities results in

$$\int_0^{\mathbf{l}} A(x,\lambda)A(x,\omega)d\mathbf{m}(x) = \frac{A(\mathbf{l},\omega)A^+(\mathbf{l},\lambda)-A(\mathbf{l},\lambda)A^+(\mathbf{l},\omega)}{\omega^2-\lambda^2} \tag{2.5}$$

which is a special case of so-called *Lagrange identity* ([17], Lemma 1.1; see also [5], p. 189, Exercise 3). Now we easily see that

$$\int_0^{\mathbf{l}} A(x,\omega_n)A(x,\omega_k)d\mathbf{m}(x) = \|A(\cdot,\omega_n)\|^2_{d\mathbf{m}}\delta_{kn}\,, \quad k,n=1,2,\ldots$$

It is also true that the family (2.4) spans the function space $L^2(\mathbf{m})$. To show that, let us suppose that there exists $f\in L^2(\mathbf{m})$ such this for all $n\in\mathbb{N}$ we have $f\perp A(\cdot,\omega_n)$. It means that

$$\hat{f}_{\text{even}}(\omega_n) = \langle f, A(\cdot,\omega_n)\rangle_{d\mathbf{m}} = 0, \qquad n=1,2,\ldots$$

Recall that the principal spectral measure of the short string has atoms only at the points $\pm\omega_n$ so that

$$\int_{\mathbb{R}} \left|\hat{f}_{\text{even}}(\lambda)\right|^2 \mu(d\lambda) = \sum_{n\in\mathbb{Z}} \left|\hat{f}_{\text{odd}}(\omega_n)\right|^2 \mu(\{\omega_n\}) = 0.$$

According to Theorem 2.2, $\|f\|^2_{d\mathbf{m}} = 1/\pi\|\hat{f}_{\text{even}}\|^2_{L^2(\mu)} = 0$. Hence, $f=0$ in $L^2(\mathbf{m})$. So, we have proved

Lemma 2.5. *If $\mathbf{l}+\mathbf{m}(\mathbf{l}-)<\infty$, $k=0$ and ω_n's ($n=1,2,\dots$) are all positive, real zeros of $A(\mathbf{l},\cdot)$ then the family of functions*

$$\varphi_n(x) := \frac{A(x,\omega_n)}{\|A(\cdot,\omega_n)\|_{d\mathbf{m}}}, \quad x\in[0,\mathbf{l}],\ n=1,2,\dots \tag{2.6}$$

form an orthonormal basis of the function space $L^2(\mathbf{m})$.

It is easy to guess that we also would like to have the basis of the corresponding space $\mathscr{X}$. To achieve this goal we will use the Christoffel-Darboux type relation (cf. [5], Section 6.3, p. 234)

$$\begin{aligned}&\int_0^{\mathbf{l}} A(x,\omega)A(x,\lambda)d\mathbf{m}(x) + \int_0^{\mathbf{l}} B(x,\omega)B(x,\lambda)dx \\ =\ & \frac{A(\mathbf{l},\omega)B(\mathbf{l},\lambda)-B(\mathbf{l},\omega)A(\mathbf{l},\lambda)}{\lambda-\omega}.\end{aligned} \tag{2.7}$$

Combined with (2.5), it yields the corresponding relation for B, i.e.

$$\int_0^{\mathbf{l}} B(x,\lambda)B(x,\omega)dx = \frac{\omega A(\mathbf{l},\omega)B(\mathbf{l},\lambda)-\lambda A(\mathbf{l},\lambda)B(\mathbf{l},\omega)}{\lambda^2-\omega^2}. \tag{2.8}$$

Now, we can prove the following

Lemma 2.6. *If $\mathbf{l}+\mathbf{m}(\mathbf{l}-)<\infty$, $k=0$ and ω_n's ($n=1,2,\dots$) are all positive, real zeros of $A(\mathbf{l},\cdot)$, then the family of functions*

$$\psi_n(x) := \frac{B(x,\omega_n)}{\|B(\cdot,\omega_n)\|_{dx}}, \quad x\in[0,\mathbf{l}],\ n=1,2,\dots \tag{2.9}$$

form an orthonormal basis of the function space $\mathscr{X}$.

Proof. The orthonormality is self-evident by virtue of (2.8). The completeness is shown in the same manner as for (2.4) by using odd transform instead of even one. □

As we will see further on, the norms occurred in the basis functions (2.6) and (2.9) will also occur in the series expansions. Therefore we will derive a simpler representation of these norms.

Lemma 2.7. *If $\mathbf{l}+\mathbf{m}(\mathbf{l}-)<\infty$, $k=0$ and $\omega_1<\omega_2<\omega_3<\dots$ are positive real zeros of $A(\mathbf{l},\cdot)$, then the norms of the functions $A(\cdot,\omega_n)$ and $B(\cdot,\omega_n)$ in the spaces $L^2([0,\mathbf{l}],d\mathbf{m})$ and $L^2([0,\mathbf{l}],dx)$, respectively, simplify to*

$$\|A(\cdot,\omega_n)\|^2_{d\mathbf{m}} = \|B(\cdot,\omega_n)\|^2_{dx} = -\frac{1}{2}B(\mathbf{l},\omega_n)\left.\frac{\partial A(\mathbf{l},\omega)}{\partial\omega}\right|_{\omega=\omega_n}.$$

Proof. We begin with showing the continuity of the function $A(\cdot, \lambda)$ in the space $L^2([0, \mathbf{l}], d\mathbf{m})$ in case of short string, i.e. $\mathbf{l} + \mathbf{m}(\mathbf{l}-) < \infty$. In other words, we have to prove that $A(\cdot, \lambda) \to A(\cdot, \omega)$ in $L^2(d\mathbf{m})$, as $\lambda \to \omega$. The mean value theorem ensures existence of such γ_0 between λ and ω that

$$\int_0^{\mathbf{l}} |A(x,\lambda) - A(x,\omega)|^2 \, d\mathbf{m}(x) \le |\lambda - \omega|^2 \int_0^{\mathbf{l}} \left| \frac{\partial A(x,\gamma)}{\partial \gamma}\bigg|_{\gamma=\gamma_0} \right|^2 d\mathbf{m}(x).$$

Using the representation (2.1) of $A(x, \lambda)$ we can establish the upper bound

$$\int_0^{\mathbf{l}} \left| \frac{\partial A(x,\gamma)}{\partial \gamma}\bigg|_{\gamma=\gamma_0} \right|^2 d\mathbf{m}(x) \le 4 \sum_{n,k\ge 1} nk\gamma_0^{2(n+k)-2} \int_0^{\mathbf{l}} p_n(x) p_k(x) d\mathbf{m}(x).$$

In view of the property $p_n(x) \le (n!)^{-2}[x\mathbf{m}(x)]^n$ (see [5], p. 162), we can bound the above integral using

$$\begin{aligned} & \sum_{n,k\ge 1} \frac{nk}{(n!k!)^2} \gamma_0^{2(n+k)-2} \int_0^{\mathbf{l}} x^{n+k} \mathbf{m}(x)^{n+k} d\mathbf{m}(x) \\ \le & \sum_{n,k\ge 1} \frac{nk}{(n!k!)^2} \gamma_0^{2(n+k)-2} (\mathbf{l}\,\mathbf{m}(\mathbf{l}))^{n+k+1} < \infty, \end{aligned}$$

since $\mathbf{l}\,\mathbf{m}(\mathbf{l}) < \infty$ by assumption. Hence, we have proved that with some positive finite constant c

$$\int_0^{\mathbf{l}} |A(x,\lambda) - A(x,\omega)|^2 \, d\mathbf{m}(x) \le c\, |\lambda - \omega|^2 .$$

The same property holds for the function $B(\cdot, \lambda)$. Now, according to formulas (2.5) and (2.8) we can write

$$\begin{aligned} \|A(\cdot,\omega)\|^2_{d\mathbf{m}} &= \lim_{\lambda\to\omega} \frac{\omega A(\mathbf{l},\lambda) B(\mathbf{l},\omega) - \lambda A(\mathbf{l},\omega) B(\mathbf{l},\lambda)}{\omega^2 - \lambda^2}, \\ \|B(\cdot,\omega)\|^2_{dx} &= \lim_{\lambda\to\omega} \frac{\omega A(\mathbf{l},\omega) B(\mathbf{l},\lambda) - \lambda A(\mathbf{l},\lambda) B(\mathbf{l},\omega)}{\lambda^2 - \omega^2}. \end{aligned}$$

Since both limits are $\frac{0}{0}$, application of the *l'Hospital's rule* (knowing from (2.1) that involved functions are smooth enough) gives us, for $\omega \ne 0$,

$$\begin{aligned} \|A(\cdot,\omega)\|^2_{d\mathbf{m}} &= \frac{\omega \left[A(\mathbf{l},\omega) \frac{\partial}{\partial\omega} B(\mathbf{l},\omega) - B(\mathbf{l},\omega) \frac{\partial}{\partial\omega} A(\mathbf{l},\omega) \right] + A(\mathbf{l},\omega) B(\mathbf{l},\omega)}{2\omega} \\ \|B(\cdot,\omega)\|^2_{dx} &= \frac{\omega \left[A(\mathbf{l},\omega) \frac{\partial}{\partial\omega} B(\mathbf{l},\omega) - B(\mathbf{l},\omega) \frac{\partial}{\partial\omega} A(\mathbf{l},\omega) \right] - A(\mathbf{l},\omega) B(\mathbf{l},\omega)}{2\omega} \end{aligned}$$

Recall $A(\mathbf{1}, \omega_n) = 0$ to complete the proof. □

So, we have not only found simple expression for the norms (simple derivative instead of an integral) but also showed that they are, in fact, the same numbers for A and B.

3 Spherical harmonics and spherical Bessel functions

This section provides the link between the spectral theory of random fields (as developed in [19], [31], [32] and [33]) and the central topic of mathematical physics (see e.g. [1], [10], [11], [16], [24], [29], [30]) treating the propagation of sound in a homogeneous medium in $\mathbb{R}^N$. It is described by the hyperbolic equation

$$\frac{\partial^2 p}{\partial t^2} = \Delta p \qquad\qquad t \in \mathbb{R}, x \in \mathbb{R}^N \tag{3.1}$$

where $p(t, x)$ is the induced pressure and $\Delta = \sum_{i=1}^{N} (\partial^2/\partial x_j^2)$ denotes the *Laplace operator*. Separate the time and space variables t and x by considering $p(t, x) = \Theta(t)\Xi(x)$ and using the separation constant λ^2. As a result we obtain the equation in time

$$\Theta''(t) = \lambda^2 \Theta(t),$$

that is easily solved, and the equations in space

$$\Delta \Xi(x) = \lambda^2 \Xi(x), \tag{3.2}$$

the so-called Helmholtz equation, whose solutions are convenient to describe in terms of the spherical coordinates $(r, \theta_1, \theta_2, \ldots, \theta_{N-2}, \phi)$ defined for $N \geq 2$ as

$$\begin{aligned}
x_1 &= r \cos\theta_1 \\
x_2 &= r \sin\theta_1 \cos\theta_2 \\
\cdots & \quad \cdots \\
x_{N-1} &= r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-2} \cos\phi \\
x_N &= r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-2} \sin\phi
\end{aligned} \tag{3.3}$$

where $r = \|x\|$. In this setup the Laplace operator takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0\,,$$

where Δ_0 is the *Laplace–Beltrami operator* on the sphere:

$$\Delta_0 = \sum_{j=1}^{N-2} \frac{1}{q_j \sin^{N-j-1}\theta_j} \frac{\partial}{\partial\theta_j}\left(\sin^{N-j-1}\theta_j \frac{\partial}{\partial\theta_j}\right) + \frac{1}{q_{N-1}} \frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial\phi}\right),$$
$$q_1 = 1, \quad q_j = (\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{j-1})^2, \quad j \geq 2.$$

The eigenvalues of the Laplace–Beltrami operator are $-l(l+N-2)$, $l = 0, 1, \ldots$, and to each eigenvalue $-l(l+N-2)$ there corresponds the set of eigenfunctions $\{S_l^m;\ m = 1, \ldots, h(l,N)\}$ of multiplicity

$$h(l,N) = \frac{(2l+N-2)(l+N-3)!}{(N-2)!l!}.$$

The whole system of these eigenfunctions presents a complete orthonormal basis in the space of square integrable functions on the unit sphere. The functions S_l^m are called *spherical harmonics*. Thus, being the eigenfunctions of the Laplace–Beltrami operator, these spherical harmonics satisfy the characteristic equation

$$\Delta_0 S_l^m = -l(l+N-2) S_l^m \tag{3.4}$$

and possess the orthonormal property

$$\langle S_l^m(\cdot), S_{l'}^{m'}(\cdot)\rangle_{L^2(s^{N-1})} = \delta_l^{l'} \delta_m^{m'}. \tag{3.5}$$

The explicit expressions of the spherical harmonics for arbitrary N are rather complicated, see e.g. [11], [29], but the special cases $N = 1, 2, 3$ of obvious physical meaning are simply described.

If $N = 1$, then the unit sphere degenerates to the set $s^0 = \{-1, 1\}$ and the only spherical harmonics are $S_0^1(x) = 1/\sqrt{2}$ and $S_1^1(x) = x/\sqrt{2}$.

If $N = 2$, the angular part of the Laplace operator reduces to $\Delta_0 = \frac{\partial^2}{\partial\phi^2}$. Hence, as a solution of

$$\frac{\partial^2 U}{\partial\phi^2} = -l^2 U$$

we obtain $h(l,2) = 2$ real orthonormal spherical harmonics

$$S_l^1(\phi) = \frac{\cos(l\phi)}{\sqrt{2\pi}}, \qquad S_l^2(\phi) = \frac{\sin(l\phi)}{\sqrt{2\pi}}.$$

If $N = 3$, the characteristic equation $\Delta_0 U = -l(l+1)U$ with $U(\theta, \phi) = \Theta(\theta)\Phi(\phi)$ takes the form

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d^2\Theta}{d\theta^2}\right) + l(l+1)\sin^2\theta = \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}. \tag{3.6}$$

We separate the variables again, with separation constant $-m^2$. The equation for Φ is then

$$\frac{d^2\Phi}{d\phi^2} - m^2\Phi = 0$$

and has an orthonormal set of solutions $\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}$. The equation for Θ is the so-called *Legendre equation*

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d^2\Theta}{d\theta^2}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0.$$

The latter equation has solutions $\Theta(\theta) = P_l^m(\cos\theta)$ $(m = -l, \ldots, l)$ where P_l^m is a *Legendre polynomial* defined by

$$P_l^m(x) = \frac{1}{2^l l!}(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^{l+m}(x^2-1)^l.$$

Hence, the set of the orthonormal solutions of the characteristic equation (3.6), consists of $2l + 1 = h(l, 3)$ real functions given by

$$A_0 P_l^0(\cos\theta), \qquad A_m P_l^m(\cos\theta)\cos m\phi, \qquad A_m P_l^m(\cos\theta)\sin m\phi,$$

with the normalizing constant $A_m = \sqrt{\frac{(l-m)!(2l+1)}{2\pi(l+m)!}}$, $m = 1, \ldots, 2l$.

Let us turn back to the problem of solving the equation (3.2), and let us separate the radial and angular coordinates r and $(\theta_1, \ldots, \theta_{N-1}, \phi)$ by substituting $\Xi(x) = u(r)U(\theta_1, \ldots, \theta_{N-1}, \phi)$ and choosing particular separation constant $k^2 = l(l + N - 2)$. The reason for this choice is the following. The separation results in two equations: $\Delta_0 U + k^2 U = 0$ and

$$r^{3-N}(r^{N-2}u')' + \left(\lambda^2 r^2 - k^2\right) = 0. \tag{3.7}$$

In the first one we recognize equation (3.4), the characteristic equation for the Laplace–Beltrami operator with the eigenvalues $-l(l + N - 2)$ and the corresponding eigenfunctions $S_l^m, m = 1, \ldots, h(l, m)$ of multiplicity $h(l, m)$.

Since the second equation (3.7) can be reduced to the Bessel differential equation, the function

$$g_l(u) := 2^{(N-2)/2}\Gamma(N/2)\frac{J_{l+(N-2)/2}(u)}{u^{(N-2)/2}}, \qquad l \geq 0 \tag{3.8}$$

satisfies (3.7) with initial condition $g_l(0) = \delta_0^l$. Here J_ν denotes the Bessel function of the first kind of order ν. Hence, according to (3.2), the eigenfunctions of the Laplace operator corresponding to eigenvalue λ^2 are given by

$$\Xi(x) = 2^{(N-2)/2}\Gamma(N/2)\frac{J_{l+(N-2)/2}(\lambda\|x\|)}{(\lambda\|x\|)^{(N-2)/2}} S_l^m\left(\frac{x}{\|x\|}\right)$$

for $l \geq 0$, $m = 1, \ldots, h(l, N)$.

In the forthcoming sections we will make use of the following notation

$$G_l(r, \lambda) := \frac{g_l(0) - g_l(r\lambda)}{\lambda} \tag{3.9}$$

where g_l is given by (3.8). We conclude this section by indicating some useful properties of this auxiliary function (3.9).

Using the well-known property of the Bessel function

$$\frac{d}{dz}\left[z^{-\nu}J_\nu(z)\right] = -z^{-\nu}J_{\nu+1}(z),$$

one can easily verify the following recurrence relation for all $l \geq 0$

$$G_{l+1}(r, \lambda) = l\frac{G_l(r, \lambda)}{r\lambda} - \frac{1}{\lambda}\frac{\partial}{\partial r}G_l(r, \lambda). \tag{3.10}$$

By applying the integral representation of the Bessel function, so-called *Poisson formula* (see 8.411.8 in [14]), we arrive at the following representation

$$G_0(r, \lambda) = \frac{2}{B(\frac{1}{2}, \frac{N-1}{2})}\int_0^r \left(1 - \frac{u^2}{r^2}\right)^{\frac{N-3}{2}} \frac{1 - \cos(u\lambda)}{r\lambda} du. \tag{3.11}$$

This shows that G_0 is odd as a function of λ. Furthermore, according to the recurrence relation (3.10), G_l is alternately odd ($l = 0, 2, \ldots$) and even ($l = 1, 3, \ldots$) function of λ. Along with the integral representation of G_0 we also have for $l > 0$

$$i^{l-1}G_l(r, \lambda) = \frac{2^{N-1}\Gamma(l)\Gamma^2(N/2)}{\pi\Gamma(N+l-1)}\int_0^r \left(1 - \frac{u^2}{r^2}\right)^{\frac{N-1}{2}} C_{l-1}^{N/2}\left(\frac{u}{r}\right) e^{i\lambda u} du, \tag{3.12}$$

where C_l^γ is the *Gegenbauer polynomial*

$$C_l^\gamma(x) = \frac{(-1)^l(l+2\gamma-1)(l+2\gamma-2)\cdots(2\gamma)}{l!(\gamma+l-\frac{1}{2})(\gamma+l-\frac{3}{2})\cdots(\gamma+\frac{1}{2})}(1-x^2)^{\frac{1}{2}-\gamma}\left(\frac{d}{dx}\right)^l(1-x^2)^{l+\gamma-\frac{1}{2}}$$

(cf. [29], Section XI.3, formula (7) or [14], formula 7.321).

Taking real and imaginary parts of the equation (3.12) we see by virtue of the Paley–Wiener theorem (cf. [3], [5]) that all functions $G_l(r,\cdot), l = 1, 2, \ldots$ are of exponential type at most r. The same argument holds for G_0 if we use (3.11). We will summarize the above in the following lemma.

Lemma 3.1. *For each $r \in \mathbb{R}_+$, the function $G_l(r,\lambda)$ of $\lambda \in \mathbb{R}$ is an odd function for $l = 0, 2, \ldots$ and an even function for $l = 1, 3, \ldots$. Moreover, it is an analytic function of finite exponential type less or equal r.*

4 Representation of the covariance

As was mentioned in the introduction, the increments of our random field are translation free in the sense that $X_t - X_s$ and X_{t-s} have the same finite dimensional distributions, i.e. $X_t - X_s \overset{d}{=} X_{t-s}$. So the corresponding variances are equal $\mathbb{E}\,|X_t - X_s|^2 = \mathbb{E}\,|X_{t-s}|^2$ and therefore

$$\mathbb{E}X_tX_s = \frac{1}{2}\left(\mathbb{E}\,|X_t|^2 + \mathbb{E}\,|X_s|^2 - \mathbb{E}\,|X_{t-s}|^2\right).$$

Since, in addition, our field is isotropic, the variance $\mathbb{E}\,|X_t|^2$ is a function only of a length of t. Denoting this function (called by Yaglom [33] the structure function) by D we thus write $D(\|t\|) = \mathbb{E}\,|X_t|^2$. With this notation the covariance can be rewritten as

$$\mathbb{E}X_tX_s = \frac{1}{2}\left(D(\|t\|) + D(\|s\|) - D(\|t-s\|)\right). \tag{4.1}$$

By putting $t = s$ in (1.1), we get the following spectral representation for the structure function

$$D(\|t\|) = 2\int_{\mathbb{R}^N}\left(1 - e^{i\langle v,t\rangle}\right)\varrho(dv) = 2\int_{\mathbb{R}^N}\left(1 - \cos\langle v,t\rangle\right)\varrho(dv) \tag{4.2}$$

(the imaginary part vanishes, since our field X is real, cf. [33], p. 435). It will be useful to associate with the spectral measure ϱ a bounded non-decreasing function (cf. [19], [31], [33])

$$\Phi(y) = \int_{\|v\|\le y} d\varrho(dv), \qquad y \in \mathbb{R}_+.$$

In fact $d\Phi(y) = \Phi(y + dy) - \Phi(y)$ may be viewed as a ϱ-measure of the spherical shell $y \leq \|v\| \leq y + dy$. Clearly,

$$\int_0^\infty d\Phi(y) = \varrho(\mathbb{R}^N) < \infty.$$

By rewriting the variable $v = (v_1, \ldots, v_n)$ in the polar coordinates (3.3) with radius $\lambda = \|v\|$, we get

$$\varrho(dv) = \frac{d\sigma_N(v) d\Phi(\lambda)}{|s^{N-1}(\lambda)|}$$

where

$$d\sigma_N(v) = \lambda^{N-1} \sin^{N-2}\theta_1 \cdots \sin\theta_{N-2} d\theta_1 \cdots d\theta_{N-2} d\phi$$

is the surface area element of the sphere $s^{N-1}(\lambda)$ in $\mathbb{R}^N$ and

$$\left|s^{N-1}(\lambda)\right| = \frac{2\pi^{N/2}}{\Gamma(N/2)} \lambda^{N-1}$$

is its surface area. Note that condition (1.2) implies

$$\int_0^\infty \frac{\lambda^2}{1+\lambda^2} d\Phi(\lambda) < \infty. \tag{4.3}$$

Recall the well-known formula

$$\int_{s^{N-1}(\lambda)} e^{i\langle v,t\rangle} d\sigma_N(v) = \left|s^{N-1}(\lambda)\right| g_0(\lambda\|t\|) \tag{4.4}$$

(cf. [31], Chapter I or [29], Section XI.3, formula (4)). Due to this formula, the representation (4.2) rewritten in polar coordinates can be given the form

$$D(r) = 2\int_0^\infty (1 - g_0(r\lambda))\, d\Phi(\lambda)$$

and the formula (4.1) for the covariance function becomes

$$\mathbb{E}X_t X_s = \int_0^\infty [1 - g_0(\lambda\|t\|) - g_0(\lambda\|s\|) + g_0(\lambda\|t-s\|)]\, d\Phi(\lambda). \tag{4.5}$$

Theorem 4.1. *The covariance function of the isotropic Gaussian random field X with homogeneous increments can be represented as follows*

$$\begin{aligned} \mathbb{E}X_t X_s &= \frac{2\pi^{N/2}}{\Gamma(N/2)} \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,N)} S_l^m\left(\tfrac{t}{\|t\|}\right) S_l^m\left(\tfrac{s}{\|s\|}\right) \\ &\times \int_0^\infty G_l(\|t\|,\lambda) G_l(\|s\|,\lambda) \lambda^2 d\Phi(\lambda). \end{aligned} \tag{4.6}$$

Proof. Note that $h(0,N)=1$, $S_0^1(\cdot)$ is a constant function for every N and in view of (3.5) this constant is given by $S_0^1(\cdot)\equiv 1/\sqrt{|s^{N-1}(1)|}$. Hence, (4.6) is equivalent to

$$\begin{aligned}\mathbb{E}X_tX_s &- \int_0^\infty (1-g_0(\lambda\|t\|))(1-g_0(\lambda\|s\|))d\Phi(\lambda) \qquad (4.7)\\ &= \frac{2\pi^{N/2}}{\Gamma(N/2)}\sum_{l=1}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\left(\tfrac{t}{\|t\|}\right) S_l^m\left(\tfrac{s}{\|s\|}\right)\int_0^\infty g_l(\lambda\|t\|)g_l(\lambda\|s\|)d\Phi(\lambda)\end{aligned}$$

which we are now going to prove. Recall the addition formula for Bessel functions, as is given on p. 370 of Yaglom [33]

$$g_0(\lambda\|t-s\|)=\frac{2\pi^{N/2}}{\Gamma(N/2)}\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\left(\tfrac{t}{\|t\|}\right) S_l^m\left(\tfrac{s}{\|s\|}\right) g_l(\lambda\|t\|)g_l(\lambda\|s\|).$$

It implies that

$$\begin{aligned}&g_0(\lambda\|t-s\|)-g_0(\lambda\|t\|)g_0(\lambda\|s\|)\\ &=\frac{2\pi^{N/2}}{\Gamma(N/2)}\sum_{l=1}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\left(\tfrac{t}{\|t\|}\right) S_l^m\left(\tfrac{s}{\|s\|}\right) g_l(\lambda\|t\|)g_l(\lambda\|s\|).\end{aligned}$$

Taking the integral with respect to $d\Phi(\lambda)$ from the both sides we see that the expression on the right in (4.7) equals to the integral

$$\int_0^\infty \left[g_0(\lambda\|t-s\|)-g_0(\lambda\|t\|)g_0(\lambda\|s\|)\right]d\Phi(\lambda).$$

But in view of (4.5) we see that also the left hand side of (4.7) equals to the latter integral. Thus (4.7) holds true. □

We want now to specify the spectral measure μ by

$$\mu(d\lambda)=\frac{\Gamma(N/2)}{2\pi^{N/2}}\lambda^2 d\Phi(\lambda) \qquad (4.8)$$

and to apply Theorem 2.1. Its condition (2.3) is ensured due to (4.3). So we can associate with the measure μ a unique string and derive the following corollary to Theorem 4.1.

Corollary 4.2. *The covariance function of the isotropic Gaussian random field X with homogeneous increments can be represented as follows*

$$\mathbb{E}X_tX_s \tag{4.9}$$
$$= \frac{4\pi^{N+1}}{\Gamma^2(N/2)} \sum_{l=0,2,\dots} \sum_{m=1}^{h(l,N)} S_l^m\left(\tfrac{t}{\|t\|}\right) S_l^m\left(\tfrac{s}{\|s\|}\right) \int_0^{\mathbf{l}+k} \check{G}_l(\|t\|,x)\check{G}_l(\|s\|,x)dx$$
$$+ \frac{4\pi^{N+1}}{\Gamma^2(N/2)} \sum_{l=1,3,\dots} \sum_{m=1}^{h(l,N)} S_l^m\left(\tfrac{t}{\|t\|}\right) S_l^m\left(\tfrac{s}{\|s\|}\right) \int_0^{\mathbf{l}} \check{G}_l(\|t\|,x)\check{G}_l(\|s\|,x)d\mathbf{m}(x),$$

where

$$\check{G}_l(r,x) := \frac{1}{\pi}\int_{\mathbb{R}} G_l(r,\lambda)A(x,\lambda)d\mu(\lambda), \quad l = 1,3,\dots \tag{4.10}$$
$$\check{G}_l(r,x) := \frac{1}{\pi}\int_{\mathbb{R}} G_l(r,\lambda)B(x,\lambda)d\mu(\lambda), \quad l = 0,2,\dots \tag{4.11}$$

and the functions $A(x,\lambda)$ and $B(x,\lambda)$ are the eigenfunctions associated with the mass function $\mathbf{m}$ whose principal spectral measure $\mu(d\lambda)$ is given by (4.8).

Proof. Condition (2.3) ensures that the measure μ satisfies assumptions of Theorem 2.1. By virtue of this theorem there exists an unique associated string with mass $\mathbf{m}(x)$ and length $\mathbf{l} \le \infty$. Note that the function $\check{G}_l(r,x)$ is defined as the even or odd (for appropriate l's) inverse transform of the function $G_l(r,\lambda)$. Since transforms are isometries, we have

$$\langle G_l(r_1,\cdot), G_l(r_2,\cdot)\rangle_{L^2(\mu)} = \pi\langle \check{G}_l(r_1,\cdot), \check{G}_l(r_2,\cdot)\rangle_{L^2(\mathbf{m})}, \quad l = 1,3,\dots$$
$$\langle G_l(r_1,\cdot), G_l(r_2,\cdot)\rangle_{L^2(\mu)} = \pi\langle \check{G}_l(r_1,\cdot), \check{G}_l(r_2,\cdot)\rangle_{L^2(dx)}, \quad l = 0,2,\dots$$

The proof is completed by applying this to representation (4.6). □

Remark 4.3. Recall the assertion of Lemma 3.1 that function $G_l(r,\cdot)$ is of finite exponential type at most r. Combined with Theorem 2.4, this implies that such functions are supported on the finite interval $[0, x(r+)]$ and that the representation (4.9) is in fact of the form

$$\mathbb{E}X_tX_s \tag{4.12}$$
$$= \frac{4\pi^{N+1}}{\Gamma^2(N/2)} \sum_{l=0,2,\dots} \sum_{m=1}^{h(l,N)} S_l^m\left(\tfrac{t}{\|t\|}\right) S_l^m\left(\tfrac{s}{\|s\|}\right) \int_0^{n(s,t)} \check{G}_l(\|t\|,y)\check{G}_l(\|s\|,y)dy$$
$$+ \frac{4\pi^{N+1}}{\Gamma^2(N/2)} \sum_{l=1,3,\dots} \sum_{m=1}^{h(l,N)} S_l^m\left(\tfrac{t}{\|t\|}\right) S_l^m\left(\tfrac{s}{\|s\|}\right) \int_0^{n(s,t)} \check{G}_l(\|t\|,y)\check{G}_l(\|s\|,y)d\mathbf{m}(y)$$

with $n(s,t) := x(\|t\|+) \wedge x(\|s\|+)$. This immediately allows us to write down the following moving average-type representation of the random field X:

$$X_t = \frac{4\pi^{N+1}}{\Gamma^2(N/2)} \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,N)} S_l^m\left(\tfrac{t}{\|t\|}\right) \int_0^{x(\|t\|+)} \check{G}_l(\|t\|, y) dM_l^m(y) \quad (4.13)$$

where $\{M_l^m\}$ is a set of independent Gaussian processes with independent increments and the variances

$$\mathbb{E}\,|M_l^m(y)|^2 = \begin{cases} y & l = 0, 2, \dots \\ \mathbf{m}(y) & l = 1, 3, \dots \end{cases}$$

In section 6 we will return to this subject.

5 Series expansion

In this section we restrict our considerations to the parameter t taking values from the ball of radius T, i.e.

$$t \in \mathcal{B}_T := \left\{ u \in \mathbb{R}^N \ : \ \|u\| \le T \right\}. \quad (5.1)$$

We consider a string with the same mass function $\mathbf{m}$ (associated via Theorem 2.1 with μ defined by (4.8)) but we cut it at the point $\mathbf{l} := x(T+)$ (which we assume to be finite) with tying constant $k = 0$ and $\mathbf{m}(\mathbf{l}-) < \infty$.

Let us concentrate for a moment on the odd l's. Since $\check{G}_l(\|t\|, \cdot)$ belongs then to the space $L^2(\mathbf{m})$, we can expand it in basis (2.6) so that

$$\check{G}_l(\|t\|, x) = \sum_{n=0}^{\infty} \langle \check{G}_l(\|t\|, \cdot), \varphi_n \rangle_{d\mathbf{m}} \varphi_n(x).$$

Having this, we can write

$$\begin{aligned} &\int_0^{\mathbf{l}} \check{G}_l(\|t\|, x) \check{G}_l(\|s\|, x) d\mathbf{m}(x) \\ = &\sum_{n=0}^{\infty} \left(\int_0^{\mathbf{l}} \check{G}_l(\|t\|, x) \varphi_n(x) d\mathbf{m}(x) \right) \left(\int_0^{\mathbf{l}} \check{G}_l(\|s\|, x) \varphi_n(x) d\mathbf{m}(x) \right), \end{aligned}$$

which is same as

$$\int_0^{\mathbf{l}} \check{G}_l(\|t\|, x) \check{G}_l(\|s\|, x) d\mathbf{m}(x) = \sum_{n=0}^{\infty} \frac{G_l(\|t\|, \omega_n) G_l(\|s\|, \omega_n)}{\|A(\cdot, \omega_n)\|_{d\mathbf{m}}^2}$$

since

$$\int_0^{\mathbf{l}} \check{G}_l(\|t\|,x)\varphi_n(x)d\mathbf{m}(x) = \frac{G_l(\|t\|,\omega_n)}{\|A(\cdot,\omega_n)\|_{d\mathbf{m}}}.$$

Exactly the same argument for even l's results in corresponding formula

$$\int_0^{\mathbf{l}} \check{G}_l(\|t\|,x)\check{G}_l(\|s\|,x)dx = \sum_{n=0}^{\infty} \frac{G_l(\|t\|,\omega_n)G_l(\|s\|,\omega_n)}{\|B(\cdot,\omega_n)\|_{dx}^2}.$$

Then, keeping in mind Lemma 2.7, we can rewrite representation (4.9) as follows

$$\mathbb{E}X_tX_s=\frac{4\pi^{N+1}}{\Gamma^2(N/2)}\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\left(\tfrac{t}{\|t\|}\right) S_l^m\left(\tfrac{s}{\|s\|}\right)\sum_{n=0}^{\infty}\frac{G_l(\|t\|,\omega_n)G_l(\|s\|,\omega_n)}{\|A(\cdot,\omega_n)\|_{d\mathbf{m}}^2}. \tag{5.2}$$

Now we can prove the following

Theorem 5.1. *Let X be a centered mean square continuous Gaussian isotropic random field with homogenous increments on $\mathbb{R}^N$. If the mass function associated with μ (cf. (4.8)) is such that $x(T+) + \mathbf{m}(x(T+)-) < \infty$ for some $T > 0$, then we have the following representation on the ball $\mathcal{B}_T$ of radius T (cf. (5.1)):*

$$X_t \quad = \quad \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\left(\tfrac{t}{\|t\|}\right) G_l(\|t\|,\omega_n)\xi_{l,n}^m \tag{5.3}$$

where $\xi_{l,n}^m$ are independent mean zero Gaussian random variables with variances

$$\sigma_n^2 = \frac{-8\pi^{N+1}}{\Gamma^2(N/2)\, B(x(T+),\omega_n)\, \frac{\partial}{\partial\omega}A(x(T+),\omega)\big|_{\omega=\omega_n}} \tag{5.4}$$

and ω_n's are zeros of $A(x(T+),\cdot)$. This series converges in mean square sense for any fixed t from the closed ball $\mathcal{B}_T$. Moreover, if the process $(X_t)_{\|t\|<T}$ is continuous, it converges with probability one in the space of continuous functions $C(\mathcal{B}_T)$ endowed with the supremum norm.

Proof. Consider the partial sum of the series by

$$X_t^M \quad = \quad \sum_{n,l=0}^{M}\sum_{m=1}^{h(l,N)} S_l^m\left(\tfrac{t}{\|t\|}\right) G_l(\|t\|,\omega_n)\xi_{l,n}^m.$$

The covariance representation (5.2) ensures mean square convergence of X_t^M to the process X_t, for every $t \in \mathcal{B}_T$, which implies weak convergence of the finite dimensional distributions. It is assumed that X is continuous, so if we manage to prove the asymptotic tightness of $(X_t^M)_{\|t\|<T}$ we would be able to use Theorem 1.5.4 of [28] which states that weak convergence of finite dimensional distributions combined with asymptotic tightness is sufficient for the sequence to converge weakly in $C(\mathcal{B}_T)$. By virtue of the Itô-Nisio theorem (see for instance [28]) it is equivalent to convergence with probability one in $C(\mathcal{B}_T)$. Now we will prove the asymptotic tightness of X_t^M in the space $C(\mathcal{B}_T)$.

Asymptotic tightness is equivalent (see for instance Theorem 1.5.7 of [28]) to the following two conditions

- X_t^M is asymptotically tight in $\mathbb{R}$ for every fixed $t \in \mathcal{B}_T$;
- there exists semi-metric d on $\mathcal{B}_T$ such that $(\mathcal{B}_T, d)$ is totally bounded and $(X_t^M)_{\|t\|<T}$ is *asymptotically uniformly d-equicontinuous in probability*, i.e. $\forall\, \varepsilon, \eta > 0$, $\exists\, \delta > 0$ such that

$$\limsup_{M\to\infty} \mathbb{P}\left(\sup_{d(s,t)<\delta} \left| X_t^M - X_s^M \right| > \varepsilon \right) < \eta.$$

The first condition is automatically satisfied by virtue of the weak convergence of the partial sums for every t. It suffices to prove the second one. Let us first define the sequence of semi-metrics on $\mathcal{B}_T$

$$d_M^2(s,t) := \mathbb{E}\left| X_t^M - X_s^M \right|^2 \le \mathbb{E}\left| X_t - X_s \right|^2 =: d^2(s,t).$$

It can be proved that (see for instance [28], p.446) for any M, any Borel probability measure ν on $(\mathcal{B}_T, d_M)$ and every $\delta, \eta > 0$ it holds that

$$\mathbb{E} \sup_{d_M(s,t)<\delta} \left| X_t^M - X_s^M \right| \lesssim \sup_t \int_0^\eta \sqrt{\log \frac{1}{\nu\left(\mathcal{B}_\varepsilon(t, d_M)\right)}} d\varepsilon + \delta \sqrt{N(\eta, \mathcal{B}_T, d_M)}, \tag{5.5}$$

where $\mathcal{B}_\varepsilon(t,d)$ denotes the ball of radius ε around point t in metric d and $N(\eta, \mathbf{Y}, d)$ is so-called η-covering number, i.e. the minimal number of balls of radius η needed to cover $\mathbf{Y}$. Since $d_M(s,t) \le d(s,t)$ we have

$$\mathbb{E} \sup_{d(s,t)<\delta} \left| X_t^M - X_s^M \right| \le \mathbb{E} \sup_{d_M(s,t)<\delta} \left| X_t^M - X_s^M \right|. \tag{5.6}$$

Proposition A.2.17 of [28] applied to the process X itself (uniform continuity and boundness with respect to standard deviation semi-metric $d(s,t)$ is equivalent to the continuity of almost all sample paths with respect to Euclidean distance and continuity of the map $t \to \mathbb{E}\,|X_t|^2$ (cf. [28], Lemma 1.5.9), the latter being satisfied by virtue of the mean square continuity) yields that there exists some Borel probability measure ν^* on $(\mathcal{B}_T, d)$ such that

$$\sup_{t\in\mathcal{B}_T}\int_0^\eta \sqrt{\log\frac{1}{\nu^*\left(\mathcal{B}_\varepsilon(t,d)\right)}}d\varepsilon \xrightarrow{\eta\searrow 0} 0. \tag{5.7}$$

From relation $d_M \leq d$ we can easily see that d_M-open sets are also d-open sets. It implies that σ-algebras of Borel sets satisfy $\mathscr{B}(\mathcal{B}_T, d_M) \subset \mathscr{B}(\mathcal{B}_T, d)$. Hence, the measure ν^* is also a Borel measure on $(\mathcal{B}_T, d_M)$. Choosing the measure ν^* in (5.5) and combining it with (5.6) gives

$$\begin{aligned}
&\mathbb{E}\sup_{d(s,t)<\delta}\left|X_t^M - X_s^M\right| \\
\lesssim\;& \sup_{t\in\mathcal{B}_T}\int_0^\eta \sqrt{\log\frac{1}{\nu^*\left(\mathcal{B}_\varepsilon(t,d_M)\right)}}d\varepsilon + \delta\sqrt{N(\eta,\mathcal{B}_T,d_M)} \\
\leq\;& \sup_{t\in\mathcal{B}_T}\int_0^\eta \sqrt{\log\frac{1}{\nu^*\left(\mathcal{B}_\varepsilon(t,d)\right)}}d\varepsilon + \delta\sqrt{N(\eta,\mathcal{B}_T,d)};
\end{aligned}$$

the last inequality being justified by the fact that if greater the metric, smaller the balls. The first term on the right hand side can be made arbitrarily small by (5.7). It is not difficult to see that condition (5.7) is sufficient for the space $(\mathcal{B}_T, d)$ to be totally bounded (see for instance [28], p. 446). Hence, the number $N(\eta, \mathcal{B}_T, d)$ is finite and also the second term on the right hand side can be arbitrarily small. This proves the desired equicontinuity of $(X_t^M)_{\|t\|<T}$. $\square$

Remark 5.2. Notice that our expansion (5.3) is of different form than the one derived in Malyarenko [21], Theorem 1. Conditions of this theorem are difficult to verify, except perhaps in the case of Lévy's fractional Brownian motion. Moreover, the series (5.3) converges with probability one in the space of continuous functions on the unit $\mathbb{R}^N$-ball while the series in Theorem 1 of [21] is proved to converge only in the mean square sense for any fixed point of the ball.

6 Moving average for smooth strings

In this section we will show how the representation (4.13) does simplify when the random field has a smooth mass function of the associated string, and we will obtain the integral representation in the time domain, which can be viewed as a multivariate moving average representation.

To this end, we have to invert function $t(x) = \int_0^x \sqrt{\mathbf{m}'(y)}dy$ defined in section 2. Therefore we need require the mass function that is continuously differentiable with a positive derivative. This admits the following representation of the covariance function in the time domain.

Theorem 6.1. *If the mass function* $\mathbf{m}$ *of a Gaussian isotropic random field* X *with homogeneous increments is continuously differentiable and* $\mathbf{m}' > 0$, *then for every* $s, t \in \mathbb{R}^N$ *we have*

$$\begin{aligned} \mathbb{E}X_t X_s &= \frac{8\pi^{N+2}}{\Gamma^2(N/2)} \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right) S_l^m\left(\frac{s}{\|s\|}\right) \\ &\times \int_0^{\|s\|\wedge\|t\|} k_l(\|t\|,u) k_l(\|s\|,u) dV(2u) \end{aligned} \tag{6.1}$$

where $V(2u) := \pi^{-1}\mathbf{m}(x(u))$ *and for* $u \le \|t\|$ *the kernels are given by*

$$k_l(\|t\|,u) = \check{G}_l(\|t\|,x(u))x'(u), \qquad l = 0,2,\dots, \tag{6.2}$$

$$k_l(\|t\|,u) = \check{G}_l(\|t\|,x(u)), \qquad l = 1,3,\dots. \tag{6.3}$$

Proof. Let us first derive some useful relations between functions $\mathbf{m}$, x and V. Differentiating $t = \int_0^{x(t)} \sqrt{\mathbf{m}'(y)}dy$ we obtain

$$x'(t) = 1/\sqrt{\mathbf{m}'(x(t))}. \tag{6.4}$$

Since $\mathbf{m}'(x(t)) = 2\pi\, V'(2t)/x'(t)$, from (6.4) we get

$$2\pi\, V'(2t)\, x'(t) = 1. \tag{6.5}$$

To prove the representation (6.1) we apply the change of variable $y = x(u)$ to both terms on the right side of (4.12). Due to (6.5), the measure dy in the integral of the first term becomes

$$x'(u)du = 2\pi x'(u)^2 dV(2u).$$

Hence,

$$\begin{aligned} & \int_0^{n(s,t)} \check{G}_l(\|t\|, y)\check{G}_l(\|s\|, y)dy \qquad (6.6) \\ = \quad & 2\pi \int_0^{\|s\|\wedge\|t\|} \check{G}_l(\|t\|, x(u))x'(u)\check{G}_l(\|s\|, x(u))x'(u)dV(2u)\,. \end{aligned}$$

The same change of variable allows us to write in the following manner the measure $d\mathbf{m}(y) = \mathbf{m}'(y)dy$ in the integral of the second term in (4.12):

$$\mathbf{m}'(x(u))x'(u)du = 2\pi V'(2u)/x'(u)x'(u)du = 2\pi dV(2u).$$

Thus

$$\begin{aligned} & \int_0^{n(s,t)} \check{G}_l(\|t\|, y)\check{G}_l(\|s\|, y)d\mathbf{m}(y) \qquad (6.7) \\ = \quad & 2\pi \int_0^{\|s\|\wedge\|t\|} \check{G}_l(\|t\|, x(u))\check{G}_l(\|s\|, x(u))dV(2u)\,. \end{aligned}$$

Due to (6.6) and (6.7) the representation (4.12) turns into (6.1). □

Corollary 6.2. *Under assumptions of Theorem 6.1 for every $s, t \in \mathbb{R}^N$ we have*

$$X_t = \frac{2\sqrt{2}\pi^{N/2+1}}{\Gamma(N/2)} \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,N)} S_l^m \left(\frac{t}{\|t\|}\right) \int_0^{\|t\|} k_l(\|t\|, u)dM_l^m(u) \qquad (6.8)$$

where $\{M_l^m(u)\}$ denotes a family of independent Gaussian martingales, each with zero mean and variance function

$$E|M_l^m(u)|^2 = V(2u).$$

Remark 6.3. The representation (6.1) may be compared with similar result by A. Malyarenko [21], that is derived under a number of conditions on the spectral measure, listed in his Theorem 1.

7 Examples

This section is devoted to applications of our general results first to Lévy's Brownian motion and then to Lévy's fractional Brownian motion of arbitrary Hurst index.

7.1 Lévy's Brownian motion

Paul Lévy [20] defined the Brownian motion on $\mathbb{R}^N$ as a centered Gaussian random field with a covariance structure

$$\mathbb{E}X_tX_s = \frac{1}{2}\left(\|t\| + \|s\| - \|t-s\|\right).$$

Properties of this field were investigated by several authors, see for instance Chentsov [4], McKean [22], Molchan [23] or Samorodnitsky and Taqqu [27]. In this case we have (cf. [23], [32])

$$d\Phi(\lambda) = c_N^2 \frac{2\pi^{N/2}}{\Gamma(N/2)} \frac{d\lambda}{\lambda^2}, \qquad c_N^2 = \frac{\Gamma((N+1)/2)}{2\pi^{(N+1)/2}}.$$

Hence $\mu(d\lambda) = c_N^2 d\lambda$. It is well known (cf. [5], [9]) that the mass function of the string associated with the Lebesgue measure is the identity. Furthermore, according to "rule 1" of [5], p. 265, the measure multiplied by constant c corresponds to the mass function $c^{-1}\mathbf{m}(c^{-1}x)$ and functions $A(c^{-1}x, \lambda)$, $c^{-1}B(c^{-1}x, \lambda)$. Hence the mass function associated with Lévy's Brownian motion is

$$\mathbf{m}(x) = c_N^{-4} x$$

and the solutions of the eigenvalue problem are

$$A(x,\lambda) = \cos(c_N^{-2}\lambda x), \quad B(x,\lambda) = c_N^{-2}\sin(c_N^{-2}\lambda x).$$

Since $x(t) = c_N^2 t$, in this case we have

$$A(x(t),\lambda) = \cos(\lambda t), \quad B(x(t),\lambda)x'(t) = \sin(\lambda t). \tag{7.1}$$

7.1.1 Moving average representation

In order to obtain the moving average representation for this particular field we will use Theorem 6.1, according to which we need compute inverse transform of the function G_l with respect to the measure $\mu(d\lambda) = c_N^2 d\lambda$. In view of (4.11), (6.2) and (7.1), the kernel k_l for $l = 2, 4, 6, \ldots$ is given by the sine transform of $\frac{2}{\pi}c_N^2 G_l(r,\lambda)$, i.e.

$$k_l(r,u) = -c_N^2 \frac{2^{N/2}\Gamma(N/2)}{\pi r^{\frac{N-2}{2}}} \int_0^\infty J_{l+(N-2)/2}(r\lambda)\sin(\lambda u)\frac{d\lambda}{\lambda^{N/2}}.$$

By virtue of formula 2.12(10) of [12] the integral can be taken and we get

$$k_l(r,u) = c_N^2 \frac{(-1)^{l/2} 2^{N-1}(l-1)!\Gamma^2(N/2)}{\pi\Gamma(N+l-1)} \left(1-\tfrac{u^2}{r^2}\right)^{\frac{N-1}{2}} C_{l-1}^{N/2}\left(\tfrac{u}{r}\right) 1_{(0,r)}(u). \quad (7.2)$$

For odd l's we need compute the cosine transform with the help of formula 1.12(10) of [12]. We obtain

$$k_l(r,u) = c_N^2 \frac{(-1)^{\frac{l+1}{2}} 2^{N-1}(l-1)!\Gamma^2(N/2)}{\pi\Gamma(N+l-1)} \left(1-\tfrac{u^2}{r^2}\right)^{\frac{N-1}{2}} C_{l-1}^{N/2}\left(\tfrac{u}{r}\right) 1_{(0,r)}(u) \quad (7.3)$$

(note that formula (3.12) provides for inverses of these sine and cosine transforms).

To compute k_0 integrate (3.11) with respect to $\pi^{-1}\sin(\lambda u)d\lambda$ over $\mathbb{R}$. Since

$$\int_0^\infty (1-\cos(\lambda w))\sin(\lambda u)\frac{d\lambda}{\lambda} = \frac{\pi}{2} 1_{(u,r)}(w) \quad (7.4)$$

(see [14], formulas 3.721.1 and 3.741.2) we obtain

$$k_0(r,u) = \frac{2\, c_N^2}{B(\frac{1}{2}, \frac{N-1}{2})} \int_{u/r}^1 \left(1-y^2\right)^{\frac{N-3}{2}} dy. \quad (7.5)$$

Alternative representations for the kernels are obtained via formulas 8.932.2-3 of [14] that express Gegenbauer's polynomials in (7.2) and (7.3) in terms of the hypergeometric function

$$F(\alpha,\beta;\gamma;z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+(n-1))\beta(\beta+1)\cdots(\beta+(n-1))}{n!\gamma(\gamma+1)\cdots(\gamma+(n-1))} z^n.$$

Applying, in addition, formula 9.131.1 of [14], we arrive at the expressions (7.6) and (7.7) below. Corollary 6.2 yields the following

Theorem 7.1. *Let X be Lévy's Brownian motion on $\mathbb{R}^N$. It can be represented as*

$$X_t = \frac{2\sqrt{2}\pi^{N/2+1}}{\Gamma(N/2)} \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,N)} S_l^m\left(\tfrac{t}{\|t\|}\right) \int_0^{\|t\|} k_l(\|t\|,u)\, dM_l^m(u)$$

where k_0 is given by (7.5),

$$k_l(r,u) = -\frac{2c_N^2}{\pi} \frac{\Gamma(N/2)\Gamma((l+1)/2)}{\Gamma(\frac{l+N-1}{2})} \frac{u}{r} F\left(\frac{l+1}{2}, \frac{3-l-N}{2}; \frac{3}{2}; \frac{u^2}{r^2}\right) \quad (7.6)$$

for $l = 2, 4, \ldots$ and

$$k_l(r,u) = -\frac{c_N^2}{\pi}\frac{\Gamma(N/2)\Gamma(l/2)}{\Gamma(\frac{l+N}{2})}F\left(\frac{l}{2},\frac{2-l-N}{2};\frac{1}{2};\frac{u^2}{r^2}\right) \tag{7.7}$$

for $l = 1, 3, 5, \ldots$ Here $\{M_l^m\}$ is a family of independent Gaussian martingales, each with mean zero and the variance function

$$\mathbb{E}|M_l^m(u)|^2 = \frac{4\pi^N}{\Gamma^2((N+1)/2)}u.$$

7.1.2 Series expansion

In the case of Lévy's Brownian motion Theorem 5.1 implies

Theorem 7.2. *If X is the Lévy's Brownian motion on $\mathbb{R}^N$ then it can be represented on the ball $\mathcal{B}_T$ of radius T (cf. (5.1)) as follows*

$$X_t = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right)G_l(\|t\|,\omega_n)\xi_{l,n}^m \tag{7.8}$$

where

$$\omega_n = \frac{(2n+1)\pi}{2T}$$

and $\xi_{l,n}^m$ are independent mean zero Gaussian random variables with variances

$$\sigma_n^2 = \frac{4\pi^{(N+1)/2}\Gamma((N+1)/2)}{T\Gamma^2(N/2)}.$$

This series converges in mean square for any fixed t from the closed ball $\mathcal{B}_T$ and with probability one in the space of continuous functions $C(\mathcal{B}_T)$ endowed with the supremum norm.

Remark 7.3. Note that if $N = 1$ we obtain a series representation of standard Brownian motion on $[0, 1]$

$$W_t = \sqrt{2}\sum_{n=0}^{\infty}\frac{1-\cos(t(n+\frac{1}{2})\pi)}{(n+\frac{1}{2})\pi}\xi_n^0 - \sqrt{2}\sum_{n=0}^{\infty}\frac{\sin(t(n+\frac{1}{2})\pi)}{(n+\frac{1}{2})\pi}\xi_n^1$$

where $\{\xi_n^0\}$ and $\{\xi_n^1\}$ are independent sequences of standard Gaussian independent random variables, so that (7.8) can be viewed as a multivariate version of the classical Paley–Wiener expansion.

7.2 The Lévy's fractional Brownian motion

Lévy's fractional Brownian motion is defined on $\mathbb{R}^N$ as a centered Gaussian random field with covariance function

$$\mathbb{E}X_t X_s = \frac{1}{2}\left(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H}\right)$$

where $H \in (0,1)$ is called the Hurst index. Observe that for $H = 1/2$ it reduces to Lévy's Brownian motion considered in the preceding section. Let us recall few well known facts about this process. In this case we have (cf. [21], [32])

$$d\Phi(\lambda) = c_{HN}^2 \frac{2\pi^{N/2}}{\Gamma(N/2)} \frac{d\lambda}{\lambda^{1+2H}}$$

and

$$\mu(d\lambda) = c_{HN}^2 \lambda^{1-2H} d\lambda \tag{7.9}$$

where

$$c_{HN}^2 = \pi^{-(N+2)/2} 2^{2H-1} \Gamma(H + N/2)\Gamma(H+1)\sin(\pi H).$$

We have to associate with this measure μ the mass function and the eigenfunctions A and B. But we already know from [9] that the mass function of the string associated with the measure $\mu_0(d\lambda) = c_{H1}^2 \lambda^{1-2H} d\lambda$ is of the form

$$\mathbf{m}_0(x) = \frac{\kappa_H^{1/H}}{4H(1-H)} x^{\frac{1-H}{H}}, \qquad \kappa_H = \frac{2\pi^{3/2}}{\Gamma(1-H)\Gamma(1/2+H)}.$$

The corresponding eigenfunctions are

$$A_0(x,\lambda) = \Gamma(1-H)\Big(\frac{\lambda}{2}\Big)^H \sqrt{\kappa_H x}\, J_{-H}\Big(\lambda(\kappa_H x)^{\frac{1}{2H}}\Big)$$

and

$$B_0(x,\lambda) = \frac{\kappa_H \Gamma(1-H)}{2H} \Big(\frac{\lambda}{2}\Big)^H (\kappa_H x)^{\frac{1-H}{2H}} J_{1-H}\Big(\lambda(\kappa_H x)^{\frac{1}{2H}}\Big).$$

So, evoking "rule 1" of [5], p. 265, we easily write down the mass function $\mathbf{m}$ and eigenfunctions A and B associated with the present μ:

$$\begin{aligned}
\mathbf{m}(x) &= \frac{\kappa_{HN}^{1/H}}{4H(1-H)} x^{\frac{1-H}{H}}, \\
A(x,\lambda) &= \Gamma(1-H)\Big(\frac{\lambda}{2}\Big)^H \sqrt{\kappa_{HN} x}\, J_{-H}\Big(\lambda(\kappa_{HN} x)^{\frac{1}{2H}}\Big), \\
B(x,\lambda) &= \frac{\kappa_{HN}\Gamma(1-H)}{2H} \Big(\frac{\lambda}{2}\Big)^H (\kappa_{HN} x)^{\frac{1-H}{2H}} J_{1-H}\Big(\lambda(\kappa_{HN} x)^{\frac{1}{2H}}\Big)
\end{aligned}$$

where

$$\kappa_{HN} = \frac{2\pi^{(N+2)/2}}{\Gamma(H+N/2)\Gamma(1-H)}.$$

In this case $x(t) = t^{2H}/\kappa_{HN}$, hence

$$A(x(t),\lambda) = \Gamma(1-H)\Big(\frac{\lambda t}{2}\Big)^H J_{-H}(\lambda t), \tag{7.10}$$

$$B(x(t),\lambda)x'(t) = \Gamma(1-H)\Big(\frac{\lambda t}{2}\Big)^H J_{1-H}(\lambda t). \tag{7.11}$$

7.2.1 Moving average representation

In this section we extend the results of section 7.1.1 to the case of arbitrary Hurst index. We need the inverse transforms of function G_l with respect to the measure (7.9) obtained by using the eigenfunctions (7.10) and (7.11). First fix an odd positive l. Then the kernel k_l is given by

$$\begin{aligned} k_l(r,u) &= -\frac{c_{HN}^2}{\pi}\Gamma(N/2)\Gamma(1-H)\frac{2^{N/2}}{r^{(N-2)/2}}\left(\frac{u}{2}\right)^H \\ &\times \int_0^\infty J_{l+(N-2)/2}(r\lambda)J_{-H}(u\lambda)\frac{d\lambda}{\lambda^{H+\frac{N-2}{2}}}. \end{aligned}$$

The integral is taken by the help of formula 6.574.1 of [14] which yields

$$\begin{aligned} k_l(r,u) &= -\frac{c_{HN}^2}{\pi}\frac{\Gamma(\frac{N}{2})\Gamma\left(\frac{1-2H+l}{2}\right)}{\Gamma\left(\frac{2H-1+l+N}{2}\right)}\left(\frac{r}{2}\right)^{2H-1} \\ &\times F\left(\frac{1-2H+l}{2}, \frac{3-2H-l-N}{2}; 1-H; \frac{u^2}{r^2}\right). \end{aligned} \tag{7.12}$$

For even $l > 0$, the same formula yields

$$\begin{aligned} k_l(r,u) &= -\frac{c_{HN}^2}{\pi}\frac{\Gamma(\frac{N}{2})\Gamma\left(\frac{2-2H+l}{2}\right)}{(1-H)\Gamma\left(\frac{2H-2+l+N}{2}\right)}\left(\frac{r}{2}\right)^{2H-1} \\ &\times \frac{u}{r}F\left(\frac{2-2H+l}{2}, \frac{4-2H-l-N}{2}; 2-H; \frac{u^2}{r^2}\right). \end{aligned} \tag{7.13}$$

Finally, we will show that

$$\begin{aligned} k_0(r,u) &= \frac{2c_{HN}^2\Gamma(\frac{N}{2})\Gamma(1-H)}{\pi\Gamma(H+\frac{N-2}{2})}\left(\frac{u}{2}\right)^{2H-1} \\ &\times \int_{u/r}^1 y^{1-2H}(1-y^2)^{H-1+\frac{N-2}{2}}\,dy. \end{aligned} \tag{7.14}$$

Indeed, this kernel k_0 is computed as the sum of (7.13) with $l = 0$ and

$$c_{HN}^2 \Gamma(1-H) \left(\frac{u}{2}\right)^H \frac{2}{\pi} \int_0^\infty J_{1-H}(u\lambda) \frac{d\lambda}{\lambda^H}.$$

The latter integral is taken by means of formulas 6.561.14 and 8.391 of [14] and we get

$$k_0(r,u) = \frac{c_{HN}^2 \Gamma^2(1-H)}{\pi} \left(\frac{u}{2}\right)^{2H-1} \left[1 - \frac{B_{u^2/r^2}(1-H, H+\frac{N-2}{2})}{B(1-H, H+\frac{N-2}{2})}\right]$$

where

$$B_x(\alpha,\beta) = \int_0^x y^{\alpha-1}(1-y)^{\beta-1} dy$$

is the *incomplete beta function*. Thus (7.14) is confirmed.

We have proved the following result

Theorem 7.4. *Let X be Lévy's fractional Brownian motion on $\mathbb{R}^N$ with Hurst index $H \in (0,1)$. Then it is represented as follows*

$$X_t = \frac{2\sqrt{2}\pi^{N/2+1}}{\Gamma(N/2)} \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right) \int_0^{\|t\|} k_l(\|t\|,u) dM_l^m(u) \tag{7.15}$$

where k_l is given by (7.12)–(7.14) and $\{M_l^m\}$ is the family of independent Gaussian martingales, each with mean zero and the variance function

$$\mathbb{E}|M_l^m(u)|^2 = \frac{\pi^{N/2}}{2H\Gamma(2-H)\Gamma(H+N/2)} u^{2-2H}.$$

Remark 7.5. Note that in the scalar case $N = 1$ the representation (7.15) takes the form

$$X_t = 2\pi d_H \sum_{l \in \{0,1\}} \int_0^t k_l(t,u) u^{1/2-H} dW_l(u), \qquad d_H^2 = \frac{\pi^{N/2}(1-H)}{H\Gamma(2-H)\Gamma(H+\frac{N}{2})},$$

where W_0 and W_1 are independent standard Brownian motions. It is not hard to verify that the latter representation does coincide with that of Dzhaparidze and van Zanten [6]:

$$X_t = \sqrt{2} c_{H1} \int_0^t k_t^e(u) dW_0(u) - \sqrt{2} c_{H1} \int_0^t k_t^o(u) dW_1(u)$$

where

$$k_t^o(u) = \int_0^\infty \frac{\sin \lambda t}{\lambda^{H+1/2}} J_{-H}(\lambda u)\sqrt{\lambda u}\, d\lambda,$$
$$k_t^e(u) = \int_0^\infty \frac{1-\cos \lambda t}{\lambda^{H+1/2}} J_{1-H}(\lambda u)\sqrt{\lambda u}\, d\lambda.$$

7.2.2 Series expansion

Theorem 7.6. *Let $\omega_0 < \omega_1 < \omega_2 < \cdots$ be the non-negative real-valued zeros of the Bessel function J_{-H}. Then Lévy's fractional Brownian motion X with Hurst index H restricted to the ball $\mathcal{B}_T$ of radius T (cf. (5.1)) can be represented as follows*

$$X_t = \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,N)} \sum_{n=0}^{\infty} S_l^m\left(\frac{t}{\|t\|}\right) G_l\left(\|t\|, \frac{\omega_n}{T}\right) \xi_{l,n}^m \qquad (7.16)$$

where $\xi_{l,n}^m$ are independent mean zero Gaussian random variables with variances

$$\sigma_n^2 = \frac{2^{2H+3} H \pi^{N/2} \Gamma(H+N/2)}{T^{2-2H}\Gamma(1-H)\Gamma^2(N/2)} \omega_n^{-2H} J_{1-H}^{-2}(\omega_n).$$

This series converges with probability 1 in the space of continuous functions on $\mathcal{B}_T$.

Proof. By (7.10) we have $A(x(t),\lambda) = 0$ if and only if $\lambda = \omega_n/T$ and

$$\left.\frac{\partial}{\partial \omega} A(x(T),\omega)\right|_{\omega=\omega_n/T} = -\Gamma(1-H) T \left(\frac{\omega_n}{2}\right)^H J_{1-H}(\omega_n).$$

By (7.11)

$$B(x(T),\omega_n) = \frac{\Gamma(1-H)\left(\frac{\omega_n}{2}\right)^H J_{1-H}(\omega_n)\kappa_{HN}}{2HT^{2H-1}}.$$

The required expression for σ_n^2 is now verified via (5.4) and the assertion of the present theorem follows from Theorem 5.1. □

8 Concluding remarks

The examples of the preceding section were meant, at first instance, to illustrate our general statements in the sections 4–6. But they also bring about some ideas for further development that we want to discuss.

The method of vibrating strings have determined the form of the moving average representations (4.13) and (6.8), as well as of the series expansion (5.3). For, the kernels in (4.13) and (6.8) are obtained by means of the even and odd transforms with respect to the eigenfunctions A and B of the related string, and (5.3) does involve the zero's of A. However, on studding Lévy's Brownian motion, H. McKean [22] has pointed out that our kernels (7.2)–(7.3) are in fact singular in the sense that nontrivial square integrable function can be found that are orthogonal to k_l when $l > 2$. For instance

$$\int_0^r u^p C_l^{N/2}\left(\frac{u}{r}\right)\left(1-\frac{u^2}{r^2}\right)^{\frac{N-1}{2}} du = 0$$

if $l > 2$ and $0 < p < l$. Therefore, McKean proposed to work with the different kernels $k_l(r,u) = d_l(u/r)$ where

$$d_l(x) = \begin{cases} B(\frac{N-1}{2},\frac{1}{2})^{-1}\int_x^1 (1-z^2)^{\frac{N-3}{2}} dz & l = 0, \\ \frac{1}{2}B(\frac{N-1}{2},\frac{1}{2})^{-1} x^{l-1}\left(1-x^2\right)^{\frac{N-1}{2}} & l > 0, \end{cases}$$

that are more appropriate for most purposes, being nonsingular. To see this, represent the function G_l, $l > 0$ given by (3.9) in an alternative way, not as the Fourier transform (3.12) but as the Hankel transform

$$G_l(r,\lambda) = \frac{\Gamma(\frac{N}{2})}{\sqrt{2}\Gamma(\frac{N+1}{2})}\int_0^r \left(\frac{u}{r}\right)^{l-1}\left(1-\frac{u^2}{r^2}\right)^{\frac{N-1}{2}} J_{l-3/2}(\lambda u)\sqrt{\lambda u}\, du$$

(see formula (33) on p. 26 of [13], that is invertible due to formula (7) on p. 48 of [13]), which can actually be written as a fractional integral of Erdélyi–Kober type, namely up to a constant c

$$G_l(r,\lambda) = c\, I^{\frac{N+1}{2}}_{0+;2,\frac{l-2}{2}}\left(J_{l-3/2}(\lambda\cdot)\sqrt{\lambda\cdot}\right)(r) \tag{8.1}$$

(see Samko at al. [26], p. 322). Hence the operators with McKean's new kernels transforming $J_{l-3/2}(\lambda\cdot)\sqrt{\lambda\cdot}$ into $G_l(\cdot,\lambda)$, are non-singular for every $l > 0$. Note that in the case of odd dimensions, say $N = 2n+1$, we have just $n-$fold repeated integrals and the inversion is carried out by the $n-$fold differentiation. By the way, due to this fact McKean [22] confirmed Lévy's conjecture that the Brownian motion possesses Markov property in odd dimensional spaces, but not in even dimensional spaces since the corresponding inverse differential operators are fractional.

Nonsingular kernels in the case of Lévy's fractional Brownian motion can be found in Malyarenko [21]. They are of the form $r^{\frac{1}{2}-H}\, k_l(r,u) = d_l(u/r)$ with

$$d_l(x) = \begin{cases} c_0 x^{H-\frac{1}{2}} \int_x^1 z^{1-2H}(1-z^2)^{H-1+\frac{N-2}{2}} dz & l = 0, \\ c_l x^{l-H-\frac{1}{2}} \left(1-x^2\right)^{H+\frac{N-2}{2}}, & l > 0, \end{cases}$$

where c_l are some constants. Note that in this case we have the similar to (8.1) relation but with the fractional integral $I^{H+\frac{N}{2}}_{0+;2,\frac{l-1}{2}-H}$.

Being aware of the advantages caused by the non-singularity of these new kernels, we intend in our forthcoming work to extend our methods of vibrating strings so as to obtain non-singular modifications to the general results of the sections 4–6, and to provide aforementioned applications to the ordinary and fractional Brownian motions as special cases.

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