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# Tandem Brownian queues

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ABSTRACT
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# Tandem Brownian Queues

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**Abstract**

We analyze a two-node tandem queue with Brownian input. We first derive an explicit expression for the joint distribution function of the workloads of the first and second queue, which also allows us to calculate their exact large-buffer asymptotics. The nature of these asymptotics depends on the model parameters, i.e., there are different regimes. By using sample-path large-deviations (Schilder's theorem) these regimes can be interpreted: we explicitly characterize the most likely way the buffers fill. [1]

[1]This research has been funded by the Dutch BSIK/BRICKS (Basic Research in Informatics for Creating the Knowledge Society) project. M. Mandjes is also affiliated with the Korteweg-de Vries Institute, University of Amsterdam, the Netherlands, and EURANDOM, Eindhoven, the Netherlands.

# 1 Introduction

Consider $\{B(t) - ct, t \geq 0\}$, where $B(t)$ is a standard Brownian motion, and $c > 0$ is a scalar. The distribution of the supremum $\overline{B}_c$ of such a Brownian motion with drift is known: $\mathbb{P}(\overline{B}_c > b) = \exp(-2bc)$. The reflection of $\{B(t) - ct, t \geq 0\}$ at 0 could be called a *Brownian queue*. It can be argued [11] that the steady-state workload $Q$ of such a Brownian queue is distributed as $\overline{B}_c$, i.e., also exponentially with mean $1/(2c)$.

The case of *networks* of Brownian queues is considerably less studied. In [8] and [3] a two-node tandem queue is analyzed: [8] derives the joint distribution function of the first and total queue length, whereas [3] focuses on the distribution function of the second queue. Also, several papers consider the more general case of tandem systems with Lévy input, i.e., arrival processes with stationary, independent increments (this class comprises, besides Brownian motion, also compound Poisson input). We remark that the solution presented in [7] and [4] is in terms of a joint Laplace transform; no explicit expression for the joint distribution function is given.

In this paper we analyze a two-node tandem queue with Brownian input. Building on the work of [8], we explicitly derive the joint distribution function $\mathbb{P}(Q_1 > b_1, Q_2 > b_2)$, where $Q_i$ is the steady-state workload of node $i$. By setting $b_1 = \alpha b$, $b_2 = (1-\alpha)b$, with $\alpha \in [0,1]$, and letting $b \to \infty$, we also obtain exact large-buffer asymptotics, i.e., we find a function $f(\cdot)$ such that $\mathbb{P}(Q_1 > \alpha b, Q_2 > (1-\alpha)b)/f(b) \to 1$ as $b \to \infty$. It turns out that the nature of the asymptotics depends on the value of $\alpha$ and the service rates of both queues, i.e., there are different regimes. These regimes can be further interpreted relying on Schilder's sample-path large-deviations theorem. In particular, we obtain the so-called most probable path, i.e., the most likely way for the buffers to fill.

The remainder of the paper is organized as follows. In Section 2 we present a detailed description of the two-node tandem queue, as well as a closely related two-node parallel queue. We also give formal implicit expressions for the overflow probabilities, and we briefly discuss Schilder's sample-path large-deviations theorem. In Section 3 the two-node parallel queue is analyzed: we derive an exact expression of the joint distribution function, large-buffer asymptotics, and the most probable path. Then we argue that the two-node parallel queue is closely related to the two-node tandem queue. Exploiting this property we obtain in Section 4 the desired results for the tandem system. Finally, in Section 5 we further discuss our results, and identify some open research questions.

# 2 Preliminaries

In this section we first describe our queueing models: the two-node parallel queue and the two-node tandem queue. For each of the models we present an implicit expression for the joint overflow probability. We conclude by briefly discussing some large-deviations results, which will be needed in the next sections.

## 2.1 Two-node parallel queue

Section 3 considers a two-node parallel queue with service rate $c_{\rm I}$ at queue I, and $c_{\rm II}$ at queue II. Traffic that enters the system has to be served at both queue I and II, which is done in

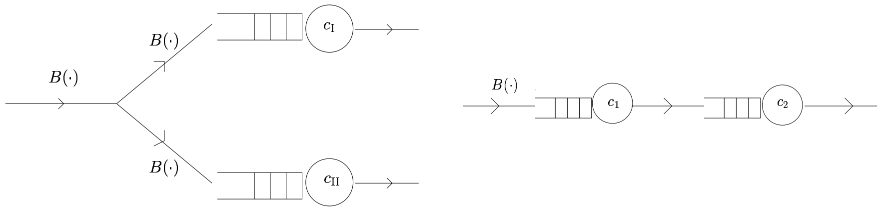


Figure 1: Left: Two-node parallel queue. Right: Two-node tandem queue

parallel; see Figure 1 for an illustration. The case $c_{\rm I} = c_{\rm II}$ being trivial, we assume without loss of generality that $c_{\rm I} > c_{\rm II}$. Let $Q_{\rm I}$ and $Q_{\rm II}$ denote the steady-state workload of queue I and queue II, respectively.

We assume that the input process is a standard Brownian motion $\{B(t), t \in \mathbb{R}\}$. It can be verified that $\Gamma(s,t) := \mathbb{C}\mathrm{ov}(B(s), B(t)) = \min\{s,t\}$, with $s,t \geq 0$.

We study the joint distribution of the steady-state workloads of queue I and queue II:

$$\mathbb{P}(Q_{\rm I} > b_{\rm I}, Q_{\rm II} > b_{\rm II}). \tag{1}$$

Note that if $b_{\rm II} < b_{\rm I}$, then (due to $c_{\rm I} > c_{\rm II}$) the event $\{Q_{\rm I} > b_{\rm I}\}$ automatically implies $\{Q_{\rm II} > b_{\rm II}\}$. Hence, we concentrate on $b_{\rm II} \geq b_{\rm I}$. Reich's formula [11] states that

$$Q_{\rm I} = \sup_{s\geq 0}\{-B(-s) - c_{\rm I}s\} \quad \text{and} \quad Q_{\rm II} = \sup_{t\geq 0}\{-B(-t) - c_{\rm II}t\}. \tag{2}$$

Let $s^*$ and $t^*$ denote an optimizing $s$ and $t$ in (2). Now, $-s^*$ $(-t^*)$ can be interpreted as the beginning of the busy period of queue I (queue II) containing time 0. Hence, $c_{\rm I} > c_{\rm II}$ implies that $s^* \leq t^*$, and therefore (1) can be rewritten as $\mathbb{P}(B(\cdot) \in S)$, with

$$S := \{f \in F | \exists t \geq 0 : \exists s \in [0,t] : -f(-s) > b_{\rm I} + c_{\rm I}s, -f(-t) > b_{\rm II} + c_{\rm II}t\}, \tag{3}$$

where $F := \{f | f(0) = 0\}$.

## 2.2 Two-node tandem queue

In Section 4 we consider a two-node tandem queue, again with standard Brownian input. Thus, the output of the first queue is fed into the second queue; see Figure 1. Assume constant service rates $c_1$ and $c_2$, respectively. To avoid the trivial situation of the second queue remaining empty, it is assumed that $c_1 > c_2 > 0$.

We focus on the joint probability that the stationary workloads of the first and second queue, $Q_1$ and $Q_2$, respectively, exceed thresholds $b_1$ and $b_2$, with $b_1, b_2 \geq 0$. For any queue in which traffic leaves the first queue as fluid, the steady-state *total* workload $Q_{\rm T}$ in the two-node tandem queue behaves as single queue emptied at rate $c_2$, see e.g. [9] and references therein. As a consequence,

$$Q_1 = \sup_{s\geq 0}\{-B(-s) - c_1 s\} \quad \text{and} \quad Q_{\rm T} = \sup_{t\geq 0}\{-B(-t) - c_2 t\}. \tag{4}$$

As for the parallel system, we have that the optimizing $s$ is not larger than the optimizing $t$ in (4). Hence, for $b_{\rm T} \geq b_1 \geq 0$, $\mathbb{P}(Q_1 > b_1, Q_{\rm T} > b_{\rm T})$ equals $\mathbb{P}(B(\cdot) \in T)$, with

$$T := \{f \in F | \exists t \geq 0 : \exists s \in [0,t] : -f(-s) > b_1 + c_1 s, -f(-t) > b_{\rm T} + c_2 t\}. \tag{5}$$

Note that (3) and (5) coincide if $c_1 = c_{\rm I}$, $c_2 = c_{\rm II}$, $b_1 = b_{\rm I}$, and $b_{\rm T} = b_{\rm II}$. We will exploit this property in Section 4. Evidently, the distribution of $(Q_1, Q_{\rm T})$ uniquely determines the distribution of $(Q_1, Q_2)$. Using that $Q_2 = Q_{\rm T} - Q_1$, we obtain that $\mathbb{P}(Q_1 > b_1, Q_2 > b_2)$, with $b_1, b_2 \geq 0$, equals $\mathbb{P}(B(\cdot) \in U)$, where

$$U := \left\{ f \in F \,\middle|\, \exists t \geq 0 : \exists s \in [0,t] : \forall u \in [0,s] : \begin{array}{l} -f(-s) > b_1 + c_1 s, \\ f(-u) - f(-t) > b_2 + c_2 t - c_1 u \end{array} \right\}. \tag{6}$$

### 2.3 Large deviations

In this subsection we recall two key large-deviations theorems, which are needed in the analysis of Sections 3.3 and 4.3.

**Theorem 2.1** *Let $(X, Y) \sim$ Norm$(0,\Sigma)$, for a non-degenerate 2-dimensional covariance-matrix $\Sigma$. Then,*

$$\begin{array}{ll} (i) & -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i \geq x\right) = \frac{1}{2} x^2/(\Sigma_{11})^2; \\ (ii) & -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i \geq x, \frac{1}{n}\sum_{i=1}^n Y_i \geq y\right) = \inf_{a\geq x} \inf_{b \geq y} \Lambda(a,b), \end{array}$$

*where $\Lambda(a,b) := \frac{1}{2}(a\ b)\Sigma^{-1}(a\ b)^T$.*

We continue with a description of the framework of Schilder's sample-path LDP (see [2], and also Thm. 1.3.27 of [5] for a more detailed treatment). Define the path space $\Omega$ as

$$\Omega := \left\{ \omega : \mathbb{R} \to \mathbb{R}, \text{continuous}, \omega(0) = 0, \lim_{t\to\infty} \frac{\omega(t)}{1+|t|} = \lim_{t\to-\infty} \frac{\omega(t)}{1+|t|} = 0 \right\}.$$

Then one can construct a reproducing kernel Hilbert space $R \subseteq \Omega$, consisting of elements that are roughly as smooth as the covariance function $\Gamma(s,\cdot)$; for details, see [1]. Now we can define the rate function:

$$I(\omega) := \left\{ \begin{array}{ll} \frac{1}{2}\int (\omega'(t))^2 {\rm d}t & \text{if} \quad \omega \in R; \\ \infty & \text{otherwise.} \end{array} \right.$$

For standard Brownian inputs the following sample-path large deviations principle (LDP) holds.

**Theorem 2.2** **[Schilder]** *With $G \subset \Omega$,*

$$-\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^n B_i(\cdot) \in G \right) = \inf_{\omega \in G} I(\omega).$$

*Remark:* Intentionally, Theorem 2.2 has been formulated slightly imprecise. In fact, the LDP consists of an upper and lower bound, which apply to closed and open sets, respectively. However, we will use Theorem 2.2 for the sets $U$, $S$ and $T$ as defined before. It can be verified that

$$\inf_{\omega \in U^o} I(\omega) = \inf_{\omega \in \overline{U}} I(\omega),$$

where $U^o$ ($\overline{U}$) is the interior (closure) of $U$; the same holds for $S$ and $T$. For a proof, this can be done completely analogously to [10] and Appendix A of [9].

# 3 Analysis of the two-node parallel queue

In this section we focus on the two-node parallel queue. We derive the joint distribution function of queue I and queue II, large-buffer asymptotics, and the most probable path leading to overflow.

## 3.1 Joint distribution function

In this subsection we derive an exact expression for $p(\bar{b}) := \mathbb{P}(Q_{\rm I} > b_{\rm I}, Q_{\rm II} > b_{\rm II})$, with $\bar{b} \equiv (b_{\rm I}, b_{\rm II})$. For the sake of brevity, write $\chi \equiv \chi(\bar{b}) := (b_{\rm II} - b_{\rm I})/(c_{\rm I} - c_{\rm II})$. Furthermore, let $\Phi(\cdot)$ denote the distribution function of a standard Normal random variable, $\phi(\cdot) := \Phi'(\cdot)$, and $\Psi(\cdot) := 1 - \Phi(\cdot)$. We first present the main theorem of this subsection.

**Theorem 3.1** *For each $b_{\rm II} \geq b_{\rm I} \geq 0$,*

$$p(\bar{b}) = -\Psi(k_1(\bar{b})) + \Psi(k_2(\bar{b}))e^{-2b_{\rm I}c_{\rm I}} + \Psi(k_3(\bar{b}))e^{-2b_{\rm II}c_{\rm II}} + (1 - \Psi(k_4(\bar{b})))e^{-2(b_{\rm I}(c_{\rm I}-2c_{\rm II})+b_{\rm II}c_{\rm II})},$$

*where*

$$k_1(\bar{b}) := \frac{b_{\rm I} + c_{\rm I}\chi}{\sqrt{\chi}};\ k_2(\bar{b}) := \frac{-b_{\rm I} + c_{\rm I}\chi}{\sqrt{\chi}};\ k_3(\bar{b}) := \frac{b_{\rm I} + (c_{\rm I} - 2c_{\rm II})\chi}{\sqrt{\chi}};\ k_4(\bar{b}) := \frac{-b_{\rm I} + (c_{\rm I} - 2c_{\rm II})\chi}{\sqrt{\chi}}.$$

*Proof:* In [8] an expression was derived for $\mathbb{P}(Q_{\rm I} \leq b_{\rm I}, Q_{\rm II} \leq b_{\rm II})$ in case of standard Brownian input. We give a short sketch of the proof. First note that, due to time-reversibility arguments,

$$\mathbb{P}(Q_{\rm I} \leq b_{\rm I}, Q_{\rm II} \leq b_{\rm II}) = \mathbb{P}(\forall t \geq 0 : B(t) \leq \min\{b_{\rm I} + c_{\rm I}t, b_{\rm II} + c_{\rm II}t\}).$$

Let $y \equiv y(\bar{b}) := b_{\rm I} + c_{\rm I}\chi$. Hence, $(\chi, y)$ is the point where $b_{\rm I} + c_{\rm I}t$ and $b_{\rm II} + c_{\rm II}t$ intersect. For $t \in [0, \chi]$ the minimum is given by $b_{\rm I} + c_{\rm I}t$, whereas for $t \in [\chi, \infty)$ the minimum is $b_{\rm II} + c_{\rm II}t$. Now, conditioning on the value of $B(\chi)$, being normally distributed with mean 0 and variance $\chi$, one obtains that $\mathbb{P}(Q_{\rm I} \leq b_{\rm I}, Q_{\rm II} \leq b_{\rm II})$ equals

$$\int_{-\infty}^{y} \frac{1}{\sqrt{\chi}}\phi\left(\frac{x}{\sqrt{\chi}}\right)\mathbb{P}(\forall t \in [0, \chi] : B(t) \leq b_{\rm I} + c_{\rm I}t | B(\chi) = x)\mathbb{P}(\forall t \geq 0 : B(t) \leq y - x + c_{\rm II}t){\rm d}x.$$

The first probability can be expressed (after some rescaling) in terms of the Brownian bridge:

$$\mathbb{P}(\forall t \in [0,1] : B(t) \leq b + ct | B(1) = 0) = 1 - \exp(-2b(b+c)),$$

whereas the second translates into the supremum of a Brownian motion: $1-\exp(-2(y-x)c_{\rm II})$. After substantial calculus we obtain that $\mathbb{P}(Q_{\rm I}\leq b_{\rm I}, Q_{\rm II}\leq b_{\rm II})$ equals

$$\Phi(k_1(\bar{b}) - \Phi(k_2(\bar{b}))e^{-2b_{\rm I}c_{\rm I}} - \Phi(k_3(\bar{b}))e^{-2b_{\rm II}c_{\rm II}} + \Phi(k_4(\bar{b}))e^{-2(b_{\rm I}(c_{\rm I}-2c_{\rm II})+b_{\rm II}c_{\rm II})}.$$

Furthermore, it is well known that $\mathbb{P}(Q_i > b_i) = e^{-2b_ic_i}$, $i = {\rm I, II}$. The stated follows from

$$p(\bar{b}) = 1 - \mathbb{P}(Q_{\rm I}\leq b_{\rm I}) - \mathbb{P}(Q_{\rm II}\leq b_{\rm II}) + \mathbb{P}(Q_{\rm I}\leq b_{\rm I}, Q_{\rm II}\leq b_{\rm II}).$$ □

### 3.2 Exact large-buffer asymptotics

In this subsection we derive the exact asymptotics of the joint buffer content distribution. We write $f(u)\sim g(u)$ when $f(u)/g(u)\to 1$ if $u\to\infty$. Define $\zeta(x) := \left(\sqrt{2\pi}x\right)^{-1}\exp(-x^2/2)$. Also,

$$\alpha_+ := \frac{c_{\rm I}}{2c_{\rm I}-c_{\rm II}};\ \alpha_0 := \frac{2c_{\rm II}-c_{\rm I}}{c_{\rm II}};\ \alpha_- := \frac{c_{\rm I}-2c_{\rm II}}{2c_{\rm I}-3c_{\rm II}}.$$

It can be verified that $\alpha_0 < 0 < \alpha_- < \alpha_+ < 1$ if $c_{\rm I} > 2c_{\rm II}$, whereas $0\leq\alpha_0<\alpha_+<1$ if $c_{\rm I}\leq 2c_{\rm II}$. Let us first present the following useful lemma.

**Lemma 3.2** *Let $b_{\rm I}=\alpha b$ and $b_{\rm II}=b$, with $\alpha\in[0,1]$. If $b\to\infty$, then*

$$\Psi(k_1(\bar{b}))\sim\zeta(k_1(\bar{b}));$$

$$\Psi(k_2(\bar{b}))\sim\begin{cases}\zeta(k_2(\bar{b})) & \text{if } \alpha<\alpha_+;\\ 1/2 & \text{if } \alpha=\alpha_+;\\ 1 & \text{otherwise};\end{cases}$$

$$\Psi(k_3(\bar{b}))\sim\begin{cases}\zeta(k_3(\bar{b})) & \text{if } \alpha>\alpha_0;\\ 1/2 & \text{if } \alpha=\alpha_0;\\ 1 & \text{otherwise};\end{cases}$$

$$1-\Psi(k_4(\bar{b}))\sim\begin{cases}1 & \text{if } \alpha<\alpha_- \text{ and } c_{\rm I}>2c_{\rm II};\\ 1/2 & \text{if } \alpha=\alpha_- \text{ and } c_{\rm I}\geq 2c_{\rm II};\\ -\zeta(k_4(\bar{b})) & \text{otherwise}.\end{cases}$$

*Proof:* First determine for which values of $b_{\rm I}/b_{\rm II}=\alpha$, $k_i(\bar{b})$, $i\in\{1,2,3,4\}$, is positive or negative. Note that $k_1(\bar{b})$ is always positive, given that $b_{\rm II}\geq b_{\rm I}\geq 0$. Also, $k_4(\bar{b})$ is always negative if $c_{\rm I}\leq 2c_{\rm II}$ and $b_{\rm I}>0$. Hence, we obtain $\alpha_+,\alpha_0$ and $\alpha_-$ as critical values from $k_i(\bar{b})$, $i=2,3,4$, respectively. Next use the fact that $\Psi(u)\sim\zeta(u)$ and $\Psi(-u)\sim 1$ as $u\to\infty$. Observe that $\Psi(0)=1/2$. □

Define

$$\beta(\bar{b}) := \frac{1}{\sqrt{2\pi}}\left(-\frac{1}{k_1(\bar{b})}+\frac{1}{k_2(\bar{b})}+\frac{1}{k_3(\bar{b})}-\frac{1}{k_4(\bar{b})}\right) \text{ and } \gamma(\bar{b}) := \frac{(b_{\rm II}c_{\rm I}-b_{\rm I}c_{\rm II})^2}{2(b_{\rm II}-b_{\rm I})(c_{\rm I}-c_{\rm II})}.$$

Straightforward calculus also shows the following equalities:

$$\exp\left(-\frac{k_1(\bar{b})^2}{2}\right) = \exp\left(-\frac{k_2(\bar{b})^2}{2}\right)\exp\left(-2b_{\mathrm{I}}c_{\mathrm{I}}\right) = \exp\left(-\frac{k_3(\bar{b})^2}{2}\right)\exp\left(-2b_{\mathrm{II}}c_{\mathrm{II}}\right) =$$
$$\exp\left(-\frac{k_4(\bar{b})^2}{2}\right)\exp\left(-2(b_{\mathrm{I}}(c_{\mathrm{I}} - 2c_{\mathrm{II}}) + b_{\mathrm{II}}c_{\mathrm{II}})\right) = \exp\left(-\gamma(\bar{b})\right). \tag{7}$$

**Theorem 3.3** *Let* $b_{\mathrm{I}} = \alpha b$ *and* $b_{\mathrm{II}} = b$, *with* $\alpha \in [0,1]$. *Suppose* $c_{\mathrm{I}} > 2c_{\mathrm{II}}$. *For* $b \to \infty$,

$$p(\bar{b}) \sim \begin{cases} e^{-2(b_{\mathrm{I}}(c_{\mathrm{I}}-2c_{\mathrm{II}})+b_{\mathrm{II}}c_{\mathrm{II}})} & \textit{if } \alpha \in [0,\alpha_-); \\ \frac{1}{2}e^{-2(b_{\mathrm{I}}(c_{\mathrm{I}}-2c_{\mathrm{II}})+b_{\mathrm{II}}c_{\mathrm{II}})} & \textit{if } \alpha = \alpha_-; \\ \beta(\bar{b})e^{-\gamma(\bar{b})} & \textit{if } \alpha \in (\alpha_-,\alpha_+); \\ \frac{1}{2}e^{-2b_{\mathrm{I}}c_{\mathrm{I}}} & \textit{if } \alpha = \alpha_+; \\ e^{-2b_{\mathrm{I}}c_{\mathrm{I}}} & \textit{if } \alpha \in (\alpha_+,1]. \end{cases}$$

*Proof:* We only prove the first statement, as the other four statements follow in a similar way. We have to prove that

$$p(\bar{b})\exp(2(b_{\mathrm{I}}(c_{\mathrm{I}} - 2c_{\mathrm{II}}) + b_{\mathrm{II}}c_{\mathrm{II}})) \to 1 \text{ as } b \to \infty, \text{ for } \alpha \in [0,\alpha_-).$$

From Lemma 3.2 we obtain that for $\alpha \in [0,\alpha_-)$,

$$\Psi(k_1(\bar{b})) \sim \zeta(k_1(\bar{b})); \quad \Psi(k_2(\bar{b})) \sim \zeta(k_2(\bar{b})); \quad \Psi(k_3(\bar{b})) \sim \zeta(k_3(\bar{b})); \quad 1 - \Psi(k_4(\bar{b})) \sim 1 - \zeta(k_4(\bar{b})).$$

Now it can be checked from (7) that, as $b \to \infty$,

$$\Psi(k_1(\bar{b})) = o\left(e^{-2(b_{\mathrm{I}}(c_{\mathrm{I}}-2c_{\mathrm{II}})+b_{\mathrm{II}}c_{\mathrm{II}}))}\right),$$

and the same applies for $\Psi(k_2(\bar{b}))e^{-2b_{\mathrm{I}}c_{\mathrm{I}}}$ and $\Psi(k_3(\bar{b}))e^{-2b_{\mathrm{II}}c_{\mathrm{II}}}$. With $1 - \Psi(k_4(\bar{b})) \sim 1$, Theorem 3.1 implies the stated. □

**Theorem 3.4** *Let* $b_{\mathrm{I}} = \alpha b$ *and* $b_{\mathrm{II}} = b$, *with* $\alpha \in [0,1]$. *Suppose* $c_{\mathrm{I}} < 2c_{\mathrm{II}}$. *For* $b \to \infty$,

$$p(\bar{b}) \sim \begin{cases} e^{-2b_{\mathrm{II}}c_{\mathrm{II}}} & \textit{if } \alpha \in [0,\alpha_0); \\ \frac{1}{2}e^{-2b_{\mathrm{II}}c_{\mathrm{II}}} & \textit{if } \alpha = \alpha_0; \\ \beta(\bar{b})e^{-\gamma(\bar{b})} & \textit{if } \alpha \in (\alpha_0,\alpha_+); \\ \frac{1}{2}e^{-2b_{\mathrm{I}}c_{\mathrm{I}}} & \textit{if } \alpha = \alpha_+; \\ e^{-2b_{\mathrm{I}}c_{\mathrm{I}}} & \textit{if } \alpha \in (\alpha_+,1]. \end{cases}$$

*Proof:* The proof is similar to that of Theorem 3.3. □

*Remark:* Note that for $c_{\mathrm{I}} = 2c_{\mathrm{II}}$, one obtains $\alpha_0 = 0$. It can be verified that in this special case Theorem 3.4 reduces to

$$p(\bar{b}) \sim \begin{cases} e^{-2b_{\mathrm{II}}c_{\mathrm{II}}} & \text{if } \alpha = 0; \\ \beta(\bar{b})e^{-\gamma(\bar{b})} & \text{if } \alpha \in (0,\alpha_+); \\ \frac{1}{2}e^{-2b_{\mathrm{I}}c_{\mathrm{I}}} & \text{if } \alpha = \alpha_+; \\ e^{-2b_{\mathrm{I}}c_{\mathrm{I}}} & \text{if } \alpha \in (\alpha_+,1]. \end{cases}$$

### 3.3 Most probable path

In the previous subsection it was shown that the nature of the large-buffer asymptotics strongly depends on the model parameters $\alpha$, $c_{\rm I}$ and $c_{\rm II}$, i.e., there are different regimes. In this subsection we will interpret these regimes by using sample-path large deviations. Schilder's theorem (Theorem 2.2) implies that the exponential decay rate of the joint overflow probability in the parallel system is characterized by the path in $S$ that minimizes the decay rate. Among all paths such that queue I exceeds $b_{\rm I}$ and queue II exceeds $b_{\rm II}$, this is the so-called most probable path (MPP): informally speaking, given that this rare event occurs, with overwhelming probability $(b_{\rm I}, b_{\rm II})$ is reached by a path 'close to' the MPP. The goal of this subsection is to find the MPP in $S$, and to relate its form to the regimes identified in Section 3.2.

Consider the two-node parallel queue as described before. Now, in order to apply 'Schilder', we feed this network by $n$ i.i.d. standard Brownian sources. The link rates and buffer thresholds are also scaled by $n$: $nc_{\rm I}$, $nc_{\rm II}$, $nb_{\rm I}$ and $nb_{\rm II}$ respectively. Now, $p_n(\bar{b}) := \mathbb{P}(Q_{\rm I,n} > nb_{\rm I}, Q_{\rm II,n} > nb_{\rm II})$ can be expressed as

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} B_i(\cdot) \in S\right).$$

From 'Schilder' it follows that

$$J(\bar{b}) := -\lim_{n\to\infty}\frac{1}{n}\log p_n(\bar{b}) = \inf_{f\in S} I(f) = \inf_{t\geq 0}\inf_{s\in[0,t]} \Upsilon(s,t), \tag{8}$$

with

$$\Upsilon(s,t) := \inf_{f\in S^{s,t}} I(f) \text{ and } S^{s,t} := \{f\in F \mid -f(-s) > b_{\rm I} + c_{\rm I}s, -f(-t) > b_{\rm II} + c_{\rm II}t\},$$

using the fact that the decay rate of a union of events is the minimum of the decay rates of the individual events. As mentioned in Section 2.3, we can replace '$>$' by '$\geq$' in $S^{s,t}$, without any impact on the decay rate.

We first show how, for fixed $s, t$, the infimum of $\Upsilon(s,t)$ over $S^{s,t}$ can be computed. Define

$$g_1(s) := \frac{b_{\rm II}s}{b_{\rm I} + (c_{\rm I} - c_{\rm II})s} \quad \text{and} \quad g_2(s) := s\frac{c_{\rm I}}{c_{\rm II}} + \frac{b_{\rm I} - b_{\rm II}}{c_{\rm II}}.$$

Note that $g_1(\cdot)$ is a concave function, whereas $g_2(\cdot)$ is a linear function. Furthermore, $g_1(s) > g_2(s)$ if $s < \chi$, $g_1(s) = g_2(s)$ if $s = \chi$, and otherwise $g_1(s) < g_2(s)$. Also, define

$$A_1 := \{(s,t) \mid s \leq t \leq g_1(s)\};$$
$$A_2 := \{(s,t) \mid s \leq t \leq g_2(s)\};$$
$$A_3 := \{(s,t) \mid t \geq \max\{g_1(s), g_2(s)\}, s \geq 0\}.$$

(See Figure 2 for an illustration.)

**Lemma 3.5** *For $t \geq 0$, and $s \in [0,t]$,*

$$\Upsilon(s,t) = \begin{cases} h_1(t) := \frac{(b_{\rm II}+c_{\rm II}t)^2}{2t} & \text{if } (s,t)\in A_1; \\ h_2(s) := \frac{(b_{\rm I}+c_{\rm I}s)^2}{2s} & \text{if } (s,t)\in A_2; \\ h_3(s,t) := \frac{(b_{\rm I}+c_{\rm I}s)^2}{2s} + \frac{(b_{\rm II}+c_{\rm II}t - b_{\rm I} - c_{\rm I}s)^2}{2(t-s)} & \text{if } (s,t)\in A_3. \end{cases}$$

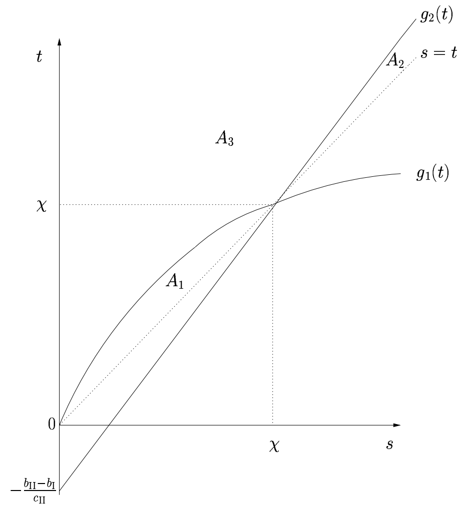


Figure 2: Line $g_1(\cdot)$ and $g_2(\cdot)$.

*Proof:* The proof is analogous to Lemma 3.4 of [9]. The paths in $S^{s,t}$ correspond to a bivariate Normal random variable $(-B(-s), -B(-t))$. Now, by using Theorem 2.1, we find for $y, z \in \mathbb{R}$ and $t \geq 0$, $s \in [0, t]$,

$$\Upsilon(s,t) = \inf_{y \geq b_{\rm I} + c_{\rm I} s} \inf_{z \geq b_{\rm II} + c_{\rm II} t} \Lambda(y,z), \text{ with } \Lambda(y,z) = \frac{1}{2} \begin{pmatrix} y & z \end{pmatrix} \begin{pmatrix} s & s \\ s & t \end{pmatrix}^{-1} \begin{pmatrix} y \\ z \end{pmatrix}. \quad (9)$$

One can show that if

$$y_0 := \mathbb{E}[-B(-s) | -B(-t) = b_{\rm II} + c_{\rm II} t] \geq b_{\rm I} + c_{\rm I} s,$$

or, equivalently, $t \leq g_1(s)$, then the optimum in (9) is attained at $(y^*, z^*) = (y_0, b_{\rm II} + c_{\rm II} t)$. Hence, the rate function is independent of $s$, and given by $\Lambda(y_0, b_{\rm II} + c_{\rm II} t) = h_1(t)$.

In a similar way, if

$$z_0 := \mathbb{E}[-B(-t) | -B(-s) = b_{\rm I} + c_{\rm I} s] \geq b_{\rm II} + c_{\rm II} t,$$

or, after rewriting, $t \leq g_2(s)$, then the optimum in (9) is attained at $(y^*, z^*) = (b_{\rm I} + c_{\rm I} s, z_0)$. The rate function is then given by $\Lambda(b_{\rm I} + c_{\rm I} s, z_0) = h_2(s)$ (independently of $t$).

If $y_0 < b_{\rm I} + c_{\rm I} s$ and $z_0 < b_{\rm II} + c_{\rm II} t$, then the optimum in (9) is attained at $(y^*, z^*) = (b_{\rm I} + c_{\rm I} s, b_{\rm II} + c_{\rm II} t)$. It is readily verified that this yields $h_3(s,t)$ for $t \geq \max\{g_1(s), g_2(s)\}$. □

In view of (8), the next step is to optimize $\Upsilon(s,t)$ over all $t \geq 0$, $s \in [0,t]$.

### 3.3.1 Area $A_1$

The optimization over $A_1$ reduces to

$$\inf_{(s,t) \in A_1} \Upsilon(s,t) = \inf_{t \in [0, \chi]} h_1(t). \quad (10)$$

It can be verified that $h_1(t)$ is strictly decreasing in the interval $[0, b_{\mathrm{II}}/c_{\mathrm{II}})$, and strictly increasing in the interval $(b_{\mathrm{II}}/c_{\mathrm{II}}, \infty)$. Therefore, if $b_{\mathrm{II}}/c_{\mathrm{II}} \leq \chi$ then $t^* = b_{\mathrm{II}}/c_{\mathrm{II}}$ and $s^* \in [g_1^{-1}(t^*), t^*]$, whereas otherwise $t^* = s^* = \chi$.

**Lemma 3.6** *Expression (10) equals* $2b_{\mathrm{II}}c_{\mathrm{II}}$ *if* $b_{\mathrm{I}}/b_{\mathrm{II}} \in [0, \alpha_0]$, *otherwise* $\gamma(\bar{b})$.

*Proof:* The condition $b_{\mathrm{II}}/c_{\mathrm{II}} \leq \chi$ is equivalent to $b_{\mathrm{I}}/b_{\mathrm{II}} \leq (2c_{\mathrm{II}} - c_{\mathrm{I}})/c_{\mathrm{II}} = \alpha_0$. Evaluation of (10) for $t^* = b_{\mathrm{II}}/c_{\mathrm{II}}$ proves the first statement. Similarly, evaluation of (10) for $t^* = \chi$ proves the second statement. □

### 3.3.2 Area $A_2$

The approach is very similar to above. We are to solve the following optimization problem:

$$\inf_{(s,t)\in A_2} \Upsilon(s,t) = \inf_{s\in[\chi,\infty)} h_2(s). \tag{11}$$

The function $h_2(s)$ has a global minimum that is attained at $s = b_{\mathrm{I}}/c_{\mathrm{I}}$. Thus, if $b_{\mathrm{I}}/c_{\mathrm{I}} \geq \chi$, then $s^* = b_{\mathrm{I}}/c_{\mathrm{I}}$ and $t^* \in [s^*, g_2(s^*)]$, whereas otherwise $s^* = t^* = \chi$. The following lemma is proven analogously to Lemma 3.6.

**Lemma 3.7** *Expression (11) equals* $2b_{\mathrm{I}}c_{\mathrm{I}}$ *if* $b_{\mathrm{I}}/b_{\mathrm{II}} \in [\alpha_+, 1]$, *otherwise* $\gamma(\bar{b})$.

### 3.3.3 Area $A_3$

We divide area $A_3$ in two parts. Let us start with the part $s \in [\chi, \infty)$ and $t \in [g_2(s), \infty)$:

$$\inf_{s\in[\chi,\infty)} \inf_{t\in[g_2(s),\infty)} h_3(s,t). \tag{12}$$

Clearly, (12) is bounded from below by

$$\inf_{s\in[\chi,\infty)} \inf_{t\in[g_2(s),\infty)} h_2(s). \tag{13}$$

One can show that $h_3(s,t)$ reduces to $h_2(s)$ if $t = g_2(s)$ $(s \in [\chi, \infty))$. Therefore, analogously to the area $A_2$, if $b_{\mathrm{I}}/c_{\mathrm{I}} \geq \chi$, then $s^* = b_{\mathrm{I}}/c_{\mathrm{I}}$ and $t^* = g_2(s^*) = (2b_{\mathrm{I}} - b_{\mathrm{II}})/c_{\mathrm{II}}$, whereas otherwise $s^* = t^* = \chi$. We thus obtain the following result.

**Lemma 3.8** *Expression (12) equals* $2b_{\mathrm{I}}c_{\mathrm{I}}$ *if* $b_{\mathrm{I}}/b_{\mathrm{II}} \in [\alpha_+, 1]$, *otherwise* $\gamma(\bar{b})$.

We now turn to the last part: $s \in [0, \chi]$ and $t \in [g_1(s), \infty)$:

$$\inf_{s\in[0,\chi]} \inf_{t\in[g_1(s),\infty)} h_3(s,t). \tag{14}$$

First concentrate on the minimum of $h_3(s,t)$ over $t \geq 0$, which is attained at

$$t = \frac{b_{\rm II} - b_{\rm I}}{c_{\rm II}} + s\frac{2c_{\rm II} - c_{\rm I}}{c_{\rm II}} =: g_3(s)$$

if $s \in [0, \chi]$ (for $s > \chi$ it is attained at $t = g_2(s)$, but this case is irrelevant here). Note that $g_3(s)$ is linearly decreasing (increasing) if $c_{\rm I} > 2c_{\rm II}$ ($c_{\rm I} < 2c_{\rm II}$). Also, $g_3(\chi) = \chi$. Hence, we have to distinguish between two cases:

- First concentrate on $c_{\rm I} > 2c_{\rm II}$. Then $g_3(s) > g_1(s)$ for all $s \in [0, \chi)$ (as $g_3(s)$ is non-increasing and $g_3(\chi) = \chi$). Substituting $t = g_3(s)$ in (14) gives

$$\inf_{s\in[0,\chi]} \frac{b_{\rm I}^2 + 2b_{\rm I}(c_{\rm I} - 2c_{\rm II})s + 4b_{\rm II}c_{\rm II}s + (c_{\rm I} - 2c_{\rm II})^2 s^2}{2s}. \tag{15}$$

  This is minimized for $s^* = b_{\rm I}/(c_{\rm I} - 2c_{\rm II})$ and $t^* = g_3(s^*) = (b_{\rm II} - 2b_{\rm I})/c_{\rm II}$ if $b_{\rm I}/(c_{\rm I} - 2c_{\rm II}) \leq \chi$, whereas otherwise $s^* = \chi = t^*$.

- Next consider $c_{\rm I} \leq 2c_{\rm II}$. In this case it is not clear a priori whether $g_3(s) \geq g_1(s)$. For the moment assume that this is true. Then (15) is again appropriate, and this is minimized for $s^* = b_{\rm I}/(2c_{\rm II} - c_{\rm I})$ and $t^* = g_3(s) = b_{\rm II}/c_{\rm II}$ if $b_{\rm I}/(2c_{\rm II} - c_{\rm I}) \leq \chi$, whereas otherwise $s^* = \chi = t^*$. Now, in the former case it can be checked that $g_3(s^*) = g_1(s^*) = b_{\rm II}/c_{\rm II}$, and in the latter case we find $g_3(s^*) = g_1(s^*) = \chi$, i.e., the minimizers satisfy $g_3(s^*) \geq g_1(s^*)$, and hence we are done.

This reasoning leads to the following result.

**Lemma 3.9** *Expression (14) equals*

$$\begin{cases} 2(b_{\rm I}(c_{\rm I} - 2c_{\rm II}) + b_{\rm II}c_{\rm II}) & \textit{if } c_{\rm I} > 2c_{\rm II} \textit{ and } b_{\rm I}/b_{\rm II} \in [0, \alpha_-]; \\ \gamma(\bar{b}) & \textit{if } c_{\rm I} > 2c_{\rm II} \textit{ and } b_{\rm I}/b_{\rm II} \in (\alpha_-, 1]; \\ 2b_{\rm II}c_{\rm II} & \textit{if } c_{\rm I} \leq 2c_{\rm II} \textit{ and } b_{\rm I}/b_{\rm II} \in [0, \alpha_0]; \\ \gamma(\bar{b}) & \textit{if } c_{\rm I} \leq 2c_{\rm II} \textit{ and } b_{\rm I}/b_{\rm II} \in (\alpha_0, 1]. \end{cases}$$

### 3.3.4 Exponential decay rate

We now present an exact expression for the rate function $J(\bar{b})$. We start with the case $c_{\rm I} > 2c_{\rm II}$.

**Theorem 3.10** *Suppose $c_{\rm I} > 2c_{\rm II}$. Then it holds that*

$$J(\bar{b}) = \begin{cases} 2(b_{\rm I}(c_{\rm I} - 2c_{\rm II}) + b_{\rm II}c_{\rm II}) & \textit{if } b_{\rm I}/b_{\rm II} \in [0, \alpha_-]; \\ \gamma(\bar{b}) & \textit{if } b_{\rm I}/b_{\rm II} \in (\alpha_-, \alpha_+); \\ 2b_{\rm I}c_{\rm I} & \textit{if } b_{\rm I}/b_{\rm II} \in [\alpha_+, 1]. \end{cases}$$

*Proof:* Combine Lemmas 3.6-3.9. There exist two critical values of $b_{\rm I}/b_{\rm II}$, given that $c_{\rm I} > 2c_{\rm II}$. Recall from Section 3.2 that $0 < \alpha_- < \alpha_+ < 1$ if $c_{\rm I} > 2c_{\rm II}$. Now, if $b_{\rm I}/b_{\rm II} \in [0, \alpha_-]$, then $J(\bar{b}) = \min\left\{2(b_{\rm I}(c_{\rm I} - 2c_{\rm II}) + b_{\rm II}c_{\rm II}), \gamma(\bar{b})\right\}$. Straightforward calculus shows that the first argument is smaller for these values of $b_{\rm I}/b_{\rm II}$. Similarly, if $b_{\rm I}/b_{\rm II} \in (\alpha_-, \alpha_+)$, then $J(\bar{b}) = \gamma(\bar{b})$.

Finally, if $b_{\mathrm{I}}/b_{\mathrm{II}} \in [\alpha_+, 1]$, then $J(\bar{b}) = \min\{2b_{\mathrm{I}}c_{\mathrm{I}}, \gamma(\bar{b})\}$. Applying straightforward calculus yields that the first argument is smaller if $b_{\mathrm{I}}/b_{\mathrm{II}} \in (\alpha_+, 1]$. □

In addition to the exponential decay rates of Theorem 3.10, we also (implicitly) obtained the corresponding MPPs. In the MPP of the first regime, queue I starts to build up at $-s^* = -b_{\mathrm{I}}/(c_{\mathrm{I}} - 2c_{\mathrm{II}})$, whereas queue II starts to build up at $-t^* = -(b_{\mathrm{II}} - 2b_{\mathrm{I}})/c_{\mathrm{II}}$. The MPP is such that, for $r \in [-t^*, 0]$,

$$f^*(r) = \mathbb{E}\left(B(r)|B(-s^*) - B(-t^*) = b_{\mathrm{II}} - b_{\mathrm{I}} + c_{\mathrm{II}}t^* - c_{\mathrm{I}}s^*\right) \quad \text{if } r \in [-t^*, -s^*);$$
$$f^*(r) = \mathbb{E}\left(B(r)| - B(-s^*) = b_{\mathrm{I}} + c_{\mathrm{I}}s^*\right) \quad \text{if } r \in [-s^*, 0].$$

That is, traffic enters the network at a constant rate $2c_{\mathrm{II}}$ in the interval $[-t^*, -s^*)$, and at a constant rate $2(c_{\mathrm{I}} - c_{\mathrm{II}})$ in the interval $[-s^*, 0]$. Given service rates $c_{\mathrm{I}}$ and $c_{\mathrm{II}}$ at queue I and queue II respectively, this indeed results in $Q_{\mathrm{I}}(0) = b_{\mathrm{I}}$ and $Q_{\mathrm{II}}(0) = b_{\mathrm{II}}$. Applying 'Schilder', one can verify that, as expected, $I(f^*) = 2(b_{\mathrm{I}}(c_{\mathrm{I}} - 2c_{\mathrm{II}}) + b_{\mathrm{II}}c_{\mathrm{II}})$.

In the second regime, queue I and queue II start to build up at $-t^* = -(b_{\mathrm{II}} - b_{\mathrm{I}})/(c_{\mathrm{I}} - c_{\mathrm{II}})$. The MPP is such that, for $r \in [-t^*, 0]$,

$$f^*(r) = \mathbb{E}\left(B(r)| - B(-t^*) = b_{\mathrm{I}} + c_{\mathrm{I}}t^*\right). \tag{16}$$

Thus, traffic enters the network with constant rate $(b_{\mathrm{I}}/(b_{\mathrm{II}} - b_{\mathrm{I}}))(c_{\mathrm{I}} - c_{\mathrm{II}}) + c_{\mathrm{I}}$ in the interval $[-t^*, 0]$, and this yields $Q_{\mathrm{I}}(0) = b_{\mathrm{I}}$ and $Q_{\mathrm{II}}(0) = b_{\mathrm{II}}$.

In the third regime, both queues start to build up at $-t^* = -b_{\mathrm{I}}/c_{\mathrm{I}}$. The MPP is such that, for $r \in [-t^*, 0]$,

$$f^*(r) = \mathbb{E}\left(B(r)| - B(-t^*) = b_{\mathrm{I}} + c_{\mathrm{I}}t^*\right).$$

The MPP produces traffic at constant rate $2c_{\mathrm{I}}$ in the interval $[-t^*, 0]$, and this gives $Q_{\mathrm{I}}(0) = b_{\mathrm{I}}$ and $Q_{\mathrm{II}}(0) = (b_{\mathrm{I}}/c_{\mathrm{I}})(2c_{\mathrm{I}} - c_{\mathrm{II}})$. Note that $Q_{\mathrm{II}}(0)$ is larger than $b_{\mathrm{II}}$ if $b_{\mathrm{I}}/b_{\mathrm{II}} \in (\alpha_+, 1]$, so there is indeed exceedance of $b_{\mathrm{II}}$.

**Theorem 3.11** *Suppose $c_{\mathrm{I}} \leq 2c_{\mathrm{II}}$. Then it holds that*

$$J(\bar{b}) = \begin{cases} 2b_{\mathrm{II}}c_{\mathrm{II}} & \text{if } b_{\mathrm{I}}/b_{\mathrm{II}} \in [0, \alpha_0]; \\ \gamma(\bar{b}) & \text{if } b_{\mathrm{I}}/b_{\mathrm{II}} \in (\alpha_0, \alpha_+); \\ 2b_{\mathrm{I}}c_{\mathrm{I}} & \text{if } b_{\mathrm{I}}/b_{\mathrm{II}} \in [\alpha_+, 1]. \end{cases}$$

*Proof:* The proof is similar to that of Theorem 3.10. □

In the MPP corresponding to the first regime of Theorem 3.11, both queues start to build up at $-t^* = -b_{\mathrm{II}}/c_{\mathrm{II}}$. The MPP is such that, for $r \in [-t^*, 0]$,

$$f^*(r) = \mathbb{E}\left(B(r)| - B(-t^*) = b_{\mathrm{II}} + c_{\mathrm{II}}t^*\right).$$

Hence, traffic is generated at a constant rate $2c_{\mathrm{II}}$ in the interval $[-t^*, 0]$, and this results in $Q_{\mathrm{I}}(0) = (b_{\mathrm{II}}/c_{\mathrm{II}})(2c_{\mathrm{II}} - c_{\mathrm{I}}) > b_{\mathrm{I}}$ and $Q_{\mathrm{II}}(0) = b_{\mathrm{II}}$. The MPPs corresponding to the second and third regime are similar to the MPPs corresponding to the second and third regime of Theorem 3.10.

### 3.4 Discussion

Using Theorems 3.3 and 3.4, also the logarithmic large-buffer asymptotics follow directly. To this end, define

$$J^*(\bar{b}_\alpha) := -\lim_{b\to\infty}\frac{1}{b}\log\mathbb{P}\left(Q_{\rm I} > \alpha b, Q_{\rm II} > b\right) \quad \text{with} \quad \alpha\in[0,1],\ b\geq 0,$$

where $\bar{b}_\alpha \equiv (\alpha b, b)$. With $\alpha b = b_{\rm I}$ and $b = b_{\rm II}$, i.e., $\bar{b}_\alpha = \bar{b}$, it is not hard to see that $J^*(\bar{b}_\alpha)$ equals $J(\bar{b})$; compare Theorems 3.10 and 3.11 with Theorems 3.3 and 3.4, respectively. Indeed, since we assumed that in the many-sources framework the standard Brownian sources are i.i.d., and because a standard Brownian motion is characterized by independent increments, $J^*(\bar{b}_\alpha)$ and $J(\bar{b})$ should match, see for instance Example 7.4 of [6].

In the analysis of the two-node parallel queue we assumed that the input process was a standard Brownian motion, i.e., no drift and $v(t) = t$. We now show how the results can be extended to general Brownian input, which have drift $\mu > 0$ and variance $v(t) = \lambda t$, $\lambda > 0$. Clearly, we should have that $c_{\rm I} > c_{\rm II} > \mu > 0$ to ensure stability. We denote the input process of a general Brownian motion by $\{B^*(t), t\in\mathbb{R}\}$. Then, analogously to (3), $p(\bar{b}) = \mathbb{P}(B^*(\cdot)\in S) = \mathbb{P}(B(\cdot)\in S^*)$, with

$$S^* := \left\{ f\in F \,\middle|\, \exists t\geq 0 : \exists s\in[0,t] : -f(-s) > \frac{b_{\rm I} + (c_{\rm I}-\mu)s}{\sqrt{\lambda}}, -f(-t) > \frac{b_{\rm II} + (c_{\rm II}-\mu)t}{\sqrt{\lambda}} \right\}.$$

Hence in order to generalize the results of Section 3 to general Brownian input, we have to set $c_i \leftarrow (c_i-\mu)/\sqrt{\lambda}$ and $b_i \leftarrow b_i/\sqrt{\lambda}$, $i = {\rm I}, {\rm II}$.

## 4 Analysis of the two-node tandem queue

In this section we focus on the two-node tandem queue. Exploiting the results of the two-node parallel queue in Section 3, we derive similar results for the two-node tandem queue.

### 4.1 Joint distribution function

In this subsection we derive an exact expression for $q(\bar{b}) := \mathbb{P}(Q_1 > b_1, Q_2 > b_2)$, with $\bar{b}\equiv(b_1,b_2)$. In Section 2.2 we argued that $p(b_{\rm I}, b_{\rm II})$ equals $q(\bar{b}_{\rm T}) := \mathbb{P}(Q_1 > b_1, Q_{\rm T} > b_{\rm T})$, with $\bar{b}_{\rm T}\equiv(b_1, b_{\rm T})$, given that $b_{\rm I} = b_1$, $b_{\rm II} = b_{\rm T}$, $c_{\rm I} = c_1$ and $c_{\rm II} = c_2$. In a first step to obtain $q(\bar{b})$, we derive $q_f(\bar{b}_{\rm T}) := -\partial q(\bar{b}_{\rm T})/\partial b_1$. With mild abuse of notation, we also write $q_f(\bar{b}_{\rm T}) = \mathbb{P}(Q_1 = b_1, Q_{\rm T} > b_{\rm T})$. Define $\tau_{\rm T}\equiv\tau(\bar{b}_{\rm T}) := (b_{\rm T}-b_1)/(c_1-c_2)$ and $\tau\equiv\tau(b_2) := b_2/(c_1-c_2)$.

**Lemma 4.1** *For each $b_{\rm T}\geq b_1\geq 0$,*

$$\begin{aligned} q_f(\bar{b}_{\rm T}) \;=\; & -\frac{\partial\ell_1(\bar{b}_{\rm T})}{\partial b_1}\phi(\ell_1(\bar{b}_{\rm T})) + 2c_1\Psi(\ell_2(\bar{b}_{\rm T}))e^{-2b_1c_1} \\ & +\frac{\partial\ell_2(\bar{b}_{\rm T})}{\partial b_1}\phi(\ell_2(\bar{b}_{\rm T}))e^{-2b_1c_1} + \frac{\partial\ell_3(\bar{b}_{\rm T})}{\partial b_1}\phi(\ell_3(\bar{b}_{\rm T}))e^{-2b_{\rm T}c_2} \\ & +2(c_1-2c_2)(1-\Psi(\ell_4(\bar{b}_{\rm T})))e^{-2(b_1(c_1-2c_2)+b_{\rm T}c_2)} \\ & -\frac{\partial\ell_4(\bar{b}_{\rm T})}{\partial b_1}\phi(l_4(\bar{b}_{\rm T}))e^{-2(b_1(c_1-2c_2)+b_{\rm T}c_2)}, \end{aligned}$$

*where*

$$\ell_1(\bar{b}_{\rm T}) := \frac{b_1 + c_1\tau_{\rm T}}{\sqrt{\tau_{\rm T}}};\ \ell_2(\bar{b}_{\rm T}) := \frac{-b_1 + c_1\tau_{\rm T}}{\sqrt{\tau_{\rm T}}};$$

$$\ell_3(\bar{b}_{\rm T}) := \frac{b_1 + (c_1 - 2c_2)\tau_{\rm T}}{\sqrt{\tau_{\rm T}}};\ \ell_4(\bar{b}_{\rm T}) := \frac{-b_1 + (c_1 - 2c_2)\tau_{\rm T}}{\sqrt{\tau_{\rm T}}}.$$

*Proof:* Use Theorem 3.1, with $b_{\rm I} = b_1$, $b_{\rm II} = b_{\rm T}$, $c_{\rm I} = c_1$ and $c_{\rm II} = c_2$, to obtain $q(\bar{b}_{\rm T})$. Then recall that $q_f(\bar{b}_{\rm T}) = -\partial q(\bar{b}_{\rm T})/\partial b_1$. We extensively use the chain rule:

$$\frac{\partial \Psi(f(u))}{\partial u} = -\frac{\partial f(u)}{\partial u}\phi(f(u)).$$

Applying straightforward calculus now gives the desired result. □

Note that

$$q(\bar{b}) = \mathbb{P}(Q_1 > b_1, Q_{\rm T} > b_2 + Q_1) = \int_{b_1}^{\infty} \mathbb{P}(Q_1 = x, Q_{\rm T} > b_2 + x){\rm d}x = \int_{b_1}^{\infty} q_f(\bar{x}){\rm d}x, \quad (17)$$

where $\bar{x} \equiv (x, b_2 + x)$. Define

$$m_1(\bar{b}) := \frac{b_1 + c_1\tau}{\sqrt{\tau}};\ m_2(\bar{b}) := \frac{-b_1 + c_1\tau}{\sqrt{\tau}};\ m_4(\bar{b}) := \frac{-b_1 + (c_1 - 2c_2)\tau}{\sqrt{\tau}}.$$

We directly present the main theorem on tandem queues.

**Theorem 4.2** *For each $b_1, b_2 \geq 0$,*

$$q(\bar{b}) = \frac{c_2}{c_1 - c_2}\Psi(m_1(\bar{b})) + \Psi(m_2(\bar{b}))e^{-2b_1c_1} + \frac{c_1 - 2c_2}{c_1 - c_2}(1 - \Psi(m_4(\bar{b})))e^{-2(b_1(c_1-c_2)+b_2c_2)}.$$

*Proof:* Use (17) in combination with Lemma 4.1. Note that $q_f(\bar{x})$ consists of 6 terms. Let us start with the first term:

$$\int_{b_1}^{\infty} -\frac{\partial \ell_1(\bar{x})}{\partial x}\phi(\ell_1(\bar{x})){\rm d}x = \Psi(\ell_1(\bar{x}))\Big|_{b_1}^{\infty} = -\Psi(m_1(\bar{b})). \quad (18)$$

Similarly, for the second and third term:

$$\int_{b_1}^{\infty}\left(2c_1\Psi(\ell_2(\bar{x}))e^{-2c_1x} + \frac{\partial \ell_2(\bar{x})}{\partial x}\phi(\ell_2(\bar{x}))e^{-2c_1x}\right){\rm d}x = -\Psi(\ell_2(\bar{x}))e^{-2c_1x}\Big|_{b_1}^{\infty} = \Psi(m_2(\bar{b}))e^{-2b_1c_1}. \quad (19)$$

Proceeding with the fourth term:

$$\int_{b_1}^{\infty}\frac{\partial \ell_3(\bar{x})}{\partial x}\phi(l_3(\bar{x}))e^{-2c_2(b_2+x)}{\rm d}x = \int_{b_1}^{\infty}\frac{\partial \ell_3(\bar{x})}{\partial x}\frac{1}{\sqrt{2\pi}}e^{-\frac{\ell_1(\bar{x})^2}{2}}{\rm d}x =$$
$$\int_{b_1}^{\infty}\frac{\partial \ell_1(\bar{x})}{\partial x}\frac{1}{\sqrt{2\pi}}e^{-\frac{\ell_1(\bar{x})^2}{2}}{\rm d}x = -\Psi(\ell_1(\bar{x}))\Big|_{b_1}^{\infty} = \Psi(m_1(\bar{b})); \quad (20)$$

here the first equality in (20) follows from the fact that $\exp(-\ell_3(\overline{x})^2/2)\exp(-2c_2(b_2+x)) = \exp(-\ell_1(\overline{x})^2/2)$, whereas the second equality holds due to $\partial\ell_3(\overline{x})/\partial x = \partial\ell_1(\overline{x})/\partial x$. We decompose the fifth term into two parts:

$$2(c_1-2c_2)(1-\Psi(\ell_4(\overline{x})))e^{-2(x(c_1-c_2)+b_2c_2)} =$$
$$2(c_1-c_2)(1-\Psi(\ell_4(\overline{x})))e^{-2(x(c_1-c_2)+b_2c_2)} + 2c_2(\Psi(\ell_4(\overline{x}))-1)e^{-2(x(c_1-c_2)+b_2c_2)}.$$

Now, taking the first decomposed fifth term and the sixth term:

$$\int_{b_1}^{\infty}\left(2(1-\Psi(\ell_4(\overline{x})))(c_1-c_2)e^{-2(x(c_1-c_2)+b_2c_2)} - \frac{\partial\ell_4(\overline{x})}{\partial x}\phi(\ell_4(\overline{x}))e^{-2(x(c_1-c_2)+b_2c_2)}\right)\mathrm{d}x =$$
$$-(1-\Psi(\ell_4(\overline{x})))e^{-2(x(c_1-c_2)+b_2c_2)}\Big|_{b_1}^{\infty} = (1-\Psi(m_4(\overline{b})))e^{-2(b_1(c_1-c_2)+b_2c_2)}. \tag{21}$$

We are left with the second decomposed fifth term:

$$\int_{b_1}^{\infty} 2c_2(\Psi(\ell_4(\overline{x}))-1)e^{-2(x(c_1-c_2)+b_2c_2)}\mathrm{d}x =$$
$$\frac{c_2}{c_1-c_2}\int_{b_1}^{\infty} 2(c_1-c_2)(\Psi(\ell_4(\overline{x}))-1)e^{-2(x(c_1-c_2)+b_2c_2)}\mathrm{d}x =$$
$$\frac{c_2}{c_1-c_2}\Psi(m_1(\overline{b})) - \frac{c_2}{c_1-c_2}(1-\Psi(m_4(\overline{b})))e^{-2(b_1(c_1-c_2)+b_2c_2)}; \tag{22}$$

here the second equality in (22) is obtained by applying integration by parts, but requires tedious calculus. Adding up (18), (19), (20), (21), and (22) yields the stated. □

*Remark:* For $b_1 > 0$ and $b_2 = 0$, we find $q(b_1,0) = \mathbb{P}(Q_1 > b_1) = \exp(-2b_1c_1)$ in Theorem 4.2, i.e., the well-known exponential distribution with mean $1/(2c_1)$. For $b_1 = 0$ and $b_2 > 0$, Theorem 4.2 yields

$$q(0,b_2) = \mathbb{P}(Q_2 > b_2) = \frac{c_1}{c_1-c_2}\Psi\left(\frac{c_1}{\sqrt{c_1-c_2}}\sqrt{b_2}\right)+\frac{c_1-2c_2}{c_1-c_2}e^{-2b_2c_2}\left(1-\Psi\left(\frac{c_1-2c_2}{\sqrt{c_1-c_2}}\sqrt{b_2}\right)\right),$$

which is in line with Theorem 4.3 in [3].

## 4.2 Exact large-buffer asymptotics

In this subsection we derive the exact asymptotics of the joint buffer content distribution. Define

$$\alpha_+ := \frac{c_1}{2c_1-c_2};\ \alpha_- := \frac{c_1-2c_2}{2c_1-3c_2}.$$

It can be verified that $0 < \alpha_- < \alpha_+ < 1$ if $c_1 > 2c_2$, and $0 < \alpha_+ < 1$ if $c_1 \le 2c_2$. Recall that $\zeta(x) = \left(\sqrt{2\pi}x\right)^{-1}\exp(-x^2/2)$. First we present the counterpart of Lemma 3.2.

**Lemma 4.3** *Let* $b_1 = \alpha b$ *and* $b_2 = (1-\alpha)b$, *with* $\alpha \in [0,1]$. *If* $b \to \infty$, *then*

$$\Psi(m_1(\overline{b})) \sim \zeta(m_1(\overline{b}));$$

$$\Psi(m_2(\bar{b})) \sim \begin{cases} \zeta(m_2(\bar{b})) & \textit{if } \alpha < \alpha_+; \\ 1/2 & \textit{if } \alpha = \alpha_+; \\ 1 & \textit{otherwise}; \end{cases}$$

$$1 - \Psi(m_4(\bar{b})) \sim \begin{cases} 1 & \textit{if } \alpha < \alpha_- \textit{ and } c_1 > 2c_2; \\ 1/2 & \textit{if } \alpha = \alpha_- \textit{ and } c_1 \geq 2c_2; \\ -\zeta(m_4(\bar{b})) & \textit{otherwise}. \end{cases}$$

*Proof:* The proof is as in Lemma 3.2. □

Define

$$\theta(\bar{b}) := \frac{1}{\sqrt{2\pi}}\left(\frac{c_2}{c_1 - c_2}\frac{1}{m_1(\bar{b})} + \frac{1}{m_2(\bar{b})} - \frac{c_1 - 2c_2}{c_1 - c_2}\frac{1}{m_4(\bar{b})}\right) \text{ and } \delta(\bar{b}) := \frac{(b_1(c_1 - c_2) + b_2 c_1)^2}{2b_2(c_1 - c_2)}. \tag{23}$$

The following equalities can shown to hold true:

$$\exp\left(-\frac{m_1(\bar{b})^2}{2}\right) = \exp\left(-\frac{m_2(\bar{b})^2}{2}\right)\exp\left(-2b_1c_1\right) =$$
$$\exp\left(-\frac{m_4(\bar{b})^2}{2}\right)\exp\left(-2(b_1(c_1 - c_2) + b_2c_2)\right) = \exp\left(-\delta(\bar{b})\right). \tag{24}$$

The proof of the following two theorems is similar to the proof of Theorem 3.3, but now requires Lemma 4.3 and Equations (23) and (24). We omit the proofs.

**Theorem 4.4** *Let $b_1 = \alpha b$ and $b_2 = (1-\alpha)b$, with $\alpha \in [0,1]$. Suppose that $c_1 > 2c_2$. For $b \to \infty$,*

$$q(\bar{b}) \sim \begin{cases} \frac{c_1 - 2c_2}{c_1 - c_2} e^{-2(b_1(c_1-c_2)+b_2c_2)} & \textit{if } \alpha \in [0, \alpha_-); \\ \frac{1}{2}\frac{c_1 - 2c_2}{c_1 - c_2} e^{-2(b_1(c_1-c_2)+b_2c_2)} & \textit{if } \alpha = \alpha_-; \\ \theta(\bar{b})e^{-\delta(\bar{b})} & \textit{if } \alpha \in (\alpha_-, \alpha_+); \\ \frac{1}{2}e^{-2b_1c_1} & \textit{if } \alpha = \alpha_+; \\ e^{-2b_1c_1} & \textit{if } \alpha \in (\alpha_+, 1]. \end{cases}$$

**Theorem 4.5** *Let $b_1 = \alpha b$ and $b_2 = (1-\alpha)b$, with $\alpha \in [0,1]$. Suppose that $c_1 \leq 2c_2$. For $b \to \infty$,*

$$q(\bar{b}) \sim \begin{cases} \theta(\bar{b})e^{-\delta(\bar{b})} & \textit{if } \alpha \in [0, \alpha_+); \\ \frac{1}{2}e^{-2b_1c_1} & \textit{if } \alpha = \alpha_+; \\ e^{-2b_1c_1} & \textit{if } \alpha \in (\alpha_+, 1]. \end{cases}$$

*Remark:* We note that for $c_1 < 2c_2$ and $b_1 = 0$ ($\alpha = 0$) the asymptotics are not given by $\theta(\bar{b})\exp(-\delta(\bar{b}))$, as it can be verified that $\theta(\bar{b})$ equals 0 in this special case. Therefore we have to rely here on the sharper asymptotic $\left(\sqrt{2\pi}u\right)^{-1}\exp(-u^2/2) - \Psi(u) \sim \left(\sqrt{2\pi}u^3\right)^{-1}\exp(-u^2/2)$. Using this, it can be shown [3] that

$$q(0, b_2) \sim \frac{1}{\sqrt{2\pi}}\left(\frac{c_1 - c_2}{b_2}\right)^{3/2}\frac{4c_2}{c_1^2(c_1 - 2c_2)^2}e^{-\frac{c_1^2}{2(c_1-c_2)}b_2}.$$

### 4.3 Most probable path

Similar to the parallel system, the large-buffer asymptotics now depend on the model parameters $\alpha$, $c_1$ and $c_2$. Again, we will interpret the corresponding regimes by determining the structure of the MPPs.

We feed $n$ i.i.d. standard Brownian sources into the tandem system, and also scale the link rates and buffer thresholds by $n$: $nc_1$, $nc_2$, $nb_1$ and $nb_2$ respectively. By using (6), we can write

$$q_n(\bar{b}) := \mathbb{P}(Q_{1,n} > nb_1, Q_{2,n} > nb_2) = \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} B_i(\cdot) \in U\right).$$

Clearly, $U \subseteq U^* \subseteq V$, with

$$\begin{aligned} U^* &:= \{f \in F | \exists t \geq 0 : \exists s \in (0,t] : -f(-s) > b_1 + c_1 s, f(-s) - f(-t) > b_2 + c_2 t - c_1 s\}; \\ V &:= \{f \in F | \exists t \geq 0 : \exists s \in [0,t] : -f(-s) > b_1 + c_1 s, -f(-t) > b_1 + b_2 + c_2 t\}. \end{aligned}$$

Hence, 'Schilder' gives

$$K(\bar{b}) := -\lim_{n\to\infty} \frac{1}{n} \log q_n(\bar{b}) = \inf_{f\in U} I(f) \geq \inf_{f \in V} I(f). \tag{25}$$

Let the MPP in set $V$ be denoted by $f^*$. If $f^* \in U$, then there is clearly equality in (25).

**Theorem 4.6** *Suppose $c_1 > 2c_2$. Then it holds that*

$$K(\bar{b}) = \begin{cases} 2(b_1(c_1 - c_2) + b_2 c_2) & \textit{if } b_1/(b_1+b_2) \in [0, \alpha_-]; \\ \delta(\bar{b}) & \textit{if } b_1/(b_1+b_2) \in (\alpha_-, \alpha_+); \\ 2b_1 c_1 & \textit{if } b_1/(b_1+b_2) \in [\alpha_+, 1]. \end{cases}$$

*Proof:* Consider Theorem 3.10 with $c_{\mathrm{I}} = c_1$, $c_{\mathrm{II}} = c_2$, $b_{\mathrm{I}} = b_1$ and $b_{\mathrm{II}} = b_1 + b_2$, i.e., we have $U \subseteq V = S$. The MPPs (in $S = V$) corresponding to each of the regimes of Theorem 3.10 were derived in Section 3.3. It can easily be checked that these MPPs are also contained in $U$, and consequently they are the MPPs in $U$. This implies that $K(\bar{b})$ is given by Theorem 3.10. □

**Theorem 4.7** *Suppose $c_1 \leq 2c_2$. Then it holds that*

$$K(\bar{b}) = \begin{cases} \delta(\bar{b}) & \textit{if } b_1/(b_1+b_2) \in [0, \alpha_+); \\ 2b_1 c_1 & \textit{if } b_1/(b_1+b_2) \in [\alpha_+, 1]. \end{cases}$$

*Proof:* Consider Theorem 3.11 with $c_{\mathrm{I}} = c_1$, $c_{\mathrm{II}} = c_2$, $b_{\mathrm{I}} = b_1$ and $b_{\mathrm{II}} = b_1 + b_2$. Again, the MPPs corresponding to the second and third regime of Theorem 3.11, are also contained in set $U$. However, the MPP corresponding to the first regime is not contained in $U$, so we need a different approach here. In order to obtain a workload in queue 2 at least as large as $b_2$ at time 0, queue 2 needs to start building up at $-\tau = -b_2/(c_1 - c_2)$ at the latest. Set $U$ can now be rewritten as

$$\left\{ f \in F \middle| \exists t \geq \tau : \exists s \in (0,t] : \forall u \in (0,s] : \begin{array}{c} -f(-s) > b_1 + c_1 s, \\ f(-u) - f(-t) > b_2 + c_2 t - c_1 u \end{array} \right\},$$

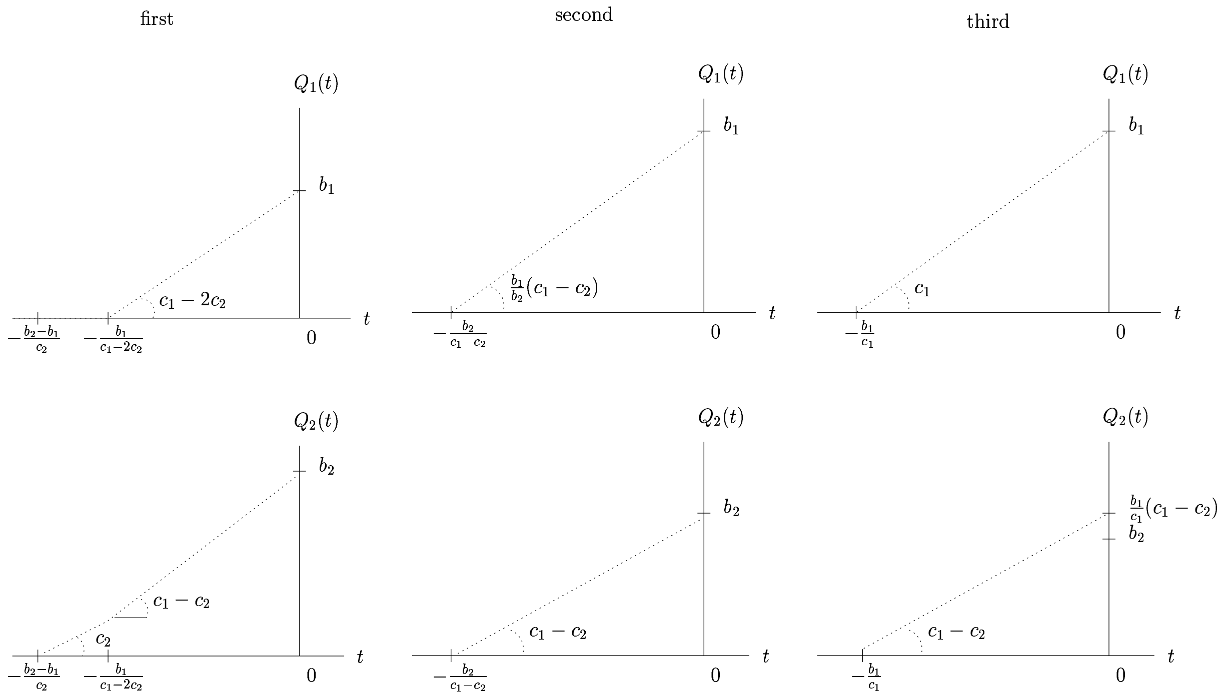


Figure 3: The most probable storage path in $\{Q_1 \geq b_1, Q_2 \geq b_2\}$ corresponding to each of the regimes of Theorem 4.6. The most probable storage path corresponding to each of the two regimes of Theorem 4.7, is also given by the most probable storage paths of the last two regimes of Theorem 4.6.

which is contained in

$$\{f \in F | \exists t \geq \tau : \exists s \in (0,t] : -f(-s) > b_1 + c_1 s, -f(-t) > b_1 + b_2 + c_2 t\} =: W.$$

Using the results of Section 3.3, with $b_{\rm I} = b_1$, $b_{\rm II} = b_1 + b_2$, $c_{\rm I} = c_1$ and $c_{\rm II} = c_2$, one can show that if $b_1/(b_1 + b_2) \in [0, \alpha_+)$ and $c_1 \leq 2c_2$, then the MPP in $W$ is given by (16). As (16) is contained in $U$, it is also the MPP in $U$, implying that $K(\bar{b}) = \delta(\bar{b})$. □

Figure 3 depicts for each of the regimes of Theorem 4.6 the most likely way the buffers fill. Clearly, the most likely way the buffers fill for each of the two regimes of Theorem 4.7, coincides with the most probable storage paths of the last two regimes of Theorem 4.6.

*Remark:* If we set $b_1 > 0$ and $b_2 = 0$, then Theorems 4.6 and 4.7 give $K(\bar{b}) = 2b_1c_1$, which indeed is the exponential decay rate of the overflow probability in single queue with standard Brownian input, emptied at rate $c_1$. For $b_1 = 0$ and $b_2 > 0$, Theorems 4.6 and 4.7 yield

$$K(\bar{b}) = \begin{cases} 2b_2c_2 & \text{if } c_1 > 2c_2; \\ \frac{c_1^2}{2(c_1-c_2)}b_2 & \text{otherwise,} \end{cases}$$

which is in line with Section 4.1 in Mandjes & Van Uitert [9].

### 4.4 Discussion

As in the two-node parallel queue, we can derive the logarithmic large-buffer asymptotics by using Theorems 4.4 and 4.5. That is,

$$-\lim_{b\to\infty}\frac{1}{b}\log\mathbb{P}(Q_1>\alpha b, Q_2>(1-\alpha)b) =: K^*(\bar{b}_\alpha) \quad \text{with} \quad a\in[0,1],\ b\geq 0,$$

where $\bar{b}_\alpha \equiv (\alpha b, (1-\alpha)b)$. With $b_1=\alpha b$ and $b_2=(1-\alpha)b$, i.e., $\bar{b}_\alpha=\bar{b}$, it is not hard to see that $K^*(\bar{b}_\alpha)$ and $K(\bar{b})$ coincide; compare Theorems 4.6 and 4.7 with Theorems 4.4 and 4.5, respectively.

Again the results can also be generalized immediately to general Brownian input. Assuming that $c_1>c_2>\mu>0$, this is done by setting $c_i \leftarrow (c_i-\mu)/\sqrt{\lambda}$ and $b_i \leftarrow b_i/\sqrt{\lambda}$, $i=1,2$.

## 5 Conclusions

In this paper we analyzed a two-node tandem queue with Brownian input. We obtained the joint distribution function of the workload of the first and second queue, large-buffer asymptotics, and the most probable path leading to overflow. These results were derived by first considering the closely related two-node parallel queue, for which similar results were obtained.

Future research directions include: (1) Analysis of the joint overflow probability in a two-class Generalized Processor Sharing (GPS) system with Brownian inputs. (2) Extending the results obtained in this paper to other input processes. The main approach used in this paper relies on the fact that Brownian motions are characterized by independent increments. Therefore, we expect our approach to be also valid for other input processes that have independent increments (and an LDP), e.g., light-tailed Lévy processes.

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