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REPORTRAPPORT

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Campbell and moment measures for finite sequential spatial processes

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REPORT PNA-R0601 JANUARY 2006

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ISSN 1386-3711

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ABSTRACT
We define moment and Campbell measures for sequential spatial processes, prove a Campbell-Mecke theorem, and relate the results to their counterparts in the theory of point processes. In particular, we show that any finite sequential spatial process model can be derived as the vector obtained by sorting the points in a suitably chosen (unordered) spatio-temporal point process by increasing time mark. The measures are used to formally define the dual concepts of interior and exterior conditioning through Palm distributions and Papangelou conditional intensities.

2000 Mathematics Subject Classification: 60G55, 60D05.
Keywords and Phrases: Campbell measure, Campbell-Mecke theorem, conditional intensity, Janossy density, moment measure, Palm distribution, sequential spatial process.
Note: This research is supported by the Technology Foundation SSW, applied science division of NWO, and the technology programme of the Ministry of Economic Affairs (project CWI.6156 `Markov sequential point processes for image analysis and statistical physics').

Campbell and Moment Measures for Finite Sequential Spatial Processes

M.N.M. van Lieshout

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1 Preliminaries

This paper is concerned with finite sequential spatial point processes $\vec{\Pi}$ whose value space N^{f} consists of finite sequences

$$\vec{\mathbf{x}} = (x_1, \ldots, x_n) \qquad n \in \mathbb{N}_0$$

of points in some compact subset D of the plane [2]. The length n of the sequence is an arbitrary integer. Clearly, $\mathbb{R}^2$ may be replaced by $\mathbb{R}^d$; in fact D could be any Polish space [1]. A mark in some complete separable metric space, say L, may be attached to each point. Often, the mark is used for specifying shape parameters in an object processes, or type labels in a mixed species model. Thus, each x_i can be written as $(d_i, m_i) \in D \times L$. The metric defines a topology and Borel σ-algebra $\mathcal{B}_L$ on L, and one may equip the spaces $(D \times L)^n$ with their product σ-algebras.

Throughout this paper, we shall assume that $\vec{\Pi}$ is absolutely continuous, so that its distribution can be defined as follows. Given a finite diffuse Borel measure $\mu(\cdot)$ on $(D, \mathcal{B}_D)$, and a mark probability measure μ_L, specify

1. a probability mass function q_n, $n \in \mathbb{N}_0$, for the number of points in D;
2. for each n, a Borel measurable and $(\mu \times \mu_L)^n$-integrable joint probability density $p_n(x_1, \ldots, x_n)$ for the locations $x_1, \ldots, x_n \in D$, given that the sequence has length n.

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Alternatively, a probability density $f(\cdot)$ on N^{f} may be specified with respect to the reference measure

$$\nu(F) = \sum_{n=0}^{\infty} \frac{e^{-\mu(D)}}{n!} \int_{(D\times L)^n} \mathbf{1}\left\{(x_1,\ldots,x_n) \in F\right\} d\mu \times \mu_L(x_1) \ldots d\mu \times \mu_L(x_n)$$

defined for F in the σ-algebra $\mathcal{N}^{\mathrm{f}}$ on finite marked sequences generated by the product Borel σ-fields on $(D \times L)^n$. In words, $\nu(\cdot)$ corresponds to random sequences of Poisson length with independent and uniformly distributed components. It is not hard to see that

$$q_n = \frac{e^{-\mu(D)}}{n!} \int_{(D\times L)^n} f(x_1,\ldots,x_n)\, d\mu \times \mu_L(x_1) \ldots d\mu \times \mu_L(x_n)$$

$$p_n(x_1,\ldots,x_n) = \frac{e^{-\mu(D)}}{n!\; q_n} f(x_1,\ldots,x_n)$$

$$f(x_1,\ldots,x_n) = e^{\mu(D)}\, n!\, q_n\, p_n(x_1,\ldots,x_n)$$

hold for each $n \in \mathbb{N}_0$. Last but not least, we may consider the likelihood of a sequence of exactly n objects located successively at each of the infinitesimal regions d_1 up to d_n and marked $m_1,\ldots,m_n$, which is known as a Janossy density and denoted by $j_n(x_1,\ldots,x_n)$. Thus,

$$j_n(\vec{\mathbf{x}}) = p_n(\vec{\mathbf{x}})\, q_n = \frac{e^{-\mu(D)}}{n!}\, f(\vec{\mathbf{x}})$$

for $\vec{\mathbf{x}} = (x_1,\ldots,x_n) = ((d_1,m_1),\ldots,(d_n,m_n))$. Neither $f(\cdot)$, nor the $p_n(\cdot,\ldots,\cdot)$ or $j_n(\cdot,\ldots,\cdot)$ are required to be symmetric, as they would for point processes. For further details, we refer to the comprehensive textbook by Daley and Vere–Jones [1].

2 Relationship with marked point process

Since the mapping

$$i_n : (D \times L)^n \to (D \times L)^n/\equiv \;\text{ defined by } i_n(x_1,\ldots,x_n) = \{x_1,\ldots,x_n\}$$

is Borel measurable for each $n \in \mathbb{N}$ with respect to the trace of the σ-algebra on unordered point configurations generated by the family of mappings $N(A \times B)$, the number of points in Borel subsets A of D with marks in $B \in \mathcal{B}_L$, any absolutely continuous finite sequential spatial point process $\vec{\Pi}$ can be mapped on a unique finite marked point process, denoted by Π. The mapping does not affect the distribution of the total number of marked points q_n. The Janossy density of Π with respect to $(\mu \times \mu_L)^n$ is the sum of the sequential Janossy densities j_n evaluated at the $n!$ permutations $\varphi(\mathbf{x})$ of $\mathbf{x} = \{x_1, \ldots, x_n\}$,

$$j_n^s(x_1, \ldots, x_n) = \sum_{\varphi} j_n(\varphi(\mathbf{x})), \tag{1}$$

cf. [1]. Clearly, $j_n^s(\cdot, \ldots, \cdot)$ is symmetric in its arguments, and one may write $j_n^s(\mathbf{x})$ for convenience. Consequently, the joint symmetric conditional probability density for the marked points of Π is given by

$$p_n^s(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\varphi} p_n(\varphi(\mathbf{x})),$$

the average of the sequential joint conditional probability density p_n over permutations, and Π has Radon–Nikodym derivative

$$f^s(\{x_1, \ldots, x_n\}) = \frac{1}{n!} \sum_{\varphi} f(\varphi(\mathbf{x}))$$

with respect to a Poisson process on D with intensity measure μ randomly labelled according to μ_L.

The goal of this section is to show the dual property that any finite sequential spatial process can be derived as the time-ordered vector of points in a classic spatio-temporal marked point process.

Proposition 2.1. *Let Y be a sequential spatial process on $D \times L$ with Janossy densities $j_n(\cdot)$, $n \in \mathbb{N}_0$. Then there exists a spatio-temporal marked point process X on $D \times L \times [0,1]$ such that Y has the same distribution as the vector of marked points in X ordered by their time mark in $[0,1]$.*

Proof. Define a marked point process Z on $D \times L$ by (1). Conditional on marked point configuration $\mathbf{x} = \{x_1, \ldots, x_n\}$, time marks must be assigned such that, say, the ordering $(x_1, \ldots, x_n)$ occurs with probability

$$\frac{j_n(x_1, \ldots, x_n)}{j_n^s(x_1, \ldots, x_n)}. \tag{2}$$

To do so, let $\pi_i : \{1, \ldots, n\} \to \{1, \ldots, n\}$, $i = 1, \ldots, n!$, be the $n!$ permutations of the integers up to n, and fix the labels of the points in the configuration $\mathbf{x}$ arbitrarily. Thus, a unique correspondence exists between orderings of $\mathbf{x}$ and the family of permutations π_i. Write $\pi_i(\mathbf{x})$ for the ordering of $\mathbf{x}$ identified with π_i. The space $(0,1)^n$ can similarly be divided into $n!$ open cells $C_i = \{t_{\pi_i(1)} < t_{\pi_i(2)} < \cdots < t_{\pi_i(n)}\}$, to which we assign mass $j_n(\pi_i(\mathbf{x}))/j_n^s(\mathbf{x})$. In other words, if marks in C_i were assigned to the marked point configuration $\mathbf{x}$ such that components x_k of $\pi_i(\mathbf{x})$ has time mark t_k, the time ordered vector of marked points would be $\pi_i(\mathbf{x})$.

The distribution of the time marks conditional on $\mathbf{x}$ is as follows. First a cell C_i is selected according to its probability mass, then a uniformly distributed point

$(t_1, \ldots, t_n)$ in the chosen C_i, and next mark t_k is assigned to the k-th component of $\pi_i(\mathbf{x})$. With slight abuse of notation, we use the notation j_n^s for a Janossy distribution of the resulting spatio-temporal point process X on $D \times L \times [0,1]$ too.

Under the above assumptions, a Janossy density for the vector of marked points in X ordered by time is

$$\tilde{j}_n(\pi_i(\mathbf{x})) = \int_{C_i} j_n^s(\{(x_1,t_1),\ldots,(x_n,t_n)\})dt_1\ldots dt_n =$$

$$j_n^s(\mathbf{x}) \int_{C_i} \frac{j_n(\pi_i(\mathbf{x}))}{j_n^s(\mathbf{x})} \frac{j_n^s(\mathbf{x})}{j_n(\pi_i(\mathbf{x}))} dt_1 \ldots dt_n = j_n^s(\mathbf{x}) \frac{j_n(\pi_i(\mathbf{x}))}{j_n^s(\mathbf{x})} = j_n(\pi_i(\mathbf{x}))$$

for $i = 1, \ldots, n$, and the result is proved. □

In the symmetric case, (2) reduces to $1/n!$, so that each permutation is equally likely.

3 Campbell and moment measures

By the definition of the σ-algebra $\mathcal{N}^{\mathrm{f}}$, the number of sequence components that fall in Borel product sets is a random variable. The same statement is true when attention is restricted to sequences with specific properties. The expectations of such random variables are used to define moment and Campbell measures, in analogy with classic marked point processes [1].

Write $N(A) := \sum_{x_i \in X} \mathbf{1}\{x_i \in A\}$ for all Borel sets $A \subseteq D \times L$. In words, $N(A)$ is the number of marked points in the random sequence X that fall in A.

Definition 3.1. Let $\vec{\Pi}$ be a sequential point process on a compact subset D of the plane with marks in the complete separable metric space L. For Borel sets $A \subseteq D$ and $B \subseteq L$, write $N(A \times B)$ for the number of components of $\vec{\Pi}$ with location in A and mark in B. Define,

$$M(A \times B) := \mathbb{E}N(A \times B).$$

For $F \in \mathcal{N}^{\mathrm{f}}$, set

$$C((A \times B) \times F) := \mathbb{E}\left[N(A \times B)\mathbf{1}\left\{\vec{\Pi} \in F\right\}\right].$$

In general, the set function M need not be finite; if it is, it defines a measure on the Borel product σ-algebra on $D \times L$. The set function C can always be extended to a σ-finite measure.

Proposition 3.2. *If the function $M(\cdot)$ is finite it can be extended uniquely to a finite measure on the product σ-algebra generated by $\mathcal{B}_D$ and $\mathcal{B}_L$, the* first order moment measure.

The function $C(\cdot \times \cdot)$ can be extended uniquely to a σ-finite measure on the product σ-algebra of the Borel sets and $\mathcal{N}^{\mathrm{f}}$, the first order Campbell measure.

Note that C is finite if and only if M is.

Proof. To show that M is a measure, suppose that $A \times B \in \mathcal{B}_D \times \mathcal{B}_L$ can be written as a countable union of disjoint sets $A_i \times B_i \in \mathcal{B}_D \times \mathcal{B}_L$ in the semi-ring of product sets. Then

$$\mathbb{E}N(A \times B) = \mathbb{E}\left[\sum_{i=1}^{\infty} N(A_i \times B_i)\right] = \sum_{i=1}^{\infty} \mathbb{E}N(A_i \times B_i) = \sum_{i=1}^{\infty} M(A_i \times B_i)$$

by the monotone convergence theorem. Hence M is countably additive. By the measure extension theorem and the assumption that M is finite, there exists a unique measure on the σ-field generated by product sets that extends M, and this measure is σ-finite.

Next, turn to the set function C, and consider a product set of the form $A \times B \times F$ that can be written as a countable union of disjoint sets $A_i \times B_i \times F_i \in \mathcal{B}_D \times \mathcal{B}_L \times \mathcal{N}^{\mathrm{f}}$. As for the moment measure,

$$C((A \times B) \times F) = \mathbb{E}\left[\sum_{i=1}^{\infty} N(A_i \times B_i)\mathbf{1}\left\{\vec{\Pi} \in F\right\}\right] = \sum_{i=1}^{\infty} C((A_i \times B_i) \times F_i).$$

To show that $C(\cdot \times \cdot)$ is σ-finite, by the topological structure imposed on D and L, countable coverings exist, say A_i, B_i, $i = 1 \ldots, \infty$. Let $F_{ijn} := \{\#(\vec{\Pi} \cap A_i \times B_j) \leq n\}$. Then the $(A_i \times B_j) \times F_{ijn}$, $i, j, n \in \mathbb{N}$ form a countable covering of $(D \times B) \times N^{\mathrm{f}}$, and $C((A_i \times B_j) \times F_{ijn}) \leq n < \infty$. Hence, another appeal to the measure extension theorem concludes the proof. □

If all the conditional joint probability densities of a sequential spatial process $\vec{\Pi}$ are symmetric, the unordered marked point process Π obtained from $\vec{\Pi}$ by ignoring permutations (cf. section 2) has the same first order moment measure, since

$$M^s(A) = \sum_{n=1}^{\infty} q_n \frac{1}{n!} \sum_{\pi} \mathbb{E}_n N(A) = M(A).$$

Furthermore, for any F_s in the σ-algebra on marked point sets generated by $N(A \times B)$ for $A \in \mathcal{B}_D$, $B \in \mathcal{B}_L$, the Campbell measure C^s of Π satisfies the equation

$$C^s(A \times F_s) = \sum_{n=0}^{\infty} q_n \frac{1}{n!} \sum_{\pi} \mathbb{E}_n \left[N(A)\ \mathbf{1}\left\{i_n(X_1, \ldots, X_n) \in F_s\right\}\right] = C(A \times i^{-1}(F_s))$$

where $i^{-1}(F) = \cup_{n=0}^{\infty} i_n^{-1}(F_s) \in \mathcal{N}^{\mathrm{f}}$.

Next, turn to componentwise measures, as defined below.

Definition 3.3. Let $\vec{\Pi}$ be a sequential point process on a compact subset D of the plane with marks in the complete separable metric space L. Write $\mathbb{E}_n$ for the expectation with respect to the conditional joint n-point probability density p_n,

$n \in \mathbb{N}_0$, and X_k, $k = 1, \ldots, n$, for the k-th component of the random vector. For Borel sets $A \subseteq D$ and $B \subseteq L$, $F \in \mathcal{N}^{\mathrm{f}}$, and $k \in \mathbb{N}$, define,

$$\begin{aligned} M_k(A \times B) &:= \sum_{n=k}^{\infty} q_n \mathbb{E}_n \left[\mathbf{1} \{ X_k \in A \times B \} \right] \\ C_k((A \times B) \times F) &:= \sum_{n=k}^{\infty} q_n \mathbb{E}_n \left[\mathbf{1} \{ X_k \in A \times B \} \mathbf{1} \{ (X_1, \ldots, X_n) \in F \} \right]. \end{aligned}$$

Note that the set functions M_k and C_k are measures as they are a countable sum of finite measures. Both are finite with values bounded by $\sum_{n \geq k} q_n <= 1$, hence can be extended uniquely to finite measures on the appropriate σ-algebras, cf. Proposition 3.2, the *componentwise first order moment and Campbell measures.*

If p_n is symmetric,

$$M_k(A) = \sum_{n=k}^{\infty} q_n \mathbb{E}_n \left[\mathbf{1} \{ X_1 \in A \} \right] = M_1(A) - \sum_{n=1}^{k-1} q_n \mathbb{E}_n \mathbf{1} \{ X_1 \in A \}$$

is decreasing in the component index.

The next result expresses the moment and Campbell measures as the sum of their componentwise sisters.

Proposition 3.4. *Provided $M(\cdot)$ is finite, $M(A) = \sum_{k=1}^{\infty} M_k(A)$ for any A in the product σ-algebra of $\mathcal{B}_D$ and $\mathcal{B}_L$. For all B in the product σ-algebra of $\mathcal{B}_D$, $\mathcal{B}_L$, and $\mathcal{N}^{\mathrm{f}}$, $C(B) = \sum_{k=1}^{\infty} C_k(B)$.*

Proof. To prove the first equality, observe that

$$\begin{aligned} \sum_{k=1}^{\infty} M_k(A) &= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} q_n \mathbb{E}_n \left[\mathbf{1} \{ X_k \in A \} \right] = \sum_{n=1}^{\infty} q_n \sum_{k=1}^{n} \mathbb{E}_n \left[\mathbf{1} \{ X_k \in A \} \right] \\ &= \sum_{n=1}^{\infty} q_n \mathbb{E}_n \left[N(A) \right] = M(A) \end{aligned}$$

by Fubini. Similarly, for any Borel subset $A \subseteq D \times L$ and $F \in \mathcal{N}^{\mathrm{f}}$,

$$\begin{aligned} \sum_{k=1}^{\infty} C_k(A \times F) &= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} q_n \mathbb{E}_n \left[\mathbf{1} \{ X_k \in A \} \; \mathbf{1} \{ (X_1, \ldots, X_n) \in F \} \right] \\ &= \sum_{n=1}^{\infty} q_n \mathbb{E}_n \left[\sum_{k=1}^{n} \mathbf{1} \{ X_k \in A \} \mathbf{1} \{ (X_1, \ldots, X_n) \in F \} \right] \\ &= \sum_{n=1}^{\infty} q_n \mathbb{E}_n \left[N(A) \; \mathbf{1} \{ (X_1, \ldots, X_n) \in F \} \right] = C(A \times F). \end{aligned}$$

Since both C and $\sum_k C_k$ possess unique σ-finite extensions on the product σ-algebra, $\sum_k C_k(B) = C(B)$ for any B. □

4 Campbell–Mecke theorems

Campbell measures are important tools in the theory of sequential spatial processes. Indeed, suppose that measurements are made at each component of a data sequence, possibly depending on the component index as well as other components. By the *Campbell–Mecke* theorem stated below, the expected sum of measurements may be expressed in terms of integrals with respect to the componentwise Campbell measures. This integral representation will form the theoretical basis for the conditional intensities to be discussed in the next section.

Proposition 4.1. *Let $\vec{\Pi}$ be a sequential point process on a compact subset D of the plane with marks in the complete separable metric space L, C_k its componentwise Campbell measures. Let, for $k \in \mathbb{N}$, $f_k : (D \times L) \times N^{\mathrm{f}} \to \mathbb{R}$ be measurable functions that are either all non-negative or integrable. Then*

$$\mathbb{E}\left[\sum_{x_i \in \vec{\Pi}} f_i(x_i, \vec{\Pi})\right] = \sum_{k=1}^{\infty} \int\int f_k(x, \vec{\mathbf{x}}) dC_k(x, \vec{\mathbf{x}}). \tag{3}$$

Proof. Let $f_k(x, \vec{\mathbf{x}}) := \mathbf{1}\{x \in A_k \times B_k; \vec{\mathbf{x}} \in F_k\}$ for $A_k \in \mathcal{B}_D$, $B_k \in \mathcal{B}_L$, and $F_k \in \mathcal{N}^{\mathrm{f}}$. Then the left hand side of (3) equals

$$\mathbb{E}\left[\sum_{X_i \in \vec{\Pi}} \mathbf{1}\{X_i \in A_i \times B_i\} \mathbf{1}\left\{\vec{\Pi} \in F_k\right\}\right] =$$

$$\sum_{n=1}^{\infty} q_n \sum_{k=1}^{n} \mathbb{E}_n\left[\mathbf{1}\{X_k \in A_k \times B_k; (X_1, \dots, X_n) \in F_k\}\right] =$$

$$\sum_{k=1}^{\infty} \sum_{n=k}^{\infty} q_n \mathbb{E}_n\left[\mathbf{1}\{X_k \in A_k \times B_k; (X_1, \dots, X_n) \in F_k\}\right] = \sum_{k=1}^{\infty} C_k((A_k \times B_k) \times F_k)$$

by Fubini, the right hand side of (3). Because product sets generate the σ-algebra on $(D \times L) \times N^{\mathrm{f}}$, (3) holds for $f_k(x, \vec{\mathbf{x}}) = \mathbf{1}\{(x, \vec{\mathbf{x}}) \in C_k\}$ for any family $(C_k)_{k \in \mathbb{N}}$ in the product σ-algebra by the monotone convergence theorem. The proof is completed by the usual measure theoretic route from indicator functions via step functions to arbitrary ones of the desired form. □

Several special cases are of interest:

- if all $f_k \equiv f$ are equal,

$$\mathbb{E}\left[\sum_{x \in \vec{\Pi}} f(x, \vec{\Pi})\right] = \int\int f(x, \vec{\mathbf{x}}) dC(x, \vec{\mathbf{x}}) = \sum_{k=1}^{\infty} \int\int f(x, \vec{\mathbf{x}}) dC_k(x, \vec{\mathbf{x}});$$

- if $M(\cdot)$ is finite and $f_k(x, \vec{\mathbf{x}}) = f(x)$ does not depend on its second argument,

$$\mathbb{E}\left[\sum_{x\in\vec{\Pi}} f(x)\right] = \int f(x)dM(x) = \sum_{k=0}^{\infty}\int f(x)dM_k(x).$$

5 Interior and exterior conditioning

In this section, we define coordinatewise Palm distributions and conditional intensities in the sequential setting. The conditional intensity quantifies the behaviour of a sequential spatial process at a particular marked point given all other locations and marks as well as their ordering (exterior conditioning), and is important in the definition of Markovianity [2]. Interior conditioning, in contrast, considers the conditional distribution of the whole sequence seen from a fixed location and mark.

From now on, assume the first order moment measure is finite. Then

$$C((A\times B)\times F) = \mathbb{E}\left[N(A\times B)\mathbf{1}\left\{\vec{\Pi}\in F\right\}\right] \le M(A\times B)$$

for every fixed $F \in \mathcal{N}^{\mathrm{f}}$ and $A\times B \in \mathcal{B}_D \times \mathcal{B}_L$. Hence, for fixed F, $C(\cdot\times F) << M(\cdot)$, so the Radon–Nikodym theorem implies existence of a non-negative Borel measurable and M-integrable function $P_x(F)$, $x \in D\times L$, so that

$$C((A\times B)\times F) = \int_{A\times B} P^x(F)dM(x).$$

Note that $P^x(F)$, which shall be called *Palm distribution*, is defined uniquely up to x in some M-null set. Analogously $C_k(\cdot\times F) << M_k(\cdot)$ for any given $F\in\mathcal{N}^{\mathrm{f}}$, so that *componentwise Palm distributions* $P_k^x(F)$ may be defined by

$$C_k(A\times F) = \int_A P_k^x(F)\, dM_k(x).$$

In order to define conditional intensities, we need the concept of *reduced Campbell measure* defined by

$$C^!((A\times B)\times F) := \mathbb{E}\left[\sum_{x_i\in\vec{\Pi}\cap(A\times B)} \mathbf{1}\left\{\vec{\Pi}_{(-i)}\in F\right\}\right]$$

for $A\in\mathcal{B}_D$, $B\in\mathcal{B}_L$, and $F\in\mathcal{N}^{\mathrm{f}}$. Here, the notation $\vec{\Pi}_{(-i)}$ is used for the sequence obtained from $\vec{\Pi}$ by deletion of the i-th component. Similarly, for $k = 1, 2, \ldots,$

$$C_k^!((A\times B)\times F) := \sum_{n=k}^{\infty} q_n\mathbb{E}_n\left[\mathbf{1}\left\{X_k\in A\times B\right\}\mathbf{1}\left\{(X_1,\ldots,X_{k-1},X_k,\ldots,X_n)\in F\right\}\right].$$

Integral relations similar to (3) hold, for example

$$\mathbb{E}\left[\sum_{x_i\in\vec{\Pi}} f_i(x_i,\vec{\Pi}_{(-i)})\right] = \sum_{k=1}^{\infty}\int\int f_k(x,\vec{\mathbf{x}})dC_k^!(x,\vec{\mathbf{x}}).$$

for measurable, non-negative or integrable functions $f_k : (D \times L) \times N^{\mathrm{f}} \to \mathbb{R}$, $k \in \mathbb{N}$. In particular, if $f_i \equiv 0$ except for $i = k$, the right hand side reduces to $\int\int f_k(x;\vec{\mathbf{x}})\, dC_k^!(x,\vec{\mathbf{x}})$, which can be used as an alternative definition of $C_k^!$.

Definition 5.1. Let $\vec{\Pi}$ be a sequential point process on a compact subset D of the plane with marks in the complete separable metric space L with finite first order moment measure $M(\cdot)$ and componentwise reduced Campbell measures $C_k^!$ such that $C_k^!((A \times B) \times \cdot) << P$ for all $A \in \mathcal{B}_D$, $B \in \mathcal{B}_L$. Then

$$C_k^!((A\times B)\times F) = \mathbb{E}\left[\mathbf{1}\left\{\vec{\Pi}\in F\right\}\Lambda_k(A\times B;\vec{\Pi})\right]$$

for all $F \in \mathcal{N}^{\mathrm{f}}$, with Radon–Nikodym derivatives $\Lambda_k(A \times B; \cdot)$ defined up to a null set, the *Papangelou kernels.*

There exist regular versions [1, p. 166] for which $\Lambda_k(A \times B; \cdot)$ is a measurable function for any fixed A and B, and a Borel measure $\Lambda_k(\cdot;\vec{\mathbf{x}})$ for all fixed marked point sequences $\vec{\mathbf{x}}$. If the latter has a density $\lambda_k(\cdot;\vec{\mathbf{x}})$ with respect to $\mu \times \mu_M$, λ_k will be called a *conditional intensity*. In that case, the integral relation

$$\mathbb{E}\left[\int f_k(x;\vec{\Pi})\;\lambda_k(x;\vec{\Pi})\;d\mu\times\mu_L(x)\right] = \int\int f_k(x;\vec{\mathbf{x}})\;dC_k^!(x,\vec{\mathbf{x}}) \tag{4}$$

holds for any measurable, non-negative or integrable function $f_k : (D{\times}L){\times}N^{\mathrm{f}} \to \mathbb{R}$. The overall conditional probability of finding a marked point at $dx = d\mu \times \mu_L(x)$ in any position given that the remainder of the sequence equals $\vec{\mathbf{x}}$ is given by $\sum_{i=1}^{n+1} \lambda_i(dx;\vec{\mathbf{x}})$, where n is the length of $\vec{\mathbf{x}}$.

Proposition 5.2. *Let $\vec{\Pi}$ be a sequential point process on a compact subset D of the plane with marks in the complete separable metric space L with finite first order moment measure $M(\cdot)$ and componentwise reduced Campbell measures $C_k^!$, $k \in \mathbb{N}$. Suppose the Janossy densities are positive everywhere. Then componentwise conditional intensities exist and are given by*

$$\lambda_k(x;\vec{\mathbf{x}}) = \frac{j_{n(\vec{\mathbf{x}})+1}(s_k(x,\vec{\mathbf{x}}))}{j_{n(\vec{\mathbf{x}})}(\vec{\mathbf{x}})} = \frac{f(s_k(x,\vec{\mathbf{x}}))}{f(\vec{\mathbf{x}})(n(\vec{\mathbf{x}})+1)}, \quad k\in\mathbb{N}.$$

Here $s_k(x,\vec{\mathbf{x}})$ is the vector obtained from $\vec{\mathbf{x}}$ by inserting $x \notin \vec{\mathbf{x}}$ at position k, $n(\vec{\mathbf{x}}) \geq k-1$ the length of $\vec{\mathbf{x}}$.

Proof. We shall prove the integral relation (4). Now, for our putative λ_k,

$$\mathbb{E}\left[\int f_k(x;\vec{\Pi})\;\lambda_k(x;\vec{\Pi})\;d\mu\times\mu_L(x)\right] = \sum_{n=k-1}^{\infty}\frac{e^{-\mu(D)}}{n!}\int_{D\times L}\int_{(D\times L)^n}$$

$$f_k(x;\vec{\mathbf{x}})\frac{1}{n+1}\frac{f(s_k(x,\vec{\mathbf{x}}))}{f(\mathbf{x})}f(\mathbf{x})\;d(\mu\times\mu_L)^n(\vec{\mathbf{x}})\;d\mu\times\mu_L(x) =$$

$$\sum_{n=k-1}^{\infty}\frac{e^{-\mu(D)}}{(n+1)!}\int_{D\times L}\int_{(D\times L)^n} f_k(x;\vec{\mathbf{x}})f(s_k(x,\vec{\mathbf{x}}))\;d(\mu\times\mu_L)^n(\vec{\mathbf{x}})\;d\mu\times\mu_L(x) =$$

$$\sum_{n=k}^{\infty}\frac{e^{-\mu(D)}}{n!}\int_{(D\times L)^n} f_k(x_k;\vec{\mathbf{x}}_{(-k)})f(\vec{\mathbf{x}})\;d(\mu\times\mu_L)^n(\vec{\mathbf{x}}),$$

which concludes the proof. □

References

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